

Equilibrium Kawasaki dynamics and determinantal point processes

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Abstract

Let μ be a point process on a countable discrete space \mathfrak{X} . Under assumption that μ is quasi-invariant with respect to any finitary permutation of \mathfrak{X} , we describe a general scheme for constructing an equilibrium Kawasaki dynamics for which μ is a symmetrizing (and hence invariant) measure. We also exhibit a two-parameter family of point processes μ possessing the needed quasi-invariance property. Each process of this family is determinantal, and its correlation kernel is the kernel of a projection operator in $\ell^2(\mathfrak{X})$.

1 Introduction

1.1 Determinantal point processes

Let \mathfrak{X} be a locally compact topological space and let $\mathcal{B}(\mathfrak{X})$ be the Borel σ -algebra on \mathfrak{X} . The *configuration space* $\Gamma := \Gamma_{\mathfrak{X}}$ over \mathfrak{X} is defined as the set of all subsets $\gamma \subset \mathfrak{X}$ which are locally finite. Such subsets are called *configurations*. The space Γ can be endowed with the vague topology, i.e., the weakest topology on Γ with respect to which all maps $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x)$, $f \in C_0(\mathfrak{X})$, are continuous. Here $C_0(\mathfrak{X})$ is the space of all continuous real-valued functions on \mathfrak{X} with compact support. We will denote by $\mathcal{B}(\Gamma)$ the Borel σ -algebra on Γ . A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a *point process* on \mathfrak{X} . For more detail, see, e.g., [10], [16].

A point process μ can be described with the help of *correlation functions*. Let m be a reference Radon measure on $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$. The n th correlation function ($n = 1, 2, \dots$) is a non-negative measurable symmetric function $k_\mu^{(n)}(x_1, \dots, x_n)$ on \mathfrak{X}^n such that, for any measurable symmetric function $f^{(n)} : \mathfrak{X}^n \rightarrow [0, \infty]$, one has

$$\begin{aligned} & \int_{\Gamma} \sum_{\{x_1, \dots, x_n\} \subset \gamma} f^{(n)}(x_1, \dots, x_n) \mu(d\gamma) \\ &= \frac{1}{n!} \int_{\mathfrak{X}^n} f^{(n)}(x_1, \dots, x_n) k_\mu^{(n)}(x_1, \dots, x_n) m(dx_1) \cdots m(dx_n) \end{aligned} \quad (1)$$

Under a mild condition on the growth of correlation functions as $n \rightarrow \infty$, they determine the point process uniquely [10].

A point process μ is called *determinantal* if there exists a function $K(x, y)$ on \mathfrak{X}^2 , called the *correlation kernel*, such that

$$k_\mu^{(n)}(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n, \quad n = 1, 2, \dots,$$

see e.g. [16], [1]. Assume that $K(x, y)$ is the integral kernel of a selfadjoint, locally trace class operator K in the (real or complex) space $L^2(\mathfrak{X}, m)$. Then, by [16], the corresponding determinantal point process exists if and only if $\mathbf{0} \leq K \leq \mathbf{1}$. (Note, however, that there are natural examples of determinantal point processes whose correlation kernel $K(x, y)$ is non-Hermitian, see, e.g., [2], [1].)

If we additionally assume that $K < \mathbf{1}$, i.e., 1 does not belong to the spectrum of K , then, as shown in [7], the corresponding determinantal point process μ is Gibbsian in a weak sense. More precisely, there exists a measurable function $r : \mathfrak{X} \times \Gamma \rightarrow [0, +\infty]$ such that

$$\int_{\Gamma} \mu(d\gamma) \sum_{x \in \gamma} F(x, \gamma) = \int_{\Gamma} \mu(d\gamma) \int_{\mathfrak{X}} m(dx) r(x, \gamma) F(x, \gamma \cup x) \quad (2)$$

for any measurable function $F : \mathfrak{X} \times \Gamma \rightarrow [0, +\infty]$. Here and below, for simplicity of notation, we just write x instead of $\{x\}$. Note that, in the theory of point processes, (2) is called *condition Σ'_m* , see [12].

It should be, however, emphasized that, in most applications, the selfadjoint operator K appears to be an orthogonal projection in $L^2(\mathfrak{X}, m)$, which is why the condition $K < \mathbf{1}$ is not satisfied.

1.2 Kawasaki dynamics

Informally, by a *Kawasaki dynamics* we mean a continuous time Markov process on Γ in which “particles” occupying positions $x \in \gamma$ randomly hop over the space

\mathfrak{X} . Such a dynamics should be described by the rate $c(\gamma, x, y)$ at which a particle occupying position x of configuration γ jumps to a new position y . We will be interested in *equilibrium dynamics*, which means that the process admits a symmetrizing (and hence invariant) measure μ , and we want to consider the time-reversible evolution preserving μ .

