

POWER CONVERGENCE OF ABEL AVERAGES

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ABSTRACT. Necessary and sufficient conditions are presented for the Abel averages of discrete and strongly continuous semigroups, T^k and T_t , to be power convergent in the operator norm in a complex Banach space. These results cover also the case where T is unbounded and the corresponding Abel average is defined by means of the resolvent of T . They complement the classical results by Michael Lin establishing sufficient conditions for the corresponding convergence for a bounded T .

1. POSING THE PROBLEM

For a bounded linear operator T on a Banach space X , the Abel average of the discrete semigroup $\{T^k\}_{k \in \mathbb{N}_0}$ is defined as

$$(1.1) \quad A_\alpha = (1 - \alpha) \sum_{k=0}^{\infty} \alpha^k T^k = (1 - \alpha)[I - \alpha T]^{-1},$$

where α is a suitable numerical parameter, i.e., such that A_α belongs to $\mathcal{L}(X)$ – the Banach algebra of all bounded linear operators¹ on X .

Likewise, for a strongly continuous semigroup $\{T_t\}_{t \geq 0}$, the Abel average is defined by the formula

$$(1.2) \quad \tilde{A}_\lambda = \lambda \int_0^\infty e^{-\lambda s} T_s ds,$$

with a suitable parameter λ , which is to be understood point-wise, as an improper Riemann integral; see, e.g., [5, page 42].

In this note, we establish necessary and sufficient conditions which ensure that the averages (1.1) and (1.2) are power convergent in the operator norm. Our main result (Theorem 2.1 below) covers also the case where T in (1.1) is unbounded.

The study of the Abel averages goes back to at least E. Hille [7] and W.F. Eberlein [4]. They are presented in the books [5, 8, 11, 18]. Uniform ergodic theorems for Abel and Cesàro averages were established by M. Lin in [13] and [14]. The following assertions can be deduced from the corresponding nowadays classical results of [14].

Assertion 1.1. *Let T be such that*

$$(1.3) \quad \|T^n/n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

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¹Actually, the operators A_α have a natural geometric origin, even in a more general nonlinear setting, see [19, page 154] and [20].

Then, for each $\alpha \in (0, 1)$, the operator A_α in (1.1) belongs to $\mathcal{L}(X)$, and the following statements are equivalent:

- (i) $(I - T)X$ is closed;
- (ii) the net $\{A_\alpha\}_{\alpha \in (0, 1)}$ converges in $\mathcal{L}(X)$, as $\alpha \rightarrow 1^-$;
- (iii) the Cesàro averages $N^{-1} \sum_{n=0}^{N-1} T^n$ converge, as $N \rightarrow \infty$.

The (operator-norm) limit in (ii) and (iii) is the same – the projection E of X onto $\text{Ker}(I - T)$ along $\text{Im}(I - T)$, that is, the Riesz projection corresponding to the (at most) simple pole 1 of the resolvent of T .

Assertion 1.2. Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup of bounded linear operators such that

$$(1.4) \quad \|T_t/t\| \rightarrow 0, \quad \text{as } t \rightarrow +\infty,$$

and let B be its generator. Then, for all $\lambda > 0$, the operator \tilde{A}_λ in (1.2) is in $\mathcal{L}(X)$ and the following statements are equivalent:

- (i) B has closed range;
- (ii) the net $\{\tilde{A}_\lambda\}_{\lambda > 0}$ converges in $\mathcal{L}(X)$, as $\lambda \rightarrow 0^+$;
- (iii) for each $\lambda > 0$, the operator \tilde{A}_λ is uniformly ergodic, that is, the sequence of its Cesàro averages $N^{-1} \sum_{n=0}^{N-1} \tilde{A}_\lambda^n$ converges in $\mathcal{L}(X)$.

The limits in (ii) and (iii) coincide; their common value is the projection \tilde{E} of X onto $\text{Ker} B$ along $\text{Im} B$, given by the Riesz decomposition²

$$(1.5) \quad X = \text{Ker} B \oplus \text{Im} B,$$

corresponding to the (at most) simple pole 0 of the resolvent of B .

Note that

$$\text{Ker} B = \bigcap_{t \geq 0} \text{Ker}(I - T_t),$$

where the inclusion “ \subset ” follows by, e.g., [18, Theorem 1.8.3, page 33].

