

GREEN FUNCTION ESTIMATES FOR SUBORDINATE BROWNIAN MOTIONS : STABLE AND BEYOND

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ABSTRACT. A subordinate Brownian motion X is a Lévy process which can be obtained by replacing the time of the Brownian motion by an independent subordinator. In this paper, when the Laplace exponent ϕ of the corresponding subordinator satisfies some mild conditions, we first prove the scale invariant boundary Harnack inequality for X on arbitrary open sets. Then we give an explicit form of sharp two-sided estimates on the Green functions of these subordinate Brownian motions in any bounded $C^{1,1}$ open set. As a consequence, we prove the boundary Harnack inequality for X on any $C^{1,1}$ open set with explicit decay rate. Unlike [KSV12b, KSV12c], our results cover geometric stable processes and relativistic geometric stable process, i.e. the cases when the subordinator has the Laplace exponent

$$\phi(\lambda) = \log(1 + \lambda^{\alpha/2}) \quad (0 < \alpha \leq 2, d > \alpha)$$

and

$$\phi(\lambda) = \log(1 + (\lambda + m^{\alpha/2})^{2/\alpha} - m) \quad (0 < \alpha < 2, m > 0, d > 2).$$

1. INTRODUCTION

Let d be a positive integer, let $W = (W_t, \mathbb{P}_x)$ be a Brownian motion in \mathbb{R}^d starting at x and let $S = (S_t : t \geq 0)$ be a subordinator independent of W , i.e. a Lévy process taking values in $[0, \infty)$ and starting at 0.

The Laplace exponent of a subordinator is a Bernstein function and hence has the representation

$$\phi(\lambda) = b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda t}) \mu(dt), \tag{1.1}$$

where $b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0, \infty)} (1 \wedge t) \mu(dt) < \infty$, usually called the Lévy measure of ϕ . If the measure μ has a completely monotone density, the Laplace exponent ϕ is called a complete Bernstein function.

2000 *Mathematics Subject Classification.* Primary 60J45, Secondary 60J75, 60G51.

Key words and phrases. geometric stable process, Green function, Harnack inequality, Poisson kernel, harmonic function, potential, subordinator, subordinate Brownian motion.

Research supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(0409-20110087).

Research supported in part by German Science Foundation DFG via IGK "Stochastics and real world models" and SFB 701.

We define the subordinate Brownian motion $X = (X_t, \mathbb{P}_x)$ by $X_t = W_{S_t}$.

The aim of this paper is to obtain the following two-sided estimates of the Green function $G_D(x, y)$ of X in a bounded $C^{1,1}$ open set $D \subset \mathbb{R}^d$ in terms of the Laplace exponent ϕ of the subordinator:

$$G_D(x, y) \asymp \left(1 \wedge \frac{\phi(|x - y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right) \frac{\phi'(|x - y|^{-2})}{|x - y|^{d+2}\phi(|x - y|^{-2})^2},$$

where $\delta_D(x)$ denotes the distance of the point x to D^c and $a \wedge b := \min\{a, b\}$. Here and in the sequel, $f \asymp g$ means that the quotient $\frac{f}{g}$ stays bounded between two positive numbers on their common domain of definition.

The process X is, in particular, a rotationally symmetric Lévy process. Recently there are many interests in studying potential theory of such processes. See, for example, [KMR, KSV12a, KSV12b, KSV12c, RSV06] and references therein. The purpose of this paper is to extend recent results in [KSV12b, KSV12c] by covering geometric stable processes and much more.

Estimates of Green function for discontinuous Markov processes were first studied for rotationally symmetric α -stable processes in [CS98] and in [Kul97] independently. These results were extended later to relativistic α -stable processes and to sums of two independent stable processes in [Ryz02] and [CKS12] respectively. Recently, the first named author with R. Song and Z. Vondraček succeeded to obtain such estimates for a large class of subordinate Brownian motions in [KSV12b].

Still, the class considered in [KSV12b] does not include some interesting cases like geometric stable processes or, more generally, the class of subordinate Brownian motions with Laplace exponent that varies slowly at infinity. Our approach covers a large class of such processes.

Another feature of our approach is that it is unifying in the following sense: the sharp estimates of the Green function are given only in terms of the Laplace exponent ϕ and its derivative.

Let us give a few examples of transient processes that are covered by our approach.

Example 1 (Geometric stable processes)

$$\phi(\lambda) = \log(1 + \lambda^{\beta/2}), \quad (0 < \beta \leq 2, d > \beta).$$

Example 2 (Iterated geometric stable processes)

$$\begin{aligned} \phi_1(\lambda) &= \log(1 + \lambda^{\beta/2}) \quad (0 < \beta \leq 2) \\ \phi_{n+1} &= \phi_1 \circ \phi_n \quad n \in \mathbb{N}, \end{aligned}$$

with an additional condition $d > 2^{1-n}\beta^n$.

Example 3 (Relativistic geometric stable processes)

$$\phi(\lambda) = \log \left(1 + \left(\lambda + m^{\beta/2} \right)^{2/\beta} - m \right) \quad (m > 0, 0 < \beta < 2, d > 2).$$

In order to obtain the sharp Green function estimates we first obtain the uniform boundary Harnack principle, with constant not depending on the open set itself. Such uniform boundary Harnack principle was first proved in [BKK08] and very recently generalized to a larger class of rotationally symmetric Lévy processes in [KSV12c]. We adapt the approach in the latter paper in order to cover the class of subordinate Brownian motions with slowly varying Laplace exponents. Unlike the approach in [KSV12c], instead of the use of the Harnack inequality, we use estimates of the Green function of balls near boundary obtained in [KM].

Further, our uniform boundary Harnack principle can be used to prove sharp Green function estimates for bounded $C^{1,1}$ open sets by adapting the method in [CKS12]. Even though we follow the roadmap in [CKS12], we needed to make significant changes due to the fact that now we do not have necessarily regularly varying Laplace exponents.

To overcome such difficulties we use new types of estimates (not only in terms of the Laplace exponent itself, but also in terms of its derivative) of the jumping kernel and the potential kernel of the subordinate Brownian motions, which were obtained for the first time in [KM]. This type of estimates is essential in our approach.

Let us be more precise now. In this paper we will always assume the following three conditions on the Laplace exponent ϕ of the subordinator S :

(A-1) ϕ is a complete Bernstein function;

(A-2) the Lévy density μ of ϕ is infinite, i.e. $\mu(0, \infty) = \infty$;

(A-3) there exist constants $\sigma > 0$, $\lambda_0 > 0$ and $\delta \in (0, 1]$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq \sigma x^{-\delta} \quad \text{for all } x \geq 1 \text{ and } \lambda \geq \lambda_0. \quad (1.2)$$

In the cases $d \leq 2$ and $0 < \delta \leq \frac{1}{2}$ we will sometimes further assume the following technical conditions:

(A-4) If $d \leq 2$, we assume that the constant δ in (A-3) satisfies $d + 2\delta - 2 > 0$ and that there are $\sigma_0 > 0$ and

$$\delta_0 \in \left(1 - \frac{d}{2}, \left(1 + \frac{d}{2} \right) \wedge \left(2\delta + \frac{d-2}{2} \right) \right) \quad (1.3)$$

such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \geq \sigma_0 x^{-\delta_0} \quad \text{for all } x \geq 1 \text{ and } \lambda \geq \lambda_0. \quad (1.4)$$

(A-5) If $d \geq 2$ and the constant δ in **(A-3)** satisfies $0 < \delta \leq \frac{1}{2}$, then we assume that there exist constants $\sigma_1 > 0$ and $\delta_1 \in [\delta, 2\delta)$ such that

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \geq \sigma_1 x^{1-\delta_1} \quad \text{for all } x \geq 1 \text{ and } \lambda \geq \lambda_0. \quad (1.5)$$

Remark 1.1. (a) Note that **(A-3)** implies $b = 0$ in (1.1), by letting $x \rightarrow \infty$.
 (b) The condition **(A-3)** is implied by the following stronger condition

$$\forall x > 0 \quad \lim_{\lambda \rightarrow \infty} \frac{\phi'(\lambda x)}{\phi'(\lambda)} = x^{\frac{\alpha}{2}-1} \quad (0 \leq \alpha < 2). \quad (1.6)$$

In other words, (1.6) says that ϕ' varies regularly at infinity with index $\frac{\alpha}{2} - 1$. A novelty here is the case $\alpha = 0$.

(c) The condition **(A-4)** is used only to obtain Green function estimates.

Now we state the main result of this paper. By $\text{diam}(D)$ we denote the diameter of D .

Theorem 1.2. Suppose that $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a transient subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$ satisfying **(A-1)**–**(A-5)**.

Then for every bounded $C^{1,1}$ open set D (see Definition 3.4) in \mathbb{R}^d with characteristics (R, Λ) , there exists $c = c(\text{diam}(D), R, \Lambda, \phi, d) > 1$ such that the Green function $G_D(x, y)$ of X in D satisfies

$$c^{-1}g_D(x, y) \leq G_D(x, y) \leq cg_D(x, y) \quad (1.7)$$

with

$$g_D(x, y) = \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right) \frac{\phi'(|x-y|^{-2})}{|x-y|^{d+2}\phi(|x-y|^{-2})^2}. \quad (1.8)$$

In particular, we extend the main result in [KSV12b].

Corollary 1.3. Suppose that $X = (X_t : t \geq 0)$ is a subordinate Brownian motion whose characteristic exponent is given by $\Phi(\theta) = \phi(|\theta|^2)$, $\theta \in \mathbb{R}^d$, where $\phi : (0, \infty) \rightarrow [0, \infty)$ is a complete Bernstein function such that

$$c_1 x^{\alpha/2} \leq \frac{\phi(\lambda x)}{\phi(\lambda)} \leq c_2 x^{\beta/2} \quad \text{for all } x \geq 1 \text{ and } \lambda \geq \lambda_1. \quad (1.9)$$

for some constants $c_1, c_2, \lambda_1 > 0$, $\alpha, \beta \in (0, 2)$ and $\alpha \leq \beta$. We further assume that $2\beta - \alpha < 1$ if $d \geq 2$ and $\beta \geq 1$, and that **(A-5)** hold with $\delta = 1 - \beta/2$.

Then for every bounded $C^{1,1}$ open set D in \mathbb{R}^d with characteristics (R, Λ) , there exists $c = c(\text{diam}(D), R, \Lambda, \phi, d) > 1$ such that the Green function $G_D(x, y)$ of X in D satisfies the following estimates:

$$c^{-1}\widehat{g}_D(x, y) \leq G_D(x, y) \leq c\widehat{g}_D(x, y) \quad (1.10)$$

with

$$\widehat{g}_D(x, y) = \left(1 \wedge \frac{\phi(|x - y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right) \frac{1}{|x - y|^d \phi(|x - y|^{-2})}.$$

In [KSV12b], the above theorem is proved when, instead of (1.9), ϕ satisfies

$$\phi(\lambda) \asymp \lambda^{\alpha/2} \ell(\lambda), \quad \lambda \rightarrow \infty \quad (0 < \alpha < 2) \quad (1.11)$$

where ℓ varies slowly at infinity, i.e.

$$\forall x > 0 \quad \lim_{\lambda \rightarrow \infty} \frac{\ell(\lambda x)}{\ell(\lambda)} = 1.$$

By Potter's theorem ([BGT87, Theorem 1.5.6(i)]), (1.11) clearly implies (1.9).

Using Green function estimates, we prove the boundary Harnack principle for subordinate Brownian motions satisfying **(A-1)**, **(A-2)**, **(A-3)** and **(A-5)** in $C^{1,1}$ open set. We emphasize that in the next theorem we do not assume neither the transience nor **(A-4)**.

Theorem 1.4. *Suppose that $X = (X_t, \mathbb{P}_x : t \geq 0, x \in \mathbb{R}^d)$ is a (not necessarily transient) subordinate Brownian motion satisfying **(A-1)**, **(A-2)**, **(A-3)** and **(A-5)** and that D is a (possibly unbounded) $C^{1,1}$ open set in \mathbb{R}^d with characteristics (R, Λ) . Then there exists $c = c(R, \Lambda, \phi) > 0$ such that for every $r \in (0, \frac{R\Lambda 1}{4}]$, $z \in \partial D$ and any nonnegative function u in \mathbb{R}^d that is harmonic in $D \cap B(z, r)$ with respect to X and vanishes continuously on $D^c \cap B(z, r)$, we have*

$$\frac{u(x)}{u(y)} \leq c \sqrt{\frac{\phi(\delta_D(y)^{-2})}{\phi(\delta_D(x)^{-2})}} \quad \text{for every } x, y \in D \cap B(z, \frac{r}{2}). \quad (1.12)$$

We remark that Theorem 1.4 cover the processes in Examples 1-3 without the assumptions on transience.

Our paper is organized as follows. In Section 2 we record some preliminary results concerning subordinate Brownian motions obtained in [KM]. We start Section 3 by analyzing special harmonic functions in half-space and use these results to obtain key probabilistic estimates on $C^{1,1}$ open sets. Section 4 contains estimates of Poisson kernel on balls which are used in Section 5 to obtain the uniform boundary Harnack principle on arbitrary open sets. After proving sharp Green function estimates in Lipschitz domains in Section 6, we finally obtain in Section 7 the boundary Harnack principle and sharp Green function estimates in $C^{1,1}$ open sets.

Notation. Throughout the paper we use the notation $f(r) \asymp g(r)$, $r \rightarrow a$ to denote that $\frac{f(r)}{g(r)}$ stays between two positive constants as $r \rightarrow a$. We say that $f: \mathbb{R} \rightarrow \mathbb{R}$ is increasing if $s \leq t$ implies $f(s) \leq f(t)$ and analogously for a decreasing function. For $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure. We will use “:=” to denote a definition, which is read as “is defined to be”.