In the statistical mechanics literature one usually takes as \mathfrak{X} the lattice \mathbb{Z}^d , x and y are assumed to be neighboring sites of the lattice, and μ is a Gibbs measure.

Using the theory of Dirichlet forms, Kondratiev *et al.* [9] constructed an equilibrium Kawasaki dynamics with a continuous space \mathfrak{X} and a classical double-potential Gibbs measure of Ruelle type as the symmetrizing measure μ . This approach was extended in [11] to the case when μ is a determinantal point process. However, since the authors of [11] heavily used formula (2), their construction of the Kawasaki dynamics was restricted to the case of a selfadjoint operator K with $K < \mathbf{1}$.

Let us also note that, in [15] (in the case where \mathfrak{X} is a discrete space) and in [11], an equilibrium Glauber dynamics (i.e., a spacial birth-and-death process) was constructed which has a determinantal point process as symmetrizing measure. To this end, one again needed that $K < \mathbf{1}$. Under the same assumption, an equilibrium diffusion process for a determinantal measure was constructed in [17].

The purpose of the present note is to describe a general scheme for constructing an equilibrium Kawasaki dynamics in the case of a discrete space \mathfrak{X} . The crucial property of a measure μ on Γ which makes it possible to construct the dynamics, is the quasi-invariance of μ with respect to finitary permutations of \mathfrak{X} . We show that the construction can be applied to a family of determinantal measures μ whose correlation kernels K are projection kernels. Thus, at least in a concrete case we can remove the undesirable restriction $K < \mathbf{1}$.

More precisely, we will deal with the *gamma kernel measures* which were introduced and studied by Borodin and Olshanski in [3]. As shown there, these determinantal point processes arise from several models of representation-theoretic origin through certain limit transitions. The quasi-invariance property of the gamma kernel measures is established in [14]. It would be interesting to find other natural examples of discrete determinantal point processes possessing the quasi-invariance property.

It is worth noting that although, on abstract level, one can find some similarity between determinantal point processes and Gibbs measures, the Gibbs measure technique seems to be hardly applicable to determinantal measures. The main reason is that, in determinantal point processes, the interaction between “particles” is non-local. Note also that for lattice spin Gibbs measures (at least in the case of their uniqueness), the needed quasi-invariance property is obvious from the very definition, which is not the case for determinantal measures.

In the present note we employ the Dirichlet form approach, but, with the exception of a reference to the nontrivial abstract existence theorem for Hunt processes associated with regular Dirichlet forms [6], we manage with fairly easy and standard arguments.

2 Discrete point processes

From now on, we will assume that \mathfrak{X} is a countable set with discrete topology. Thus, a configuration in \mathfrak{X} is an arbitrary subset of \mathfrak{X} . We can therefore identify Γ with $\{0, 1\}^{\mathfrak{X}}$, so that a subset γ of \mathfrak{X} is identified with its indicator function. Then the vague topology on Γ is nothing else but the product topology on $\{0, 1\}^{\mathfrak{X}}$. Thus, Γ is a compact topological space.

Let m be the counting measure on \mathfrak{X} : $m(\{x\}) = 1$ for each $x \in \mathfrak{X}$. Let μ be a point process on \mathfrak{X} . Then, by (1),

$$k_{\mu}^{(n)}(x_1, \dots, x_n) = \mu(\gamma \in \Gamma : \{x_1, \dots, x_n\} \subset \gamma)$$

for distinct points $x_1, \dots, x_n \in \mathfrak{X}$, otherwise $k_{\mu}^{(n)}(x_1, \dots, x_n) = 0$. In this situation, the correlation functions uniquely identify the corresponding point process. If μ is determinantal, then its correlation kernel is simply a matrix with the row and columns indexed by points of \mathfrak{X} .

A permutation $\sigma : \mathfrak{X} \rightarrow \mathfrak{X}$ is said to be *finitary* if it fixes all but finitely many points in \mathfrak{X} . The simplest example of a nontrivial finitary permutation is the transposition $\sigma_{x,y}$, where x, y are distinct points in \mathfrak{X} ; by definition $\sigma_{x,y}$ permutes x and y and leaves invariant all other points. The finitary permutations form a countable group, which we denote as $\mathfrak{S}(\mathfrak{X})$. The transpositions $\sigma_{x,y}$ constitute a set of generators for $\mathfrak{S}(\mathfrak{X})$. The tautological action of the group $\mathfrak{S}(\mathfrak{X})$ on \mathfrak{X} gives rise to a natural action of this group on the space $\Gamma = \{0, 1\}^{\mathfrak{X}}$ by homeomorphisms:

$$(\sigma(\gamma))(x) := \gamma(\sigma^{-1}(x)), \quad \sigma \in \mathfrak{S}(\mathfrak{X}), \quad \gamma \in \Gamma, \quad x \in \mathfrak{X}.$$