In the discrete case, an analog of claim (iii) of Assertion 1.2 can also be obtained. As follows from (1.3), the spectrum of T is contained in the closure of the open unit disk Δ . By the spectral mapping theorem, the spectrum of A_α is then contained in $\Delta \cup \{1\}$. Since

$$\text{Ker}(I - A_\alpha) = \text{Ker}(I - T) \quad \text{and} \quad \text{Im}(I - A_\alpha) = \text{Im}(I - T),$$

cf. the proof of Theorem 2.1 below, we have the Riesz decomposition

$$\text{Ker}(I - A_\alpha) \oplus \text{Im}(I - A_\alpha) = \text{Ker}(I - T) \oplus \text{Im}(I - T) = X,$$

and thus the point 1 is at most a simple pole of A_α . In particular, it is at most an isolated point of the spectrum of A_α . Hence, $\|A_\alpha^n/n\| \rightarrow 0$, as $n \rightarrow +\infty$; see, e.g., [16]. Therefore, all the operators A_α , $\alpha \in (0, 1)$, are uniformly ergodic, even power convergent to the same limit E as above. This complements Assertion 1.1 in the spirit of Assertion 1.2.

²See, e.g., [8, Theorem 18.8.1, pages 521–522] and [22, Theorems 5.8-A and 5.8-D, pages 306–311].

As we shall see in Assertions 1.3 and 1.4 below, both claims (ii) above are equivalent to the power convergence of the corresponding Abel averages; see also Remark 2.2 below. Indeed, under the conditions of Assertions 1.1 and 1.2, by the technique used in [14] one can show that, for α close to 1^- and λ close to 0^+ , the operators A_α and \tilde{A}_λ , respectively, are power convergent in $\mathcal{L}(X)$. As we shall see later, if X is a complex Banach space, the assumptions of Assertions 1.1 and 1.2 allow one to prove the corresponding power convergence of the operators A_α and \tilde{A}_λ , for all $\alpha \in (0, 1)$ and all $\lambda > 0$, respectively. More precisely, the following extensions of Assertions 1.1 and 1.2 hold. See also [15].

Assertion 1.3. *Let T be a bounded linear operator in a complex Banach space X obeying (1.3), and let A_α , $\alpha \in (0, 1)$, be its Abel average (1.1). Then the following statements are equivalent:*

- (i) $(I - T)X$ is closed;
- (ii) for some $\alpha \in (0, 1)$, the sequence $\{A_\alpha^n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(X)$;
- (iii) for each $\alpha \in (0, 1)$, the sequence $\{A_\alpha^n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(X)$.

The limits in (ii) and (iii) coincide with the projection E from Assertion 1.1.

Assertion 1.4. *Let $\{T_t\}_{t \geq 0} \subset \mathcal{L}(X)$ be a strongly continuous semigroup of bounded linear operators in a complex Banach space X such that (1.4) holds. Let B be its generator and \tilde{A}_λ , $\lambda > 0$, be its Abel average (1.2). Then the following statements are equivalent:*

- (i) B has closed range;
- (ii) for some $\lambda > 0$, the sequence $\{\tilde{A}_\lambda^n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(X)$;
- (iii) for each $\lambda > 0$, the sequence $\{\tilde{A}_\lambda^n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(X)$.

The limits in (ii) and (iii) coincide with the projection \tilde{E} from Assertion 1.2.

In fact, the conditions (1.3) and (1.4) are quite far from being necessary for the corresponding Abel averages to converge as stated above. For example, the former one can be replaced by the dissipativity condition used in the classical Lumer–Phillips theorem; see, e.g., [23, page 250]. The next assertion, which provides an example of this sort, might be useful in the study of the sets of fixed points of some nonlinear operators; see [19] and [20].