We will use the following conventions in this paper. The values of the constants C_1, C_2, C_3, C_4 and ε_1 will remain the same throughout this paper, while c_1, c_2, \dots stand for constants whose values are unimportant and which may change from one appearance to another. All constants are positive finite numbers. The labeling of the constants c, c_1, c_2, \dots starts anew in the proof of each result. The dependence of the constant c on the dimension d will not be mentioned explicitly.

2. PRELIMINARIES

By concavity, we see that every Bernstein function ψ satisfies

$$\psi(t\lambda) \leq \lambda\psi(t) \quad \text{for all } \lambda \geq 1, t > 0. \quad (2.1)$$

Thus $\lambda \mapsto \frac{\psi(\lambda)}{\lambda}$ is decreasing, which implies

$$\lambda\psi'(\lambda) \leq \psi(\lambda) \quad \text{for all } \lambda > 0. \quad (2.2)$$

We first recall the following results from [KM].

Lemma 2.1. [KM, Lemma 3.1] *Suppose that ψ is a special Bernstein function, i.e., $\lambda \mapsto \frac{\lambda}{\psi(\lambda)}$ is also a Bernstein function. Then the functions $\eta_1, \eta_2: (0, \infty) \rightarrow (0, \infty)$ given by*

$$\eta_1(\lambda) = \lambda^2\psi'(\lambda) \quad \text{and} \quad \eta_2(\lambda) = \lambda^2 \frac{\psi'(\lambda)}{\psi(\lambda)^2}$$

are increasing. □

Recall that we will always assume that the Laplace exponent ϕ of S satisfies **(A-1)**–**(A-3)**. We also recall the following elementary fact from [KM] which says that **(A-3)** controls the growth of ϕ .

Lemma 2.2. [KM, Lemma 2.2(ii)] *For every $\varepsilon > 0$ there exists $c(\varepsilon, \sigma) > 1$ such that*

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \leq c x^{1-\delta+\varepsilon} \quad \text{for all } x \geq 1 \quad \text{and} \quad \lambda \geq \lambda_0; \quad (2.3)$$

□

The analysis of 1-dimensional subordinate Brownian motions will be crucial in this approach. Therefore we now consider an one-dimensional subordinate Brownian motion (Z_t, \mathbb{P}_x) with the characteristic exponent $\phi(\theta^2)$.

Let

$$\bar{Z}_t := \sup\{0 \vee Z_s : 0 \leq s \leq t\}$$

be the supremum process of Z and let $L = (L_t : t \geq 0)$ be a local time of $\bar{Z} - Z$ at 0. The right continuous inverse L_t^{-1} of L is a subordinator and it is called the ladder time process of Z . The process $\bar{Z}_{L_t^{-1}}$ is also a subordinator, called the ladder height process of Z . (For the basic properties of the ladder time and ladder height processes, we refer the reader to [Ber96, Chapter 6].)

Let κ be the Laplace exponent of the ladder height process of Z . It follows from [Fri74, Corollary 9.7] that

$$\kappa(\lambda) = \exp \left\{ \frac{1}{\pi} \int_0^\infty \frac{\log(\phi(\lambda^2 \theta^2))}{1 + \theta^2} d\theta \right\}, \quad \forall \lambda > 0. \quad (2.4)$$

By our assumptions and [KSV12a, Proposition 3.7] or [KMR, Proposition 2.1] we see that the ladder height process of Z has no drift and is not compound Poisson, and so the process Z does not creep upwards. Since Z is symmetric, we know that Z also does not creep downwards.

Denote by V the potential measure of the ladder height process of Z . We will slightly abuse notation and use the same letter V to denote the renewal function of the ladder height process of Z , that is $V(t) = V((0, t))$. V is a smooth function by [KSV12a, Corollary 3.8].

Combining [KSV12a, Proposition 3.7] and [Ber96, Proposition III.1] the following result holds.

Proposition 2.3. *There exists a constant $c > 1$ such that for all $r > 0$*

$$\frac{c^{-1}}{\sqrt{\phi(r^{-2})}} \leq V(r) \leq \frac{c}{\sqrt{\phi(r^{-2})}}.$$

□

We next consider multidimensional subordinate Brownian motions. Let $W = (W_t = (W_t^1, \dots, W_t^d) : t \geq 0)$ be a Brownian motion in \mathbb{R}^d with

$$\mathbb{E} \left[e^{i\theta \cdot (W_t - W_0)} \right] = e^{-t|\theta|^2}, \quad \forall \theta \in \mathbb{R}^d, t > 0,$$

and let S be a subordinator independent of W with Laplace exponent ϕ . In the remainder of this paper, we always assume that $X = (X_t, \mathbb{P}_x)$ is a subordinate process defined by $X_t = W_{S_t}$. This process is a pure-jump symmetric Lévy process with the characteristic exponent $\Phi(\xi) = \phi(|\xi|^2)$, i.e.

$$\mathbb{E}_0 \left[e^{i\xi \cdot X_t} \right] = e^{-t\Phi(\xi)} = e^{-\phi(|\xi|^2)}.$$

Moreover, Φ has the representation

$$\Phi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) \Pi(dy)$$

with the Lévy measure of the form $\Pi(dx) = j(|x|) dx$

For any open set D , let us denote by τ_D to denote the first exit time of D , i.e.

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

Using Proposition 2.3, the proof of the next result is the same as the one of [KSV12c, Proposition 3.2]. So we skip the proof.

Lemma 2.4. *There exists $c > 0$ such that for any $r \in (0, \infty)$ and $x_0 \in \mathbb{R}^d$,*

$$\mathbb{E}_x[\tau_{B(x_0, r)}] \leq cV(r)V(r - |x - x_0|) \asymp \frac{1}{\sqrt{\phi(r^{-2})\phi((r - |x - x_0|)^{-2})}} \text{ for } x \in B(x_0, r).$$

□

The process X has a transition density $p(t, x, y)$ given by

$$p(t, x, y) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \mathbb{P}(S_t \in ds). \quad (2.5)$$

When X is transient, we can define the Green function (potential) by

$$G(x, y) = g(|y - x|) = \int_0^\infty p(t, x, y) dt.$$

Note that g and j are decreasing.

The following result is proved in [KM].

Proposition 2.5. *Suppose ϕ satisfies (A-1)–(A-4). Then we have*

$$j(r) \asymp r^{-d-2}\phi'(r^{-2}), \quad r \rightarrow 0+. \quad (2.6)$$

If X is transient, then

$$g(r) \asymp r^{-d-2} \frac{\phi'(r^{-2})}{\phi(r^{-2})^2}, \quad r \rightarrow 0+. \quad (2.7)$$

□

As a consequence of (2.6) it follows that if ϕ satisfies (A-1)–(A-4) then for any $K > 0$, there exists $c = c(K) > 1$ such that

$$j(r) \leq c j(2r), \quad \forall r \in (0, K). \quad (2.8)$$

The function j also enjoys the following property: if ϕ satisfies (A-1)–(A-4) then there is a constant $c > 0$ such that

$$j(r+1) \leq j(r) \leq c j(r+1) \text{ for all } r \geq 1 \quad (2.9)$$

(see [KM]).

Let $D \subset \mathbb{R}^d$ be an open subset. The killed process X^D is defined by

$$X_t^D = X_t \text{ if } t < \tau_D \quad \text{and} \quad X_t^D = \Delta \text{ otherwise,}$$

where Δ is an extra point adjoined to D (usually called cemetery).

The transition density of X^D is given by

$$p_D(t, x, y) = p(t, x, y) - \mathbb{E}_x[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t]$$

A subset D of \mathbb{R}^d is said to be Greenian (for X) if X^D is transient. When $d \geq 3$, any non-empty open set $D \subset \mathbb{R}^d$ is Greenian. An open set $D \subset \mathbb{R}^d$ is Greenian if and only if

D^c is non-polar for X (or equivalently, has positive capacity with respect to X). For any Greenian open set D in \mathbb{R}^d let $G_D(x, y) = \int_0^\infty p_D(t, x, y) dt$ be the Green function of X^D . $G_D(x, y)$ is symmetric and, for fixed $y \in D$, $G_D(\cdot, y)$ is harmonic (with respect to X) in $D \setminus \{y\}$.

The next two results are the key estimates in [KM].

Proposition 2.6. *Suppose X is transient and ϕ satisfies (A-1)–(A-4). There exist constants $c_1, c_2 > 0$ and $b_1, b_2 \in (0, \frac{1}{2})$, $2b_1 < b_2$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0, 1)$ we have*

$$c_1 \frac{r^{-d-2} \phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_y \tau_{B(x_0, r)} \leq G_{B(x_0, r)}(x, y) \leq c_2 \frac{r^{-d-2} \phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_y \tau_{B(x_0, r)}$$

for all $x \in B(x_0, b_1 r)$ and $y \in B(x_0, r) \setminus B(x_0, b_2 r)$. \square

Proposition 2.7. *Suppose X is transient and ϕ satisfies (A-1)–(A-4). There exist constants $c_1 > 0$ and $a \in (0, \frac{1}{3})$ so that for $x_0 \in \mathbb{R}^d$ and $r \in (0, 1)$ we have*

$$\mathbb{E}_x [\tau_{B(x_0, r)}] \geq \frac{c_1}{\phi(r^{-2})} \quad \text{for any } x \in B(x_0, ar).$$

\square

Before we state the Harnack inequality, we recall the definition of harmonic functions.

Definition 2.8. *Let D be an open subset of \mathbb{R}^d . A function u defined on \mathbb{R}^d is said to be*

(i) *harmonic in D with respect to X if*

$$\mathbb{E}_x [|u(X_{\tau_B})|] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x [u(X_{\tau_B})], \quad x \in B,$$

for every open set B whose closure is a compact subset of D ;

(ii) *regular harmonic in D with respect to X if it is harmonic in D with respect to X and*

$$u(x) = \mathbb{E}_x [u(X_{\tau_D})] \quad \text{for any } x \in D.$$

The following Harnack inequality is the main result of [KM].

Theorem 2.9 (Harnack inequality). *There exists a constant $c > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0, 1)$ we have*

$$h(x_1) \leq c h(x_2) \quad \text{for all } x_1, x_2 \in B(x_0, r/2)$$

and for every non-negative function $h: \mathbb{R}^d \rightarrow [0, \infty)$ which is harmonic in $B(x_0, r)$. \square

By the result of Ikeda and Watanabe (see [IW62, Theorem 1]) the following formula is true

$$\mathbb{P}_x(X_{\tau_D} \in F) = \int_F \int_D G_D(x, y) j(|z - y|) dy dz \quad (2.10)$$

for any $F \subset \overline{D}^c$. We define the Poisson kernel of the set D by

$$K_D(x, z) = \int_D G_D(x, y) j(|z - y|) dy, \quad (2.11)$$

so that $\mathbb{P}_x(X_{\tau_D} \in F) = \int_F K_D(x, z) dz$ for any $F \subset \overline{D}^c$.

Proposition 2.10. *Suppose X is transient and ϕ satisfies **(A-1)**–**(A-4)**. There exists $c_1 = c_1(\phi) > 0$ and $c_2 = c_2(\phi) > 0$ such that for every $r \in (0, 1)$ and $x_0 \in \mathbb{R}^d$,*

$$K_{B(x_0, r)}(x, y) \leq c_1 \frac{j(|y - x_0| - r)}{\sqrt{\phi(r^{-2})\phi((r - |x - x_0|)^{-2})}} \quad (2.12)$$

$$\leq c_1 \frac{j(|y - x_0| - r)}{\phi(r^{-2})} \quad (2.13)$$

for all $(x, y) \in B(x_0, r) \times \overline{B(x_0, r)}^c$ and

$$K_{B(x_0, r)}(x_0, y) \geq c_2 \frac{j(|y - x_0|)}{\phi(r^{-2})} \quad \text{for all } y \in \overline{B(x_0, r)}^c. \quad (2.14)$$

□

Proof. First using (2.8) and (2.9) to (2.11), then applying Lemma 2.4 and Proposition 2.7, (2.12) and (2.14) follow easily (see the proof of [KSV12a, Proposition 4.10] for the details). (2.13) follows from (2.12) and the fact that ϕ is increasing. □

3. ANALYSIS ON HALF-SPACE AND $C^{1,1}$ OPEN SETS

In this section we establish key estimates which will be used in sections later in this paper.

Recall that $X = (X_t : t \geq 0)$ is the d -dimensional subordinate Brownian motion defined by $X_t = W_{S_t}$ where $W = (W^1, \dots, W^d)$ is a (not necessarily transient) d -dimensional Brownian motion and $S = (S_t : t \geq 0)$ an independent subordinator with the Laplace exponent ϕ satisfying **(A-1)**–**(A-3)**. In this section, we further assume that **(A-4)** holds.

Let $Z = (Z_t : t \geq 0)$ be the one-dimensional subordinate Brownian motion defined by $Z_t := W_{S_t}^d$.

Recall that V is denoted the renewal function of the ladder height process of Z . We use the notation

$$\mathbb{R}_+^d := \{x = (x_1, \dots, x_{d-1}, x_d) := (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$$

for the half-space.

Set $w(x) := V((x_d)^+)$. Since $Z_t = W_{S_t}^d$ has a transition density, by using [Sil80, Theorem 2], the proof of the next result is the same as the one of [KSV12b, Theorem 4.1]. We omit the proof.

Theorem 3.1. *The function w is harmonic in \mathbb{R}_+^d with respect to X and, for any $r > 0$, regular harmonic in $\mathbb{R}^{d-1} \times (0, r)$ for X . □*

Using Theorem 3.1, (2.8) and (2.9), the proof of the next result is the same as the one of [KSV, Proposition 3.3].

