Differential calculus on algebras and graphs

Alexander Grigor'yan*
Department of Mathematics
University of Bielefeld
33501 Bielefeld, Germany

Yuri Muranov†
Department of Mathematics
Grodno State University
Grodno 230023, Republic of Belarus

September 2012

Abstract

We give a detailed survey on abstract differential calculi on associative algebras and its application for construction of a formalism of differential forms on finite directed graphs

Contents

*Partially supported by SFB 701 of German Research Council
†Partially supported by the CONACyT Grant 98697 and SFB 701 of German Research Council

1 Introduction

This paper deals with the notion of a differential calculus on an abstract associative unital algebra $\mathcal{A}$ over a commutative unital ring $\mathbb{K}$. The classical differential calculus of first order is defined by the algebra $\mathcal{A} = C^{\infty}(\mathbb{R})$ of smooth functions over $\mathbb{R}$ and the operation of ordinary differentiation. An example of a differential calculus of higher order is provided by the graded algebra of differential forms on a smooth manifold with the operation of exterior differentiation. A distinctive feature of differentiation in the both cases is the product rule that becomes a defining property for an abstract differential calculus.

In fact, our interest to this notion is motivated by a modest desire to develop a calculus of differential forms on finite graphs (rather, *directed* graphs). A direct construction of such calculus was given in [6], where it was used to develop a homology theory on finite graphs. In this paper we stay on more algebraic point of view of [1], [2], which makes some constructions more functorial and gives an opportunity to use more effectively methods of homological algebra [10].

This approach to homologies on digraph is not new and was used by A.Dimakis and F.Müller-Hoissen in a series of publications, in particular, in [4] and [5]. However, a detailed account of the necessary algebraic background for this approach seems to be missing in the literature.

In Section 2 of this paper we give a detailed survey of classical results on abstract differential calculi on associative algebras in the form that is adapted to further application to graphs. We give a systematical description of the subject and provide several technical theorems which are based on the classical algebraic results (see [1], [8], and [10]) which will be helpful in the next sections. Starting with a standard construction of a first order calculus from [1], we give two methods for construction of higher order universal differential calculi and prove their equivalence. We also show that all possible differential calculi on a given algebra $\mathcal{A}$ can be obtained by taking a quotient of the universal differential calculus over a graded ideal.

In Section 3, we define the differential calculus on the algebra of functions on a finite set and describe its basic properties.

In Section 4, we define the calculus on simple finite digraphs. We use the algebraic machinery developed in previous sections and prove that we have a functor from the category of digraphs with inclusion maps to the category of differential calculi with morphisms of the calculi. We describe some homology properties of these calculi and prove among others the following cohomology realization theorem:

> *for any finite collection of nonnegative integers $k_0, k_1, \dots k_n$ with $k_0 \geq 1$, there exists a finite digraph G such that the cohomology groups of its differential calculus satisfies the conditions*
>
> $$\dim H^i(\Omega_G) = k_i \quad \text{for all } 0 \leq i \leq n.$$

In Section 5 we consider a category of monotonic graphs and transfer to this case the results of previous section. We describe a sufficiently wide class of graphs for which the differential calculus admits geometrical realization in terms of simplicial complexes.

2 Differential calculus on algebras

In this section we provide a self-contained introduction to differential calculus on associative algebras. Most of the topics of this section can be found in a number of textbooks, such as [1], [10]. We present this material in the form that is convenient for further applications.

2.1 Associative unital algebras

Let $\mathbb{K}$ be a commutative unital ring and $\mathcal{A}$ be an associative unital algebra over $\mathbb{K}$. The latter means that $\mathcal{A}$ is a unital ring with operations addition $a, b \mapsto a + b$ and multiplication $a, b \mapsto ab$, where $a, b \in \mathcal{A}$, and at the same time $\mathcal{A}$ is a left module over $\mathbb{K}$ with the same operation addition and with multiplication by a scalar $k, a \mapsto ka$, for $k \in \mathbb{K}$ and $a \in \mathcal{A}$; the two multiplications are compatible in the following sense:

$$(k_1 a_1)(k_2 a_2) = (k_1 k_2)(a_1 a_2) \tag{2.1}$$

for all $k_1, k_2 \in \mathbb{K}$ and $a_1, a_2 \in \mathcal{A}$. The unity of $\mathbb{K}$ will be denoted by $1_{\mathbb{K}}$ and the unity of $\mathcal{A}$ – by $1_{\mathcal{A}}$.

For example, if X is any set and $\mathcal{A}$ is the set of all $\mathbb{K}$-valued functions on X, then $\mathcal{A}$ has a natural structure of an associative unital algebra over $\mathbb{K}$, where addition and multiplication on functions are defined pointwise.

Our purpose is to construct the notion of exterior derivation on $\mathcal{A}$. We will define a graded algebra $\Omega_{\mathcal{A}} = \oplus_{p \geq 0} \Omega^p_{\mathcal{A}}$ with $\Omega^0_{\mathcal{A}} = \mathcal{A}$ and operators $d : \Omega^p_{\mathcal{A}} \to \Omega^{p+1}_{\mathcal{A}}$ that possess natural properties of differentiation. For example, if $\mathcal{A}$ is the algebra of smooth real-valued functions on a smooth manifold, then $\Omega^p_{\mathcal{A}}$ amounts to the space of differential forms of order p, and d is the exterior derivative.

The core of this program – construction of $\Omega^1_{\mathcal{A}}$, is taken from [1]. Construction of $\Omega^p_{\mathcal{A}}$ for all p was sketched in [2], [3], [9] as well as in [4], [5], but without details that we add here.

This construction is based on the notion of a tensor product of bimodules that is described below.

2.2 Tensor product of bimodules

Let R be a ring. We say that A is a R-bimodule if A is both left and right R-module, and the left and right multiplication by the elements of R are related by the identity

$$(sa)\, r = s\, (ar) \tag{2.2}$$

for all $a \in A$ and $r, s \in R$.

Let A and B be R-bimodules, and let us define the notion of their tensor product $A \otimes_R B$.

Definition 2.1 Denote by F a free abelian group generated by the ordered pairs $a \otimes b$ with $a \in A$ and $b \in B$. The tensor product $A \otimes_R B$ is defined as the quotient group

$$A \otimes_R B = F/G,$$

where G is the subgroup of F generated by the elements of the form

$$\begin{array}{l} (a_1 + a_2) \otimes b - (a_1 \otimes b + a_2 \otimes b) \\ a \otimes (b_1 + b_2) - (a \otimes b_1 + a \otimes b_2) \\ (ar) \otimes b - a \otimes (rb) \end{array} \tag{2.3}$$

for all $a, a_1, a_2 \in A,\ b, b_1, b_2 \in B,\ \ r \in R.$

The elements of $A \otimes_R B$ are also the finite sums of the terms $a \otimes b$, but subject to the following relations

$$\begin{array}{c} (a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b \\ a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 \\ (ar) \otimes b = a \otimes (rb)\,, \end{array} \tag{2.4}$$

for all $a, a_1, a_2 \in A,\ b, b_1, b_2 \in B,\ \ r \in R.$

It follows from the third identity in (2.4) that

$$a \otimes 0 = a \otimes 00 = a0 \otimes 0 = 0 \otimes 0$$

and similarly $0 \otimes b = 0 \otimes 0$. Consequently, we have

$$a \otimes b + 0 \otimes 0 = a \otimes b + a \otimes 0 = a \otimes (b + 0) = a \otimes b$$

so that $0 \otimes 0$ is a neutral element of the abelian group $A \otimes_R B$.

Define in F the left and right multiplication by the elements of R as follows. Set first

$$r\,(a \otimes b) := (ra) \otimes b \quad \text{and} \quad (a \otimes b)\, r = a \otimes (br) \tag{2.5}$$

for all $r \in R, a \in A, b \in B$, and then extend this operation to all the elements of F by additivity. It is easy to verify that $rG \subset G$ and $Gr \subset G$. Hence, multiplications (2.5) are well-defined on $A \otimes_R B$, and thus making $A \otimes_R B$ into R-bimodule.

It is clear from (2.4), (2.5) that the mapping

$$(a, b) \mapsto a \otimes_R b, \quad a \in A, b \in B,$$

defines a R-bilinear mapping from $A \times B$ to $A \otimes_R B$.

We shall call an R-bimodule A by an R-module, if the left and right R-module structures of A are identical, that is,

$$ra = ar \tag{2.6}$$

for all $a \in A$ and $r \in R$. If A, B are R-modules then also $A \otimes_R B$ is a R-module since by (2.4), (2.5), (2.6)

$$r\,(a \otimes b) = (ra) \otimes b = (ar) \otimes b = a \otimes (rb) = a \otimes (br) = (a \otimes b)\, r$$

for all $a \in A, b \in B,\ r \in R$.

Let S be another ring, and assume that A is both R- and S-bimodule. We say that the two bimodule structures of A are compatible if for all $a \in A, r \in R$ and $s \in S$

$$(sa)\, r = s\,(ar) \quad \text{and} \quad (ra)\, s = r\,(as)\,.$$

If each A, B is R- and S-bimodules with compatible structures then $A \otimes_R B$ can be regarded also as a S-bimodule, where S-bimodule structure on $A \otimes_R B$ is defined by the identities (2.5) for all $r \in S, a \in A, b \in B$. It follows from (2.5) that the structures of R- and S-bimodules of $A \otimes_R B$ are compatible.

Let each A, B, C be R- and S-bimodules and compatible structures. Then the following associative law for tensor products is satisfied:

$$(A \otimes_R B) \otimes_S C = A \otimes_R (B \otimes_S C)\,.$$

Indeed, the elements of the both sides are the finite sums of the terms $a \otimes b \otimes c$ with $a \in A, b \in B, c \in C$ subject to the following generating relations generalizing (2.4): the distributive laws with respect to a, b, c and the identity

$$ar \otimes bs \otimes c = a \otimes rb \otimes sc \tag{2.7}$$

for all $r \in R, s \in S, a \in A, b \in B, c \in C$.

By induction one defines the tensor product of any finite number $A_1, ..., A_n$ of bimodules. Let $R_1, ..., R_{n-1}$ be rings, and assume that each A_i is an R_j-bimodule for all $i = 1, ..., n$ and $j = 1, ..., n-1$, and that all R_j-bimodule structures are compatible. Then $A_1 \otimes_{R_1} A_2 \otimes_{R_2} ... \otimes_{R_{n-1}} A_n$ is again a R_j-bimodule for each R_j, and all R_j-bimodule structures are compatible. The elements of $A_1 \otimes_{R_1} A_2 \otimes_{R_2} ... \otimes_{R_{n-1}} A_n$ are finite sums of the expressions $a_1 \otimes a_2 \otimes ... \otimes a_n$ with $a_i \in A_i$ subject to the obvious generating relations generalizing (2.4) and (2.7).

2.3 A first order differential calculus

From now on $\mathbb{K}$ is a commutative unital ring and $\mathcal{A}$ is an associative unital algebra over $\mathbb{K}$.

Definition 2.2 A *first order differential calculus* on the algebra $\mathcal{A}$ is a pair (Γ, d) where Γ is an $\mathcal{A}$-bimodule, and $d \colon \mathcal{A} \to \Gamma$ is a $\mathbb{K}$-linear map such that

(*i*) $d(ab) = (da) \cdot b + a \cdot (db)$ for all $a, b \in \mathcal{A}$ (where $\cdot$ denotes multiplication between the elements of $\mathcal{A}$ and Γ).

(*ii*) The minimal left $\mathcal{A}$-module containing $d\mathcal{A}$, coincides with Γ, that is, any element $\gamma \in \Gamma$ can be written in the form

$$\gamma = \sum_i a_i \cdot db_i \tag{2.8}$$

with $a_i, b_i \in \mathcal{A}$, where i run over any finite set of indexes.

By [1, III, §10.2], a mapping d satisfying (*i*) is called *a derivation of $\mathcal{A}$ into Γ*. The condition (*i*) implies

$$d1_{\mathcal{A}} = d(1_{\mathcal{A}}1_{\mathcal{A}}) = (d1_{\mathcal{A}})\, 1_{\mathcal{A}} + 1_{\mathcal{A}}\, (d1_{\mathcal{A}}) = 2d1_{\mathcal{A}}$$

and hence $d1_{\mathcal{A}} = 0$. The $\mathbb{K}$-linearity implies then that $d(k1_{\mathcal{A}}) = 0$ for any $k \in \mathbb{K}$.

Example 2.3 Set $\mathcal{A} = C^m(\mathbb{R})$ for some $m \geq 1$, $\Gamma = C^{m-1}(\mathbb{R})$, and let $\cdot$ be a usual multiplication of functions from $\mathcal{A}$ and Γ. Then the ordinary derivative $df := f'$ defines a first order differential calculus on $\mathcal{A}$. Indeed, the condition (*i*) is the classical product rule, while (*ii*) follows from the following observation: any function $\gamma \in C^{m-1}(\mathbb{R})$ can be represented in the form $f \cdot dg$ with C^m-functions $f = 1$ and $g = \int \gamma(x)\, dx$.

Example 2.4 Let $\mathcal{A} = \Gamma = C(\mathbb{R})$ and define the product $\cdot$ between $f \in \mathcal{A}$ and $\gamma \in \Gamma$ as follows:

$$(f \cdot \gamma)(x) = f(x)\gamma(x) \quad \text{and} \quad (\gamma \cdot f)(x) = \gamma(x) f(x+l) \tag{2.9}$$

where $l \in \mathbb{R}$ is a fixed non-zero number. It is easy to see that Γ is indeed a $\mathcal{A}$-bimodule, but its left- and right $\mathcal{A}$-modules are different. The operator

$$df(x) := f(x+l) - f(x)$$

is a derivation of $\mathcal{A}$ into Γ because

$$\begin{aligned} d(fg)(x) &= f(x+l)\, g(x+l) - f(x)\, g(x) \\ &= (f(x+l) - f(x))\, g(x+l) + f(x)\, (g(x+l) - g(x)) \\ &= df(x)\, g(x+l) + f(x)\, dg(x) \\ &= (df \cdot g + f \cdot dg)(x)\,. \end{aligned}$$

Moreover, the operator d satisfies also the condition (*ii*) as any function $\gamma \in C(\mathbb{R})$ can be represented in the form $f \cdot dg$ with C-functions $f = \gamma$ and $g(x) = \dfrac{x}{l}$. Hence, (Γ, d) is a first order differential calculus on $\mathcal{A}$.

Let us describe a construction of the first order differential calculus for a general algebra $\mathcal{A}$. It follows from (2.1) that the mapping

$$\varepsilon \colon \mathbb{K} \to \mathcal{A}, \quad \varepsilon(k) = k1_{\mathcal{A}} \tag{2.10}$$

is an injective homomorphism of algebras $\mathbb{K}$ and $\mathcal{A}$, so that $\mathbb{K}$ can be identified as a subalgebra of $\mathcal{A}$. Taking $a_2 = 1_{\mathcal{A}}$ and $k_1 = 1_{\mathbb{K}}$ in (2.1) we obtain $ka = ak$, that is, the elements from $\mathbb{K}$ commute with elements from $\mathcal{A}$. In particular, $\mathcal{A}$ can be regarded as a $\mathbb{K}$-module. Then the tensor product $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$ is also defined as a $\mathbb{K}$-module.

Note that $\mathcal{A}$ has a natural structure of $\mathcal{A}$-bimodule using the algebra multiplication in $\mathcal{A}$, and the $\mathcal{A}$- and $\mathbb{K}$-bimodule structures of $\mathcal{A}$ are obviously compatible. Therefore, $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$ has also a compatible structure of an $\mathcal{A}$-bimodule. We will denote by $\cdot$ the product of the elements of $\mathcal{A}$ by those of $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$, so that

$$c \cdot (a \otimes b) = (ca) \otimes b \quad \text{and} \quad (a \otimes b) \cdot c = a \otimes (bc) \tag{2.11}$$

for all $a, b, c \in \mathcal{A}$.

Notation 2.5 *In what follows we will always denote $\otimes_{\mathbb{K}}$ simply by $\otimes$. Later on we will consider also tensor product over $\mathcal{A}$ where the full notation $\otimes_{\mathcal{A}}$ will be used.*

Define the following operator

$$d\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \qquad da := 1_{\mathcal{A}} \otimes a - a \otimes 1_{\mathcal{A}}, \tag{2.12}$$

and observe that it satisfies the product rule. Indeed, it follows from (2.11) and (2.13) that

$$\begin{aligned} da \cdot b &= (1_{\mathcal{A}} \otimes a - a \otimes 1_{\mathcal{A}}) \cdot b = 1_{\mathcal{A}} \otimes (ab) - a \otimes b \\ a \cdot db &= a \cdot (1_{\mathcal{A}} \otimes b - b \otimes 1_{\mathcal{A}}) = a \otimes b - (ab) \otimes 1_{\mathcal{A}} \end{aligned}$$

whence

$$da \cdot b + a \cdot db = 1_{\mathcal{A}} \otimes (ab) - (ab) \otimes 1_{\mathcal{A}} = d\,(ab)\,.$$

Hence, d is a derivation from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{A}$. Now we reduce the $\mathcal{A}$-bimodule $\mathcal{A} \otimes \mathcal{A}$ to obtain a first order differential calculus.

Definition 2.6 Define $\Omega^1_{\mathcal{A}}$ as the minimal left $\mathcal{A}$-submodule of $\mathcal{A} \otimes \mathcal{A}$ containing $d\mathcal{A}$. In other words, $\Omega^1_{\mathcal{A}}$ consists of all finite sums of the elements of $\mathcal{A} \otimes \mathcal{A}$ of the form $a \cdot db$ with $a, b \in \mathcal{A}$ (cf. (2.8))

In particular, we have $d\mathcal{A} \subset \Omega^1_{\mathcal{A}}$ so that d can be regarded as a mapping from $\mathcal{A}$ to $\Omega^1_{\mathcal{A}}$.

Proposition 2.7 *$\Omega^1_{\mathcal{A}}$ is a $\mathcal{A}$-bimodule and, hence, $(\Omega^1_{\mathcal{A}}, d)$ is a first order differential calculus on $\mathcal{A}$.*

Proof. Let $u \in \Omega^1_{\mathcal{A}}$ and $c \in \mathcal{A}$. We need to prove that $c \cdot u$ and $u \cdot c$ belong to $\Omega^1_{\mathcal{A}}$. By definition of $\Omega^1_{\mathcal{A}}$, it suffices to verify this for $u = a \cdot db$ where $a, b \in \mathcal{A}$. Then

$$c \cdot u = (ca) \cdot db \in \Omega^1_{\mathcal{A}}$$

and

$$u\cdot c=(a\cdot db)\cdot c=a\cdot(db\cdot c)=a\cdot(d\,(bc)-b\cdot dc)=a\cdot d\,(bc)-(ab)\cdot dc\in\Omega^1_{\mathcal{A}}.$$

Hence, $\Omega^1_{\mathcal{A}}$ satisfies all the requirements of Definition 2.2. ■

Let us give an alternative equivalent description of $\Omega^1_{\mathcal{A}}$. Define a $\mathbb{K}$-linear map

$$\mu:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A},\quad \mu\left(\sum_i a_i\otimes b_i\right)=\sum_i a_ib_i \tag{2.13}$$

where i run over a finite index set. By (2.11) the map μ is a homomorphism of $\mathcal{A}$-bimodules. It follows from (2.11), (2.12) and (2.13) that, for all $a,b\in\mathcal{A}$,

$$\mu\,(a\cdot db)=\mu\,(a\otimes b)-\mu\,(ab\otimes 1_{\mathcal{A}})=ab-ab=0,$$

so that $a\cdot db\in\ker\mu$ and, hence, $\Omega^1_{\mathcal{A}}\subset\ker\mu$. In fact, the following is true.

Theorem 2.8 *[1, III, §10.10]*
(i) We have the identity $\Omega^1_{\mathcal{A}}=\ker\mu$, *where* μ *is defined by* (2.13).
(ii) For every differential calculus of first order (Γ,d') *over the algebra* $\mathcal{A}$ *there exists exactly one epimorphism* p *of* $\mathcal{A}$*-bimodules*

$$p\colon\Omega^1_{\mathcal{A}}\to\Gamma$$

such that the following diagram is commutative

$$\begin{array}{ccc}\mathcal{A} & \xrightarrow{d} & \Omega^1_{\mathcal{A}}\\ \updownarrow^{\mathrm{id}} & & \downarrow^{p}\\ \mathcal{A} & \xrightarrow{d'} & \Gamma\end{array} \tag{2.14}$$

that is, $d'=p\circ d$.

Definition 2.9 The pair $(\Omega^1_{\mathcal{A}},d)$ is called the *universal* first order differential calculus on $\mathcal{A}$.

Example 2.10 As in Example 2.3, consider the $\mathbb{R}$-algebras $\mathcal{A}=C^m(\mathbb{R})$ and $\Gamma=C^{m-1}(R)$ and the usual derivative of functions f from $\mathcal{A}$ that will be denoted by $d'f$. Let us describe explicitly the epimorphism $p:\Omega^1_{\mathcal{A}}\to\Gamma$ from Theorem 2.8(ii). Define a mapping $p:\mathcal{A}\otimes\mathcal{A}\to\Gamma$ by

$$p\,(f\otimes g)=\frac{1}{2}\,(fg'-f'g)$$

and extend it additively to all elements of $\mathcal{A}\otimes\mathcal{A}$. Let us show that $p|_{\Omega^1_{\mathcal{A}}}$ is a $\mathcal{A}$-bimodule homomorphism. By Theorem 2.8(i) any element ω of $\Omega^1_{\mathcal{A}}$ has the form

$$\omega=\sum_i\,(f_i\otimes g_i) \tag{2.15}$$

where $f_i, g_i \in \mathcal{A}$ and

$$\sum_i f_i g_i = 0.$$

Therefore, for any $a \in \mathcal{A}$ we have

$$\begin{aligned} p(a \cdot \omega) &= p\left(\sum_i (af_i \otimes g_i)\right) = \frac{1}{2}\sum_i \left(af_i g_i' - (af_i)' g_i\right) \\ &= \frac{1}{2}\sum_i (af_i g_i' - af_i' g_i) - \frac{1}{2}\sum_i a' f_i g_i = ap(\omega) \end{aligned}$$

and similarly $p(\omega \cdot a) = p(\omega) a$.

