

Substitution rules and topological properties of the Robinson tilings

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Abstract A relatively simple substitution for the Robinson tilings is presented, which requires only 56 tiles up to translation. In this substitution, due to Joan M. Taylor, neighboring tiles are substituted by partially overlapping patches of tiles. We show that this overlapping substitution gives rise to a normal primitive substitution as well, implying that the Robinson tilings form a model set and thus have pure point diffraction. This substitution is used to compute the Čech cohomology of the hull of the Robinson tilings via the Anderson-Putnam method, and also the dynamical zeta function of the substitution action on the hull. The dynamical zeta function is then used to obtain a detailed description of the structure of the hull, relating it to features of the cohomology groups.

1 Introduction

Robinson's aperiodic set of tiles [8] was the first reasonably small such set which could tile the plane only aperiodically. The local matching rules enforce a hierarchical structure into the tilings, which is used to prove that only aperiodic tilings are admitted. Despite this hierarchical structure, for a long time it was not known whether the Robinson tilings can be generated also by a substitution, which would have enormous advantages for a more detailed study. Only very recently, a substitution for the Robinson tilings could be constructed explicitly [4], albeit a rather complicated one. The Robinson tilings therefore remain an interesting example, not only for historical reasons. In this paper, we present a much simpler substitution, derived from an overlapping substitution due to Joan M. Taylor, which we then use to analyse the structure of the hull of the Robinson tilings in more detail, and relate it to some of the topological invariants of the hull.

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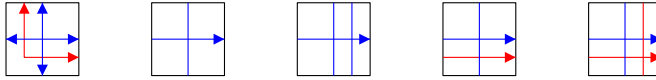
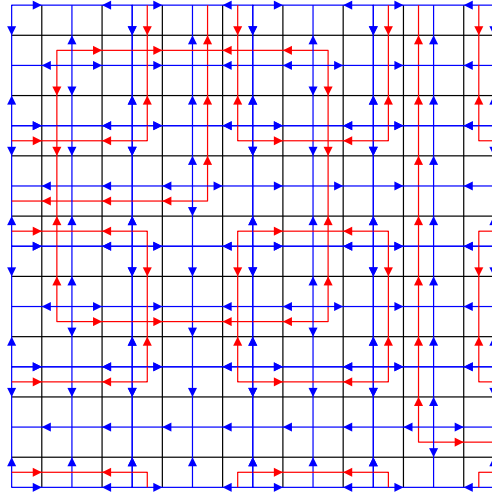


Fig. 1 The Robinson tiles. The one on the left is called a cross and plays a special role. All tiles can also be rotated and reflected.

Fig. 2 A patch of a Robinson tiling. Note the red square frames with corners at cross tiles, which occur at all sizes of the form 2^n , proving the aperiodicity of the Robinson tilings. The corners of the smallest square frames are at crosses forming the odd/odd sublattice of tiles.



2 A simple substitution for the Robinson tilings

Robinson tilings consist of the five square tiles shown in Fig. 1. As the tiles are allowed to be rotated and reflected, there are 28 tiles up to translation. In a legal Robinson tiling, the tiles must obey some local rules. Firstly, the decoration lines must continue across edges, with exactly one arrow head at each line join. Secondly, there must be a square sublattice of index 4 whose tiles are all cross tiles. Apart from this lattice of cross tiles, there may be other crosses as well. We assume in the following, that this sublattice of cross tiles is at the odd/odd position. All tilings satisfying the two rules (which are both local) are called Robinson tilings. In any Robinson tiling, the decoration lines form a hierarchy of square frames of all sizes 2^n (see Fig. 2), which proves that Robinson tilings cannot be periodic.

The local rules given above admit also some tilings with defect lines, which are not repetitive (for details, see [8, 5]). As we are heading at primitive substitution rules, by which we can reach only repetitive tilings, we want to discard these defective tilings. We therefore confine ourselves to the minimal subspace of repetitive tilings which is closed and invariant under translations and substitutions. The tilings which we discard form a set of measure zero. In particular, their exclusion does not change any spectral properties.