Proposition 3.2. *For all positive constants r_0 and L , we have*

$$\sup_{x \in \mathbb{R}^d: 0 < x_d < L} \int_{B(x, r_0)^c \cap \mathbb{R}_+^d} w(y) j(|x - y|) dy < \infty.$$

□

Define an operator $(\mathcal{A}, \mathfrak{D}(\mathcal{A}))$ by

$$\begin{aligned} \mathcal{A}f(x) &:= \text{p.v.} \int_{\mathbb{R}^d} (f(y) - f(x)) j(|y - x|) dy \\ &:= \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |x - y| > \varepsilon\}} (f(y) - f(x)) j(|y - x|) dy \\ \mathfrak{D}(\mathcal{A}) &:= \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} : \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |x - y| > \varepsilon\}} (f(y) - f(x)) j(|y - x|) dy \right. \\ &\quad \left. \text{exists and it is finite} \right\}. \end{aligned} \tag{3.1}$$

Let C_0^2 be the collection of C^2 functions in \mathbb{R}^d vanishing at infinity. It is well known that $C_0^2 \subset \mathfrak{D}(\mathcal{A})$ and that by the rotational symmetry of X , \mathcal{A} restricted to C_0^2 coincides with the infinitesimal generator \mathcal{L} of the process X (see e.g. [Sat99, Theorem 31.5]).

Since V is smooth by [KSV12a, Corollary 3.8], using our Theorem 3.1, (2.8) and (2.9), the proof of the next result is the same as [KSV, Proposition 3.3] or [KSV12b, Proposition 4.2], so we skip the proof.

Theorem 3.3. *$\mathcal{A}w(x)$ is well defined and $\mathcal{A}w(x) = 0$ for all $x \in \mathbb{R}_+^d$.* □

In the rest of this section we aim to prove two key estimates of the exit probability and the exit time for $C^{1,1}$ open sets. Let us recall the definition of a $C^{1,1}$ open set.

Definition 3.4. *An open set D in \mathbb{R}^d ($d \geq 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$ -function $\psi = \psi_z : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ satisfying $\psi(0) = 0$, $\nabla \psi(0) = (0, \dots, 0)$,*

$$\|\nabla \psi\|_\infty \leq \Lambda, \quad |\nabla \psi(x) - \nabla \psi(w)| \leq \Lambda|x - w|, \quad x, w \in \mathbb{R}^{d-1}$$

and an orthonormal coordinate system CS_z : $y = (y_1, \dots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ with origin at z such that

$$B(z, R) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R) \text{ in } CS_z : y_d > \psi(\tilde{y})\}.$$

The pair (R, Λ) is called the characteristics of the $C^{1,1}$ open set D . By a $C^{1,1}$ open set in \mathbb{R} we mean an open set which can be expressed as the union of disjoint intervals so

that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive.

Remark 3.5. In some literature, the $C^{1,1}$ open set defined above is called a uniform $C^{1,1}$ open set since (R, Λ) is universal for all $z \in \partial D$.

For $x \in \mathbb{R}^d$, let $\delta_{\partial D}(x)$ denote the Euclidean distance between x and ∂D . Recall that for $x \in \mathbb{R}^d$, $\delta_D(x)$ is the Euclidean distance between x and ∂D . It is well known that any $C^{1,1}$ open set D with characteristics (R, Λ) there exists $r_1 > 0$ so that it satisfies:

- (i) *uniform interior ball condition*, i.e. for every $x \in D$ with $\delta_D(x) < r_1$ there exists $z_x \in \partial D$ so that

$$|x - z_x| = \delta_{\partial D}(x) \quad \text{and} \quad B(x_0, r_1) \subset D,$$

for $x_0 = z_x + r_1 \frac{x - z_x}{|x - z_x|}$;

- (ii) *uniform exterior ball condition*, i.e. for every $y \in \mathbb{R}^d \setminus D$ with $\delta_{\partial D}(y) < r_1$ there exists $z_y \in \partial D$ so that

$$|y - z_y| = \delta_{\partial D}(y) \quad \text{and} \quad B(y_0, r_1) \subset \mathbb{R}^d \setminus D,$$

for $y_0 = z_y + r_1 \frac{y - z_y}{|y - z_y|}$.

Assume for the rest of this section that D is a $C^{1,1}$ open set with characteristics (R, Λ) satisfying the uniform interior ball condition and the uniform exterior ball condition with the radius $R \leq 1$ (by choosing R smaller if necessary).

Lemma 3.6. Assume additionally that **(A-5)** holds. Fix $Q \in \partial D$ and let

$$h(y) = \begin{cases} V(\delta_D(y)) & y \in B(Q, R) \cap D \\ 0 & \text{otherwise.} \end{cases}$$

There exists $C_1 = C_1(\Lambda, R, \phi) > 0$ independent of the point $Q \in \partial D$ such that $\mathcal{A}h$ is well defined in $D \cap B(Q, \frac{R}{4})$ and

$$|\mathcal{A}h(x)| \leq C_1 \quad \text{for all } x \in D \cap B(Q, \frac{R}{4}). \quad (3.2)$$

Proof. We first note that when $d = 1$, the lemma follows from Proposition 3.2 and Theorem 3.3 by following the same proof as the one in [KSV12b, Lemma 4.4].

Assume now that $d \geq 2$. Fix $x \in D \cap B(Q, \frac{R}{4})$ and let $x_0 \in \partial D$ such that $\delta_D(x) = |x - x_0|$.

Denote by ψ a $C^{1,1}$ function and by $CS = CS_{x_0}$ an orthonormal coordinate system with x_0 chosen so that $x = (\tilde{0}, x_d)$ and

$$B(x_0, R) \cap D = \{y = (\tilde{y}, y_d) \text{ in } CS: y \in B(0, R), y_d > \psi(\tilde{y})\}.$$

We fix such ψ and the coordinate system CS .

Define two auxiliary functions $\psi_1, \psi_2: B(\tilde{0}, R) \rightarrow \mathbb{R}$ by

$$\psi_1(\tilde{y}) = R - \sqrt{R^2 - |\tilde{y}|^2} \quad \text{and} \quad \psi_2(\tilde{y}) = -\left(R - \sqrt{R^2 - |\tilde{y}|^2}\right).$$

By the interior/exterior uniform ball conditions (with radius R) it follows that

$$\psi_2(\tilde{y}) \leq \psi(\tilde{y}) \leq \psi_1(\tilde{y}) \quad \text{for any } y \in D \cap B(x, \frac{R}{4}). \quad (3.3)$$

Now we define a function $h_x(y) = V(\delta_{H^+}(y))$, where

$$H^+ = \{y = (\tilde{y}, y_d) \text{ in } CS: y_d > 0\}$$

denote the half-space in CS .

Since $\delta_{H^+}(y) = (y_d)^+$ in CS , we can use Theorem 3.3 to deduce that

$$\mathcal{A}h_x(y) = 0, \quad \forall y \in H^+.$$

Now the idea is to show that $\mathcal{A}(h - h_x)(x)$ is well defined and that there exists a constant $C_1 = C_1(\Lambda, R, \phi) > 0$ so that

$$\int_{\{y \in D \cup H^+ : |y-x| > \varepsilon\}} |h(y) - h_x(y)| j(|y-x|) dy \leq C_1 \quad \text{for any } \varepsilon > 0. \quad (3.4)$$

To do this we estimate the integral in (3.4) by the sum of the following three integrals:

$$I_1 = \int_{B(x, \frac{R}{4})^c} (h(y) + h_x(y)) j(|y-x|) dy$$

$$I_2 = \int_A (h(y) + h_x(y)) j(|y-x|) dy, \quad \text{where}$$

$$A := \{y \in (D \cup H^+) \cap B(x, \frac{R}{4}) : \psi_2(\tilde{y}) \leq y_d \leq \psi_1(\tilde{y})\}$$

$$I_3 = \int_E |h(y) - h_x(y)| j(|y-x|) dy, \quad \text{where } E := \{y \in B(x, \frac{R}{4}) : y_d > \psi_1(\tilde{y})\}$$

and prove that $I_1 + I_2 + I_3 \leq C_1$.

To estimate I_1 note that, by definition of h , $h = 0$ on $B(Q, R)^c$ which gives

$$I_1 \leq \sup_{\substack{z \in \mathbb{R}^d \\ 0 < z_d < R}} \int_{B(z, \frac{R}{4})^c} V(y_d) j(|z-y|) dy + c_1 \int_{B(0, \frac{R}{4})^c} j(|y|) dy < \infty.$$

Here we have used Proposition 3.2 and the fact that the Lévy measure is a finite measure away from the origin.

Now we estimate I_2 . Denoting by $m_{d-1}(dy)$ the surface measure, we obtain

$$I_2 \leq \int_0^{\frac{R}{4}} \int_{|\tilde{y}|=r} \mathbf{1}_A(y) (h_x(y) + h(y)) j \left(\sqrt{r^2 + |y_d - x_d|^2} \right) m_{d-1}(dy) dr.$$

Since V is increasing and

$$R - \sqrt{R^2 - |\tilde{y}|^2} \leq \frac{|\tilde{y}|^2}{R} \leq |\tilde{y}|,$$

we can use (3.3) to deduce

$$h_x(y) + h(y) \leq 2V(\psi_1(\tilde{y}) - \psi_2(\tilde{y})) \leq 2V(2|\tilde{y}|).$$

Then, by the fact that j decreases, Proposition 2.3 and (2.6), we get

$$\begin{aligned} I_2 &\leq 2 \int_0^{\frac{R}{4}} \int_{|\tilde{y}|=r} \mathbf{1}_A(y) V(2|\tilde{y}|) j(r) m_{d-1}(dy) dr \\ &\leq c_2 \int_0^{\frac{R}{4}} r^{-d-2} \frac{\phi'(r^{-2})}{\sqrt{\phi(r^{-2})}} m_{d-1}(\{y \in A: |\tilde{y}| = r\}) dr. \end{aligned}$$

Noting that $|\psi_2(\tilde{y}) - \psi_1(\tilde{y})| \leq \frac{2|\tilde{y}|^2}{R} = \frac{2r^2}{R}$ for $|\tilde{y}| = r$, we obtain

$$m_{d-1}(\{y: |\tilde{y}| = r, \psi_2(\tilde{y}) < y_d < \psi_1(\tilde{y})\}) \leq c_3 r^d \quad \text{for } r \leq \frac{R}{4}.$$

Thus, by the previous observation and the integration by parts we get

$$\begin{aligned} I_2 &\leq c_4 \int_0^{\frac{R}{4}} r^{-2} \frac{\phi'(r^{-2})}{\sqrt{\phi(r^{-2})}} dr = c_4 \int_0^{\frac{R}{4}} r \left(-\sqrt{\phi(r^{-2})} \right)' dr \\ &\leq c_4 \left[\lim_{r \downarrow 0} r \sqrt{\phi(r^{-2})} + \int_0^{\frac{R}{4}} \sqrt{\phi(r^{-2})} dr \right]. \end{aligned}$$

By Lemma 2.2 applied to a fixed $\varepsilon < \delta$ we see that there is a constant $c = c(\varepsilon) > 0$ so that

$$\phi(r^{-2}) \leq c r^{-2(1-\delta+\varepsilon)},$$

which gives

$$I_2 \leq c_4 \int_0^{\frac{R}{4}} \sqrt{\phi(r^{-2})} dr \leq c_5 \int_0^{\frac{R}{4}} r^{-1+\delta-\varepsilon} dr < \infty.$$

In order to estimate I_3 , we consider two cases. First, if $0 < y_d = \delta_{H^+}(y) \leq \delta_D(y)$,

$$h(y) - h_x(y) \leq V(y_d + R^{-1}|\tilde{y}|^2) - V(y_d) = \int_{y_d}^{y_d + R^{-1}|\tilde{y}|^2} v(z) dz \leq R^{-1}|\tilde{y}|^2 v(y_d), \quad (3.5)$$

since v is decreasing.