Observe that $f \otimes g - g \otimes f \in \Omega^1_{\mathcal{A}}$ and

$$p(f \otimes g - g \otimes f) = \frac{1}{2}(fg' - f'g) - \frac{1}{2}(gf' - g'f) = (fg)'. \tag{2.16}$$

Since $(fg)'$ can be any function from Γ, we see that $p : \Omega^1_{\mathcal{A}} \to \Gamma$ is an epimorphism.

Finally, for any $f \in \mathcal{A}$ we have by (2.16)

$$(p \circ d) f = p(1 \otimes f - f \otimes 1) = f'$$

so that $p \circ d$ is the ordinary first order derivative on $\mathcal{A}$.

Example 2.11 Generalizing the previous example, let M be a smooth manifold, $\mathcal{A}$ be the $\mathbb{R}$-algebra of C^∞ functions on M, and Γ be linear space of all first order differential forms on M. Clearly, Γ is also a $\mathcal{A}$-module. Denote by D the exterior derivative acting from $\mathcal{A}$ to Γ, so that (Γ, D) is a first order differential calculus on $\mathcal{A}$. Then the mapping $p : \Omega^1_{\mathcal{A}} \to \Gamma$ that satisfies (2.14), is given by

$$p(f \otimes g) = \frac{1}{2}(fDg - (Df) g),$$

which is proved in the same way as in Example 2.10.

Example 2.12 As in Example 2.4, consider $\mathbb{R}$-algebras $\mathcal{A} = \Gamma = C(\mathbb{R})$ and the operator $d' : \mathcal{A} \to \Gamma$ defined by

$$d' f(x) = f(x + l) - f(x)$$

for some non-zero real l. Let us describe explicitly the epimorphism $p : \Omega^1_{\mathcal{A}} \to \Gamma$ from Theorem 2.8. Indeed, first define a linear mapping $p : \mathcal{A} \otimes \mathcal{A} \to \Gamma$ by

$$p(f \otimes g)(x) = f(x) g(x + l)$$

for all $f, g \in \mathcal{A}$. Observe that p is a $\mathcal{A}$-bimodule homomorphism as for any $a \in \mathcal{A}$

$$p(a \cdot (f \otimes g)) = p((af) \otimes g) = a(x) f(x) g(x + l) = a \cdot p(f \otimes g)$$

and

$$p\left((f\otimes g)\cdot a\right)=p\left(f\otimes(ga)\right)=f\left(x\right)g\left(x+l\right)a\left(x+l\right)=p\left(f\otimes g\right)\cdot a.$$

Now we restrict p to $\Omega^1_{\mathcal{A}}$ and show that $p\left(\Omega^1_{\mathcal{A}}\right)=\Gamma$. Indeed, for the element $f\cdot dg\in\Omega^1_{\mathcal{A}}$ we have

$$\begin{aligned} f\cdot dg &= f\cdot\left(1_{\mathcal{A}}\otimes g-g\otimes 1_{\mathcal{A}}\right)\\ &= f\otimes g-(fg)\otimes 1_{\mathcal{A}} \end{aligned}$$

and

$$p\left(f\cdot df\right)=f\left(x\right)g\left(x+l\right)-f\left(x\right)g\left(x\right).$$

Setting $g\left(x\right)=\dfrac{x}{l}$ we obtain $p\left(f\cdot dg\right)=f$ whence the claim follows.

Finally, we have for any $f\in\mathcal{A}$

$$\left(p\circ d\right)f=p\left(1\otimes f-f\otimes 1\right)=1\left(x\right)f\left(x+l\right)-f\left(x\right)1\left(x+l\right)=d'f$$

whence $d'=f\circ d$ follows.

2.4 Higher order differential calculus

Let us pass to construction of a higher order differential calculus on $\mathcal{A}$. We start with the following two definitions.

Definition 2.13 A graded unital algebra Λ over a commutative unital ring $\mathbb{K}$ is an associative unital $\mathbb{K}$-algebra that can be written as a direct sum

$$\Lambda=\bigoplus_{p=0,1,\ldots}\Lambda^p$$

of $\mathbb{K}$-modules Λ^p with the following conditions: the unity 1_Λ of Λ belongs to Λ^0 and

$$u\in\Lambda^p,\quad v\in\Lambda^q\quad\Rightarrow\quad u*v\in\Lambda^{p+q},$$

where $*$ denotes multiplication in Λ. If $u\in\Lambda^p$ then p is called the *degree* of u and is denoted by $\deg u$.

The operation of multiplication in a graded algebra is called an *exterior* (or a graded) multiplication. A homomorphism $f\colon\Lambda'\to\Lambda''$ of two graded unital $\mathbb{K}$-algebras Λ' and Λ'' is a homomorphism of $\mathbb{K}$-algebras that preserves degree of elements.

Definition 2.14 A differential calculus on an associative unital $\mathbb{K}$-algebra $\mathcal{A}$ is a couple (Λ,d), where Λ is a graded algebra

$$\Lambda=\bigoplus_{p=0,1,\ldots}\Lambda^p$$

over $\mathbb{K}$ such that $\Lambda^0=\mathcal{A}$, and $d:\Lambda\to\Lambda$ is a $\mathbb{K}$-linear map, such that (*i*) $d\Lambda^p \subset \Lambda^{p+1}$

(*ii*) $d^2 = 0$

(*iii*) $d(u * v) = (du) * v + (-1)^p u * (dv)$, for all $u \in \Lambda^p$, $v \in \Lambda^q$, where $*$ is the exterior multiplication in Λ;

(*iv*) the minimal left $\mathcal{A}$-submodule of Λ^{p+1} containing $d\Lambda^p$ coincides with Λ^{p+1}, that is, any $w \in \Lambda^{p+1}$ can be represented as a finite sum of the form

$$w = \sum_k a_k * dv_k \tag{2.17}$$

for some $a_k \in \mathcal{A}$ and $v_k \in \Lambda^p$.

The property (*iii*) in Definition 2.14 is called the *Leibniz rule* or the *product rule.*

A classical example of a differential calculus is the calculus of exterior differential forms on a smooth manifold with the wedge product and with the exterior derivation. This calculus is based on the algebra $\mathcal{A}$ of smooth functions on the manifold.

The following property of a differential calculus will be frequently used.

Lemma 2.15 *Let* (Λ, d) *be differential calculus on* $\mathcal{A}$*. Then for any* $p \geq 0$ *any element* $w \in \Lambda^p$ *can be written as a finite sum*

$$w = \sum_j a_0^j * da_1^j * da_2^j * \cdots * da_p^j, \tag{2.18}$$

where $a_i^j \in \mathcal{A}$ *for all* $0 \leq i \leq p$ *and* $*$ *is the exterior multiplication in* Λ.

Proof. Representation (2.18) for $p = 0$ is true by $\Lambda^0 = \mathcal{A}$. Let us make an inductive step from $p-1$ to p. By part (*iv*) of Definition 2.14, it suffices to show the existence of the representation (2.18) for $w = a * dv$ with $a \in \mathcal{A}$ and $v \in \Lambda^{p-1}$. By the inductive hypothesis, v admits the representation in the form

$$v = \sum_j a_1^j * da_2^j * da_2^j * \cdots * da_p^j$$

where all $a_i^j \in \mathcal{A}$. Using the associative law, the Leibniz rule and $d^2 = 0$, we obtain

$$dv = \sum_j da_1^j * da_2^j * da_2^j * \cdots * da_p^j,$$

whence (2.18) follows with $a_0^i = a$. ∎

2.4.1 First construction

The first method of construction a differential calculus on $\mathcal{A}$ uses multiple tensor products $\otimes_{\mathbb{K}}$ of $\mathcal{A}$ by itself as in the following definition.

Definition 2.16 Given an arbitrary associative unital $\mathbb{K}$-algebra $\mathcal{A}$, define a graded $\mathbb{K}$-algebra T as follows:

$$T = \bigoplus_{p=0,1,\dots} T^p,$$

where

$$T^p = \begin{cases} \mathcal{A}, & p = 0 \\ \underbrace{\mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{p \text{ times } \otimes}, & p \geq 1, \end{cases} \tag{2.19}$$

and the exterior multiplication $T^p \bullet T^q \longrightarrow T^{p+q}$ is defined by

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_p) \bullet (b_0 \otimes b_1 \otimes \cdots \otimes b_j) := a_0 \otimes a_1 \otimes \cdots \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_q, \tag{2.20}$$

for all $a_i, b_j \in \mathcal{A}$.

It is a trivial exercise to check that the multiplication $\bullet$ is well-defined and that T is indeed a graded associative unital $\mathbb{K}$-algebra with the unity $1_T = 1_{\mathcal{A}}$. The multiplication $\bullet$ by elements of $\mathcal{A} = T^0$ endows each $\mathbb{K}$-module T^p by a structure of $\mathcal{A}$-bimodule.

Note, that the original multiplication in the algebra $\mathcal{A}$ coincides with the exterior multiplication $T^0 \bullet T^0 \to T^0$, and the multiplication $\cdot$ of the elements of $\mathcal{A} = T_0$ and $\mathcal{A} \otimes \mathcal{A} = T_1$ defined in (2.11), coincides with exterior multiplication $T^0 \bullet T^1 \to T^1$.

Define a $\mathbb{K}$-linear map $d\colon T^p \to T^{p+1}$ $(p \geq 0)$ by a formula

$$d(a_0 \otimes \cdots \otimes a_p) = \sum_{i=0}^{p+1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes 1_{\mathcal{A}} \otimes a_i \otimes \cdots \otimes a_p, \tag{2.21}$$

for all $a_i \in \mathcal{A}$. In particular, for any $a \in \mathcal{A} = T_0$ we have

$$da = 1_{\mathcal{A}} \otimes a - a \otimes 1_{\mathcal{A}}$$

that matches the previous definition (2.12). Also, for all $a, b \in \mathcal{A}$ we have

$$d\,(a \otimes b) = 1_{\mathcal{A}} \otimes a \otimes b - a \otimes 1_{\mathcal{A}} \otimes b + a \otimes b \otimes 1_{\mathcal{A}}. \tag{2.22}$$

It is easy to verify that the operator d is well-defined.

Proposition 2.17 *For the operator* (2.21) *we have* $d^2 = 0$. *In particular,* d *determines the following cochain complex*

$$0 \longrightarrow T^0 \xrightarrow{d} T^1 \xrightarrow{d} T^2 \longrightarrow \dots$$

of $\mathbb{K}$*-modules.*

Proof. Straightforward computation. ■

Remark 2.18 *The homomorphism $\varepsilon\colon \mathbb{K} \to \mathcal{A}$ defined in (2.10) evidently satisfies the property $d \circ \varepsilon = 0$. Hence we can equip the cochain complex T^* by the augmentation ε. We shall denote this cochain complex with the augmentation ε by $\widetilde{T}^{\varepsilon}$.*

Proposition 2.19 *The map d defined in (2.21) satisfies the following product rule:*

$$d(u \bullet v) = du \bullet v + (-1)^p u \bullet dv \tag{2.23}$$

for all $u \in T^p$ and $v \in T^q$.

Proof. It suffices to prove (2.23) for $u = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in T^p$ and $v = b_0 \otimes b_1 \otimes \cdots \otimes b_q \in T^q$. We have

$$\begin{aligned} d(u \bullet v) &= d(a_0 \otimes a_1 \otimes \cdots \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_q) \\ &= \sum_{j=0}^{p} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes 1_{\mathcal{A}} \otimes a_j \otimes \cdots \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_q \\ &\quad + (-1)^{p+1} a_0 \otimes \cdots \otimes a_{p-1} \otimes a_p b_0 \otimes 1_{\mathcal{A}} \otimes b_1 \otimes \cdots \otimes b_q \\ &\quad + \sum_{i=2}^{q+1} (-1)^{p+i} a_0 \otimes \cdots \otimes \otimes a_{p-1} \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes 1_{\mathcal{A}} \otimes b_i \cdots \otimes b_q \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & du \bullet v + (-1)^p u \bullet dv \\ = & \sum_{j=0}^{p} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes 1_{\mathcal{A}} \otimes a_j \otimes \cdots \otimes a_p) \bullet (b_0 \otimes b_1 \otimes \cdots \otimes b_q) \\ & + (-1)^{p+1} (a_0 \otimes \cdots \otimes a_p \otimes 1_{\mathcal{A}}) \bullet (b_0 \otimes b_1 \otimes \cdots \otimes b_q) \qquad [\text{term with } j = p+1] \\ & + (-1)^p (a_0 \otimes \cdots \otimes a_p) \bullet (1_{\mathcal{A}} \otimes b_0 \otimes \cdots \otimes b_q) \qquad [\text{term with } i = 0] \\ & + (-1)^p (a_0 \otimes \cdots \otimes a_p) \bullet (-1) (b_0 \otimes 1_{\mathcal{A}} \otimes b_1 \otimes \cdots \otimes b_q) \qquad [\text{term with } i = 1] \\ & + (-1)^p (a_0 \otimes \cdots \otimes a_p) \bullet \sum_{i=2}^{q+1} (-1)^i b_0 \otimes \cdots \otimes b_{i-1} \otimes 1_{\mathcal{A}} \otimes b_i \otimes \cdots \otimes b_q \ . \end{aligned}$$

Noticing that the terms with $j = p+1$ and $i = 0$ cancel out, we obtain the required identity. ■

Now we reduce the graded algebra T introduced above, to obtain a differential calculus in the sense of Definition 2.14.

Definition 2.20 Set $\Omega^0_{\mathcal{A}} = \mathcal{A} = T^0$. For all integers $p \geq 0$, define inductively $\Omega^{p+1}_{\mathcal{A}}$ as the minimal left $\mathcal{A}$-submodule of T^{p+1} containing $d\Omega^p_{\mathcal{A}}$, that is, $\Omega^{p+1}_{\mathcal{A}}$ consists of all the elements of the form *(2.17)* for some $a_k \in \mathcal{A}$ and $v_k \in \Omega^p_{\mathcal{A}}$.

It follows from this definition that each $\Omega^p_{\mathcal{A}}$ is a $\mathbb{K}$-module and that $d\Omega^p_{\mathcal{A}} \subset \Omega^{p+1}_{\mathcal{A}}$. Clearly, for $p = 1$ Definition 2.20 is consistent with previous Definition 2.6.

Theorem 2.21 *For all $p, q \geq 0$*

$$u \in \Omega^p_{\mathcal{A}},\ v \in \Omega^q_{\mathcal{A}} \ \Rightarrow \ u \bullet v \in \Omega^{p+q}_{\mathcal{A}}. \tag{2.24}$$

Consequently, the direct sum

$$\Omega_{\mathcal{A}} = \bigoplus_{p=0,1,...} \Omega^p_{\mathcal{A}},$$

with the multiplication $\bullet$ and with differential d is a differential calculus on $\mathcal{A}$.

Applying (2.24) with $q = 0$, we obtain that $\Omega^p_{\mathcal{A}}$ is also a right $\mathcal{A}$-module, that is, $\Omega^p_{\mathcal{A}}$ is an $\mathcal{A}$-bimodule.

Proof. The proof by induction in p. For $p = 0$ the statement is trivial, as by definition $\Omega^q_{\mathcal{A}}$ is a left $\mathcal{A}$-module. Let us make an inductive step from $p-1$ to p. It suffices to prove that $u \bullet v \in \Omega^{p+q}_{\mathcal{A}}$ for $u = a \bullet db$ where $a \in \mathcal{A}$ and $b \in \Omega^{p-1}_{\mathcal{A}}$. We have by the associative law and by the Leibniz rule

$$\begin{aligned} u \bullet v &= (a \bullet db) \bullet v = a \bullet ((db) \bullet v) \\ &= a \bullet [d(b \bullet v) + (-1)^p\, b \bullet dv] \\ &= a \bullet d(b \bullet v) + (-1)^p (a \bullet b) \bullet dv. \end{aligned}$$

By the inductive hypothesis we have $b \bullet v \in \Omega^{p+q-1}_{\mathcal{A}}$ whence $d(b \bullet v) \in \Omega^{p+q}_{\mathcal{A}}$ and $a \bullet d(b \bullet v) \in \Omega^{p+q}_{\mathcal{A}}$. Also, we have $a \bullet b \in \Omega^{p-1}_{\mathcal{A}}$ and $dv \in \Omega^{q+1}_{\mathcal{A}}$, whence by the inductive hypothesis $(a \bullet b) \bullet dv \in \Omega^{p+q}_{\mathcal{A}}$. It follows that $u \bullet v \in \Omega^{p+q}_{\mathcal{A}}$.

Finally, $(\Omega_{\mathcal{A}}, d)$ satisfies all the conditions of Definition 2.14 by Propositions 2.17, 2.19, Definition 2.20 and by (2.24). Hence, $(\Omega_{\mathcal{A}}, d)$ is a differential calculus on $\mathcal{A}$. ■

2.4.2 Second construction

Now let us describe a different construction of the differential calculus on $\mathcal{A}$ that is based on the first order differential calculus $\Omega^1_{\mathcal{A}}$ from Definition 2.6. Define for each $p \geq 0$ a $\mathcal{A}$-bimodule $\widetilde{\Omega}^p_{\mathcal{A}}$ by

$$\widetilde{\Omega}^0_{\mathcal{A}} = \mathcal{A}, \qquad \widetilde{\Omega}^p_{\mathcal{A}} = \underbrace{\Omega^1_{\mathcal{A}} \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}} \otimes_{\mathcal{A}} ... \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}}}_{p-1 \text{ times } \otimes_{\mathcal{A}}} \text{ for } p \geq 1. \tag{2.25}$$

In particular, $\widetilde{\Omega}^1_{\mathcal{A}} = \Omega^1_{\mathcal{A}}$. Clearly, each $\widetilde{\Omega}^p_{\mathcal{A}}$ is also a $\mathbb{K}$-module. Define the following multiplication $\star$ between the elements $u \in \widetilde{\Omega}^p_{\mathcal{A}}$ and $v \in \widetilde{\Omega}^q_{\mathcal{A}}$:

$$u \star v = \begin{cases} u \cdot v, & \text{if } p = 0 \text{ or } q = 0 \\ u \otimes_{\mathcal{A}} v, & \text{if } p, q \geq 1, \end{cases} \tag{2.26}$$

where $\cdot$ denotes the multiplication in $\widetilde{\Omega}^k_{\mathcal{A}}$ by the elements of $\mathcal{A}$ that comes from the $\mathcal{A}$-bimodule structure of $\widetilde{\Omega}^k_{\mathcal{A}}$. Clearly, multiplication $\star$ is associative, has a unity $1_{\mathcal{A}}$, and makes the direct sum

$$\widetilde{\Omega}_{\mathcal{A}} = \bigoplus_{p=0,1,...} \widetilde{\Omega}^p_{\mathcal{A}}$$

into a graded $\mathbb{K}$-algebra. It turns out that the graded algebras $\widetilde{\Omega}_{\mathcal{A}}$ and $\Omega_{\mathcal{A}}$ (cf. Definition 2.20) are isomorphic as is stated below.

Theorem 2.22 *(i) There exists an unique isomorphism*

$$f\colon \widetilde{\Omega}_{\mathcal{A}} \to \Omega_{\mathcal{A}} \tag{2.27}$$

of graded $\mathbb{K}$-algebras given by $\mathcal{A}$-bimodule isomorphisms

$$f_p\colon \widetilde{\Omega}^p_{\mathcal{A}} \to \Omega^p_{\mathcal{A}}, \quad p \geq 0,$$

where $f_0\colon \mathcal{A} \to \mathcal{A}$ and $f_1\colon \Omega^1_{\mathcal{A}} \to \widetilde{\Omega}^1_{\mathcal{A}}$ are identical maps.

(ii) Define an operator $\widetilde{d} : \widetilde{\Omega}^p_{\mathcal{A}} \to \widetilde{\Omega}^{p+1}_{\mathcal{A}}$ to make the following diagram commutative:

$$\begin{array}{ccc} \widetilde{\Omega}^p_{\mathcal{A}} & \xrightarrow{\widetilde{d}} & \widetilde{\Omega}^{p+1}_{\mathcal{A}} \\ \downarrow f_p & & \downarrow f_{p+1} \\ \Omega^p_{\mathcal{A}} & \xrightarrow{d} & \Omega^{p+1}_{\mathcal{A}} \end{array} \tag{2.28}$$

Then $(\widetilde{\Omega}_{\mathcal{A}}, \widetilde{d})$ is a differential calculus that is isomorphic to $(\Omega_{\mathcal{A}}, d)$.

Clearly, the operators d and $\widetilde{d}$ on $\mathcal{A}$ are the same. As in the proof of Lemma 2.15 we obtain that any element of $\widetilde{\Omega}^p_{\mathcal{A}}$ can be represented as a finite sum of the terms

$$a_0 \star \widetilde{d}a_1 \star ... \star \widetilde{d}a_p,$$

and the following identity holds:

$$\widetilde{d}\left(a_0 \star \widetilde{d}a_1 \star ... \star \widetilde{d}a_p\right) = \widetilde{d}a_0 \star \widetilde{d}a_1 \star ... \star \widetilde{d}a_p.$$

Proof. We will use the following property of the tensor product:

$$\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathcal{A} \tag{2.29}$$

where $\cong$ stands for a $\mathcal{A}$-bimodule isomorphism. Indeed, consider the mapping

$$\varphi : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}, \quad \varphi(a) = a \otimes 1_{\mathcal{A}} \tag{2.30}$$

that is clearly homomorphism of $\mathcal{A}$-bimodules. This mapping is injective for the following reason. By Definition 2.1, $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}$ is the quotient group F/G where F is a free abelian group generated by all symbols $a \otimes b$, and G is the subgroup generated by the relations (2.3). Consider the mapping $\mu : F \to \mathcal{A}$ given by

$$\mu\left(\sum_i a_i \otimes b_i\right) = \sum_i a_i b_i.$$

It is clear from the inspection of (2.3) that $\mu(u) = 0$ for all $u \in G$. However, $\mu(a \otimes 1_{\mathcal{A}}) = a \neq 0$ for any non-zero $a \in \mathcal{A}$. Hence, $a \otimes 1_{\mathcal{A}} \notin G$ and, hence, $a \otimes 1_{\mathcal{A}}$ represents a non-zero element of F/G. Next, the mapping φ is surjective as for all $a, b \in \mathcal{A}$ we have

$$a \otimes b = a \otimes (b 1_{\mathcal{A}}) = (ab) \otimes 1_{\mathcal{A}} = \varphi(ab).$$

Hence, φ is a $\mathcal{A}$-bimodule isomorphism between $\mathcal{A}$ and $\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A}$.

