

Distinguished bases of exceptional modules.

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An indecomposable representation M of a quiver $Q = (Q_0, Q_1)$ is said to be exceptional provided $\text{Ext}^1(M, M) = 0$. And it is called a tree module provided one can choose a set \mathcal{B} of bases of the vector spaces M_x ($x \in Q_0$) such that the coefficient quiver $\Gamma(M, \mathcal{B})$ is a tree quiver; we call \mathcal{B} a tree basis of M . It is known that exceptional modules are tree modules. Now a tree module usually has many tree bases and the corresponding coefficient quivers may look quite differently. The aim of this note is to introduce a class of exceptional modules which have a distinguished tree basis, the “radiation modules”. For a Dynkin quiver Q , nearly all indecomposable representations turn out to be radiation modules, the only exception is the maximal indecomposable module in case \mathbb{E}_8 . Also, the exceptional representation of the generalized Kronecker quivers are given by radiation modules. Consequently, with the help of Schofield induction one can display all the exceptional modules of an arbitrary quiver in a nice way.

Let $Q = (Q_0, Q_1)$ be a locally finite quiver. We will consider finite-dimensional representations of Q (thus kQ -modules, where kQ is the path algebra of Q). An indecomposable representation M of Q is said to be *exceptional* provided $\text{Ext}^1(M, M) = 0$. And it is called a *tree module* provided one can choose a set \mathcal{B} of bases of the vector spaces M_x ($x \in Q_0$) such that the coefficient quiver $\Gamma(M, \mathcal{B})$ (see [R2]) is a tree quiver; we call \mathcal{B} a *tree basis* of M . It is known that exceptional modules are tree modules. But even if we know that a module M is a tree module, it often seems to be difficult to find directly a tree basis. The aim of this note is to provide for the exceptional modules an algorithm for obtaining a tree basis. The first modules which we will consider are what we call the radiation modules, see section 2. They are inductively defined, in any step one constructs an exceptional module with a thin vertex. Note that we say that a vertex x is thin for the module M provided the vector space M_x is one-dimensional. In section 1 we are analyzing exceptional modules with a thin vertex.

As we will see in section 3, nearly all indecomposable representation of a Dynkin quiver are radiation modules, the only exception is the maximal indecomposable module in case \mathbb{E}_8 . The radiation modules which are our main concern are the preprojective and preinjective representation of a tree without leafs with bipartite orientation (a leaf of a tree is a vertex with a single neighbor). These modules are studied in sections 4. Typical examples of trees without leafs are the n -regular trees with $n \geq 2$, note that they occur

as the universal cover of a generalized Kronecker quiver. Thus, using covering theory, one obtains a distinguished tree basis for any exceptional representation of a generalized Kronecker module. With the help of Schofield induction we then get a nice tree basis for any exceptional module, see the outline in the last section 5.

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1. Some exceptional representations M with a thin vertex.

If M is a representation of M and $\dim M_x = 1$ for some vertex x , we say that x is a *thin* vertex for M .

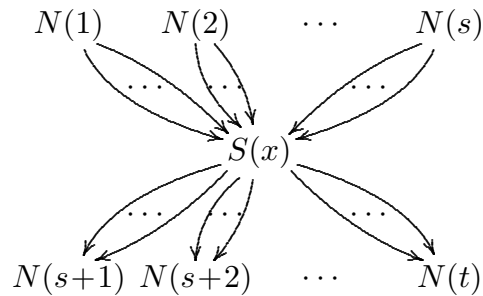
Let Q be a quiver with underlying graph \overline{Q} being a tree. Let x be a vertex of Q and let Q^x be obtained from Q by deleting the vertex x and all the arrows involving x .

We say that a family of indecomposable modules $N(1), \dots, N(t)$ is an *exceptional family*, provided the modules are pairwise non-isomorphic and $\text{Ext}^1(N(i), N(j)) = 0$ for all i, j . The family $N(1), \dots, N(t)$ is an *orthogonal family* provided $\text{Hom}(N(i), N(j)) = 0$ for all $i \neq j$.

Proposition 1. (a) *Let $N(1), \dots, N(t)$ be an orthogonal exceptional family of indecomposable modules with $N(i)_x = 0$ and assume that for any index i , there is a neighbor $y(i)$ of x with $\dim N(i)_{y(i)} = n(i) > 0$. Then there is an exceptional representation M of Q , unique up to isomorphism, with thin vertex x such that the restriction of M to Q^x is $M' = \bigoplus_{i=1}^t N(i)^{n(i)}$.*

(b) *Conversely, let M be an exceptional representation of Q , and let x be a vertex of Q with $M_x \neq 0$. Then the restriction M' of M to Q^x is of the form $M' = \bigoplus_{i=1}^t N(i)^{n(i)}$ with $n(i) > 0$, where $N(1), \dots, N(t)$ is an exceptional family of indecomposable modules. For any index i , there is a unique neighbor $y(i)$ of x with $\dim N(i)_{y(i)} \neq 0$. If $\dim M_x = 1$, then $n(i) \leq \dim N(i)_{y(i)}$ for all $1 \leq i \leq t$. If $\dim M_x = 1$ and $\dim N(i)_{y(i)} = 1$ for $i \in I$, where I is a subset of $\{1, 2, \dots, t\}$, then the modules $N(i)$ with $i \in I$ form an orthogonal family.*

Proof: (a) Let $N(1), \dots, N(t)$ be an orthogonal exceptional family of indecomposable modules, with $N(i)_x = 0$, and assume that for any index i , there is a neighbor $y(i)$ of x with $\dim N(i)_{y(i)} = n(i) > 0$. We can assume that there are arrows $y(i) \rightarrow x$ for $1 \leq i \leq s$ and arrows $x \rightarrow y(i)$ for $s+1 \leq i \leq t$. Since $N(i)_x = 0$, we see that the modules $S(x), N(1), \dots, N(t)$ are orthogonal bricks with the following Ext-quiver:



with $n(i)$ arrows between $N(i)$ and $S(x)$, for $1 \leq i \leq t$.

Now such a quiver has the following real root:

$$\begin{array}{ccccccc}
n(1) & n(2) & \cdots & n(s) & & & \\
& & & & 1 & & \\
n(s+1) & n(s+2) & \cdots & n(t) & & &
\end{array}$$

and the process of simplification shows that the category of representations of Q which have a filtration with factors in $\{S(x), N(1), \dots, N(t)\}$ has precisely one indecomposable object with one factor $S(x)$ and $n(i)$ factors $N(i)$, for $1 \leq i \leq t$, and that this is an exceptional module.

(b) We use the following lemma.

Lemma 1. *Let X' be a submodule of X and $X'' = X/X'$. We assume that $\text{Ext}^1(X, X) = 0$ and $\text{Hom}(X', X'') = 0$. Then $\text{Ext}^1(X', X') = 0$ and $\text{Ext}^1(X'', X'') = 0$.*

Proof: The inclusion map $X' \rightarrow X$ yields an epimorphism $\text{Ext}^1(X, X) \rightarrow \text{Ext}^1(X', X)$, thus $\text{Ext}^1(X', X) = 0$. Applying $\text{Hom}(X', -)$ to the exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ yields an exact sequence

$$\text{Hom}(X', X'') \rightarrow \text{Ext}^1(X', X') \rightarrow \text{Ext}^1(X', X).$$

Since the end terms are zero, also the middle term is zero. This shows that $\text{Ext}^1(X', X') = 0$. By duality, also $\text{Ext}^1(X'', X'') = 0$.