If $y_d = \delta_{H^+}(y) > \delta_D(y)$ and $y \in E$, using the fact that $\delta_D(y)$ is greater than or equal to the distance between y and the graph of ψ_1 and

$$y_d - R + \sqrt{|\tilde{y}|^2 + (R - y_d)^2} = \frac{|\tilde{y}|^2}{\sqrt{|\tilde{y}|^2 + (R - y_d)^2} + (R - y_d)} \leq \frac{|\tilde{y}|^2}{2(R - y_d)} \leq \frac{|\tilde{y}|^2}{R},$$

we obtain

$$h_x(y) - h(y) \leq \int_{R - \sqrt{|\tilde{y}|^2 + (R - y_d)^2}}^{y_d} v(z) dz \leq R^{-1}|\tilde{y}|^2 v\left(R - \sqrt{|\tilde{y}|^2 + (R - y_d)^2}\right). \quad (3.6)$$

By (3.5) and (3.6),

$$\begin{aligned} I_3 &\leq R^{-1} \int_{E \cap \{y: y_d \leq \delta_D(y)\}} |\tilde{y}|^2 v(y_d) j(|x - y|) dy \\ &\quad + R^{-1} \int_{E \cap \{y: y_d > \delta_D(y)\}} |\tilde{y}|^2 v\left(R - \sqrt{|\tilde{y}|^2 + (R - y_d)^2}\right) j(|x - y|) dy \\ &=: R^{-1}(L_1 + L_2). \end{aligned}$$

Since

$$E \subset \{z = (\tilde{z}, z_d) \in \mathbb{R}^d : |\tilde{z}| < \frac{R}{4} \wedge \sqrt{2Rz_d - z_d^2} \text{ and } 0 < z_d \leq \frac{R}{2}\},$$

changing to polar coordinates for \tilde{y} and using (2.1), (2.2), (2.6) and Proposition 2.3, yields

$$\begin{aligned} L_1 &\leq c_6 \int_0^{\frac{R}{2}} v(y_d) \left(\int_0^{\frac{R}{4} \wedge \sqrt{2Ry_d - y_d^2}} \frac{r^d \phi'((r^2 + |y_d - x_d|^2)^{-1})}{(r^2 + |y_d - x_d|^2)^{(d+2)/2}} dr \right) dy_d \\ &\leq c_7 \int_0^{\frac{R}{2}} v(y_d) \left(\int_0^{\frac{R}{4} \wedge \sqrt{2Ry_d - y_d^2}} \frac{r^d \phi'((r + |y_d - x_d|)^{-2})}{(r + |y_d - x_d|)^{d+2}} dr \right) dy_d \end{aligned}$$

If $\delta \neq \frac{1}{2}$, by **(A-3)**

$$\begin{aligned}
& \int_0^R \frac{\phi'((r + |y_d - x_d|)^{-2})}{(r + |y_d - x_d|)^2} dr \\
&= \phi'((R + |y_d - x_d|)^{-2}) \int_0^R \frac{\phi'((r + |y_d - x_d|)^{-2})}{\phi'((R + |y_d - x_d|)^{-2})} \frac{dr}{(r + |y_d - x_d|)^2} \\
&\leq c_8 \phi'((R + |y_d - x_d|)^{-2}) \int_0^R \left(\frac{(r + |y_d - x_d|)^{-2}}{(R + |y_d - x_d|)^{-2}} \right)^{-\delta} \frac{dr}{(r + |y_d - x_d|)^2} \\
&= c_8 \phi'((R + |y_d - x_d|)^{-2}) (R + |y_d - x_d|)^{-2\delta} \int_0^R (r + |y_d - x_d|)^{-2+2\delta} dr \\
&\leq c_9 \phi'((R + |y_d - x_d|)^{-2}) (R + |y_d - x_d|)^{-2\delta} |y_d - x_d|^{-(1-2\delta)_+} \tag{3.7}
\end{aligned}$$

Thus, in the case $\delta > \frac{1}{2}$, (3.7) implies

$$L_1 \leq c_{11} \int_0^{\frac{R}{2}} v(y_d) dy_d \leq c_{12} V\left(\frac{R}{2}\right) < \infty. \tag{3.8}$$

If we apply Proposition 2.3 and use **(A-5)**, we deduce

$$v(y_d) \leq \frac{1}{y_d} \int_0^{y_d} v(s) ds = \frac{1}{y_d} V(y_d) \leq c_{13} \frac{1}{y_d} \phi((y_d)^{-2})^{-1/2} \leq c_{14} y_d^{-\delta_1}.$$

Therefore, in the case $0 < \delta < 1/2$ we use $\delta_1 < 2\delta < 1$, $\delta_1 + 1 - 2\delta < 1$ ($\delta_1 \in [\delta, 2\delta)$) and the dominated convergence theorem to see that

$$x_d \mapsto \int_0^{\frac{R}{2}} \frac{1}{(y_d)^{\delta_1} |y_d - x_d|^{1-2\delta}} dy_d \tag{3.9}$$

is a strictly positive continuous function in $x_d \in [0, \frac{R}{4}]$. Hence it is bounded and, consequently, $L_1 < \infty$.

If $\delta = \frac{1}{2}$, we can calculate

$$\int_0^R \frac{\phi'((r + |y_d - x_d|)^{-2})}{(r + |y_d - x_d|)^2} dr \leq c_{10} \ln \frac{R}{|y_d - x_d|}. \tag{3.10}$$

Since in this case $\delta_1 \in [\frac{1}{2}, 1)$, the dominated convergence theorem

implies that

$$x_d \mapsto \int_0^{\frac{R}{2}} \frac{1}{(y_d)^{\delta_1}} \ln \frac{R}{|y_d - x_d|} dy_d \quad (3.11)$$

is a strictly positive continuous function in $x_d \in [0, \frac{R}{4}]$ and hence it is bounded. (3.10) and this imply that $L_1 < \infty$.

Let us estimate L_2 . Switching to polar coordinates for \tilde{y} , and by the use of (2.6), Proposition 2.3 and (A-3) we get

$$\begin{aligned} L_2 &\leq c_{15} \int_0^{x_d + \frac{R}{4}} \left(\int_0^{\sqrt{2Ry_d - y_d^2}} v(R - \sqrt{r^2 + (R - y_d)^2}) r^d j((r^2 + |y_d - x_d|^2)^{1/2}) dr \right) dy_d \\ &\leq c_{16} \int_0^{x_d + \frac{R}{4}} \left(\int_0^{\sqrt{2Ry_d - y_d^2}} \frac{v(R - \sqrt{r^2 + (R - y_d)^2}) \phi'((r^2 + |y_d - x_d|^2)^{-1})}{(r^2 + |y_d - x_d|^2)^{(d+2)/2}} r^d dr \right) dy_d \\ &\leq c_{17} \int_0^{x_d + \frac{R}{4}} \left(\int_0^{\sqrt{2Ry_d - y_d^2}} \frac{v(R - \sqrt{r^2 + (R - y_d)^2}) \phi'((r + |y_d - x_d|)^{-2})}{(r + |y_d - x_d|)^2} r^d dr \right) dy_d. \end{aligned}$$

Since, for $0 < r < R$,

$$R - \sqrt{r^2 + (R - y_d)^2} = \frac{(\sqrt{2Ry_d - y_d^2} + r)(\sqrt{2Ry_d - y_d^2} - r)}{R + \sqrt{r^2 + (R - y_d)^2}} \geq \frac{\sqrt{y_d}}{3\sqrt{R}} \left(\sqrt{2Ry_d - y_d^2} - r \right)$$

and $\sqrt{2Ry_d - y_d^2} < \sqrt{R/2} \sqrt{2R - y_d} < R$ for $0 < y_d < x_d + \frac{R}{4}$, we have

$$L_2 \leq c_{17} \int_0^{x_d + \frac{R}{4}} \int_0^{\sqrt{2Ry_d - y_d^2}} \frac{v \left(\sqrt{y_d} (\sqrt{2Ry_d - y_d^2} - r) \right) \phi'((r + |y_d - x_d|)^{-2})}{(r + |y_d - x_d|)^2} dr dy_d.$$

Using **(A-3)**, we see that with $a := \sqrt{2Ry_d - y_d^2}$ and $b := |y_d - x_d|$,

$$\begin{aligned}
& \int_0^a \frac{v(\sqrt{y_d}(a-r))\phi'((r+b)^{-2})}{(r+b)^2} dr \\
& \leq \int_0^{a/2} \frac{v(\sqrt{y_d}(a-r))\phi'((r+b)^{-2})}{(r+b)^2} dr + c_{18} \int_{a/2}^a \frac{v(\sqrt{y_d}(a-r))}{(r+b)^{2(1-\delta)}} dr \\
& \leq v(\sqrt{y_d}(a/2)) \int_0^{a/2} \frac{\phi'((r+b)^{-2})}{(r+b)^2} dr + \frac{c_{18}}{(b+a/2)^{2(1-\delta)}} \int_{a/2}^a v(\sqrt{y_d}(a-r)) dr \\
& \leq v(\sqrt{y_d}(a/2)) \int_0^R \frac{\phi'((r+b)^{-2})}{(r+b)^2} dr + \frac{c_{19}}{a^{2(1-\delta)}} \frac{1}{\sqrt{y_d}} V(\sqrt{y_d}(a/2)) \\
& := B_1(y_d) + B_2(y_d).
\end{aligned}$$

Using **(A-5)**, Proposition 2.3, the estimate $a > \sqrt{y_d R}$ and the assumption $\delta_1 - \delta < \delta \leq \frac{1}{2}$ for $0 < \delta \leq \frac{1}{2}$, we deduce

$$\int_0^{\frac{R}{2}} B_2(y_d) dy_d \leq c_{20} \begin{cases} \int_0^{\frac{R}{2}} (y_d)^{-\delta_1 + \delta - \frac{1}{2}} dy_d < \infty & \text{when } 0 < \delta \leq \frac{1}{2} \\ V(R) \int_0^{\frac{R}{2}} (y_d)^{\delta - \frac{3}{2}} dy_d < \infty & \text{when } \frac{1}{2} < \delta \leq 1. \end{cases}$$

By (3.7)–(3.11), we also have that

$$\int_0^{\frac{R}{2}} B_1(y_d) dy_d \leq c_{21} \int_0^{\frac{R}{2}} v(y_d) \int_0^R \frac{\phi'((r+|y_d-x_d|)^{-2})}{(r+|y_d-x_d|)^2} dr dy_d < \infty.$$

Thus $L_2 < \infty$.

Now we see that $\mathcal{A}(h - h_x)(x)$ is well defined. Indeed, since $h_x(x) = h(x)$ and

$$\begin{aligned}
& \mathbf{1}_{\{y \in D \cup H^+ : |y-x| > \varepsilon\}} |h(y) - h_x(y)| j(|y-x|) \\
& \leq \mathbf{1}_{A \cup B(x, \frac{R}{4})^c} (h(y) + h_x(y)) j(|y-x|) + \mathbf{1}_E |h(y) - h_x(y)| j(|y-x|) \in L^1(\mathbb{R}^d),
\end{aligned}$$

we can use the dominated convergence theorem to deduce that limit

$$\lim_{\varepsilon \downarrow 0} \int_{\{y \in D \cup H^+ : |y-x| > \varepsilon\}} (h(y) - h_x(y)) j(|y-x|) dy$$

exists. Moreover, $\mathcal{A}h(x)$ is then also well defined and satisfies $|\mathcal{A}h(x)| \leq C_1$. \square

For $a, b > 0$, we define $D_Q(a, b) := \{y \in D : a > \rho_Q(y) > 0, |\tilde{y}| < b\}$.

Lemma 3.7. *Assume additionally that (A-5) holds. There are constants $R_1 = R_1(R, \Lambda, \phi) \in (0, \frac{R}{16\sqrt{1+(1+\Lambda)^2}})$ and $c_i = c_i(R, \Lambda, \phi) > 0$, $i = 1, 2$, such that for every $r \leq R_1$, $Q \in \partial D$ and $x \in D_Q(r, r)$,*

$$\mathbb{P}_x \left(X_{\tau_{D_Q(r,r)}} \in D \right) \geq c_1 V(\delta_D(x)) \quad (3.12)$$

and

$$\mathbb{E}_x \left[\tau_{D_Q(r,r)} \right] \leq c_2 V(\delta_D(x)). \quad (3.13)$$

Proof. Without loss of generality we may assume that $Q = 0$ and that $\psi: \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a $C^{1,1}$ function such that in the coordinate system CS_0

$$B(0, R) \cap D = \{(\tilde{y}, y_d) \in B(0, R) \text{ in } CS_0 : y_d > \psi(\tilde{y})\}.$$

The function ρ defined by $\rho(y) = y_d - \psi(\tilde{y})$ satisfies

$$\frac{\rho(y)}{\sqrt{1+\Lambda^2}} \leq \delta_D(y) \leq \rho(y) \quad \text{for all } y \in B(0, R) \cap D. \quad (3.14)$$

Define for $a > 0$,

$$D_a = \{y \in D : 0 < \rho(y) < a, |\tilde{y}| < a\}$$

and a function

$$h(y) = \begin{cases} V(\delta_D(y)) & y \in B(0, R) \cap D \\ 0 & \text{otherwise.} \end{cases}$$

Using Dynkin formula and the same approximation argument as in the proof of the Lemma 4.5 in [KSV12b], from our Lemma 3.6 we have the following estimate for any open set $U \subset B(0, \frac{R}{4}) \cap D$:

$$h(x) - C_1 \mathbb{E}_x \tau_U \leq \mathbb{E}_x h(X_{\tau_U}) \leq h(x) + C_1 \mathbb{E}_x \tau_U \quad (3.15)$$

where $C_1 > 0$ is the constant from Lemma 3.6.

By choosing $A := \frac{R}{4\sqrt{1+(1+\Lambda)^2}}$ we obtain

$$D_r \subset D_A \subset D(0, \frac{R}{4}) \cap D \quad \text{for all } r \leq A.$$

Indeed, $y \in D_r$ and $r > 0$ the following is true

$$|y|^2 = |\tilde{y}|^2 + |y_d|^2 \leq r^2 + (|y_d - \psi(\tilde{y})| + |\psi(\tilde{y})|)^2 \leq (1 + (1 + \Lambda)^2)r^2. \quad (3.16)$$

In particular, for $r \leq A$

$$|y| \leq \sqrt{1 + (1 + \Lambda)^2} A = \frac{R}{4}.$$

The idea is to choose $\lambda_0 \geq 1$ large enough so that (3.12) and (3.13) hold for $r \leq \lambda_0^{-1} A$ and $x \in D_r$.