In order to construct a mapping f as in (2.27), define first a $\mathcal{A}$-bimodule $\widetilde{T}^p$ by

$$\begin{aligned}\widetilde{T}^0 &= \mathcal{A}\\ \widetilde{T}^p &= \underbrace{(\mathcal{A}\otimes\mathcal{A})\otimes_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})\otimes_{\mathcal{A}}...\otimes_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})}_{p-1 \text{ times } \otimes_{\mathcal{A}}}, \quad p\geq 1.\end{aligned}$$

Since $\Omega^1_{\mathcal{A}}$ is a sub-module of $\mathcal{A}\otimes\mathcal{A}$, it follows that $\widetilde{\Omega}^p_{\mathcal{A}}$ is a sub-module of $\widetilde{T}^p$.

Recall that $\Omega^p_{\mathcal{A}}$ is a sub-module of T^p where T^p was defined by (2.19). Let us show that, for all $p\geq 0$,

$$\widetilde{T}^p \cong T^p. \tag{2.31}$$

For $p=0$ and $p=1$ it is obvious as

$$\widetilde{T}^0=\mathcal{A}=T^0 \quad \text{and} \quad \widetilde{T}^1=\mathcal{A}\otimes\mathcal{A}=T^1.$$

If (2.31) is already proved for some $p\geq 1$ then we have by the associative law of tensor product, (2.29) and the inductive hypothesis

$$\begin{aligned}\widetilde{T}^{p+1} &= \widetilde{T}^p\otimes_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})\\ &\cong T^p\otimes_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})\\ &= (T^{p-1}\otimes\mathcal{A})\otimes_{\mathcal{A}}(\mathcal{A}\otimes\mathcal{A})\\ &= T^{p-1}\otimes(\mathcal{A}\otimes_{\mathcal{A}}\mathcal{A})\otimes\mathcal{A}\\ &\cong T^{p-1}\otimes\mathcal{A}\otimes\mathcal{A}\\ &= T^{p+1},\end{aligned}$$

which proves the inductive step.

Denote by f_p the mapping from $\widetilde{T}^p$ to T^p that provides the isomorphism (2.31). For $p=0,1$ the mappings f_p are identity mappings. It follows from the above computation and (2.30) that for $p\geq 2$ and for

$$u=(a_1\otimes b_1)\otimes_{\mathcal{A}}(a_2\otimes b_2)\otimes_{\mathcal{A}}...\otimes_{\mathcal{A}}(a_p\otimes b_p)\in\widetilde{T}^p \tag{2.32}$$

where $a_i,b_i\in\mathcal{A}$, we have

$$f_p(u)=a_1\otimes b_1a_2\otimes b_2a_3\otimes...\otimes b_{p-1}a_p\otimes b_p\in T^p. \tag{2.33}$$

Set

$$\widetilde{T}=\bigoplus_{p=0,1,...}^{p}\widetilde{T}^p$$

and define the exterior multiplication $\star$ in $\widetilde{T}$ by (2.26), so that $\widetilde{T}$ becomes a graded $\mathbb{K}$-algebra. Set $f=\oplus_{p\geq 0}f_p$ and show that the mapping $f:\widetilde{T}\to T$ is an isomorphism of the graded algebras $\widetilde{T}$ and T (cf. Definition 2.16). It suffices to verify that

$$f(u\star v)=f(u)\bullet f(v) \tag{2.34}$$

for all $u, v \in \widetilde{T}$. Let $u \in \widetilde{T}^p$ and $v \in \widetilde{T}^q$. If $p = 0$, that is, $u \in \mathcal{A}$, then $u \star v = u \cdot v$ and

$$f(u \star v) = f(u \cdot v) = u \cdot f(v) = f(u) \bullet f(v).$$

The same argument works for $q = 0$. For $p = 1$ it suffices to prove Assume now that $p \geq 1$ and $q \geq 1$. It suffices to verify (2.34) for u as in (2.32) and for

$$v = (\alpha_1 \otimes \beta_1) \otimes_{\mathcal{A}} (\alpha_2 \otimes \beta_2) \otimes_{\mathcal{A}} ... \otimes_{\mathcal{A}} \left(\alpha_q \otimes \beta_q\right)$$

where $\alpha_j, \beta_j \in \mathcal{A}$. Then by (2.26) and (2.33)

$$f(u \star v) = a_1 \otimes b_1 a_2 \otimes ... \otimes b_{p-1} a_p \otimes b_p \alpha_1 \otimes \beta_1 \alpha_2 \otimes ... \otimes \beta_q$$

whereas by (2.20)

$$\begin{aligned} f(u) \bullet f(v) &= (a_1 \otimes b_1 a_2 \otimes ... \otimes b_{p-1} a_p \otimes b_p) \bullet \left(\alpha_1 \otimes \beta_1 \alpha_2 \otimes ... \otimes \beta_{q-1} \alpha_q \otimes \beta_q\right) \\ &= a_1 \otimes b_1 a_2 \otimes ... \otimes b_{p-1} a_p \otimes b_p \alpha_1 \otimes \beta_1 \alpha_2 \otimes ... \otimes \beta_q, \end{aligned}$$

which proves (2.34).

Let us prove that the restriction of f to $\widetilde{\Omega}_{\mathcal{A}}$ provides an isomorphism of the graded algebras $\widetilde{\Omega}_{\mathcal{A}}$ and $\Omega_{\mathcal{A}}$, that is,

$$f(\widetilde{\Omega}^p_{\mathcal{A}}) = \Omega^p_{\mathcal{A}}.$$

For $p = 0, 1$ it is clear. Assume $p \geq 2$. By Lemma 2.15 any element of $\Omega^p_{\mathcal{A}}$ can be written as a finite sum of the terms

$$w = v_1 \bullet v_2 \bullet \cdots \bullet v_p$$

where $v_i \in \Omega^1_{\mathcal{A}}$. For the element

$$v := v_1 \star v_2 \star ... \star v_p \in \widetilde{\Omega}^p_{\mathcal{A}}$$

we have by (2.34) and $f|_{\Omega^1_{\mathcal{A}}} = \mathrm{id}$

$$f(v) = f(v_1) \bullet f(v_2) \bullet ... \bullet f(v_p) = v_1 \bullet v_2 \bullet \cdots \bullet v_p = w,$$

which implies the inclusion

$$f(\widetilde{\Omega}^p_{\mathcal{A}}) \supset \Omega^p_{\mathcal{A}}.$$

Let us prove the opposite inclusion. By definition (2.25) of $\widetilde{\Omega}^p_{\mathcal{A}}$, any element of $\widetilde{\Omega}^p_{\mathcal{A}}$ is a finite sum of the terms

$$v = v_1 \star v_2 \star ... \star v_p$$

where $v_i \in \Omega^1_{\mathcal{A}}$. As above we have

$$f(v) = v_1 \bullet v_2 \bullet ... \bullet v_p \tag{2.35}$$

that belongs to $\Omega^p_{\mathcal{A}}$ by Theorem 2.21, whence

$$f(\widetilde{\Omega}^p_{\mathcal{A}}) \subset \Omega^p_{\mathcal{A}}.$$

The last argument proves also the uniqueness of the isomorphism (2.27) of the graded algebras $\widetilde{\Omega}_{\mathcal{A}}$ and $\Omega_{\mathcal{A}}$. Indeed, since f_0 and f_1 must be the identical maps, they are uniquely determined, and the uniqueness of f_p follows from (2.35).

Finally, the claim (ii) is a trivial consequence of (i). ∎

2.4.3 Universality

Theorem 2.23 *The differential calculus*

$$(\Omega_{\mathcal{A}}, d) \cong (\widetilde{\Omega}_{\mathcal{A}}, \widetilde{d})$$

has the following universal property. For any other differential calculus (Λ, d') *over* $\mathcal{A}$*, there exists one and only one epimorphism* $p : \Omega_{\mathcal{A}} \to \Lambda$ *of graded* $\mathcal{A}$*-algebras given by*

$$p = \bigoplus_k p_k, \quad p_k \colon \Omega^k_{\mathcal{A}} \to \Lambda_k$$

with $p_0 = \mathrm{id}$ *and such that, for all* $k \geq 0$*, the following diagram is commutative:*

$$\begin{array}{ccc} \Omega^k_{\mathcal{A}} & \xrightarrow{d} & \Omega^{k+1}_{\mathcal{A}} \\ \downarrow^{p_k} & & \downarrow^{p_{k+1}} \\ \Lambda^k & \xrightarrow{d'} & \Lambda^{k+1} \end{array}$$

Proof. Denote by $*$ the exterior multiplication in Λ. By Lemma 2.15 any element $w \in \Lambda^k$ with $k \geq 1$ can be written as a finite sum

$$w = \sum_j a^j_0 * da^j_1 * da^j_2 * \cdots * da^j_k \tag{2.36}$$

where $a^j_l \in \mathcal{A}$ for all $0 \leq l \leq k$. Consider a graded algebra

$$\widetilde{\Lambda} = \bigoplus_{k=0,1,\dots} \widetilde{\Lambda}^k,$$

where $\widetilde{\Lambda}^k$ for $k \leq 1$ is defined by

$$\widetilde{\Lambda}^0 = \mathcal{A}, \qquad \widetilde{\Lambda}^1 = \Lambda^1$$

and for $k \geq 2$ by

$$\widetilde{\Lambda}^k = \underbrace{\widetilde{\Lambda}^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \widetilde{\Lambda}^1}_{k \text{ factors}}.$$

The exterior multiplication $\star$ in $\widetilde{\Lambda}$ is defined as in (2.26). The condition (2.36) implies that the maps p_0 and p_1 induce an epimorphism $q : \widetilde{\Lambda} \longrightarrow \Lambda$ of the graded algebras, where

$$q = \bigoplus_{k=0,1,\dots} q_k$$

and $q_k \colon \widetilde{\Lambda}^k \longrightarrow \Lambda^k$ are defined as follows: q_0 and q_1 are the identity mappings, while for $k \geq 2$ the mapping q_k is defined by

$$q_k(w_1 \star w_2 \star \cdots \star w_k) = q_1(w_1) * q_1(w_2) * \cdots * q_1(w_k) \in \Lambda^k$$

for all $w_i \in \widetilde{\Lambda}^1$. Let $f_0 = \mathrm{Id}$. By Theorems 2.8 and 2.22 we have an unique epimorphism $f_1 = p$ of $\mathcal{A}$-bimodules making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\widetilde{d}} & \widetilde{\Omega}^1_{\mathcal{A}} \\ \downarrow^{f_0} & & \downarrow^{f_1} \\ \mathcal{A} & \xrightarrow{d'} & \widetilde{\Lambda}^1 \end{array} \tag{2.37}$$

The diagram (2.37) induces an epimorphism $f : \widetilde{\Omega}_{\mathcal{A}} \to \widetilde{\Lambda}$ of graded algebras given by

$$f = \bigoplus_{k=0,1,2\ldots} f_k$$

where for $k \geq 2$ the mapping

$$f_k \colon \underbrace{\widetilde{\Omega}^1_{\mathcal{A}} \star \cdots \star \widetilde{\Omega}^1_{\mathcal{A}}}_{k \text{ factors}} \longrightarrow \underbrace{\widetilde{\Lambda}^1 \star \cdots \star \widetilde{\Lambda}^1}_{k \text{ factors}}$$

is defined by

$$f_k(w_1 \star \cdots \star w_k) = f_1(w_1) \star \cdots \star f_1(w_k) \in \widetilde{\Lambda}^k$$

for all $w_i \in \widetilde{\Omega}^1_{\mathcal{A}}$.Thus, we obtain an epimorphism $p : \widetilde{\Omega}_{\mathcal{A}} \longrightarrow \Lambda$ of graded algebras defined by

$$p = \bigoplus_{k=0,1,\ldots} p_k = \bigoplus_{k=0,1,\ldots} q_k \circ f_k \colon$$

such that $p_0 = \mathrm{Id}$ and $p_1 = p$.

To finish the prove of the theorem we must check the commutativity of the diagram

$$\begin{array}{ccc} \Omega^k_{\mathcal{A}} & \xrightarrow{d} & \Omega^{k+1}_{\mathcal{A}} \\ \downarrow^{p_k} & & \downarrow^{p_{k+1}} \\ \Lambda^k & \xrightarrow{d'} & \Lambda^{k+1} \end{array} \tag{2.38}$$

for all $k \geq 0$. By Theorem 2.22 we can identify in the first line of (2.38) the graded algebra $\Omega_{\mathcal{A}}$ with $\widetilde{\Omega}_{\mathcal{A}}$ and d with $\widetilde{d}$. Let us prove by induction in k that this diagram is commutative. For $k = 0$ this is true by Theorem 2.8. Inductive step from $k-1$ to k assuming $k \geq 1$. It suffices to check the commutativity of (2.38) only on the elements $w \in \Omega^k_{\mathcal{A}}$ of the form $w = a \bullet dv$, where $a \in \mathcal{A}$ and $v \in \Omega^{k-1}_{\mathcal{A}}$. Since p is a homomorphism of graded algebras, the inductive hypothesis and $d'^2 = 0$, we obtain

$$d'p_k(a \bullet dv) = d'(a \star p_k(dv)) = d'(a \star d'p_{k-1}(v)) = d'a \star d'p_{k-1}(v).$$

On the other side, using the Leibniz rule and the inductive hypothesis, we obtain

$$p_{k+1}d(a \bullet dv) = p_{k+1}\left(da \bullet dv\right) = p_1\left(da\right) \star p_k\left(dv\right) = d'a \star d'p_{k-1}(v).$$

The comparison the above two lines proves that the diagram (2.38) is commutative. ■

Corollary 2.24 *Under the hypotheses of Theorem 2.23, there exists a two-sided graded ideal*

$$\mathcal{J} = \bigoplus_{l=1,2,\ldots} \mathcal{J}^l, \quad \mathcal{J}_l \subset \Omega_{\mathcal{A}}$$

of the graded algebra $\Omega_{\mathcal{A}}$ *such that*

$$\Lambda^k = \Omega^k_{\mathcal{A}} / \mathcal{J}^k\,, \quad \Omega_{\mathcal{A}}\mathcal{J}\Omega_{\mathcal{A}} \subseteq \mathcal{J}, \quad d\mathcal{J}^k \subset \mathcal{J}^{k+1} \text{ for all } \ k \geq 0, \quad \text{and } \mathcal{J}^0 = \{0\}. \tag{2.39}$$

Furthermore, the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \ldots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & \mathcal{J}^1 & \xrightarrow{d} & \mathcal{J}^2 & \xrightarrow{d} \ldots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega^0_{\mathcal{A}} & \xrightarrow{d} & \Omega^1_{\mathcal{A}} & \xrightarrow{d} & \Omega^2_{\mathcal{A}} & \xrightarrow{d} \ldots \\
& \downarrow^{p_0} & & \downarrow^{p_1} & & \downarrow^{p_2} & \\
0 \longrightarrow & \Lambda^0 & \xrightarrow{d'} & \Lambda^1 & \xrightarrow{d'} & \Lambda^2 & \xrightarrow{d'} \ldots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \ldots
\end{array} \tag{2.40}$$

where the mappings $\mathcal{J}^k \to \Omega^k_{\mathcal{A}}$ *are the identical inclusions. In diagram (2.40) the rows are chain complexes of* $\mathbb{K}$*-modules, and the columns are exact sequences of* $\mathbb{K}$*-modules.*

Proof. Indeed, define $\mathcal{J}$ by

$$\mathcal{J}^k = \mathrm{Ker}\{p_k : \Omega^k_{\mathcal{A}} \to \Lambda^k\}.$$

Since p is an epimorphism of graded algebras, the statement follows from the commutativity of the diagram (2.38). ∎

Definition 2.25 The differential calculus $(\Omega_{\mathcal{A}}, d)$ is called *the universal differential calculus* on algebra $\mathcal{A}$.

Proposition 2.26 *Let* $(\Omega_{\mathcal{A}}, d)$ *be the universal differential calculus on algebra* $\mathcal{A}$ *and* $\mathcal{J} \subset \Omega_{\mathcal{A}}$ *be a graded ideal, that satisfies the property* $d\mathcal{J} \subset \mathcal{J}$*. Denote by* $d_{\mathcal{J}}$ *the map of degree one*

$$\Omega_{\mathcal{A}}/\mathcal{J} \to \Omega_{\mathcal{A}}/\mathcal{J}$$

that is induced by d*. Then* $(\Omega_{\mathcal{A}}/\mathcal{J}, d_{\mathcal{J}})$ *is a differential calculus on algebra* $\mathcal{A}$.

Proof. It is easy to check that $d^2_{\mathcal{J}} = 0$ and $d_{\mathcal{J}}$ satisfies the Leibniz rule. ∎

Corollary 2.27 *Under assumptions of Corollary 2.24, we have the following cohomology long exact sequence:*

$$0 \longrightarrow H^0(\Omega_{\mathcal{A}}) \longrightarrow H^0(\Lambda) \longrightarrow H^1(\mathcal{J}) \longrightarrow H^1(\Omega_{\mathcal{A}}) \longrightarrow H^1(\Lambda) \longrightarrow \ldots$$

Proof. This follows from the commutative diagram (2.40) by means of the standard homology algebra [10]. ∎

2.5 Quotient calculi

Theorem 2.28 *Let for all $p \geq 1$ we have a $\mathbb{K}$-linear subspace*

$$\mathcal{E}^p \subset \Omega^p_{\mathcal{A}},$$

such that

$$\mathcal{E} = \bigoplus_{p\geq 1} \mathcal{E}^p$$

is a graded ideal of the exterior algebra $\Omega_{\mathcal{A}}$. Consider a subspace

$$\mathcal{J} = \bigoplus_{p\geq 1} \mathcal{J}^p \subset \Omega_{\mathcal{A}} = \bigoplus_{p\geq 0} \Omega^p_{\mathcal{A}}$$

defined by

$$\mathcal{J}^p = \begin{cases} \mathcal{E}^p, & \text{for } p = 1 \\ \mathcal{E}^p + d\mathcal{E}^{p-1}, & \text{for } p \geq 2. \end{cases}$$

Then

$$\mathcal{J} = \bigoplus_{p\geq 1} \mathcal{J}^p \subset \Omega_{\mathcal{A}}$$

is a graded ideal of algebra $\Omega_{\mathcal{A}}$ such that $d\mathcal{J} \subset \mathcal{J}$. In particular, the inclusion

$$\mathcal{J} \longrightarrow \Omega_{\mathcal{A}}$$

is a morphism of cochain complexes.