The hierarchy of square frames of all sizes (Fig. 2) suggests a hierarchical structure in the tilings, and it would only be natural if the Robinson tilings could be constructed also by a substitution rule. The construction of such a substitution rule

was achieved only recently [4], and the substitution proved to be rather complicated, with 208 tiles up to translation. The reason is, that the self-similarity inherent in the Robinson tilings scales around the tile centers, not the vertices. For the substitution, one therefore had to dissect and reassemble the original tiles to new ones, having their vertices at the original tile centers, which results in the rather large number of tiles.

Here, we want to follow a different route, starting from a proposal of Joan M. Taylor (private communication). Recall that the self-similarity scales about tile centers. The idea now is to replace a tile by a 3×3 -patch of tiles under the substitution. This patch is larger than the original tile inflated by a factor of 2, so that there are consistency conditions to be obeyed: the substitutions of neighboring tiles have an overlap, on which they must agree. A relatively simple solution is obtained if we pass to new tiles which are larger by a factor 2. These new tiles have their centers at the tiles at even/even positions (recall that the tiles at odd/odd positions are all crosses). If we add to those even/even tiles a layer of thickness one half, all the remaining tiles are consumed, and we end up with new square tiles of edge length 2 at even/even positions. It turns out that the 28 translation classes of tiles at even/even positions split up into two classes each, so that we now have 56 tile types up to translation. Moreover, these tiles admit a well-defined overlapping substitution, as shown in Fig. 3.

The overlapping substitution of Fig. 3 is considerably simpler than the one found previously [4]. The set of translation classes of tiles has been cut to a mere 56, from 208 previously. For certain applications, however, such as the computation of the cohomology via the Anderson-Putnam method [1], an overlapping substitution is not suitable. To avoid this problem, we observe that we can always pass to a normal substitution by replacing a tile not by a full 3×3 -patch, but by the 2×2 -subpatch at the upper right corner, say. Note that we always have to take the subpatch at the same corner, also for the rotated tiles, so that each tile is assigned to a unique supertile. As a result, this assignment breaks the rotation/reflection covariance of the substitution rules, but this is a small price to pay.

Having derived our substitution from an overlapping substitution has yet another advantage. Since the 3×3 -patches cover more than the inflated tiles, the overlapping substitution obviously forces the border [6], a property which is inherited also by the normal substitution derived from it. This allows to avoid the use of collared tiles [1] in the Anderson-Putnam method, which is a tremendous advantage, as it also helps to keep the number of tile types small.

3 The structure of the hull

Due to the repetitivity, the translation group acts minimally on the space of all repetitive Robinson tilings: every translation orbit is dense. The tiling space is therefore the hull of any of its member tilings. Having a substitution, the hull can now be constructed as an inverse limit space [1], and having a *simple* substitution which