We are going to show that there are constants $c_1, c_2 > 0$ such that for any $\lambda \geq 4$ and $x \in D_{\lambda^{-1}A}$ the following two inequalities hold:

$$\mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] \geq c_1 \left(\sqrt{\phi(16\lambda^2 R^{-2})} - \sqrt{\phi(R^{-2})} \right) \mathbb{E}_x \tau_{D_{\lambda^{-1}A}} \quad (3.17)$$

$$\mathbb{P}_x \left(X_{\tau_{D_{\lambda^{-1}A}}} \in D \right) \geq c_2 \left(\phi(16\lambda^2 R^{-2}) - \phi(R^{-2}) \right) \mathbb{E}_x \tau_{D_{\lambda^{-1}A}} \quad (3.18)$$

Once we prove this, we can choose $\lambda_0 > 4$ so that

$$\sqrt{\phi(16\lambda_0^2 R^{-2})} > \sqrt{\phi(R^{-2})} + \frac{2C_1}{c_1}.$$

Then, for any $\lambda \geq \lambda_0$ and $x \in D_{\lambda^{-1}A}$ we can use

$$c_1 \left(\sqrt{\phi(16\lambda^2 R^{-2})} - \sqrt{\phi(R^{-2})} \right) - C_1 > C_1$$

and (3.15) to get

$$V(\delta_D(x)) = h(x) \geq \mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] - C_1 \mathbb{E}_x \tau_{D_{\lambda^{-1}A}} \geq C_1 \mathbb{E}_x \tau_{D_{\lambda^{-1}A}},$$

which proves (3.13) with $R_1 = \lambda_0^{-1}A$.

Similarly, by (3.15), for any $\lambda \geq \lambda_0$ and $x \in D_{\lambda^{-1}A}$ we have

$$\begin{aligned} V(\delta_D(x)) = h(x) &\leq \mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] + C_1 \mathbb{E}_x \tau_{D_{\lambda^{-1}A}} \\ &\leq V(R) \mathbb{P}_x \left(X_{\tau_{D_{\lambda^{-1}A}}} \in D \right) + C_1 c_2^{-1} \left(\phi(16\lambda_0^2 R^{-2}) - \phi(R^{-2}) \right)^{-1} \mathbb{E}_x \tau_{D_{\lambda^{-1}A}}, \end{aligned}$$

which together with (3.18) yields

$$\mathbb{P}_x \left(X_{\tau_{D_{\lambda^{-1}A}}} \in D \right) \geq \frac{V(\delta_D(x))}{V(R) + C_4 c_2^{-1} \left(\phi(16\lambda_0^2 R^{-2}) - \phi(R^{-2}) \right)^{-1}}.$$

This proves (3.12) with $R_1 = \lambda_0^{-1}A$.

Now we prove (3.17). Note that for $z \in D_{\lambda^{-1}A}$ and $y \notin B(0, \lambda^{-1} \frac{R}{r})$,

$$|z| \leq \sqrt{1 + (1 + \lambda^2)} \lambda^{-1} A = \lambda^{-1} \frac{R}{4} \leq |y|$$

implies

$$j(|z - y|) \geq j(2|y|) \geq c_3 j(|y|).$$

Then the Ikeda-Watanabe formula implies

$$\begin{aligned}
\mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] &\geq \int_{B(0,r) \cap D \setminus D_{\lambda^{-1}A}} \int_{D_{\lambda^{-1}A}} G_{D_{\lambda^{-1}A}}(x,z) j(|z-y|) V(\delta_D(y)) dz dy \\
&\geq c_3 \left(\int_{D_{\lambda^{-1}A}} G_{D_{\lambda^{-1}A}}(x,z) dz \right) \int_{B(0,R) \cap D \setminus D_{\lambda^{-1}A}} V(\delta_D(y)) j(|y|) dy \\
&\geq c_3 \mathbb{E}_x \tau_{D_{\lambda^{-1}A}} \int_{B(0,R) \cap D \setminus D_{\lambda^{-1}A}} j(|y|) V\left(\frac{y_d - \psi(\tilde{y})}{\sqrt{1+\lambda^2}}\right) dy,
\end{aligned}$$

since $\frac{y_d - \psi(\tilde{y})}{\sqrt{1+\lambda^2}} \leq \delta_D(y)$ by (3.14).

On the set $E := \{(\tilde{y}, y_d) : 2\Lambda|\tilde{y}| < y_d, \lambda^{-1}\frac{R}{4} < |y| < R\}$ we have

$$|y| \leq \sqrt{1+4\Lambda^2} y_d \quad \text{and} \quad y_d - \psi(\tilde{y}) \geq y_d - \Lambda|\tilde{y}| \geq \frac{|y|}{2\sqrt{1+4\Lambda^2}}.$$

Since $E \subset B(0,R) \setminus D_{\lambda^{-1}A}$, changing to polar coordinates gives

$$\mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] \geq c_4 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \int_{\lambda^{-1}\frac{R}{4}}^R j(r) V\left(\frac{r}{2\sqrt{1+4\Lambda^2}}\right) r^{d-1} dr$$

with constant $c_4 > 0$ depending on Λ and d .

Then (2.6) and Proposition 2.3 imply

$$\begin{aligned}
\mathbb{E}_x[h(X_{\tau_{D_{\lambda^{-1}A}}})] &\geq c_5 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \int_{\lambda^{-1}\frac{R}{4}}^R r^{-3} \frac{\phi'(r^{-2})}{\sqrt{\phi(r^{-2})}} dr \\
&= c_5 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \left(\sqrt{\phi(16\lambda^2 R^{-2})} - \sqrt{\phi(R^{-2})} \right).
\end{aligned}$$

We prove (3.18) similarly:

$$\begin{aligned}
\mathbb{P}_x \left(X_{\tau_{D_{\lambda^{-1}A}}} \in D \right) &\geq \mathbb{P}_x \left(X_{\tau_{D_{\lambda^{-1}A}}} \in B(0,R) \cap D \setminus B(0, \lambda^{-1}\frac{R}{4}) \right) \\
&\geq c_6 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \int_{\lambda^{-1}\frac{R}{4}}^R j(r) r^{d-1} dr \\
&\geq c_7 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \int_{\lambda^{-1}\frac{R}{4}}^R r^{-3} \phi'(r^{-2}) dr \\
&= 2^{-1} c_7 \mathbb{E}_x[\tau_{D_{\lambda^{-1}A}}] \left(\phi(16\lambda^2 R^{-2}) - \phi(R^{-2}) \right).
\end{aligned}$$

□

4. ANALYSIS OF POISSON KERNEL

In this section we always assume that the Laplace exponent ϕ of the subordinator $S = (S_t : t \geq 0)$ satisfies **(A-1)**–**(A-4)** and corresponding subordinate Brownian motion $X = (X_t, \mathbb{P}_x)$ is transient.

First we record an inequality.

Lemma 4.1. *For every $R_0 > 0$, there exists a constant $c(R_0, \phi) > 0$ such that*

$$\lambda^2 \int_0^{\lambda^{-1}} r^{-1} \phi'(r^{-2}) dr + \int_{\lambda^{-1}}^{R_0} r^{-3} \phi'(r^{-2}) dr \leq c \phi(\lambda^2), \quad \forall \lambda \geq \frac{1}{R_0}. \quad (4.1)$$

Proof. Assume $\lambda \geq \frac{1}{R_0}$. By (1.2), $\phi'(r^{-2}) \leq c_1 r^{2\delta} \lambda^{2\delta} \phi'(\lambda^2)$ for $r \leq \lambda^{-1}$. Thus

$$\begin{aligned} & \lambda^2 \int_0^{\lambda^{-1}} r^{-1} \phi'(r^{-2}) dr + \int_{\lambda^{-1}}^{R_0} r^{-3} \phi'(r^{-2}) dr \\ &= \lambda^2 \phi'(\lambda^2) \int_0^{\lambda^{-1}} r^{-1} \frac{\phi'(r^{-2})}{\phi'(\lambda^2)} dr - 2 \int_{\lambda^{-1}}^{R_0} (\phi(r^{-2}))' dr \\ &\leq c_2 \phi'(\lambda^2) \lambda^{2+2\delta} \int_0^{\lambda^{-1}} r^{-1+2\delta} dr + c_2 \phi(\lambda^2) \leq c_3 (\phi'(\lambda^2) \lambda^2 + \phi(\lambda^2)) \leq 2c_3 \phi(\lambda^2) \end{aligned}$$

where we have used (2.2) in the last inequality. □

Recall that the infinitesimal generator \mathcal{L} of X is given by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x) - y \cdot \nabla f(x) \mathbf{1}_{\{|y| \leq \varepsilon\}}) j(|y|) dy \quad (4.2)$$

for every $\varepsilon > 0$ and $f \in C_b^2(\mathbb{R}^d)$ where C_b^2 is the collection of bounded C^2 functions in \mathbb{R}^d .

Using Lemma 4.1, we now prove [KSV12c, Lemma 4.2] under a weaker assumption.

Lemma 4.2. *There exists a constant $c(\phi) > 0$ such that for every $f \in C_b^2(\mathbb{R}^d)$ with $0 \leq f \leq 1$,*

$$|\mathcal{L}f_r(x)| \leq c\phi(r^{-2}) \left(1 + \sup_y \sum_{j,k} \left| \frac{\partial^2 f}{\partial y_j \partial y_k}(y) \right| \right) + b_0, \quad \text{for every } x \in \mathbb{R}^d \text{ and } r \in (0, 1],$$

where $f_r(y) := f(\frac{y}{r})$ and $b_0 := 2 \int_{|z|>1} j(|z|) dz < \infty$.

Proof. Set $L_1 = \sup_{y \in \mathbb{R}^d} \sum_{j,k} |\frac{\partial^2 f(y)}{\partial y_j \partial y_k}|$. Then

$$|f(z+y) - f(z) - y \cdot \nabla f(z)| \leq \frac{1}{2} L_1 |y|^2,$$

which implies the following estimate

$$|f_r(z+y) - f_r(z) - y \cdot \nabla f_r(z) \mathbf{1}_{\{|y| \leq r\}}| \leq \frac{L_1}{2} \frac{|y|^2}{r^2} \mathbf{1}_{\{|y| \leq r\}} + 2 \cdot \mathbf{1}_{\{|y| \geq r\}}.$$

Now, (2.6) and (4.1) yield

$$\begin{aligned} & |\mathcal{L}f_r(z)| \\ & \leq \int_{\mathbb{R}^d} |f_r(z+y) - f_r(z) - y \cdot \nabla f_r(z) \mathbf{1}_{\{|y| \leq r\}}| j(|y|) dy \\ & \leq \frac{L_1}{2} \int_{\mathbb{R}^d} \mathbf{1}_{\{|y| \leq r\}} \frac{|y|^2}{r^2} j(|y|) dy + 2 \int_{\mathbb{R}^d} \mathbf{1}_{\{r \leq |y| \leq 1\}} j(|y|) dy + 2 \int_{\mathbb{R}^d} \mathbf{1}_{\{|y| \geq 1\}} j(|y|) dy \\ & \leq c\phi(r^{-2}) \left(2 + \frac{L_1}{2}\right) + 2 \int_{\{|y| \geq 1\}} j(|y|) dy, \end{aligned}$$

where the constant c is independent of $r \in (0, 1]$. \square

Lemma 4.3. *For every $a \in (0, 1)$, there exists a positive constant $c = c(a, \phi) > 0$ such that for any $r \in (0, 1)$ and any open set D with $D \subset B(0, r)$*

$$\mathbb{P}_x(X_{\tau_D} \in B(0, r)^c) \leq c\phi(r^{-2}) \mathbb{E}_x[\tau_D] \quad \text{for all } x \in D \cap B(0, ar).$$

Proof. Using Lemma 4.2, the proof of the lemma is similar to that of [KSV12a, Lemma 4.15]. We omit the details. \square

Let $A(x, a, b) := \{y \in \mathbb{R}^d : a \leq |y - x| < b\}$ and recall that the Poisson kernel $K_D(x, z)$ of X in D is defined in (2.11).

Unlike [KSV12c], instead of Harnack inequality we use Proposition 2.6 in the next proposition.

Proposition 4.4. *Let $p \in (0, 1)$. Then there exists a constant $c(\phi, p) > 0$ such that for any $r \in (0, 1)$ we have*

$$\int_{\frac{1+p}{2}r}^{|z|} K_{B(0,s)}(x, z) ds \leq c \frac{r}{\phi(r^{-2})} j(|z|)$$

for all $x \in B(0, pr)$ and $z \in A(0, \frac{1+p}{2}r, r)$.

Proof. We split the Poisson kernel into two parts:

$$K_{B(0,s)}(x, z) = \int_{B(0,s)} G_{B(0,s)}(x, y) j(|z - y|) dy = I_1(s) + I_2(s)$$

where

$$\begin{aligned} I_1(s) &= \int_{B(0,3s/4)} G_{B(0,s)}(x,y)j(|z-y|) dy \\ I_2(s) &= \int_{A(0,3s/4,s)} G_{B(0,s)}(x,y)j(|z-y|) dy. \end{aligned}$$

First we consider $I_1(s)$. Since $|z-y| \geq \frac{1}{4}|z|$, we conclude from (2.7) that

$$\begin{aligned} I_1(s) &\leq j\left(\frac{|z|}{4}\right) \int_{B(0,3s/4)} G(x,y) dy \leq c_1 j(|z|) \int_0^{2s} t^{d-1} g(t) dt \\ &\leq c_2 \frac{j(|z|)}{\phi(s^{-2})}. \end{aligned}$$

Then, since $|z| \leq r$,

$$\begin{aligned} \int_{\frac{1+p}{2}r}^{|z|} I_1(s) ds &\leq c_2 j(|z|) \int_{\frac{1+p}{2}r}^{|z|} \frac{ds}{\phi(s^{-2})} \\ &\leq c_2 j(|z|) \frac{|z| - \frac{1+p}{2}r}{\phi(r^{-2})} \leq c_2 j(|z|) \frac{r}{\phi(r^{-2})}. \end{aligned}$$

On the other hand, by Proposition 2.6 and Lemma 2.4,

$$\begin{aligned} I_2(s) &\leq c_3 s^{-d-2} \frac{\phi'(s^{-2})}{\phi(s^{-2})} \int_{A(0,3s/4,s)} \mathbb{E}_y[\tau_{B(0,s)}] j(|z-y|) dy \\ &\leq c_4 s^{-d-2} \frac{\phi'(s^{-2})}{\phi(s^{-2})} \int_{A(0,3s/4,s)} \frac{j(|z-y|)}{\sqrt{\phi(s^{-2})\phi((s-|y|)^{-2})}} dy \\ &\leq c_4 s^{-d-2} \frac{\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \int_{A(0,3s/4,s)} \frac{j(|z-y|)}{\sqrt{\phi(|z-y|^{-2})}} dy, \end{aligned}$$

since $s - |y| \leq |z - y|$.