Proof. Any element $w \in \mathcal{J}^p$ can be represented in the form

$$w = w_1 + w_2 \tag{2.41}$$

where $w_1 \in \mathcal{E}^p$ and $w_2 = d(v),\ v \in \mathcal{E}^{p-1}$. For $x \in \Omega^i_{\mathcal{A}}, y \in \Omega^j_{\mathcal{A}}$ we have

$$xwy = xw_1y + xw_2y = xw_1y + x\,(dv)\,y.$$

The element xw_1y lies in $\mathcal{E}$, since by our assumption $\mathcal{E}$ is an ideal. Now, using the Leibniz rule, we have

$$d(xvy) = (dx)\,vy + (-1)^i xd(vy) = (dx)\,vy + (-1)^i x\,(dv)\,y + (-1)^i(-1)^{p-1}xv\,(dy)\,,$$

and hence

$$\begin{aligned} x(dv)y &= (-1)^i[d(xvy) - (dx)vy + (-1)^{i+p}xv(dy)] \\ &= (-1)^i d(xvy) + (-1)^{i+1}(dx)vy + (-1)^p xv(dy) \end{aligned}$$

In the last sum

$$(-1)^i d(xvy) \in d\Omega^{i+j+p-1}$$

and two others element lie in Ω^{i+j+p}, since $\mathcal{E}$ is an ideal. Thus we proved that $\mathcal{J}$ is an ideal. For an element w with decomposition (2.41) we have

$$w = dw_1 + dw_2 = dw_1 + d\,(dv) = dw_1 \in d\mathcal{E}^p \in \mathcal{J}^{p+1},$$

which finishes the proof. ■

Corollary 2.29 *Under assumptions of Theorem 2.28 we have a commutative diagram of cochain complexes*

$$\begin{array}{ccccccccc}
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & \mathcal{J}^1 & \stackrel{d}{\longrightarrow} & \mathcal{J}^2 & \stackrel{d}{\longrightarrow} \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega^0_{\mathcal{A}} & \stackrel{d}{\longrightarrow} & \Omega^1_{\mathcal{A}} & \stackrel{d}{\longrightarrow} & \Omega^2_{\mathcal{A}} & \stackrel{d}{\longrightarrow} \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega^0_{\mathcal{A}} & \stackrel{d'}{\longrightarrow} & \Omega^1_{\mathcal{A}}/\mathcal{J}^1 & \stackrel{d'}{\longrightarrow} & \Omega^2_{\mathcal{A}}/\mathcal{J}^2 & \stackrel{d'}{\longrightarrow} \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots
\end{array} \tag{2.42}$$

where the columns are exact sequences of $\mathbb{K}$-modules and the differentials d' are induced by d. Commutative diagram (2.42) induces a long exact sequence

$$0 \longrightarrow H^0(\Omega_{\mathcal{A}}) \longrightarrow H^0(\Omega_{\mathcal{A}}/\mathcal{J}) \longrightarrow H^1(\mathcal{J}) \longrightarrow H^1(\Omega_{\mathcal{A}}) \longrightarrow H^1(\Omega_{\mathcal{A}}/\mathcal{J}) \longrightarrow \dots$$

Corollary 2.30 *Let for any $p \geq 1$, $\mathcal{E}^p \subset \mathcal{F}^p$ be $\mathbb{K}$-linear subspaces of $\Omega^p_{\mathcal{A}}$ such that $\mathcal{E} = \bigoplus_{p\geq 1} \mathcal{E}^p$ and $\mathcal{F} = \bigoplus_{p\geq 1} \mathcal{F}^p$ are graded ideals of the exterior algebra $\Omega_{\mathcal{A}}$. Define $\mathcal{J}^p$ and $\mathcal{I}^p$ by*

$$\mathcal{J}^p = \begin{cases} \mathcal{E}^p, & \text{for } p = 1 \\ \mathcal{E}^p + d\left(\mathcal{E}^{p-1}\right), & \text{for } p \geq 2 \end{cases},$$

and

$$\mathcal{I}^p = \begin{cases} \mathcal{F}^p, & \text{for } p = 1 \\ \mathcal{F}^p + d\left(\mathcal{F}^{p-1}\right), & \text{for } p \geq 2, \end{cases},$$

and set

$$\mathcal{J} = \bigoplus_{p\geq 1} \mathcal{J}^p, \qquad \mathcal{I} = \bigoplus_{p\geq 1} \mathcal{I}^p.$$

Then

$$\mathcal{J}^p \subset \mathcal{I}^p \subset \Omega^p_{\mathcal{A}} ,$$

which induces inclusions of cochain complexes

$$\mathcal{J} \longrightarrow \mathcal{I} \longrightarrow \Omega_{\mathcal{A}}. \tag{2.43}$$

Proof. Indeed, it is easy to check that we have the inclusions $\mathcal{J}^p \subset \mathcal{I}^p \subset \Omega^p_{\mathcal{A}}$ that commute with differentials. ■

Corollary 2.31 *Under assumptions of Corollary 2.30 we have the following short exact sequence of cochain complexes of $\mathbb{K}$-modules*

$$0 \longrightarrow \mathcal{I}/\mathcal{J} \longrightarrow \Omega_{\mathcal{A}}/\mathcal{J} \longrightarrow \Omega_{\mathcal{A}}/\mathcal{I} \longrightarrow 0 \tag{2.44}$$

which can be written in the form of a commutative diagram of $\mathbb{K}$-modules

$$\begin{array}{ccccccccc} 0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & 0 & \longrightarrow & \mathcal{I}^1/\mathcal{J}^1 & \longrightarrow & \mathcal{I}^2/\mathcal{J}^2 & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \Omega^0_{\mathcal{A}} & \longrightarrow & \Omega^1_{\mathcal{A}}/\mathcal{J}^1 & \longrightarrow & \Omega^2_{\mathcal{A}}/\mathcal{J}^2 & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \Omega^0_{\mathcal{A}} & \longrightarrow & \Omega^1_{\mathcal{A}}/\mathcal{I}^1 & \longrightarrow & \Omega^2_{\mathcal{A}}/\mathcal{I}^2 & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \end{array} \tag{2.45}$$

In (2.45) all columns are exact and rows are cochain complexes. All the differentials in (2.45) are induced by differential d. The diagram (2.45) induces a cohomology long exact sequence

$$0 \longrightarrow H^0(\Omega_{\mathcal{A}}/\mathcal{J}) \longrightarrow H^0(\Omega_{\mathcal{A}}/\mathcal{I}) \longrightarrow H^1(\mathcal{I}/\mathcal{J}) \longrightarrow H^1(\Omega_{\mathcal{A}}/\mathcal{J}) \longrightarrow \dots$$

Proof. For $k \geq 1$, by [8, III, §1] we obtain short exact sequences of $\mathbb{K}$-modules

$$0 \longrightarrow \mathcal{I}^k/\mathcal{J}^k \longrightarrow \Omega^k_{\mathcal{A}}/\mathcal{J}^k \longrightarrow \Omega^k_{\mathcal{A}}/\mathcal{I}^k \longrightarrow 0,$$

that are the vertical maps in (2.45). Differentials in (2.45)

$$\mathcal{I}^k/\mathcal{J}^k \longrightarrow \mathcal{I}^{k+1}/\mathcal{J}^{k+1},$$

$$\Omega^k_{\mathcal{A}}/\mathcal{J}^k \longrightarrow \Omega^{k+1}_{\mathcal{A}}/\mathcal{J}^{k+1},$$

and

$$\Omega^k_{\mathcal{A}}/\mathcal{I}^k \longrightarrow \Omega^{k+1}_{\mathcal{A}}/\mathcal{I}^{k+1}$$

are induced by d by passing to quotient space. Checking the commutativity is trivial. ■

2.6 Functorial properties

Now we discuss functorial properties of differential calculi. Consider a category ALG in which objects are associative unital $\mathbb{K}$-algebras and morphisms are homomorphisms of $\mathbb{K}$-algebras.

Definition 2.32 Define a category DC of differential calculi by the following way. An object of DC is a differential calculus $(\Lambda_{\mathcal{A}}, d_{\mathcal{A}})$ on unital associative algebra $\mathcal{A}$ (see Definition 2.14). A morphism

$$\lambda\colon (\Lambda_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow (\Lambda_{\mathcal{B}}, d_{\mathcal{B}})$$

in the category DC is given by a preserving degrees morphism of graded algebras

$$\lambda = \bigoplus_{i=0,1,\dots} \lambda_i\colon \Lambda_{\mathcal{A}} \to \Lambda_{\mathcal{B}},$$

where

$$\lambda_i \colon \Lambda^i_{\mathcal{A}} \to \Lambda^i_{\mathcal{B}},\ i \geq 0,$$
$$\lambda_0 \colon \mathcal{A} \longrightarrow \mathcal{B}$$

is a morphism in the category ALG, and the maps λ_i ($i \geq 0$) are homomorphisms of $\mathbb{K}$-modules which commutes with the differentials, that is

$$\lambda_{i+1} d_{\mathcal{A}} = d_{\mathcal{B}} \lambda_i,\ i \geq 0.$$

The composition of morphisms from category DC is evidently a morphism in DC.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital associative algebras over a commutative unital ring $\mathbb{K}$ and $g \colon \mathcal{A} \to \mathcal{B}$ be a homomorphism. Now we would like to define an induced by g morphism

$$\lambda = \bigoplus_{0,1,\ldots} \lambda_i = \mathcal{U}(g) \colon (\Omega_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow (\Omega_{\mathcal{B}}, d_{\mathcal{B}})$$

of the universal differential calculus $(\Omega_{\mathcal{A}}, d_{\mathcal{A}})$ to the universal differential calculus $(\Omega_{\mathcal{B}}, d_{\mathcal{B}})$.

Let $T_{\mathcal{A}}, T_{\mathcal{B}}$ be graded algebras defined by algebras $\mathcal{A}$, and $\mathcal{B}$ as in Definition 2.16. Let

$$\phi_k \colon T^k_{\mathcal{A}} \to T^k_{\mathcal{B}}, \qquad k \geq 0,$$

be a homomorphism of $\mathbb{K}$-modules (see [1, II, §3.2]) defined by

$$\phi_k(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = g(a_0) \otimes g(a_1) \otimes \cdots \otimes g(a_k).$$

Denote by

$$\phi = \bigoplus_{k=0,1,\ldots} \phi_k \colon T_{\mathcal{A}} = \bigoplus_k T^k_{\mathcal{A}} \longrightarrow T_{\mathcal{B}} = \bigoplus_k T^k_{\mathcal{B}}$$

a graded homomorphism of graded $\mathbb{K}$-modules. The map ϕ is a degree preserving homomorphism of graded algebras, since

$$\begin{aligned} & \phi_{k+l}[(a_0 \otimes a_1 \otimes \cdots \otimes a_k) \bullet (b_0 \otimes b_1 \otimes \cdots \otimes b_l)] \\ = \ & g(a_0) \otimes g(a_1) \otimes \cdots \otimes g(a_k b_0) \otimes g(b_1) \otimes \cdots \otimes g(b_l) \\ = \ & g(a_0) \otimes g(a_1) \otimes \cdots \otimes g(a_k) g(b_0) \otimes g(b_1) \otimes \cdots \otimes g(b_l) \\ = \ & \phi_k(a_0 \otimes a_1 \otimes \cdots \otimes a_k) \bullet \phi_l(b_0 \otimes b_1 \otimes \cdots \otimes b_l\ . \end{aligned}$$

The maps ϕ_k commutes with differentials, since $g(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

Let λ_i ($i \geq 0$) be a restriction

$$\lambda_i = \phi_i|_{\Omega^i_{\mathcal{A}}} \colon \Omega^i_{\mathcal{A}} \longrightarrow T^i_{\mathcal{B}},$$

and set $\lambda = \oplus \lambda_i$.

Proposition 2.33 *The homomorphism of $\mathbb{K}$-modules λ_k is a morphism of differential calculi*

$$(\Omega_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow (\Omega_{\mathcal{B}}.d_{\mathcal{B}})$$

Proof. We must check only that

$$\lambda_k(\Omega^k_{\mathcal{A}}) \subset \Omega^k_{\mathcal{B}}.$$

This follows from the fact that ϕ_k commute with the differentials and from the inductive definition of $\Omega^k_{\mathcal{A}}$, $\Omega^k_{\mathcal{B}}$ as in Definition 2.20. ■

Theorem 2.34 *We can assign to any associative unital $\mathbb{K}$-algebra $\mathcal{A}$ a universal differential calculus $\mathcal{U}(\mathcal{A}) = (\Omega_{\mathcal{A}}, d_{\mathcal{A}})$ and to homomorphism $g\colon \mathcal{A} \to \mathcal{B}$ of such algebras a morphism*

$$\lambda = \mathcal{U}(g)\colon (\Omega_{\mathcal{A}}, d_{\mathcal{A}}) \longrightarrow (\Omega_{\mathcal{B}}, d_{\mathcal{B}})$$

of the universal differential calculi. Thus, $\mathcal{U}$ is a functor from the category of associative unital $\mathbb{K}$-algebras to the category of differential calculi.

Proof. It is easy to see, that

$$\mathcal{U}(\mathrm{Id}) = \mathrm{Id},$$

and for homomorphisms of algebras

$$g\colon \mathcal{A} \to \mathcal{B}, \quad h\colon \mathcal{B} \to \mathcal{C}$$

we have

$$\mathcal{U}(h \circ g) = \mathcal{U}(h) \circ \mathcal{U}(g).$$

■

Theorem 2.35 *Let (Ω, d) be a differential calculus on an algebra $\mathcal{A}$ with an exterior multiplication $\bullet$. The multiplication $\bullet$ induces a well-defined multiplication*

$$H^p(\Omega) \bullet H^q(\Omega) \longrightarrow H^{p+q}(\Omega)$$

that is associative.

Proof. Let $w, v \in \Omega$ and $dw = 0, dv = 0$. Then $d(wv) = 0$ by Leibniz rule. Now, let

$$w_1 = w + dx, \quad v_1 = v + dy,$$

where $dw = 0$ and $dv = 0$. Then we have

$$\begin{aligned} w_1 \bullet v_1 &= (w + dx) \bullet (v + dy) \\ &= w \bullet v + w \bullet dy + (dx) \bullet v + (dx) \bullet (dy) \\ &= w \bullet v + d(\pm w \bullet y) + d(x \bullet v) + d(x \bullet d(y)) \end{aligned}$$

where we have used the Leibniz rule

$$d(w \bullet y) = dw \bullet dy \pm w \bullet dy = \pm w \bullet dy,$$

$$d(x \bullet v) = dx \bullet v \pm x \bullet dv = (dx) \bullet v,$$

and $d(x \bullet dy) = (dx) \bullet (dy)$. ■

3 Differential calculus on finite sets

From now on let $\mathbb{K}$ be a field. We apply the general constructions of the previous sections to the algebra $\mathcal{A}$ of functions $V \to \mathbb{K}$ defined on a finite set $V = \{0, 1, \dots, n\}$. The algebra $\mathcal{A}$ has a $\mathbb{K}$-basis

$$e^i : V \to \mathbb{K}, \quad i = 0, 1, \dots, n$$

where

$$e^i(j) = \delta_i^j := \begin{cases} 1_{\mathbb{K}}, & i = j, \\ 0, & i \neq j, \end{cases}$$

for all $0 \leq i, j \leq n$, and the following relations are satisfied:

$$e^i e^j = \delta_i^j e^j, \quad \sum_{i=0}^{n} e^i = 1_{\mathcal{A}} \tag{3.46}$$

Denote by (Ω_V^1, d) the first order differential calculus $(\Omega_{\mathcal{A}}^1, d)$ defined in Section 2 with the exterior multiplication $\bullet$.

Theorem 3.1 *The $\mathbb{K}$-module Ω_V^1 has a basis $\{e^i \otimes e^j\}$ where $0 \leq i, j \leq n,\ i \neq j$. The differential*

$$d\colon \mathcal{A} \to \Omega_V^1$$

on the basis elements e^i of $\mathcal{A}$ is given by the formula

$$de^i = \sum_{0 \leq j \leq n,\ j \neq i} (e^j \otimes e^i - e^i \otimes e^j). \tag{3.47}$$

Also, the following identity is satisfied:

$$e^i \bullet de^j = \begin{cases} e^i \otimes e^j, & i \neq j \\ -\sum_{k \neq i} e^i \otimes e^k, & i = j \end{cases} \tag{3.48}$$

Proof. For $0 \leq i, j \leq n$, we have by (2.13)

$$\mu(e^i \otimes e^j) = e^i e^j = \delta_j^i = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

Hence $e^i \otimes e^j \in \Omega_V^1$ for $i \neq j$ and $e^i \otimes e^i \notin \Omega_V^1$ for $0 \leq i \leq n$. The finite dimensional $\mathbb{K}$-module $\mathcal{A} \otimes \mathcal{A}$ has basis $\{e^i \otimes e^j\}$ for $0 \leq i, j \leq n$ (see [1, II §7.7 Remark]), whence the first statement follows.

By definition d we have

$$\begin{aligned} de^i &= 1_{\mathcal{A}} \otimes e^i - e^i \otimes 1_{\mathcal{A}} \overset{\text{by (3.46)}}{=} \left(\sum_{0 \leq j \leq n} e^j \right) \otimes e^i - e^i \otimes \left(\sum_{0 \leq j \leq n} e^j \right) \\ &= \sum_{0 \leq j \leq n} e^j \otimes e^i - \sum_{0 \leq j \leq n} e^i \otimes e^j = \sum_{0 \leq j \leq n, j \neq i} (e^j \otimes e^i - e^i \otimes e^j), \end{aligned}$$

which proves (3.47). Next, we have

$$\begin{aligned} e^i \bullet de^j &= e^i \bullet \left(1_{\mathcal{A}} \otimes e^j - e^j \otimes 1_{\mathcal{A}}\right) \\ &= e^i \otimes e^j - \delta^i_j e^j \otimes 1_{\mathcal{A}} = \begin{cases} e^i \otimes e^j, & i \neq j \\ e^i \otimes e^i - e^i \otimes \left(\sum_k e^k\right), & i = j. \end{cases} \end{aligned}$$

Noticing that

$$e^i \otimes e^i - e^i \otimes \left(\sum_k e^k\right) = -\sum_{k\neq i} e^i \otimes e^k,$$

we obtain (3.48). ■

Let $\Omega^k_V = \Omega^k_{\mathcal{A}} \subset T^k_{\mathcal{A}}$ $(k \geq 0)$, and

$$\Omega_V = \Omega_{\mathcal{A}}$$

be the graded algebra defined in Section 2 with the multiplication $\bullet$. Let us introduce the following notation:

$$e^{i_0\dots i_k} := e^{i_0} \otimes e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_k}$$

assuming that $i_m \neq i_{m+1}$ for all $0 \leq m \leq k-1$. Clearly, $e^{i_0\dots i_k}$ are the elements of Ω^k_V.

Theorem 3.2 *(i) The elements $\{e^{i_0\dots i_k}\}$ form a $\mathbb{K}$-basis in Ω^k_V.*

(ii) The exterior multiplication $\bullet$ of the basis elements is given by the following formula

$$e^{i_0\dots i_k} \bullet e^{j_0\dots j_l} = \begin{cases} 0, & i_k \neq j_0 \\ e^{i_0\dots i_k j_1 \dots j_l}, & i_k = j_0. \end{cases}$$

(iii) The differential d is given on the basis elements by

$$de^{i_0\dots i_k} = \sum_{m=0}^{k+1} \sum_{j\neq i_{m-1}, i_m} (-1)^m \, e^{i_0\dots i_{m-1} j i_m \dots i_k}.$$

Proof. (i) The elements $e^{i_0\dots i_k}$ with $i_m \neq i_{m+1}$ for all $0 \leq m \leq k-1$ are linearly independent in the $\mathbb{K}$-module T^k (see [1, II §7.7 Remark]). We must only prove that such elements lie in $\Omega^k_V \subset T^k$. By Theorem 2.22 we have an isomorphism of graded algebras

$$f\colon \widetilde{\Omega}_{\mathcal{A}} \to \Omega_V = \Omega_{\mathcal{A}}$$

with an isomorphism of $\mathbb{K}$-modules

$$f_k\colon \widetilde{\Omega}^k_{\mathcal{A}} \to \Omega^k_V, \quad k \geq 0,$$

which is identity isomorphism for $k = 0, 1$. Hence the statement (i) is true for $k = 0, 1$ by definition of algebra $\mathcal{A}$ and by Theorem 3.1. For $k \geq 2$, consider an element $w = e^{i_0 \dots i_k} \in T^k$ with $i_m \neq i_{m+1}$ for all $0 \leq m \leq k-1$. Then the elements $e^{i_0 i_1}, e^{i_1 i_2}, \dots, e^{i_{k-1} i_k}$ lie in $\widetilde{\Omega}^1_{\mathcal{A}}$ and hence their $\star$-product

$$\omega = e^{i_0 i_1} \star e^{i_1 i_2} \star \dots \star e^{i_{k-1} i_k}$$

is contained in $\widetilde{\Omega}^k_{\mathcal{A}}$, and, hence, $f_k(\omega) \in \Omega^k_V$. By definition of f_k we have

$$\begin{aligned} f_k\left(e^{i_0 i_1} \star e^{i_1 i_2} \star \dots \star e^{i_{k-1} i_k}\right) &= f_1\left(e^{i_0 i_1}\right) \bullet f_1\left(e^{i_1 i_2}\right) \bullet \dots \bullet f_1\left(e^{i_{k-1} i_k}\right) \\ &= f_1\left(e^{i_0} \otimes e^{i_1}\right) \bullet f_1(e^{i_1} \otimes e^{i_2}) \bullet \dots \bullet f_1(e^{i_{k-1}} \otimes e^{i_k}) \\ &= e^{i_0} \otimes \left(e^{i_1} e^{i_1}\right) \otimes \dots \left(e^{i_{k-1}} e^{i_{k-1}}\right) \otimes e^{i_k} \\ &= e^{i_0 \dots i_k} \end{aligned}$$

so that $f_k(\omega) \in \Omega^k_V$.

(ii) This follows from definition of multiplication $\bullet$ in Definition 2.16 and (3.46).

(iii) We prove this by induction on k. For $k = 0$ it is proved in Theorem 3.1. For $k = 1$, let $i \neq j$. We have using (3.46)

$$\begin{aligned} d\left(e^i \otimes e^j\right) &= 1_{\mathcal{A}} \otimes e^i \otimes e^j - e^i \otimes 1_{\mathcal{A}} \otimes e^j + e^i \otimes e^j \otimes 1_{\mathcal{A}} \\ &= \sum_k (e^k \otimes e^i \otimes e^j - e^i \otimes e^k \otimes e^j + e^i \otimes e^j \otimes e^k) \\ &= (e^i \otimes e^i \otimes e^j - e^i \otimes e^i \otimes e^j - e^i \otimes e^j \otimes e^j + e^i \otimes e^j \otimes e^j) \\ &\quad + \sum_{k \neq i} e^k \otimes e^i \otimes e^j - \sum_{k \neq i, k \neq j} e^i \otimes e^k \otimes e^j + \sum_{k \neq j} e^i \otimes e^j \otimes e^k \end{aligned}$$

The sum in the brackets is equal to zero, and we obtain the result for $k = 1$. For $k \geq 2$ we have, using the Leibniz rule,

$$\begin{aligned} & d\left(e^{i_0} \otimes e^{i_1} \otimes \dots \otimes e^{i_k}\right) \\ = \ & d\left((e^{i_0} \otimes e^{i_1}) \bullet (e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_k})\right) \\ = \ & (e^{i_0} \otimes e^{i_1}) \bullet (e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_k}) - (e^{i_0} \otimes e^{i_1}) \bullet d(e^{i_1} \otimes e^{i_2} \otimes \dots \otimes e^{i_k}) \end{aligned}$$

The result then follows by the inductive hypotheses and elementary transformations. ■

As in Remark 2.18 we have a homomorphism

$$\varepsilon \colon \mathbb{K} \to \mathcal{A}$$

such that $d\varepsilon \colon \mathbb{K} \to \Omega^1_V$ is trivial. Consider a cochain complex Ω^ε_V with the augmentation ε

$$0 \longrightarrow \mathbb{K} \longrightarrow \Omega^0_V \longrightarrow \Omega^1_V \longrightarrow \dots \longrightarrow \Omega^n_V. \longrightarrow \dots \tag{3.49}$$

Proposition 3.3 *The first cohomology group $H^0(\Omega^\varepsilon_V)$ of the cochain complex (3.49) is trivial.*

Proof. Let $w \in \Omega_V^0$ be such that $dw = 0$. The element w can be written in the form

$$w = \sum_{i=0}^{n} f_i e^i$$

where $f_i \in \mathbb{K}$. By Theorem 3.1 we have

$$dw = \sum_i f_i \left(\sum_{k \neq i} (e^k \otimes e^i - e^i \otimes e^k) \right) = \sum_{k \neq i} (f_i - f_k) e^k \otimes e^i.$$

Since for $i \neq k$, $e^i \otimes e^j$ are the basic elements, the last sum is trivial if and only if $f_i = f_k$ for all i, k. Then we obtain

$$w = f_0 \left(\sum_{i=0}^{n} e^i \right) = f_0 \varepsilon(1) = \varepsilon(f_0),$$

that is, w belongs to the image of ε. ■

Let SET be a category in which objects are finite sets and morphisms are the maps of finite sets. Let V and W be finite sets, and $A(V) = \mathcal{A}_V, A(W) = \mathcal{A}_W$ be algebras of $\mathbb{K}$-valued functions respectively. For any map

$$F \colon V \to W$$

define an induced homomorphism of algebras

$$A(F) = F^* \colon \mathcal{A}_W \to \mathcal{A}_V$$

by

$$F^*(f) = f \circ F, \ f \in \mathcal{A}_W, \quad f \circ F \in \mathcal{A}_V. \tag{3.50}$$

Proposition 3.4 *The map A is a contravariant functor from the category SET to the category ALG of associative unital algebras.*

Proof. It is trivially to check that all conditions are satisfied. ■

Corollary 3.5 *For a finite set V, let $\mathcal{U}(\mathcal{A}_V) = (\Omega_V, d)$ be the universal differential calculus (Ω_V, d) on algebra $\mathcal{A}_V$ of $\mathbb{K}$-valued functions on V. Let us assign to any map $F \colon V \to W$ of finite sets a morphism*

$$\mathcal{U}(A(F)) = \mathcal{U}(F^*) \colon (\Omega_W, d) \to (\Omega_V, d)$$

where F^* is defined in (3.50). Then the composition $\mathcal{U} \circ A$ defines a contravariant functor from the category SET to the category DC of differential calculi.