Therefore, $\mathfrak{S}(\mathfrak{X})$ also acts on the set of all probability measures on Γ . A measure μ on Γ is said to be *quasi-invariant* with respect to the action of $\mathfrak{S}(\mathfrak{X})$ if for any element $\sigma \in \mathfrak{S}(\mathfrak{X})$ the measure μ is equivalent to $\sigma(\mu)$. As easily seen, it suffices to require that, for any transposition $\sigma_{x,y}$, the measure $\sigma_{x,y}(\mu)$ is absolutely continuous with respect to μ . (Note that, since $\sigma_{x,y}^2$ is the identity, the latter condition implies that the measure $\sigma_{x,y}(\mu)$ is equivalent to μ .)

If μ is a determinantal point process with correlation kernel $K(x, y)$, then, for any $\sigma \in \mathfrak{S}(\mathfrak{X})$, the measure $\sigma(\mu)$ is also determinantal, with correlation kernel $K^{\sigma}(x, y) = K(\sigma^{-1}(x), \sigma^{-1}(y))$. However, in the general case, it is not clear how to decide whether μ is equivalent to $\sigma(\mu)$ by looking at the kernels K and K^{σ} .

One can raise a more general question [14]: Given two determinantal measures on $\{0, 1\}^{\mathfrak{X}}$, how to test their equivalence (or, on the contrary, disjointness) by inspection of their correlation kernels? Note that a product measure μ on $\{0, 1\}^{\mathfrak{X}}$ is the determinantal point process with the correlation kernel $K(x, y)$ given by

$$K(x, y) = \begin{cases} \mu(\gamma : \gamma(x) = 1), & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

For product measures, the answer to the above question is well known: it is given by the classical Kakutani theorem [8].

3 Gamma kernel measures

The quickest way of introducing the gamma kernel is as follows (see [13]). We say that a couple (z, z') of complex numbers is *admissible* if

$$(z + n)(z' + n) > 0 \quad \text{for all } n \in \mathbb{Z}. \quad (3)$$

This condition is satisfied if

$$\begin{aligned} &\text{either } z \in \mathbb{C} \setminus \mathbb{Z} \text{ and } z' = \bar{z} \\ &\text{or there exists } m \in \mathbb{Z} \text{ such that } m < z, z' < m + 1. \end{aligned}$$

In what follows, we fix an admissible couple of parameters, (z, z') .

Next, we identify \mathfrak{X} with the lattice $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ of half-integers and consider the following second-order difference operation on the lattice \mathbb{Z}' :

$$\begin{aligned} (\mathcal{D}_{z, z'} f)(x) &= \sqrt{\left(z + x + \frac{1}{2}\right) \left(z' + x + \frac{1}{2}\right)} f(x + 1) - (2x + z + z') f(x) \\ &\quad + \sqrt{\left(z + x - \frac{1}{2}\right) \left(z' + x - \frac{1}{2}\right)} f(x - 1), \end{aligned}$$

where $x \in \mathbb{Z}'$ and $f(x)$ is a test function on \mathbb{Z}' . Note that $x \pm \frac{1}{2}$ is an integer for any $x \in \mathbb{Z}'$. Consequently, by virtue of (3), the quantities under the sign of square root are strictly positive, so that we may extract the positive square root.

Let $D_{z, z'}$ stand for the operator in $\ell^2(\mathbb{Z}')$ which is defined by the operation $\mathcal{D}_{z, z'}$ on the domain consisting of all $f \in \ell^2(\mathbb{Z}')$ such that $\mathcal{D}_{z, z'} f \in \ell^2(\mathbb{Z}')$. One can prove that $D_{z, z'}$ is selfadjoint and has simple, purely continuous spectrum filling the whole real axis.

Let $K_{z,z'}$ be the spectral projection associated with the selfadjoint operator $D_{z,z'}$ and corresponding to the positive part of the spectrum. That is, denoting by $Q(\cdot)$ the projection-valued measure on \mathbb{R} that governs the spectral decomposition of $D_{z,z'}$, we set $K_{z,z'} := Q((0, +\infty))$. We define $\mu_{z,z'}$ as the determinantal measure on Γ with the correlation kernel $K_{z,z'}(x, y)$ —the integral kernel of the operator $K_{z,z'}$.