Assertion 1.5. *For a complex Banach space X and $T \in \mathcal{L}(X)$, let $W(T)$ denote the numerical range of T ; see [2, page 81] or [18, page 12], and let $\overline{W}(T)$ be its closure. Suppose that T is such that $\operatorname{Re}W(T) \subset (-\infty, 1]$. Then, for each $\alpha \in (0, 1)$, the Abel averages (1.1) of T obey the estimate $\|A_\alpha\| \leq 1$ and statements (i), (ii), and (iii) of Assertion 1.3 are equivalent. Furthermore, if $\operatorname{Re}\overline{W}(T) \subset (-\infty, 1)$, then $I - T$ is invertible on X and $\lim_{n \rightarrow +\infty} A_\alpha^n = 0$.*

Also, the assumption (1.3) in Assertion 1.3 can be relaxed to

$$(1.6) \quad \sup_N \left\| \frac{1}{N} \sum_{n=0}^{N-1} T^n \right\| < \infty,$$

which, by [17, Theorem 3.1], is equivalent to

$$(1.7) \quad \sup_{\alpha \in (0,1)} \sup_{N \in \mathbb{N}_0} \left\| (1 - \alpha) \sum_{k=0}^N \alpha^k T^k \right\| < \infty.$$

Indeed, by [6], condition (1.6) and the closedness of $(I - T)X$ yield the existence of $\lim_{\alpha \rightarrow 1^-} A_\alpha$, which is equivalent to the fact that the point 1 is at most a simple pole of the resolvent of T ; see [8, Theorem 18.8.1, pages 521–522]. Hence, by the Koliha–Li characterization of the power convergence [9, 10, 12], statements (i), (ii), and (iii) of Assertion 1.3 are again equivalent, this time under the weaker assumption (1.6) in place of (1.3).

In the light of the above facts, it would be interesting to find an analogous characterization of the norm-boundedness in $t > 0$ of the integral averages

$$\frac{1}{t} \int_0^t T_s ds,$$

assuming, e.g., the uniform boundedness of the partial integrals in (1.2).

Of course, the uniform Abel boundedness (1.7) is by no means necessary for the existence of $\lim_{\alpha \rightarrow 1^-} A_\alpha$. Relevant matrix examples can easily be constructed by using [25, Theorem 8, page 378].

2. THE RESULTS

In this section, we derive the conditions that are necessary and sufficient for the statements of Assertions 1.3, 1.4, and 1.5 to hold. Moreover, our results cover also the case where T in (1.1) is unbounded, and hence (1.3) is not applicable. The key observation which allowed us to get them is that the principal thing one needs is the spectrum $\sigma(T)$ lying merely in the half-plane

$$\Pi = \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \leq 1\}.$$

Note also that (1.3) and statement (i) in Assertion 1.3 imply that

$$(2.1) \quad \operatorname{Ker}(I - T) \oplus \operatorname{Im}(I - T) = X;$$

see, e.g., [16] and [19, pages 40–43]. In the sequel, for a closed densely defined linear operator T in a complex Banach space X , by $\mathcal{D}(T)$ and $\rho(T)$ we denote the domain and the resolvent set of T , respectively. For such an operator with $(1, +\infty) \subset \rho(T)$, the Abel average can be defined as the following bounded linear operator

$$(2.2) \quad A_\alpha = (1 - \alpha)[I - \alpha T]^{-1}, \quad \alpha \in (0, 1).$$

Finally, by $\operatorname{Im}(I - T)$ we mean the $(I - T)$ -image of $\mathcal{D}(T)$.

Theorem 2.1. *Let T be a densely defined closed linear operator in a complex Banach space X such that $(1, +\infty) \subset \rho(T)$. Then the following statements are equivalent:*

- (i) *for each $\alpha \in (0, 1)$, the sequence $\{A_\alpha^n\}_{n \in \mathbb{N}}$ of powers of its Abel average (2.2) converges in $\mathcal{L}(X)$;*
- (ii) *$\sigma(T) \subset \Pi$ and (2.1) holds.*

For every $\alpha \in (0, 1)$, the limit in (i) is the projection of X onto $\operatorname{Ker}(I - T)$ along $\operatorname{Im}(I - T)$.