Proof. Since the composition of functors is again a functor, the result follows from Proposition 3.4 and Theorem 2.34. ■

Now let

$$F \colon V \to W$$

be a identical inclusion of a set $V = \{0, 1, \ldots, k\}$ into a set $W = \{0, 1, \ldots, n\}$ where $k < n$. As before, let $\mathcal{A}$ and $\mathcal{B}$ be the algebras of $\mathbb{K}$-valued functions on V and W, respectively. Define a $\mathbb{K}$-linear subspace $\mathcal{J}$ of Ω_W by

$$\mathcal{J} = \oplus_{m\geq 0}\mathcal{J}^m$$

where

$$\mathcal{J}^0 = \text{span}\{e^{k+1}, \ldots, e^n\} \subset \Omega_W^0 = \mathcal{B}$$

and for $m \geq 1$, a subspace $\mathcal{J}^m$ of Ω_W^m is generated by the elements $e^{i_0\cdots i_m}$ such that the set $\{i_0, i_1, \ldots, i_m\}$ contains at least one number from the set $\{k+1, k+2, \ldots, n\}$.

Proposition 3.6 (i) *The subspace $\mathcal{J} \subset \Omega_W$ is an graded ideal in the graded algebra Ω_W such that*

$$d\mathcal{J}^m \subset \mathcal{J}^{m+1} \text{ for all } m \geq 0.$$

Thus, the restriction of the differential d to $\mathcal{J}$ induces a cochain complex

$$0 \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \longrightarrow \mathcal{J}^2 \longrightarrow \ldots$$

of $\mathbb{K}$-modules such that the natural inclusion

$$\mathcal{J} \longrightarrow \Omega_W$$

is a morphism of cochain complexes.

(ii) *The factor algebra $\Omega_W/\mathcal{J}$ endowed with the induced differential is a differential calculus which is isomorphic to the differential calculus Ω_V.*

Proof. (i) Let $e^{i_0\cdots i_p} \in \mathcal{J}^p$, $e^{j_0\cdots j_q} \in \Omega_V^q$, $e^{l_0\cdots l_r} \in \Omega_W^r$. Then by Theorem 3.2 the product

$$e^{j_0\cdots j_q} \bullet e^{i_0\cdots i_p} \bullet e^{l_0\cdots l_r}$$

lies in $\mathcal{J}^{p+q+r}$. The condition $d\mathcal{J}^m \subset \mathcal{J}^{m+1}$ is satisfied by definition of $\mathcal{J}$ and Theorem 3.2.

(ii) Any element $[w] \in \Omega_W^p/\mathcal{J}^p$ has a unique representative

$$w = \sum w_{i_0\ldots i_p} e^{i_0\cdots i_p}$$

where $w_{i_0\ldots i_p} \in \mathbb{K}$ and the sum goes over indices $i_j \in \{1, \ldots, k\}$ for $0 \leq j \leq p$.Define a map

$$s_p \colon \Omega_W^p/\mathcal{J}^p \to \Omega_V^p$$

by $s_p[w] = w$ and set

$$s = \bigoplus_p s_p \colon \Omega_W/\mathcal{J} \to \Omega_V.$$

Then the map s is a well-defined homomorphism of graded algebras that commutes with differential. It is easy to see that it an epimorphism with a trivial kernel. Hence it is an isomorphism. ∎

Remark 3.7 The composition

$$\Omega_W \longrightarrow \Omega_W/\mathcal{J} \xrightarrow{s} \Omega_V,$$

where the first map is a natural projection, coincides with the morphism of $\mathcal{U}(A(F))$ from Corollary 3.5 for the inclusion $F\colon V \to W$.

Corollary 3.8 *Under the hypotheses of Proposition 3.6 we have a cohomology long exact sequence*

$$0 \longrightarrow H^0(\mathcal{J}) \longrightarrow H^0(\Omega_W) \longrightarrow H^0(\Omega_V) \longrightarrow H^1(\mathcal{J}) \longrightarrow \dots$$

Theorem 3.9 *For any finite set V the cohomology group $H^p(\Omega_V)$ is trivial for $p \geq 1$.*

Proof. Follows from Theorem 5.4 in [6]. ■

Corollary 3.10 *Under assumptions of Proposition 3.6*

$$H^p(\mathcal{J}) = 0, \quad \textit{for} \ \ p \geq 0.$$

4 Differential calculus on graphs

A directed graph G is couple (V, E) where V is any set and E is a subset of $V \times V$. Elements of V are called the vertices and the elements of E – directed edges. Sometimes, to avoid misunderstanding, we shall use the extended notations V_G and E_G instead of V and E, respectively.

All graphs considered in this paper are directed graphs with a finite set of vertices.

Let $H = (V, E)$ be a simple complete graph consisting of $n+1$ vertices $\{0, 1, \dots, n\}$ and all directed edges $\{i, j\}$ $(i \neq j)$. The "simple" means that we have no edges $\{i, i\}$. Let $\mathcal{A}$ be the algebra of $\mathbb{K}$-valued functions on the set V, where $\mathbb{K}$ is a field.

Definition 4.1 The differential calculus on a full finite simple digraph H is the universal differential calculus (Ω_V, d) on the algebra $\mathcal{A}$ constructed in Section 3.

By Theorem 3.2 the multiplication $\bullet$ in (Ω_V, d) is given on the basic elements by

$$e^{i_0 i_1 \dots i_k} \bullet e^{j_0 j_1 \dots i_l} = \begin{cases} 0, & i_k \neq j_0 \\ e^{i_0 i_1 \dots i_k j_1 \dots i_l}, & i_k = j_0. \end{cases} \tag{4.51}$$

Now let G be a subgraph of the graph H with the same set of vertices $V = \{0, 1, 2, \dots, n\}$ and a set $E_G \subset E_H$ of edges. Denote by $g\colon G \to H$ the natural inclusion.

Definition 4.2 (i) A basic element $e^{i_0 i_1 \dots i_k} \in \Omega_V^k$ is called *allowed* if $\{i_j, i_{j+1}\} \in E_G$ for all $0 \le j \le k-1$, and non-allowed otherwise.

(ii) Let $\mathcal{E}_g^k$ be a $\mathbb{K}$-submodule of Ω_V^k generated by non-allowed elements (in particular, $\mathcal{E}_g^0 = \{0\}$), and set

$$\mathcal{E}_g = \bigoplus_{k \ge 0} \mathcal{E}_g^k \subset \Omega_V.$$

Proposition 4.3 *The set $\mathcal{E}_g$ is a graded ideal of algebra Ω_V.*

Proof. The result follows from (4.51). ■

Definition 4.4 Denote by

$$\mathcal{J}_g^k = \begin{cases} \mathcal{E}_g^k, & \text{for } k = 0, 1, \\ \mathcal{E}_g^k + d\mathcal{E}_g^{k-1}, & \text{for } k \ge 2, \end{cases}$$

a $\mathbb{K}$-submodule of Ω_V^k, and set

$$\mathcal{J}_g = \bigoplus_{k \ge 0} \mathcal{J}_g^k.$$

Proposition 4.5 *The set $\mathcal{J}_g$ is a graded ideal of algebra Ω_V, $d\mathcal{J}_g \subset \mathcal{J}_g$, and the inclusion*

$$\mathcal{J}_g \longrightarrow \Omega_V$$

is a morphism of cochain complexes. In particular, we have an exact sequence of cochain complexes

$$0 \longrightarrow \mathcal{J}_g \longrightarrow \Omega_V \longrightarrow \Omega_V / \mathcal{J}_g \longrightarrow 0.$$

Proof. Follows from Proposition 4.3 and Theorem 2.28. ■

Definition 4.6 Let $g \colon G \to H$ be an inclusion of a graph G into the simple complete graph H with the same set V of vertices. The differential calculus on G is the calculus

$$(\Omega_G, d) = (\Omega_V / \mathcal{J}_g, d)$$

on the algebra $\mathcal{A}$ with a differential that is induced from differential d on Ω_V.

In particular,

$$\Omega_G^0 = \Omega_V^0 = \mathcal{A},$$

and we have a cochain complex

$$0 \longrightarrow \Omega_G^0 \longrightarrow \Omega_G^1 \longrightarrow \Omega_G^2 \longrightarrow \dots \longrightarrow \Omega_G^n \longrightarrow \dots$$

and a cochain complex with an augmentation

$$0 \longrightarrow \mathbb{K} \longrightarrow \Omega_G^0 \longrightarrow \Omega_G^1 \longrightarrow \Omega_G^2 \longrightarrow \dots \longrightarrow \Omega_G^n \longrightarrow \dots$$

Proposition 4.7 *Under assumption above, there is a short exact sequence*

$$0 \longrightarrow \mathbb{K} \longrightarrow H^0(\Omega_G) \longrightarrow H^1(\mathcal{J}_g) \longrightarrow 0$$

and there are isomorphisms

$$H^i(\Omega_G) \cong H^{i+1}(\mathcal{J}_g)$$

for $i \geq 1$.

Proof. A short exact sequence of cochain complexes from Proposition 4.5 provides a cohomology long exact sequence

$$0 \longrightarrow H^0(\Omega_V) \longrightarrow H^0(\Omega_G) \longrightarrow H^1(\mathcal{J}_g) \longrightarrow H^1(\Omega_V) \longrightarrow H^1(\Omega_G) \longrightarrow \dots$$

By Theorem 3.9 $H^k(\Omega_V) = 0$ for $k \geq 1$ and $H^0(\Omega_V) \cong \mathbb{K}$ by Proposition 3.3. Now the result follows. ■

Consider a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{s} & G \\ & \searrow^{f} \quad {}^{g}\swarrow & \\ & H & \end{array} \tag{4.52}$$

of inclusions of graphs F and G into H with the same number of vertices. Let $\mathcal{E}_f$ and $\mathcal{E}_g$ be the subspaces generated by non-allowed elements for the inclusions f and g correspondingly, and $\mathcal{J}_f \subset \Omega_V$, $\mathcal{J}_g \subset \Omega_V$ are the graded ideals defined above.

Theorem 4.8 *We have the inclusions of the chain complexes*

$$\mathcal{J}_g \subset \mathcal{J}_f \subset \Omega_V,$$

which induce a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{J}_f/\mathcal{J}_g \longrightarrow \Omega_V/\mathcal{J}_g \xrightarrow{s^*} \Omega_V/\mathcal{J}_f \longrightarrow 0. \tag{4.53}$$

The cohomology long exact sequence of (4.53) *has the following form*

$$0 \longrightarrow H^0(\Omega_G) \longrightarrow H^0(\Omega_F) \longrightarrow H^1(\mathcal{J}_f/\mathcal{J}_g) \longrightarrow H^1(\Omega_G) \longrightarrow H^1(\Omega_F) \longrightarrow \dots \tag{4.54}$$

Proof. Any non-allowed element from $\mathcal{E}_g$ is evidently non-allowed in $\mathcal{E}_f$. Now the result follows from Corollaries 2.30 and 2.31. ■

Now consider an arbitrary inclusion of graphs $\sigma\colon F \to G$, where $F = (V_F, E_F)$ and $G = (V_G, E_G)$. Let H_1 and H_2 be complete simple digraphs with the same number of vertices as F and G, respectively. Consider a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\sigma} & G \\ \downarrow f & & \downarrow g \\ H_1 & \xrightarrow{\sigma_v} & H_2 \end{array} \tag{4.55}$$

where vertical maps are natural inclusions, and

$$\sigma_v\colon H_1 \to H_2$$

is the inclusion defined by σ. By Corollary 3.5 and Definition 4.1, the map σ_v induces a morphism

$$\mathcal{U}(\sigma_v)\colon \Omega_{V_G} \longrightarrow \Omega_{V_F}.$$

Thus by Proposition 4.5 we can write down the following diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathcal{J}_g & & \mathcal{J}_f \\ \downarrow & & \downarrow \\ \Omega_{V_G} & \overset{\mathcal{U}(\sigma_v)}{\longrightarrow} & \Omega_{V_F} \\ \downarrow & & \downarrow \\ \Omega_G & & \Omega_F \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \tag{4.56}$$

where the vertical columns are exact sequences of cochain complexes.

Lemma 4.9 *In diagram* (4.56) *we have*

$$\mathcal{U}(\mathcal{J}_g) \subset \mathcal{J}_f \subset \Omega_{V_F}$$

and hence the induced morphism

$$\mathcal{U}(\sigma)\colon \Omega_G \longrightarrow \Omega_F$$

of differential calculus is defined.

Proof. Let

$$e^{i_0\dots i_k} \in \mathcal{E}_g^k \subset \Omega_{H_2}^k$$

be a non-allowed element for the graph G. If all i_j for $j = 0, 1, \dots, k$ are contained in the image of $\sigma|_{V_F}\colon V_F \to V_G$, then $\mathcal{U}(e^{i_0\dots i_k}) = e^{i_0\dots i_k} \subset \mathcal{E}_f$ by diagram (4.55) since the map σ is an inclusion. In the opposite case by Proposition 3.6 and Remark 3.7 we obtain $\mathcal{U}(e^{i_0\dots i_k}) = 0$. Hence $\mathcal{U}(\mathcal{E}_g) \subset \mathcal{E}_f$. From now the result follows from definition $\mathcal{J}$, since vertical maps in diagram (4.56) are morphisms of cochain complexes. ■

Denote by GR the category in which objects are simple finite directed graphs and the morphisms are inclusions.

Theorem 4.10 *Let $\mathcal{U}(G)$ be a differential calculus (Ω_G, d) defined in Definition 4.6, and for an inclusion $\sigma\colon F \to G$ of graphs let*

$$\mathcal{U}(\sigma)\colon \Omega_G \longrightarrow \Omega_F$$

be a morphism of differential calculi defined in Lemma 4.9. Then $\mathcal{U}$ is a contravariant functor from category GR to the category DC.

Proof. We must only check that for two inclusions of graphs

$$\sigma\colon F \to G \quad \text{and} \quad \tau\colon G \to M$$

we have

$$\mathcal{U}(\tau \circ \sigma) = \mathcal{U}(\sigma) \circ \mathcal{U}(\tau).$$

By Lemma 4.9 we have a commutative diagram

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_m & \to & \mathcal{J}_g & \to & \mathcal{J}_f \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{V_M} & \xrightarrow{\mathcal{U}(\tau_v)} & \Omega_{V_G} & \xrightarrow{\mathcal{U}(\sigma_v)} & \Omega_{V_F} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_M & \xrightarrow{\mathcal{U}(\tau)} & \Omega_G & \xrightarrow{\mathcal{U}(\sigma)} & \Omega_F \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array} \tag{4.57}$$

By Corollary 3.5 we have $\mathcal{U}(\sigma_v) \circ \mathcal{U}(\tau_v) = \mathcal{U}(\tau_v \sigma_v)$. The commutativity of diagram

$$\begin{array}{ccc} \mathcal{J}_m & \longrightarrow & \mathcal{J}_g \\ & \searrow \quad \swarrow & \\ & \mathcal{J}_f & \end{array}$$

follows from Lemma 4.9. This implies the claim, since the vertical columns in (4.57) are exact sequences. ■

Remark 4.11 *Let $s\colon F \to G$ be an inclusion of graphs with the same number of vertices. Then $\mathcal{U}(s)$ coincides with the morphism s^* from Theorem 4.8.*

Definition 4.12 Let G be a simple graph with the set of vertices V and the set of edges E_G. Define a simple graph $\overline{G}$ with the same set of vertices V and with the set of inverse-directed edges

$$E_{\overline{G}} = \{\{i, j\} : \{j, i\} \in E_G\}.$$

Note that the mapping $G \longrightarrow \overline{G}$ is an involution on the set of simple graphs.

Theorem 4.13 *Let G be a simple graph. Then we have an isomorphism of cochain complexes*

$$\Omega_G \longrightarrow \Omega_{\overline{G}}$$

which is given on the basic elements by the following map

$$e^{i_0 i_1 \ldots i_{p-1} i_p} \longrightarrow (-1)^k e^{i_p i_{p-1} \ldots i_1 i_0},$$

where $k = 1$ for $p = 1, 2 \bmod 4$ and $k = 0$ for $p = 0, 3 \bmod 4$.

Proof. Let H be a full simple digraph with the same number of vertices $V = \{0, 1, \ldots, n\}$ as the graph G, and

$$g\colon G \to H, \qquad \overline{g}\colon \overline{G} \to H$$

be the natural inclusions. Define a $\mathbb{K}$-linear map

$$\tau : \Omega_V \longrightarrow \Omega_V,$$

on the basic elements by the following way:

$$\tau\left(e^{i_0 i_1 \ldots i_{p-1} i_p}\right) = e^{i_p i_{p-1} \ldots, i_1 i_0}.$$

The map τ an anti-automorphism of the graded algebra Ω_V since

$$\tau(vw) = \tau(w)\tau(v), \quad \tau(v+w) = \tau(v) + \tau(w).$$

It is easy to see that the diagram

$$\begin{array}{ccc} \Omega_V^{2k+1} & \xrightarrow{d} & \Omega_V^{2k+2} \\ \downarrow \tau & & \downarrow \tau \\ \Omega_V^{2k+1} & \xrightarrow{d} & \Omega_V^{2k+2} \end{array}$$

is commutative, that is $\tau d = d\tau$, and the diagram

$$\begin{array}{ccc} \Omega_V^{2k} & \xrightarrow{d} & \Omega_V^{2k+1} \\ \downarrow \tau & & \downarrow \tau \\ \Omega_V^{2k} & \xrightarrow{d} & \Omega_V^{2k+1} \end{array}$$

is anti-commutative that is $\tau d = -d\tau$. Thus, we obtain a commutative diagram of chain complexes

$$\begin{array}{ccccccccccc} 0 \to & \mathbb{K} & \longrightarrow & \Omega_V^0 & \xrightarrow{d} & \Omega_V^1 & \xrightarrow{d} & \Omega_V^2 & \xrightarrow{d} & \Omega_V^3 & \to \ldots \\ & \downarrow = & & \downarrow \tau & & \downarrow -\tau & & \downarrow -\tau & & \downarrow \tau & \\ 0 \to & \mathbb{K} & \longrightarrow & \Omega_V^0 & \xrightarrow{d} & \Omega_V^1 & \xrightarrow{d} & \Omega_V^2 & \xrightarrow{d} & \Omega_V^3 & \to \ldots \end{array} \tag{4.58}$$

where all the vertical maps are isomorphisms of $\mathbb{K}$-modules.

Since

$$\tau|_{\mathcal{E}_g} \colon \mathcal{E}_g \to \mathcal{E}_{\overline{g}}$$

is clearly bijective, $\mathcal{E}_g = -\mathcal{E}_g$ and $\mathcal{E}_{\overline{g}} = -\mathcal{E}_{\overline{g}}$, the restriction of the vertical maps in (4.58) to $\mathcal{J}_g$ provides isomorphisms $\mathcal{J}_g^p \longrightarrow \mathcal{J}_{\overline{g}}^p$, whence the result follows. ∎

By a graph with a pointed vertex we mean a couple (G, v) where G is a graph and v is one of its vertices.

Definition 4.14 Let $\{(G_i, v_i)\}_{i\in A}$ be a finite family of graphs $G_i = (V_i, E_i)$ with pointed vertices. Assume that all the vertex sets V_i are disjoint. The wedge sum (or bouquet) (G, v) of the graphs (G_i, v_i) is a graph with the set V of vertices that is obtained from the disjoint union

$$U = \bigcup_{i\in A} V_i$$

by identification of all pointed vertices v_i, $i \in A$, with one vertex v, and with the following set of edges:

$$E = \bigcup_{i\in A} E_i$$

with the same identification of the endpoint.