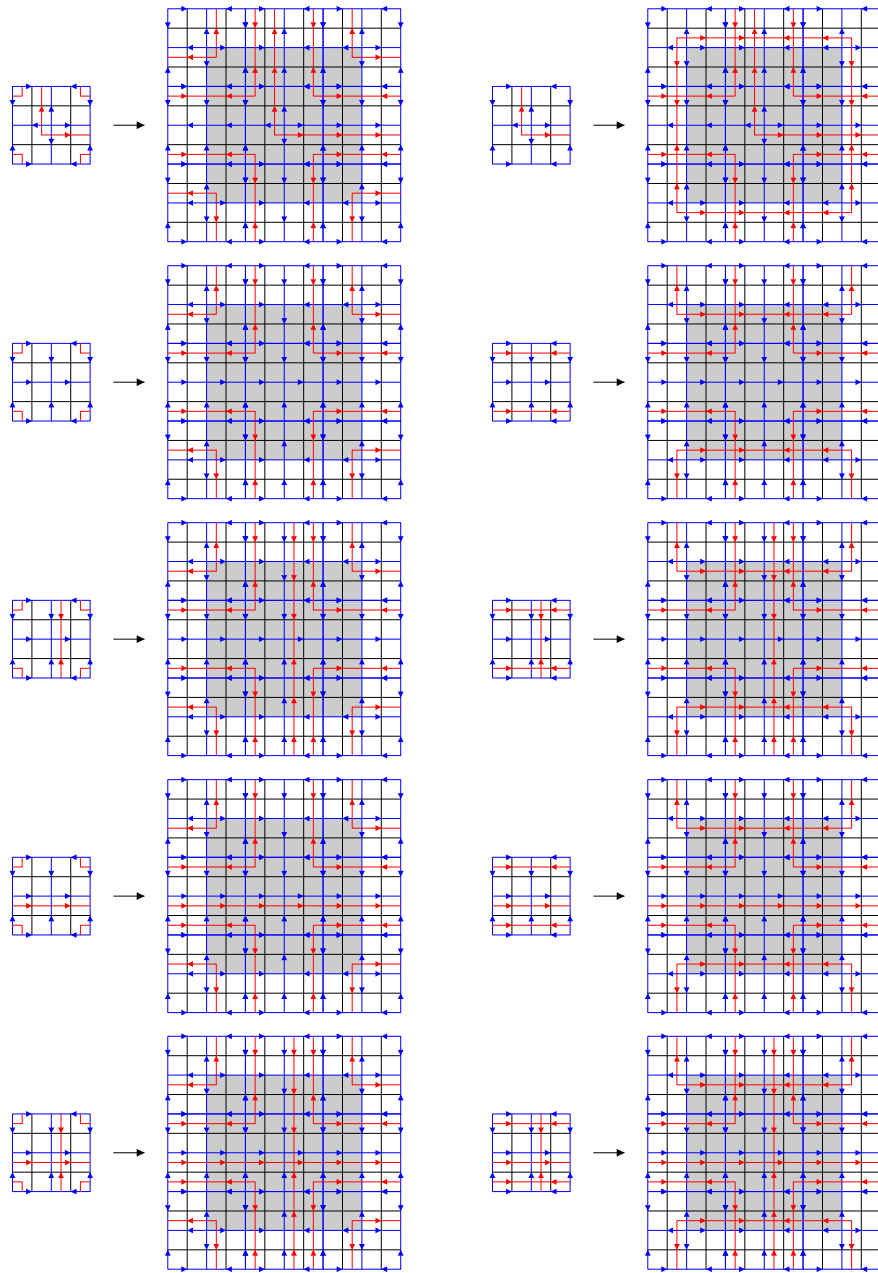


Fig. 3 Overlapping substitution for the Robinson tiling. Each tile is replaced by a 3×3 -patch of tiles. Rotated/reflected tiles are substituted by the corresponding rotated/reflected patches. The inflated tiles cover only the area shaded in gray. The substitutions of neighboring tiles have thus an overlap, on which they agree.

forces the border and requires only 56 tiles up to translation simplifies the task considerably. The mere fact of having a lattice substitution tiling has some immediate consequences. Since the crosses at odd/odd positions form a lattice-periodic subset of tiles (with period 4 in each direction), results of Lee and Moody [7] allow to conclude that the Robinson tilings form a model set and are thus pure-point diffractive. Since the defective Robinson tilings are a subset of measure zero, the pure-point diffractiveness extends even to all Robinson tilings.

As a limit-periodic model set, the space of Robinson tilings must project 1-to-1 almost everywhere to an underlying 2d, 2-adic solenoid \mathbb{S}_2^2 via the torus parametrisation [2]. In the following, we will analyse the structure of the set where this projection fails to be 1-to-1, and try to connect it to the Čech cohomology of the hull. The latter was obtained in [4] via the Anderson-Putnam method [1] as

$$H^2 = \mathbb{Z}[\frac{1}{4}] \oplus \mathbb{Z}[\frac{1}{2}]^{10} \oplus \mathbb{Z}^8 \oplus \mathbb{Z}_4, \quad H^1 = \mathbb{Z}[\frac{1}{2}]^2 \oplus \mathbb{Z}, \quad H^0 = \mathbb{Z}, \quad (1)$$

which is confirmed using our new, simpler substitution. There is a natural substitution action on the hull, whose Artin-Mazur zeta function is defined as