Assume now that M is an exceptional representation of Q . Let y_1, \dots, y_n be the neighbors of x , say with arrow $y_i \rightarrow x$ for $1 \leq i \leq r$ and $x \rightarrow y_i$ for $r+1 \leq i \leq n$. The quiver Q^x is the disjoint union of two parts Q^+ and Q^- , where Q^- is the union of the connected components of Q^x which contain the vertices y_1, \dots, y_r , and Q^+ is the union of the connected components of Q^x which contain the vertices y_{r+1}, \dots, y_n . Let M^+ be the restriction of M to Q^+ and M^- the restriction of M to Q^- . Note that M^+ is a submodule of M , whereas M^- is a factor module of M , say $M^- = M/X$. Thus, M has the following chain of submodules $M^+ \subseteq X \subseteq M$ and X/M^+ is a direct sum of copies of $S(x)$. Now we apply Lemma 1 to the submodule X of M , this is possible, since X and M/X has disjoint support, thus $\text{Hom}(X, M/X) = 0$. We see that $\text{Ext}^1(X, X) = 0$ and $\text{Ext}^1(M^-, M^-) = 0$. Next, we apply Lemma 1 to the submodule M^+ of X , again using that we deal with modules M^+ and X/M^+ with disjoint support. We conclude that $\text{Ext}^1(M^+, M^+) = 0$. Since there is no arrow between vertices of Q^- and Q^+ , thus $\text{Ext}^1(M^-, M^+) = 0 = \text{Ext}^1(M^+, M^-)$. Note that $M' = M^+ \oplus M^-$. Altogether we have shown that $\text{Ext}^1(M', M') = 0$, thus the modules $N(1), \dots, N(t)$ form an exceptional family.

Now consider a module $N(i)$. Since the support of M is connected, and M is indecomposable, at least one of the vertices y_j must belong to the support of $N(i)$. Since \overline{Q} is a tree, there is just one such vertex. This shows that there is a unique neighbor $y(i)$ of x such that $N(i)_{y(i)} \neq 0$, namely $y(i) = y_j$.

From now on, let us assume that $\dim M_x = 1$. Let us show that $n(i) \leq \dim N(i)_{y(i)}$ for all $1 \leq i \leq t$. We assume that for $1 \leq i \leq s$, the arrow between x and $y(i)$ ends in x , whereas for $s+1 \leq i \leq t$, it starts in x . First, consider some i with $s+1 \leq i \leq t$,

say $i = r + 1$. Then $\dim \text{Ext}^1(S(x), N(r + 1)) = \dim N(r + 1)_{y(r+1)}$. There is the exact sequence

$$0 \rightarrow N(r + 1)^{n(r+1)} \oplus X' \rightarrow X \rightarrow S(x) \rightarrow 0 \quad \text{with} \quad X' = \bigoplus_{i=r+2}^t N(i)^{n(i)}.$$

If $n(i) > \dim \text{Ext}^1(S(x), N(r + 1))$, then X splits off a copy of $N(r + 1)$ (see the Ext-Lemma in [R3]). But this is impossible, since X is indecomposable. The dual argument works for $1 \leq i \leq r$. This shows that $n(i) \leq \dim N(i)_{y(i)}$ for all indices $1 \leq i \leq t$.

In particular, we see that $\dim N(i)_{y(i)} = 1$ implies $n(i) = 1$. Now assume in addition that there are two different indices i, j with $\dim N(i)_x = 1 = \dim N(j)_x$, say $i = 1, j = 2$. We want to show that $\text{Hom}(N(1), N(2)) = 0$. Thus, assume that $\text{Hom}(N(1), N(2)) \neq 0$. Then the support of $N(1)$ and $N(2)$ is contained in the same connected component of Q^x , thus $y(1) = y(2)$. Let $y = y(1) = y(2)$.

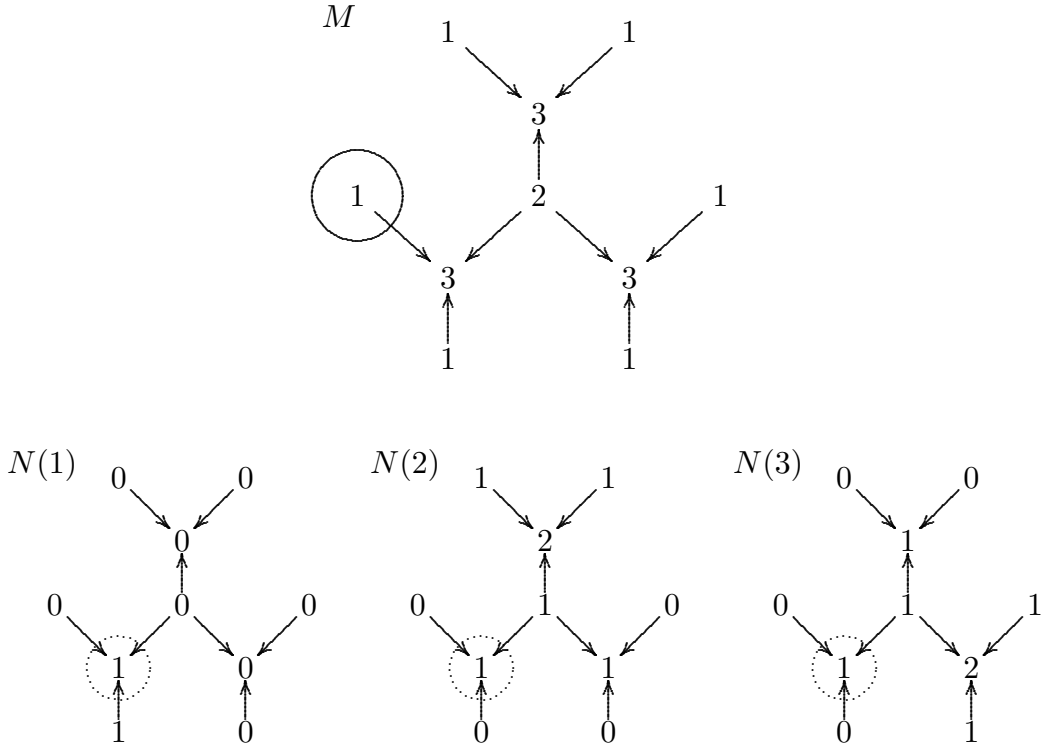
Let us assume that the arrow α between x and y starts in x . Since $\dim M_x = 1$, we can assume that $M_x = k$ and consider the element $M_\alpha(1) \in M_y$. Note that

$$M_y = M'_y = \bigoplus_{i=1}^t N(i)^{n(i)}_y = N(1)_y \oplus N(2)_y \oplus N'_y \quad \text{with} \quad N' = \bigoplus_{i=2}^t N(i)^{n(i)}_y,$$

thus we can write $M_\alpha(1) = (a_1, a_2, a_3)$ with $a_1 \in N(1)_y$, $a_2 \in N(2)_y$, $a_3 \in N'_y$. Note that both a_1, a_2 are non-zero, since otherwise M would split off $N(1)$ or $N(2)$, respectively. Let $\phi: N(1) \rightarrow N(2)$ be a non-zero homomorphism. Since $\text{Ext}^1(N(2), N(1)) = 0$, we know that ϕ has to be a monomorphism or an epimorphism. It follows from $\dim N(1)_y = 1 = \dim N(2)_y$, that ϕ_y is bijective. Replacing if necessary ϕ by a scalar multiple, we can assume that $\phi_y(a_1) = a_2$. Let $N'' \subseteq N(1) \oplus N(2)$ be the graph of ϕ , thus $N''_z = \{(b, \phi(b)) \mid b \in N(1)_z\}$ for all vertices z of Q . The module N'' is a submodule of $N(1) \oplus N(2)$ which has $0 \oplus N(2)$ as a direct complement inside $N(1) \oplus N(2)$. Since we see that M_α maps M_x into $N''_y \oplus N'_y$, it follows that M splits off $N(2)$, namely $M = Y \oplus N(2)$, where $Y_x = M_x$ and the restriction of Y to Q^x is equal to $N'' \oplus N'$. This contradicts the indecomposability of M .