As shown in [3], the kernel $K_{z,z'}(x, y)$ admits an explicit expression in terms of the classical Γ -function:

$$K_{z,z'}(x, y) = \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z - z'))} \cdot \frac{A(x)B(y) - B(x)A(y)}{x - y}, \quad x, y \in \mathbb{Z}',$$

where

$$A(x) = \frac{\Gamma(z + x + \frac{1}{2})}{\sqrt{\Gamma(z + x + \frac{1}{2})\Gamma(z' + x + \frac{1}{2})}}, \quad B(x) = \frac{\Gamma(z' + x + \frac{1}{2})}{\sqrt{\Gamma(z + x + \frac{1}{2})\Gamma(z' + x + \frac{1}{2})}}.$$

Note that the quantity under the sign of square root is always strictly positive. The above expression is well defined provided that $x \neq y$, $z \neq z'$. For $x = y$, one takes the formal limit as $y \rightarrow x$, which leads to

$$K_{z,z'}(x, x) = \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z - z'))} \left(\psi \left(z + x + \frac{1}{2} \right) - \psi \left(z' + x + \frac{1}{2} \right) \right), \quad x \in \mathbb{Z}',$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function. The definition for the case $z = z' \in \mathbb{R} \setminus \mathbb{Z}$ is obtained by taking the limits as $z' \rightarrow z$.

We call $K_{z,z'}(x, y)$ and $\mu_{z,z'}$ the *gamma kernel* (with parameters z, z') and the *gamma kernel measure*, respectively. For more detail about the gamma kernel measures and related measures on partitions (the so-called *z-measures*), see [2, 3, 4, 5].

Theorem 3.1 ([14]). *All gamma kernel measures $\mu_{z,z'}$ are quasi-invariant with respect to the action of the group $\mathfrak{S}(\mathbb{Z}')$.*

As shown in [14], the Radon–Nikodým derivative of $\sigma_{x,y}(\mu_{z,z'})$ relative to $\mu_{z,z'}$ admits an explicit expression. This expression involves an infinite product which converges only when γ belongs to a relatively meager subset of the whole configuration space Γ . Fortunately, this subset has full measure.

4 Construction of dynamics

Let again \mathfrak{X} be as in Section 2. Let μ be a point process on \mathfrak{X} which is quasi-invariant with respect to the action of $\mathfrak{S}(\mathfrak{X})$. Thus,

$$\text{for any distinct } x, y \in \mathfrak{X}, \text{ the measures } \mu \text{ and } \sigma_{x,y}(\mu) \text{ are equivalent} \quad (4)$$

Let \mathcal{C} denote the space of all cylinder functions on Γ , i.e., a function $F : \Gamma \rightarrow \mathbb{R}$ is in \mathcal{C} if and only if there exists a finite subset $\Lambda \subset \mathfrak{X}$ and a function $\tilde{F} : \{0, 1\}^\Lambda \rightarrow \mathbb{R}$ such that $F(\gamma) = \tilde{F}(\gamma_\Lambda)$, $\gamma \in \Gamma$, where γ_Λ is the restriction of γ to Λ . Note that each $F \in \mathcal{C}$ is continuous on Γ . Let $\tilde{\mathcal{C}}$ stand for the dense subspace in $L^2(\Gamma, \mu)$ formed by the images of the cylinder functions. If $\text{supp } \mu = \Gamma$, then $\tilde{\mathcal{C}}$ can be identified with \mathcal{C} . (Here and below $\text{supp } \mu$, the topological support of μ , is the smallest closed subset of full measure.)

Let $\tilde{\mathfrak{X}}^2 := \{(x, y) \in \mathfrak{X}^2 \mid x \neq y\}$. Let $c : \Gamma \times \tilde{\mathfrak{X}}^2 \rightarrow [0, \infty)$ be a measurable function satisfying the symmetry relation $c(\gamma, x, y) = c(\gamma, y, x)$. That is, given γ , $c(\gamma, x, y)$ depends on the *unordered* couple $\{x, y\}$. (Here and below all relations involving $c(\gamma, x, y)$ are assumed to hold for μ -a.a. $\gamma \in \Gamma$.) As will be clear from the formulas below, we will actually exploit only the restriction of the function c to the subset of those triples (γ, x, y) for which γ contains precisely one of the points x, y ; then, informally, $c(\gamma, x, y)$ is the rate of jump from position $\gamma \cap \{x, y\}$ to the new position $\{x, y\} \setminus (\gamma \cap \{x, y\})$. For this reason one can call $c(\gamma, x, y)$ the *rate function*.