We shall denote the wedge sum by

$$G = \bigvee_{i\in A} G_i.$$

From now we shall consider a wedge sum of two graphs

$$G = G_1 \bigvee G_2,$$

with pointed vertexes $v_1 \in G_1$, $v_2 \in G_2$ and $v \in G$. Denote

$$W_1 = V_1 \setminus \{v_1\}, \quad W_2 = V_2 \setminus \{v_2\}.$$

Let H_1, H_2, and H be complete simple graphs with the set of vertices V_1, V_2, and V, respectively. Let

$$g_1 \colon G_1 \to H_1,\ g_2 \colon G_2 \to H_2,\ g \colon G \to H$$

be natural inclusions. The graded ideals

$$\mathcal{E}_{g_1} \subset \Omega_{V_1},\ \mathcal{E}_{g_2} \subset \Omega_{V_2},\ \mathcal{E}_g \subset \Omega_V$$

are defined as above as well as the graded ideals

$$\mathcal{J}_{g_1} \subset \Omega_{V_1},\ \mathcal{J}_{g_2} \subset \Omega_{V_2},\ \mathcal{J}_g \subset \Omega_V.$$

Lemma 4.15 *Let $G = G_1 \bigvee G_2$ as above. Let a basis element*

$$e^{i_0 i_1 \dots i_p} \in \Omega_V^p$$

be such that the multiindex $\{i_0, i_1, \dots, i_p\}$ contains at least one vertex $i_k \in W_1$ and at least one vertex $i_m \in W_2$. Then

$$e^{i_0 i_1 \dots i_p} \in \mathcal{J}_g.$$

Proof. Let $e^{i_0 i_1 \dots i_p}$ be as in the hypotheses. Consider two cases. If the pointed vertex v does not belong to the sequence $\{i_0, i_1, \dots, i_p\}$ then, by definition of the wedge sum, $e^{i_0 i_1 \dots i_p} \in \mathcal{E}_g \subset \mathcal{J}_g$, since it is non-allowed. Now assume that

$$v \in \{i_0, i_1, \dots, i_p\}.$$

In this case we have necessarily $p \geq 2$. For $p = 2$ the element $e^{i_0 i_1 \dots i_p}$ can be written as

$$e^{i_1 v i_2} \in \Omega_V^2 \quad \text{where} \quad i_1 \in W_1, \ i_2 \in W_2$$

(or $i_1 \in W_2$ and $i_2 \in W_1$). By definition of the wedge sum,

$$e^{i_1 i_2} \in \mathcal{E}_g.$$

Hence

$$de^{i_1 i_2} = \left(\left[\sum_i e^{i i_1 i_2} \right] - \left[\sum_{i \neq v} e^{i_1 i i_2} \right] + \left[\sum_i e^{i_1 i_2 i} \right] - e^{i_1 v i_2} \right) \in \mathcal{J}_g.$$

Here first three summands lie in $\mathcal{E}_g$, whence

$$e^{i_1 v i_2} \in \mathcal{J}_g.$$

Now consider the case $p \geq 3$. Then there exists $1 \leq l \leq p-1$ such that $v = i_l$. Then either

$$i_{l-1} \in W_1 \text{ and } i_{l+1} \in W_2$$

(or $i_{l-1} \in W_2$ and $i_{l+1} \in W_1$). Using the case $p = 2$ we conclude that

$$e^{i_{l-1} v i_{l+1}} \in \mathcal{J}_g,$$

which implies

$$e^{i_0 i_1 \dots i_p} \in \mathcal{J}_g$$

since J_g is a two-sided ideal in Ω_V. ■

Theorem 4.16 *Let $G = G_1 \bigvee G_2$ where G_i $(i = 1, 2)$ are connected graphs. Then*

$$H^0(\Omega_G) = \mathbb{K}$$

and, for any $k \geq 1$,

$$H^k(\Omega_G) = H^k(\Omega_{G_1}) \oplus H^k(\Omega_{G_2}).$$

Proof. Let V_i $(i = 1, 2)$ be the set of vertexes of the graphs G_i, and we recall that these sets are disjoint. Let V be the set of vertexes of the graph G. Let H_i $(i = 1, 2)$ be complete simple graph with the set of vertexes V_i, and H be a complete simple graph with the set of vertexes V. Let

$$f_i \colon H_i \longrightarrow H, \quad i = 1, 2$$

be the natural inclusions of complete simple graphs. Define for any mapping f_i the graded ideal $\mathcal{J}_i$ of Ω_V as in Proposition 3.6. By definition, $\mathcal{J}_1^k$ is a $\mathbb{K}$-linear subspace of Ω_V^k that is generated by the elements $e^{i_0 i_1 \dots i_k} \in \Omega_V^k$ such that the set $\{i_0, i_1, \dots, i_k\}$ contains at least one vertex from $V \setminus V_1$. The subspace $\mathcal{J}_2^k \subset \Omega_V^k$ is defined similarly. By Proposition 3.6 the graded ideals $\mathcal{J}_i$ of Ω_V induce short exact sequences of $\mathbb{K}$-modules

$$\begin{aligned} &0 \longrightarrow \mathcal{J}_1 \xrightarrow{\hat{f}_1} \Omega_V \xrightarrow{p_1} \Omega_{V_1} \longrightarrow 0, \\ &0 \longrightarrow \mathcal{J}_2 \xrightarrow{\hat{f}_2} \Omega_V \xrightarrow{p_2} \Omega_{V_2} \longrightarrow 0 \end{aligned} \tag{4.59}$$

where p_i and $\hat{f}_i$ are chain maps. Then

$$p = (p_1, p_2) \colon \Omega_V \longrightarrow \Omega_{V_1} \oplus \Omega_{V_2}$$

is a chain map. We denote by p^k a restriction of p to Ω_V^k. In dimension 0, the map

$$p^0 \colon \Omega_V^0 \longrightarrow \Omega_{V_1}^0 \oplus \Omega_{V_2}^0$$

is a monomorphism with a one-dimensional cokernel generated by $e^{v_1} \oplus 0$ (or $0 \oplus e^{v_2}$), since

$$p^0(e^v) = e^{v_1} \oplus e^{v_2}.$$

The map p is an epimorphism in dimensions $k \geq 1$. Indeed, consider an arbitrary element

$$e^{i_0 i_1 \dots i_k} \oplus e^{j_0 j_1 \dots j_k} \in \Omega_{V_1}^k \oplus \Omega_{V_2}^k.$$

Set $\alpha = \{i_0 i_1 \dots i_k\}$ and define a new multiindex α' by the following rule: if α does not contain the pointed vertex v_1 then $\alpha' = \alpha$, otherwise α' is obtained from α by changing v_1 to v. Similarly, using multiindex $\beta = \{j_0 j_1 \dots j_k\}$ we define a multiindex β'. Then we have

$$p_1(e^{\alpha'}) = e^{i_0 i_1 \dots i_k}, \quad p_2(e^{\beta'}) = e^{j_0 j_1 \dots j_k}$$

and it is clear that

$$p_2(e^{\alpha'}) = 0, \quad p_1(e^{\beta'}) = 0.$$

It follows that

$$p\left(e^{\alpha'} + e^{\beta'}\right) = e^{i_0 i_1 \dots i_k} \oplus e^{j_0 j_1 \dots j_k} \in \Omega_{V_1}^k \oplus \Omega_{V_2}^k,$$

which proves that p is an epimorphism in dimensions $k \geq 1$.

Observe that

$$\mathcal{J}_{12}^k := \mathcal{J}_1^k \cap \mathcal{J}_2^k$$

is a graded ideal in Ω_V and $\operatorname{Ker} p = \mathcal{J}_{12}$. Note, that $\mathcal{J}_{12}^0 = \{0\}$ and we have a short exact sequence

$$0 \longrightarrow \Omega_V^0 \xrightarrow{p} \Omega_{V_1}^0 \oplus \Omega_{V_2}^0 \longrightarrow \langle e^{v_1} \oplus 0 \rangle \longrightarrow 0. \tag{4.60}$$

Consider now the case $k \geq 1$. In this case we obtain a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{J}_{12}^k \xrightarrow{\hat{f}} \Omega_V^k \xrightarrow{p} \Omega_{V_1}^k \oplus \Omega_{V_2}^k \longrightarrow 0, \tag{4.61}$$

where $\hat{f} = \hat{f}_1|_{\mathcal{J}_{12}} = \hat{f}_2|_{\mathcal{J}_{12}}$ and p are chain maps. Recall, that the ideal $\mathcal{J}_{12}$ is generated by $e^{i_0 i_1 \dots i_k} \in \Omega_V$ such that the set $\{i_0, i_1, \dots, i_k\}$ contains $i_l \in W_1$ and $i_m \in W_2$.

Let us introduce the following notations:

$$\mathcal{J}'_{12} = \oplus_{k\geq 1}\mathcal{J}_{12}^k, \quad \mathcal{J}'_g = \oplus_{k\geq 1}\mathcal{J}_g^k, \quad \mathcal{J}'_{g_i} = \oplus_{k\geq 1}\mathcal{J}_{g_i}^k$$

and

$$\Omega'_V = \oplus_{k\geq 1}\Omega_V^k \quad \text{and} \quad \Omega'_{V_i} = \oplus_{k\geq 1}\Omega_{V_i}^k.$$

Lemma 4.17 *Under the above assumptions ($k \geq 1$) there is a commutative diagram of chain complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}'_{12} & \longrightarrow & \mathcal{J}'_g & \xrightarrow{q} & \mathcal{J}'_{g_1} \oplus \mathcal{J}'_{g_2} & \longrightarrow & 0 \\ & & \| & & \downarrow^{mono} & & \downarrow^{mono} & & \\ 0 & \longrightarrow & \mathcal{J}'_{12} & \longrightarrow & \Omega'_V & \xrightarrow{p} & \Omega'_{V_1} \oplus \Omega'_{V_2} & \rightarrow & 0 \end{array} \tag{4.62}$$

where the rows are exact sequences and two right vertical maps are natural inclusions.

Proof. The bottom exact sequence is the exact sequence (4.61). The inclusion $\mathcal{J}'_{12} \to \mathcal{J}'_g$ follows from Lemma 4.15 and we obtain that the left square is commutative. Set

$$q = p|_{\mathcal{J}'_g} : \mathcal{J}'_g \to \Omega'_{V_1} \oplus \Omega'_{V_2}$$

where we identify $\mathcal{J}'_g$ with a subspace of Ω'_V. It remains to prove that the image of q is $\mathcal{J}'_{g_1} \oplus \mathcal{J}'_{g_2}$.

Let us first prove that

$$\mathcal{J}'_{g_1} \oplus \mathcal{J}'_{g_2} \subset q\left(\mathcal{J}'_g\right) \tag{4.63}$$

For $i = 1, 2$ we shall define the grade preserving homomorphisms of $\mathbb{K}$-modules

$$s_i \colon \Omega_{V_i} \longrightarrow \Omega_V.$$

For any basic elements $e^\alpha = e^{i_0 \dots i_k} \in \Omega_{V_1}$, let α' be the multiindex that is equal to α if α does not contain the pointed vertex v_1, and otherwise α' is obtained from α by changing all occurrences of v_1 in α to v. Similarly define $e^{\beta'}$ for $e^\beta = e^{j_0 \dots j_k} \in \Omega_{V_2}$. Then set

$$s_1(e^\alpha) = e^{\alpha'} \quad \text{and} \quad s_2(e^\beta) = e^{\beta'}$$

and extend s_1 and s_2 by linearity to all the spaces Ω_{V_1} and Ω_{V_2}, respectively. It follows immediately from this definition, that

$$s_1\left(e^\alpha\right) \in \mathcal{E}_g, \text{ if } e^\alpha \in \mathcal{E}_{g_1}$$

and

$$s_2\left(e^\beta\right) \in \mathcal{E}_g, \text{ if } e^\beta \in \mathcal{E}_{g_2}.$$

Note, that the maps s_i do not commutes with the differentials, but they satisfy the following properties:

$$p_i s_j = \begin{cases} \mathrm{Id}\colon \Omega^k_{V_i} \to \Omega^k_{V_i}, & i = j \\ 0\colon \Omega^k_{V_j} \to \Omega^k_{V_i}, & i \neq j \end{cases} \tag{4.64}$$

and, for $e^\alpha \in \Omega^k_{V_1}$,

$$d_H(s_1(e^\alpha)) = s_1(d_{H_1}(e^\alpha)) + \sum_\gamma f_\gamma e^\gamma, \text{ where } \sum_\gamma f_\gamma e^\gamma \in \mathcal{J}^k_{12} \tag{4.65}$$

and a similar identity holds for $e^\beta \in \Omega^k_{V_2}$. Let

$$u = u_1 \oplus u_2 \in \mathcal{J}^k_{g_1} \oplus \mathcal{J}^k_{g_2} \subset \Omega^k_{V_1} \oplus \Omega^k_{V_2}.$$

By definition of $\mathcal{J}^k_{g_1}$ we have

$$u_1 = w_1 + d_{H_1}(w_1') \in \mathcal{J}^k_{g_1} \subset \Omega^k_{V_1} \quad \text{where} \quad w_1, w_1' \in \mathcal{E}_{g_1}.$$

Note that $s_1(w_1), s_1(w_1') \in \mathcal{E}_g$ whence it follows that

$$s_1(w_1) + d_H(s_1(w_1')) \in \mathcal{J}_g.$$

Using by (4.64) and (4.65) we obtain that

$$\begin{aligned} p_1\left(s_1(w_1) + d_H(s_1(w_1'))\right) &= w_1 + p_1\left(d_H(s_1(w_1'))\right) \\ &= w_1 + p_1\left(s_1(d_{H_1}(w_1')) + \sum_\gamma f_\gamma e^\gamma\right) \\ &= w_1 + d_{H_1}(w_1') \\ &= u_1 \end{aligned}$$

where we have used that

$$\sum_\gamma f_\gamma e^\gamma \in \mathcal{J}_{12} \text{ and } p(\mathcal{J}_{12}) = 0.$$

By the same line of arguments we obtain

$$p_2\left(s_1(w_1) + d_H(s_1(w_1'))\right) = w_1 + p_2\left(d_H(s_1(w_1'))\right) = 0,$$

and

$$\begin{aligned} p_2\left(s_2(w_2) + d_H(s_2(w_2'))\right) &= u_2, \\ p_1\left(s_2(w_2) + d_H(s_2(w_2'))\right) &= 0. \end{aligned}$$

Hence,

$$p((s_1(w_1) + d_H(s_1(w_1')) + s_2(w_2) + d_H(s_2(w_2'))) = u_1 \oplus u_2,$$

which proves the inclusion (4.63).

Let us prove the opposite inclusion. Any element of $\mathcal{J}_g$ has the form $w + dw'$ where

$$w = s_1(u_1) + s_2(u_2) + u_3$$
$$w' = s_1(u_1') + s_2(u_2') + u_3'$$

where $u_1, u_1' \in \mathcal{E}_{g_1}$, $u_2, u_2' \in \mathcal{E}_{g_2}$ and $u_3, u_3' \in \mathcal{J}_{12}$. As above we obtain for $i = 1, 2$

$$p_i(w + dw') = u_i + d_{H_i}u_i' \in \mathcal{J}_{g_i}$$

which finishes the proof of Lemma. ∎

By Lemma 4.17 we obtain a commutative diagram of chain complexes in which rows and columns are exact (in dimensions $k \geq 1$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_{12} & \longrightarrow & \mathcal{J}_g & \xrightarrow{q} & J_{g_1} \oplus \mathcal{J}_{g_2} & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}_{12} & \longrightarrow & \Omega_V & \xrightarrow{p} & \Omega_{V_1} \oplus \Omega_{V_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Omega_G & \xrightarrow{\cong} & \Omega_{G_1} \oplus \Omega_{G_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.66}$$

In dimension 0 we have isomorphisms

$$\Omega_G^0 = \Omega_V^0, \quad \Omega_{G_1}^0 = \Omega_{V_1}^0, \quad \Omega_{G_2}^0 = \Omega_{V_2}^0$$

and an exact sequence

$$0 \longrightarrow \Omega_V^0 \xrightarrow{p} \Omega_{V_1}^0 \oplus \Omega_{V_2}^0 \longrightarrow \langle e^{v_1} \oplus 0 \rangle \longrightarrow 0.$$

Now from (4.66) and the last exact sequence we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega_G^0 & \longrightarrow & \Omega_G^1 & \xrightarrow{d} & \Omega_G^2 & \xrightarrow{d} \dots \\
 & \downarrow & & \downarrow = & & \downarrow = & \\
0 \longrightarrow & \Omega_{G_1}^0 \oplus \Omega_{G_2}^0 & \xrightarrow{d \oplus d} & \Omega_{G_1}^1 \oplus \Omega_{G_2}^1 & \xrightarrow{d \oplus d} & \Omega_{G_1}^2 \oplus \Omega_{G_2}^2 & \xrightarrow{d \oplus d} \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \langle e^{v_1} \oplus 0 \rangle & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \dots
\end{array} \tag{4.67}$$

where the columns are exact sequences, and the rows are chain complexes. Using the obvious identity

$$H^*(\Omega_{G_1} \oplus \Omega_{G_2}) = H^*(\Omega_{G_1}) \oplus H^*(\Omega_{G_2})$$

and a cohomology long exact sequence of (4.67) we finish the proof of the theorem.
∎

Corollary 4.18 *Let*

$$G = \bigvee_{i \in A} G_i$$

be a finite wedge sum, and all G_i are connected graphs. Then

$$H^0(\Omega_G) = \mathbb{K}$$

and

$$H^i(\Omega_G) = \oplus_{i \in A} H^i(\Omega_{G_i}).$$

Proof. Induction by i. ■

Let G be a graph with the set of vertices $V = \{0, 1, \dots, n\}$.

Definition 4.19 [6] (i)The cone CG of the graph G is obtained by adding a new vertex v to the set of vertexes V and new edges $\{i, v\}$ for all $0 \le i \le n$.

(ii) The suspension SG of the graph G is obtained from the graph G by adding two new vertices v and w and new edges $\{i, v\}$, $\{i, w\}$ for all $0 \le i \le n$.

We recall here the following result from [6].

Theorem 4.20 *For any graph G we have*

$$H^p(\Omega_{CG}) \cong \begin{cases} \mathbb{K}, & p = 0 \\ 0, & p \ge 1, \end{cases}$$

$$H^{p+1}(\Omega_{SG}) \cong \begin{cases} \mathbb{K}, & p = -1 \\ H^0(\Omega_G^{\varepsilon}), & p = 0 \\ H^p(\Omega_G), & p \ge 1 \end{cases}$$

where Ω_G^{ε} is a cochain complex with the augmentation.

One of the main results of this paper is the following theorem.

Theorem 4.21 *For any finite collection of nonnegative integers $k_0, k_1, \dots k_n$ such that $k_0 \ge 1$ there exists a digraph G such that the cohomology groups of its differential calculus satisfies the conditions*

$$\dim H^i(\Omega_G) = k_i, \ \textit{for all } 0 \le i \le n. \tag{4.68}$$

In particular, if $k_0 = 1$ then the graph G is connected.

Proof. At first we construct a connected graph G^m ($m \ge 1$) such that

$$\dim H^p(\Omega_{G^m}) = \begin{cases} 1, & p = 0, m \\ 0, & \text{otherwise.} \end{cases} \tag{4.69}$$

For $m = 1$ this is the graph with the set of vertexes $V = \{0, 1, 2\}$ and the set of edges $E = \{\{0, 1\}, \{1, 2\}, \{2, 1\}\}$. Then, by induction we define $G^{m+1} = SG^m$. By Theorem 4.20 it satisfies (4.69).

For any $m \geq 1$, define the graph F^m as follows. If $k_m = 0$ then F^m consists of a single vertex, and if $k_m \geq 1$ then F^m is equal to the wedge some of k_m copies of G^m. By Theorem 4.20, we have

$$H^p(\Omega_{F^m}) = \begin{cases} 1, & p = 0, \\ k_m, & p = m, \\ 0, & \text{otherwise}. \end{cases} \tag{4.70}$$

Let F_0 be a graph, consisting of k_0 vertexes and no edges. Now define

$$G = \bigvee_{m=0,1,2,\ldots,k_n} F^m.$$

Then (4.68) follows from Theorem 4.20. ■

The next result can be helpful for computational purposes.

Corollary 4.22 *Under assumption of Theorem 4.21, there exists a graph G with*

$$k_0 + 2k_1 + 4k_2 + 6k_3 + \cdots + (2n)k_n$$

vertices such that

$$\dim H^i(\Omega_G) = k_i, \ \forall \ 0 \leq i \leq n.$$

Proof. This follows from a direct computation of the number of vertices of the graphs in the proof of Theorem 4.21. ■

The number of vertices of G in Corollary 4.22 can be easily improved. An interesting open question is to find the minimum number of vertices of the graph satisfying (4.68).

5 Differential calculus on monotonic graphs

Definition 5.1 (i) A complete monotonic graph Γ is a finite simple graph with a set of vertices $V = \{0, 1, 2, \ldots, n\}$ and the set of directed edges

$$E = \{\{i, j\} : i < j; i, j = 0, 1, 2, \ldots, n\}.$$

(ii) A monotonic graph G is any subgraph of a complete monotonic graph Γ.

We have a natural inclusion

$$\gamma \colon \Gamma \longrightarrow H,$$

where H is a full finite simple digraph with the set V of vertices defined in Section 4.

By Definition 4.2 and Proposition 4.3 of Section 4 we have a graded ideal

$$\mathcal{E}_\gamma = \bigoplus_{p \geq 0} \mathcal{E}_\gamma^p \subset \bigoplus_{p \geq 0} \Omega_V^p = \Omega_V, \tag{5.71}$$

where $\mathcal{E}_\gamma^0 = \{0\}$ and $\mathcal{E}_\gamma^p$ for $p \geq 1$ is generated by non-allowed elements.

Recall that $\mathcal{A}$ is an algebra of $\mathbb{K}$-valued functions on the set V.