We use the dual considerations in case the arrow between x and y starts in y and terminates in x . This concludes the proof.

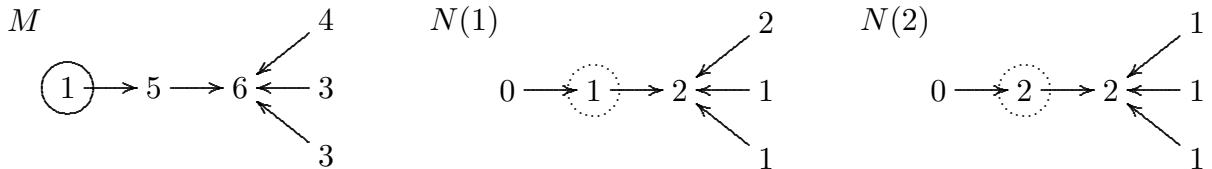
Example 1. Consider the following exceptional representation M , encircled is a vertex x with $\dim M_x = 1$, note that x has a unique neighbor y . The lower line exhibits the dimension vectors of the modules $N(i)$, here the vertex y is encircled by a dotted line.



(A related representation, namely the representation $P(c, 3)$ for the 3-regular tree with bipartite orientation and c a source, will be discussed in detail at the end of section 4. The representation M considered above is the restriction of $P(c, 3)$ to the ball with center c and radius 2.)

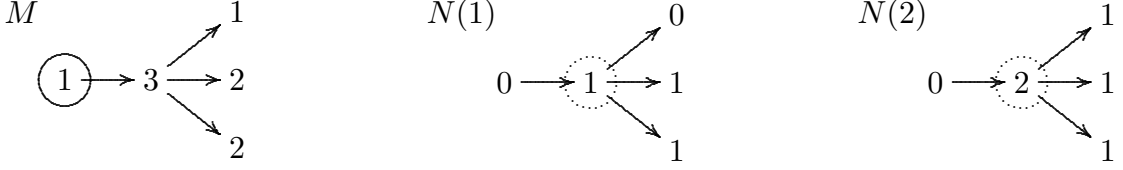
We mainly will be interested in exceptional modules M with a thin vertex x such that the restriction M' to Q^x decomposes as the direct sum of indecomposable modules $N(i)$ with $\dim N(i)_{y(i)} = 1$, so that also $n(i) = 1$. But let us exhibit here two examples with $\dim N(i)_{y(i)} > 1$ for some index i .

Example 2. An exceptional module M with a thin vertex x such that the restriction M' of M to Q^x is of the form $N(1) \oplus N(2)^2$ for an exceptional orthogonal family $N(1), N(2)$.



Example 3. An exceptional module M with a thin vertex x such that the restriction M' of M to Q^x is of the form $N(1) \oplus N(2)$ for an exceptional family $N(1), N(2)$ which is

not orthogonal.



Here, we have $\dim \text{Hom}(N(1), N(2)) = 1$ and $\dim N(1)_y = 1$, but $\dim N(2)_y = 2$.

2. Radiation modules.

Let us assume that we deal with a quiver Q which is a tree. We consider pairs (M, x) where M is an indecomposable module and x is a vertex with $\dim M_x = 1$ (called the *origin* of M or better of the pair (M, x)). We define the class of *radiation modules* as well as the corresponding *radiation quivers* inductively as follows.

First of all, for any vertex x , the pair $(S(x), x)$ is a radiation module, its radiation quiver $R(S(x), x)$ is the quiver with a single vertex (and no arrow), the vertex being labeled x . Second, if M is an indecomposable module of length at least 2 and x is a vertex with $\dim M_x = 1$, then (M, x) is a radiation module provided the restriction of M to the quiver obtained from Q by deleting the vertex x is the direct sum $\bigoplus N(i)$ of indecomposable modules $N(i)$ such that for any index i there is a neighbor $y(i)$ of x with $(N(i), y(i))$ being a radiation module. The radiation quiver $R(M, x)$ of (M, x) is obtained from the disjoint union of the radiation quivers $R(N(i), y(i))$ by adding a vertex with label x and for every index i we connect x and $y(i) \in R(N(i), y(i))_0$ by an arrow which points in the same direction as the arrow in Q between x and $y(i)$, and we use this arrow of Q also as the label of the arrow in $R(M, x)$. Note that the labels of the vertices and the arrows of $R(M, x)$ provide a quiver homomorphism $R(M, x) \rightarrow Q$.

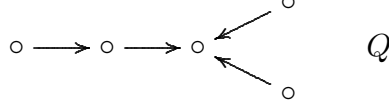
Let (M, x) be a radiation module, and b a non-zero element of M_x (note that M_x is one-dimensional). For any radiation module (M, x) we define a basis $\mathcal{B}(M, x)$ as follows, we call it a *radiation basis* containing b . Assume that the restriction of M to the quiver obtained by deleting the vertex x is $\bigoplus N(i)$ with indecomposable representations $N(i)$. For every index i , there is a vertex $y(i)$ which is a neighbor of x such that $(N(i), y(i))$ is a radiation module. Since $x, y(i)$ are neighbors, there is an arrow say α which connects them. If $\alpha: x \rightarrow y(i)$, let $\mathcal{B}(i)$ be a radiation basis of $(N(i), y(i))$ containing the element $M_\alpha(b)$. If $\alpha: y(i) \rightarrow x$, let $\mathcal{B}(i)$ be a radiation basis of $(N(i), y(i))$ containing the element $M_\alpha^{-1}(b)$. Finally, let $\mathcal{B}(M, x)$ be the set containing the element b as well as all the elements in the (disjoint!) sets $\mathcal{B}(i)$.

Proposition 2. *If (M, x) is a radiation module, then M is exceptional. A radiation basis $\mathcal{B}(M, x)$ of M is a tree basis and the corresponding coefficient quiver is $R(M, x)$.*

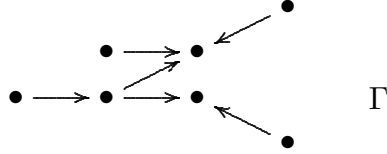
Proof. The first assertion follows from the proposition 1(a). Also the remaining assertions are straight-forward: Namely, assume that there is given a finite set of trees $T(i)$ and for every i a vertex $y(i)$ in $T(i)$. If we take the disjoint union of the graphs $T(i)$, an additional vertex x and connect x with the vertices $y(i)$ by a single edge, then we obtain

again a tree. It remains to show that $R(M, x)$ is a coefficient quiver of M . By induction we see that $\mathcal{B}(M, x)$ is a basis of M and the corresponding coefficient quiver is just $R(M, x)$.

Example 4. A tree basis which is not a radiation basis. We deal with the quiver Q of type \mathbb{D}_5 with subspace orientation.