For any $(x, y) \in \tilde{\mathfrak{X}}^2$ we define the operator $\nabla_{x,y}$ acting on functions $F(\gamma)$ according to formula

$$(\nabla_{x,y}F)(\gamma) := F(\sigma_{x,y}(\gamma)) - F(\gamma).$$

In accordance with the intuitive meaning of the rate function, we would like to define the generator of the future dynamics by the formula

$$-(AF)(\gamma) := \sum_{(x,y) \in \tilde{\mathfrak{X}}^2} c(\gamma, x, y) (\nabla_{x,y}F)(\gamma), \quad (5)$$

where F ranges over an appropriate space of functions on Γ . (We put the minus sign in the left-hand side for convenience, because we want A to be a nonnegative operator.) To make the definition rigorous we have to specify the domain of the operator, and we also have to impose suitable conditions on the rate function. Let us consider the following two conditions:

“Symmetry”: for any fixed $(x, y) \in \tilde{\mathfrak{X}}^2$, the measure $c(\gamma, x, y)\mu(d\gamma)$ is $\sigma_{x,y}$ -invariant. (6)

“ L^2 -condition”: for any fixed $x \in \mathfrak{X}$, $\sum_{y \in \mathfrak{X}, y \neq x} c(\cdot, x, y) \in L^2(\Gamma, \mu)$. (7)

The “symmetry condition” is analogous to the “detailed balance condition” for lattice spin systems of statistical mechanics; such a condition is necessary if we want the future Markov process to be symmetric (that is, reversible with respect to μ). The “ L^2 -condition” is a technical assumption; below we will also introduce a weaker condition, see (9).

Lemma 4.1. *Assume the rate function $c(\gamma, x, y)$ satisfies (6) and (7). Then the formula (5) correctly determines a nonnegative symmetric operator A in $L^2(\Gamma, \mu)$ with dense domain $\tilde{\mathcal{C}}$.*

Proof. Let F range over \mathcal{C} . If $F = 0$ μ -a.e., then, due to the quasi-invariance of μ , the same holds for $\nabla_{x,y}F$, so that the right-hand side of (5) also equals 0 μ -a.e. Thus, AF depends only on the image of F in $\tilde{\mathcal{C}}$.

Next, because of (7), $AF \in L^2(\Gamma, \mu)$. Indeed, write $F(\gamma) = \tilde{F}(\gamma_\Lambda)$ as above, with an appropriate finite subset $\Lambda \subset \mathfrak{X}$. Then $\nabla_{x,y}F$ vanishes when both x and y are outside Λ . Therefore, we may assume that at least one of the points x, y (say, x) is inside Λ . Since $|(\nabla_{x,y}F)(\gamma)|$ is bounded from above by a constant not depending on x, y and γ , we see from (7) that, for any fixed $x \in \Lambda$, the sum over y of the functions $c(\cdot, x, y)(\nabla_{x,y}F)(\cdot)$ is in $L^2(\Gamma, \mu)$. This is sufficient, because there are only finitely many $x \in \Lambda$.

Let us set

$$\mathcal{E}(F, G) := \frac{1}{2} \int_{\Gamma} \mu(d\gamma) \sum_{(x,y) \in \tilde{\mathfrak{X}}^2} c(\gamma, x, y) (\nabla_{x,y}F)(\gamma) (\nabla_{x,y}G)(\gamma), \quad F, G \in \tilde{\mathcal{C}}. \quad (8)$$

Using (6), it is easy to check that

$$(AF, G) = \mathcal{E}(F, G), \quad F, G \in \tilde{\mathcal{C}},$$

where (\cdot, \cdot) denotes the inner product in $L^2(\Gamma, \mu)$.

Finally, the fact that A is symmetric and nonnegative is evident, because the bilinear form (8) clearly possesses these properties. \square

Let us examine the expression (8). Observe that it still makes sense and correctly defines a symmetric bilinear form on $\tilde{\mathcal{C}} \times \tilde{\mathcal{C}}$ if we replace (7) by the weaker condition

$$\text{“}L^1\text{-condition”}: \text{ for each } x \in \mathfrak{X}, \quad \sum_{y \in \mathfrak{X}, y \neq x} c(\cdot, x, y) \in L^1(\Gamma, \mu); \quad (9)$$

the proof is the same as above. As for the symmetry condition (5), it is actually not restrictive: one can always modify the rate function, without changing $\mathcal{E}(F, G)$, in such a way that (5) will be fulfilled: For each (x, y) , we simply take the average of the measure $c(\cdot, x, y)\mu$ and its image under $\sigma_{x,y}$ (the resulting measure will remain absolutely continuous with respect to μ).

Lemma 4.2. *Under the “ L^1 -condition” (9), the form $(\mathcal{E}, \tilde{\mathcal{C}})$ defined by (8) is closable on $L^2(\Gamma, \mu)$.*

Proof. Note that under the stronger “ L^2 -condition” (7), the claim is evident, because the quadratic form corresponding to a symmetric operator is always closable. Without (7), the argument is slightly lengthier (cf. [6, Example 1.2.4]).

