Cluster point processes on manifolds

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Abstract

The probability distribution $\mu_{\rm cl}$ of a general cluster point process in a Riemannian manifold X (with independent random clusters attached to points of a configuration with distribution μ) is studied via the projection of an auxiliary measure $\hat{\mu}$ in the space of configurations $\hat{\gamma} = \{(x, \bar{y})\} \subset X \times \mathfrak{X}$, where $x \in X$ indicates a cluster "centre" and $\bar{y} \in \mathfrak{X} := \bigsqcup_n X^n$ represents a corresponding cluster relative to x. We show that the measure $\mu_{\rm cl}$ is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$ of compactly supported diffeomorphisms of X, and prove an integration-by-parts formula for $\mu_{\rm cl}$. The associated equilibrium stochastic dynamics is then constructed using the method of Dirichlet forms. General constructions are illustrated by examples including Euclidean spaces, Lie groups, homogeneous spaces, Riemannian manifolds of non-positive curvature and metric spaces. The paper is an extension of our earlier results for Poisson cluster measures [J. Funct. Analysis 256 (2009) 432–478] and for Gibbs cluster measures [http://arxiv.org/abs/1007.3148], where different projection constructions were utilised.

Keywords: Cluster point process; Configuration space; Riemannian manifold; Poisson measure; Projection; Quasi-invariance; Integration by parts; Dirichlet form; Stochastic dynamics

MSC 2010: Primary 58J65; Secondary 31C25, 46G12, 60G55, 70F45

Contents

1. Introduction

The concept of a *configuration space* (over a suitable Riemannian manifold) is instrumental in the description of various types of multi-particle structures and naturally appears in many areas of mathematics and mathematical physics (e.g., theory of random point processes, statistical mechanics, quantum field theory, representation theory) and applied sciences (e.g., chemical physics, image processing, spatial ecology, astronomy, etc.).

Despite not possessing any Banach manifold structure, configuration spaces have many features of proper manifolds and can indeed be endowed with "manifold-like" structures (see [31] and also [3, 4]). As it turns out, the way to do it depends heavily on the choice of a suitable probability measure μ on the configuration space $\varGamma_X$ (over the manifold X). Such a choice is often suggested by a physical system under study, but, in order to furnish a meaningful analytical framework, the measure μ must satisfy certain regularity properties, such as the Diff_0-quasi-invariance with respect to the action of certain diffeomorphism group and/or an integration-by-parts (IBP) formula. Hence, it is not surprising that the study of the configuration space as a measure space $(\varGamma_X, \mu)$ requires tools and techniques at the interface of geometric analysis and measure theory. According to this paradigm, it is important (i) to prove the quasi-invariance and IBP formulae for a wide class of measures μ arising in applications, and (ii) to study the dependence of the properties of the measure μ on the topology and geometry of the underlying manifold X and their interplay with the multi-particle structure of the space $\varGamma_X$.

Such a programme has been implemented for the Poisson and Gibbs measures on $\varGamma_X$ in the case where $X = \mathbb{R}^d$ (see [3, 4, 5, 2, 1] and further references therein). The present paper is a step towards realisation of this programme for the important class of (in general, non-Gibbsian) measures on $\varGamma_X$ emerging as distributions of *cluster point processes* in X. Intuitively, a cluster point process is obtained by generating random clusters around points of the background configuration of cluster "centres" [15]. Cluster models are well known in the general theory of random point processes [14, 15] and are widely used (both in temporal and spatial domains) in numerous applications ranging from neurophysiology (nerve impulses) and ecology (spatial aggregation of species) to seismology (earthquakes) and cosmology (constellations and galaxies); see [10, 14, 15] for some references to original papers.

In our earlier papers [8, 9, 10, 11], we have developed a projection construction of Poisson and Gibbs cluster processes in a Euclidean space $X = \mathbb{R}^d$, based on the representation of their probability distributions (i.e., the corresponding measures on the configuration space Γ_X) as the push-forward (image) of suitable auxiliary measures on a more complex configuration space $\Gamma_{\mathfrak{X}}$ over the disjoint-union space $\mathfrak{X} := \bigsqcup_n X^n$, with "droplet" points $\bar{y} \in \mathfrak{X}$ representing individual clusters (of variable size). Such an approach allows one to adapt the ideas of analysis and geometry on configuration spaces developed earlier by Albeverio, Kondratiev and Röckner [3, 4] for plain (i.e., non-cluster) Poisson and Gibbs measures in Γ_X, and to obtain results including the Diff_0-quasi-invariance and IBP formula.

In the present paper, we extend the projection approach to the case of cluster measures on general Riemannian manifolds X and with arbitrary centre processes. In so doing, suitable smoothness properties of the distribution of individual cluster are required, but it should be stressed that *no smoothness of the centre process is needed.* That is to say, attaching "nice" clusters to points of a centre configuration acts as smoothing of the entire cluster process. To an extent, this may be thought of as an infinite-dimensional analogue of the well-known fact that the convolution of two measures in $\mathbb{R}^d$ is absolutely continuous provided that at least one of those measures is such. However, this analogy should not be taken too far, since the relationship between centres and clusters is asymmetric (the latter are attached to the former, but not vice versa); in particular, as it turns out, smoothness of the centre process alone is not sufficient for the smoothness of the resulting cluster process. Let us point out that the results of the present paper are new even in the case of Poisson and Gibbs cluster point processes in $\mathbb{R}^d$, where our new approach allows one to handle more general models, for instance with the probability distribution of centres given by a Poisson measure with a *non-smooth intensity*, or by a Gibbs measure with a *non-smooth interaction potential*.

The structure of the paper is as follows. In Section 2 we introduce the general framework of configuration spaces and measures on them and discuss a "fibre bundle" structure of the configuration space over a product manifold. Section 3 is devoted to the projection construction of cluster measures μ_{cl}. Here we derive necessary conditions for the cluster measure μ_{cl} to be well defined (i.e., with no multiple and accumulation points) and study the existence of moments. In Section 4 we prove the Diff_0-quasi-invariance and an IBP formula for μ_{cl}. Furthermore, for a general cluster measure we are able to construct the corresponding Dirichlet form and to prove its closedness, which implies in a standard way the existence of the corresponding equilibrium stochastic dynamics. In Section 5 we discuss examples of cluster distributions that can be generated in a natural way via certain manifold structures, such as the group action in the case of homogeneous manifolds and a metric structure for general Riemannian manifolds. Finally, the Appendix contains some additional or supporting material.

2. Configuration spaces and measures

2.1. General setup: probability measures on configuration spaces

Let X be a Polish space equipped with the Borel σ-algebra $\mathcal{B}(X)$ generated by the open sets. Let $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, and consider the space $\mathfrak{X}$ built from all Cartesian powers of X, that is, the disjoint union

$$\mathfrak{X} := \bigsqcup_{n\in\mathbb{Z}_+} X^n, \tag{2.1}$$

including $X^0 = \{\emptyset\}$. That is, $\bar{x} = (x_1, x_2, \dots) \in \mathfrak{X}$ if and only if $\bar{x} \in X^n$ for some $n \in \mathbb{Z}_+$. We take the liberty to write $x_i \in \bar{x}$ if x_i is a coordinate of the "vector" $\bar{x}$. The space $\mathfrak{X}$ is endowed with the natural disjoint union topology induced by the topology in X.

Remark 2.1. Note that a set $B \subset \mathfrak{X}$ is compact if and only if $B = \bigsqcup_{n=0}^{N} B_n$, where $N < \infty$ and B_n are compact subsets of X^n, respectively.

Remark 2.2. $\mathfrak{X}$ is a Polish space as a disjoint union of Polish spaces.

Denote by $\mathcal{N}(X)$ the space of $\mathbb{Z}_+$-valued measures on $\mathcal{B}(X)$ with countable (i.e., finite or countably infinite) support. Consider the natural projection

$$\mathfrak{X} \ni \bar{x} \mapsto \mathfrak{p}(\bar{x}) := \sum_{x_i\in\bar{x}} \delta_{x_i} \in \mathcal{N}(X), \tag{2.2}$$

where δ_x is the Dirac measure at point $x \in X$. That is to say, under the map $\mathfrak{p}$ each vector from $\mathfrak{X}$ is "unpacked" into its components to yield a countable aggregate of (possibly multiple) points in X, which can be interpreted as a *generalised configuration* γ,

$$\mathfrak{p}(\bar{x}) \leftrightarrow \gamma := \bigsqcup_{x_i\in\bar{x}} \{x_i\}, \qquad \bar{x} = (x_1, x_2, \dots) \in \mathfrak{X}. \tag{2.3}$$

In what follows, we interpret the notation γ either as an aggregate of points in X or as a $\mathbb{Z}_+$-valued measure or both, depending on the context. Even though generalised configurations are not, strictly speaking, subsets of X (because of possible multiplicities), it is convenient to use set-theoretic notation, which should not cause any confusion. For instance, we write $\gamma \cap B$ for the restriction of configuration γ to a subset $B \in \mathcal{B}(X)$. For a function $f\colon X \to \mathbb{R}$ we denote

$$\langle f, \gamma\rangle := \sum_{x_i\in\gamma} f(x_i) \equiv \int_X f(x)\, \gamma(\mathrm{d}x). \tag{2.4}$$

In particular, if $\mathbf{1}_B(x)$ is the indicator function of a set $B \in \mathcal{B}(X)$ then $\langle \mathbf{1}_B, \gamma\rangle = \gamma(B)$ is the total number of points (counted with their multiplicities) in $\gamma \cap B$.

Definition 2.1. The *configuration space* $\varGamma_X^\sharp$ is the set of all generalised configurations γ in X, endowed with the *cylinder σ-algebra* $\mathcal{B}(\varGamma_X^\sharp)$ generated by the class of cylinder sets $C_B^n := \{\gamma \in \varGamma_X^\sharp \colon \gamma(B) = n\}$, $B \in \mathcal{B}(X)$, $n \in \mathbb{Z}_+$.

Remark 2.3. It is easy to see that the map $\mathfrak{p}\colon \mathfrak{X} \to \varGamma_X^\sharp$ defined by formula (2.3) is measurable.

Let us denote by $\mathrm{M}_+(X)$ the class of non-negative measurable functions on X.

Lemma 2.1. *For any* $f \in \mathrm{M}_+(X)$*, the pairing* $\langle f, \cdot\rangle$ *defined in* (2.4) *is a measurable function on the configuration space* $\varGamma_X^\sharp$*.*

Proof. For functions of the form $f(x) = \sum_{i=1}^k c_i \mathbf{1}_{B_i}(x)$ (with $c_i \ge 0$, $B_i \in \mathcal{B}(X)$, we have

$$\langle f, \gamma\rangle = \sum_{x_i\in\gamma} \sum_{i=1}^k c_i \mathbf{1}_{B_i}(x_i) = \sum_{i=1}^k c_i\,\gamma(B_i), \qquad \gamma \in \varGamma_X^\sharp,$$

which is a $\mathcal{B}(\varGamma_X^\sharp)$-measurable function of γ since each of the functions $\gamma(B_i)$ is measurable by definition of the cylinder σ-algebra $\mathcal{B}(\varGamma_X)$. The general case is then derived by the usual approximation and monotone class argument (see, e.g., [15, §A1.1]). □

Definition 2.2. A configuration $\gamma \in \varGamma_X^\sharp$ is said to be *locally finite* if $\gamma(B) < \infty$ for any compact set $B \subset X$. A configuration $\gamma \in \varGamma_X^\sharp$ is called *simple* if $\gamma(\{x\}) \le 1$ for each $x \in X$. A configuration $\gamma \in \varGamma_X^\sharp$ is called *proper* if it is both locally finite and simple. The set of all proper configurations is denoted by $\varGamma_X$ and called the *proper configuration space* over X. The corresponding σ-algebra $\mathcal{B}(\varGamma_X)$ is generated by the cylinder sets $C_B^n = \{\gamma \in \varGamma_X \colon \gamma(B) = n\}$ $(B \in \mathcal{B}(X),\ n \in \mathbb{Z}_+)$.

Conventional theory of point processes (see, e.g., [15]) is usually built on the assumption that the sample configurations of the process almost surely (a.s.) have no accumulation or multiple points, so that its distribution μ is a probability measures on the proper configuration space $(\varGamma_X, \mathcal{B}(\varGamma_X))$. However, many elements of standard theory can be extended to the case of generalised configurations without much trouble. In particular, it still holds by the Kolmogorov extension theorem (see, e.g., [27, Theorem 5.1, [page 144] or [15, Theorem A1.5.IV, page 381]) that any probability measure μ on $(\varGamma_X^\sharp, \mathcal{B}(\varGamma_X^\sharp))$ is uniquely determined by a family of its finite-dimensional distributions $\mu(C_{B_1}^{n_1} \cap \cdots \cap C_{B_k}^{n_k})$ (which is necessarily consistent). Furthermore, any probability measure μ on $\varGamma_X^\sharp$ is uniquely characterised by its Laplace functional

$$L_\mu[f] := \int_{\varGamma_X^\sharp} \mathrm{e}^{-\langle f, \gamma\rangle}\,\mu(\mathrm{d}\gamma), \qquad f \in \mathrm{M}_+(X) \tag{2.5}$$

(note that the integral in (2.5) is well defined since $0 \le \mathrm{e}^{-\langle f,\gamma\rangle} \le 1$). To see why $L_\mu[\cdot]$ completely determines a measure μ on $\mathcal{B}(\varGamma_X^\sharp)$, note that if $B \in \mathcal{B}(X)$ then $L_\mu[s\mathbf{1}_B]$ as a function of $s > 0$ gives the Laplace–Stieltjes transform of the distribution of the random variable $\gamma(B)$ and as such determines the values of the measure μ on the cylinder sets $C_B^n \in \mathcal{B}(\varGamma_X^\sharp)$ $(n \in \mathbb{Z}_+)$. In particular, $L_\mu[s\mathbf{1}_B] = 0$ if and only if $\gamma(B) = \infty$ (μ-a.s.). Similarly, using linear combinations $\sum_{i=1}^k s_i \mathbf{1}_{B_i}$ we can recover the values of μ on the cylinder sets

$$C_{B_1,\dots,B_k}^{n_1,\dots,n_k} := \bigcap_{i=1}^k C_{B_i}^{n_i} = \{\gamma \in \varGamma_X^\sharp : \gamma(B_i) = n_i,\ i = 1, \dots, k\},$$

and hence on the ring $\mathcal{C}(X)$ of finite disjoint unions of such sets. Since the ring $\mathcal{C}(X)$ generates the cylinder σ-algebra $\mathcal{B}(\varGamma_X^\sharp)$, the Kolmogorov extension theorem ensures that the measure μ on $\mathcal{B}(\varGamma_X^\sharp)$ is determined uniquely.

2.2. Cluster point processes

Let us recall the notion of a general cluster point process with independent clusters (see, e.g., [14, 15]). Heuristically, its spatial realisations are built in two stages: (i) a background random configuration of (invisible) "centres" is obtained as a realisation of some point process γ_{c} governed by a probability measure μ on $\varGamma_X$, and (ii) relative to each centre $x \in \gamma_{\mathrm{c}}$, a set of observable secondary points (referred to as a *cluster* centred at x) is generated, independently of all other clusters, according to a point process γ_x' with distribution μ_x on $\varGamma_X$ ($x \in X$). The resulting (countable) assembly of random points, called the *cluster point process* (*CPP*), can be expressed symbolically as

$$\gamma = \bigsqcup_{x\in\gamma_{\mathrm{c}}} \gamma_x' \in \varGamma_X^\sharp,$$

where the disjoint union signifies that possible multiplicities of points are taken into account. More precisely, assuming a suitable measurability of the family of secondary processes γ_x' with respect to $x \in X$, the integer-valued measure corresponding to a CPP realisation γ is represented as

$$\gamma(B) = \int_X \gamma_x'(B)\,\gamma_{\mathrm{c}}(\mathrm{d}x) = \sum_{x\in\gamma_{\mathrm{c}}} \gamma_x'(B), \qquad B \in \mathcal{B}(X). \tag{2.6}$$

The distribution μ_x of the inner-cluster point process γ_x' determines a probability measure η_x on the space $\mathfrak{X}$ symmetric with respect to permutations of coordinates. Conversely, μ_x is a push-forward of the measure η_x under the projection map $\mathfrak{p}\colon \mathfrak{X} \to \varGamma_X^\sharp$ defined by (2.3),

$$\mu_x = \mathfrak{p}^*\eta_x \equiv \eta_x \circ \mathfrak{p}^{-1}. \tag{2.7}$$

Note that for any $x \in X$ and for η_x-a.a. $\bar{y} \in \mathfrak{X}$ the projection set $\mathfrak{p}(\bar{y}) \subset X$ is locally finite and simple.

We assume that the family of measures $\{\eta_x(\cdot),\ x \in X\}$ satisfies the following "weak measurability" condition.

Condition 2.1. For any Borel set $B \in \mathcal{B}(\mathfrak{X})$, the function

$$X \ni x \mapsto \eta_x(B) \in [0,1]$$

is measurable with respect to $\mathcal{B}(X)$.

This has the following useful corollary.

Lemma 2.2. *Under Condition* 2.1, *for any bounded function* $f \in \mathrm{M}_+(\mathfrak{X})$ *the map*

$$X \ni x \mapsto \int_{\mathfrak{X}} f(\bar{y})\,\eta_x(\mathrm{d}\bar{y}) \tag{2.8}$$

is $\mathcal{B}(X)$*-measurable.*

Proof. For a linear combination $f(\bar{y}) = \sum_{i=1}^k c_i \mathbf{1}_{A_i}(\bar{y})$ (with some constants $c_i \ge 0$ and Borel sets $A_i \in \mathcal{B}(\mathfrak{X})$), the integral in (2.8) is reduced to

$$\sum_{i=1}^k c_i \int_{\mathfrak{X}} \mathbf{1}_{B_i}(\bar{y})\,\eta_x(\mathrm{d}\bar{y}) = \sum_{i=1}^k c_i\,\eta_x(B_i),$$

which is a measurable function in x owing to Condition 2.1. The general case then follows by the standard approximation and monotone class argument (see, e.g., [15, §A1.1]). □

We are now ready to give a more formal definition of the cluster process distribution as a measure $\mu_{\rm cl}$ on the configuration space $\varGamma_X^\sharp$. In view of the two-stage description of the cluster process, it is natural to construct the measure $\mu_{\rm cl}$ by first conditioning on the background configuration $\gamma_{\rm c} = \gamma \in \varGamma_X$ of centres and then averaging with respect to its distribution μ. This can be expressed symbolically as an intuitively appealing decomposition

$$\mu_{\rm cl}(A) = \int_{\varGamma_X} \mu_{\rm cl}(A \,|\, \gamma)\, \mu(\mathrm{d}\gamma), \qquad A \in \mathcal{B}(\varGamma_X^\sharp), \tag{2.9}$$

however to make formula (2.9) well defined one needs to ensure that the integrand $\mu_{\rm cl}(A \,|\, \gamma)$ is measurable in γ with respect to the cylinder σ-algebra $\mathcal{B}(\varGamma_X)$.

Taking advantage of the independent structure of the family of secondary processes, it is more convenient to work with the Laplace functionals of the measures involved, leading from (2.9) to an equivalent decomposition

$$L_{\mu_{\rm cl}}[f] = \int_{\varGamma_X} L_{\mu_{\rm cl}}[f\,|\,\gamma]\, \mu(\mathrm{d}\gamma), \qquad f \in \mathrm{M}_+(X). \tag{2.10}$$

Now, adapting a general method used to construct the distributions of conditioned-based point processes (see [15, §6]), it follows that in order to make sure that formula (2.10) determines the CPP distribution, one needs (i) to specify the conditional Laplace functional $L_{\mu_{\rm cl}}[f\,|\,\gamma]$ of the hypothetical cluster measure $\mu_{\rm cl}$ conditioned on the configuration $\gamma \in \varGamma_X$ of centres, and (ii) to check that this is a measurable function of γ (see [15, Section 6.1, Proposition 6.1.II, page 165, and Lemma 6.1.III, page 166]). To this end, using the independence of the in-cluster configurations γ'_x $(x \in \gamma)$, we obtain

$$\begin{aligned} L_{\mu_{\rm cl}}[f\,|\,\gamma] &= \int_{\mathfrak{X}^\infty} \exp\left\{-\sum_{x\in\gamma}\sum_{y_i\in\bar{y}_x} f(y_i)\right\} \bigotimes_{x\in\gamma} \eta_x(\mathrm{d}\bar{y}_x) \\ &= \prod_{x\in\gamma} \int_{\mathfrak{X}} \exp\left\{-\sum_{y_i\in\bar{y}_x} f(y_i)\right\} \eta_x(\mathrm{d}\bar{y}_x) \\ &= \exp\left\{-\sum_{x\in\gamma} \bar{f}(x)\right\} = \exp\{-\langle \bar{f}, \gamma\rangle\}, \end{aligned}$$

where

$$\bar{f}(x) := -\log\left(\int_{\mathfrak{X}} \exp\left\{-\sum_{y_i\in\bar{y}} f(y_i)\right\} \eta_x(\mathrm{d}\bar{y})\right) \ge 0, \qquad x \in X.$$

By Lemma 2.2, the function $\bar{f}(\cdot)$ is $\mathcal{B}(X)$-measurable, and Lemma 2.1 then implies that $L_{\mu_{\rm cl}}[f\,|\,\gamma] = \exp\{-\langle \bar{f}, \gamma\rangle\}$ is $\mathcal{B}(\varGamma_X^\sharp)$-measurable, as required.

Thus, we have established that the cluster measure $\mu_{\rm cl}$ exists, and in particular its Laplace functional is given by

$$L_{\mu_{\rm cl}}[f] = \int_{\varGamma_X} \prod_{x\in\gamma} \left(\int_{\mathfrak{X}} \exp\Bigl(-\sum_{y_i\in\bar{y}} f(y_i)\Bigr)\, \eta_x(\mathrm{d}\bar{y})\right) \mu(\mathrm{d}\gamma), \qquad f \in \mathrm{M}_+(X). \tag{2.11}$$

Remark 2.4. Formula (2.11) is well know in the case of CPPs without accumulation points (see, e.g., [15, §6.3]).

Remark 2.5. Unlike the standard CPP theory where the sample configurations are *presumed* to be a.s. locally finite (see, e.g., [15, Definition 6.3.I]), the CPP constructed above may still have accumulation and/or multiple points arising due to contributions from remote clusters, even though both background point process $\gamma_{\rm c}$ and the inner-cluster processes γ_x' are proper. However, developing the differential analysis on configuration spaces in the spirit of Albeverio, Kondratiev and Röckner [3, 4] demands that measures under study are actually supported on the proper configuration space Γ_X. We shall address this issue for the general cluster measure $\mu_{\rm cl}$ in Section 3.2 below and give sufficient conditions in order that $\mu_{\rm cl}$-almost all (a.a.) configurations are proper (see our earlier papers [10, 11] for the cases of the Poisson and Gibbs CPPs, respectively).

2.3. Measures on marked configuration spaces

In this section, we develop a special construction of measures on marked configuration spaces, which will be useful below.

Consider the product space $Z := X \times \mathfrak{X}$ endowed with the product σ-algebra $\mathcal{B}(Z) = \mathcal{B}(X) \otimes \mathcal{B}(\mathfrak{X})$, and the corresponding configuration space Γ_Z. Let

$$p_X(z) := x, \qquad p_{\mathfrak{X}}(z) := \bar{y}, \qquad z = (x, \bar{y}) \in X \times \mathfrak{X},$$

denote the natural projections onto the spaces X and $\mathfrak{X}$, respectively. The maps p_X and $p_{\mathfrak{X}}$ can be extended to the configuration space Γ_Z:

$$\Gamma_Z \ni \hat{\gamma} \mapsto p_X(\hat{\gamma}) := \bigsqcup_{z\in\hat{\gamma}} p_X(z) \in \Gamma_X^\sharp,$$
$$\Gamma_Z \ni \hat{\gamma} \mapsto p_{\mathfrak{X}}(\hat{\gamma}) := \bigsqcup_{z\in\hat{\gamma}} p_{\mathfrak{X}}(z) \in \Gamma_{\mathfrak{X}}^\sharp.$$

We will work on a smaller space, the so-called *marked configuration space*

$$\Gamma_X(\mathfrak{X}) := \{\hat{\gamma} \in \Gamma_Z : \, p_X(\hat{\gamma}) \in \Gamma_X\},$$

endowed with the topology induced from Γ_Z. Furthermore, for each $\gamma \in \Gamma_X$, consider the space $\mathfrak{X}^\gamma := p_X^{-1}(\gamma)$ (i.e., the *fibre* at γ), which can be identified with the corresponding Cartesian product of identical copies of the space $\mathfrak{X}$,

$$\mathfrak{X}^\gamma = \prod_{x\in\gamma} \mathfrak{X}_x, \qquad \mathfrak{X}_x = \mathfrak{X}.$$

Therefore, each marked configuration $\hat{\gamma} \in \Gamma_X(\mathfrak{X})$ can be represented in the form

$$\hat{\gamma} = (\gamma, \bar{y}^\gamma) = \bigsqcup_{x\in\gamma} \{(x, \bar{y}_x)\}, \tag{2.12}$$

with

$$\gamma = p_X(\hat{\gamma}) \in \Gamma_X, \qquad \bar{y}^\gamma := (\bar{y}_x)_{x\in\gamma} \in \mathfrak{X}^\gamma. \tag{2.13}$$

More formally, a one-to-one correspondence between $x \in \gamma$ and $\bar{y}_x \in y^\gamma$ is described by the relations

$$x = p_X\big(p_{\mathfrak{X}}^{-1}(\bar{y}_x) \cap \hat{\gamma}\big), \qquad \bar{y}_x = p_{\mathfrak{X}}\big(p_X^{-1}(x) \cap \hat{\gamma}\big). \tag{2.14}$$

Recall that the probability measure μ on Γ_X and the (measurable) family of probability measures $\{\eta_x, \, x \in X\}$ on $\mathfrak{X}$ were defined in Section 2.2.