Proof. For $\lambda \in \rho(T)$, let $R(\lambda, T)$ denote the resolvent of T . Thus, we have

$$(a) \quad (\lambda I - T)R(\lambda, T)x = x, \quad x \in X;$$

$$(b) \quad R(\lambda, T)(\lambda I - T)x = x, \quad x \in \mathcal{D}(T).$$

For $\alpha \in (0, 1)$, we have $1/\alpha \in \rho(T)$; hence, $A_\alpha = (\alpha^{-1} - 1)R(\alpha^{-1}, T)$, which yields

$$(2.3) \quad (a) \quad (I - \alpha T)A_\alpha x = (1 - \alpha)x, \quad x \in X;$$

$$(b) \quad A_\alpha(I - \alpha T)x = (1 - \alpha)x, \quad x \in \mathcal{D}(T).$$

Take now an $x \in \text{Ker}(I - A_\alpha)$, that is, $A_\alpha x = x$. As $\text{Im}A_\alpha$ lies in $\mathcal{D}(T)$, our x is in $\mathcal{D}(T)$, and by (a) in (2.3) we have that $x - \alpha Tx = x - \alpha x$. Thus, $x = Tx$, and hence $\text{Ker}(I - A_\alpha) \subset \text{Ker}(I - T)$. Conversely, choose $x \in \text{Ker}(I - T) \subset \mathcal{D}(T)$. Then by (b) in (2.3), we have that $A_\alpha(x - \alpha x) = (1 - \alpha)x$. Hence, $x = A_\alpha x$, and thereby

$$(2.4) \quad \text{Ker}(I - A_\alpha) = \text{Ker}(I - T).$$

Let now x be in $\text{Im}(I - T)$, that is, $x = y - Ty$ for some $y \in \mathcal{D}(T)$. By (b) in (2.3), we then get $\alpha(I - T)y = (I - A_\alpha)(I - \alpha T)y$, which yields $x = (I - A_\alpha)z$ for

$$z = \frac{1}{\alpha}(I - \alpha T)y.$$

Therefore, $\text{Im}(I - T) \subset \text{Im}(I - A_\alpha)$. Conversely, let $x \in \text{Im}(I - A_\alpha)$, i.e., $x = y - A_\alpha y$, for some $y \in X$. Note that $z = \alpha(1 - \alpha)^{-1}A_\alpha y$ is in $\mathcal{D}(T)$. For this z , by (a) in (2.3) we have

$$(I - T)z = \frac{\alpha}{1 - \alpha}(I - T)A_\alpha y = y - A_\alpha y = (I - A_\alpha)y = x,$$

which finally yields

$$(2.5) \quad \text{Im}(I - A_\alpha) = \text{Im}(I - T).$$

Let us stress that both (2.4) and (2.5) hold for any $\alpha \in (0, 1)$. Moreover, the subspaces in (2.4) and (2.5) are closed whenever A_α is power convergent.

For an $\alpha \in (0, 1)$, consider the following univalent analytic function

$$(2.6) \quad f_\alpha(\zeta) = \frac{1 - \alpha}{1 - \alpha\zeta}, \quad \zeta \in \mathbb{C} \setminus \{\alpha^{-1}\}.$$

It maps the domain $\Omega_\alpha = \{\zeta \in \mathbb{C} : |\alpha\zeta - 1| > 1 - \alpha\}$ onto the open unit disk $\Delta \subset \mathbb{C}$, and $f_\alpha(1) = 1$. Obviously, $A_\alpha = f_\alpha(T)$, and $\sigma(A_\alpha)$ lies in the closure of Δ (actually, it lies in $\Delta \cup \{1\}$ by the Koliha–Li characterization of the power convergence). Thus, by the spectral mapping theorem (see, e.g., [22, Theorem 5.71-A, page 302]) and our assumption $(1, +\infty) \subset \rho(T)$, we obtain that $\sigma(T)$ lies in $\overline{\Omega}_\alpha$ – the closure of Ω_α . Therefore,

$$\sigma(T) \subset \bigcap_{\alpha \in (0, 1)} \overline{\Omega}_\alpha = \Pi.$$

Moreover, (2.4) and (2.5) yield (2.1), by the Koliha–Li characterization of the power convergence [9, 10, 12]. Thus, (i) \Rightarrow (ii).