Let M be the maximal indecomposable module. We denote by Γ the following coefficient quiver of M :



There are three vertices x with $\dim M_x = 1$, namely the leaves of the quiver Q . Removing any such vertex x , the module $M|(Q \setminus \{x\})$ is the direct sum of two indecomposable modules, thus if $R(M, x)$ is a radiation quiver of M , then there are two arrows starting in x . But note that dealing with Γ , only one arrow starts at any leaf. This shows that Γ cannot occur as a radiation quiver.

Properties of a radiation quiver. Let us start with a general (and well-known) property of exceptional modules:

Lemma 2. *Let M be a representation of the quiver Q with $\text{Ext}^1(M, M) = 0$. Then for every arrow α in Q , the linear map M_α is injective or surjective.*

Proof: Let $\alpha: x \rightarrow y$ be an arrow such that M_α is neither injective nor surjective. Write $M_x = M'_x \oplus M''_x$ with M'_x the kernel of M_α and $M_y = M'_y \oplus M''_y$ with M'_y the image of M_α . By assumption M'_x and M''_y are non-zero, thus there is a non-zero linear map $N: M_x \rightarrow M_y$ such that $N(M'_x) \subseteq M''_y$ and $N(M''_x) = 0$.

In order to construct a non-trivial self-extension of M , start with the direct sum $M \oplus M$ and replace the map $(M \oplus M)_\alpha = \begin{bmatrix} M_\alpha & 0 \\ 0 & M_\alpha \end{bmatrix}$ by the map $\begin{bmatrix} M_\alpha & N \\ 0 & M_\alpha \end{bmatrix}$. The properties of N show that the rank of this map is larger than the rank of $(M \oplus M)_\alpha$, thus the new representation cannot be isomorphic to $M \oplus M$, and obviously, it is an extension of M by itself.

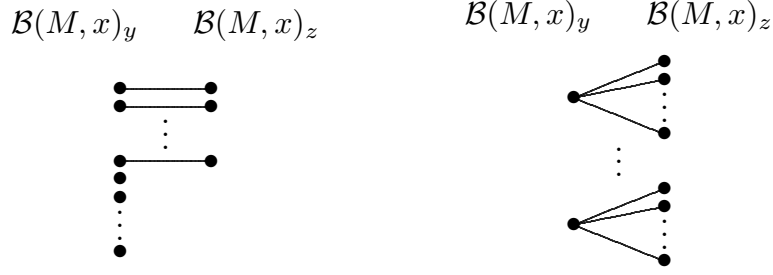
Proposition 3. *Let (M, x) be a radiation module with radiation basis $\mathcal{B}(M, x)$. Consider an edge $\{y, z\}$ in \overline{Q} such that $d(x, y) + 1 = d(y, z)$. Then any element of $\mathcal{B}(M, x)_z$ is connected to precisely one element of $\mathcal{B}(M, x)_y$ by an arrow.*

If $\dim M_y \leq \dim M_z$, then any element of $\mathcal{B}(M, x)_y$ is connected to at least one element of $\mathcal{B}(M, x)_z$. If $\dim M_y \geq \dim M_z$, then any element of $\mathcal{B}(M, x)_y$ is connected to at most one element of $\mathcal{B}(M, x)_z$.

Note that here we consider a path

$$x \text{ --- } \dots \text{ --- } y \xrightarrow{\alpha} z$$

in the tree \overline{Q} . It follows that the edges between $\mathcal{B}(M, x)_y$ and $\mathcal{B}(M, x)_z$ are of one of the following form:



Corollary. *Let (M, x) be a radiation module. Consider an edge $\{y, z\}$ in \overline{Q} such that $d(x, y) + 1 = d(y, z)$. Then the number of arrows between $\mathcal{B}(M, x)_y$ and $\mathcal{B}(M, x)_z$ is precisely $\dim M_z = |\mathcal{B}(M, x)_z|$.*

Proof of Proposition 3. Write $\mathcal{B} = \mathcal{B}(M, x)$. First, we claim that any element of \mathcal{B}_z is connected to precisely one element of \mathcal{B}_y by an arrow. This is an immediate consequence of the way the radiation quiver is constructed. Namely, any vertex b of $R(M, x)$ occurs at a certain step as an origin. In the next step, this vertex b is connected by an arrow to the new origin, say b' . In later steps, no further arrows are attached which involve b . Thus, let $d(x, z) = i$ and let D be the maximal distance of vertices to x . We start with step 0, namely with the disjoint union of radiation quivers consisting of a single vertex, one for each vertex with distance D to x . The basis \mathcal{B}_z is constructed in step $D - i$, all the elements of \mathcal{B}_z are origins of radiation quivers. In the step $D - i + 1$, the basis \mathcal{B}_y is constructed: here, any vertex $b \in \mathcal{B}_z$ is connected by an edge to precisely one vertex $b' \in \mathcal{B}_y$.

For the second assertion, we have to take into account the orientation of the arrow α which connects y and z , as well as whether M_α is injective or surjective.

(1) *If α starts at y and M_α is injective, then any vertex in \mathcal{B}_y is connected by an edge to at least one vertex in \mathcal{B}_z .* Proof: If $b \in \mathcal{B}_y$ is not connected by an edge to a vertex in \mathcal{B}_z , then b belongs to the kernel of M_α .

(2) *If α starts at y and M_α is surjective, then any vertex in \mathcal{B}_y is connected by an edge to at most one vertex in \mathcal{B}_z .* Proof: If there are vertices $b' \neq b''$ in \mathcal{B}_z which are connected by edges to $b \in \mathcal{B}_y$, then neither b' nor b'' belong to the image of M_α .

(3) *If α starts at z and M_α is injective, then any vertex in \mathcal{B}_y is connected by an edge to at most one vertex in \mathcal{B}_z .* Proof: If there are vertices $b' \neq b''$ in \mathcal{B}_z which are connected by edges to $b \in \mathcal{B}_y$, then $b' - b''$ belongs to the kernel of M_α .

(4) *If α starts at z and M_α is surjective, then any vertex in \mathcal{B}_y is connected by an edge to at least one vertex in \mathcal{B}_z .* Proof: If $b \in \mathcal{B}_y$ is not connected by an edge to a vertex in \mathcal{B}_z , then b is not in the image of M_α .

Example 4, revisited. Observe that there is an arrow $\alpha: y \rightarrow z$ such that $\dim M_y = \dim M_z = 2$. The matrix presentation given by Γ involves 3 non-zero coefficients for M_α , whereas Proposition 3 and its Corollary assert that for a radiation basis $\mathcal{B}(M, x)$ only two non-zero coefficients do occur.

3. Dynkin quivers.

Proposition 4. *Let Q be a Dynkin quiver and M an indecomposable module with a thin vertex x . Then (M, x) is a radiation module.*

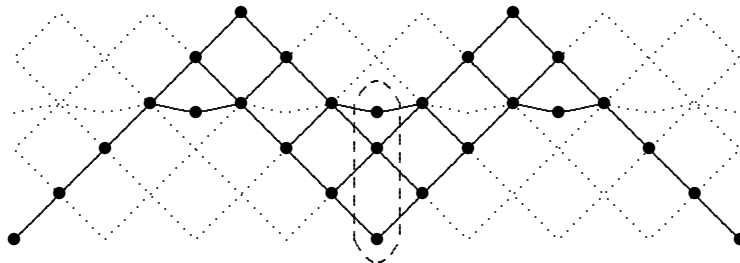
This is an immediate consequence of the following lemma.