Lemma 2.3. *Under Condition* 2.1, *for any bounded function* $f \in \mathrm{M}_+(Z)$ *the map*

$$X \ni x \mapsto \int_{\mathfrak{X}} f(x, \bar{y})\, \eta_x(\mathrm{d}\bar{y}) \tag{2.15}$$

is $\mathcal{B}(X)$*-measurable.*

Proof. By a standard monotone class argument, it suffices to consider functions of the form $f(x, \bar{y}) = g(x) \cdot \mathbf{1}_B(\bar{y})$, with a measurable function $g(\cdot)$ and a Borel set $B \subset \mathfrak{X}$. Then the integral in (2.15) is reduced to

$$g(x) \int_B \eta_x(\mathrm{d}y) = g(x)\, \eta_x(B),$$

which is obviously a measurable function of x owing to Condition 2.1. □

For $\gamma \in \Gamma_X$, define the corresponding product measure on the space $\mathfrak{X}^\gamma$,

$$\eta^\gamma(\mathrm{d}\bar{y}^\gamma) := \bigotimes_{x\in\gamma} \eta_x(\mathrm{d}\bar{y}_x). \tag{2.16}$$

Consider a probability measure $\hat{\mu}$ on $\Gamma_X(\mathfrak{X})$ defined as the distribution of the marked point process with configurations (2.12), governed by the measures μ and η_x, $x \in X$. This process exists under Condition 2.1 (see [15, §6.1 and §6.4]). The measure $\hat{\mu}$ can be expressed symbolically as a "skew product"

$$\hat{\mu}(\mathrm{d}\hat{\gamma}) = \eta^\gamma(\mathrm{d}y^\gamma)\, \mu(\mathrm{d}\gamma), \qquad \hat{\gamma} = (\gamma, y^\gamma) \in \Gamma_X(\mathfrak{X}). \tag{2.17}$$

To be more precise, we can rewrite a heuristic expression (2.17) in integral form

$$\int_{\Gamma_X(\mathfrak{X})} F(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_X} \left(\int_{\mathfrak{X}^\gamma} F(\gamma, \bar{y}^\gamma)\, \eta^\gamma(\mathrm{d}\bar{y}^\gamma) \right) \mu(\mathrm{d}\gamma), \qquad F \in \mathrm{M}_+(\Gamma_X(\mathfrak{X})). \tag{2.18}$$

In particular, it follows that the internal integral is measurable as a function of $\gamma \in \Gamma_X$.

Remark 2.6. A direct proof of the measurability of the internal integral on the right-hand side of equation (2.18) based on a *measurable* indexation of the (proper) ground configurations $\gamma \in \Gamma_X$

Remark 2.7. Given the probability measure μ on Γ_X and the (measurable) family of probability measures $\{\eta_x,\ x \in X\}$, we can construct two point processes, the *cluster process* (with distribution μ_{cl}) and the *marked process* (with distribution $\hat{\mu}$). The crucial difference between them is that sample configurations of the latter are always proper (a.s.), in contrast to the former process which, in general, is supported on the space of generalised configurations. Moreover, many analytical properties of $\hat{\mu}$ are simpler than those of μ_{cl}. In fact, as will be explained in Section 3 below, there is a natural link between the two processes, in that the cluster measure μ_{cl} can be represented as a certain "projection" of the marked measure $\hat{\mu}$, which paves the way for the study of μ_{cl} using $\hat{\mu}$. This is the main idea of our approach.

3. Cluster measures on configuration spaces

3.1. Projection construction of the cluster measure

Recall that the "unpacking" map $\mathfrak{p} : \mathfrak{X} \to \Gamma_X^\sharp$ is defined in (2.3), and consider a map $\mathfrak{q} : Z \to \Gamma_X^\sharp$ acting by the formula

$$\mathfrak{q}(x, \bar{y}) := \mathfrak{p}(\bar{y}) = \bigsqcup_{y_i \in \bar{y}} \{y_i\}, \qquad (x, \bar{y}) \in Z. \tag{3.1}$$

In the usual "diagonal" way, the map $\mathfrak{q}$ can be lifted to the configuration space $\Gamma_X(\mathfrak{X})$:

$$\Gamma_X(\mathfrak{X}) \ni \hat{\gamma} \mapsto \mathfrak{q}(\hat{\gamma}) := \bigsqcup_{z \in \hat{\gamma}} \mathfrak{q}(z) \in \Gamma_X^\sharp. \tag{3.2}$$

Proposition 3.1. *The map* $\mathfrak{q} : \Gamma_X(\mathfrak{X}) \to \Gamma_X^\sharp$ *defined by* (3.2) *is measurable.*

Proof. Observe that $\mathfrak{q}$ can be represented as a composition

$$\mathfrak{q} = \mathfrak{p} \circ p_{\mathfrak{X}} : \Gamma_X(\mathfrak{X}) \xrightarrow{p_{\mathfrak{X}}} \Gamma_{\mathfrak{X}}^\sharp \xrightarrow{\mathfrak{p}} \Gamma_X^\sharp, \tag{3.3}$$

where the maps $p_{\mathfrak{X}}$ and $\mathfrak{p}$ are defined, respectively, by

$$\Gamma_X(\mathfrak{X}) \ni \hat{\gamma} \mapsto p_{\mathfrak{X}}(\hat{\gamma}) := \bigsqcup_{(x,\bar{y}) \in \hat{\gamma}} \{\bar{y}\} \in \Gamma_{\mathfrak{X}}^\sharp, \tag{3.4}$$

$$\Gamma_{\mathfrak{X}}^\sharp \ni \bar{\gamma} \mapsto \mathfrak{p}(\bar{\gamma}) := \bigsqcup_{\bar{y} \in \bar{\gamma}} \mathfrak{p}(\bar{y}) \in \Gamma_X^\sharp. \tag{3.5}$$

For $p_{\mathfrak{X}} : \Gamma_X(\mathfrak{X}) \to \Gamma_{\mathfrak{X}}^\sharp$ (see (3.4)) we have

$$p_{\mathfrak{X}}^{-1}(C_{\bar{B}}^n) = C_{X \times \bar{B}}^n = \{\hat{\gamma} \in \Gamma_X(\mathfrak{X}) : \hat{\gamma}(X \times \bar{B}) = n\} \in \mathcal{B}(\Gamma_X(\mathfrak{X})),$$

since $X \times \bar{B} \in \mathcal{B}(Z)$. Furthermore, the measurability of the projection $\mathfrak{p} : \Gamma_{\mathfrak{X}}^\sharp \to \Gamma_X^\sharp$ (see (3.5)) was shown in [10, §3.3, p. 455]. As a result, the composition of maps in (3.3) is measurable, as claimed. □

Let us define a measure on $\Gamma_X^\sharp$ as the push-forward of $\hat{\mu}$ (see (2.17)) under the map $\mathfrak{q}$ defined in (3.1), (3.2):

$$\mathfrak{q}^*\hat{\mu}(A) \equiv \hat{\mu}(\mathfrak{q}^{-1}(A)), \qquad A \in \mathcal{B}(\Gamma_X^\sharp), \tag{3.6}$$

or, equivalently,

$$\int_{\Gamma_X^\sharp} F(\gamma)\, \mathfrak{q}^*\hat{\mu}(\mathrm{d}\gamma) = \int_{\Gamma_X(\mathfrak{X})} F(\mathfrak{q}(\hat{\gamma}))\, \hat{\mu}(\mathrm{d}\hat{\gamma}), \qquad F \in \mathrm{M}_+(\Gamma_X^\sharp). \tag{3.7}$$

The next general result shows that this measure may be identified with the original cluster measure μ_{cl}.

Theorem 3.2. *Measure* (3.6) *coincides with the cluster measure* μ_{cl},

$$\mu_{\mathrm{cl}} = \mathfrak{q}^*\hat{\mu} \equiv \hat{\mu} \circ \mathfrak{q}^{-1}. \tag{3.8}$$

Proof. Let us evaluate the Laplace transform of the measure $\mathfrak{q}^*\hat{\mu}$. For any function $f \in \mathrm{M}_+(X)$, we obtain, on using (2.17), (3.2) and(3.7),

$$\begin{aligned}
L_{\mathfrak{q}^*\hat{\mu}}[f] &= \int_{\Gamma_X^\sharp} \exp(-\langle f, \gamma\rangle)\, \mathfrak{q}^*\hat{\mu}(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X(\mathfrak{X})} \exp(-\langle f, \mathfrak{q}(\hat{\gamma})\rangle)\, \hat{\mu}(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_X} \biggl(\int_{\Gamma_{\mathfrak{X}}^\sharp} \exp\biggl(-\sum_{x\in\gamma} f(\mathfrak{p}(\bar{y}_x + x))\biggr) \bigotimes_{x\in\gamma} \eta(\mathrm{d}\bar{y}_x)\biggr)\, \mu(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X} \biggl(\int_{\Gamma_{\mathfrak{X}}^\sharp} \prod_{x\in\gamma} \exp(-f(\bar{y}_x)) \bigotimes_{x\in\gamma} \eta_x(\mathrm{d}\bar{y}_x)\biggr)\, \mu(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X} \prod_{x\in\gamma} \biggl(\int_{\mathfrak{X}} \exp\biggl(-\sum_{y\in\bar{y}} f(\bar{y})\biggr)\, \eta_x(\mathrm{d}\bar{y})\biggr)\, \mu(\mathrm{d}\gamma),
\end{aligned}$$

which coincides with the Laplace transform (2.11) of the cluster measure μ_{cl}. $\square$

An alternative description of the cluster measure μ_{cl} can be given as follows. Consider a natural map $\mathfrak{r}_\gamma : \mathfrak{X}^\gamma \to \Gamma_{\mathfrak{X}}^\sharp$ defined by

$$\mathfrak{X}^\gamma \ni (\bar{y}_x)_{x\in\gamma} \stackrel{\mathfrak{r}_\gamma}{\longmapsto} \bigsqcup_{x\in\gamma} \{\bar{y}_x\} \in \Gamma_{\mathfrak{X}}^\sharp$$

(see (2.12)). The map $\mathfrak{r}_\gamma$ is measurable, which can be shown by repeating the arguments used in [10, §3.3, p. 455] in the proof of measurability of $\mathfrak{p}$. Further, define the map (cf. (3.5))

$$p_\gamma := \mathfrak{p} \circ \mathfrak{r}_\gamma :\ \mathfrak{X}^\gamma \xrightarrow{\mathfrak{r}_\gamma} \Gamma_{\mathfrak{X}}^\sharp \xrightarrow{\mathfrak{p}} \Gamma_X^\sharp. \tag{3.9}$$

Clearly, p_γ is measurable as a composition of measurable maps. Note that the projection $p_{\mathfrak{X}}$ defined in (3.4) can be represented as

$$p_{\mathfrak{X}}(\hat{\gamma}) = \mathfrak{r}_\gamma(\bar{y}^\gamma), \qquad \hat{\gamma} = (\gamma, \bar{y}^\gamma) \in \Gamma_X(\mathfrak{X}). \tag{3.10}$$

Furthermore, applying $\mathfrak{p}$ to both sides of equality (3.10) and using relations (3.3) and (3.9), we obtain the representation

$$\mathfrak{q}(\hat{\gamma}) = p_\gamma(\bar{y}^\gamma), \qquad \hat{\gamma} = (\gamma, \bar{y}^\gamma) \in \Gamma_X(\mathfrak{X}). \tag{3.11}$$

Consider the probability measures ϖ^γ and μ^γ on $\Gamma_{\mathfrak{X}^\gamma}^\sharp$ and $\Gamma_X^\sharp$, respectively, defined by

$$\varpi^\gamma := \mathfrak{r}_\gamma^*\eta^\gamma, \tag{3.12}$$

$$\mu^\gamma := p_\gamma^*\eta^\gamma. \tag{3.13}$$

Theorem 3.3. *The cluster measure μ_{cl} on $\Gamma_X^\sharp$ is represented in either of the following two forms*

$$\mu_{\mathrm{cl}}(\mathrm{d}\gamma) = \int_{\Gamma_X} \mu^\zeta(\mathrm{d}\gamma)\, \mu(\mathrm{d}\zeta), \tag{3.14}$$

$$\mu_{\mathrm{cl}}(\mathrm{d}\gamma) = \mathfrak{p}^*\varpi(\mathrm{d}\gamma), \tag{3.15}$$

where ϖ is a measure on $\Gamma^{\sharp}_{\mathfrak{X}}$ defined by

$$\varpi(\mathrm{d}\bar{\gamma}) := \int_{\Gamma_X} \varpi^{\gamma}(\mathrm{d}\bar{\gamma})\,\mu(\mathrm{d}\gamma). \tag{3.16}$$

Proof. By the change of measure (3.13) and relations (2.17) and (3.11), we have, for any Borel function $F \in \mathrm{M}_{+}(\Gamma^{\sharp}_{X})$,

$$\begin{aligned}
\int_{\Gamma_X} \left(\int_{\Gamma^{\sharp}_X} F(\zeta)\,\mu^{\gamma}(\mathrm{d}\zeta) \right) \mu(\mathrm{d}\gamma) &= \int_{\Gamma_X} \left(\int_{\mathfrak{X}} F(p_{\gamma}(\bar{y}^{\gamma}))\,\eta^{\gamma}(\mathrm{d}\bar{y}^{\gamma}) \right) \mu(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X(\mathfrak{X})} F(\mathfrak{q}(\hat{\gamma}))\,\hat{\mu}(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_X(\mathfrak{X})} F(\hat{\gamma})\,\mathfrak{q}^{*}\hat{\mu}(\mathrm{d}\gamma),
\end{aligned}$$

according to (3.8). Thus, formula (3.14) is proved.

Similarly, using relation (3.16) and the change of measure (3.12), we get

$$\begin{aligned}
\int_{\Gamma^{\sharp}_{\mathfrak{X}}} F(\zeta)\,\mathfrak{p}^{*}\varpi(\mathrm{d}\zeta) &= \int_{\Gamma_X} \left(\int_{\Gamma^{\sharp}_{\mathfrak{X}}} F(\mathfrak{p}(\bar{\gamma}))\,\varpi^{\gamma}(\mathrm{d}\bar{\gamma}) \right) \mu(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X} \left(\int_{\mathfrak{X}} F(\mathfrak{p} \circ \mathfrak{r}_{\gamma}(\bar{y}^{\gamma}))\,\eta^{\gamma}(\mathrm{d}\bar{y}^{\gamma}) \right) \mu(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X} \left(\int_{\mathfrak{X}} F(p_{\gamma}(\bar{y}^{\gamma}))\,\eta^{\gamma}(\mathrm{d}\bar{y}^{\gamma}) \right) \mu(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X(\mathfrak{X})} F(\mathfrak{q}(\hat{\gamma}))\hat{\mu}(\mathrm{d}\hat{\gamma}),
\end{aligned}$$

which proves formula (3.15). □

Remark 3.1. In the case where $X = \mathbb{R}^d$, $\eta_x(\mathrm{d}\bar{y}) = \eta_0(\mathrm{d}\bar{y} - x)$ and $\mu(\mathrm{d}\gamma)$ is a Poisson measure $\pi_{\theta}(\mathrm{d}\gamma)$ with intensity θ, the measure ϖ coincides with the auxiliary Poisson measure π_{σ} considered in [10], with intensity measure $\sigma(\bar{B}) = \int_X \eta_x(\bar{B})\,\theta(\mathrm{d}x)$, $\bar{B} \in \mathcal{B}(\mathfrak{X})$.

The relationships between various measure spaces introduced above are succinctly illustrated by the following commutative diagrams:

$$\begin{array}{ccc}
(\Gamma_X(\mathfrak{X}), \hat{\mu}) & \xrightarrow{\ \mathrm{id}\ } & (\Gamma^{\sharp}_X \times \mathfrak{X}^{\gamma},\, \mu \otimes \eta^{\gamma}) \\
\mathfrak{q}\big\downarrow & & \big\downarrow \mathrm{id} \otimes p_{\gamma} \\
(\Gamma^{\sharp}_X, \mu_{\mathrm{cl}}) & \xleftarrow{\ \int \mathrm{d}\mu\ } & (\Gamma^{\sharp}_X \times \Gamma^{\sharp}_{\mathfrak{X}},\, \mu \otimes \mu^{\gamma})
\end{array}
\qquad
\begin{array}{ccc}
(\Gamma_X(\mathfrak{X}), \hat{\mu}) & \xrightarrow{\ \mathrm{id} \otimes \mathfrak{r}_{\gamma}\ } & (\Gamma^{\sharp}_X \times \Gamma^{\sharp}_{\mathfrak{X}},\, \mu \otimes \varpi^{\gamma}) \\
\mathfrak{q}\big\downarrow & & \big\downarrow \int \mathrm{d}\mu \\
(\Gamma^{\sharp}_X, \mu_{\mathrm{cl}}) & \xleftarrow{\ \mathfrak{p}\ } & (\Gamma^{\sharp}_{\mathfrak{X}}, \varpi)
\end{array}$$

3.2. Conditions for absence of accumulation and multiple points

Let us now give sufficient conditions for the cluster point process to be proper, so that $\mu_{\mathrm{cl}}(\Gamma_X) = 1$. For any Borel subset $B \subset X$, consider the set

$$\mathfrak{X}_B := \{\bar{y} \in \mathfrak{X} :\ \mathfrak{p}(\bar{y}) \cap B \neq \emptyset\} \in \mathcal{B}(\mathfrak{X}), \tag{3.17}$$

where the projection map $\mathfrak{p}$ is defined in (2.3). That is to say, the set $\mathfrak{X}_B$ consists of all points $\bar{y} \in \mathfrak{X}$ with at least one coordinate $y_i \in \bar{y}$ belonging to B.

Condition 3.1. For any compact set $B \subset X$,

$$\int_{\Gamma_X} \sum_{x\in\gamma} \eta_x(\mathfrak{X}_B)\, \mu(\mathrm{d}\gamma) < \infty. \tag{3.18}$$

Remark 3.2. In view of definition (3.17), the left-hand side of (3.18) equals the expected number (under the measure μ_{cl}) of clusters that contribute at least one point to the set B.

We introduce the set

$$\tilde{\mathfrak{X}} := \{\bar{y} \in \mathfrak{X} :\ \forall\, y_i, y_j \in \bar{y},\ y_i \neq y_j\}.$$

Condition 3.2. For μ-a.a. configurations $\gamma \in \Gamma_X$, the probability measure η^γ on $\mathfrak{X}^\gamma$ (see (2.16)) is concentrated on the set

$$\widetilde{\mathfrak{X}^\gamma} := \{\bar{y}^\gamma \in (\tilde{\mathfrak{X}})^\gamma :\ \forall\,\{x_1, x_2\} \subset \gamma,\ \mathfrak{p}(\bar{y}_{x_1}) \cap \mathfrak{p}(\bar{y}_{x_2}) = \emptyset\}, \tag{3.19}$$

that is, $\eta^\gamma(\widetilde{\mathfrak{X}^\gamma}) = 1$.

Remark 3.3. The set $\widetilde{\mathfrak{X}^\gamma}$ ensures that different clusters attached to the ground configuration γ do not have common points.

Remark 3.4. A sufficient condition for (3.19) is that for any $x \in X$ the measure η_x is absolutely continuous with respect to the volume measure in $\mathfrak{X}$.

Theorem 3.4. *Let μ_{cl} be a cluster measure on the generalised configuration space $\Gamma^\sharp_X$. Then*

(a) *under Condition* 3.1*, μ_{cl}-a.a configurations $\gamma \in \Gamma^\sharp_X$ are locally finite*;

(b) *under Condition* 3.2*, μ_{cl}-a.a configurations $\gamma \in \Gamma^\sharp_X$ are simple.*

Therefore, if both Conditions 3.1 *and* 3.2 *are met then the cluster measure μ_{cl} is concentrated on the proper configuration space Γ_X.*

Proof. (a) Let $B \subset X$ be a compact set. From formula (3.15) and definition (3.17) of the set $\mathfrak{X}_B$, it is clear that $\gamma(B) < \infty$ for μ_{cl}-a.a. configurations $\gamma \in \Gamma^\sharp_X$ if and only if

$$\bar{\gamma}(\mathfrak{X}_B) < \infty \quad \text{for } \varpi\text{-a.a. } \bar{\gamma} \in \Gamma^\sharp_{\mathfrak{X}}, \tag{3.20}$$

where the measure ϖ is defined in (3.16). Let $f(\bar{y}) := \mathbf{1}_{\mathfrak{X}_B}(\bar{y}),\ \bar{y} \in \mathfrak{X}$. Recalling definitions (2.16), (3.14), (3.15) and using Condition 3.1, we obtain

$$\begin{aligned}
\int_{\Gamma^\sharp_{\mathfrak{X}}} \langle f, \bar{\gamma}\rangle\, \varpi(\mathrm{d}\bar{\gamma}) &= \int_{\Gamma_X} \left(\int_{\Gamma^\sharp_{\mathfrak{X}}} \langle f, \bar{\gamma}\rangle\, \varpi^\gamma(\mathrm{d}\bar{\gamma}) \right) \mu(\mathrm{d}\gamma)\\
&= \int_{\Gamma_X} \left(\int_{\mathfrak{X}^\gamma} \sum_{\bar{y}\in\bar{y}^\gamma} f(\bar{y})\, \eta^\gamma(\mathrm{d}\bar{y}^\gamma) \right) \mu(\mathrm{d}\gamma)\\
&= \int_{\Gamma_X} \left(\sum_{x\in\gamma} \int_{\mathfrak{X}_B} \eta_x(\mathrm{d}\bar{y}) \right) \mu(\mathrm{d}\gamma)\\
&= \int_{\Gamma_X} \left(\sum_{x\in\gamma} \eta_x(\mathfrak{X}_B) \right) \mu(\mathrm{d}\gamma) < \infty,
\end{aligned}$$

which implies (3.20). Thus, the absence of accumulation points is proved.

(b) Let $\Gamma_X^{\ddagger}$ be the set of all generalised configurations in X that have multiple points. By definition (3.9) of the map p_γ there is the inclusion

$$p_\gamma^{-1}(\Gamma_X^{\ddagger}) \subset \mathfrak{X}^\gamma \setminus \widetilde{\mathfrak{X}^\gamma},$$

where the set $\widetilde{\mathfrak{X}^\gamma}$ is introduced in (3.19). Hence, from (3.13) we get, for μ-a.a. $\gamma \in \Gamma_X$,

$$\mu^\gamma(\Gamma_X^{\ddagger}) = \eta^\gamma(p_\gamma^{-1}(\Gamma_X^{\ddagger})) \le 1 - \eta^\gamma(\widetilde{\mathfrak{X}^\gamma}) = 0$$

according to Condition 3.2, and by formula (3.14) this implies that $\mu_{\mathrm{cl}}(\Gamma_X^{\ddagger}) = 0$. □

3.3. Existence of moments

Definition 3.1. For $r \ge 1$ and a Borel measure θ on X, a measure μ on the configuration space Γ_X is said to be in the *moment class* $\mathcal{M}_\theta^r = \mathcal{M}_\theta^r(\Gamma_X)$ if for any measurable function f on X the following holds:

(i) if $f(x) = 0$ for θ-a.a. $x \in X$ then $\langle f, \gamma\rangle = 0$ for μ-a.a. $\gamma \in \Gamma_X$;

(ii) if $\int_X |f(x)|^\kappa\, \theta(\mathrm{d}x) < \infty$ for all $1 \le \kappa \le r$ then

$$\int_{\Gamma_X} |\langle f, \gamma\rangle|^r\, \mu(\mathrm{d}\gamma) < \infty. \tag{3.21}$$

Remark 3.5. Equivalently, $\mu \in \mathcal{M}_\theta^r$ if and only if for any $f \in \bigcap_{1\le\kappa\le r} L^\kappa(X,\theta)$ it holds that $\langle f, \cdot\,\rangle \in L^r(\Gamma_X, \mu)$. Note that due to condition (i) of Definition 3.1, the function $g(\gamma) := \langle f, \gamma\rangle$ on Γ_X is well defined up to a set of μ-measure zero (i.e., is independent of the choice of a representative of the equivalence class f).