Observing that $A(z, 3s/4, s) \subset B(z, s) \subset A(0, |z| - s, 2r)$ we arrive at

$$\begin{aligned} I_2(s) &\leq c_4 s^{-d-2} \frac{\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \int_{A(0,|z|-s,2r)} \frac{j(|v|)}{\sqrt{\phi(|v|^{-2})}} dv \\ &= c_5 s^{-d-2} \frac{\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \int_{|z|-s}^{2r} t^{-3} \frac{\phi'(t^{-2})}{\sqrt{\phi(t^{-2})}} dv \\ &\leq c_6 s^{-d-2} \frac{\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \sqrt{\phi((|z|-s)^{-2})}. \end{aligned}$$

Then using the fact that $s \mapsto \phi'(s^{-2})$ and $s \mapsto \phi(s^{-2})^{-1}$ are increasing we obtain

$$\begin{aligned} \int_{\frac{1+p}{2}r}^{|z|} I_2(s) ds &\leq c_6 \int_{\frac{1+p}{2}r}^{|z|} \frac{s^{-d-2}\phi'(s^{-2})}{\phi(s^{-2})^{3/2}} \sqrt{\phi((|z|-s)^{-2})} ds \\ &\leq c_6 \frac{\left(\frac{1+p}{2}r\right)^{-d-2}\phi'(|z|^{-2})}{\phi(|z|^{-2})^{3/2}} \int_0^{|z|-\frac{1+p}{2}r} \sqrt{\phi(t^{-2})} dt. \end{aligned} \quad (4.3)$$

By Lemma 2.2 with $\varepsilon = \frac{\delta}{2} > 0$ for any $a \in (0, 1)$ we have

$$\begin{aligned} \int_0^a \sqrt{\phi(s^{-2})} ds &= \int_0^a \frac{\sqrt{\phi(s^{-2})}}{\sqrt{\phi(a^{-2})}} ds \sqrt{\phi(a^{-2})} \\ &\leq c_7 a^{1-\delta/2} \sqrt{\phi(a^{-2})} \int_0^a s^{-1+\delta/2} ds \leq c_8 a \sqrt{\phi(a^{-2})} \end{aligned}$$

This together with (4.3) gives

$$\begin{aligned} \int_{\frac{1+p}{2}r}^{|z|} I_2(s) ds &\leq c_9 \frac{\left(\frac{1+p}{2}r\right)^{-d-2}\phi'(|z|^{-2})}{\phi(|z|^{-2})^{3/2}} \left(|z| - \frac{1+p}{2}r\right) \phi\left(\left(|z| - \frac{1+p}{2}r\right)^{-2}\right)^{1/2} \\ &\leq c_9 \frac{\left(\frac{1+p}{2}r\right)^{-d-2}\phi'(|z|^{-2})}{\phi(|z|^{-2})^{3/2}} |z| \sqrt{\phi(|z|^{-2})} \leq c_{10} j(|z|) \frac{r}{\phi(r^{-2})}. \end{aligned}$$

□

5. UNIFORM BOUNDARY HARNACK PRINCIPLE

In this section we give a proof of the uniform boundary Harnack principle for X in an arbitrary open set with the constant not depending on the open set itself. This type of the boundary Harnack principle was first obtained in [BKK08] for rotationally symmetric stable processes. Since, using results of previous section, the proofs in this section are almost identical to the one in [KSV12c, Section 5], we give details only on parts that require extra explanation.

Recall that $X = (X_t, \mathbb{P}_x)$ is a subordinate process defined by $X_t = W_{S_t}$ where $W = (W_t, \mathbb{P}_x)$ is a Brownian motion in \mathbb{R}^d independent of the subordinator S and the Laplace exponent ϕ of the subordinator S satisfies **(A-1)**–**(A-3)**.

Using (2.8), (2.9), Proposition 2.10, Lemma 4.4 and the fact that for $U \subset D$

$$K_D(x, z) = K_U(x, z) + \mathbb{E}_x[K_D(X_{\tau_U}, z)], \quad (x, z) \in U \times D^c, \quad (5.1)$$

the proof of the next result is the same as the one of [KSV12c, Lemma 5.2].

Lemma 5.1. *Assume that X is transient and satisfies (A-1)–(A-4). For every $p \in (0, 1)$, there exists $c = c(\phi, p) > 0$ such that for every $r \in (0, 1)$, $z_0 \in \mathbb{R}^d$, $U \subset B(z_0, r)$ and for any $(x, y) \in (U \cap B(z_0, pr)) \times B(z_0, r)^c$,*

$$K_U(x, y) \leq c \frac{1}{\phi(r^{-2})} \left(\int_{U \setminus B(z_0, \frac{(1+p)r}{2})} j(|z - z_0|) K_U(z, y) dz + j(|y - z_0|) \right).$$

The process X satisfies the hypothesis **H** in [Szt00]. Therefore, by [Szt00, Theorem 1], for a Lipschitz open set $V \subset \mathbb{R}^d$ and an open subset $U \subset V$

$$\mathbb{P}_x(X_{\tau_U} \in \partial V) = 0 \quad \text{and} \quad \mathbb{P}_x(X_{\tau_U} \in dz) = K_U(x, z) dz \quad \text{on } V^c. \quad (5.2)$$

Using (5.2) and Lemma 5.1, the proof of the next result is the same as the one of [KSV12c, Lemma 5.3].

Lemma 5.2. *Assume that X is transient and satisfies (A-1)–(A-4). For every $p \in (0, 1)$, there exists $c = c(\phi, p) > 0$ such that for every $r \in (0, 1)$, for every $z_0 \in \mathbb{R}^d$, $U \subset B(z_0, r)$ and any nonnegative function u in \mathbb{R}^d which is regular harmonic in U with respect to X and vanishes in $U^c \cap B(z_0, r)$ we have*

$$u(x) \leq c \frac{1}{\phi(r^{-2})} \int_{(U \setminus B(z_0, \frac{(1+p)r}{2})) \cup B(z_0, r)^c} j(|y - z_0|) u(y) dy, \quad x \in U \cap B(z_0, pr).$$

We give a detailed proof of the next result.

Lemma 5.3. *Assume that X is transient and satisfies (A-1)–(A-4). There exists $C_2 = C_2(d, \phi) > 1$ such that for every $r \in (0, 1)$, for every $z_0 \in \mathbb{R}^d$, $U \subset B(z_0, r)$ and for any $(x, y) \in U \cap B(z_0, \frac{r}{2}) \times B(z_0, r)^c$,*

$$\begin{aligned} & C_2^{-1} \mathbb{E}_x[\tau_U] \left(\int_{U \setminus B(z_0, \frac{r}{2})} j(|z - z_0|) K_U(z, y) dz + j(|y - z_0|) \right) \\ & \leq K_U(x, y) \leq C_2 \mathbb{E}_x[\tau_U] \left(\int_{U \setminus B(z_0, \frac{r}{2})} j(|z - z_0|) K_U(z, y) dz + j(|y - z_0|) \right). \end{aligned}$$

Proof. Without loss of generality, we assume $z_0 = 0$. Fix $r \in (0, 1)$ and let $U_1 := U \cap B(0, \frac{1}{2}r)$, $U_2 := U \cap B(0, \frac{2}{3}r)$ and $U_3 := U \cap B(0, \frac{3}{4}r)$. Let $x \in U \cap B(0, \frac{r}{2})$, $y \in B(0, r)^c$.

By (5.1),

$$\begin{aligned}
K_U(x, y) &= \mathbb{E}_x[K_U(X_{\tau_{U_2}}, y)] + K_{U_2}(x, y) \\
&= \int_{U \setminus U_2} K_U(z, y) \mathbb{P}_x(X_{\tau_{U_2}} \in dz) + K_{U_2}(x, y) \\
&= \int_{U_3 \setminus U_2} K_U(z, y) \mathbb{P}_x(X_{\tau_{U_2}} \in dz) + \int_{U \setminus U_3} K_U(z, y) K_{U_2}(x, z) dz + K_{U_2}(x, y) \\
&= \int_{U_3 \setminus U_2} K_U(z, y) \mathbb{P}_x(X_{\tau_{U_2}} \in dz) + \int_{U \setminus U_3} K_U(z, y) \int_{U_2} G_{U_2}(x, w) j(|z - w|) dw dz \\
&\quad + \int_{U_2} G_{U_2}(x, w) j(|y - w|) dw =: I_1 + I_2 + I_3.
\end{aligned}$$

From Lemma 4.3 and Lemma 5.1, we see that there exist c_1 and c_2 such that

$$I_1 \leq c_1 \left(\sup_{z \in U_3} K_U(z, y) \right) \phi(r^{-2}) \mathbb{E}_x[\tau_{U_2}] \leq c_2 \mathbb{E}_x[\tau_{U_2}] \left(\int_{U \setminus U_3} j(|z|) K_U(z, y) dz + j(|y|) \right). \quad (5.3)$$

Now using (2.8) and (2.9) one can check as in [KSV12c] that there exists $c_5 = c_5(d, \phi) > 1$ such that

$$c_5^{-1} \mathbb{E}_x[\tau_{U_2}] \int_{U \setminus U_3} j(|z|) K_U(z, y) dz \leq I_2 \leq c_5 \mathbb{E}_x[\tau_{U_2}] \int_{U \setminus U_3} j(|z|) K_U(z, y) dz \quad (5.4)$$

and

$$c_5^{-1} \mathbb{E}_x[\tau_{U_2}] j(|y|) \leq I_3 \leq c_5 \mathbb{E}_x[\tau_{U_2}] j(|y|). \quad (5.5)$$

The upper bound follows from (5.3)–(5.5).

Using the strong Markov property, we get

$$\begin{aligned}
\mathbb{E}_x[\tau_U] &= \mathbb{E}_x[\tau_{U_2}] + \mathbb{E}_x \left[\mathbb{E}_{X_{\tau_{U_2}}}[\tau_U] \right] \\
&\leq \mathbb{E}_x[\tau_{U_2}] + \left(\sup_{z \in U} \mathbb{E}_z[\tau_U] \right) \mathbb{P}_x(X_{\tau_{U_2}} \in B(0, \frac{2r}{3})^c) \\
&\leq \mathbb{E}_x[\tau_{U_2}] + c_6 \phi(r^{-2})^{-1} \phi((\frac{2r}{3})^{-2}) \mathbb{E}_x[\tau_{U_2}] \leq c_7 \mathbb{E}_x[\tau_{U_2}],
\end{aligned}$$

where in the second inequality we have used Lemma 2.4 and Lemma 4.3 and in last inequality we have used (2.1).

Since

$$\begin{aligned} \int_{U \setminus U_1} j(|z|)K_U(z, y)dz &= \int_{U \setminus U_3} j(|z|)K_U(z, y)dz + \int_{U_3 \setminus U_1} j(|z|)K_U(z, y)dz \\ &\leq \int_{U \setminus U_3} j(|z|)K_U(z, y)dz + \left(\sup_{z \in U_3} K_U(z, y) \right) \int_{A(0, r/2, 3r/4)} j(|y|)dy, \end{aligned}$$

by (2.6) and Lemma 5.1,

$$\begin{aligned} &\int_{U \setminus U_1} j(|z|)K_U(z, y)dz \\ &\leq \left(1 + \frac{c_8}{\phi(r^{-2})} \int_{\frac{r}{2}}^{\frac{3r}{4}} s^{-3} \phi'(s^{-2}) ds \right) \left(\int_{U \setminus U_3} j(|z|)K_U(z, y)dz + j(|y|) \right) \\ &= \left(1 - 2 \frac{c_8}{\phi(r^{-2})} \int_{\frac{r}{2}}^{\frac{3r}{4}} (\phi(s^{-2}))' ds \right) \left(\int_{U \setminus U_3} j(|z|)K_U(z, y)dz + j(|y|) \right) \\ &\leq \left(1 + c_9 \frac{\phi(4r^{-2})}{\phi(r^{-2})} \right) \left(\int_{U \setminus U_3} j(|z|)K_U(z, y)dz + j(|y|) \right). \end{aligned} \quad (5.6)$$

Combining (2.1) and (5.4)–(5.6), we finish the proof of the lower bound. \square

Using Lemmas 5.2 and 5.3, the proof of the next result is the same as the one of [KSV12c, Lemma 5.5].

Lemma 5.4. *Assume that X is transient and satisfies (A-1)–(A-4). For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0, r)$ and for any nonnegative function u in \mathbb{R}^d which is regular harmonic in U with respect to X and vanishes a.e. on $U^c \cap B(z_0, r)$*

$$C_2^{-1} \mathbb{E}_x[\tau_U] \int_{B(z_0, \frac{r}{2})^c} j(|y - z_0|)u(y)dy \leq u(x) \leq C_2 \mathbb{E}_x[\tau_U] \int_{B(z_0, \frac{r}{2})^c} j(|y - z_0|)u(y)dy$$

for every $x \in U \cap B(z_0, \frac{r}{2})$ (where C_2 is the constant from Lemma 5.3).

As [KSV12c, Corollary 5.6], the last two lemmas immediately imply the following approximate factorization of the Poisson kernel.