Lemma 3. *Let Q be a Dynkin quiver and M an indecomposable module with thin vertex x . Let M' be the restriction of M to the quiver obtained by deleting the vertex x . Then M' decomposes as the direct sum of orthogonal indecomposable modules $N(i)$. For every index i , there is a unique neighbor $y(i)$ of x with $\dim N(i)_{y(i)} \neq 0$ and we have $\dim N(i)_y = 1$.*

Proof of lemma 3. For a Dynkin quiver, all indecomposable representations are exceptional, thus we can apply proposition 1(b) and decompose $M' = \bigoplus N(i)^{n(i)}$ where $N(1), \dots, N(t)$ is an exceptional family. For every i , there is a unique neighbor y of x with $\dim N(i)_y \neq 0$. If $\dim N(i)_{y(i)} \geq 2$ for some i , then the process of simplification shows that the full subcategory of all modules with a filtration with factors $S(x)$ and $N(i)$ is representation infinite, impossible. This shows that the modules $N(i)$ form an orthogonal exceptional family.

Note that for Δ a Dynkin quiver, the only exceptional modules without a thin vertex are the maximal indecomposable modules for the quivers of type \mathbb{E}_8 .

Example 5. Consider the quiver of type \mathbb{E}_8 with subspace orientation and let M be the maximal indecomposable representation with a thin vertex, say the vertex x . Then x has a unique neighbor y and $\dim M_y = 3$. It follows that the restriction M' of M to Q^x is the direct sum of 3 indecomposable modules $N(1), N(2), N(3)$. The easiest way to determine these modules $N(i)$ explicitly is as follows: Consider the hammock H_y for the path algebra kQ^x , this is the support of the functor $\text{Hom}(P(y), -)$ and has the following shape



(it is a poset with partial ordering going from left to right). There is only one triple of pairwise incomparable elements, it is surrounded by a dashed line. The corresponding

modules $N(1), N(2), N(3)$ are the direct summands of M' which we are looking for:

$$\begin{array}{ccc}
 & & N(1) \\
 & & \begin{array}{ccccccc} & & & & 1 & & \\ & & & & \downarrow & & \\ 0 & \rightarrow & \textcircled{1} & \rightarrow & 1 & \rightarrow & 2 \rightarrow 2 \leftarrow 1 \leftarrow 1 \end{array} \\
 \\
 M & & N(2) \\
 \begin{array}{ccccccc} & & & & 3 & & \\ & & & & \downarrow & & \\ \textcircled{1} & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 \rightarrow 6 \leftarrow 4 \leftarrow 2 \end{array} & & \begin{array}{ccccccc} & & & & 2 & & \\ & & & & \downarrow & & \\ 0 & \rightarrow & \textcircled{1} & \rightarrow & 2 & \rightarrow & 2 \rightarrow 3 \leftarrow 2 \leftarrow 1 \end{array} \\
 \\
 & & N(3) \\
 & & \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \rightarrow & \textcircled{1} & \rightarrow & 1 & \rightarrow & 1 \rightarrow 1 \leftarrow 1 \leftarrow 0 \end{array}
 \end{array}$$

4. Bipartite trees without leaves

Let us now draw the attention to infinite quivers, but we will assume that the quivers are locally finite without infinite paths. For a general discussion of the Auslander-Reiten components of infinite quiver we may refer to [BLP].

The quiver Q is said to be *bipartite* provided that every vertex is a sink or a source. Note that any graph which is a tree can be endowed with precisely two orientations so that we obtain a bipartite tree quiver. A vertex x of a tree quiver is called a *leaf* provided there is at most one arrow α which starts or ends in x . Note that a tree quiver without leaves has to be infinite. As typical examples one should take the n -regular tree with bipartite orientation, where $n \geq 2$. These are the quivers which we later will use when dealing with the generalized Kronecker quivers. We should stress that the case $n = 3$ was already discussed in [FR].

If Q is a quiver, let Q^* be the opposite quiver: it has the same vertices, but every arrow $\alpha: x \rightarrow y$ is replaced by an arrow $\alpha^*: y \rightarrow x$.

Let Q be a bipartite tree quiver. Given two vertices x, y , let $d(x, y)$ be the distance between x and y . We denote by $N(x)$ the set of neighbors of x (thus the set of vertices y with $d(x, y) = 1$).

If $x \in Q_0, t \in \mathbb{N}_0$, let $B(x, t)$ be the set of vertices y with $d(x, y) \leq t$, we may call it the *ball* with center x and radius t . If necessary, we will consider $B(x, t)$ as a full subquiver of Q . We will be interested in the balls $B(x, t)$ where x is a sink and t is even, as well in those where x is a source and t is odd. Note that for these pairs x, t , the boundary $\{y \mid d(x, y) = t\}$ of $B(x, y)$ consists of sinks (if x is a sink, then y is also a sink iff $d(x, y)$ is even. If x is a source, then y is a sink iff $d(x, y)$ is odd).

We denote by ρ^- the composition of all BGP-reflection functors at all the sources of Q , this is a functor $\rho^-: \text{mod } kQ \rightarrow \text{mod } kQ^*$. Note that $\rho^- \rho^-$ is just the Auslander-Reiten translation τ^- , see Gabriel [G] (here we use that \overline{Q} has no cyclic paths of odd length).

Given a bipartite tree quiver Q without leaves, let us define indecomposable representations $P(x, t) = P_Q(x, t)$ for certain $x \in Q_0, t \in \mathbb{N}_0$ as follows: If x is a sink, let $P(x, 0) = S(x) = P(x)$, the one-dimensional representation with support $\{x\}$. If x is a source, let $P(x, 1) = P(x)$, the indecomposable projective module with top $S(x)$; it is the

unique thin module with support $B(x, 1)$. For $t \geq 2$, define $P(x, t) = \tau^- P(x, t - 2)$. Thus, by definition

$$\begin{aligned} P(x, 2t) &= \tau^{-t} P(x, 0), \\ P(x, 2t + 1) &= \tau^{-t} P(x, 1). \end{aligned}$$

Looking at the same time at the quivers Q and Q^* , we see that for all $t \geq 1$

$$P_Q(x, t) = \rho^- P_{Q^*}(x, t - 1).$$

We call the modules of the form $P(x, t)$ the *preprojective* modules.

Lemma 4.

- (a) *The support of $P(x, t)$ is $B(x, t)$.*
- (b) *If $t \geq 1$ and $d(x, y) \in \{t - 1, t\}$, then $\dim P(x, t)_y = 1$.*
- (c) *Let X be an indecomposable representation of Q . Then $\text{Hom}(X, P(x, t)) \neq 0$ if and only if X is preprojective and the support of X is contained in the support of $P(x, t)$.*
- (d) *Let $P = P(x, t)$ and z a vertex with $P_z = 0$. Then $d(x, z) = t + 1$ if and only if $\text{Ext}^1(S(z), P) \neq 0$ and then $\dim \text{Ext}^1(S(z), P) = \dim P_y = 1$.*

Proof: (b) We use induction, the case $t = 0$ being trivial. Let $t \geq 1$. We know that $P_Q(x, t) = \rho^- P_{Q^*}(x, t - 1)$. Let y be a vertex of Q with $d(x, y) = t$. There is a unique neighbor y' of y with $d(x, y') = t - 1$ and $\dim P_{Q^*}(x, t - 1)_{y'} = 1$ (this is clear for $t = 1$, whereas for $t > 1$ this is the induction hypothesis). The definition of ρ^- shows that $P(x, t)_y = P_{Q^*}(x, t - 1)_{y'}$, thus $\dim P(x, t)_y = 1$. If $t \geq 1$ and y' is any vertex of Q with $d(x, y') = t - 1$, then ρ^- does not change the vector space at the position y' , thus $\dim P(x, t)_{y'} = \dim P_{Q^*}(x, t - 1)_{y'} = 1$.