For each $\alpha \in (0, 1)$, the homographic transformation (2.6) maps Π onto the closed disk $\{\zeta \in \mathbb{C} : |\zeta - 1/2| \leq 1/2\}$; see, e.g., [21, page 84]. This yields $\sigma(A_\alpha) \subset \Delta \cup \{1\}$. Since $\text{Ker}(I - A_\alpha) = \text{Ker}(I - T)$ and $\text{Im}(I - A_\alpha) = \text{Im}(I - T)$, it follows by (2.1)

that, for each $\alpha \in (0, 1)$, the powers A_α^n converge to the projection E of X onto $\text{Ker}(I - T)$ along $\text{Im}(I - T)$, where E is as in Assertion 1.1. \square

Remark 2.2. Condition (2.1) in (ii) of Theorem 2.1 can be replaced by the existence of $\lim_{\alpha \rightarrow 1^-} A_\alpha$. In view of (2.4) and (2.5), the latter limit is equal to the Riesz projection E of X onto $\text{Ker}(I - T)$ along $\text{Im}(I - T)$, given by the decomposition in (2.1). The point 1 is simultaneously at most a simple pole of the resolvents of both T and A_α .

The theorem just proven obviously extends Assertion 1.3. Since the closure of the numerical range of a bounded linear operator contains its spectrum (see, e.g., [2, Theorem 10.1, page 88], [18, Theorem 1.3.9, page 12], or [24, Proposition, page 217]), Theorem 2.1 is also a generalization of Assertion 1.5. In the same spirit, we obtain the following extension of Assertion 1.4.

Theorem 2.3. *Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup in a complex Banach space X . Let B be its generator and \tilde{A}_λ be its Abel average (1.2). Additionally, assume that $\rho(B)$ contains the positive real axis. Then the following statements are equivalent:*

- (i) $\rho(B)$ contains the whole open right half-plane and $\text{Ker}B \oplus \text{Im}B = X$;
- (ii) for some $\lambda > 0$, the sequence $\{\tilde{A}_\lambda^n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(X)$ and $\rho(B)$ contains the whole open right half-plane;
- (iii) for each $\lambda > 0$, the sequence $\{\tilde{A}_\lambda^n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(X)$.

For each $\lambda > 0$, the limit above is the projection of X onto $\text{Ker}B$ along $\text{Im}B$.

Proof. For $\lambda > 0$, we have $(1 + \lambda)^{-1} =: \alpha \in (0, 1)$. Set $T = I + B$. Then

$$\begin{aligned} \tilde{A}_\lambda &= \lambda(\lambda I - B)^{-1} = \frac{1 - \alpha}{\alpha} \left(\frac{1 - \alpha}{\alpha} I - (T - I) \right)^{-1} \\ &= (1 - \alpha)[(1 - \alpha)I - \alpha(T - I)]^{-1} \\ &= (1 - \alpha)[I - \alpha T]^{-1} = A_\alpha. \end{aligned}$$

Now the proof follows directly from Theorem 2.1. \square

With the help of [3, Theorem VIII.1.11, page 622] we get the following generalization of Assertion 1.4. Recall that the Abel average \tilde{A}_λ was defined in (1.2) and its n -th power can be written as (see, e.g., [5, page 43])

$$(2.7) \quad \tilde{A}_\lambda^n = \lambda^n [R(\lambda, A)]^n = \frac{\lambda^n}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} T_t dt, \quad n = 1, 2, \dots$$

As mentioned just after Assertion 1.2, we have

$$\text{Ker}B = \bigcap_{t \geq 0} \{x \in X : T_t x = x\}.$$

Corollary 2.4. *Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup in a complex Banach space X . Let B be its generator and \tilde{A}_λ be its Abel average (1.2). Assume also that*

$$(2.8) \quad \lim_{t \rightarrow +\infty} \frac{\log \|T_t\|}{t} = 0.$$

Then the following statements are equivalent:

- (i) $\text{Ker}B \oplus \text{Im}B = X$;
- (ii) for some $\lambda > 0$, the sequence $\{\tilde{A}_\lambda^n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(X)$;
- (iii) for each $\lambda > 0$, the sequence $\{\tilde{A}_\lambda^n\}_{n \in \mathbb{N}}$ converges in $\mathcal{L}(X)$.

For each $\lambda > 0$, the limit in (ii) and (iii) is the projection of X onto $\text{Ker}B$ along $\text{Im}B$.

Remark 2.5. In Theorem 2.3 and Corollary 2.4, the convergence in claims (ii) and (iii) is based either on the condition $(0, +\infty) \subset \rho(B)$ or on (2.8). Both are weaker than (1.4) used in Assertion 1.4; see, e.g., [3, Theorem VIII.1.11, page 622] or [5, Corollary II.1.3, page 44]. Like in Remark 2.2, either (ii) or (iii) can be replaced by the existence of $\lim_{\lambda \rightarrow 0^+} \tilde{A}_\lambda$, due to [8, Theorem 18.8.1, pages 521–522].