Corollary 5.5. *Assume that X is transient and satisfies (A-1)–(A-4). Let $z_0 \in \mathbb{R}^d$ and $D \subset \mathbb{R}^d$ be open. Then for every $r \in (0, 1)$ and all $(x, y) \in (D \cap B(z_0, \frac{r}{2})) \times (D^c \cap B(z_0, r)^c)$ it holds that*

$$C_2^{-1} \mathbb{E}_x[\tau_{D \cap B(z_0, r)}] A_D(z_0, r, y) \leq K_D(x, y) \leq C_2 \mathbb{E}_x[\tau_{D \cap B(z_0, r)}] A_D(z_0, r, y), \quad (5.7)$$

where

$$\begin{aligned} A_D(z_0, r, y) &:= \int_{(D \cap B(z_0, r)) \setminus B(z_0, \frac{r}{2})} j(|z - z_0|) K_{D \cap B(z_0, r)}(z, y) dz \\ &\quad + j(|y - z_0|) + \int_{B(z_0, \frac{r}{2})^c} j(|z - z_0|) \mathbb{E}_z \left[K_D(X_{\tau_{D \cap B(z_0, r)}}, y) \right] dz. \end{aligned}$$

As the proof of [KSV12c, Theorem 1.1], Lemma 5.4 and (5.7) imply the uniform boundary Harnack principle with the constant not depending on the open set itself. That is why this type of result is called the uniform boundary Harnack principle.

Theorem 5.6. *Suppose that ϕ satisfies (A-1)–(A-3). There exists a constant $c = c(\phi) > 0$ such that*

- (i) *For every $z_0 \in \mathbb{R}^d$, every open set $D \subset \mathbb{R}^d$, every $r \in (0, 1)$ and for any nonnegative functions u, v in \mathbb{R}^d which are regular harmonic in $D \cap B(z_0, r)$ with respect to X and vanish a.e. on $D^c \cap B(z_0, r)$, we have*

$$\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)}$$

for all $x, y \in D \cap B(z_0, \frac{r}{2})$.

- (ii) *If X is, additionally, transient and satisfies (A-4), then for every $z_0 \in \mathbb{R}^d$, every Greenian open set $D \subset \mathbb{R}^d$, every $r \in (0, 1)$, we have*

$$K_D(x_1, y_1) K_D(x_2, y_2) \leq c K_D(x_1, y_2) K_D(x_2, y_1)$$

for all $x_1, x_2 \in D \cap B(z_0, \frac{r}{2})$ and all $y_1, y_2 \in \overline{D}^c \cap B(z_0, r)^c$.

Proof. Under the assumption of transience and (A-1)–(A-4) the result follows from Lemma 5.4 and Corollary 5.5 (see proof of [KSV12c, Theorem 1.1]).

If the process X is not transient, we can use argument similar as in the proof of [KM, Theorem 1.2, p. 17] where it is shown how to deduce Harnack inequality in dimensions $d = 1, 2$ from Harnack inequality in dimension $d \geq 3$ (since in this case the process is always transient). Since we will use the argument in the proof of Theorem 1.4 again. Here we provide the detail for the readers' convenience.

We use the notation $\tilde{x} = (x^1, \dots, x^{d-1})$ for $x = (x^1, \dots, x^{d-1}, x^d) \in \mathbb{R}^d$ and $X = ((\tilde{X}_t, X_t^d), \mathbb{P}_{(\tilde{x}, x^d)})$. As in the proof of [KM, Theorem 1.2, p. 17], we have that for every $x^d \in \mathbb{R}$, $\tilde{X} = (\tilde{X}_t, \mathbb{P}_{\tilde{x}})$ is a $(d-1)$ -dimensional subordinate Brownian motion with characteristic exponent $\tilde{\Phi}(\tilde{\xi}) = \phi(|\tilde{\xi}|^2)$ for $\tilde{\xi} \in \mathbb{R}^{d-1}$.

Suppose (i) is true for for some $d \geq 2$ and let D be an open subset of \mathbb{R}^{d-1} and $u, v: \mathbb{R}^{d-1} \rightarrow [0, \infty)$ be functions that are regular harmonic in $D \cap B(\tilde{x}_0, r)$ with respect to \tilde{X} and vanish on $D^c \cap B(\tilde{x}_0, r)$ a.e. with respect to $(d-1)$ -dimensional Lebesgue measure.

Let $f: \mathbb{R}^d \rightarrow [0, \infty)$ such that $f(\tilde{x}, x^d) = u(\tilde{x})$ and $g: \mathbb{R}^d \rightarrow [0, \infty)$ such that $g(\tilde{x}, x^d) = v(\tilde{x})$, which are regular harmonic in $B(\tilde{x}_0, r) \times \mathbb{R}$ with respect to X by the strong Markov property since $\tau_{(B(\tilde{x}_0, s) \cap D) \times \mathbb{R}} = \inf\{t > 0 : \tilde{X}_t \notin B(\tilde{x}_0, s) \cap D\}$. Clearly they vanish on $(B(\tilde{x}_0, r) \times \mathbb{R}) \cap (D \times \mathbb{R})^c$ a.e. with respect to d -dimensional Lebesgue measure.

Thus, by applying the result to f, g , we see that there exists a constant $c > 0$ such that for all $\tilde{x}_0 \in \mathbb{R}^{d-1}$, open set $D \subset \mathbb{R}^{d-1}$ and $r \in (0, 1)$

$$\frac{u(\tilde{x}_1)}{v(\tilde{x}_1)} = \frac{f((\tilde{x}_1, 0))}{g((\tilde{x}_1, 0))} \leq c \frac{f((\tilde{x}_2, 0))}{g((\tilde{x}_2, 0))} = c \frac{u(\tilde{x}_2)}{v(\tilde{x}_2)} \quad \text{for all } \tilde{x}_1, \tilde{x}_2 \in D \cap B(\tilde{x}_0, \frac{r}{2}).$$

Applying this argument first to $d = 3$ and then to $d = 2$ we finish the proof of the theorem. \square

6. GREEN FUNCTION ESTIMATES ON BOUNDED LIPSCHITZ DOMAIN

The purpose of this section is to establish sharp two-sided Green function estimates for X in any bounded Lipschitz domain D of \mathbb{R}^d .

Recall that we have assumed that $X = (X_t, \mathbb{P}_x)$ is the subordinate process defined by $X_t = W_{S_t}$ where $W = (W_t, \mathbb{P}_x)$ is a Brownian motion in \mathbb{R}^d independent of the subordinator S and the Laplace exponent ϕ of the subordinator S satisfies **(A-1)**–**(A-3)**. In this section we further assume that X is transient and that **(A-4)** also holds.

We will first establish the interior estimates using Proposition 2.5 and Theorem 2.9. As in [KSV12b], once we have the interior estimates, we can apply Theorem 2.9 and the boundary Harnack principle (Theorem 5.6), and use the arguments of [Bog00, Han05] to get the full estimates for bounded Lipschitz domain D .

Lemma 6.1. *For every bounded domain $D \subset \mathbb{R}^d$, there exists a constant $C_2 = C_2(d, \phi, \text{diam}(D)) > 0$ such that*

$$G_D(x, y) \leq C_3 \frac{|x - y|^{-d-2} \phi'(|x - y|^{-2})}{\phi(|x - y|^{-2})^2} \quad \text{for all } x, y \in D, \quad (6.1)$$

and for all $x, y \in D$ with $b_2^{-1}|x - y| \leq \delta_D(x) \wedge \delta_D(y)$

$$G_D(x, y) \geq C_3^{-1} \frac{|x - y|^{-d-2} \phi'(|x - y|^{-2})}{\phi(|x - y|^{-2})^2} \quad (6.2)$$

where $b_2 \in (0, \frac{1}{2})$ is the constant from Proposition 2.6.

Proof. Since $G_D(x, y) \leq g(|x - y|)$ and D is bounded, (6.1) is an immediate consequence of Proposition 2.5.

Now we show (6.2). We have two cases:

Case 1: $|x - y| \leq b_2$.

Since $B(x, b_2^{-1}|x-y|) \subset D$ and $y \in A(x, |x-y|, b_2^{-1}|x-y|)$, we can use Proposition 2.6 to get

$$\begin{aligned} G_D(x, y) &\geq G_{B(x, b_2^{-1}|x-y|)}(x, y) \geq c_1 \frac{b_2^{d+2}|x-y|^{-d-2}\phi'(b_2^2|x-y|^{-2})}{\phi(b_2^2|x-y|^{-2})} \mathbb{E}_x[\tau_{B(x, b_2^{-1}|x-y|)}] \\ &\geq c_2 \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}, \end{aligned}$$

where in the last inequality we have used Proposition 2.7, **(A-3)** and the facts that $b_2 \in (0, \frac{1}{2})$ and that the function $r \mapsto \frac{1}{\phi(r)}$ is decreasing.

Case 2: $|x-y| > b_2$.

In this case it follows that $\delta_D(x) \wedge \delta_D(y) > 1$. Let $x_0 \in \partial B(y, b_2)$. Then

$$b_2^{-1}|x_0 - y| = 1 < \delta_D(x) \wedge \delta_D(y)$$

and so, by the Case 1, we obtain

$$G_D(x_0, y) \geq c_2 \frac{b_2^{-d-2}\phi'(b_2^{-2})}{\phi(b_2^{-2})^2}. \quad (6.3)$$

Since $G_D(\cdot, y)$ is harmonic in $B(x_0, \frac{b_2}{2}) \cup B(x, \frac{b_2}{2})$ (with respect to X), we can use Proposition 2.9 to deduce

$$\begin{aligned} G_D(x, y) &= \mathbb{E}_x[G_D(X_{\tau_{B(x, b_2/4)}}, y)] \geq \mathbb{E}_x[G_D(X_{\tau_{B(x, b_2/4)}}, y); X_{\tau_{B(x, b_2/4)}} \in B(x_0, \frac{b_2}{4})] \\ &\geq c_3 G_D(x_0, y) \mathbb{P}_x(X_{\tau_{B(x, b_2/4)}} \in B(x_0, \frac{b_2}{4})). \end{aligned} \quad (6.4)$$

By Proposition 2.10, (2.10) and (2.11) we get

$$\begin{aligned} \mathbb{P}_x(X_{\tau_{B(x, b_2/4)}} \in B(x_0, \frac{b_2}{4})) &= \int_{B(x_0, \frac{b_2}{4})} K_{B(x, \frac{b_2}{4})}(x_0, z) dz \\ &\geq \frac{c_4}{\phi(16b_2^{-2})} \int_{B(x_0, \frac{b_2}{4})} j(|z-x|) dz. \end{aligned} \quad (6.5)$$

Since $|z-x| \leq \text{diam}(D)$, by the monotonicity of j we deduce

$$\mathbb{P}_x(X_{\tau_{B(x, b_2/4)}} \in B(x_0, \frac{b_2}{4})) \geq c_5 \frac{b_2^d j(\text{diam}(D))}{\phi(16b_2^{-2})}.$$

Therefore, using (6.3)–(6.5) we deduce

$$\begin{aligned} G_D(x, y) &\geq c_6 \frac{b_2^{-d-2}\phi'(b_2^{-2})}{\phi(b_2^{-2})^2} \\ &\geq \frac{c_6 b_2}{\text{diam}(D)} \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}, \end{aligned}$$

where in the last inequality we have used Lemma 2.1 (by considering the cases $d = 1$ and $d \geq 2$ separately) and the fact that $b_2 < |x-y| \leq \text{diam}(B)$. \square

An open set D is said to be Lipschitz domain if there is a localization radius $R_1 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$, there is a Lipschitz function $\phi_z : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ satisfying

$$|\phi_z(x) - \phi_z(w)| \leq \Lambda|x - w|,$$

and an orthonormal coordinate system CS_z with origin at z such that

$$B(z, R_1) \cap D = B(z, R_1) \cap \{y = (\tilde{y}, y_d) \text{ in } CS_z : y_d > \phi_z(\tilde{y})\}.$$

The pair (R_1, Λ) is called the characteristics of the Lipschitz domain D .

Lemma 6.2. *For every $L > 0$ and bounded Lipschitz domain D with the characteristic (R_1, Λ) , domain $D \subset \mathbb{R}^d$ there exists $c = c(L, d, \phi, R_1, \Lambda, \text{diam}(D)) > 0$ such that for every $x, y \in D$ with $|x - y| \leq L(\delta_D(x) \wedge \delta_D(y))$,*

$$G_D(x, y) \geq c \frac{|x - y|^{-d-2} \phi'(|x - y|^{-2})}{\phi(|x - y|^{-2})^2}. \quad (6.6)$$

Proof. By symmetry of G_D we may assume $\delta_D(x) \leq \delta_D(y)$. Moreover, by Lemma 6.1 we can assume that $L > b_2$ and so we only need to show (6.6) for $b_2\delta_D(x) \leq |x - y| \leq L\delta_D(x)$.

Choose a point $w \in \partial B(x, b_2\delta_D(x))$. Then Lemma 6.1 gives

$$G_D(x, w) \geq c_1 \frac{(b_2\delta_D(x))^{-d-2} \phi'((b_2\delta_D(x))^{-2})}{\phi((b_2\delta_D(x))^{-2})^2}.$$

Since $|y - w| \leq |x - y| + |x - w| \leq (L + 1)\delta_D(x)$ and $G_D(x, \cdot) = G_D(\cdot, x)$ is harmonic with respect to X in $B(y, b_2\delta_D(x)) \cup B(w, b_2\delta_D(x))$, using the assumption that D is a bounded Lipschitz domain, by Lemma 2.1 (by considering the cases $d = 1$ and $d \geq 2$ separately), Theorem 2.9 and Harnack chain argument obtain

$$\begin{aligned} G_D(x, y) &\geq c_2 G_D(x, w) \geq c_3 \frac{(b_2\delta_D(x))^{-d-2} \phi'((b_2\delta_D(x))^{-2})}{\phi((b_2\delta_D(x))^{-2})^2} \\ &\geq c_4 \frac{|x - y|^{-d-2} \phi'(|x - y|^{-2})}{\phi(|x - y|^{-2})^2}. \end{aligned}$$

□

For the remainder of this section, we assume that D is a bounded Lipschitz domain with characteristics (R_1, Λ) .