Lemma 3.5. *The family of the classes* $\{\mathcal{M}_\theta^r,\ r \ge 1\}$ *is nested, that is,* $\mathcal{M}_\theta^{r+\delta} \subset \mathcal{M}_\theta^r$ *for all* $r \ge 1$ *and any* $\delta > 0$.

Proof. Indeed, let $\mu \in \mathcal{M}_\theta^{r+\delta}$, then for any $f \in \bigcap_{1\le q\le r+\delta} L^q(X,\theta) \subset \bigcap_{1\le q\le r} L^q(X,\theta)$ we have, by the Lyapunov inequality,

$$\int_{\Gamma_X} |\langle f, \gamma\rangle|^r\, \mu(\mathrm{d}\gamma) \le \left(\int_{\Gamma_X} |\langle f, \gamma\rangle|^{r+\delta}\, \mu(\mathrm{d}\gamma)\right)^{r/(r+\delta)} < \infty,$$

hence $\mu \in \mathcal{M}_\theta^r$, as claimed. □

Condition 3.3. There is a locally finite measure θ on X (i.e., $\theta(B) < \infty$ for any compact $B \subset X$), referred to as the *reference measure*, such that $\mu \in \mathcal{M}_\theta^1(\Gamma_X)$.

Remark 3.6. The condition $\mu \in \mathcal{M}_\theta^1(\Gamma_X)$ implies that $\gamma(B) < \infty$ (μ-a.s.) for any Borel set B such that $\theta(B) < \infty$. Indeed, choose $f(x) = \mathbf{1}_B(x) \in L^1(X,\theta)$, then $\langle f, \gamma\rangle = \gamma(B)$ and Definition 3.1 yields $\int_{\Gamma_X} \gamma(B)\, \mu(\mathrm{d}\gamma) < \infty$, hence the integrand is μ-a.s. finite.

Example 3.1. Condition 3.3 holds for a Poisson measure π_θ with a locally finite intensity θ, as well as for a wide class of Gibbs perturbations of π_θ (see, e.g., [3, 4]). More generally, any measure μ with bounded correlation functions up to order n (with respect to θ) belongs to $\mathcal{M}_\theta^n(\Gamma_X)$ (see Appendix B).

Example 3.2. Example of a different type is given by $\mu = \delta_{\gamma_0}$, the Dirac measure on Γ_X concentrated on a given configuration $\gamma_0 \in \Gamma_X$ (e.g., if $X = \mathbb{R}^d$ then we can set $\gamma_0 = \mathbb{Z}^d$). Here we have

$$\int_{\Gamma_X} |\langle f, \gamma\rangle|^n \, \mu(\mathrm{d}\gamma) = |\langle f, \gamma_0\rangle|^n,$$

which implies that $\mu \in \mathcal{M}^n_\theta(\Gamma_X)$ with $\theta = \sum_{x\in\gamma_0} \delta_x$.

Definition 3.2. Introduce the measures $\hat\sigma$ on $Z = X \times \mathfrak{X}$ and $\bar\sigma$ on $\mathfrak{X}$ as follows

$$\hat\sigma(\mathrm{d}x \times \mathrm{d}\bar y) := \eta_x(\mathrm{d}\bar y)\, \theta(\mathrm{d}x), \tag{3.22}$$

$$\bar\sigma(\mathrm{d}\bar y) := \int_X \eta_x(\mathrm{d}\bar y)\, \theta(\mathrm{d}x). \tag{3.23}$$

Lemma 3.6. *Suppose that* $\mu \in \mathcal{M}^n_\theta(\Gamma_X)$ *for some integer* $n \ge 1$. *Then* $\hat\mu \in \mathcal{M}^n_{\hat\sigma}(\Gamma_Z)$.

Proof. We need to check the two conditions in Definition 3.1.

(i) Let f be a measurable function on Z such that $f(x,\bar y) = 0$ for $\hat\sigma$-a.a. $(x,\bar y) \in Z$. It follows that, for θ-a.a. $x \in X$,

$$\bar f(x) := \int_{\mathfrak{X}} |f(x,\bar y)|\, \eta_x(\mathrm{d}\bar y) = 0, \tag{3.24}$$

which implies that $\langle \bar f, \gamma\rangle = 0$ for μ-a.a. $\gamma \in \Gamma_X$. Then, by formula (3.23), definition of the measure $\hat\mu$ (see (2.17) and (2.18)) and inclusion $\mu \in \mathcal{M}^n_\theta(\Gamma_X)$, we have

$$\int_{\Gamma_Z} |\langle f, \hat\gamma\rangle|\, \hat\mu(\mathrm{d}\hat\gamma) \le \int_{\Gamma_X} |\langle \bar f, \gamma\rangle|\, \mu(\mathrm{d}\gamma) = 0,$$

so that $\langle f, \hat\gamma\rangle = 0$ for $\hat\mu$-a.a. $\hat\gamma \in \Gamma_Z$, as required by condition (i) of Definition 3.1.

(ii) Let us show that, for any measurable function f on Z such that $\int_Z |f(z)|^\kappa \, \hat\sigma(\mathrm{d}z) < \infty$ for all $1 \le \kappa \le r$, we have

$$\int_{\Gamma_Z} |\langle f, \hat\gamma\rangle|^n \, \hat\mu(\mathrm{d}\hat\gamma) < \infty. \tag{3.25}$$

To this end, let us first observe using definition (3.22) that

$$\int_X \left(\int_{\mathfrak{X}} |f(x,\bar y)|^\kappa \, \eta_x(\mathrm{d}\bar y)\right) \theta(\mathrm{d}x) = \int_Z |f(z)|^\kappa \, \hat\sigma(\mathrm{d}z) < \infty,$$

which means that the function

$$\bar f_\kappa(x) := \int_{\mathfrak{X}} |f(x,\bar y)|^\kappa \, \eta_x(\mathrm{d}\bar y), \qquad x \in X, \tag{3.26}$$

belongs to $L^1(X,\theta)$. Moreover, by the Lyapunov inequality we have, for $q \ge 1$,

$$\begin{aligned}\int_X \bar f_\kappa(x)^q \, \theta(\mathrm{d}x) &= \int_X \left(\int_{\mathfrak{X}} |f(x,\bar y)|^\kappa \, \eta_x(\mathrm{d}\bar y)\right)^q \theta(\mathrm{d}x)\\ &\le \int_X \left(\int_{\mathfrak{X}} |f(x,\bar y)|^{\kappa q} \, \eta_x(\mathrm{d}\bar y)\right) \theta(\mathrm{d}x)\\ &= \int_Z |f(z)|^{\kappa q} \, \hat\sigma(\mathrm{d}z) < \infty,\end{aligned}$$

as long as $1 \le \kappa q \le n$. In other words,

$$\bar{f}_\kappa(x) \in L^q(X,\theta), \qquad 1 \le q \le n/\kappa. \tag{3.27}$$

Now, using the multinomial expansion we can write for an integer $n \ge 1$

$$\begin{aligned}\int_{\Gamma_Z} |\langle f, \hat{\gamma}\rangle|^n \,\hat{\mu}(\mathrm{d}\hat{\gamma}) &\le \int_{\Gamma_Z} \left(\sum_{z\in\hat{\gamma}} |f(z)|\right)^n \hat{\mu}(\mathrm{d}\hat{\gamma}) \\ &= \sum_{m=1}^{n} \int_{\Gamma_Z} \sum_{\{z_1,\dots,z_m\}\subset\hat{\gamma}} \phi_n(z_1,\dots,z_m)\,\hat{\mu}(\mathrm{d}\hat{\gamma}),\end{aligned} \tag{3.28}$$

where $\phi_n(z_1,\dots,z_m)$ is a symmetric function given by

$$\phi_n(z_1,\dots,z_m) := \sum_{\substack{i_1,\dots,i_m\ge 1\\ i_1+\cdots+i_m=n}} \frac{n!}{i_1!\cdots i_m!}\,|f(z_1)|^{i_1}\cdots|f(z_m)|^{i_m}. \tag{3.29}$$

By definition of the measure $\hat{\mu}$ (see (2.17) and (2.18)), the integral on the right-hand side of (3.28) is reduced to

$$\begin{aligned}&\int_{\Gamma_X} \sum_{\{x_1,\dots,x_m\}\subset\gamma} \left(\int_{\mathfrak{X}^\gamma} \phi_n(x_1,\bar{y}_1;\dots;x_m,\bar{y}_m)\,\eta^\gamma(\mathrm{d}\bar{\gamma})\right)\mu(\mathrm{d}\gamma) \\ &\qquad = \sum_{\substack{i_1,\dots,i_m\ge 1\\ i_1+\cdots+i_m=n}} \frac{n!}{i_1!\cdots i_m!} \int_{\Gamma_X} \sum_{\{x_1,\dots,x_m\}\subset\gamma} \prod_{j=1}^{m} \bar{f}_{i_j}(x_j)\,\mu(\mathrm{d}\gamma),\end{aligned} \tag{3.30}$$

where we used notation (3.26). Furthermore, with the help of the Jensen inequality the integral on the right-hand side of (3.30) may be estimated from above by

$$\begin{aligned}\int_{\Gamma_X} \prod_{j=1}^{m} \sum_{x_j\in\gamma} \bar{f}_{i_j}(x_j)\,\mu(\mathrm{d}\gamma) &= \int_{\Gamma_X} \prod_{j=1}^{m} \langle \bar{f}_{i_j},\gamma\rangle\,\mu(\mathrm{d}\gamma) \\ &\le \prod_{j=1}^{m} \left(\int_{\Gamma_X} \langle \bar{f}_{i_j},\gamma\rangle^{n/i_j}\,\mu(\mathrm{d}\gamma)\right)^{i_j/n}.\end{aligned} \tag{3.31}$$

To summarise, by inspection of relations (3.28), (3.29), (3.30) and (3.31) we see that in order to verify (3.25) it suffices to check that, for any $k = 1,\dots,n$,

$$\int_{\Gamma_X} \langle \bar{f}_k,\gamma\rangle^{n/k}\,\mu(\mathrm{d}\gamma) < \infty. \tag{3.32}$$

But we already know (see (3.27)) that $\bar{f}_k \in L^q(X,\theta)$ for $1 \le q \le n/k$. On the other hand, by the hypothesis of the lemma we have $\mu \in \mathcal{M}^n_\theta(\Gamma_X) \subset \mathcal{M}^{n/k}_\theta(\Gamma_X)$ (see Remark 3.5), and now the required bound (3.32) follows by condition (3.21) with $r = n/k$. □

The next condition on the measure $\bar{\sigma}$ will play an important part in our analysis.

Condition 3.4. For any compact set $B \subset X$, it holds that

$$\bar{\sigma}(\mathfrak{X}_B) < \infty, \tag{3.33}$$

where the set $\mathfrak{X}_B$ is defined in (3.17).

Remark 3.7. Conditions 3.3 and 3.4 taken together imply Condition 3.1. Indeed, from (3.23) and (3.33) it follows that $\eta_x(\mathfrak{X}_B)$ as a function of $x \in X$ belongs to the space $L^1(X, \theta)$. Thus, according to Definition 3.1 we can apply Condition 3.3 to obtain

$$\sum_{x\in\gamma} \eta_x(\mathfrak{X}_B) \in L^1(\Gamma_X, \mu),$$

which is nothing else but condition (3.18). Hence, by Theorem 3.4(a), the measure μ_{cl} is concentrated on configurations without accumulation points.

Denote by $N_X(\bar{y})$ the "dimension" of vector $\bar{y} \in \mathfrak{X}$, that is, the total number of its components

$$N_X(\bar{y}) := \sum_{n\in\mathbb{Z}_+} n\,\mathbf{1}_{X^n}(\bar{y}), \qquad \bar{y} \in \mathfrak{X} = \bigsqcup_{n\in\mathbb{Z}_+} X^n. \tag{3.34}$$

Lemma 3.7. *Suppose that, in addition to Conditions* 3.3 *and* 3.4*, the function* $N_X(\bar{y})$ *satisfies, for any compact set* $B \subset X$*, an integrability condition*

$$\int_X \int_{\mathfrak{X}_B} N_X(\bar{y})^n\, \eta_x(\mathrm{d}\bar{y})\, \theta(\mathrm{d}x) < \infty, \tag{3.35}$$

Then the cluster measure μ_{cl} *belongs to the class* $\mathcal{M}_\theta^n(\Gamma_X)$*.*

Proof. Using the change of measure (3.8), for any $\phi \in C_0(X)$ we obtain

$$\int_{\Gamma_X} |\langle \phi, \gamma\rangle|^n\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) = \int_{\Gamma_Z} |\langle \phi, \mathfrak{q}(\hat{\gamma})\rangle|^n\, \hat{\mu}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_Z} |\langle \mathfrak{q}^*\phi, \hat{\gamma}\rangle|^n\, \hat{\mu}(\mathrm{d}\hat{\gamma}),$$

where

$$\mathfrak{q}^*\phi(x, \bar{y}) := \sum_{y_i\in\bar{y}} \phi(y_i), \qquad (x, \bar{y}) \in Z. \tag{3.36}$$

It suffices to show that $\mathfrak{q}^*\phi \in L^m(Z, \hat{\sigma})$ for any $m = 1, \dots, n$. By the elementary inequality $(a_1 + \dots + a_k)^m \le k^{m-1}(a_1^m + \dots + a_k^m)$, from (3.36) we have

$$\int_Z |\mathfrak{q}^*\phi(z)|^m\, \hat{\sigma}(\mathrm{d}z) \le \int_Z N_X(\bar{y})^{m-1} \sum_{y_i\in\bar{y}} |\phi(y_i)|^m\, \hat{\sigma}(\mathrm{d}x \times \mathrm{d}\bar{y}). \tag{3.37}$$

Recalling that $\hat{\sigma}(\mathrm{d}x \times \mathrm{d}\bar{y}) = \eta_x(\mathrm{d}\bar{y})\, \theta(\mathrm{d}x)$ and denoting $C_\phi := \sup_{x\in X} |\phi(x)| < \infty$ and $K_\phi := \operatorname{supp} \phi \subset X$, the right-hand side of (3.37) is dominated by

$$(C_\phi)^m \int_X \int_{\mathfrak{X}_{K_\phi}} N_X(\bar{y})^m\, \eta_x(\mathrm{d}\bar{y})\, \theta(\mathrm{d}x),$$

which proves the result. □

3.4. "Translations" and the droplet cluster

Let us describe a general setting that may be used to construct the family of measures $\{\eta_x(\mathrm{d}\bar{y})\}_{x\in X}$ on $\mathfrak{X}$ via suitable push-forwards ("translations") of a pattern measure Q defined on some auxiliary space. Examples of application of such an approach will be given in Section 5 below.

More precisely, let $(W, \mathcal{B}(W))$ be a measurable space, with a Borel σ-algebra $\mathcal{B}(W)$ generated by the open subsets of W. Consider the corresponding space (cf. (2.1))

$$\mathfrak{W} := \bigsqcup_{n=0}^{\infty} W^n, \tag{3.38}$$

and let Q be a probability measure on $\mathcal{B}(\mathfrak{W})$. For any map $\varphi : W \to X$, define as usual its diagonal lifting $\bar{\varphi} : \mathfrak{W} \to \mathfrak{X}$ by

$$\mathfrak{W} \ni \bar{w} \mapsto \bar{\varphi}(\bar{w}) := (\varphi(w_i))_{w_i\in\bar{w}} \in \mathfrak{X}. \tag{3.39}$$

Like in Condition 3.3, it is assumed that the reference measure θ on X is locally finite.

The main assumption in this section is as follows.

Condition 3.5. Suppose there is a measurable map

$$W \times X \ni (w, x) \mapsto \varphi_x(w) \in X$$

such that the measures η_x on $\mathfrak{X}$ are representable as $\eta_x = \bar{\varphi}_x^* Q$; that is, for all $x \in X$,

$$\eta_x(\bar{B}) = Q(\bar{\varphi}_x^{-1}(\bar{B})), \qquad \bar{B} \in \mathcal{B}(\mathfrak{X}). \tag{3.40}$$

Remark 3.8. In view of formula (3.40), we shall often consider $\{\varphi_x(\cdot)\}_{x\in X}$ as a *family* of the maps $W \ni w \mapsto \varphi_x(w) \in X$ (indexed by $x \in X$).

Remark 3.9. Fubini's theorem implies that, for each $x \in X$, the map $\mathfrak{W} \ni \bar{w} \mapsto \bar{\varphi}_x(\bar{w}) \in \mathfrak{X}$ is measurable and hence $\bar{\varphi}_x^{-1}(\bar{B})$ is a Borel subset of $\mathfrak{W}$, so that the right-hand side of formula (3.40) is well defined. We also have that, for any fixed Borel set $\bar{B} \subset \mathfrak{X}$, the function $\eta_x(\bar{B}) : X \to [0, 1]$ is measurable.

Definition 3.3. Given a map $\varphi_x(w)$ as above, the set

$$D_B(w) := \{x \in X : \varphi_x(w) \in B\} \subset X, \qquad w \in W,\ B \in \mathcal{B}(X), \tag{3.41}$$

is called a *droplet* of shape B anchored at w. Furthermore, the set

$$\bar{D}_B(\bar{w}) := \big\{x \in X : \bar{\varphi}_x(\bar{w}) \in \mathfrak{X}_B\big\} \subset X, \qquad \bar{w} \in \mathfrak{W}, \tag{3.42}$$

is referred to as the *droplet cluster* (of shape B) anchored at $\bar{w}$.

Note that the droplet $D_B(w)$ is a Borel subset of X for each $w \in W$; moreover, by Remark 3.9 the same is true for the droplet cluster $\bar{D}_B(\bar{w})$. On account of definition (3.17), formula (3.42) can be rewritten in the form

$$\bar{D}_B(\bar{w}) = \bigcup_{w_i\in\bar{w}} D_B(w_i), \qquad \bar{w} \in \mathfrak{W}. \tag{3.43}$$

The following identity enlightens the geometric meaning of Condition 3.4 stated above.

Lemma 3.8. *For any Borel set $B \subset X$, there is the equality*

$$\bar{\sigma}(\mathfrak{X}_B) = \int_{\mathfrak{W}} \theta(\bar{D}_B(\bar{w}))\, Q(\mathrm{d}\bar{w}). \tag{3.44}$$

In particular, $\bar{\sigma}(\mathfrak{X}_B) < \infty$ if and only if the right-hand side of equation (3.44) *is finite.*

Proof. According to (3.23), (3.40) and (3.41), we have

$$\begin{aligned}
\bar{\sigma}(\mathfrak{X}_B) &= \int_X \int_{\mathfrak{X}_B} \eta_x(\mathrm{d}\bar{y})\, \theta(\mathrm{d}x) \\
&= \int_X \left(\int_{\mathfrak{W}} \mathbf{1}_{\mathfrak{X}_B}(\bar{\varphi}_x(\bar{w}))\, Q(\mathrm{d}\bar{w}) \right) \theta(\mathrm{d}x) \\
&= \int_{\mathfrak{W}} \left(\int_X \mathbf{1}_{\bar{D}_B(\bar{w})}(x)\, \theta(\mathrm{d}x) \right) Q(\mathrm{d}\bar{w}) \\
&= \int_{\mathfrak{W}} \theta(\bar{D}_B(\bar{w}))\, Q(\mathrm{d}\bar{w}),
\end{aligned}$$

as claimed. □

Due to formula (3.44), Condition 3.4 can be rewritten as follows.

Condition 3.4′. For any compact set $B \in \mathcal{B}(X)$, the mean θ-measure of the droplet cluster $\bar{D}_B(\bar{w}) \subset X$ is finite,

$$\int_{\mathfrak{W}} \theta(\bar{D}_B(\bar{w}))\, Q(\mathrm{d}\bar{w}) < \infty. \tag{3.45}$$

Building on Lemma 3.8, let us give two simple criteria, either of which is sufficient for Condition 3.4′ and hence for Condition 3.4. The first criterion below (Proposition 3.9) bounds the growth of the droplet volume and also assumes a finite mean number of points in the cluster, while the second criterion (Proposition 3.10) requires the continuity and separability of the maps $\varphi_x(w)$ and puts a restriction on the range of the parent cluster.

Proposition 3.9. *Suppose that the following two conditions hold*:

(i) (finite range of "translations") *for any compact set $B \subset X$, the map $\varphi_x(w)$ and the measure θ satisfy the bound*

$$C_B := \sup_{w \in W} \theta(D_B(w)) < \infty; \tag{3.46}$$

(ii) (finite mean of the cluster size) *the total number of components in $\bar{w} \in \mathfrak{W}$* (*cf.* (3.34)) *satisfies the integrability condition* (*cf.* (3.35))

$$\int_{\mathfrak{W}} N_W(\bar{w})\, Q(\mathrm{d}\bar{w}) < \infty. \tag{3.47}$$

Then Condition 3.4′ *is satisfied.*

Proof. By Lemma 3.8, it suffices to show that the integral on the right-hand side of (3.44) is finite. From (3.43) and (3.46) we readily obtain

$$\theta(\bar{D}_B(\bar{w})) \le \sum_{w_i \in \bar{w}} \theta(D_B(w_i)) \le C_B\, N_W(\bar{w}), \qquad \bar{w} \in \mathfrak{W}, \tag{3.48}$$

and by condition (3.47) it follows

$$\int_{\mathfrak{W}} \theta(\bar{D}_B(\bar{w}))\, Q(\mathrm{d}\bar{w}) \le C_B \int_{\mathfrak{W}} N_W(\bar{w})\, Q(\mathrm{d}\bar{w}) < \infty,$$

as required. □

Remark 3.10. Bound (3.46) holds, for example, if for every $w \in W$ the map $X \ni x \mapsto \varphi_x(w) \in X$ is an isometry and the measure θ is absolutely continuous with respect to the volume measure on X, with a bounded Radon–Nikodym density.

Proposition 3.10. *Suppose that the family of measurable maps $\varphi_x(w)$ described above satisfies in addition the following two conditions*:

(i) (continuity in x) *the map $\varphi_x(w)$ is continuous in $x \in X$; that is, for any open subset $U \subset X$ and each $w \in W$, the set $\{x \in X : \varphi_x(w) \in U\}$ is open in X;*

(ii) (separability) *for any compact set $B \subset X$ and each $w \in W$, there exists a compact $B_w \subset X$ such that for any $x \notin B_w$ we have $\varphi_x(w) \notin B$.*

Assume also that there is a compact set $E_0 \in \mathcal{B}(W)$ such that $Q(\mathfrak{E}_0) = 1$, where $\mathfrak{E}_0 := \bigsqcup_{n=0}^{\infty} E_0^n$ (cf. (3.38)); that is, all components of Q-a.a. vectors $\bar{w} \in \mathfrak{W}$ lie in $E_0 \subset W$. Then Condition 3.4′ is satisfied.