Proposition 5.2 *For $p \geq 0$*

$$d\mathcal{E}_\gamma^p \subset \mathcal{E}_\gamma^{p+1},$$

and differential calculus $(\Omega_\Gamma, d_\Gamma)$ from Definition 4.6 coincides with the calculus $(\Omega_V/\mathcal{E}_\gamma, d)$. In particular, we have an exact sequence of cochain complexes

$$0 \longrightarrow \mathcal{E}_\gamma \longrightarrow \Omega_V \longrightarrow \Omega_\Gamma \longrightarrow 0.$$

Proof. For $p \geq 1$, an element $e^{i_0 i_1 \dots i_p} \in \mathcal{E}_\gamma^p$ is non-allowed if and only the sequence $\{i_0, i_1, \dots, i_p\}$ is non monotonic increasing. Now the result follows from description of differential on basic elements in Theorem 3.2. ■

Corollary 5.3 (i) *The basic elements of the differential calculus $(\Omega_\Gamma, d_\Gamma)$ of the graph Γ can be represented by classes of elements $e^{i_0 i_1 \dots i_p} \in \Omega_V^p$ such that $0 \leq i_0 < i_1 < \dots < i_p \leq n$.*

(ii) *For $0 \leq k, l \leq n$, the exterior multiplication $\bullet$ of basis elements is given by the following formula*

$$(e^{i_0 i_1 \dots i_k}) \bullet (e^{j_0 j_1 \dots j_l}) = \begin{cases} 0, & i_k \neq j_0 \\ e^{i_0 i_1 \dots i_k j_1 \dots j_l}, & i_k = j_0. \end{cases}$$

(iii) *The differential d_Γ is given on basis elements by*

$$d\left(e^{i_0 i_1 \dots i_k}\right) = \widetilde{\sum_{j \neq i_0}} e^{j i_0 i_1 \dots i_k} - \widetilde{\sum_{j \neq i_0; j \neq i_1}} e^{i_0 j i_1 \dots i_k} + \dots$$

$$+(-1)^{l+1} \widetilde{\sum_{j \neq i_l; j \neq i_{l+1}}} e^{i_0 i_1 \dots i_l j i_{l+1} \dots i_k} + \dots + (-1)^{k+1} \widetilde{\sum_{j \neq i_k}} e^{i_0 i_1 \dots i_k j},$$

where $\widetilde{\ }$ over the sign $\sum$ means that in summation are present only the elements with strongly monotonic increasing sets of indices.

Proof. Follows from Theorem 3.2 and the proof of Proposition 5.2. ■

We shall omit subscript Γ in the differential, if it is clear from context what cochain complex we consider.

Corollary 5.4 *For $k > n$,*

$$\Omega_\Gamma^k = 0$$

and the maps

$$\mathcal{E}_\gamma^k \longrightarrow \Omega_H^k$$

are isomorphisms.

Proof. The space Ω_Γ^k is generated by basic elements $e^{i_0 i_1 \dots i_k}$, where $0 \leq i_0 < i_1 < \dots < i_k \leq n$. Any finite sequences of more than $n+1$-elements from the set $V = \{0, 1, 2, \dots, n\}$ has at least two equal elements. Now the statement follows from Corollary 5.3. ■

Now we recall a definition of simplicial homology groups of a simplicial complex [11].

Let S be a simplicial complex, and let every simplex $\Delta^n \in S$ is given by a fixed ordered collection of vertexes $\Delta^n = [i_0, i_1, \dots, i_n]$. We suppose also, that every face $\Delta^k \in \Delta^n$ is given by a subset of vertexes having the induced order. The simplexes $[i_0, i_1, \dots, i_n]$ and $[i_{\sigma(0)}, i_{\sigma 1}, \dots, i_{\sigma(n)}]$ where σ is a permutation are oriented equally if $\mathrm{sign}(\sigma) = 1$, and the have an opposite orientation if $\mathrm{sign}(\sigma) = -1$. Consider a chain complex $C^n(S), n = -1, 0, 1, 2, \dots$ over the ring $\mathbb{K}$ generated in dimension $n \geq 0$ by all simplexes $\Delta^n \in S$ and $C^{-1}(S) = 0$. The equally oriented simplexes with the same vertexes we will not distinguish and opposite oriented simplexes with Δ we shall denote by $-\Delta$, such that the sum of these two simplexes is zero. Thus for $n \geq 0$, $C^n(S)$ is a $\mathbb{K}$-module consisting of elements

$$\sum_{finite} k_i \Delta_i^n, \quad k_i \in \mathbb{K}, \; \Delta_i^n \in S.$$

Define a boundary map

$$C^n(S) \to C^{n-1}(S),$$

by

$$\partial[i_0, i_1, \dots, i_n] = \sum (-1)^j [i_0, i_1, \dots, \hat{i_j}, \dots i_n], \text{ for } n \geq 1; \; \partial(\Delta^0) = 0 \text{ for all } \Delta^0.$$

For example,

$$\partial[i, j] = [j] - [i], \quad \partial[i, j, k] = [j, k] - [i, k] + [i, j].$$

It is easy to check, that $\partial^2 = 0$.

Example 5.5 *Let Δ^n be a simplicial complex consisting of one simplex $\Delta^n = [0, 1, \dots, n]$. Every k-dimensional face $(0 \leq k \leq n)$ is given by a simplex $\Delta^k = [i_0, i_1, \dots, i_k]$ in which $0 \leq i_0 < i_1 < \dots < i_k \leq n$ is an increasing subsequence of $0 < 1 < 2 < \dots < n$. Let $C(\Delta^n)$ be a chain complex with k-dimensional modules $C_k(\Delta^n)$ generated by k-simplexes of Δ^n. The homology groups of the chain complex $C(\Delta^n)$ are non-trivial only in dimension zero by (see [11]), and $H^0(C(\Delta^n)) = \mathbb{K}$.*

Let Λ be a cochain complex

$$0 \longrightarrow \Lambda^0 \longrightarrow \Lambda^1 \longrightarrow \dots \longrightarrow \Lambda^n \longrightarrow \dots$$

of $\mathbb{K}$-modules with a differential d. Recall a definition of adjoint (dual) chain complex to Λ consisting of $\mathbb{K}$-modules

$$\Lambda_p^* = \mathrm{Hom}_{\mathbb{K}}(\Lambda^p, \mathbb{K})$$

and the boundary maps

$$\delta_p \colon \Lambda_p^* \longrightarrow \Lambda_{p-1}^*, \; p \geq 0,$$

where $\delta_0 = 0$ and δ_p for $p \geq 1$ is defined by the rule

$$[\delta_p(f_p)](w) = f_p(dw), \; f_p \in \Lambda_p^*, \; f_p \colon \Lambda^p \to \mathbb{K}, \; w \in \Lambda^{p-1}.$$

Note that $\delta_{p-1} \circ \delta_p = 0$ (see, for example, [10]). We shall usually omit subscript in differential δ if this does not lead to confusion.

Definition 5.6 The chain complex

$$0 \longleftarrow \Lambda_0^* \longleftarrow \Lambda_1^* \longleftarrow \ldots \longleftarrow \Lambda_n^* \longleftarrow \ldots$$

with the boundary map δ is called *dual* to the cochain complex Λ and will be denoted by Λ^*.

If $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_k}$ is a basis in the $\mathbb{K}$-linear space Λ^p then Λ_p^* is a $\mathbb{K}$-linear space with the dual basis $e_{\alpha_1}^*, e_{\alpha_2}^*, \ldots, e_{\alpha_k}^*$ defined by the rule

$$e_\alpha^*(e^\beta) = \begin{cases} 1, & \text{for } \alpha = \beta \\ 0, & \text{for } \alpha \neq \beta. \end{cases} \tag{5.72}$$

In particular, the linear spaces $H^p(\Lambda)$ and $H_p(\Lambda^*)$ are dual and hence isomorphic.

Theorem 5.7 *Let $(\Omega_V/\mathcal{E}_\gamma, d) = (\Omega_\Gamma, d)$ be the differential calculus on the complete monotonic graph Γ with the set of vertexes $V = \{0, 1, \ldots, n\}$, and Ω_Γ^* be the dual chain complex with the basis $e_{i_0 i_1 \ldots i_p}^*$ $(0 \leq i_0 < i_1 < \cdots < i_p \leq n)$ which is dual to the basis described in Corollary 5.3. Then the boundary operator*

$$\delta \colon [\Omega_\Gamma]_p^* \longrightarrow [\Omega_\Gamma]_{p-1}^*$$

is given on the basic elements by the rule

$$\delta(e_{i_0 i_1 \ldots i_p}^*) = \sum_{0 \leq k \leq p} (-1)^k e_{i_0 \ldots i_{k-1} \hat{i}_k i_{k+1} \ldots i_p}^*$$

where $\hat{i}_k$ means omitting the symbol i_k from the multiindex.

Proof. Let $e^{j_0 j_1 \ldots j_{p-1}}$ be a basic element of Ω_Γ^{p-1}. Then

$$\begin{aligned}
[\delta(e_{i_0 i_1 \ldots i_p}^*)](e^{j_0 j_1 \ldots j_{p-1}}) &= e_{i_0 i_1 \ldots i_p}^*(d e^{j_0 j_1 \ldots j_{p-1}}) \\
&= e_{i_0 i_1, \ldots i_p}^* \left[\sum_{q=0}^{p} \widetilde{\sum_k} (-1)^q e^{j_0 j_1 \ldots, j_{q-1} k j_q \ldots j_{p-1}} \right] \\
&= \sum_{q=0}^{p} \widetilde{\sum_k} (-1)^q e_{i_0 i_1 \ldots i_p}^* \left[e^{j_0 j_1 \ldots, j_{q-1} k j_q \ldots j_{p-1}} \right]
\end{aligned}$$

where $\widetilde{\quad}$ means that only elements with monotonic multiindices are used in the summation. We have

$$e_{i_0 i_1 \ldots i_p}^* \left[e^{j_0 j_1 \ldots j_{q-1} k j_q \ldots j_{p-1}} \right] = 1$$

only if

$$\{i_0 i_1 \ldots i_p\} = \{j_0 j_1 \ldots j_{q-1} k j_q \ldots j_{p-1}\}$$

for some place q. This means that the sequence $\{j_0, j_1, \ldots, j_{p-1}\}$ is obtained from the sequence $\{i_0, i_1, \ldots, i_p\}$ by deleting a term $i_q = k$. For such basic elements we have

$$[\delta(e_{i_0 i_1 \ldots i_p}^*)](e^{i_1 \ldots i_p}) = 1, \ q = 0;$$

$$[\delta(e^*_{i_0i_1\dots i_p})](e^{i_0i_2\dots i_p}) = -1,\ q=1;$$

and generally

$$[\delta(e^*_{i_0i_1\dots i_p})](e^{i_0i_1\dots i_{q-1}i_{q+1}\dots i_p}) = (-1)^q,\ 0\le q\le p.$$

Hence, we obtain

$$\delta(e^*_{i_0i_1\dots i_p}) = (-1)^q \sum_{0\le q\le p} e^*_{i_0i_1\dots i_{q-1}\hat{i}_q i_{q+1}\dots i_p}$$

which finishes the proof. ■

Corollary 5.8 *Let Δ^n is a simplicial complex from Example 5.5. Under assumptions of Theorem 5.7, for $k\ge 0$ consider the maps*

$$T_k\colon [\Omega_\Gamma]^*_k \to C_k(\Delta^n),$$

of $\mathbb{K}$-modules given on basic elements by formulas

$$e^*_{i_0i_1\dots i_p} \longrightarrow [i_0, i_1, \dots, i_p],\ 0\le i_0 < i_1 < \dots < i_p \le n.$$

The maps T_k commutes with differentials, that is

$$T_{k-1}\delta = dT_k,$$

and define an isomorphism between the chain complexes

$$T = \bigoplus_k T_k\colon \Omega^*_\Gamma \to C(\Delta^n).$$

Proof. Follows from Theorem 5.7, Corollary 5.3, and Example 5.5. ■

Corollary 5.9 *Under the assumption of Corollary 5.8,*

$$H^p(\Omega_\Gamma) = \begin{cases} \mathbb{K}, & p=0 \\ 0, & p\ge 1. \end{cases}$$

Proof. Follows from Corollary 5.8 and Example 5.5. ■

Now let G be a monotonic graph with a set of vertices $V=\{0,1,2,\dots,n\}$, Γ be the complete monotonic graph with the same set of vertices V and H be the complete simple graph with the set of vertices V.

We have a commutative diagram of inclusions of graph as (4.52)

$$\begin{array}{ccccc} G & & \overset{s}{\longrightarrow} & & \Gamma \\ & \searrow^{g} & & {}^{\gamma}\swarrow & \\ & & H & & \end{array} \tag{5.73}$$

The exact sequence of chain complexes (4.53) has the following form

$$0 \longrightarrow \mathcal{J}_g/\mathcal{E}_\gamma \longrightarrow \Omega_\Gamma \longrightarrow \Omega_G \longrightarrow 0 \tag{5.74}$$

where

$$\Omega_V/\mathcal{E}_\gamma = \Omega_\Gamma,\ \Omega_V/\mathcal{J}_g = \Omega_G.$$

Recall, that $\mathcal{J}^0_g = 0$ and $\mathcal{E}^0_\gamma = 0$.

Theorem 5.10 *For $p \geq 1$, we have an isomorphism*

$$H^p(\Omega_G) \cong H^{p+1}(\mathcal{J}_g/\mathcal{E}_\gamma)$$

and an exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow H^0(\Omega_G) \longrightarrow H^1(\mathcal{J}_g/\mathcal{E}_\gamma) \longrightarrow 0.$$

Proof. By Corollary 5.9

$$H^p(\Omega_\Gamma) = \begin{cases} \mathbb{K}, & \text{for } p = 0 \\ 0, & \text{for } p \geq 1. \end{cases}$$

Now the result follows from exact sequence (5.74). ∎

Definition 5.11 Let $\mathcal{E}_s^0 = 0$ and $\mathcal{E}_s^p$, $p \geq 1$ be a subspace of Ω_Γ^p generated by those $e^{i_0 i_1 \dots i_p} \in \Omega_\Gamma^p$ that are non-allowed elements for the graph G. Let

$$\mathcal{J}_s^p = \mathcal{E}_s^p + d_\Gamma \mathcal{E}_s^{p-1} \subset \Omega_\Gamma^p$$

where d_Γ is the differential in Ω_Γ described in Corollary 5.3. Denote

$$\mathcal{E}_s = \bigoplus_{0 \leq i \leq n} \mathcal{E}_s^i, \ \mathcal{J}_s = \bigoplus_{0 \leq i \leq n} \mathcal{J}_s^i.$$

Proposition 5.12 *The submodule*

$$\mathcal{J}_s \subset \Omega_\Gamma$$

is an graded ideal such that the inclusion is a morphism of chain complexes and the exact sequence

$$0 \longrightarrow \mathcal{J}_s \longrightarrow \Omega_\Gamma \longrightarrow \Omega_\Gamma/\mathcal{J}_s \longrightarrow 0 \tag{5.75}$$

is isomorphic to exact sequence in (5.74). *In particular, we obtain an isomorphism*

$$\Omega_\Gamma/\mathcal{J}_s \cong \Omega_G.$$

Proof. The subspace $\mathcal{E}_s \in \Omega_\Gamma$ is evidently a graded ideal. Now, as in proof of Theorem 2.28, we can see that $\mathcal{J}_s \subset \Omega_\Gamma$ is a graded ideal and the inclusion is a morphism of chain complexes. By definition, we have a graded isomorphism

$$\Omega_V/\mathcal{E}_\gamma \longrightarrow \Omega_\Gamma.$$

It follows from Definition 5.11 and Corollary 5.3 that a restriction of this map to

$$\mathcal{J}_g/\mathcal{E}_\gamma \subset \Omega_V/\mathcal{E}_\gamma$$

correctly defines a graded isomorphism

$$\mathcal{J}_g/\mathcal{E}_\gamma \longrightarrow \mathcal{J}_s$$

which agrees with the differentials. The Proposition is proved. ■

Let Γ be a complete monotonic graph with the set $V = \{0, 1, 2, \dots, n\}$ of vertices and the set of edges

$$E = \{\{i, j\} : i < j; i, j = 0, 1, 2, \dots, n\}.$$

Let F and G be monotonic graphs with the same number of vertices such that we have a commutative diagram of inclusions

$$\begin{array}{ccccc} F & & \xrightarrow{r} & & G \\ & \searrow^{t} & & {}^{s}\swarrow & \\ & & \Gamma & & \end{array} \tag{5.76}$$

where the horizontal map is an inclusion $r\colon F \longrightarrow G$ and swallow maps are inclusions into Γ.

Theorem 5.13 *Let $\mathcal{E}_s$ and $\mathcal{E}_t$ be the subspaces generated by non-admissible elements in Ω_Γ for the inclusions s and t respectively,, and $\mathcal{J}_s \subset \Omega_\Gamma$, $\mathcal{J}_t \subset \Omega_\Gamma$ are the ideals defined in Definition 5.11. Then we have the inclusions of chain complexes*

$$\mathcal{J}_s \subset \mathcal{J}_t \subset \Omega_\Gamma,$$

which induce a short exact sequence of chain complexes

$$0 \longrightarrow \mathcal{J}_t/\mathcal{J}_s \longrightarrow \Omega_\Gamma/\mathcal{J}_s \longrightarrow \Omega_\Gamma/\mathcal{J}_t \longrightarrow 0 \tag{5.77}$$

where $(\Omega_\Gamma/\mathcal{J}_s, d) = (\Omega_G, d)$ is a differential calculus of the graph G and $(\Omega_\Gamma/\mathcal{J}_t, d) = (\Omega_F, d)$ is a differential calculus of the graph F. The cohomology long exact sequence of (5.77) has the following form

$$0 \longrightarrow H^0(\Omega_G) \longrightarrow H^0(\Omega_F) \longrightarrow H^1(\mathcal{J}_t/\mathcal{J}_s) \longrightarrow H^1(\Omega_G) \longrightarrow \dots \tag{5.78}$$

Proof. Similarly to the proof of Theorem 4.8. ■

Remark 5.14 *In the case of an inclusion $r\colon F \to G$ from (5.76) we can write down a commutative diagram similarly to (4.52)*

$$\begin{array}{ccccc} F & & \xrightarrow{r} & & G \\ & \searrow^{f} & & {}^{g}\swarrow & \\ & & H & & \end{array}$$

where H is a simple complete graph. Theorem 4.8 is applicable in this situation, as well. The advantage of Theorem 5.13 is in simplification of all computations, since we can work with very small number of basic elements directly described in Corollary 5.3.

Remark 5.15 *The monotonic graphs with inclusion maps give a subcategory of the category GR. The realization Theorem 4.21 is true in the category of monotonic graphs. It is easy to see, that we can define suspension and a wedge sum in the category of monotonic graphs. Now let D_2 be a monotonic graph that has two vertices and no edges. Then*

$$H^k(D_2) \cong \begin{cases} \mathbb{K}^2, & k = 0 \\ 0, & k \neq 0 \end{cases},$$

$$H^k(SD_2) \cong \begin{cases} \mathbb{K}, & k = 0, 1 \\ 0, & k \geq 2. \end{cases}$$

Now we can repeat all the constructions from the proof of Theorem 4.21 in the category of monotonic graphs.

Now we describe some more sufficiently wide classes of monotonic graphs for which there exists a geometric realization of differential calculus similarly to Corollary 5.8.

Let Γ be a complete monotonic graph with the set $V = \{0, 1, \dots, n\}$ and Θ be a complete monotonic graph with the set $W = \{0, 1, \dots, k\}$ $(k \leq n)$ of vertices. We have a natural inclusion

$$\sigma \colon \Theta \to \Gamma.$$

By Lemma 4.9 we have a morphism of cochain complexes

$$\mathcal{U}(\sigma) \colon \Omega_\Gamma \to \Omega_\Theta$$

which induces a morphism of chain complexes

$$\mathcal{U}(\sigma)^* \colon \Omega_\Theta^* \to \Omega_\Gamma *$$

by standard rule

$$\mathcal{U}(\sigma)^*(f) = f \circ \mathcal{U}(\sigma), \quad f \colon \Omega_\Theta \to \mathbb{K}.$$

Proposition 5.16 *There exists a commutative diagram of chain complexes*

$$\begin{array}{ccc} \Omega_\Theta^* & \xrightarrow{\mathcal{U}(\sigma)^*} & \Omega_\Gamma * \\ \downarrow T & & \downarrow T \\ C(\Delta^k) & \xrightarrow{\tau_*} & C(\Delta^n) \end{array}$$

where τ_* is induced by natural inclusion $\tau \colon \Delta^k \to \Delta^n$ on the first k-face.

Proof. Follows from Proposition 3.6, Corollary 5.3, and Corollary 5.8. ∎

Let Γ be a complete monotonic graph with the set $V = \{0, 1, \dots, n\}$ of vertexes and the set E_Γ of edges, and let $s \colon G_k \to \Gamma$ be the natural inclusion of the subgraph G_k with the same set of vertexes and the set of edges $E_k = E_{G_k} = E_\Gamma \setminus \{\{k, k+1\}\}$ where k is a number $0 \leq k \leq n-1$. That is the graph G_k is obtained from Γ by deleting exactly one edge $\{k, k+1\}$.

By Proposition 5.12 we have an exact sequence

$$0 \longrightarrow \mathcal{J}_s \longrightarrow \Omega_\Gamma \longrightarrow \Omega_{G_k} \longrightarrow 0$$

where $\mathcal{J}_s$ is defined in Definition 5.11.

Proposition 5.17 *For the inclusion* s *we have* $\mathcal{J}_s^p = \mathcal{E}_s^p$ *where*

$$\mathcal{E}_s^0 = \{0\},\ \mathcal{E}_s^1 = \langle e^{ij} : i = k, j = k+1 \rangle,$$

and for $p \geq 2$

$$\mathcal{E}_s^p = \left\langle e^{i_0 i_1 \dots i_p} : (0 \leq i_0 < \dots < i_p \leq) \text{ and } \{k, k+1\} \subset \{i_0, i_1, \dots, i_p\} \right\rangle \subset \Omega_\Gamma^p,$$

where $\langle A \rangle$ *means a subspace generated by* A.