(a) The induction procedure mentioned in (b) shows that the support of $P(x, t)$ is contained in $B(x, t)$, and as we have seen, it contains all the vertices y with $d(x, y) = t$. But the support of an indecomposable module has to be connected, thus it has to be all of $B(x, t)$.

(c) Let X be an indecomposable representation of Q with $\text{Hom}(X, P(x, t)) \neq 0$. We write $P(x, t) = \tau^{-s} P$ for some indecomposable projective module (namely $P = \tau^s P(x, t)$ and $s = \lfloor \frac{t}{2} \rfloor$). If $\tau^s X \neq 0$, then $\text{Hom}(\tau^s X, P) \neq 0$ and therefore $\tau^s X$ is projective. Altogether, we see that $X = \tau^{-i} P'$ for some indecomposable projective module $P' = P(y)$ and $0 \leq i \leq s$. Since $\text{Hom}(P', \tau^{-s+i} P) = \text{Hom}(P', \tau^i P(x, t)) \neq 0$, it follows that y belongs to the support of $\tau^i P(x, t)$, thus to the support $B(x, t)$ of $P(x, t)$.

Conversely, consider the quiver $B(x, t)$. Since this is a tree quiver, the category of representations of $B(x, t)$ has a preprojective component. Let $\mathcal{C}(x, t)$ be the full subcategory of the indecomposable representations of $B(x, t)$ which are predecessors of $P(x, t)$, we may consider this as a subcategory of the category of representations of Q . As we have seen, any predecessor X of $P(x, t)$ in the category $\text{mod } kQ$ belongs to $\mathcal{C}(x, t)$.

(d) Let $P = P(x, t)$ and z a vertex with $P_z = 0$ and $\text{Ext}^1(S(z), P) \neq 0$. Since $P_z = 0$, we must have $d(x, z) > t$. Since $\text{Ext}^1(S(z), P) \neq 0$, we see that $d(x, z) = t + 1$. Then $\dim \text{Ext}^1(S(z), P) = \dim P_y = 1$, according to (b).

Here is a description of the Auslander-Reiten sequences starting in $P(x, t)$

$$0 \rightarrow P(x, t) \rightarrow \bigoplus_{y \in N(x)} P(y, t+1) \rightarrow P(x, t+2) \rightarrow 0.$$

We recall the inductive definition of the *reachable* objects of a length category: First of all, the simple projective objects are reachable. Second, if M is indecomposable, but not simple projective, then M is reachable provided that there exists a minimal right almost split map $M' \rightarrow M$ such that all the indecomposable direct summands of M' are reachable.

Proposition 5. *The preprojective modules are reachable and they form a component of the Auslander-Reiten quiver.*

It is of interest to stress the following property of this preprojective component \mathcal{P} : for any indecomposable object X in \mathcal{P} , there are sectional paths starting in a simple projective module S and ending in X , namely, there is a sectional paths from $S = P(y, 0)$ to $X = P(x, t)$, provided $d(x, y) = t$.

Proposition 6. *For $t \geq 1$, the pairs $(P(x, t), y)$ with $d(x, y) \in \{t, t-1\}$ are radiation modules.*

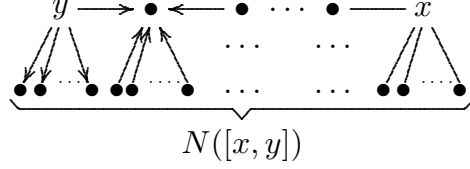
Proof. According to Lemma 4(b), we know that the vertices y with $d(x, y) \in \{t, t-1\}$ are thin vertices for $P(x, t)$. For the proof of the proposition, we use induction with respect to t , the case of $t = 1$ being obvious.

Thus, consider for $t \geq 2$ the pairs $(P(x, t), y)$ with $d(x, y) \in \{t, t-1\}$. The essential case to deal with is $d(x, y) = t-1$. Namely, if y is a vertex with $d(x, y) = t$, then there is a unique neighbor y' of y inside the support of $P(x, y)$, and we have both $\dim P(x, t)_y = 1$ and $\dim P(x, t)_{y'} = 1$. Note that $d(x, y') = t-1$. If we know that $(P(x, t), y')$ is a radiation module, then clearly also $(P(x, t), y)$ is a radiation module.

Let us assume now that $d(x, y) = t-1$. Let U be the submodule of $P(x, t)$ with $P(x, t)/U = S(y)$, this is just the restriction of $P(x, t)$ to Q^y . We write $U = \bigoplus_{i \in I} N(i)^{n(i)}$, with pairwise non-isomorphic indecomposable modules $N(i)$ and integers $n(i) \geq 1$. Since U is a submodule of $P(x, t)$, we know that these submodules $N(i)$ are again preprojective modules, see Lemma 4(c). Also, any such module $N(i)$ satisfies $N(i)_y = 0$. Of course, we also have $\text{Ext}^1(S(y), N(i)) \neq 0$, and the extensions in $\text{Ext}^1(S(y), N(i))$ are furnished by an arrow $y \rightarrow y(i)$ of Q , namely by the unique arrow which connects y with the support of $N(i)$. According to Lemma 4(d), we know that $\dim N(i)_{y(i)} = 1$ for all $i \in I$. Thus, we can apply Proposition 1(b) in order to see that the modules $N(i)$ form an orthogonal exceptional family. It remains to look at the modules $N(i)$. First of all, if $y(i)$ is a neighbor of y with $d(x, y(i)) = t$, then $N(i) = S(y(i))$ and the pair $(N(i), y(i))$ is a trivial radiation module. For the remaining modules $N(i)$ we have $d(x, y(i)) = t-2$. Since $N(i)$ is preprojective, it is of the form $N(i) = P(x', t')$ for some vertex x' and $t' \leq t$. Since the support of $N(i)$ is properly contained in the support of $P(x, t)$, we even have $t' < t$. The vertex $y(i)$ has to belong to the boundary of the support of $N(i)$. By induction we know that $(N(i), y(i))$ is a radiation module. Thus we see that $(P(x, t), y)$ is a radiation module.

We have seen in the proof of Proposition 6 that for a source y and $d(x, y) = t-1$, the kernel of the canonical epimorphism $P(x, t) \rightarrow S(y)$ is a direct sum of orthogonal bricks. Let us describe this kernel in more details.

Given a pair x, y of vertices of Q , we denote by $[x, y]$ the set of vertices lying on the path between x and y . If S is a set of vertices of Q , we denote by $N(S)$ the set of neighbors of S : these are the vertices z which do not belong to S but such that there is a vertex $z' \in S$ with $d(z, z') = 1$. We will be interested in the sets $N([x, y])$, where x, y are vertices with y a source:



(This picture reminds on centipedes.)

We consider vertices $y \neq z$, where y is a source, and look at the modules $S(y)$ and $P(z, d(y, z) - 1)$.