Proof. Let $B \subset X$ be an arbitrary compact set. Using formula (3.43) and definition (3.41), for any $\bar{w} \in \mathfrak{E}_0$ we have the inclusion

$$\begin{aligned} \bar{D}_B(\bar{w}) = \bigcup_{w_i \in \bar{w}} D_B(w_i) \subset \bigcup_{w \in E_0} D_B(w) \\ \equiv \{x : \varphi_x(E_0) \cap B \ne \emptyset\} =: \tilde{D}_B \subset X. \end{aligned} \tag{3.49}$$

To complete the proof, it suffices to show that $\theta(\tilde{D}_B) < \infty$, since then, by Lemma 3.8, it will follow

$$\begin{aligned} \bar{\sigma}(\mathfrak{X}_B) = \int_{\mathfrak{W}} \theta(\bar{D}_B(\bar{w}))\, Q(\mathrm{d}\bar{w}) = \int_{\mathfrak{E}_0} \theta(\bar{D}_B(\bar{w}))\, Q(\mathrm{d}\bar{w}) \\ \le \theta(\tilde{D}_B) \int_{\mathfrak{E}_0} Q(\mathrm{d}\bar{w}) = \theta(\tilde{D}_B) < \infty. \end{aligned}$$

To this end, for each $w \in E_0$ consider the set

$$A_w := \{w' \in E_0 :\ \varphi_x(w') \notin B \ \text{ for all } \ x \notin B_w\} \subset E_0, \tag{3.50}$$

where $B_w \subset X$ is a compact defined in property (ii); in particular, it follows that $w \in A_w$ and hence $E_0 \subset \bigcup_{w \in W} A_w$. Furthermore, using property (i) it can be shown that each A_w is an open set in the topology induced from W by restriction to E_0 (i.e., with open sets in E_0 defined as $U \cap E_0$ for all U open in W). Since E_0 is compact, there is a finite subcover; that is, one can choose finitely many points $w_1, \dots, w_m \in E_0$ such that $E_0 \subset \bigcup_{i=1}^{m} A_{w_i}$. Then, using (3.50), it is easy to see that for any x outside the set $B_* := \bigcup_{i=1}^{m} B_{w_i}$ we have $\varphi_x(w) \notin B$ for all $w \in E_0$. According to definition (3.49) of the set $\tilde{D}_B$, this implies that $\tilde{D}_B \subset B_*$, hence $\theta(\tilde{D}_B) \le \theta(B_*) < \infty$. The proof is complete. □

The next statement gives a criterion sufficient for Condition 3.2 (in turn, implying the simplicity of the cluster measure μ_{cl}, according to Theorem 3.4).

Proposition 3.11. *Let μ_{cl} be a cluster measure on the generalised configuration space $\varGamma_X^\sharp$. Assume that the background measure μ of cluster centres has a locally bounded second-order correlation function κ_μ^2 (see Appendix* B*). Assume also that for θ-a.a. $x \in X$ the corresponding "point" droplet cluster $\bar{D}_{\{x\}}(\bar{w})$ has a.s. zero θ-measure,*

$$\theta\bigl(\bar{D}_{\{x\}}(\bar{w})\bigr) = 0 \quad \textit{for } Q\textit{-a.a. } \bar{w} \in \mathfrak{W}. \tag{3.51}$$

Then Condition 3.2 *is satisfied and hence μ_{cl}-a.a. configurations $\gamma \in \varGamma_X^\sharp$ are simple.*

Proof. It suffices to prove that, for any compact set $\varLambda \subset X$, there are μ_{cl}-a.s. no cross-ties between the clusters whose centres belong to $\varLambda$. In view of the projection construction of the cluster measure μ_{cl} (see (3.8)), this means that if $A_\varLambda$ is the set of generalised configurations $\hat{\gamma} \in \varGamma_Z^\sharp$, each with at least two points $z = (x, \bar{y}_x)$, $z' = (x', \bar{y}_{x'})$ $(z, z' \in \hat{\gamma},\ z \neq z')$ such that $\{x, x'\} \subset \gamma \cap \varLambda$ and $\mathfrak{p}(\bar{y}_x) \cap \mathfrak{p}(\bar{y}_{x'}) \neq \emptyset$, then we must show that $\hat{\mu}(A_\varLambda) = 0$. Note that since the ground configuration $\gamma \in \varGamma_X^\sharp$ may have multiple points, the points $x = p_x(z)$, $x' = p_X(z')$ in the pair $\{x, x'\} \subset \gamma$ are allowed to coincide.

Recalling the skew-product definition (2.17) of $\hat{\mu}$, we see by inspection of all pairs $\{x, x'\} \subset \gamma := p_X(\hat{\gamma})$ that

$$\hat{\mu}(A_\varLambda) \le \int_{\varGamma_X^\sharp} \sum_{\{x,x'\}\subset\gamma} \mathbf{1}_{\varLambda^2}(x, x')\, f(x, x')\, \mu(\mathrm{d}\gamma), \tag{3.52}$$

where

$$f(x, x') := \eta_x \otimes \eta_{x'}(\mathcal{D}_{\mathfrak{X}}) = \int_{\mathfrak{X}^2} \mathbf{1}_{\mathcal{D}_{\mathfrak{X}}}(\bar{y}, \bar{y}')\, \eta_x(\mathrm{d}\bar{y})\, \eta_{x'}(\mathrm{d}\bar{y}') \tag{3.53}$$

and the set $\mathcal{D}_{\mathfrak{X}} \in \mathcal{B}(\mathfrak{X}^2)$ is defined by

$$\mathcal{D}_{\mathfrak{X}} := \{(\bar{y}, \bar{y}') \in \mathfrak{X}^2 :\ \mathfrak{p}(\bar{y}) \cap \mathfrak{p}(\bar{y}') \neq \emptyset\}. \tag{3.54}$$

By definition (B.1) of correlation functions, the right-hand side of (3.52) is reduced to

$$\frac{1}{2!} \int_{\varLambda^2} f(x, x')\, \kappa_\mu^2(x, x')\, \theta(\mathrm{d}x)\, \theta(\mathrm{d}x') \le \mathrm{const} \int_{\varLambda^2} f(x, x')\, \theta(\mathrm{d}x)\, \theta(\mathrm{d}x'), \tag{3.55}$$

since, by assumption, κ_μ^2 is bounded on $\varLambda^2$. Furthermore, substituting (3.53) and changing the variables $\bar{y} = \varphi_x(\bar{w})$, $\bar{y}' = \varphi_{x'}(\bar{w}')$ (see (3.40)), the integral on the right-hand side of (3.55) is rewritten as

$$\begin{aligned}
&\int_{\varLambda^2} \left(\int_{\mathfrak{X}^2} \mathbf{1}_{\mathcal{D}_{\mathfrak{X}}}(\varphi_x(\bar{w}), \varphi_{x'}(\bar{w})')\, Q^{\otimes 2}(\mathrm{d}\bar{w} \times \mathrm{d}\bar{w}') \right) \theta^{\otimes 2}(\mathrm{d}x \times \mathrm{d}x') \\
&\qquad = \int_{\mathfrak{X}^2} \left(\int_{\varLambda^2} \mathbf{1}_{\mathcal{D}_{\mathfrak{X}}}(\varphi_x(\bar{w}), \varphi_{x'}(\bar{w})')\, \theta^{\otimes 2}(\mathrm{d}x \times \mathrm{d}x') \right) Q^{\otimes 2}(\mathrm{d}\bar{w} \times \mathrm{d}\bar{w}') \\
&\qquad = \int_{\mathfrak{X}^2} \theta^{\otimes 2}\bigl(B_\varLambda(\bar{w}, \bar{w}')\bigr)\, Q^{\otimes 2}(\mathrm{d}\bar{w} \times \mathrm{d}\bar{w}'),
\end{aligned}$$

where the set $B_\varLambda(\bar{w}, \bar{w}') \subset \varLambda^2$ is given by (cf. (3.54))

$$B_\varLambda(\bar{w}, \bar{w}') := \{(x, x') \in \varLambda^2 :\ \varphi_x(w) = \varphi_{x'}(w') \text{ for some } w \in \bar{w},\ w' \in \bar{w}'\}.$$

It remains to note that

$$\begin{aligned}\theta^{\otimes 2}\big(B_\Lambda(\bar w,\bar w')\big) &= \int_\Lambda \theta\Big(\bigcup\nolimits_{w_i\in\bar w}\bigcup\nolimits_{w'_j\in\bar w'}\{x' : \varphi_{x'}(w'_j)=\varphi_x(w_i)\}\Big)\,\theta(\mathrm{d}x)\\ &\le \sum_{w_i\in\bar w}\int_\Lambda \theta\Big(\bigcup\nolimits_{w'_j\in\bar w'}\{x' : \varphi_{x'}(w'_j)=\varphi_x(w_i)\}\Big)\,\theta(\mathrm{d}x)\\ &= \sum_{w_i\in\bar w}\int_\Lambda \theta\big(\bar D_{\{\varphi_x(w_i)\}}(\bar w')\big)\,\theta(\mathrm{d}x) = 0 \qquad (Q^{\otimes 2}\text{-a.s.}),\end{aligned}$$

since, by assumption (3.51), $\theta\big(\bar D_{\{\varphi_x(w)\}}(\bar w')\big)=0$ for θ-a.a. $x\in\Lambda$, Q-a.a. $\bar w\in\mathfrak{W}$ and each $w_i\in\bar w$, and, moreover, $\bar w\in\mathfrak{W}$ contains at most countably many coordinates. Hence, the right-hand side of (3.55) vanishes and due to estimates (3.52) and (3.55) the claim of the proposition follows. □

It is easy to give simple sufficient criteria for condition (3.51) of Proposition 3.11. The first criterion below is set out in terms of the reference measure θ, whereas the second one exploits the in-cluster parent distribution Q.

Proposition 3.12. *Assume that for each $x\in X$, the equation $\varphi_y(w)=x$ has at most one solution $y=y(x;w)$ for every $w\in\bar w$ and Q-a.a. $\bar w\in\mathfrak{W}$. Furthermore, let the measure θ be non-atomic, that is, $\theta\{y\}=0$ for each $y\in X$. Then condition* (3.51) *is satisfied.*

Proof. Using formula (3.43) and definition (3.41), we obtain

$$\begin{aligned}0\le\theta\big(\bar D_{\{x\}}(\bar w)\big) &\le \sum_{w_i\in\bar w}\theta\big(D_{\{x\}}(w_i)\big)\\ &= \sum_{w_i\in\bar w}\theta\{y\in X:\ \varphi_y(w_i)=x\}\\ &= \sum_{w_i\in\bar w}\theta\{y(x;w_i)\}=0,\end{aligned}$$

since the measure θ is non-atomic. □

Proposition 3.13. *Suppose that the in-cluster configurations a.s. have no fixed points, that is, for any $x\in X$ and θ-a.a. $y\in X$,*

$$Q\big\{\bar w\in\mathfrak{W}:\ \exists\, w_i\in\bar w\ \textit{ such that }\ \varphi_y(w_i)=x\big\}=0. \tag{3.56}$$

Then condition (3.51) *follows.*

Proof. Observe that identity (3.44) together with the change of measure (3.40) yields, for each $x\in X$,

$$\begin{aligned}\int_{\mathfrak{W}}\theta\big(\bar D_{\{x\}}(\bar w)\big)\,Q(\mathrm{d}\bar w) &= \int_X\bigg(\int_{\mathfrak{X}_{\{x\}}}\eta_{x'}(\mathrm{d}\bar y)\bigg)\,\theta(\mathrm{d}x')\\ &= \int_X Q\big(\bar\varphi_{x'}^{-1}(\mathfrak{X}_{\{x\}})\big)\,\theta(\mathrm{d}x')\\ &= \int_X Q\{\bar w\in\mathfrak{W}:\ \varphi_{x'}(w_i)=x\ \text{ for some }\ w_i\in\bar w\}\,\theta(\mathrm{d}x')=0,\end{aligned}$$

according to (3.56). Thus, the proof is complete. □

4. Quasi-invariance and integration by parts

From now on, we restrict ourselves to the case where $\mu_{\rm cl}$-a.a. configurations $\gamma \in \varGamma_X^\sharp$ are locally finite.

Let us also assume that X is a Riemannian manifold (with a fixed Riemannian structure). Our aim in this section is to prove the quasi-invariance of the measure $\mu_{\rm cl}$ with respect to the action of compactly supported diffeomorphisms of X (Section 4.2), and to establish an IBP formula (Section 4.3). We begin in Section 4.1 with a brief description of some convenient "manifold-like" concepts and notation first introduced in [3] (see also [10, §4.1]), which furnish a suitable framework for analysis on configuration spaces.

4.1. Differentiable functions on configuration spaces

Let T_xX be the tangent space of X at point $x \in X$, with the corresponding (canonical) inner product denoted by a "fat" dot $\boldsymbol{\cdot}$. The gradient on X is denoted by ∇. Following [3], we define the "tangent space" of the configuration space $\varGamma_X$ at $\gamma \in \varGamma_X$ as the Hilbert space $T_\gamma\varGamma_X := L^2(X \to TX;\, \mathrm{d}\gamma)$, or equivalently $T_\gamma\varGamma_X = \bigoplus_{x\in\gamma} T_xX$. The scalar product in $T_\gamma\varGamma_X$ is denoted by $\langle\cdot,\cdot\rangle_\gamma$, with the corresponding norm $|\cdot|_\gamma$. A vector field V over $\varGamma_X$ is a map $\varGamma_X \ni \gamma \mapsto V(\gamma) = (V(\gamma)_x)_{x\in\gamma} \in T_\gamma\varGamma_X$. Thus, for vector fields V_1, V_2 over $\varGamma_X$ we have

$$\langle V_1(\gamma), V_2(\gamma)\rangle_\gamma = \sum_{x\in\gamma} V_1(\gamma)_x \boldsymbol{\cdot} V_2(\gamma)_x, \qquad \gamma \in \varGamma_X.$$

For $\gamma \in \varGamma_X$ and $x \in \gamma$, denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x} \cap \gamma = \{x\}$. For any measurable function $F : \varGamma_X \to \mathbb{R}$, define the function $F_x(\gamma,\cdot) : \mathcal{O}_{\gamma,x} \to \mathbb{R}$ by $F_x(\gamma, y) := F((\gamma \setminus \{x\}) \cup \{y\})$, and set

$$\nabla_x F(\gamma) := \nabla F_x(\gamma, y)\big|_{y=x}\,, \qquad x \in X,$$

provided that $F_x(\gamma,\cdot)$ is differentiable at x.

Recall that for a function $\phi : X \to \mathbb{R}$ its support $\operatorname{supp}\phi$ is defined as the closure of the set $\{x \in X\colon \phi(x) \neq 0\}$. Denote by $\mathcal{FC}(\varGamma_X)$ the class of functions on $\varGamma_X$ of the form

$$F(\gamma) = f(\langle\phi_1,\gamma\rangle, \dots, \langle\phi_k,\gamma\rangle), \qquad \gamma \in \varGamma_X, \tag{4.1}$$

where $k \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^k)$ (:= the set of C^∞-functions on $\mathbb{R}^k$ globally bounded together with all their derivatives), and $\phi_1, \dots, \phi_k \in C_0^\infty(X)$ (:= the set of C^∞-functions on X with compact support). Each $F \in \mathcal{FC}(\varGamma_X)$ is local, that is, there is a compact $B \subset X$ (e.g., $B = \cup_{j=1}^k \operatorname{supp}\phi_j$) such that $F(\gamma) = F(\gamma \cap B)$ for all $\gamma \in \varGamma_X$. Thus, for a fixed γ there are finitely many non-zero derivatives $\nabla_x F(\gamma)$.

For a function $F \in \mathcal{FC}(\varGamma_X)$ its $\varGamma$-gradient $\nabla^\varGamma F$ is defined as

$$\nabla^\varGamma F(\gamma) := (\nabla_x F(\gamma))_{x\in\gamma} \in T_\gamma\varGamma_X, \qquad \gamma \in \varGamma_X, \tag{4.2}$$

so the directional derivative of F along a vector field V is given by

$$\nabla_V^\varGamma F(\gamma) := \langle\nabla^\varGamma F(\gamma), V(\gamma)\rangle_\gamma = \sum_{x\in\gamma} \nabla_x F(\gamma) \boldsymbol{\cdot} V(\gamma)_x, \qquad \gamma \in \varGamma_X.$$

Note that the sum here contains only finitely many non-zero terms.

Further, let $\mathcal{FV}(\Gamma_X)$ be the class of cylinder vector fields V on Γ_X of the form

$$V(\gamma)_x = \sum_{i=1}^{k} G_i(\gamma)\, v_i(x) \in T_x X, \qquad x \in X, \tag{4.3}$$

where $G_i \in \mathcal{FC}(\Gamma_X)$ and $v_i \in \mathrm{Vect}_0(X)$ (:= the space of compactly supported C^∞-smooth vector fields on X), $i = 1, \dots, k$ ($k \in \mathbb{N}$). Any vector filed $v \in \mathrm{Vect}_0(X)$ generates a constant vector field V on Γ_X defined by $V(\gamma)_x := v(x)$. We shall preserve the notation v for it. Thus,

$$\nabla_v^\Gamma F(\gamma) = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot v(x), \qquad \gamma \in \Gamma_X. \tag{4.4}$$

The approach based on "lifting" the differential structure from the underlying space X to the configuration space Γ_X as described above can also be applied to the spaces $\mathfrak{X} = \bigsqcup_{n=0}^{\infty} X^n$, $Z = X \times \mathfrak{X}$ and $\Gamma_{\mathfrak{X}}$, Γ_Z, respectively. In such cases, we will use the analogous notation as above without further explanation.

4.2. Quasi-invariance

In this section, we discuss the property of quasi-invariance of the measure μ_{cl} with respect to diffeomorphisms of X. Let us start by describing how diffeomorphisms of X act on configuration spaces. For a measurable map $\varphi : X \to X$, its *support* $\operatorname{supp}\varphi$ is defined as the closure of the set $\{x \in X : \varphi(x) \neq x\}$. Let $\mathrm{Diff}_0(X)$ be the group of diffeomorphisms of X with *compact support*. For any $\varphi \in \mathrm{Diff}_0(X)$, consider the corresponding "diagonal" diffeomorphism $\bar\varphi : \mathfrak{X} \to \mathfrak{X}$ acting on each constituent space X^n ($n \in \mathbb{Z}_+$) as

$$X^n \ni \bar y = (y_1, \dots, y_n) \mapsto \bar\varphi(\bar y) := (\varphi(y_1), \dots, \varphi(y_n)) \in X^n. \tag{4.5}$$

Finally, we introduce a special class of diffeomorphisms $\hat\varphi$ on Z acting only in the $\bar y$-coordinate,

$$\hat\varphi(z) := (x, \bar\varphi(\bar y)), \qquad z = (x, \bar y) \in Z. \tag{4.6}$$

Remark 4.1. Despite $K_\varphi := \operatorname{supp}\varphi$ is compact in X, the support of the diffeomorphism $\hat\varphi$ (again defined as the closure of the set $\{z \in Z : \hat\varphi(z) \neq z\}$) is given by $\operatorname{supp}\hat\varphi = X \times \mathfrak{X}_{K_\varphi}$ (see (3.17)), where $\mathfrak{X}_{K_\varphi}$ is *not* compact in the topology of $\mathfrak{X}$ (cf. Remark 2.1).

In a standard fashion, the maps φ, $\bar\varphi$ and $\hat\varphi$ can be lifted to measurable "diagonal" transformations (denoted by the same letters) of the configuration spaces Γ_X, $\Gamma_{\mathfrak{X}}$ and Γ_Z, respectively:

$$\Gamma_X \ni \gamma \mapsto \varphi(\gamma) := \{\varphi(x),\ x \in \gamma\} \in \Gamma_X, \tag{4.7}$$

$$\Gamma_{\mathfrak{X}} \ni \bar\gamma \mapsto \bar\varphi(\bar\gamma) := \{\bar\varphi(\bar y),\ \bar y \in \bar\gamma\} \in \Gamma_{\mathfrak{X}}, \tag{4.8}$$

$$\Gamma_Z \ni \hat\gamma \mapsto \hat\varphi(\hat\gamma) := \{\hat\varphi(z),\ z \in \hat\gamma\} \in \Gamma_Z. \tag{4.9}$$

The following lemma shows that the operator $\mathfrak{q}$ commutes with the action of diffeomorphisms (4.7) and (4.9).

Lemma 4.1. *For any diffeomorphism $\varphi \in \mathrm{Diff}_0(X)$ and the corresponding diffeomorphism $\hat\varphi$, it holds*

$$\varphi \circ \mathfrak{q} = \mathfrak{q} \circ \hat\varphi. \tag{4.10}$$

Proof. The statement follows from definition (3.2) of the map $\mathfrak{q}$ in view of the structure of diffeomorphisms φ and $\hat{\varphi}$ (see (4.6), (4.7) and (4.9)). □

Assume that, for all $x \in X$, the measure η_x is absolutely continuous with respect to the Riemannian volume $\mathrm{d}\bar{y}$ on $\mathfrak{X}$ and, moreover,

$$h_x(\bar{y}) := \frac{\eta_x(\mathrm{d}\bar{y})}{\mathrm{d}\bar{y}} > 0 \qquad \text{for a.a. } \bar{y} \in \mathfrak{X}. \tag{4.11}$$

This implies that the measure η_x is quasi-invariant with respect to the action of transformations $\bar{\varphi} : \mathfrak{X} \to \mathfrak{X}$ ($\varphi \in \mathrm{Diff}_0(X)$), that is, the measure $\bar{\varphi}^*\eta_x$ is absolutely continuous with respect to η_x with the Radon–Nikodym density

$$\rho_\eta^{\bar{\varphi}}(x, \bar{y}) := \frac{\mathrm{d}(\bar{\varphi}^*\eta_x)}{\mathrm{d}\eta_x}(\bar{y}) = \frac{h_x(\bar{\varphi}^{-1}(\bar{y}))}{h_x(\bar{y})}\, J_{\bar{\varphi}}(\bar{y})^{-1} \tag{4.12}$$

(we set $\rho_\eta^{\bar{\varphi}}(x, \bar{y}) = 1$ if $h_x(\bar{y}) = 0$ or $h_x(\bar{\varphi}^{-1}(\bar{y})) = 0$). Here $J_{\bar{\varphi}}(\bar{y})$ is the Jacobian determinant of the diffeomorphism $\bar{\varphi}$; due to the diagonal structure of $\bar{\varphi}$ (see (4.5)) we have $J_{\bar{\varphi}}(\bar{y}) = \prod_{y_i \in \bar{y}} J_\varphi(y_i)$, where $J_\varphi(y)$ is the Jacobian determinant of φ.

Theorem 4.2. *The measure $\hat{\mu}$ is quasi-invariant with respect to the action of $\hat{\varphi}$ on Γ_Z defined by formula* (4.6)*, with the Radon–Nikodym density $R_{\hat{\mu}}^{\hat{\varphi}} = \mathrm{d}(\hat{\varphi}^*\hat{\mu})/\mathrm{d}\hat{\mu}$ given by*

$$R_{\hat{\mu}}^{\hat{\varphi}}(\hat{\gamma}) = \prod_{z \in \hat{\gamma}} \rho_\eta^{\bar{\varphi}}(z), \qquad \hat{\gamma} \in \Gamma_Z. \tag{4.13}$$

Moreover, $R_{\hat{\mu}}^{\hat{\varphi}} \in L^1(\Gamma_Z, \hat{\mu})$.