For any $F \in \mathcal{C}$, we abbreviate $\mathcal{E}(F) := \mathcal{E}(F, F)$. Let $(F_n)_{n=1}^\infty$ be a sequence in \mathcal{C} such that $\|F_n\|_{L^2(\Gamma, \mu)} \rightarrow 0$ as $n \rightarrow \infty$ and

$$\mathcal{E}(F_k - F_m) \rightarrow 0 \quad \text{as } k, m \rightarrow \infty. \quad (10)$$

To prove the closability of \mathcal{E} , it suffices to show that there exists a subsequence $(F_{n_k})_{k=1}^\infty$ such that $\mathcal{E}(F_{n_k}) \rightarrow 0$ as $k \rightarrow \infty$. Since $\|F_n\|_{L^2(\Gamma, \mu)} \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $(F_{n_k})_{k=1}^\infty$ such that $F_{n_k}(\gamma) \rightarrow 0$ as $k \rightarrow \infty$ for μ -a.a. $\gamma \in \Gamma$. Then, by (4), for any $(x, y) \in \tilde{\mathfrak{X}}^2$, $F_{n_k}(\sigma_{x,y}\gamma) \rightarrow 0$ as $k \rightarrow \infty$ for μ -a.a. $\gamma \in \Gamma$. Therefore, for any $(x, y) \in \tilde{\mathfrak{X}}^2$,

$$(F_{n_k}(\sigma_{x,y}\gamma) - F_{n_k}(\gamma)) \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad \text{for } \mu\text{-a.a. } \gamma \in \Gamma. \quad (11)$$

Now, by (11) and Fatou’s lemma,

$$\begin{aligned} 2\mathcal{E}(F_{n_k}) &= \sum_{(x,y) \in \tilde{\mathfrak{X}}^2} \int_{\Gamma} c(\gamma, x, y) (F_{n_k}(\sigma_{x,y}(\gamma)) - F_{n_k}(\gamma))^2 \mu(d\gamma) \\ &= \sum_{(x,y) \in \tilde{\mathfrak{X}}^2} \int_{\Gamma} c(\gamma, x, y) \left((F_{n_k}(\sigma_{x,y}(\gamma)) - F_{n_k}(\gamma)) - \lim_{m \rightarrow \infty} (F_{n_m}(\sigma_{x,y}(\gamma)) - F_{n_m}(\gamma)) \right)^2 \mu(d\gamma) \\ &\leq \liminf_{m \rightarrow \infty} \sum_{(x,y) \in \tilde{\mathfrak{X}}^2} \int_{\Gamma} c(\gamma, x, y) \left((F_{n_k}(\sigma_{x,y}(\gamma)) - F_{n_k}(\gamma)) - (F_{n_m}(\sigma_{x,y}(\gamma)) - F_{n_m}(\gamma)) \right)^2 \mu(d\gamma) \\ &= 2 \liminf_{m \rightarrow \infty} \mathcal{E}(F_{n_k} - F_{n_m}), \end{aligned}$$

which by (10) can be made arbitrarily small for k large enough. \square

We denote by $(\bar{\mathcal{E}}, \mathbb{D}(\bar{\mathcal{E}}))$ the closure of $(\mathcal{E}, \tilde{\mathcal{C}})$ on $L^2(\Gamma, \mu)$ (thus $\mathbb{D}(\bar{\mathcal{E}})$ is the domain of $\bar{\mathcal{E}}$). For the notions of a Dirichlet form and of a regular Dirichlet form, appearing in the following lemma, we refer to e.g. [6, Section 1.1].

Lemma 4.3. *Assume (9). Then the form $(\bar{\mathcal{E}}, \mathbb{D}(\bar{\mathcal{E}}))$ just defined is a regular Dirichlet form.*

Proof. For each $F \in \mathcal{C}$, we clearly have $(0 \vee F) \wedge 1 \in \mathcal{C}$ and $\mathcal{E}((0 \vee F) \wedge 1) \leq \mathcal{E}(F)$. Therefore, $(\mathcal{E}, \tilde{\mathcal{C}})$ is Markovian. Since this property is preserved under closing (see [6, Theorem 3.1.1]), the form $(\bar{\mathcal{E}}, \mathbb{D}(\bar{\mathcal{E}}))$ is Markovian, too. Hence it is a Dirichlet form. Finally, by the very construction, it is regular, because $\mathbb{D}(\bar{\mathcal{E}})$ includes $\tilde{\mathcal{C}}$, which is dense in the space of continuous functions on the compact space $\text{supp } \mu \subseteq \Gamma$. \square