Lemma 5. *Let y be a source and $z \neq y$ some other vertex. Then $S(y), P(z, d(y, z) - 1)$ is an orthogonal pair with*

$$\begin{aligned} \dim \text{Ext}^1(P(z, d(y, z) - 1), S(y)) &= 0 \\ \dim \text{Ext}^1(S(y), P(z, d(y, z) - 1)) &= 1. \end{aligned}$$

Proof. According to Lemma 4(a), we have $P(z, d(y, z) - 1)_y = 0$, thus the pair $S(y), P(z, d(y, z) - 1)$ is orthogonal. We have $\text{Ext}^1(P(z, d(y, z) - 1), S(y)) = 0$, since $S(y)$ is injective. According to Lemma 4(d), we have $\dim \text{Ext}^1(S(y), P(z, d(y, z) - 1)) = 1$.

If $d(x, y) = t - 1$ and we look at $z \in N([x, y])$ (thus z is one of the legs of the centipedes), then we have

$$d(y, z) - 1 = t - d(x, z).$$

(Namely, let z' be the neighbor of z which belongs to $[x, y]$, then $d(y, z) - 1 = d(y, z')$ and $t - d(x, z) = 1 + d(x, y) - d(x, z') - 1 = d(x, y) - d(x, z') = d(y, z')$.) This explains that we will have to consider modules $P(z, t')$ with $t' = t - d(x, z)$.

Proposition 7. *Let y be a source and x any vertex. Let $t = d(x, y) + 1$. Then there is the following exact sequence:*

$$0 \rightarrow \bigoplus_{z \in N([x, y])} P(z, t - d(x, z)) \rightarrow P(x, t) \rightarrow S(y) \rightarrow 0$$

The modules $S(y)$ and $P(z, t - d(x, z))$ with $z \in N([x, y])$ are pairwise orthogonal bricks and the map $P(x, t) \rightarrow S(y)$ is the projective cover of $S(y)$ in the full subcategory

$$\mathcal{F}\left(S(y); P(z, t - d(x, z)), z \in N([x, y])\right).$$

Proof. We use induction on t . In the case $t = 1$, we have $x = y$ and there is the exact sequence

$$0 \rightarrow \bigoplus_{z \in N(x)} P(z, 0) \rightarrow P(x, 1) \rightarrow S(x) \rightarrow 0.$$

Now assume the assertion is true for some $t \geq 1$. Consider a pair of vertices x, z_0 with $d(x, z_0) = t$. There is a unique vertex y with $d(x, y) = t - 1$ and $d(y, z_0) = 1$. By induction, there is the exact sequence

$$(*) \quad 0 \rightarrow \bigoplus_{z \in N([x, y])} P(z, t - d(x, z)) \rightarrow P(x, t) \rightarrow S(y) \rightarrow 0.$$

Choose some neighbor z_0 of y with $d(x, z_0) = t$. There is (up to isomorphism) a unique module M with top $S(y)$ and socle $S(z_0)$ and we may rearrange the factors in $(*)$ in order to obtain an exact sequence of the form

$$0 \rightarrow \bigoplus_{z \in N([x, y]), z \neq z_0} P(z, t - d(x, z)) \rightarrow P(x, t) \rightarrow M \rightarrow 0.$$

Now apply the functor ρ^- . We obtain the sequence

$$(**) \quad 0 \rightarrow \bigoplus_{z \in N([x, y]), z \neq z_0} P(z, t + 1 - d(x, z)) \rightarrow P(x, t + 1) \rightarrow \rho^- M \rightarrow 0.$$

The module $\rho^- M$ has top $S(z_0)$ and its socle is the direct sum of the modules $S(z)$ where $z \neq y$ is a neighbor of z_0 . Of course, for these vertices z , we have $S(z) = P(z, 0)$ and $d(x, z) = t + 1$. Since these modules $P(z, 0)$ are projective, we obtain from $(**)$ an exact sequence of the form

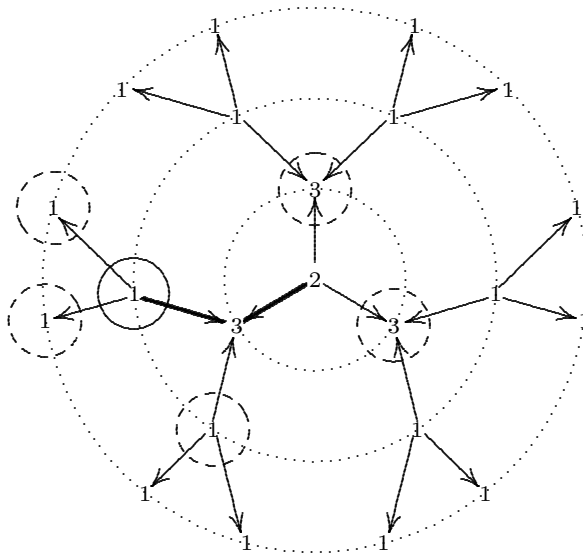
$$0 \rightarrow U \rightarrow P(x, t + 1) \rightarrow S(z_0) \rightarrow 0$$

where U is the direct sum of the modules $P(z, t + 1 - d(x, z))$ with $z \in N([x, y]), z \neq z_0$ as well as those of the form $P(z, 0) = P(z, t + 1 - d(x, z))$ with $z \neq y$ a neighbor of z_0 . It remains to observe that the elements in $N([x, z_0])$ are precisely the vertices $z \in N([x, y]), z \neq z_0$ and the neighbors $z \neq y$ of z_0 . Thus we see that we obtain for the pair x, z_0 the required sequence.

Now let us show that the modules $S(y)$ and $P(z, t - d(x, z))$ and $z \in N([x, y])$ are orthogonal and that the only non-trivial extension groups between these modules are the groups $\text{Ext}^1(S(y), P(z, t - d(x, z)))$ and these are one-dimensional. We can assume that $t \geq 2$ and we denote by y' the unique vertex with $d(y, y') = 1$ and $d(y', x) = t - 2$. If $d(x, z) = t$, then $P(z, t - d(x, z)) = P(z, 0) = S(z)$ is simple projective, not isomorphic to $S(y)$ and $\dim \text{Ext}^1(S(y), S(z)) = 1$. Also, for $z \neq z'$, the modules $S(z), S(z')$ are orthogonal. Thus, assume that $d(x, z) < t$. Let $z' \in [x, y]$ with $d(z, z') = 1$. Write $a = d(y, z')$ and $b = d(z', x)$, thus $a + b = t - 1$ and $d(z, x) = b + 1$, thus $t - d(x, z) = a$. Since $d(x, z) < t$, we see that $z' \neq y$, thus $d(z, y') = a$ and therefore $P(z, a)_y = 0$ and $\dim P(z, a)_a = 1$. According to Lemma 4(d), we have $\dim \text{Ext}^1(S(y), P(z, a)) = 1$. It follows from $P(z, a)_y = 0$ that $P(z, a)_{z'} = 0$ for $d(x, z') = t$, thus $P(z, a)$ and $P(z', 0)$ are orthogonal, and also that $\text{Ext}^1(P(z, a), P(z', 0)) = 0$. Assume now that we deal with two vertices $z_1 \neq z_2$ in $N([x, y])$ with $d(x, z_2) \leq d(x, z_1) < t$, let $a_1 = t - d(x, z_1)$, and $a_2 = t - d(x, z_2)$, thus $a_1 \leq a_2$. In order to see that there are no homomorphisms or extensions between $P(z_1, a_1)$ and $P(z_2, a_2)$, we can apply ρ^{a_1} , thus we have to consider $P(z_1, 0)$ and $P(z_2, a_2 - a_1)$. Note that $d(z_1, z_2) = a_2 - a_1 + 2$, thus z_1 is not in the support of $P(z_2, a_2 - a_1)$ and not even a neighbor of this support. This completes the proof.