Proof. First of all, note that $\rho_\varphi(z) = 1$ for any $z = (x, \bar{y}) \notin \operatorname{supp} \hat{\varphi} = X \times \mathfrak{X}_B =: Z_{K_\varphi}$, where $K_\varphi = \operatorname{supp} \varphi$ (see Remark 4.1), and $\hat{\sigma}(Z_{K_\varphi}) = \bar{\sigma}(\mathfrak{X}_{K_\varphi}) < \infty$ by Condition 3.4 (see (3.22)). Therefore, $\hat{\gamma}(Z_{K_\varphi}) < \infty$ for $\hat{\mu}$-a.a. configurations $\hat{\gamma} \in \Gamma_Z$, hence the product in (4.13) contains only finitely many terms different from 1 and so the function $R_{\hat{\mu}}^{\hat{\varphi}}(\hat{\gamma})$ is well defined. Moreover, it satisfies the "localisation" equality

$$R_{\hat{\mu}}^{\hat{\varphi}}(\hat{\gamma}) = R_{\hat{\mu}}^{\hat{\varphi}}(\hat{\gamma} \cap Z_{K_\varphi}) \qquad \text{for } \hat{\mu}\text{-a.a. } \hat{\gamma} \in \Gamma_Z. \tag{4.14}$$

Now, using definitions (4.7), (4.8) and (2.16), we obtain

$$\begin{aligned}
\int_{\Gamma_Z} F(\hat{\gamma})\, \hat{\varphi}^*\hat{\mu}(\mathrm{d}\hat{\gamma}) &= \int_{\Gamma_Z} F(\hat{\varphi}(\hat{\gamma}))\, \hat{\mu}(\mathrm{d}\hat{\gamma}) \\
&= \int_{\Gamma_X} \left(\int_{\mathfrak{X}^\gamma} F(\gamma, \bar{\varphi}(\bar{y}^\gamma))\, \eta^\gamma(\mathrm{d}\bar{y}^\gamma) \right) \mu(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X} \left(\int_{\mathfrak{X}^\gamma} F(\gamma, \bar{\varphi}(\bar{y}^\gamma)) \bigotimes_{x \in \gamma} \eta_x(\mathrm{d}\bar{y}_x) \right) \mu(\mathrm{d}\gamma) \\
&= \int_{\Gamma_X} \left(\int_{\mathfrak{X}^\gamma} F(\gamma, \bar{y}^\gamma) \bigotimes_{x \in \gamma} \bar{\varphi}^*\eta_x(\mathrm{d}\bar{y}_x) \right) \mu(\mathrm{d}\gamma).
\end{aligned} \tag{4.15}$$

Furthermore, by the quasi-invariance property of the measure η_x (see formula (4.12) for the density), the right-hand side of (4.15) is represented in the form

$$\int_{\Gamma_X} \left(\int_{\mathfrak{X}^\gamma} F(\gamma, \bar{y}^\gamma) \prod_{x \in \gamma} \rho_\eta^{\bar{\varphi}}(x, \bar{y}_x)\, \eta^\gamma(\mathrm{d}\bar{y}^\gamma) \right) \mu(\mathrm{d}\gamma) = \int_{\Gamma_Z} F(\hat{\gamma}) R_{\hat{\mu}}^{\hat{\varphi}}(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}),$$

which proves the quasi-invariance of $\hat{\mu}$. In particular, for $F \equiv 1$ this yields $\int_{\Gamma_Z} R_{\hat{\mu}}^{\hat{\varphi}}(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}) = 1$, and hence $R_{\hat{\mu}}^{\hat{\varphi}} \in L^1(\Gamma_Z, \hat{\mu})$, as claimed. $\square$

Let $\mathcal{I}_{\mathfrak{q}} : L^\infty(\Gamma_X, \mu_{\mathrm{cl}}) \to L^\infty(\Gamma_Z, \hat{\mu})$ be the isometry defined by the map $\mathfrak{q}$ (see (3.2)),

$$(\mathcal{I}_{\mathfrak{q}} F)(\hat{\gamma}) := F \circ \mathfrak{q}(\hat{\gamma}), \qquad \hat{\gamma} \in \Gamma_Z. \tag{4.16}$$

The adjoint operator $\mathcal{I}_{\mathfrak{q}}^*$ is a bounded operator on the corresponding dual spaces,

$$\mathcal{I}_{\mathfrak{q}}^* : L^\infty(\Gamma_Z, \hat{\mu})' \to L^\infty(\Gamma_X, \mu_{\mathrm{cl}})'. \tag{4.17}$$

Lemma 4.3. *The operator $\mathcal{I}_{\mathfrak{q}}^*$ defined by* (4.17) *can be restricted to the operator*

$$\mathcal{I}_{\mathfrak{q}}^* : L^1(\Gamma_Z, \hat{\mu}) \to L^1(\Gamma_X, \mu_{\mathrm{cl}}). \tag{4.18}$$

Proof. It is known (see [24]) that, for any σ-finite measure space (M, μ), the corresponding space $L^1(M, \mu)$ can be identified with the subspace V of the dual space $L^\infty(M, \mu)'$ consisting of all linear functionals on $L^\infty(M, \mu)$ continuous with respect to the bounded convergence in $L^\infty(M, \mu)$. That is, $\ell \in V$ if and only if $\ell(\psi_n) \to 0$ for any $\psi_n \in L^\infty(M, \mu)$ such that $|\psi_n| \le 1$ and $\psi_n(x) \to 0$ as $n \to \infty$ for μ-a.a. $x \in M$. Hence, to prove the lemma it suffices to show that, for any $F \in L^1(\Gamma_Z, \hat{\mu})$, the functional $\mathcal{I}_{\mathfrak{q}}^* F \in L^\infty(\Gamma_Z, \hat{\mu})'$ is continuous with respect to bounded convergence in $L^\infty(\Gamma_Z, \hat{\mu})$. To this end, for any sequence (ψ_n) in $L^\infty(\Gamma_X, \mu_{\mathrm{cl}})$ such that $|\psi_n| \le 1$ and $\psi_n(\gamma) \to 0$ for μ_{cl}-a.a. $\gamma \in \Gamma_X$, we have to prove that $\mathcal{I}_{\mathfrak{q}}^* F(\psi_n) \to 0$.

Let us first show that $\mathcal{I}_{\mathfrak{q}} \psi_n(\hat{\gamma}) \equiv \psi_n(\mathfrak{q}(\hat{\gamma})) \to 0$ for $\hat{\mu}$-a.a. $\hat{\gamma} \in \Gamma_Z$. Set

$$\begin{aligned} A_\psi &:= \{\gamma \in \Gamma_X : \psi_n(\gamma) \to 0\} \in \mathcal{B}(\Gamma_X), \\ \hat{A}_\psi &:= \{\hat{\gamma} \in \Gamma_Z : \psi_n(\mathfrak{q}(\hat{\gamma})) \to 0\} \in \mathcal{B}(\Gamma_Z), \end{aligned}$$

and note that $\hat{A}_\psi = \mathfrak{q}^{-1}(A_\psi)$; then, recalling relation (3.8), we get

$$\hat{\mu}(\hat{A}_\psi) = \hat{\mu}\big(\mathfrak{q}^{-1}(A_\psi)\big) = \mu_{\mathrm{cl}}(A_\psi) = 1,$$

as claimed. Now, by the dominated convergence theorem this implies

$$\mathcal{I}_{\mathfrak{q}}^* F(\psi_n) = \int_{\Gamma_Z} F(\hat{\gamma})\, \mathcal{I}_{\mathfrak{q}} \psi_n(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}) \to 0,$$

and the proof is complete. $\square$

Corollary 4.4. *For any measurable functions $F \in L^\infty(\Gamma_X, \mu_{\mathrm{cl}})$ and $G \in L^1(\Gamma_Z, \hat{\mu})$, we have the identity*

$$\int_{\Gamma_Z} G(\hat{\gamma})\, \mathcal{I}_{\mathfrak{q}} F(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_X} F(\gamma)\, \mathcal{I}_{\mathfrak{q}}^* G(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma). \tag{4.19}$$

Taking advantage of Theorem 4.2 and applying the projection construction, we obtain our main result in this section.

Theorem 4.5. *The cluster measure μ_{cl} is quasi-invariant with respect to the action of the diffeomorphism group $\mathrm{Diff}_0(X)$ on Γ_X. The corresponding Radon–Nikodym density is given by $R_{\mu_{\mathrm{cl}}}^{\varphi} = \mathcal{I}_{\mathfrak{q}}^* R_{\hat{\mu}}^{\hat{\varphi}} \in L^1(\Gamma_X, \mu_{\mathrm{cl}})$.*

Proof. Note that, due to (3.8) and (4.10),

$$\mu_{\rm cl} \circ \varphi^{-1} = \hat{\mu} \circ \mathsf{q}^{-1} \circ \varphi^{-1} = \hat{\mu} \circ \hat{\varphi}^{-1} \circ \mathsf{q}^{-1}.$$

That is, $\varphi^*\mu_{\rm cl} = \mu_{\rm cl} \circ \varphi^{-1}$ is a push-forward of the measure $\hat{\varphi}^*\hat{\mu} = \hat{\mu} \circ \hat{\varphi}^{-1}$ under the map q, that is, $\varphi^*\mu_{\rm cl} = \mathsf{q}^*\hat{\varphi}^*\hat{\mu}$. In particular, if $\hat{\varphi}^*\hat{\mu}$ is absolutely continuous with respect to $\hat{\mu}$ then so is $\varphi^*\mu_{\rm cl}$ with respect to $\mu_{\rm cl}$. Moreover, by formula (3.8) and Theorem 4.2, for any $F \in L^\infty(\Gamma_X, \mu_{\rm cl})$ we have

$$\int_{\Gamma_X} F(\gamma)\, \varphi^*\mu_{\rm cl}(\mathrm{d}\gamma) = \int_{\Gamma_Z} \mathcal{I}_{\mathsf{q}} F(\hat{\gamma})\, \hat{\varphi}^*\hat{\mu}(\mathrm{d}\hat{\gamma}) = \int_{\Gamma_Z} \mathcal{I}_{\mathsf{q}} F(\hat{\gamma})\, R_{\hat{\mu}}^{\hat{\varphi}}(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}). \tag{4.20}$$

By Lemma 4.3, the operator $\mathcal{I}_{\mathsf{q}}^*$ acts from $L^1(\Gamma_Z, \hat{\mu})$ to $L^1(\Gamma_X, \mu_{\rm cl})$. Therefore, again using (3.8) the right-hand side of (4.20) can be rewritten as

$$\int_{\Gamma_X} F(\gamma)\, (\mathcal{I}_{\mathsf{q}}^* R_{\hat{\mu}}^{\hat{\varphi}})(\gamma)\, \mu_{\rm cl}(\mathrm{d}\gamma),$$

which completes the proof. □

Remark 4.2. Cluster measure $\mu_{\rm cl}$ on the configuration space Γ_X can be used to construct a unitary representation U of the diffeomorphism group $\mathrm{Diff}_0(X)$ by operators in $L^2(\Gamma_X, \mu_{\rm cl})$, given by the formula

$$U_\varphi F(\gamma) = \sqrt{R_{\mu_{\rm cl}}^{\varphi}(\gamma)}\, F(\varphi^{-1}(\gamma)), \qquad F \in L^2(\Gamma_X, \mu_{\rm cl}). \tag{4.21}$$

Such representations, which can be defined for arbitrary quasi-invariant measures on Γ_X, play a significant role in the representation theory of the group $\mathrm{Diff}_0(X)$ [20, 31] and quantum field theory [17, 18]. An important question is whether the representation (4.21) is irreducible. According to [31], this is equivalent to the $\mathrm{Diff}_0(X)$-ergodicity of the measure $\mu_{\rm cl}$, which in our case is equivalent to the ergodicity of the measure $\hat{\mu}$ with respect to the group of transformations $\hat{\varphi}$ ($\varphi \in \mathrm{Diff}_0(X)$).

4.3. Integration-by-parts (IBP) formulae

In this section, we assume that the conditions of Lemma 3.7 are satisfied with $n = 1$. Thus, the measures μ, $\hat{\mu}$ and $\mu_{\rm cl}$ belong to the corresponding $\mathcal{M}^1$-classes. It is also assumed, as before, that for each $x \in X$ the measure η_x is absolutely continuous with respect to the Riemannian volume $\mathrm{d}\bar{y}$ on $\mathfrak{X}$, with the Radon–Nykodym density $h_x(\bar{y})$.

4.3.1. Integration by parts for the cluster distributions η_x. Let $v \in \mathrm{Vect}_0(X)$ (:= the space of compactly supported smooth vector fields on X), and define a "vertical" vector field $\hat{v}$ on Z by the formula

$$\hat{v}(x, \bar{y}) := (v(y_i))_{y_i \in \bar{y}}, \qquad \bar{y} = (y_i) \in \mathfrak{X}. \tag{4.22}$$

Observe that if the density $h_x(\bar{y})$ is differentiable ($\mathrm{d}\bar{y}$-a.e.) then the measure η_x satisfies the IBP formula (see, e.g., [6, §1.3, §2.4]; cf. [12, §5.1.3, p. 207])

$$\int_{\mathfrak{X}} \nabla^{\hat{v}} f(\bar{y})\, \eta_x(\mathrm{d}\bar{y}) = -\int_{\mathfrak{X}} f(\bar{y})\, \beta_\eta^{\hat{v}}(x, \bar{y})\, \eta_x(\mathrm{d}\bar{y}), \qquad f \in C_0^\infty(\mathfrak{X}), \tag{4.23}$$

where $\nabla^{\hat{v}}$ is the derivative along the vector field $\hat{v}$ and

$$\beta_\eta^{\hat{v}}(x,\bar{y}) := (\beta_\eta(x,\bar{y}),\hat{v}(x,\bar{y}))_{T_{\bar{y}}\mathfrak{X}} + \operatorname{div}\hat{v}(x,\bar{y}) \tag{4.24}$$

is the logarithmic derivative of $\eta_x(\mathrm{d}\bar{y}) = h_x(\bar{y})\,\mathrm{d}\bar{y}$ along $\hat{v}$, expressed in terms of the vector logarithmic derivative

$$\beta_\eta(x,\bar{y}) := \frac{\nabla h_x(\bar{y})}{h_x(\bar{y})} \in T_{\bar{y}}\mathfrak{X}, \qquad (x,\bar{y}) \in X \times \mathfrak{X}. \tag{4.25}$$

Denote for brevity

$$\|\bar{y}\|_1 := \sum_{y_i\in\bar{y}} |y_i|, \qquad \bar{y} \in \mathfrak{X}.$$

Lemma 4.6. *Suppose that* $\int_{Z_B} \|\beta_\eta(z)\|_1^n\,\hat{\sigma}(\mathrm{d}z) < \infty$ *for any compact* $B \subset X$*, and assume that condition* (3.35) *is satisfied. Let* $\hat{v}$ *be a vector filed on* Z *defined by* (4.22) *with* $v \in \operatorname{Vect}_0(X)$*. Then* $\beta_\eta^{\hat{v}} \in L^n(Z,\hat{\sigma})$*.*

Proof. To show that $\beta_\eta^{\hat{v}} \in L^n(Z,\hat{\sigma})$, it suffices to check that each of the two terms on the right-hand side of (4.24) belongs to $L^n(Z,\hat{\sigma})$. Setting $b_v := \sup_{x\in X}|v(x)| < \infty$ and noting that $K_v := \operatorname{supp} v$ is a compact in X, we have

$$\begin{aligned}
\int_Z |(\beta_\eta(z),\hat{v}(z))|^n\,\hat{\sigma}(\mathrm{d}z) &\le \int_X\int_{\mathfrak{X}_{K_v}} \Bigl(\sum_{y_i\in\bar{y}} |\beta_\eta(x,\bar{y})_i|\cdot|v(y_i)|\Bigr)^n \eta_x(\mathrm{d}\bar{y})\,\theta(\mathrm{d}x)\\
&\le (b_v)^n \int_X\int_{\mathfrak{X}_{K_v}} \Bigl(\sum_{y_i\in\bar{y}} |\beta_\eta(x,\bar{y})_i|\Bigr)^n \eta_x(\mathrm{d}\bar{y})\,\theta(\mathrm{d}x)\\
&= (b_v)^n \int_{Z_{K_v}} \|\beta_\eta(z)\|_1^n\,\hat{\sigma}(\mathrm{d}z) < \infty,
\end{aligned} \tag{4.26}$$

by the first hypothesis of the theorem. Similarly, denoting $d_v := \sup_{x\in X}|\operatorname{div} v(x)| < \infty$, we obtain

$$\begin{aligned}
\int_Z |\operatorname{div}\hat{v}(x,\bar{y})|^n\,\hat{\sigma}(\mathrm{d}x\times\mathrm{d}\bar{y}) &= \int_Z \Bigl(\sum_{y_i\in\bar{y}} |\operatorname{div} v(y_i)|\Bigr)^n \eta_x(\mathrm{d}\bar{y})\,\theta(\mathrm{d}x)\\
&\le (d_v)^n \int_X\int_{\mathfrak{X}_{K_v}} N_X(\bar{y})^n\,\eta_x(\mathrm{d}\bar{y})\,\theta(\mathrm{d}x) < \infty,
\end{aligned} \tag{4.27}$$

according to assumption (3.35). As a result, combining bounds (4.26) and (4.27), we see that $\beta_\eta^{\hat{v}} \in L^n(Z,\hat{\sigma})$, as claimed. □

Let us define the space $H^{1,n}_{\mathrm{loc}}(\mathfrak{X})$ ($n \ge 1$) as the set of functions $f \in L^n(\mathfrak{X},\mathrm{d}\bar{y})$ satisfying, for any compact $B \subset X$, the condition

$$\kappa_n^B(f) := \int_{\mathfrak{X}} \|\nabla f(\bar{y})\|_1^n\,\mathrm{d}\bar{y} \equiv \int_{\mathfrak{X}_B} \Bigl(\sum_{y_i\in\bar{y}} |\nabla_{y_i} f(\bar{y})|\Bigr)^n \mathrm{d}\bar{y} < \infty. \tag{4.28}$$

Due to the elementary inequality $(|a|+|b|)^n \le 2^{n-1}\bigl(|a|^n+|b|^n\bigr)$, $H^{1,n}(\mathfrak{X})$ is a linear space.

The integrability condition in Lemma 4.6 on the vector logarithmic derivative $\beta_\eta(z)$ can be characterised as follows.

Lemma 4.7. *Assume that, for some integer $n \geq 1$, $h_x^{1/n} \in H^{1,n}_{\mathrm{loc}}(\mathfrak{X})$ for θ-a.a. $x \in X$. Then $\int_{Z_B} \|\beta_\eta(z)\|_1^n \, \sigma_Z(\mathrm{d}z) < \infty$ if and only if for any compact $B \subset X$*

$$\int_X \kappa_n^B(h_x^{1/n}) \, \theta(\mathrm{d}x) < \infty. \tag{4.29}$$

Proof. Substituting formulae (4.11) and (4.25), it is easy to see that

$$\begin{aligned}
\int_{Z_B} \|\beta_\eta(z)\|_1^n \, \hat{\sigma}(\mathrm{d}z) &= \int_X \int_{\mathfrak{X}_B} \left(\sum_{y_i \in \bar{y}} \frac{|\nabla_{y_i} h_x(\bar{y})|}{h_x(\bar{y})} \right)^n h_x(\bar{y}) \, \mathrm{d}\bar{y} \, \theta(\mathrm{d}x) \\
&= \int_X \int_{\mathfrak{X}_B} \left(\sum_{y_i \in \bar{y}} \frac{|\nabla_{y_i} h_x(\bar{y})|}{h_x(\bar{y})^{1-1/n}} \right)^n \mathrm{d}\bar{y} \, \theta(\mathrm{d}x) \\
&= n^n \int_X \int_{\mathfrak{X}_B} \left(\sum_{y_i \in \bar{y}} \left| \nabla_{y_i} \big(h_x(\bar{y})^{1/n} \big) \right| \right)^n \mathrm{d}\bar{y} \, \theta(\mathrm{d}x) \\
&= n^n \int_X \kappa_n^B(h_x^{1/n}) \, \theta(\mathrm{d}x) < \infty,
\end{aligned}$$

according to (4.28) and (4.29). □

From now on, we assume the following

Condition 4.1. For any compact $B \subset X$, the vector logarithmic derivative β_η defined in (4.25) satisfies the integral bound

$$\int_{Z_B} \|\beta_\eta(z)\|_1 \, \hat{\sigma}(\mathrm{d}z) < \infty.$$

4.3.2. Integration by parts for η_x as a push-forward measure. Using the general IBP framework outlined in Appendix B, and in particular picking up on Remark C.1, let us consider the special case with $\mathcal{W} := \mathfrak{W} \equiv \bigsqcup_{n=1}^\infty W^n$, $\mathcal{Y} := \mathfrak{X} \equiv \bigsqcup_{n=1}^\infty X^n$ and $\phi := \bar{\varphi}_x$, where the maps $\bar{\varphi}_x : \mathcal{W} \to \mathfrak{X}$ ($x \in X$) are described in Section 3.4. We assume that $\varphi_x \in C_b^2(W, X)$ uniformly in $x \in X$ (i.e., with global constants bounding the first two derivatives, $\mathrm{d}\varphi_x(\bar{w})$ and $\mathrm{d}^2\varphi_x(\bar{w})$). Furthermore, given a probability measure Q on $\mathfrak{W}$, consider the family of measures $\{\eta_x\}_{x \in X}$ on $\mathfrak{X}$ defined by (cf. (3.40))

$$\eta_x := \bar{\varphi}_x^* Q, \qquad x \in X. \tag{4.30}$$

We need the following two integrability conditions on the vector logarithmic derivative $\beta_Q(\bar{w})$ and the number of components $N_W(\bar{w})$ in a (random) vector $\bar{w} \in \mathfrak{W}$, both involving the θ-measure of the droplet cluster $\bar{D}_B(\bar{w})$ for any compact $B \subset X$ (see (3.42)):

$$\int_{\mathfrak{W}} \|\beta_Q(\bar{w})\|_1^n \, \theta(\bar{D}_B(\bar{w})) \, Q(\mathrm{d}\bar{w}) < \infty, \tag{4.31}$$

$$\int_{\mathfrak{W}} N_W(\bar{w})^n \, \theta(\bar{D}_B(\bar{w})) \, Q(\mathrm{d}\bar{w}) < \infty. \tag{4.32}$$

We can now prove the following result.

Theorem 4.8. *Suppose that conditions* (4.31) *and* (4.32) *hold for some* $n \geq 1$. *Then the following statements are true*:

(a) *The function* $N_X(\bar{y})$ *satisfies the integrability condition (*3.35*), that is, for any compact set* $B \subset X$

$$\int_X \int_{\mathfrak{X}_B} N_X(\bar{y})^n \, \eta_x(\mathrm{d}\bar{y}) \, \theta(\mathrm{d}x) < \infty. \tag{4.33}$$

(b) *For any* $v \in \mathrm{Vect}_0(X)$, *the measure* η_x *satisfies the IBP formula* (4.24) *with the corresponding logarithmic derivative* $\beta_\eta^{\hat{v}} \in L^n(Z, \hat{\sigma})$.

Proof. (a) By the change of measure (4.30), we obtain (cf. the proof of Lemma 3.8)

$$\begin{aligned}\int_X \int_{\mathfrak{X}_B} N_X(\bar{y})^n \, \eta_x(\mathrm{d}\bar{y}) \, \theta(\mathrm{d}x) &= \int_X \int_{\mathfrak{W}} \mathbf{1}_{\mathfrak{X}_B}(\bar{\varphi}_x(\bar{w})) N_W(\bar{w})^n \, Q(\mathrm{d}\bar{w}) \, \theta(\mathrm{d}x) \\ &= \int_{\mathfrak{W}} N_W(\bar{w})^n \left(\int_X \mathbf{1}_{\mathfrak{X}_B}(\bar{\varphi}_x(\bar{w})) \, \theta(\mathrm{d}x) \right) Q(\mathrm{d}\bar{w}) \\ &= \int_{\mathfrak{W}} N_W(\bar{w})^n \, \theta(\bar{D}_B(\bar{w})) \, Q(\mathrm{d}\bar{w}) < \infty,\end{aligned}$$

according to condition (4.32), and so the first part of the theorem is proved.