3. AN EXAMPLE

We present an unbounded linear operator T , which has the properties described by Theorem 2.1. Here X is the complex Hilbert space $L^2(\mathbb{R})$.

$$(3.1) \quad T_0 = D^2 + (2 - t^2), \quad D = \frac{d}{dt}, \quad \mathcal{D}(T_0) = S(\mathbb{R}),$$

where $S(\mathbb{R})$ is the space of Schwartz test functions. Then T_0 is essentially self-adjoint and such that

$$(3.2) \quad T_0 x_n = \lambda_n x_n, \quad \lambda_n = 1 - 2n, \quad n \in \mathbb{N}_0.$$

The eigenvalues λ_n are simple and the eigenvectors

$$(3.3) \quad x_n(t) = h_n(t) \exp(-t^2/2), \quad t \in \mathbb{R},$$

constitute an orthonormal basis of X ; see, e.g., [1, pages 36–39]. In (3.3), for $n \in \mathbb{N}_0$, h_n is the Hermite polynomial of degree n . In particular, $h_0 = \pi^{1/4}$. Let T be the closure of (3.1). Then $X_0 := \text{Ker}(I - T)$ is the one-dimensional subspace of X spanned by x_0 . Let X_1 be the orthogonal complement of X_0 , i.e.,

$$(3.4) \quad X = X_0 \oplus X_1.$$

Take any $x \in X_1$. Then

$$(3.5) \quad x = \sum_{n=1}^{\infty} \alpha_n x_n,$$

and hence $x = (I - T)y$ for

$$y = \sum_{n=1}^{\infty} \frac{\alpha_n}{2n} x_n.$$

This immediately yields that $X_1 = \text{Im}(I - T)$, and hence

$$X = \text{Ker}(I - T) \oplus \text{Im}(I - T),$$

by (3.4). For the resolvent of T , we have

$$R(\lambda, T)x_0 = \frac{1}{\lambda - 1}x_0, \quad R(\lambda, T)x_n = \frac{1}{\lambda - \lambda_n}x_n, \quad n \in \mathbb{N}.$$

Thus, in view of (3.2), $R(\lambda, T)$ is a compact operator, positive for $\lambda > 1$. Then its spectral decomposition is

$$(3.6) \quad R(\lambda, T) = \sum_{n=0}^{\infty} \frac{1}{\lambda - \lambda_n} P_n,$$

where P_n , $n \in \mathbb{N}_0$, is the orthogonal projection onto the subspace spanned by x_n . For $\lambda > 1$ and any $x \in X$, cf. (3.5), we have

$$\begin{aligned} \|(\lambda - 1)R(\lambda, T)x - P_0x\| &= (\lambda - 1) \left[\sum_{n=1}^{\infty} \frac{|\alpha_n|^2}{(\lambda - 1 + 2n)^2} \right]^{1/2} \\ &\leq (\lambda - 1)\|x\|, \end{aligned}$$

which yields that, in $\mathcal{L}(X)$, $(\lambda - 1)R(\lambda, T) \rightarrow P_0$ as $\lambda \rightarrow 1^+$. For $m \in \mathbb{N}$, by (3.6) we have

$$[(\lambda - 1)R(\lambda, T)]^m = \sum_{n=0}^{\infty} \left(\frac{\lambda - 1}{\lambda - 1 + 2n} \right)^m P_n.$$

Then, for $m \geq 4$,

$$\begin{aligned} \|[(\lambda - 1)R(\lambda, T)]^m - P_0\| &= \left\| \sum_{n=1}^{\infty} \left(\frac{\lambda - 1}{\lambda - 1 + 2n} \right)^m P_n \right\| \\ &\leq \left(\frac{\lambda - 1}{\lambda + 1} \right)^{m-2} \sum_{n=1}^{\infty} \left(\frac{\lambda - 1}{\lambda - 1 + 2n} \right)^2 \\ &= \left(\frac{\lambda - 1}{\lambda + 1} \right)^{m-2} C(\lambda) \rightarrow 0, \quad \text{as } m \rightarrow +\infty. \end{aligned}$$

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