Theorem 4.4. *Let, as above, μ be a $\mathfrak{S}(\mathfrak{X})$ -quasi-invariant probability measure on the configuration space $\Gamma = \Gamma_{\mathfrak{X}}$ and $c(\gamma, x, y) = c(\gamma, y, x)$ be a nonnegative measurable function satisfying (6) and (9). Then the corresponding form $(\bar{\mathcal{E}}, \mathbb{D}(\bar{\mathcal{E}}))$, as defined above, gives rise to a conservative symmetric Markov semigroup $\{T_t\}_{t \geq 0}$ in $L^2(\Gamma, \mu)$, which in turn determines a symmetric Hunt process on $\text{supp } \mu \subseteq \Gamma$.*

Proof. The existence of $\{T_t\}$ follows from the fact that $(\bar{\mathcal{E}}, \mathbb{D}(\bar{\mathcal{E}}))$ is a Dirichlet form [6, Section 1.3]. Conservativity holds because $\bar{\mathcal{E}}(1) = 0$. The existence of a Hunt process is a consequence of the regularity of the form, see [6, Chapter 7]. \square

Remark 4.5. If the rate function satisfies the “ L^2 -condition” (7), then one can say more. Let \hat{A} stand for infinitesimal generator of the semigroup $\{T_t\}$, so that $-\hat{A}$ is the nonnegative selfadjoint operator associated with the form $(\bar{\mathcal{E}}, \mathbb{D}(\bar{\mathcal{E}}))$. Then, by virtue of Lemma 4.1, \hat{A} is the Friedrichs’ extension of the symmetric operator A determined by (5). In the general case, however, the generator is determined implicitly and we cannot even say whether its domain contains \mathcal{C}^{∞} .

As an illustration, in the examples below we discuss 3 variants of choice of the rate function. Let us introduce some notation. Let

$$\varphi(\gamma, x, y) := \frac{\mu(\sigma_{x,y}(d\gamma))}{\mu(d\gamma)}, \quad \gamma \in \Gamma, (x, y) \in \tilde{\mathfrak{X}}^2, \quad (12)$$

stand for the Radon–Nikodým derivative. Note that

$$\varphi(\sigma_{x,y}(\gamma), x, y) = (\varphi(\gamma, x, y))^{-1}. \quad (13)$$

Next, observe that the symmetry condition (6) can be restated in the following form:

$$c(\gamma, x, y) = \varphi^{\frac{1}{2}}(\gamma, x, y) a(\gamma, x, y) \quad \text{with} \quad a(\gamma, x, y) = a(\sigma_{x,y}(\gamma), x, y). \quad (14)$$

Finally, fix an arbitrary function $u(x, y)$ on $\tilde{\mathfrak{X}}^2$ such that $u(x, y) = u(y, x) \geq 0$ and, for any fixed x , $\sum_y u(x, y) < \infty$. For instance, if \mathfrak{X} is the vertex set of a locally finite graph, one may suppose that $u(x, y) = 0$ unless $\{x, y\}$ is an edge.

Example 4.6. Set

$$c(\gamma, x, y) = u(x, y) \min(\varphi(\gamma, x, y), 1)$$

(compare with the well-known Metropolis dynamics). Equivalently,

$$a(\gamma, x, y) = u(x, y) \min\left(\varphi^{\frac{1}{2}}(\gamma, x, y), \varphi^{-\frac{1}{2}}(\gamma, x, y)\right),$$

which satisfies the required symmetry property by virtue of (13). For any (x, y) , the L^2 -norm of the function $c(\cdot, x, y)$ is less than or equal to $u(x, y)$. Consequently, the assumption on $u(x, y)$ guarantees the fulfilment of the “ L^2 -condition” (7).

Example 4.7. Set

$$c(\gamma, x, y) = u(x, y)\varphi^{\frac{1}{2}}(\gamma, x, y),$$

which means $a(\gamma, x, y) = u(x, y)$. Since the L^2 -norm of the function $\varphi^{\frac{1}{2}}(\cdot, x, y)$ equals 1, the L^2 -norm of $c(\cdot, x, y)$ equals $u(x, y)$, which again implies the “ L^2 -condition” (7).

Example 4.8. Set

$$c(\gamma, x, y) = u(x, y)(\varphi(\gamma, x, y) + 1),$$

which is equivalent to

$$a(\gamma, x, y) = u(x, y) \left(\varphi^{\frac{1}{2}}(\gamma, x, y) + \varphi^{-\frac{1}{2}}(\gamma, x, y) \right)$$

(compare with the Glauber dynamics discussed in [15]). In this case we cannot dispose of the “ L^2 -condition” (7); we cannot even claim that the function $\varphi(\cdot, x, y)$ certainly belongs to $L^2(\Gamma, \mu)$. Instead of this, we observe that the latter function has L^1 -norm 1, which implies the fulfillment of the “ L^1 -condition” (9). This weaker condition still makes it possible to apply Theorem 5, but gives a less precise description of the generator of the process.