(b) Recall that the vector field $\hat{v}$ on $Z = X \times \mathfrak{X}$ is defined by

$$\hat{v}(x, \bar{y}) := (v(y_i))_{y_i \in \bar{y}}, \qquad \bar{y} = (y_i) \in \mathfrak{X}. \tag{4.34}$$

Then, owing to the component-wise structure of the map $\bar{\varphi}_x$ (cf. (3.39)), we have

$$(\mathcal{I}_{\bar{\varphi}_x} \hat{v})(x, \bar{w}) = \big((\mathcal{I}_{\varphi_x} v)(w_i) \big)_{w_i \in \bar{w}}, \qquad \bar{w} = (w_i) \in \mathfrak{W}.$$

It is clear that $\hat{v} \in \mathrm{Vect}_b^1(\mathfrak{X})$. Moreover, $\mathcal{I}_{\varphi_x} v \in \mathrm{Vect}_b^1(W)$, $\mathcal{I}_{\bar{\varphi}_x} \hat{v} \in \mathrm{Vect}_b^1(\mathfrak{W})$ uniformly in $x \in X$, which implies that

$$C_1 := \sup_{x \in X,\, w \in W} |\mathcal{I}_{\varphi_x} v(w)| < \infty, \tag{4.35}$$

$$C_2 := \sup_{x \in X,\, w \in W} |\operatorname{div} \mathcal{I}_{\varphi_x} v(w)| < \infty. \tag{4.36}$$

By Theorem C.1 and Remark C.1 in Appendix C, the measure $\eta_x = \bar{\varphi}_x^* Q$ satisfies the IBP formula (4.24) with the logarithmic derivative

$$\beta_\eta^{\hat{v}}(x, \bar{y}) = \big(\mathcal{I}_{\varphi_x}^* \beta_Q^{\mathcal{I}_{\bar{\varphi}_x} \hat{v}} \big)(\bar{y}), \qquad x \in X, \ \bar{y} \in \mathfrak{X}, \tag{4.37}$$

where

$$\begin{aligned}\beta_Q^{\mathcal{I}_{\bar{\varphi}_x} \hat{v}}(x, \bar{w}) &= \big(\beta_Q(\bar{w}), \mathcal{I}_{\bar{\varphi}_x} \hat{v}(\bar{w}) \big)_{T_{\bar{w}} \mathfrak{W}} + \operatorname{div} \mathcal{I}_{\bar{\varphi}_x} \hat{v}(\bar{w}) \\ &= \sum_{w_i \in \bar{w}} \big(\beta_Q(\bar{w})_i, \mathcal{I}_{\varphi_x} v(w_i) \big)_{T_{w_i} W} + \sum_{w_i \in \bar{w}} \operatorname{div} \mathcal{I}_{\varphi_x} v(w_i).\end{aligned}$$

Let us show that $\beta_\eta^{\hat{v}} \in L^n(Z, \hat{\sigma})$. Recall that the map $\mathcal{I}_{\varphi_x}^* : L^n(\mathfrak{W}, Q) \to L^n(\mathfrak{X}, \eta_x)$ is an isometry. Thus, according to (4.37) and after the change of measure (4.30), we have

$$\begin{aligned}\int_Z \big| \beta_\eta^{\hat{v}}(z) \big|^n \, \hat{\sigma}(\mathrm{d}z) &\leq \int_X \int_{\mathfrak{X}} \big\| \beta_\eta^{\hat{v}}(x, \bar{y}) \big\|_1^n \, \hat{\sigma}(\mathrm{d}x \times \mathrm{d}\bar{y}) \\ &= \int_X \int_{\mathfrak{W}} \big\| \beta_Q^{\mathcal{I}_{\bar{\varphi}_x} \hat{v}}(x, \bar{w}) \big\|_1^n \, Q(\mathrm{d}\bar{w}) \, \theta(\mathrm{d}x).\end{aligned}$$

Observe that $\operatorname{supp} \mathcal{I}_{\bar\varphi_x}\hat v = \bar\varphi_x^{-1}(\mathfrak{X}_{K_v})$, where $K_v := \operatorname{supp} v$. Then, similarly to the proof of Lemma 4.7, we obtain

$$\begin{aligned}
&\int_X\int_{\mathfrak{W}} \bigl|\bigl(\beta_Q(\bar w), \mathcal{I}_{\bar\varphi_x}\hat v(\bar w)\bigr)_{T_{\bar w}\mathfrak{W}}\bigr|^n\, Q(\mathrm{d}\bar w)\,\theta(\mathrm{d}x)\\
&\qquad \le \int_X\int_{\bar\varphi_x^{-1}(\mathfrak{X}_{K_v})}\left(\sum_{w_i\in\bar w}|\beta_Q(\bar w)_i|\cdot|\mathcal{I}_{\varphi_x}v(w_i)|\right)^n Q(\mathrm{d}\bar w)\,\theta(\mathrm{d}x)\\
&\qquad \le \sup_{x\in X,\, w\in W}|\mathcal{I}_{\varphi_x}v(w)|^n\int_{\mathfrak{W}}\left(\sum_{w_i\in\bar w}|\beta_Q(\bar w)_i|\right)^n\left(\int_X \mathbf{1}_{\mathfrak{X}_{K_v}}(\bar\varphi_x(\bar w))\,\theta(\mathrm{d}x)\right)Q(\mathrm{d}\bar w)\\
&\qquad = C_1^n\int_{\mathfrak{W}}\|\beta_Q(\bar w)\|_1^n\;\theta(\bar D_{K_v}(\bar w))\,Q(\mathrm{d}\bar w)<\infty,
\end{aligned}$$

according to condition (4.31). Similarly, using bound (4.36) and making the change of measure (4.30), we get

$$\begin{aligned}
&\int_X\int_{\mathfrak{W}}|\operatorname{div}\mathcal{I}_{\bar\varphi_x}\hat v(\bar w)|^n\,Q(\mathrm{d}\bar w)\,\theta(\mathrm{d}x)\\
&\qquad\le \int_X\int_{\mathfrak{W}}\left(\sum_{wi\in\bar w}|\operatorname{div}\mathcal{I}_{\varphi_x}v(w_i)|\right)^n Q(\mathrm{d}\bar w)\,\theta(\mathrm{d}x)\\
&\qquad\le \sup_{x\in X,\, w\in W}|\operatorname{div}\mathcal{I}_{\varphi_x}v(w)|^n\int_X\int_{\bar\varphi_x^{-1}(\mathfrak{X}_{K_v})}N_W(\bar w)^n\,Q(\mathrm{d}\bar w)\,\theta(\mathrm{d}x)\\
&\qquad= C_2^n\int_X\int_{\mathfrak{X}_{K_v}}N_X(\bar y)^n\,\eta_x(\mathrm{d}\bar y)\,\theta(\mathrm{d}x)<\infty,
\end{aligned}$$

according to part (a). Thus, part (b) of the theorem is proved. □

Remark 4.3. Recalling a simple bound (3.48) for the θ-measure of the droplet cluster $\bar D_B(\bar w)$, we observe that, under condition (3.46) (see Proposition 3.9), conditions (4.31) and (4.32) of Theorem 4.8 specialise, respectively, as follows:

$$\int_{\mathfrak{W}}\|\beta_Q(\bar w)\|_1^n\,N_W(\bar w)\,Q(\mathrm{d}\bar w)<\infty,\qquad \int_{\mathfrak{W}}N_W(\bar w)^{n+1}\,Q(\mathrm{d}\bar w)<\infty.$$

Similarly, the assumptions of Proposition 3.10 imply that $\sup_{\bar w\in\mathfrak{W}}\theta(\bar D_B(\bar w))<\infty$ (see the proof), so that conditions (4.31) and (4.32) transcribe, respectively, as

$$\int_{\mathfrak{W}}\|\beta_Q(\bar w)\|_1^n\;Q(\mathrm{d}\bar w)<\infty,\qquad \int_{\mathfrak{W}}N_W(\bar w)^n\,Q(\mathrm{d}\bar w)<\infty.$$

4.3.3. Integration by parts for the cluster measure μ_{cl}. Denote by $\mathcal{FC}_{\hat\sigma}(\varGamma_Z)$ the class of functions on $\varGamma_Z$ of the form

$$F(\hat\gamma)=f(\langle\phi_1,\hat\gamma\rangle,\dots,\langle\phi_k,\hat\gamma\rangle),\qquad \hat\gamma\in\varGamma_Z,\tag{4.38}$$

where $k\in\mathbb{N}$, $f\in C_b^\infty(\mathbb{R}^k)$ and $\phi_1,\dots,\phi_k\in C_{\hat\sigma}^\infty(Z):=$ the set of C^∞-functions on Z with $\hat\sigma$-finite support (cf. (4.1)).

For any $F\in\mathcal{FC}(\varGamma_X)$ we introduce the function $\hat F=\mathcal{I}_{\mathrm{q}}F:\varGamma_Z\to\mathbb{R}$. It follows from condition (3.33) that

$$\hat F\in\mathcal{FC}_{\hat\sigma}(\varGamma_Z).\tag{4.39}$$

Theorem 4.9. *The measure $\hat{\mu}$ satisfies the following IBP formula*

$$\int_{\Gamma_Z} \nabla_{\hat{v}}^{\Gamma} F(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}) = -\int_{\Gamma_Z} F(\hat{\gamma}) B_{\hat{\mu}}^{\hat{v}}(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}), \tag{4.40}$$

where

$$B_{\hat{\mu}}^{\hat{v}}(\hat{\gamma}) := \sum_{(x,\bar{y})\in\hat{\gamma}} \beta_{\eta}^{\hat{v}}(x,\bar{y}) \in L^1(\Gamma_Z, \hat{\mu}). \tag{4.41}$$

Proof. Let us first observe that the integral on the left-hand side of (4.40) is well defined because $\hat{\mu} \in \mathcal{M}_{\theta}^1(\Gamma_Z)$. Indeed, the inclusion (4.39) implies that the function

$$G(\hat{\gamma}) := \sum_{z\in\hat{\gamma}} \nabla_{\bar{y}} \hat{F}(\hat{\gamma}) \cdot \hat{v}(\bar{y}), \qquad z = (x,\bar{y}) \in Z, \quad \hat{\gamma} \in \Gamma_Z,$$

is bounded and has $\hat{\sigma}$-finite support, which implies that $G \in L^1(Z, \hat{\sigma})$. Thus the function

$$\Gamma_Z \ni \hat{\gamma} \mapsto \langle G, \gamma \rangle \equiv \nabla_{\hat{v}}^{\Gamma} \hat{F}(\hat{\gamma})$$

belongs to $L^2(\Gamma_Z, \hat{\mu})$ by the definition of the class $\mathcal{M}_{\theta}^1(\Gamma_Z)$.

Using decomposition (2.18) of the measure $\hat{\mu}$ and taking the notational advantage of the one-to-one association $x \leftrightarrow \bar{y}_x$ for $(x, \bar{y}_x) \in \hat{\gamma} = (\gamma, \bar{y}^{\gamma})$ (see (2.14)), we obtain

$$\begin{aligned} \int_{\Gamma_Z} \nabla_{\hat{v}}^{\Gamma} F(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}) &= \int_{\Gamma_X} \left(\int_{\mathfrak{X}^{\gamma}} \sum_{x\in\gamma} \nabla_{\bar{y}_x}^{\hat{v}} F(\gamma, \bar{y}^{\gamma})\, \eta^{\gamma}(\mathrm{d}\bar{y}^{\gamma}) \right) \mu(\mathrm{d}\gamma) \\ &= \int_{\Gamma_X} \sum_{x\in\gamma} \left(\int_{\mathfrak{X}^{\gamma}} \nabla_{\bar{y}_x}^{\hat{v}} F(\gamma, \bar{y}^{\gamma})\, \eta^{\gamma}(\mathrm{d}\bar{y}^{\gamma}) \right) \mu(\mathrm{d}\gamma) \\ &= \int_{\Gamma_X} \sum_{x\in\gamma} \left(\int_{\mathfrak{X}^{\gamma}} \nabla_{\bar{y}_x}^{\hat{v}} F(\gamma, \bar{y}^{\gamma}) \bigotimes_{x'\in\gamma} \eta_{x'}(\mathrm{d}\bar{y}_{x'}) \right) \mu(\mathrm{d}\gamma), \end{aligned} \tag{4.42}$$

by a product structure of η^{γ} (see (2.16)). Furthermore, on applying the IBP formula (4.23) the right-hand side of (4.42) is represented in the form

$$\begin{aligned} -\int_{\Gamma_X} \sum_{x\in\gamma} & \left(\int_{\mathfrak{X}^{\gamma}} F(\gamma, \bar{y}^{\gamma})\, \beta_{\eta}^{\hat{v}}(x, \bar{y}_x) \bigotimes_{x'\in\gamma} \eta_{x'}(\mathrm{d}\bar{y}_{x'}) \right) \mu(\mathrm{d}\gamma) \\ &= -\int_{\Gamma_X} \left(\int_{\mathfrak{X}^{\gamma}} \sum_{x\in\gamma} F(\gamma, \bar{y}^{\gamma})\, \beta_{\eta}^{\hat{v}}(x, \bar{y}_x)\, \eta^{\gamma}(\mathrm{d}\bar{y}^{\gamma}) \right) \mu(\mathrm{d}\gamma) \\ &= -\int_{\Gamma_Z} F(\hat{\gamma}) B_{\hat{\mu}}^{\hat{v}}(\hat{\gamma})\, \hat{\mu}(\mathrm{d}\hat{\gamma}). \end{aligned}$$

which proves formula (4.40).

Finally, in view of Condition 3.3, Lemma 3.6 implies that $\hat{\mu} \in \mathcal{M}_{\hat{\sigma}}^n(\Gamma_Z)$, and by Definition 3.1 and Condition 4.1 it follows that $B_{\hat{\mu}}^{\hat{v}} \in L^1(\Gamma_Z, \hat{\mu})$. □

The next two theorems are our main results in this section.

Theorem 4.10. *For any function $F \in \mathcal{FC}(\Gamma_X)$, the cluster measure μ_{cl} satisfies the following IBP formula*

$$\int_{\Gamma_X} \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot v(x)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) = -\int_{\Gamma_X} F(\gamma)\, B^v_{\mu_{\mathrm{cl}}}(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma), \tag{4.43}$$

where $B^v_{\mu_{\mathrm{cl}}}(\gamma) := \mathcal{I}^*_{\mathsf{q}} B^{\hat v}_{\hat\mu} \in L^1(\Gamma_X, \mu_{\mathrm{cl}})$ (see (4.18)*) and the logarithmic derivative $B^{\hat v}_{\hat\mu}(\hat\gamma)$ is defined in* (4.41).

Proof. For any function $F \in \mathcal{FC}(\Gamma_X)$ and vector field $v \in \mathrm{Vect}_0(X)$, let us denote for brevity

$$H(x,\gamma) := \nabla_x F(\gamma) \cdot v(x), \qquad x \in X,\ \ \gamma \in \Gamma_X. \tag{4.44}$$

Furthermore, setting $\hat F = \mathcal{I}_{\mathsf{q}} F : \Gamma_Z \to \mathbb{R}$ we introduce the notation

$$\hat H(z,\hat\gamma) := \nabla_{\bar y} \hat F(\hat\gamma) \cdot \hat v(\bar y), \qquad z = (x,\bar y) \in Z,\ \ \hat\gamma \in \Gamma_Z. \tag{4.45}$$

From these definitions, it is clear that

$$\mathcal{I}_{\mathsf{q}}\Biggl(\sum_{x\in\gamma} H(x,\gamma)\Biggr)(\hat\gamma) = \sum_{z\in\hat\gamma} \hat H(z,\hat\gamma), \qquad \hat\gamma \in \Gamma_Z. \tag{4.46}$$

By Theorem 4.9, the measure $\hat\mu$ satisfies the IBP formula

$$\int_{\Gamma_Z} \sum_{z\in\hat\gamma} \hat H(z,\hat\gamma)\, \hat\mu(\mathrm{d}\hat\gamma) = -\int_{\Gamma_Z} \hat F(\hat\gamma) B^{\hat v}_{\hat\mu}(\hat\gamma)\, \hat\mu(\mathrm{d}\hat\gamma), \tag{4.47}$$

where the logarithmic derivative $B^{\hat v}_{\hat\mu}(\hat\gamma) = \langle \beta^{\hat v}_\eta, \hat\gamma\rangle$ belongs to $L^1(\Gamma_Z, \hat\mu)$ by Theorem 4.9.

Now, using formulae (4.45), (4.46) and (4.47), we obtain

$$\begin{aligned}
\int_{\Gamma_X} \sum_{x\in\gamma} H(x,\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) &= \int_{\Gamma_Z} \Biggl(\sum_{(x,\bar y)\in\hat\gamma} \nabla_{\bar y} \mathcal{I}_{\mathsf{q}} F(\hat\gamma) \cdot \hat v(\bar y)\Biggr) \hat\mu(\mathrm{d}\hat\gamma) \\
&= -\int_{\Gamma_Z} \mathcal{I}_{\mathsf{q}} F(\hat\gamma)\, B^{\hat v}_{\hat\mu}(\hat\gamma)\, \hat\mu(\mathrm{d}\hat\gamma) \\
&= -\int_{\Gamma_X} F(\gamma)\, \mathcal{I}^*_{\mathsf{q}} B^{\hat v}_{\hat\mu}(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma),
\end{aligned}$$

where $\mathcal{I}^*_{\mathsf{q}} B^{\hat v}_{\hat\mu} \in L^1(\Gamma_X, \mu_{\mathrm{cl}})$ by Lemma 4.3. Thus, formula (4.43) is proved. □

Remark 4.4. Observe that the logarithmic derivative $B^{\hat v}_{\hat\mu} = \langle \beta^{\hat v}_\eta, \hat\gamma\rangle$ (see (4.47)) does not depend on the underlying measure μ, and so it is the same as, say, in the Poisson case with $\mu = \pi_\theta$. Nevertheless, the logarithmic derivative $B^v_{\mu_{\mathrm{cl}}}$ does depend on μ via the mapping $\mathcal{I}^*_{\mathsf{q}}$.

According to Theorem 4.10, $B^v_{\mu_{\mathrm{cl}}} \in L^1(\Gamma_Z, \mu_{\mathrm{cl}})$. However, under the conditions of Lemma 4.6 with $n \ge 2$, this statement can be enhanced.

Lemma 4.11. *Assume that $\int_Z |\beta_\eta(z)|^m\, \hat\sigma(\mathrm{d}z) < \infty$ for $m = 1, 2, \dots, n$ and some integer $n \ge 2$, and let condition* (3.35) *hold. Then $B^v_{\mu_{\mathrm{cl}}} \in L^n(\Gamma_Z, \mu_{\mathrm{cl}})$.*

Proof. By Lemmata 3.7 and 4.6, it follows that $\langle \beta_\eta^{\hat v}, \hat\gamma\rangle \in L^n(\Gamma_Z, \hat\mu)$. Let $r := n/(n-1)$, so that $n^{-1} + r^{-1} = 1$. Note that $\mathcal{I}_{\mathfrak{q}}$ can be treated as a bounded operator acting from $L^r(\Gamma_X, \mu_{\mathrm{cl}})$ to $L^r(\Gamma_Z, \hat\mu)$. Hence, $\mathcal{I}_{\mathfrak{q}}^*$ is a bounded operator from $L^r(\Gamma_Z, \hat\mu)' = L^n(\Gamma_Z, \hat\mu)$ to $L^r(\Gamma_X, \mu_{\mathrm{cl}})' = L^n(\Gamma_X, \mu_{\mathrm{cl}})$, which implies that $B_{\mu_{\mathrm{cl}}}^v = \mathcal{I}_{\mathfrak{q}}^* \langle \beta_\eta^{\hat v}, \hat\gamma\rangle \in L^n(\Gamma_Z, \mu_{\mathrm{cl}})$. □

Formula (4.43) can be extended to more general vector fields on Γ_X. Let $\mathcal{FV}(\Gamma_X)$ be the class of vector fields V of the form $V(\gamma) = (V(\gamma)_x)_{x\in\gamma}$,

$$V(\gamma)_x = \sum_{j=1}^k G_j(\gamma)\, v_j(x) \in T_x X,$$

where $G_j \in \mathcal{FC}(\Gamma_X)$ and $v_j \in \mathrm{Vect}_0(X),\ j = 1, \dots, k$. For any such V we set

$$B_{\mu_{\mathrm{cl}}}^V(\gamma) := (\mathcal{I}_{\mathfrak{q}}^* B_{\hat\mu}^{\mathcal{I}_{\mathfrak{q}} V})(\gamma),$$

where $B_{\hat\mu}^{\mathcal{I}_{\mathfrak{q}} V}(\hat\gamma)$ is the logarithmic derivative of $\hat\mu$ along $\mathcal{I}_{\mathfrak{q}} V(\hat\gamma) := V(\mathfrak{q}(\hat\gamma))$ (see [3]). Note that $\mathcal{I}_{\mathfrak{q}} V$ is a vector field on Γ_Z owing to the obvious equality

$$T_{\hat\gamma}\Gamma_Z = T_{\mathfrak{q}(\hat\gamma)}\Gamma_X.$$

Clearly,

$$B_{\mu_{\mathrm{cl}}}^V(\gamma) = \sum_{j=1}^k \biggl(G_j(\gamma) B_{\mu_{\mathrm{cl}}}^{v_j}(\gamma) + \sum_{x\in\gamma} \nabla_x G_j(\gamma) \cdot v_j(x) \biggr).$$

Theorem 4.12. *For any $F_1, F_2 \in \mathcal{FC}(\Gamma_X)$ and $V \in \mathcal{FV}(\Gamma_X)$, we have*

$$\begin{aligned} &\int_{\Gamma_X} \sum_{x\in\gamma} \nabla_x F_1(\gamma) \cdot V(\gamma)_x\, F_2(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) \\ &\quad = -\int_{\Gamma_X} F_1(\gamma) \sum_{x\in\gamma} \nabla_x F_2(\gamma) \cdot V(\gamma)_x\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma) - \int_{\Gamma_X} F_1(\gamma) F_2(\gamma) B_{\mu_{\mathrm{cl}}}^V(\gamma)\, \mu_{\mathrm{cl}}(\mathrm{d}\gamma). \end{aligned}$$

Proof. The proof can be obtained by a straightforward generalisation of the arguments used in the proof of Theorem 4.10. □

We define the *vector logarithmic derivative* of μ_{cl} as a linear operator

$$B_{\mu_{\mathrm{cl}}} \colon \mathcal{FV}(\Gamma_X) \to L^1(\Gamma_X, \mu_{\mathrm{cl}})$$

via the formula

$$B_{\mu_{\mathrm{cl}}} V(\gamma) := B_{\mu_{\mathrm{cl}}}^V(\gamma).$$

This notation will be used in the next section.

4.4. Dirichlet forms and equilibrium stochastic dynamics

Throughout this section, we assume that the conditions of Lemma 3.7 are satisfied with $n = 2$. Thus, the measures μ, $\hat\mu$ and μ_{cl} belong to the corresponding $\mathcal{M}^2$-classes. Our considerations will involve the Γ-gradients (see Section 4.1) on different configuration spaces, such as Γ_X, $\Gamma_{\mathfrak{X}}$ and Γ_Z; to avoid confusion, we shall denote them by ∇_X^Γ, $\nabla_{\mathfrak{X}}^\Gamma$ and ∇_Z^Γ, respectively.

Let us introduce a pre-Dirichlet form $\mathcal{E}_{\mu_{\rm cl}}$ associated with the Gibbs cluster measure $\mu_{\rm cl}$, defined on functions $F_1, F_2 \in \mathcal{FC}(\Gamma_X) \subset L^2(\Gamma_X, \mu_{\rm cl})$ by

$$\mathcal{E}_{\mu_{\rm cl}}(F_1, F_2) := \int_{\Gamma_X} \langle \nabla_X^\Gamma F_1(\gamma), \nabla_X^\Gamma F_2(\gamma)\rangle_\gamma \, \mu_{\rm cl}(\mathrm{d}\gamma). \tag{4.48}$$

Let us also consider the operator $H_{\mu_{\rm cl}}$ defined by

$$H_{\mu_{\rm cl}} F := -\Delta^\Gamma F + B_{\mu_{\rm cl}} \nabla_X^\Gamma F, \qquad F \in \mathcal{FC}(\Gamma_X), \tag{4.49}$$

where $\Delta^\Gamma F(\gamma) := \sum_{x\in\gamma} \Delta_x F(\gamma)$.

The next theorem readily follows from the general theory of (pre-)Dirichlet forms associated with measures from the class $\mathcal{M}^2(\Gamma_X)$ (see [4, 26]).

Theorem 4.13. *The pre-Dirichlet form* (4.48) *is well defined, i.e.,* $\mathcal{E}_{\mu_{\rm cl}}(F_1, F_2) < \infty$ *for all* $F_1, F_2 \in \mathcal{FC}(\Gamma_X)$. *Furthermore, expression* (4.49) *defines a symmetric operator* $H_{\mu_{\rm cl}}$ *in* $L^2(\Gamma_X, \mu_{\rm cl})$*, which is the generator of* $\mathcal{E}_{\mu_{\rm cl}}$*, that is,*

$$\mathcal{E}_{\mu_{\rm cl}}(F_1, F_2) = \int_{\Gamma_X} F_1(\gamma)\, H_{\mu_{\rm cl}} F_2(\gamma)\, \mu_{\rm cl}(\mathrm{d}\gamma), \qquad F_1, F_2 \in \mathcal{FC}(\Gamma_X). \tag{4.50}$$

Formula (4.50) implies that the form $\mathcal{E}_{\mu_{\rm cl}}$ is closable. It follows from the properties of the *carré du champ* $\sum_{x\in\gamma} \nabla_x F_1(\gamma) \cdot \nabla_x F_2(\gamma)$ that the closure of $\mathcal{E}_{\mu_{\rm cl}}$ (for which we shall keep the same notation) is a quasi-regular local Dirichlet form on a bigger state space $\ddot{\Gamma}_X$ consisting of all integer-valued Radon measures on X (see [26, condition (Q), page 298, and Subsection 4.5.1]). By the general theory of Dirichlet forms (see [25]), this implies the following result (cf. [3, 4, 10]).