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{a_m}{m} z^m\right) \quad (2)$$

where a_m is the number of points in the hull that are invariant under an m -fold substitution. Note that if the hull consists of two components for which the periodic points can be counted separately, $a_m = a'_m + a''_m$, the corresponding partial zeta functions have to be multiplied: $\zeta(z) = \zeta'(z) \cdot \zeta''(z)$.

Anderson and Putnam have given a different way to compute the dynamical zeta function, as a by-product of computing the Čech cohomology [1]. Recall that the hull is obtained as the inverse limit of the substitution acting on an approximant cell complex. As a consequence, the cohomology of the hull is the direct limit of the substitution action on the cohomology of that cell complex. Suppose $A^{(m)}$ is the matrix of the substitution action on the m -th cohomology group (with rational coefficients) of the hull of a substitution tiling. The dynamical zeta function is then given by [1]

$$\zeta(z) = \frac{\prod_{k \text{ odd}} \det(1 - zA^{d-k})}{\prod_{k \text{ even}} \det(1 - zA^{d-k})} = \frac{\prod_{k \text{ odd}} \prod_i (1 - z\lambda_i^{d-k})}{\prod_{k \text{ even}} \prod_i (1 - z\lambda_i^{d-k})} \quad (3)$$

where the latter equality holds if the matrices $A^{(m)}$ diagonalizable, and the $\lambda_i^{(m)}$ are their eigenvalues. Note that Anderson and Putnam have used the matrices of the substitution action on the cochain groups of the approximant complex, rather than the cohomology, but the additional terms in their formula cancel between numerator and denominator.

If we apply this to the Robinson tilings, and take into account the eigenvalues of the substitution action on the cohomology, we obtain for the zeta function

$$\zeta(z) = \frac{(1-2z)^2(1-z)}{(1-z)(1-4z)(1-2z)^{10}(1-z)^8} \quad (4)$$

$$= \frac{(1-2z)^2}{(1-z)(1-4z)} \cdot \left(\frac{1-z}{1-2z}\right)^{10} \cdot \frac{1}{(1-z)^{17}}, \quad (5)$$

where in the second line we have written the zeta function as the product of the zeta functions of one 2d solenoid \mathbb{S}_2^2 , ten 1d solenoids \mathbb{S}_2 , and 17 extra fixed points.

How can this be interpreted? A Robinson tiling generically consists of a single, infinite order supertile. Such tilings project 1-to-1 to the solenoid \mathbb{S}_2^2 . However, a Robinson tiling can consist also of two infinite order supertiles, which are separated by a horizontal or vertical row of tiles without any crosses. These are the tilings where the projection to \mathbb{S}_2^2 is not 1-to-1. A separating row of tiles can be decorated with a single blue line, or a double line with the second line (red) on either side of the middle blue line, and all three cases can be combined with arrows in one or the other direction. All six possibilities, everything else being the same, project to the same point on \mathbb{S}_2^2 . Moreover, if we take the translation orbit along the defect line, we obtain a whole 1d sub-solenoid \mathbb{S}_2 of such 6-tuples. So, in addition to the 2d solenoid \mathbb{S}_2^2 , the hull contains 5 extra 1d solenoids \mathbb{S}_2 in horizontal and 5 in vertical direction. Further, there are 28 fixed points of the substitution, consisting of 4 infinite order supertiles, which all project to the origin of \mathbb{S}_2^2 . The 2d solenoid and the 10 extra 1d solenoids contain one such fixed point each, so that in addition to those there must be 17 further ones, which all show up in the zeta function (5). We finally note that the structure of the hull is in line with the interpretation of [3], where terms $\mathbb{Z}[\frac{1}{2}]$ in H^2 are associated with extra 1d sub-solenoids \mathbb{S}_2 in the hull.

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