However, in the concrete case when μ is one of the gamma kernel measures $\mu_{z, z'}$, it turns out that the function $\varphi(\cdot, x, y)$ does belong to $L^2(\Gamma, \mu)$. Thus, we can satisfy the “ L^2 -condition” (7) provided that $u(x, y)$ satisfies some additional assumptions (for instance, it suffices to require that for any fixed x , $u(x, y)$ vanishes for all but finitely many y 's). The fact that $\varphi(\cdot, x, y)$ is square integrable follows from the results of [14]: There it is proved that the Radon–Nikodým derivative $\varphi(\gamma, x, y)$ belongs to the algebra of functions on Γ spanned by the so-called multiplicative functionals; on the other hand, each such functional is integrable with respect to $\mu_{z, z'}$, hence any element of the algebra is integrable, which implies that $(\varphi(\cdot, x, y))^2$ is integrable.

Remark 4.9. As seen from the above examples, there is quite a lot of flexibility about the choice of the rate function. Of course, the rate function should be specified depending on a concrete problem. Finally, note that in the lattice spin models of statistical mechanics, due to short-range interaction of spins, the function $\varphi(\gamma, x, y)$ usually takes a simple form and depends only on a small finite part of the whole spin configuration γ . For the gamma kernel measures, the structure of $\varphi(\gamma, x, y)$ is a much more sophisticated.

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References

- [1] A. Borodin, *Determinantal point processes*. In: *The Oxford Handbook on Random Matrix Theory*, Gernot Akemann, Jinho Baik, and Philippe Di Francesco, eds. Oxford University Press, 2011, Chapter 11, 231–249; arXiv:0911.1153. MR2932631
- [2] A. Borodin and G. Olshanski, *Distributions on partitions, point processes and the hypergeometric kernel*, Comm. Math. Phys. **211** (2000), 335–358. MR1754518
- [3] A. Borodin and G. Olshanski, *Random partitions and the gamma kernel*, Adv. Math. **194** (2005), 141–202. MR2141857
- [4] A. Borodin and G. Olshanski, *Markov processes on partitions*, Probab. Theory Related Fields **135** (2006), 84–152. MR2214152
- [5] A. Borodin and G. Olshanski, *Meixner polynomials and random partitions*, Moscow Math. J. **6** (2006), 629–655, 771. MR2291156
- [6] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, 1994. MR1303354
- [7] H.-O. Georgii and H.J. Yoo, *Conditional intensity and Gibbsianness of determinantal point processes*, J. Stat. Phys. **118** (2005), 55–84. MR2122549
- [8] S. Kakutani, *On equivalence of infinite product measures*, Ann. Math. **49**, (1948), 214–224. MR0023331
- [9] Y. G. Kondratiev, E. Lytvynov, and M. Röckner, *Equilibrium Kawasaki dynamics of continuous particles systems*, Infin. Dimen. Anal. Quant. Prob. Rel. Top. **10** (2007), 185–210. MR2337519
- [10] A. Lenard, *Correlation functions and the uniqueness of the state in classical statistical mechanics*, Commun. Math. Phys. **30** (1973), 35–44. MR0323270
- [11] E. Lytvynov and N. Ohlerich, *A note on equilibrium Glauber and Kawasaki dynamics for fermion point processes*, Methods Funct. Anal. Topology **14** (2008), 67–80. MR2402154
- [12] K. Matthes, W. Warmuth, and J. Mecke, *Bemerkungen zu einer Arbeit von Nguyen Xuan Xahn und Hans Zessin*, Math. Nachr. **88** (1979), 117–127. MR0543397

- [13] G. Olshanski [Ol'shanskiĭ], *Difference operators and determinantal point processes*, *Funct. Anal. Appl.* **42** (2008), 317–329. MR2492429
- [14] G. Olshanski, *The quasi-invariance property for the Gamma kernel determinantal measure*. *Adv. Math.* **226** (2011), 2305-2350. MR2739779
- [15] T. Shirai and H. J. Yoo, *Glauber dynamics for fermion point processes*, *Nagoya Math. J.* **168**, (2002), 139–166. MR1942400
- [16] A. Soshnikov, *Determinantal random point fields*, *Russian Math. Surveys* **55** (2000), 923–975. MR1799012
- [17] H. J. Yoo, *Dirichlet forms and diffusion processes for fermion random point fields*, *J. Funct. Anal.* **219** (2005), 143–160. MR2108363