Theorem 4.14. *There exists a conservative diffusion process* $\mathbf{X} = (\mathbf{X}_t,\ t \ge 0)$ *on* $\ddot{\Gamma}_X$*, properly associated with the Dirichlet form* $\mathcal{E}_{\mu_{\rm cl}}$*, that is, for any function* $F \in L^2(\ddot{\Gamma}_X, \mu_{\rm cl})$ *and all* $t \ge 0$*, the map*

$$\ddot{\Gamma}_X \ni \gamma \mapsto p_t F(\gamma) := \int_\Omega F(\mathbf{X}_t)\, \mathrm{d}P_\gamma$$

is an $\mathcal{E}_{\mu_{\rm cl}}$*-quasi-continuous version of* $\exp(-tH_{\mu_{\rm cl}})F$. *Here* Ω *is the canonical sample space (of* $\ddot{\Gamma}_X$*-valued continuous functions on* $\mathbb{R}_+$*) and* $(P_\gamma,\ \gamma \in \ddot{\Gamma}_X)$ *is the family of probability distributions of the process* $\mathbf{X}$ *conditioned on the initial value* $\gamma = \mathbf{X}_0$. *The process* $\mathbf{X}$ *is unique up to* $\mu_{\rm cl}$*-equivalence. In particular,* $\mathbf{X}$ *is* $\mu_{\rm cl}$*-symmetric (i.e.,* $\int F_1\, p_t F_2 \,\mathrm{d}\mu_{\rm cl} = \int F_2\, p_t F_1 \,\mathrm{d}\mu_{\rm cl}$ *for all measurable functions* $F_1, F_2 : \ddot{\Gamma}_X \to \mathbb{R}_+$*) and* $\mu_{\rm cl}$ *is its invariant measure.*

4.5. On the irreducibility of the Dirichlet form

Let $\mathcal{E}_{\hat\mu}$ be the pre-Dirichlet form associated with the Gibbs measure $\hat\mu$, defined on functions $F_1, F_2 \in \mathcal{FC}_{\hat\sigma}(\Gamma_Z) \subset L^2(\Gamma_Z, \hat\mu)$ by

$$\mathcal{E}_{\hat\mu}(F_1, F_2) := \int_{\Gamma_Z} \langle \nabla_Z^\Gamma F_1(\hat\gamma), \nabla_Z^\Gamma F_2(\hat\gamma)\rangle_{\hat\gamma}\, \hat\mu(\mathrm{d}\hat\gamma). \tag{4.51}$$

The integral on the right-hand side of (4.51) is well defined because $\hat{\mu} \in \mathcal{M}^2_\theta(\Gamma_Z) \subset \mathcal{M}^1_\theta(\Gamma_Z)$. Indeed, the function

$$G(z) := (\nabla_z F_1(\hat{\gamma}), \nabla_z F_2(\hat{\gamma}))$$

is bounded and has a $\hat{\sigma}$-finite support, which implies that $G \in L^1(Z, \hat{\sigma})$. Thus the function

$$\Gamma_Z \ni \hat{\gamma} \mapsto \langle G, \gamma \rangle \equiv \langle \nabla^\Gamma_Z F_1(\hat{\gamma}), \nabla^\Gamma_Z F_2(\hat{\gamma}) \rangle_{\hat{\gamma}}$$

belongs to $L^2(\Gamma_Z, \hat{\mu})$ by the definition of the class $\mathcal{M}^1_\theta(\Gamma_Z)$. It can be shown by a direct computation that

$$\mathcal{E}_{\mu_{\rm cl}}(F, F) = \mathcal{E}_{\hat{\mu}}(\mathcal{I}_{\mathsf{q}} F, \mathcal{I}_{\mathsf{q}} F), \qquad F \in \mathcal{FC}(\Gamma_X). \tag{4.52}$$

Note that the pre-Dirichlet form $(\mathcal{E}_{\hat{\mu}}, \mathcal{FC}_{\hat{\sigma}}(\Gamma_Z))$ is not necessarily closable. A sufficient condition of its closability is an IBP formula for the measure $\hat{\mu}$ with respect to *all* directions in Γ_Z rather then only in $\mathfrak{X}^\gamma$ (cf. Theorem 4.9), which requires in turn some smoothness conditions on the measure μ and also on the measure η_x as a function of $x \in X$. Such conditions are satisfied, for instance, if $X = \mathbb{R}^d$, μ is a Poisson measure or, more generally, Gibbs measure with a smooth interaction potential, and the family $\{\eta_x\}$ is defined by translations of a parent measure η_0 (i.e., $\eta_x(B) := \eta_0(B - x)$). This case has been studied in great detail in [10, 11], where formula (4.52) was extended to all functions F from the domain of $\mathcal{E}_{\mu_{\rm cl}}$ (with the closure $\bar{\mathcal{E}}_{\hat{\mu}}$ of the pre-Dirichlet form $(\mathcal{E}_{\hat{\mu}}, \mathcal{FC}_{\hat{\sigma}}(\Gamma_Z))$ on the right-hand side). In turn, this makes it possible to characterise the kernel of the Dirichlet form $\mathcal{E}_{\mu_{\rm cl}}$ via the kernels of the forms $\bar{\mathcal{E}}_{\hat{\mu}}$ and $\mathcal{E}_\mu$; in particular, it has been proved in [10, 11] that $\mathcal{E}_{\mu_{\rm cl}}$ is *irreducible* (that is, its kernel consists of constants) whenever $\mathcal{E}_\mu$ is such.

Let us remark that irreducibility is an important property closely related to the ergodicity of stochastic dynamics and extremality of invariant measures. It seems plausible that in our situation the irreducibility of $\mathcal{E}_{\mu_{\rm cl}}$ is controlled by the properties of the distribution of centres μ rather then the cluster distributions $\{\eta_x\}$, but this remains an open question.

5. Examples

In order to make tractable the general cluster model discussed above, one needs an efficient method to construct the family $\{\eta_x\}_{x\in X}$ of cluster distributions attached to centres x lying on a ground configuration γ. In the situation where X is a linear space, this is straightforward by translations of a parent distribution η_0 specified at the origin (see Section 5.1). For other classes of spaces, the linear action has to be replaced by another suitable transformation (see Sections 5.2, 5.3 and 5.4.1). Alternative, more direct methods may also be applicable based on specific properties of the space structure, for instance by confining oneself to a class of distributions with a suitable invariance property (cf. Section 5.3) or by exploiting the space metric, leading to "radially symmetric" distributions (see Section 5.4.2).

We discuss below a number of selected examples where this programme can be realised. In so doing, we will mostly be using the push-forward method of Section 3.4. Specifically, the discussion of the resulting cluster measure $\mu_{\rm cl}$ in each example will be essentially confined to the following two important aspects:

(i) verification of general sufficient conditions for the cluster process configurations to be proper, such as Condition 3.4′ in Proposition 3.9 specialised to the conditions of

Propositions 3.9 and 3.10 (local finiteness), and the conditions of Propositions 3.11 and their particular cases in Propositions 3.12 and 3.13 (simplicity); and

(ii) verification of appropriate smoothness conditions on the mapping φ_x that we imposed as a prerequisite of an IBP formula for the cluster measure μ_{cl} (see the beginning of Section 4.3.2).

5.1. Euclidean spaces

In the situation where $X = \mathbb{R}^d$, the family $\{\eta_x\}_{x\in X}$ of cluster distributions can be constructed by translations of a parent distribution η_0 specified at the origin [10, 11]. This can be formulated in terms of the construction of Section 3.4. Take $W := X$ and define the family of maps $\varphi_x : X \to X$ ($x \in X$) as translations

$$\varphi_x(y) := y + x, \qquad y \in X. \tag{5.1}$$

Then definition (3.41) of the droplet $D_B(y)$ specialises to

$$D_B(y) = B - y, \qquad y \in X,\;\; B \in \mathcal{B}(X).$$

Furthermore, formula (3.43) for the droplet cluster now reads

$$\bar{D}_B(\bar{y}) = \bigcup_{y_i \in \bar{y}} (B - y_i), \qquad \bar{y} \in \mathfrak{X},$$

which makes the notion of the droplet cluster particularly transparent as a set-theoretic union of "droplets" of shape B shifted to the centrally reflected coordinates of the vector $\bar{y} = (y_i)$. The parent measure Q on $\mathfrak{X}$ (see (3.40)) can then be interpreted as a pattern distribution η_0, and the measures η_x are obtained by translations of η_0 to points $x \in X$:

$$\eta_x(\bar{B}) := \bar{\varphi}_x^* \eta_0(\bar{B}) \equiv \eta_0(\bar{B} - x), \qquad \bar{B} \in \mathcal{B}(\mathfrak{X}). \tag{5.2}$$

Let us discuss in this context criteria of properness of the corresponding cluster measure μ_{cl} laid out in Section 3.4. First of all, conditions (3.46) and (3.47) of Proposition 3.9 (which guarantee Condition 3.4′) are reduced, respectively, to

$$\sup_{y\in X} \theta(B - y) < \infty, \qquad \int_{\mathfrak{X}} N_X(\bar{y})\, \eta_0(\mathrm{d}\bar{y}) < \infty. \tag{5.3}$$

In turn, the first condition in (5.3) is satisfied, for instance, if the measure $\theta(\mathrm{d}x)$ is absolutely continuous with respect to Lebesgue measure $\mathrm{d}x$ on X and the corresponding Radon–Nikodym density is bounded (cf. Remark 3.10). Next, condition (i) of Proposition 3.10 (i.e., continuity of φ_x in x is obviously satisfied for (5.1), while condition (ii) holds with a compact $B_y = B - y$ ($y \in X$). Finally, let us point out that the use of Propositions 3.12 and 3.13 is greatly facilitated by the fact that the equation $\varphi_y(w) = x$, reducing for (5.1) to equation $w + y = x$, has the unique solution $y = x - w$.

Regarding conditions for IBP formulae, note that map (5.1) is of course smooth, with $\mathrm{d}\varphi_x = \mathrm{id}$ (the identity operator) and $\mathrm{d}^2\varphi_x = 0$. Finally, if the probability measure $\eta_0(\mathrm{d}\bar{x})$ is absolutely continuous with respect to Lebesgue measure $\mathrm{d}\bar{x}$ on $\mathfrak{X}$, then conditions (4.31) and (4.32) in Theorem 4.8 can be easily rewritten in terms of the corresponding Radon–Nikodym density.

5.2. Lie groups

Let $X = G$ be a (non-compact) Lie group, and $\mathfrak{g}$ the corresponding Lie algebra endowed with a scalar product $(\cdot,\cdot)_{\mathfrak{g}}$ (see, e.g., [19]). This scalar product generates in a standard way a right-invariant Riemannian structure on G. The group product of elements $g_1, g_2 \in G$ is denoted by $g_1 g_2 \in G$, and $e \in G$ stands for the identity of the group G.

Let us show how a family of measures $\{\eta_x\}_{x\in G}$ on $\mathfrak{G} := \bigsqcup_{n=0}^{\infty} G^n$ can be set out using the push-forward construction of Section 3.4. Take $W := G$ and define the map $\varphi_x(g) : G \times G \to G$ as a translation

$$\varphi_x(g) := gx, \qquad g, x \in G. \tag{5.4}$$

By the properties of the Lie group multiplication, the map $\varphi_x(g)$ is continuous in $(g, x) \in G \times G$ and therefore automatically measurable. In view of (5.4), definition (3.41) of the droplet $D_B(g)$ specialises to

$$D_B(g) = g^{-1}B, \qquad B \in \mathcal{B}(G), \ \ g \in G.$$

Accordingly, by formula (3.43) the corresponding droplet cluster is represented as

$$\bar{D}_B(\bar{g}) = \bigcup_{g_i \in \bar{g}} g_i^{-1}B, \qquad \bar{g} \in \mathfrak{G} := \bigsqcup_{n=0}^{\infty} G^n.$$

If Q is a probability measure on $\mathfrak{G}$, then on substituting (5.4) into definition (3.40) we get

$$\eta_x(\bar{B}) := (\bar{\varphi}_x^* Q)(\bar{B}) \equiv Q(\bar{B}x^{-1}), \qquad \bar{B} \in \mathcal{B}(\mathfrak{G}) \quad (x \in G). \tag{5.5}$$

Observe from (5.5) that in fact the measure Q coincides with η_e; hence definition (5.5) can be rewritten in a "translation" form naturally generalising formula (5.2) in the Euclidean case, namely

$$\eta_x(\bar{B}) = \eta_e(\bar{B}x^{-1}), \qquad \bar{B} \in \mathcal{B}(\mathfrak{G}) \quad (x \in G). \tag{5.6}$$

Specialising the general criteria of properness of μ_{cl} described in Section 3.4, we have that conditions (3.46) and (3.47) of Proposition 3.9 are reduced, respectively, to

$$\sup_{g\in G} \theta(g^{-1}B) < \infty, \qquad \int_{\mathfrak{G}} N_G(\bar{g})\, Q(\mathrm{d}\bar{g}) < \infty. \tag{5.7}$$

Similarly to the previous section, the first condition in (5.7) is satisfied proviso the reference measure θ is absolutely continuous with respect to a left Haar measure on G and the corresponding Radon–Nikodym density is bounded (cf. Remark 3.10). As mentioned above, maps (5.4) automatically satisfy the continuity condition (i) of Proposition 3.10, whereas condition (ii) holds with a compact $B_g = Bg^{-1}$ $(g \in G)$. Moreover, as a natural extension of the Euclidean case, the equation $\varphi_y(g) = x$ with (5.4) takes the form $gy = x$, which has the unique solution $y = g^{-1}x$. Hence, Propositions 3.12 and 3.13 can be easily applied.

In a standard fashion, the Lie algebra $\mathfrak{g}$ of the group G can be identified with the space of right-invariant vector fields on G; moreover, all tangent spaces $T_g G$ are identified with $T_e G$ (and therefore with $\mathfrak{g}$) via right translations. Under this identification, for the map $\varphi_x(w)$ defined in (5.4) we have $\mathrm{d}\varphi_x(g) = \mathrm{id}$ for any $x, g \in G$, where $\mathrm{id} : \mathfrak{g} \to \mathfrak{g}$ is the identity operator. It follows that $\|\mathrm{d}\varphi_x(g)\| = 1$ and $\mathrm{d}^2\varphi_x(g) = 0$ for all $x, g \in G$, which automatically implies that $\varphi_x \in C_b^2(G, G)$ uniformly in $x \in G$. Thus, one can apply Theorem 4.8 provided that conditions (4.31) and (4.32) are satisfied. Finally, if the probability measure Q is absolutely continuous with respect to a left Haar measure on $\mathfrak{G}$, then conditions (4.31) and (4.32) can be easily specified in terms of the corresponding Radon–Nikodym density.

5.3. Homogeneous manifolds

5.3.1. Construction of cluster distributions η_x. Let G be a (non-compact) Lie group and X a G-homogeneous Riemannian manifold (see, e.g., [7, 19]). More precisely, G is a closed subgroup of the group of isometries of X acting on X transitively, that is, for any $x, y \in X$ there exists an element $g \in G$ such that $g \cdot x = y$ (equivalently, $G \cdot x = X$ for some, and hence for all $x \in X$), and the mapping

$$G \times X \ni (g, x) \mapsto g \cdot x \in X \tag{5.8}$$

is differentiable. Given a fixed point $x_0 \in X$, the manifold X is diffeomorphic to the quotient manifold G/H_{x_0}, where $H_{x_0} := \{g \in G : g \cdot x_0 = x_0\}$ is the isotropy subgroup of G at x_0.

Example 5.1. Take $X = \mathbb{R}^d$ and the group $G = \mathbb{R}^d$ with the natural additive structure acting on X by translations. In this case, $H_{x_0} = \{0\}$ for every $x_0 \in X$.

Example 5.2. Let $X = \mathbb{R}^d$ and consider $G = \mathcal{E}^+(d)$, the Euclidean group of isometries of $\mathbb{R}^d$ preserving orientation. In this case, $H_0 = \mathcal{SO}(d)$ and $X \cong \mathcal{E}^+(d)/\mathcal{SO}(d)$.

Example 5.3. Let $X = \mathbb{H}^d$ be a d-dimensional hyperbolic space. In this situation, $G = \mathcal{SO}_0(d, 1)$ is the connected component of the identity in the orthogonal group $\mathcal{O}(d, 1)$ of the canonical quadratic form with signature $(n, 1)$, and $X \cong \mathcal{SO}_0(d, 1)/\mathcal{SO}(d)$.

Example 5.4. If G is a Lie group and H is its compact subgroup, then one can use the quotient manifold $X = G/H$ with the natural G-action on it.

Define a family of maps $\varphi_x : G \to X$ as the group action (see (5.8))

$$\varphi_x(g) := g \cdot x, \qquad g \in G,\ \ x \in X. \tag{5.9}$$

Then definition (3.41) of the droplet $D_B(g)$ takes the form

$$D_B(g) = g^{-1} \cdot B, \qquad B \in \mathcal{B}(X),\ \ g \in G,$$

and the droplet cluster is given by

$$\bar{D}_B(\bar{g}) = \bigcup_{g_i \in \bar{g}} (g_i^{-1} \cdot B), \qquad \bar{g} \in \mathfrak{G} := \bigsqcup_{n=0}^{\infty} G^n.$$

According to Section 3.4, we can now use (5.9) to define the family of distributions

$$\eta_x := \bar{\varphi}_x^* Q \equiv Q \circ \bar{\varphi}_x^{-1}, \tag{5.10}$$

where Q is a given probability measure on $\mathfrak{G}$.

Conditions (3.46) and (3.47) take the form, respectively,

$$\sup_{g \in G} \theta(g^{-1} \cdot B) < \infty, \qquad \int_{\mathfrak{G}} N_G(\bar{g})\, Q(\mathrm{d}\bar{g}) < \infty. \tag{5.11}$$

The first of conditions (5.11) is satisfied, for instance, if θ is absolutely continuous with respect to the volume measure on X and the corresponding Radon–Nikodym density is bounded (cf. Remark 3.10).

Let us point out that a special class of measures $\{\eta_x\}_{x \in X}$ on $\mathfrak{G}$ can be constructed somewhat more naturally by essentially reproducing the group translations method for Lie groups (cf. (5.5)). More precisely, fix an arbitrary point $x_0 \in X$ and an H_{x_0}-invariant measure η_{x_0} on $\mathfrak{X}$ (i.e., $\eta_{x_0}(h\bar{B}) = \eta_{x_0}(\bar{B})$ for any $\bar{B} \in \mathcal{B}(\mathfrak{X})$ and all $h \in H_{x_0}$); such a measure always exists due to the compactness of H_{x_0}. Since the group action is transitive, the group orbit of x_0 coincides with X, hence each $x \in X$ can be represented in the form $x = g \cdot x_0$ with some $g = g_x \in G$. Let us now define the measure η_x on $\mathcal{B}(\mathfrak{X})$ by the formula

$$\eta_x := g_x^* \eta_{x_0} \equiv \eta_{x_0} \circ g_x^{-1}, \qquad x = g_x \cdot x_0. \tag{5.12}$$

It follows that η_x is H_x-invariant for each $x \in X$. Definition (5.12) does not depend on the choice of a solution g_x of the equation $g \cdot x_0 = x$; indeed, if there is another solution $\tilde{g}_x$ then

$$\eta_{x_0} \circ \tilde{g}_x^{-1} = \eta_{x_0} \circ (\tilde{g}_x^{-1} g_x)\, g_x^{-1} = \eta_{x_0} \circ g_x^{-1},$$

since $\tilde{g}_x^{-1} g_x \in H_{x_0}$ and η_{x_0} is H_{x_0}-invariant.

Remark 5.1. Choosing various subgroups G of the general group of isometries of X may lead to different representations of X as a homogeneous space. Consequently, formula (5.10) will define different cluster measures. This is illustrated in the next simple example for the Euclidean space.

Example 5.5. Let $X = \mathbb{R}^d$ $(d \geq 2)$. If G is the group of *translations* $x \mapsto x - g$ $(x, g \in \mathbb{R}^d)$, then the corresponding homogeneous space is isomorphic to $\mathbb{R}^d$ and, as described in Section 5.1, the measures η_x are obtained by translations, $\eta_x(\cdot) = \eta_0(\cdot - x)$ (see Example 5.1). Let now $G = \mathcal{E}^+(d)$ (see Example 5.2), that is, the group of *rotations* $g = (\xi, A)$ with the action $\varphi_x(g) := g \cdot x = A(x - \xi) + \xi$ $(x \in \mathbb{R}^d)$, where $\xi \in \mathbb{R}^d$ and $A \in \mathcal{SO}(d)$. It is easy to check that, for a given Borel set $B \subset \mathbb{R}^d$,

$$\begin{aligned} \varphi_x^{-1}(B \setminus \{x\}) &= \bigl\{g \in G :\ A \neq I \ \text{ and } \ \xi \in (I - A)^{-1}(B - Ax)\bigr\}, \\ \varphi_x^{-1}(\{x\}) &= \bigl\{g \in G :\ A \neq I,\ \xi = x \ \text{ or } \ A = I,\ \xi \in \mathbb{R}^d\bigr\}, \end{aligned}$$

where I is the identity matrix. Consider the simplest case where each cluster contains only one point; in other words, the measures η_x are supported on X (i.e., $\eta_x(X^n) = 0$ for $n \neq 1$). Let $Q(\mathrm{d}\xi \times \mathrm{d}A)$ be a probability measure on G; assume for simplicity that $Q\{A \neq I\} = 1$. Then definition (5.10) specialises to

$$\eta_x(B) = Q(\varphi_x^{-1}(B)) = \int_{\mathcal{SO}(d)} Q(\mathbb{R}^d \times \mathrm{d}A) \int_{(I-A)^{-1}(B-Ax)} Q(\mathrm{d}\xi \,|\, A), \tag{5.13}$$

where $Q(\mathbb{R}^d \times \mathrm{d}A)$ is the marginal distribution of A and $Q(\mathrm{d}\xi|A)$ is the conditional distribution of ξ given A. Conditionally on A, η_x is obtained from η_0 via a translation by the vector $-(I - A)^{-1}Ax$, which is different from x. If A is truly random, then averaging with respect to its distribution will further mix up the random shifts $-(I - A)^{-1}Ax$.

5.3.2. Verification of smoothness. Our next goal is to show that $\varphi_x(\cdot) \in C_b^2(G, X)$ uniformly in $x \in G$ for a special Riemannian metric on G. Following [7, Ch. 7, pp. 181–186], fix any $x \in X$ and let, as before, H_x be the isotropy subgroup at x. Then the manifold X can be identified with the quotient manifold G/H_x in such a way that the map $\varphi_x : G \to X$ coincides with the natural projection $G \to G/H_x$. Let $\mathfrak{h}_x$ be the Lie algebra of H_x. It is known that the Lie algebra $\mathfrak{g}$ of the Lie group G admits a decomposition

$$\mathfrak{g} = \mathfrak{h}_x \oplus \mathfrak{x}_x, \tag{5.14}$$

where $\mathfrak{r}_x$ is a subspace of $\mathfrak{g}$ invariant with respect to the adjoint representation $H_x \ni h \mapsto \mathrm{Ad}_h$ of H_x in $\mathfrak{g}$. Then the tangent space $T_x X$ can be identified with the space $\mathfrak{r}_x$. The Riemannian metric of X induces an Ad_h-invariant scalar product $(\cdot,\cdot)_{\mathfrak{r}_x}$ on $\mathfrak{r}_x$.

Let us choose an auxiliary Ad_h-invariant scalar product $(\cdot,\cdot)_{\mathfrak{h}_x}$ on $\mathfrak{h}_x$. Such a product always exists thanks to the compactness of H_x; for instance, we can set $(\cdot,\cdot)_{\mathfrak{h}_x} := -B(\cdot,\cdot)$, where B is the Killing–Cartan form (see, e.g., [7, Ch. 7, pp. 184–185] or [19, Ch. II, §6, p. 131]). Observe that the isotropy subgroup at $g\cdot x$ has the form $H_{g\cdot x} = g\cdot H_x g^{-1}$, therefore the corresponding Lie algebra is given by $\mathfrak{h}_{g\cdot x} = \mathrm{Ad}_g(\mathfrak{h}_x)$. We equip it with the scalar product

$$(\cdot,\cdot)_{\mathfrak{h}_{g\cdot x}} := (\mathrm{Ad}_{g^{-1}}\cdot, \mathrm{Ad}_{g^{-1}}\cdot)_{\mathfrak{h}_x}.$$

Moreover, we can set $\mathfrak{r}_{g\cdot x} = \mathrm{Ad}_g(\mathfrak{r}_x)$, so that decomposition (5.14) at $g\cdot x$ takes the form

$$\mathfrak{g} = \mathrm{Ad}_g(\mathfrak{h}_x) \oplus \mathrm{Ad}_g(\mathfrak{r}_x). \tag{5.15}$$

Now we can define a scalar product $(\cdot,\cdot)_{\mathfrak{g},g}$ on $\mathfrak{g}$ by setting for all $h \in \mathfrak{h}_{g\cdot x}$, $r \in \mathfrak{r}_{g\cdot x}$

$$(h+r, h+r)_{\mathfrak{g},g} = (h,h)_{\mathfrak{h}_{g\cdot x}} + (r,r)_{\mathfrak{r}_{g\cdot x}}.$$

The G-invariance of the Riemannian metric on X implies that

$$(\cdot,\cdot)_{\mathfrak{g},g} = (\mathrm{Ad}_{g^{-1}}\cdot, \mathrm{Ad}_{g^{-1}}\cdot)_{\mathfrak{g},e}. \tag{5.16}$$

The family of scalar products $(\cdot,\cdot)_{\mathfrak{g},g}$ ($g \in G$), defines a Riemannian metric on G. Note that this metric is neither left nor right invariant.