Proof. For $p = 0, 1$ the statement follows from Definition 5.11. The inclusion $\mathcal{E}_s^p \subset \mathcal{J}_s^p$ follows from Definition 5.11. Let

$$w = e^{i_0 i_1 \dots [k][k+1] \dots i_{p-1}} \in \mathcal{E}_s^{p-1} \subset \Omega_\Gamma^{p-1},\ p \geq 2$$

be a basic non-allowed element of Ω_Γ^{p-1}. By Corollary 5.3

$$dw = (-1)^{l+1} \widetilde{\sum_{j \neq i_l; j \neq i_{l+1}}^{p}} e^{i_0 i_1 \dots i_l j i_{l+1} \dots i_{p-1}}$$

where $\widetilde{\ }$ means that in the summing contains only elements with strongly monotonic increasing sets of indices. Since we can not put integer number between k and $k+1$ to obtain monotonic increasing sequence, the all elements e^α in the sum satisfy condition $\{k, k+1\} \subset \{\alpha\}$. Hence $dw \in \mathcal{E}_s^p$, and Proposition is proved. ∎

Corollary 5.18 (i) *The basic elements of the differential calculus* $(\Omega_{G_k}, d_{G_k}) = (\Omega_\Gamma / \mathcal{J}_s, d)$ *of the graph* G_k *can be represented by classes* $[e^{i_0 i_1 \dots i_p}] \in \Omega_{G_k}^p$ *of elements* $e^{i_0 i_1 \dots i_p} \in \Omega_\Gamma^p$ *such that* $0 \leq i_0 < i_1 < \dots < i_p \leq n$ *and* $\{k, k+1\}$ *is not a subset of* $\{i_0, i_1, \dots, i_p\}$.

(ii) *For* $0 \leq p, q \leq n$, *the exterior multiplication* $\bullet$ *of the basis elements is given by the following formula*

$$[e^{i_0 i_1 \dots i_p}] \bullet [e^{j_0 j_1 \dots j_q}] = \begin{cases} 0, & i_p \neq j_0 \\ [e^{i_0 i_1 \dots i_p j_1 \dots j_q}], & i_p = j_0. \end{cases}$$

(iii) *The differential* d_{G_k} *is given on basis elements by*

$$d_{G_k}\left[e^{i_0 i_1 \dots i_p}\right] = \widehat{\sum_{j \neq i_0}} [e^{j i_0 i_1 \dots i_p}] - \widehat{\sum_{j \neq i_0; j \neq i_1}} [e^{i_0 j i_1 \dots i_p}] + \dots$$

$$+ (-1)^{l+1} \widehat{\sum_{j \neq i_l; j \neq i_{l+1}}} [e^{i_0 i_1 \dots i_l j i_{l+1} \dots i_p}] + \dots + (-1)^{k+1} \widehat{\sum_{j \neq i_p}} [e^{i_0 i_1 \dots i_p j}],$$

where $\widehat{\ }$ *means that every element* $e^{i_0, \dots i_m}$ *in the sum satisfies the condition* $0 \leq i_0 < \dots < i_m \leq n$ *and* $\{k, k+1\}$ *is not a subset of* $\{i_0, i_1, \dots, i_m\}$.

Proof. Follows from Proposition 5.17 and Corollary 5.3. ■

Let as in Example 5.5 Δ^n be a simplicial complex given by the simplex $\Delta^n = [0,1,\dots,n]$. Let

$$\Delta^{n-1}_{k+1} = [0,1,\dots,k,k+2,\dots,n] \subset \Delta^n = [0,1,\dots,n]$$

and

$$\Delta^{n-1}_{k} = [0,1,\dots,k-1,k+1,\dots,n] \subset \Delta^n = [0,1,\dots,n]$$

be two subcomplexes of Δ^n, that are given by $(n-1)$-dimensional faces of Δ^n. Define a simplicial complex

$$\Delta^{n-1}_{k,k+1} = \Delta^{n-1}_{k} \cup \Delta^{n-1}_{k+1}$$

that geometrically corresponds to the union of two $(n-1)$-faces of Δ^n. The definition is correct, since an intersection

$$\Delta^{n-1}_{k+1} \cap \Delta^{n-1}_{k} = [0,1,\dots,k-1,k+2,\dots,n]$$

is a $(n-2)$-simplex $\Delta^{n-2}_{k,k+1}$.

Theorem 5.19 *We have a commutative diagram of chain complexes*

$$\begin{array}{ccc} \Omega^*_{G_k} & \xrightarrow{\mathcal{U}(s)^*} & \Omega_{\Gamma}* \\ \downarrow T' & & \downarrow T \\ C\left(\Delta^{n-1}_{k,k+1}\right) & \xrightarrow{\tau_*} & C(\Delta^n) \end{array} \tag{5.79}$$

where T' and T are isomorphisms, and τ_* is induced by a natural inclusion $\tau\colon \Delta^{n-1}_{k,k+1} \to \Delta^n$.

Proof. In diagram (5.79) the right vertical isomorphism and the horizontal morphisms are already defined. We must define T' and check, that

$$T \circ \mathcal{U}(s)^* = \tau_* \circ T'.$$

Consider a basic element $[e_{i_0\dots i_p}]^* \in [\Omega_{G_k}]^*_p$ that is dual to $[e^{i_0\dots i_p}] \in \Omega^p_{G_k}$ which is a basic element described in Corollary 5.18. Define T' on the basic elements by

$$T'([e_{i_0\dots i_p}]^*) = [i_0,\dots,i_p] \subset C^p(\Delta^n).$$

and extend to $[\Omega_{G_k}]^*_p$ by linearity.

The simplex $T'([e_{i_0\dots i_p}]^*)$ lies in $C^p\left(\Delta^{n-1}_{k,k+1}\right)$ since the set $\{i_0,\dots,i_p\}$ does not contain as a subset $\{k,k+1\}$, and any simplex from $C\left(\Delta^{n-1}_{k,k+1}\right)$ lies in the image of T'. From Theorem 5.7 and Corollary 5.18 we obtain, that

$$\delta_G([e_{i_0\dots i_p}]^*) = (-1)^q \sum_{0\le q\le p} [e_{i_0 i_1 \dots i_{q-1}\hat{i_q} i_{q+1}\dots i_p}]^*$$

where $[e_{i_0i_1\dots i_{q-1}\hat{i_q}i_{q+1}\dots i_p}]^*$ are the basic elements of $[\Omega_{G_k}]^*_{p-1}$. Hence, T' commutes with the differentials. By Proposition 5.17 the projection $\Omega_\Gamma \to \Omega_{G_k}$ is given on the basic elements by the formulas

$$e^{i_0\dots i_p} \longrightarrow \begin{cases} 0, & \text{if } \{k, k+1\} \subset \{i_0, ..., i_p\} \\ [e^{i_0\dots i_p}], & \text{otherwise} \end{cases}$$

Hence,

$$\mathcal{U}(s)^*([e_{i_0\dots i_p}]^*) = e^*_{i_0\dots i_p}, \quad [e_{i_0\dots i_p}]^* \in [\Omega_{G_k}]^*_p, \quad e^*_{i_0\dots i_p} \in [\Omega_\Gamma]^*_p$$

and the diagram is commutative, which finishes the proof. ■

Corollary 5.20 *Under the assumptions of Theorem 5.19, we have*

$$H^i(\Omega_{G_k}) \cong \begin{cases} \mathbb{K}, & \text{if } i = 0 \\ 0, & \text{if } i \geq 1. \end{cases}$$

Proof. Mayer-Vietorias exact sequence for simplicial complexes (see [11]). ■

Now let Γ be a complete monotonic graph with a set of vertices $V = \{0, 1, 2, \dots, n\}$. Let $K \subset V$ be a subset such that $n \notin K$. Consider a monotonic subgraph $s\colon G_K \to \Gamma$ with the same number of vertexes and the set of edges

$$E_K = E_{G_K} = E_\Gamma \setminus \{\{i, i+1\} : i \in K\}.$$

Theorem 5.21 *There exists a simplicial complex S_K with an inclusion $\tau\colon S_K \to \Delta^n$ such that the following diagram is commutative diagram*

$$\begin{array}{ccc} \Omega^*_{G_K} & \xrightarrow{\mathcal{U}(s)^*} & \Omega_\Gamma * \\ \downarrow T' & & \downarrow T \\ C\,(S_K) & \xrightarrow{\tau_*} & C(\Delta^n) \end{array} \tag{5.80}$$

where T' and T are isomorphisms, and τ_* is induced by a natural inclusion $\tau\colon S_K \to \Delta^n$.

Proof. The proof is based on the same line of arguments as the proof of Theorem 5.19, that we briefly repeat. By Proposition 5.12 we have an exact sequence

$$0 \longrightarrow \mathcal{J}_s \longrightarrow \Omega_\Gamma \longrightarrow \Omega_{G_K} \longrightarrow 0$$

where $\mathcal{J}_s$ is defined in Definition 5.11. Similarly to the proof of Proposition 5.17 we see that $d\mathcal{E}^i_s \subset \mathcal{E}^{i+1}_s$ and, hence, $\mathcal{J}^p_s = \mathcal{E}^p_s$ where

$$\mathcal{E}^0_s = \{0\}, \qquad \mathcal{E}^1_s = \langle e^{[k][k+1]} : k \in K \rangle,$$

and, for $p \geq 2$,

$$\mathcal{E}^p_s = \left\langle e^{i_0i_1\dots i_p} \in \Omega^p_\Gamma : \exists\ k \in K \text{ such that } \{k, k+1\} \subset \{i_0, i_1, \dots, i_p\} \right\rangle \subset \Omega^p_\Gamma,$$

where $\langle A\rangle$ means a subspace generated by A. The basic elements of the differential calculus $(\Omega_{G_K}, d_K) = (\Omega_\Gamma/\mathcal{J}_s, d)$ of the graph G_K can be represented by classes $[e^{i_0i_1\dots i_p}] \in \Omega^p_{G_K}$ of elements $e^{i_0i_1\dots i_p} \in \Omega^p_\Gamma$ such that $0 \le i_0 < i_1 < \dots < i_p \le n$ and, for any $k \in K$, $\{k, k+1\}$ is not a subset of $\{i_0, i_1, \dots, i_p\}$. The differential d_K is given on the basis elements by

$$\begin{aligned} d_K\left[e^{i_0i_1\dots i_p}\right] &= \widehat{\sum_{j\neq i_0}}[e^{ji_0i_1\dots i_p}] - \widehat{\sum_{j\neq i_0; j\neq i_1}}[e^{i_0ji_1\dots i_p}] + \dots \\ &\quad +(-1)^{l+1}\widehat{\sum_{j\neq i_l; j\neq i_{l+1}}}[e^{i_0i_1\dots i_lji_{l+1}\dots i_p}] + \dots + (-1)^{k+1}\widehat{\sum_{j\neq i_p}}[e^{i_0i_1\dots i_pj}] \end{aligned}$$

where $\widehat{\ }$ means that every element $e^{i_0,\dots i_m}$ of the sum satisfies the conditions $0 \le i_0 < \dots < i_m \le n$ and, for any $k \in K$, the pair $\{k, k+1\}$ is not a subset of $\{i_0, i_1, \dots, i_m\}$. Let for a number $k \in K$

$$\Delta^{n-1}_{k,k+1} = \Delta^{n-1}_k \cup \Delta^{n-1}_{k+1}$$

be a simplicial complex, defined above, that geometrically corresponds to union of two $(n-1)$-faces of Δ^n. It is given by the union of all simplexes from Δ^n that does not contain the edge $\{k, k+1\}$. Set

$$S_K = \bigcap_{k\in K} \Delta^{n-1}_{k,k+1} .$$

Equivalently, S_K can be described as the union of all the simplexes from Δ^n that do not contain an edge in the form $\{k, k+1\}$ for $k \in K$. In the diagram (5.80) the right vertical isomorphism and the horizontal morphisms are already defined. We must define T' and to check, that

$$T \circ \mathcal{U}(s)^* = \tau_* \circ T'.$$

Define T' on the basic elements by

$$T'([e_{i_0\dots i_p}]^*) = [i_0, \dots, i_p] \subset C^p(\Delta^n).$$

and extend to $[\Omega_{G_K}]^*_p$ by linearity. We obtain an isomorphism $T' \colon [\Omega_{G_K}]^* \longrightarrow C(S_K)$ of $\mathbb{K}$-modules which commutes with differentials. The projection $\Omega_\Gamma \to \Omega_{G_K}$ is given on basic elements by the condition

$$e^{i_0\dots i_p} \longrightarrow \begin{cases} 0, & \text{if } \{k, k+1\} \subset \{i_0, \dots, i_p\} \text{ for at least one } k \in K \\ [e^{i_0\dots i_p}], & \text{otherwise} \end{cases}$$

Hence

$$\mathcal{U}(s)^*([e_{i_0\dots i_p}]^*) = e^*_{i_0\dots i_p}, \quad [e_{i_0\dots i_p}]^* \in [\Omega_{G_k}]^*_p, \quad e^*_{i_0\dots i_p} \in [\Omega_\Gamma]^*_p$$

and the diagram (5.80) is commutative, which finishes the proof. ■

Let Γ be a complete monotonic graph with the set $V = \{0, 1, \dots, n\}$ of vertexes ($n \geq 2$) and the set E_Γ of edges. Let $s\colon F_k \to \Gamma$ be the natural inclusion of the subgraph F_k with the same set of vertexes and the set of edges

$$E_k = E_{F_k} = E_\Gamma \setminus \{\{k, k+2\}\}$$

where $0 \leq k \leq n-2$. That is, the graph F_k is obtained from Γ by deleting exactly one edge $\{k, k+2\}$.

Theorem 5.22 *There exists a simplicial complex $\Delta^{n-1}_{k,k+2}$ with an inclusion $\tau\colon \Delta^{n-1}_{k,k+2} \to \Delta^n$ such that the following diagram is commutative*

$$\begin{array}{ccc} \Omega^*_{F_K} & \xrightarrow{\mathcal{U}(s)^*} & \Omega_\Gamma * \\ \downarrow T' & & \downarrow T \\ C\left(\Delta^{n-1}_{k,k+2}\right) & \xrightarrow{\tau_*} & C(\Delta^n) \end{array} \tag{5.81}$$

where T' and T are isomorphisms, and τ_* is induced by a natural inclusion $\tau\colon \Delta^{n-1}_{k,k+2} \to \Delta^n$.

Proof. By Proposition 5.12 we have an exact sequence

$$0 \longrightarrow \mathcal{J}_s \longrightarrow \Omega_\Gamma \longrightarrow \Omega_{F_K} \longrightarrow 0$$

where $\mathcal{J}_s$ is defined in Definition 5.11. By definition, $\mathcal{E}^0_s = \{0\}$ and, for $p \geq 1$,

$$\mathcal{E}^p_s = \left\langle e^{i_0 i_1 \dots i_p} \in \Omega^p_\Gamma : i_m = k, i_{m+1} = k+2, \text{ for } i_m, i_{m+1} \in \{i_0, i_1, \dots, i_p\} \right\rangle \subset \Omega^p_\Gamma.$$

Now we can describe $\mathcal{J}^p_s$. For $e^{i_0 i_1 \dots [k][k+2] \dots i_p} \in \mathcal{E}^p_s$, by description of differential in Corollary 5.3, we obtain that

$$d_\Gamma(e^{i_0 i_1 \dots [k][k+2] \dots i_p}) = (\pm 1)(e^{i_0 i_1 \dots [k][k+1][k+2] \dots i_p}) + w, \quad \text{where} \quad w \in \mathcal{E}^{p+1}_s.$$

Hence,

$$\mathcal{J}^0_s = \{0\}, \qquad \mathcal{J}^1_s = \langle e^{[k][k+2]} \rangle = \mathcal{E}^1_s,$$

and, for $p \geq 2$, $\mathcal{J}^p_s$ is generated by those $e^{i_0 \dots i_p} \in \Omega^p_\Gamma$ for which

$$i_m = k, i_{m+1} = k{+}2, \text{ for } i_m, i_{m+1} \in \{i_0, i_1, \dots, i_p\} \text{ or } \{k, k{+}1, k{+}2\} \subset \{i_0, i_1, \dots, i_p\}.$$

Consider a simplicial complex $\Delta^{n-1}_{k,k+2}$ which is the union of two $(n-1)$-simplexes

$$\Delta^{n-1}_{k+2} = [0, 1, \dots, k, k+1, k+3, \dots, n] \subset \Delta^n = [0, 1, \dots, k, k+1, k+2, \dots, n]$$

and

$$\Delta^{n-1}_k = [0, 1, \dots, k-1, k+1, k+2, \dots, n] \subset \Delta^n = [0, 1, \dots, k, k+1, k+2, \dots, n]$$

where

$$\Delta^{n-1}_{k+2} \cap \Delta^{n-1}_k = \Delta^{n-2}_{k,k+2} = [0, 1, \dots, k-1, k+1, k+3, \dots, n].$$

The basic elements of $[\Omega_{F_k}]^*_m$ are given by the classes

$$[e^*_{i_0 \dots i_m}],$$

where $\{i_0, \dots, i_m\}$ does not contain pair $\{i_l, i_{l+1}\} = \{k, k+2\}$ and triple $\{k, k+1, k+2\}$. We define

$$T'([e^*_{i_0 \dots i_m}]) = \{i_0, \dots, i_m\} \subset \Delta^{n-1}_{k,k+2}.$$

Then the proof is finished similarly to that of Theorems 5.19 and 5.21. ■

Corollary 5.23 *Let F_k be a graph from Theorem 5.22. Then*

$$H^i(\Omega_{F_k}) \cong \begin{cases} \mathbb{K}, & i = 0 \\ 0, & i \geq 1. \end{cases}$$

Now let Γ be a complete monotonic graph with a set of vertices $V = \{0, 1, 2, \dots, n\}$. Let $K \subset V$ be a subset such that $n \notin K$, and $L \subset V$ such that $n \notin L, (n-1) \notin L$. Consider a monotonic subgraph $s \colon G_{K,L} \to \Gamma$ with the same number of vertexes as Γ and the set of edges

$$E_K = E_{G_K} = E_\Gamma \setminus [\{\{i, i+1\} : i \in K\} \cup \{\{j, j+2\} : j \in L\}]$$

Theorem 5.24 *There exists a simplicial complex $S_{K,L}$ with an inclusion $\tau \colon S_{K,L} \to \Delta^n$ such that there exists a commutative diagram of chain complexes*

$$\begin{array}{ccc} \Omega^*_{G_{K,L}} & \xrightarrow{\mathcal{U}(s)^*} & \Omega_\Gamma * \\ \downarrow T' & & \downarrow T \\ C\left(S_{K,L}\right) & \xrightarrow{\tau_*} & C(\Delta^n) \end{array} \tag{5.82}$$

where T' and T are isomorphisms, and τ_* is induced by a natural inclusion $\tau \colon S_{K,L} \to \Delta^n$.

Proof. The simplicial complex $S_{K,L}$ is defined as

$$S_{K,L} = \left(\bigcap_{k \in K} \Delta^{n-1}_{k,k+1} \right) \bigcap \left(\bigcap_{l \in L} \Delta^{n-1}_{l,l+2} \right).$$

The same line of arguments as in the proof of Theorems 5.19, 5.21, and 5.22 finishes the proof. ■

Theorem 5.24 obviously reduces computation of cohomologies for a wide class of graphs to that of simplicial homologies.

Example 5.25 Let Γ be a complete monotonic graph with the set of vertices $V = \{0, 1, 2, 3, 4\}$ and let G be the graph that is obtained from Γ by removing the edges $\{1, 2\}, \{2, 3\}, \{1, 3\}$. Then G satisfies the hypotheses of Theorem 5.24 and, hence, can be realized as a simplicial complex S as shown on Fig. 1.

Let G' be the graph that is obtained from G by further removing the edge $\{0, 4\}$ (see Fig. 2).

Graph G' does not satisfy the hypothesis of Theorem 5.24, and one can show that it does not admit a geometric realization as a simplicial complex. The chain complex of the graph G' was explicitly described in [7].

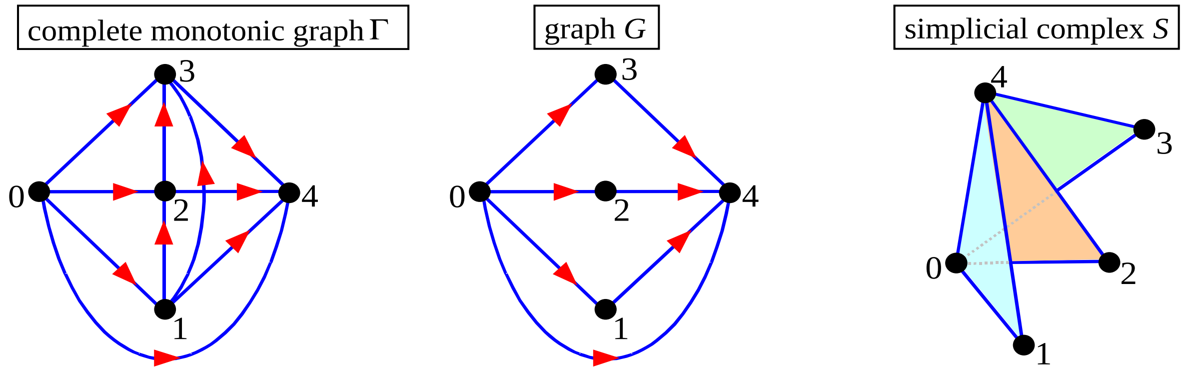

Figure 1: Graph G and its geometric realization as a simplicial complex S

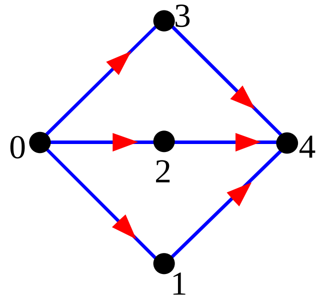

Figure 2: Graph G'

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