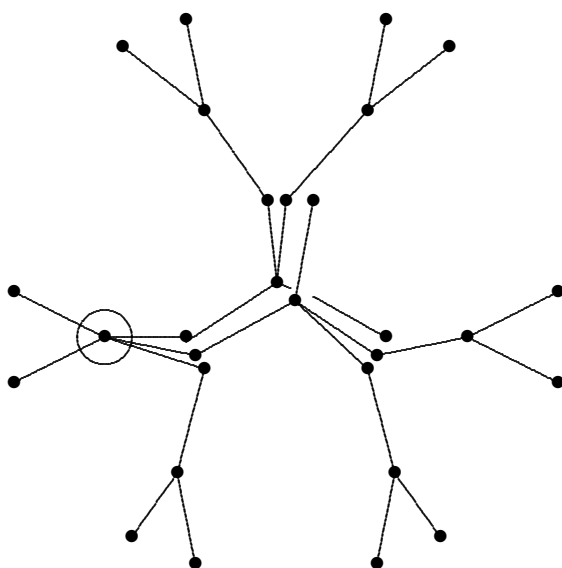
Corollary. *Let y be a source and x any vertex. The family of modules $P(z, d(y, z) - 1)$ with $z \in N([x, y])$ is an exceptional orthogonal family of bricks.*

As an example, let us consider the 3-regular tree with bipartite orientation as studied already in [FR]. Let us display the dimension vector $\mathbf{dim} P(x, 3)$ for a source x . As vertex y we take a source with $d(x, y) = 2$, thus $(P(x, 3), y)$ is a radiation module.

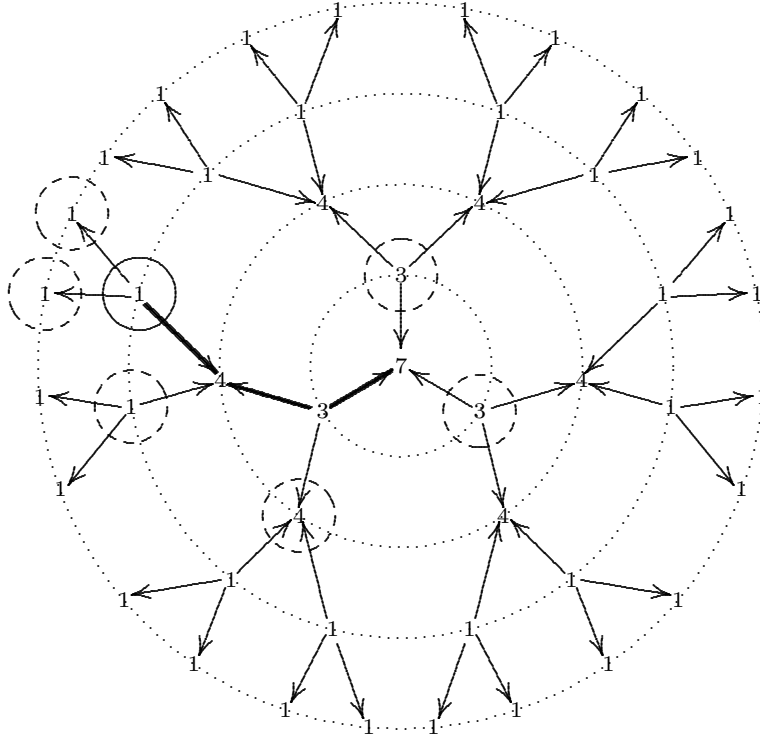


Here, x is the vertex at the center. The vertex y is encircled using a solid circle, the vertices $z \in N([x, y])$ are encircled using dashed circles.

Here is the radiation quiver $R(P(x, 3), y)$, the vertex y has been encircled.



Finally, let us show the dimension vector $\mathbf{dim} P(x, 4)$ with x a sink.



Again, x is the vertex at the center. We have chosen a source y such that $d(x, y) = 3$, the vertex y is encircled using a solid circle, whereas the dashed circles mark the vertices $z \in N([x, y])$. Altogether, we see that there is an exact sequence of the form

$$0 \rightarrow U \rightarrow P(x, 4) \rightarrow S(y) \rightarrow 0$$

with

$$U = P(z_1, 0) \oplus P(z_2, 0) \oplus P(z_3, 1) \oplus P(z_4, 2) \oplus P(z_5, 3) \oplus P(z_6, 3)$$

5. The generalized Kronecker quivers and Schofield induction.

The generalized Kronecker quivers are the quivers $K(n)$ with 2 vertices, a sink and a source, and n arrows (going from the source to the sink); the case $n = 2$ is the ordinary Kronecker quiver, its representations have been studied by Weierstraß and Kronecker. The universal cover $Q(n)$ of the quiver $K(n)$ is the n -regular tree with bipartite orientation. Using the push-down functor, any indecomposable representation of the quiver $Q(n)$ yields an indecomposable representation of $K(n)$. In particular, the representation $P_{Q(n)}(x, t)$ of $Q(n)$ defined in the last section provides an indecomposable representation $P_{K(n)}(t)$ and one obtains in this way just the preprojective $K(n)$ -modules (the special case $n = 3$ has been discussed in detail in [FR], all the considerations presented there can easily be adapted to the general case). Obviously, under the push-down functor a tree basis is sent to a tree basis. Since all the modules $P(x, t)$ are radiation modules we obtain in this way

distinguished tree bases for the preprojective $K(n)$ -modules. The dual considerations yield distinguished tree bases for the preinjective $K(n)$ -modules.

Let us consider now an arbitrary quiver Q and M an exceptional representations of Q . It has been shown in [R2] (see also [R4]) that M is a tree module. Here we want to outline in which way we can use the previous assertions in order to exhibit a nice tree basis of M . In order to exhibit a tree basis for an exceptional module M , one uses Schofield induction, thus one considers exact sequences of the form

$$0 \rightarrow Y^y \rightarrow M \rightarrow X^x \rightarrow 0$$

with indecomposable middle term, where (X, Y) is an orthogonal exceptional pair with $\dim \text{Ext}^1(X, Y) = e > 0$ and (x, y) is the dimension vector of a sincere preprojective or preinjective representation E of the e -Kronecker quiver $K(e)$. Note that the triple $(X, Y; E)$ uniquely determines M and one obtains in this way inductively all the exceptional representations of Q . In order to construct a tree basis of M we need to know tree bases of X, Y and E . For the procedure to obtain a tree basis of M from the tree bases of X, Y, E we refer to [R2] (see also [R4]).

A required tree basis for E has been exhibited already in [R2], but the present note provides an intrinsic way for obtaining such a tree basis. Let us repeat: Instead of working with the e -Kronecker quiver itself, we consider its universal cover, the e -regular tree $Q(e)$ with bipartite orientation. Let E be obtained from the representation \tilde{E} of $Q(e)$ by the push-down functor. Now $Q(e)$ is a bipartite tree quiver without leaves and \tilde{E} is a preprojective or preinjective representation of Q . Thus, there is a vertex x such that the pair (\tilde{E}, x) is a radiation quiver. But this means that \tilde{E} has a distinguished tree basis. Under the push-down functor, we obtain a distinguished tree basis for E .

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