For a fixed $x \in X$, let us compute the derivative $\mathrm{d}\varphi_x(g) : T_g G \to T_{g\cdot x} X$ of the map $G \ni g \mapsto \varphi_x(g) = g\cdot x \in X$. As in the previous section, we identify the tangent space $T_g G$ with the Lie algebra $\mathfrak{g}$ by right translations; under this identification,

$$\mathrm{d}\varphi_x(g) = P_{g\cdot x}, \qquad g \in G,\ \ x \in X. \tag{5.17}$$

Observe that $P_{g\cdot x} : \mathfrak{g} \to \mathfrak{r}_{g\cdot x}$ is an orthogonal projection (with respect to the scalar product $(\cdot,\cdot)_{\mathfrak{g},g}$ on $\mathfrak{g}$). Therefore,

$$\|\mathrm{d}\varphi_x(g)\| \le 1 \tag{5.18}$$

and $\mathrm{d}\varphi_x(g)^* : \mathfrak{r}_{g\cdot x} \to \mathfrak{g}$ is an isometry. Moreover, it follows from (5.15) that

$$\mathrm{d}\varphi_x(g) = \mathrm{Ad}_g \circ P_x \circ \mathrm{Ad}_{g^{-1}}. \tag{5.19}$$

Considering $\mathrm{d}\varphi_x(\cdot)V$ as a map from G to $\mathfrak{g}$ (via the embedding $\mathfrak{r}_x \subset \mathfrak{g}$), we obtain

$$\mathrm{d}^2\varphi_x(g)(U,V) = (\mathrm{ad}_U \circ P_{g\cdot x} - P_{g\cdot x} \circ \mathrm{ad}_U)V, \tag{5.20}$$

for any $U, V \in \mathfrak{g}$. This, together with (5.18), implies that

$$\sup_{x\in X,\, g\in G} \|\mathrm{d}^2\varphi_x(g)\| < \infty.$$

Thus, $\varphi_x \in C_b^2(G,X)$ uniformly in $x \in G$, and so one can apply Theorem 4.8 provided that conditions (4.31) and (4.32) are met. Finally, if the probability measure Q is absolutely continuous with respect to a left Haar measure on $\mathfrak{G}$, then (4.31) and (4.32) can be specified in terms of the corresponding Radon–Nikodym density. Note that the norm used in condition (4.31) is generated in this case by the special Riemannian structure (5.16) on G.

5.4. Other examples

In this section, we briefly discuss two further examples illustrating possible ways of constructing cluster distributions η_x.

5.4.1. Manifolds of non-positive curvature. Let X be a complete, path-connected manifold with non-positive sectional curvature (*Cartan–Hadamard manifold*). In this case, for every two points $x, y \in X$ there is a unique geodesic $g_{x,y}(t)$, $t \in [0, 1]$, such that $g_{x,y}(0) = x$, $g_{x,y}(1) = y$. Assume in addition that X is simply connected. It follows from the Cartan–Hadamard theorem that the exponential map $\exp_x\colon T_xX \to X$ is a diffeomorphism for every $x \in X$ (see, e.g., [13, 21]).

Choose $x_0 \in X$, and let

$$\mathrm{d}g_{x_0,x} : T_{x_0}X \to T_xX$$

be the parallel translation along the geodesic $g_{x_0,x}$. To deploy the construction of Section 3.4, we set $W := T_{x_0}X$ and

$$\varphi_x := \exp_x \circ \, \mathrm{d}g_{x_0,x} : \, W \to X.$$

For a given probability measure $Q = Q_{x_0}$ on $(T_{x_0}X)_0^\infty$, consider the corresponding translated (push-forward) measures on $(T_xX)_0^\infty$,

$$Q_x = \mathrm{d}g^*_{x_0,x} Q_{x_0} \qquad x \in X. \tag{5.21}$$

According to a general formula (3.40), we can now define a family of probability distributions on the space X by

$$\eta_x := \bar{\varphi}^*_x Q_{x_0} = \exp^*_x Q_x, \qquad x \in X. \tag{5.22}$$

Remark 5.2. In fact, X is essentially the Euclidean space $\mathbb{R}^d$ ($d = \dim X$) with a non-constant metric which defines a family of exponential maps $\exp_x : \mathbb{R}^d \to \mathbb{R}^d$ ($x \in \mathbb{R}^d$). In this interpretation, we have $W = \mathbb{R}^d$ and $\varphi_x = \exp_x \colon W \to X$.

Remark 5.3. Consider the diffeomorphism

$$i_x := \exp_x \circ \, \mathrm{d}g_{x_0,x} \circ \exp^{-1}_{x_0} : \, \mathfrak{X} \to \mathfrak{X}, \tag{5.23}$$

From (5.21), (5.22) and (5.23) it follows that the family of distributions $\{\eta_x\}_{x\in X}$ is translation invariant in the sense that $\eta_x = i^*_x \eta_{x_0}$ ($x \in X$).

5.4.2. Metric spaces. Let (X, ρ) be a metric space, endowed with the natural topology generated by the open balls $B^0_r(x) := \{x' \in X : \rho(x, x') < r\}$ ($x \in X$, $r > 0$) and equipped with a (locally finite) reference measure ϑ.

In this section, we construct an example of a family of probability measures $\{\eta_x(\mathrm{d}\bar{y})\}_{x\in X}$ on $\mathfrak{X} = \bigsqcup_n X^n$, based on a different idea that avoids using any family of maps φ_x as in Sections 5.1–5.3. To this end, note that by a radial-angular decomposition (based on Fubini's theorem) we can represent the ϑ-volume of a (closed) ball $B_r(x) := \{x' \in X : \rho(x, x') \le r\}$ ($x \in X$) as

$$\vartheta(B_r(x)) = \int_0^r \left(\int_{\partial B_s(x)} \vartheta^x_{\mathrm{ang}}(\mathrm{d}y\,|\,s) \right) \vartheta^x_{\mathrm{rad}}(\mathrm{d}s), \tag{5.24}$$

where $\partial B_r(x) = B_r(x) \setminus B_r^0(x)$ is the sphere of radius r centred at x, $\vartheta_{\rm ang}^x(\mathrm{d}y|r)$ is the uniform "surface" measure on $\partial B_r(x)$ induced by the measure $\vartheta(\mathrm{d}y)$, and $\vartheta_{\rm rad}^x(\mathrm{d}r)$ is the radial component of ϑ as seen from x. According to formula (5.24), the measure ϑ can be symbolically expressed as a skew product

$$\vartheta(\mathrm{d}y) = \vartheta_{\rm ang}^x(\mathrm{d}y|r)\,\vartheta_{\rm rad}^x(\mathrm{d}r)\Big|_{r=\rho(x,y)}.$$

For $x \in X$ and $\bar{y} \in \mathfrak{X}$, set $\bar{\rho}(x,\bar{y}) := (\rho(x,y_i))_{y_i\in\bar{y}} \in \mathfrak{X}$. As usual, the measure ϑ can be lifted to the space $\mathfrak{X}$ by setting

$$\bar{\vartheta}(\mathrm{d}\bar{y}) := \bigotimes_{y_i\in\bar{y}} \vartheta(\mathrm{d}y_i), \qquad \bar{y} \in \mathfrak{X}. \tag{5.25}$$

Similarly, for each $x \in X$ define

$$\bar{\vartheta}_{\rm rad}^x(\mathrm{d}\bar{r}) := \bigotimes_{r_i\in\bar{r}} \vartheta_{\rm rad}^x(\mathrm{d}r_i), \tag{5.26}$$

$$\bar{\vartheta}_{\rm ang}^x(\mathrm{d}\bar{y}|\bar{r}) := \bigotimes_{y_i\in\bar{y}} \vartheta_{\rm ang}^x(\mathrm{d}y_i|r_i). \tag{5.27}$$

Let us now fix a point $x_0 \in X$, and let $f : \mathbb{R}_0^\infty \to \mathbb{R}_+$ be such that $\int_{\mathfrak{X}} f(\bar{r})\,\bar{\vartheta}_{\rm rad}^{x_0}(\mathrm{d}\bar{r}) = 1$. Then we can construct a family of cluster distributions by setting, for each $x \in X$,

$$\eta_x(\mathrm{d}\bar{y}) := f(\bar{r})\,\bar{\vartheta}_{\rm rad}^{x_0}(\mathrm{d}\bar{r}) \cdot \frac{\bar{\vartheta}_{\rm ang}^x(\mathrm{d}\bar{y}|\bar{r})}{\bar{\vartheta}_{\rm ang}^x(\partial\bar{B}_{\bar{r}}(x))|\bar{r})}\Bigg|_{\bar{r}=\bar{\rho}(x,\bar{y})}, \qquad \bar{y} \in \mathfrak{X}. \tag{5.28}$$

That is to say, under the measure η_x a random vector $\bar{y}$ is sampled in two stages: first, a vector $\bar{r}$ of the distances from x to $\bar{y}$ is sampled with the probability density $f(\bar{r})$ (with respect to the measure $\bar{\vartheta}_{\rm rad}^{x_0}$), and then the components y_i of $\bar{y}$ are chosen, independently of each other, with the uniform distribution over the corresponding spheres $\partial B_{r_i}(x)$, respectively.

Remark 5.4. By definition (5.28), the measure η_x may be considered as a "translation" of the pattern measure η_{x_0} from x_0 to x; however, this is not being done by a push-forward of η_{x_0} under some mapping φ_x of the space X, as prescribed by the general recipe of Section 3.4; instead, we compensate the lack of such a mapping by using the same statistics of the distances at each point $x \in X$ (prescribed by the pattern distribution $\vartheta_{\rm rad}^{x_0}$) and by taking advantage of the uniform distribution on the corresponding spheres, which does not require any further angular information.

Remark 5.5. If there is a group G of isometries of X acting transitively, then we can use the same method as for homogeneous spaces (see Section 5.3).

Appendix

A. On a definition of the skew-product measure $\hat{\mu}$

In order to verify that the measure $\hat{\mu}$ is well defined by expression (2.17) (which requires the internal integral in (2.18) to be measurable as a function of $\gamma \in \Gamma_X$), we shall construct an auxiliary measure $\tilde{\mu}$ on $\Gamma_X \times \mathfrak{X}^\infty$ and show that $\hat{\mu}$ is its image under a certain measurable map.

Let us fix an indexation $\mathfrak{i} = \{\mathfrak{i}_\gamma,\ \gamma \in \Gamma_X\}$ in Γ_X, where $\mathfrak{i}_\gamma\colon \gamma \to \mathbb{N}$ is a bijection for each $\gamma \in \Gamma_X$. Define

$$\Gamma_{X,1} := \{(\gamma, x) \in \Gamma_X \times X\colon\ x \in \gamma\}.$$

The indexation $\mathfrak{i}$ defines a natural bijection

$$\Gamma_{X,1} \ni (\gamma, x) \mapsto (\gamma, \mathfrak{i}_\gamma(x) \in \Gamma_X \times \mathbb{N}. \tag{A.1}$$

Moreover, the indexation $\mathfrak{i}$ can be constructed so that bijection (A.1) is measurable (see [31]). This ensures that the map

$$\Gamma_X \ni \gamma \mapsto j_k(\gamma) := \mathfrak{i}_\gamma^{-1}(k) \in X \tag{A.2}$$

is measurable for each $k \in \mathbb{N}$.

Consider a family $\{\nu^\gamma,\ \gamma \in \Gamma_X\}$ of measures on $\mathfrak{X}^\infty$ defined by

$$\nu^\gamma(\mathrm{d}\bar{y}) := \bigotimes_{k\in\mathbb{N}} \eta_{j_k(\gamma)}(\mathrm{d}\bar{y}), \qquad \bar{y} \in \mathfrak{X}^\infty.$$

If $A \in \mathcal{B}(\mathfrak{X}^\infty)$ is a cylinder set, $A = A_1 \times \cdots \times A_n \times \mathfrak{X} \times \cdots$, then

$$\nu^\gamma(A) = \bigotimes_{k=1}^{n} \eta_{j_k(\gamma)}(A_k).$$

The function $\Gamma_X \ni \gamma \mapsto \nu^\gamma(A) \in \mathbb{R}$ is measurable due to the measurability of $j_k(\gamma)$ and Condition 2.1. Hence, the measure

$$\tilde{\mu}(\mathrm{d}\gamma \times \mathrm{d}\bar{y}) := \nu^\gamma(\mathrm{d}y)\,\mu(\mathrm{d}\gamma), \qquad (\gamma, \bar{y}) \in \Gamma_X \times \mathfrak{X}^\infty,$$

is well defined.

Finally, a direct calculation shows that the measure $\hat{\mu}$ defined by (2.17) can be represented as $\hat{\mu} = \mathcal{I}^*\tilde{\mu}$, where $\mathcal{I} : \Gamma_X \times \mathfrak{X}^\infty \to \Gamma_X \times \mathfrak{X}^\gamma$ is a measurable map defined by $(\gamma, (y_k)_{k\in\mathbb{N}}) \mapsto (\gamma, (y_{j_k(\gamma)})_{k\in\mathbb{N}})$. This proves the result.

B. Correlation functions

For a more systematic exposition and further details, see the classical books [16, 28, 29]; more recent useful references include [4, 22, 23].

Definition B.1. Let μ be a probability measure on the generalised configuration space $\Gamma_X^\sharp$, and let θ be a (locally finite) measure on X. Then the correlation function $\kappa_\mu^n : X^n \to \mathbb{R}_+$ of the n-th order ($n \in \mathbb{N}$) of the measure μ with respect to θ is defined by the following property: for any function $\phi \in \mathrm{M}_+(X^n)$ symmetric with respect to permutations of its arguments, it holds

$$\int_{\Gamma_X^\sharp} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \phi(x_1, \dots, x_n)\,\mu(\mathrm{d}\gamma)$$
$$= \frac{1}{n!}\int_{X^n} \phi(x_1, \dots, x_n)\,\kappa_\mu^n(x_1, \dots, x_n)\,\theta(\mathrm{d}x_1)\cdots\theta(\mathrm{d}x_n). \tag{B.1}$$

Remark B.1. Note that possible multiple points on the configuration $\gamma \in \Gamma_X^\sharp$ will lead correspondingly to some coinciding points among $\{x_1, \dots, x_n\} \subset \gamma$ on the left-hand side of formula (B.1) (cf. our convention on the use of set-theoretic notation, see Section 2.1).

By a standard approximation argument, equation (B.1) can be extended to any (symmetric) functions $\phi \in L^1(X^n, \theta^{\otimes n})$.

Condition B.1. Correlation functions $\kappa_\mu^m(x_1, \dots, x_m)$ up to the n-th order ($n \in \mathbb{N}$) of the measure μ with respect to θ exist and are bounded.

Remark B.2. Formula (B.1) with $n = 1$ and $\phi(x) = \mathbf{1}_B(x)$ for $B \in \mathcal{B}(X)$ shows that Condition B.1 automatically implies that μ-a.a. configurations γ are locally finite.

Lemma B.1. *Assume that Condition* B.1 *is satisfied with some* $n \in \mathbb{N}$*. Then* $\mu \in \mathcal{M}_\theta^n(\Gamma_X)$.

Proof. Similarly as in the proof of Lemma 3.6, we obtain (cf. (3.28))

$$\int_{\Gamma_X} |\langle f, \gamma\rangle|^n \, \mu(\mathrm{d}\gamma) \le \int_{\Gamma_X^\sharp} \left(\sum_{x\in\gamma} |f(x)| \right)^n \mu(\mathrm{d}\gamma)$$
$$= \sum_{m=1}^n \int_{\Gamma_X^\sharp} \sum_{\{x_1,\dots,x_m\}\subset\gamma} \phi_n(x_1, \dots, x_m) \, \mu(\mathrm{d}\gamma), \tag{B.2}$$

where $\phi_n(x_1, \dots, x_m)$ is a (symmetric) function given by expression (3.29). Note that, by definition of the correlation functions (see (B.1)), the integral on the right-hand side of (B.2) is reduced to

$$\frac{1}{m!} \int_{X^m} \phi_n(x_1, \dots, x_m) \, \kappa_\mu^m(x_1, \dots, x_m) \, \theta(\mathrm{d}x_1) \cdots \theta(\mathrm{d}x_m). \tag{B.3}$$

By Condition B.1, $\kappa_\mu^m \le c_m$ ($m = 1, \dots, n$) with some constant $c_m < \infty$. Hence, the integral in (B.3) is bounded by

$$\sum_{\substack{i_1,\dots,i_m\ge 1\\ i_1+\dots+i_m=n}} \frac{c_m n!}{i_1! \cdots i_m!} \prod_{j=1}^m \int_Z |f(x_j)|^{i_j} \, \theta(\mathrm{d}x_j) < \infty, \tag{B.4}$$

since each integral in (B.4) is finite owing to the assumption $f \in \bigcap_{1\le q\le n} L^q(X, \theta)$. As a result, the integral on the left-hand side of (B.2) is finite, and the lemma is proved. □

C. Integration-by-parts formula for push-forward measures

For any Riemannian manifolds $\mathcal{W}$ and $\mathcal{Y}$, denote by $C_b^2(\mathcal{W}, \mathcal{Y})$ the space of twice differentiable maps $\phi : \mathcal{W} \to \mathcal{Y}$ with globally bounded derivatives $\mathrm{d}\phi$, $\mathrm{d}^2\phi$. In particular, for any $\bar{w} \in \mathcal{W}$, the first derivative $\mathrm{d}\phi(\bar{w})$ is a bounded linear operator from the tangent space $T_{\bar{w}}\mathcal{W}$ to the tangent space $T_{\phi(\bar{w})}\mathcal{Y}$. In what follows, we fix $\phi \in C_b^2(\mathcal{W}, \mathcal{Y})$. Note that the adjoint operator $\mathrm{d}\phi(\bar{w})^* : T^*_{\phi(\bar{w})}\mathcal{Y} \to T^*_{\bar{w}}\mathcal{W}$ can be identified with a bounded operator from $T_{\phi(\bar{w})}\mathcal{Y}$ to $T_{\bar{w}}\mathcal{W}$ via the scalar products in the tangent spaces $T_{\bar{w}}\mathcal{W}$ and $T_{\phi(\bar{w})}\mathcal{Y}$ (defined by the Riemannian structure of the manifolds $\mathcal{W}$ and $\mathcal{Y}$, respectively). Furthermore, define $\mathrm{Vect}_b^1(\mathcal{W})$ as the space of differentiable vector fields on $\mathcal{W}$ with a globally bounded first derivative.

Definition C.1. We say that a probability measure $Q(\mathrm{d}\bar{w})$ on $\mathcal{W}$ satisfies an *integration-by-parts* (*IBP*) *formula* if for any vector field $V \in \mathrm{Vect}_b^1(\mathcal{W})$ there is a function $\beta_Q^V \in L^1(\mathcal{W}, Q)$ (*logarithmic derivative* of Q in the direction V) such that, for any $g \in C_b^1(\mathcal{W})$, the following identity holds

$$\int_{\mathcal{W}} (\nabla g(\bar{w}), V(\bar{w}))_{T_{\bar{w}}\mathcal{W}}\, Q(\mathrm{d}\bar{w}) = -\int_{\mathcal{W}} g(\bar{w})\, \beta_Q^V(\bar{w})\, Q(\mathrm{d}\bar{w}). \tag{C.1}$$

Whenever it exists, the function β_Q^V can be represented in the form

$$\beta_Q^V(\bar{w}) = (\beta_Q(\bar{w}), V(\bar{w}))_{T_{\bar{w}}\mathcal{W}} + \operatorname{div} V(\bar{w}), \qquad \bar{w} \in \mathcal{W}, \tag{C.2}$$

where β_Q is a vector field on $\mathcal{W}$ (called the *vector logarithmic derivative* of Q) satisfying

$$\int_{\mathcal{W}} |\beta_Q(\bar{w})|_{T_{\bar{w}}\mathcal{W}}\, Q(\mathrm{d}\bar{w}) < \infty.$$

Consider the push-forward measure $\eta := \phi^* Q$ on $\mathcal{Y}$, and denote by $\mathcal{I}_\phi$ the operator acting on functions $f : \mathcal{Y} \to \mathbb{R}$ by the formula

$$\mathcal{I}_\phi f = f \circ \phi.$$

Because of the definition of the measure η, the operator $\mathcal{I}_\phi$ is an isometry from $L^r(\mathcal{Y}, \eta)$ to $L^r(\mathcal{W}, Q)$, for any $r \in [1, \infty]$. Hence, the adjoint operator defines an isometry between the corresponding dual spaces,

$$\mathcal{I}_\phi^* : L^r(\mathcal{W}, Q)' \to L^r(\mathcal{Y}, \eta)'.$$

Furthermore, for any $r \in (1, \infty)$ we have the isomorphisms $L^r(\mathcal{W}, Q)' \cong L^n(\mathcal{W}, Q)$ and $L^r(\mathcal{Y}, \eta)' \cong L^n(\mathcal{Y}, \eta)$, where $n = r/(r-1)$ (see, e.g., [30, Ch. II, §2, p. 43]). Since $r > 1$ is arbitrary, this means that $\mathcal{I}_\phi^*$ can be treated as an isometry from $L^n(\mathcal{W}, Q)$ to $L^n(\mathcal{Y}, \eta)$ for any $n > 1$. Moreover, repeating the arguments used in the proof of Lemma 4.3, it can be shown that the same also holds for $n = 1$. To summarise, for any $n \geq 1$ the operator

$$\mathcal{I}_\phi^* : L^n(\mathcal{W}, Q) \to L^n(\mathcal{Y}, \eta)$$

is an isometry.

Theorem C.1. *Let $\phi \in C_b^2(\mathcal{W}, \mathcal{Y})$ be such that the operator*

$$\mathrm{d}\phi(\bar{w})^* : T_{\phi(\bar{w})}\mathcal{Y} \to T_{\bar{w}}\mathcal{W},\ \bar{w} \in \mathcal{W}$$

is an isometry, and suppose that the measure Q satisfies the IBP formula (C.1). *Then the push-forward measure $\eta = \phi^* Q$ satisfies an IBP formula with the logarithmic derivative $\beta_\eta^U = \mathcal{I}_\phi^* \beta_Q^V$, where $V = V_U$ is a vector field on $\mathcal{W}$ given by*

$$V(\bar{w}) = \mathrm{d}\phi(\bar{w})^*\, U(\phi(\bar{w})), \qquad U \in \mathrm{Vect}_b^1(\mathcal{Y}).$$

Proof. Note that $V \in \mathrm{Vect}_b^1(\mathcal{W})$. Applying the IBP formula (C.1), making the change of measure $\eta = \phi^* Q$ and taking into account that $\mathrm{d}\phi(\bar{w})\mathrm{d}\phi(\bar{w})^*$ is the identity operator in $T_{\phi(\bar{w})}\mathcal{Y}$, we see that (C.1) holds for η with the corresponding logarithmic derivative $\beta_\eta^U = \mathcal{I}_\phi^* \beta_Q^V$. Finally, the η-integrability of β_η^U follows by the isometry of $\mathcal{I}^*$. □

Remark C.1. All of the above remains true in the case where $\mathcal{W} = \bigsqcup_{i=0}^{\infty} \mathcal{W}_i$ and $\mathcal{Y} = \bigsqcup_{i=0}^{\infty} \mathcal{Y}_i$ are countable disjoint unions of Riemannian manifolds $(\mathcal{W}_i)$ and $(\mathcal{Y}_i)$ respectively, and the mapping ϕ acts component-wise, that is, $\phi : \mathcal{W}_i \to \mathcal{Y}_i$. Although the spaces $\mathcal{W}$ and $\mathcal{Y}$ do not possess a proper Riemannian manifold structure, all notions introduced above (including the IBP formula (C.1)) can be understood component-wise, and we use the analogous notation without further explanations.

Acknowledgements

Research of L. Bogachev was supported in part by a Leverhulme Research Fellowship. Partial support from DFG Grant 436 RUS 113/722 and SFB 701 is gratefully acknowledged. The authors would like to thank Sergio Albeverio, Yuri Kondratiev, Eugene Lytvynov, Stanislav Molchanov and Chris Wood for helpful discussions.

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