Without loss of generality we may assume that $R_1 \leq \frac{1}{4}$. Since D is Lipschitz, there exists $\kappa = \kappa(\Lambda) \in (0, \frac{1}{2})$ such that for each $Q \in \partial D$ and $r \in (0, R_1)$, there exists a point

$$A_r(Q) \in D \cap B(Q, r) \text{ satisfying } B(A_r(Q), \kappa r) \subset D \cap B(Q, r).$$

Recall that $G_D(\cdot, y)$ is regular harmonic in $D \setminus \overline{B(y, \varepsilon)}$ for every $\varepsilon > 0$ and vanishes outside D .

Fix $z_0 \in D$ with $\kappa R_1 < \delta_D(z_0) < R_1$ and set $\varepsilon_1 := \frac{\kappa R_1}{24}$. Define

$$r(x, y) := \delta_D(x) \vee \delta_D(y) \vee |x - y|, \quad x, y \in D$$

and

$$\mathcal{B}(x, y) := \begin{cases} \{A \in D : \delta_D(A) > \frac{\kappa}{2}r(x, y), |x - A| \vee |y - A| < 5r(x, y)\} & \text{if } r(x, y) < \varepsilon_1 \\ \{z_0\} & \text{if } r(x, y) \geq \varepsilon_1. \end{cases} \quad (6.7)$$

Note that for every $(x, y) \in D \times D$ with $r(x, y) < \varepsilon_1$

$$\frac{1}{6}\delta_D(A) \leq r(x, y) \leq 2\kappa^{-1}\delta_D(A), \quad A \in \mathcal{B}(x, y). \quad (6.8)$$

Set

$$C_4 := C_3(1 \wedge \text{diam}(D))\left(\frac{\delta_D(z_0)}{2}\right)^{-d-2} \frac{\phi'\left(\left(\frac{\delta_D(z_0)}{2}\right)^{-2}\right)}{\phi\left(\left(\frac{\delta_D(z_0)}{2}\right)^{-2}\right)^2}.$$

By (6.1) and Lemma 2.1 (by considering the cases $d = 1$ and $d \geq 2$ separately) we see that

$$G_D(x, z_0) \leq C_4 \quad \text{for } x \in D \setminus B(z_0, \frac{\delta_D(z_0)}{2}).$$

Now we define

$$g_D(x) := G_D(x, z_0) \wedge C_4. \quad (6.9)$$

We note that for $\delta_D(z) \leq 6\varepsilon_1$,

$$g_D(z) = G_D(z, z_0),$$

since $6\varepsilon_1 < \frac{\delta_D(z_0)}{4}$ and thus $|z - z_0| \geq \delta_D(z_0) - 6\varepsilon_1 \geq \frac{\delta_D(z_0)}{2}$.

The following lemma follows from Theorem 2.9 and the standard Harnack chain argument:

Lemma 6.3. *There exists $c > 1$ such that for every $x \in D$ satisfying $\delta_D(x) \geq \frac{\kappa^3 \varepsilon_1}{64}$ we have*

$$c^{-1} \leq g_D(x) \leq c. \quad \square$$

Theorem 6.4. *Suppose X is transient and ϕ satisfies (A-1)–(A-4). If D is a bounded Lipschitz domain with characteristics (R_1, Λ) , then there exists $c = c(\text{diam}(D), R_1, \Lambda, \phi) > 1$ such that for every $x, y \in D$,*

$$c^{-1} \frac{g_D(x)g_D(y)}{g_D(A)^2|x-y|^d\phi(|x-y|^{-2})} \leq G_D(x, y) \leq c \frac{g_D(x)g_D(y)}{g_D(A)^2|x-y|^d\phi(|x-y|^{-2})}, \quad A \in \mathcal{B}(x, y), \quad (6.10)$$

where g_D and $\mathcal{B}(x, y)$ are defined by (6.9) and (6.7) respectively.

Proof. Since the proof is an adaptation of the proofs of [Bog00, Proposition 6] and [Han05, Theorem 2.4], we only give the proof when $\delta_D(x) \leq \delta_D(y) \leq \frac{\kappa}{4}|x - y|$. In this case, we have $r(x, y) = |x - y|$

By Theorem 2.9, we see that for all $x, y \in D$ and $A_1, A_2 \in \mathcal{B}(x, y)$,

$$g_D(A_1) \text{ is comparable to } g_D(A_2).$$

Set $r = \frac{|x-y| \wedge \varepsilon_1}{2}$ and choose

$$Q_x, Q_y \in \partial D \quad \text{with} \quad |Q_x - x| = \delta_D(x) \quad \text{and} \quad |Q_y - y| = \delta_D(y).$$

Pick points $x_1 = A_{\kappa r/2}(Q_x)$ and $y_1 = A_{\kappa r/2}(Q_y)$ so that

$$x, x_1 \in B(Q_x, \kappa r/2) \quad \text{and} \quad y, y_1 \in B(Q_y, \kappa r/2).$$

Then one can easily check that $|z_0 - Q_x| \geq \kappa r$ and $|y - Q_x| \geq r$.

Then Theorem 5.6 implies

$$c_1^{-1} \frac{G_D(x_1, y)}{g_D(x_1)} \leq \frac{G_D(x, y)}{g_D(x)} \leq c_1 \frac{G_D(x_1, y)}{g_D(x_1)}$$

for some $c_1 > 1$.

Also, since $|z_0 - Q_y| \geq r$ and $|x_1 - Q_y| \geq r$, by Theorem 5.6 again,

$$c_1^{-1} \frac{G_D(x_1, y_1)}{g_D(y_1)} \leq \frac{G_D(x_1, y)}{g_D(y)} \leq c_1 \frac{G_D(x_1, y_1)}{g_D(y_1)}.$$

Therefore

$$c_1^{-2} \frac{G_D(x_1, y_1)}{g_D(x_1)g_D(y_1)} \leq \frac{G_D(x, y)}{g_D(x)g_D(y)} \leq c_1^2 \frac{G_D(x_1, y_1)}{g_D(x_1)g_D(y_1)}.$$

Now we can use Lemma 6.2 to get

$$\frac{c_2^{-1} c_1^{-2}}{g_D(x_1)g_D(y_1)} \frac{|x_1 - y_1|^{-d-2} \phi'(|x_1 - y_1|^{-2})}{\phi(|x_1 - y_1|^{-2})^2} \leq \frac{G_D(x, y)}{g_D(x)g_D(y)} \leq \frac{c_2 c_1^2}{g_D(x_1)g_D(y_1)} \frac{|x_1 - y_1|^{-d-2} \phi'(|x_1 - y_1|^{-2})}{\phi(|x_1 - y_1|^{-2})^2} \quad (6.11)$$

for some $c_2 > 1$.

Since $\frac{|x-y|}{3} < |x_1 - y_1| < 2|x - y|$, Lemma 2.1 (by considering the cases $d = 1$ and $d \geq 2$ separately) yields

$$\frac{|x_1 - y_1|^{-d-2} \phi'(|x_1 - y_1|^{-2})}{\phi(|x_1 - y_1|^{-2})^2} \leq 2 \cdot 3^{d+2} \frac{|x-y|^{-d-2} \phi'(9|x-y|^{-2})}{\phi(9|x-y|^{-2})^2} \leq 2 \cdot 3^{d+2} \frac{|x-y|^{-d-2} \phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}$$

and

$$\frac{|x_1 - y_1|^{-d-2} \phi'(|x_1 - y_1|^{-2})}{\phi(|x_1 - y_1|^{-2})^2} \geq 3^{-1} \cdot 2^{-d-2} \frac{|x-y|^{-d-2} \phi'(4^{-1}|x-y|^{-2})}{\phi(4^{-1}|x-y|^{-2})^2} \geq 3^{-1} \cdot 2^{-d-2} \frac{|x-y|^{-d-2} \phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}.$$

Therefore,

$$\frac{2^{-d-2} c_2^{-1} c_1^{-2}}{g_D(x_1)g_D(y_1)} \frac{|x-y|^{-d-2} \phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \leq \frac{G_D(x, y)}{g_D(x)g_D(y)} \leq \frac{3^{d+2} c_2 c_1^2}{g_D(x_1)g_D(y_1)} \frac{|x-y|^{-d-2} \phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}. \quad (6.12)$$

If $r = \frac{\varepsilon_1}{2}$, then $r(x, y) = |x - y| \geq \varepsilon_1$ and so

$$g_D(A) = g_D(z_0) = C_4 \quad \text{and} \quad \delta_D(x_1) \wedge \delta_D(y_1) \geq \frac{\kappa^2 r}{2} = \frac{\kappa^2 \varepsilon_1}{4}.$$

Thus, in this case, Lemma 6.3 yields

$$c_3^{-1} \leq \frac{g_D(A)^2}{g_D(x_1)g_D(y_1)} \leq c_3 \quad (6.13)$$

for some $c_3 > 1$.

In the case $r < \frac{\varepsilon_1}{2}$ we have $r(x, y) = |x - y| < \varepsilon_1$ and $r = \frac{1}{2}r(x, y)$. Hence

$$\delta_D(x_1) \wedge \delta_D(y_1) \geq \frac{\kappa^2 r}{2} = \frac{\kappa^2 r(x, y)}{4}.$$

Since $|x_1 - A| \vee |y_1 - A| \leq 5r(x, y) + |x_1 - x| + |y_1 - y| \leq 5r(x, y) + 2\kappa r \leq 6r(x, y)$, Theorem 2.9 applied to g_D gives

$$c_4^{-1} \leq \frac{g_D(A)}{g_D(x_1)} \leq c_4 \quad \text{and} \quad c_4^{-1} \leq \frac{g_D(A)}{g_D(y_1)} \leq c_4 \quad (6.14)$$

for some constant $c_4 > 0$. Combining (6.12)-(6.14), we get

$$c_5^{-1} \frac{g_D(x)g_D(y)}{g_D(A)^2} \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \leq G_D(x, y) \leq c_5 \frac{g_D(x)g_D(y)}{g_D(A)^2} \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}$$

for all $A \in \mathcal{B}(x, y)$. \square

7. EXPLICIT GREEN FUNCTION ESTIMATES ON BOUNDED $C^{1,1}$ -OPEN SETS

The purpose of this section is to establish the explicit Green function estimates from Theorem 6.4 in the case of bounded $C^{1,1}$ open sets.

Theorem 7.1. *Suppose that $X = (X_t : t \geq 0)$ is a transient d -dimensional subordinate Brownian motion where the corresponding subordinator S has the Laplace exponent ϕ satisfying (A-1)–(A-5). If D is a bounded $C^{1,1}$ domain in \mathbb{R}^d with $C^{1,1}$ characteristics (R, Λ) , then there exists $c = c(R, \Lambda, \phi, \text{diam}(D)) > 0$ such that*

$$c^{-1} (V(\delta_D(x)) \wedge 1) \leq g_D(x) \leq c (V(\delta_D(x)) \wedge 1) \quad \text{for all } x \in D. \quad (7.1)$$

Proof. The proof follows from the proof of [KSV12b, Theorem 4.6] by using our Proposition 2.5, Lemma 3.7 and Theorem 5.6. \square

Proof of Theorem 1.2. When D is connected, Theorem 1.2 follows from [KSV12b, (4.38)] and our Theorems 6.4 and 7.1.

Next we assume that D is not connected. The proof below is similar to the one in [CKSV12].

Let (R, Λ) be the $C^{1,1}$ characteristics of D . Note that D has only finitely many components and the distance between any two distinct components of D is at least $R > 0$.

Assume first that x and y are in two distinct components of D . Let $D(x)$ be the component of D that contains x . Then by the strong Markov property and (2.10) we obtain

$$G_D(x, y) = \mathbb{E}_x \left[G_D(X_{\tau_{D(x)}}, y) \right] = \mathbb{E}_x \left[\int_0^{\tau_{D(x)}} \left(\int_{D \setminus D(x)} j(|X_s - z|) G_D(z, y) dz \right) ds \right].$$

Consequently,

$$j(\text{diam}(D)) \mathbb{E}_x[\tau_{D(x)}] \int_{D \setminus D(x)} G_D(y, z) dz \leq G_D(x, y) \leq j(R) \mathbb{E}_x[\tau_{D(x)}] \int_{D \setminus D(x)} G_D(y, z) dz. \quad (7.2)$$

Applying the two-sided estimates (1.7) established in the first part of this proof to $D(x)$, we get

$$\frac{c_1^{-1}}{\sqrt{\phi(\delta_D(x)^{-2})}} = \frac{c_1^{-1}}{\sqrt{\phi(\delta_{D(x)}(x)^{-2})}} \leq \mathbb{E}_x[\tau_{D(x)}] \leq \frac{c_1^{-1}}{\sqrt{\phi(\delta_{D(x)}(x)^{-2})}} = \frac{c_1^{-1}}{\sqrt{\phi(\delta_D(x)^{-2})}}. \quad (7.3)$$

By (7.3) we get

$$\int_{D \setminus D(x)} G_D(y, z) dz \geq \int_{D(y)} G_{D(y)}(y, z) dz = \mathbb{E}_y[\tau_{D(y)}] \geq \frac{c_2}{\sqrt{\phi(\delta_D(y)^{-2})}}.$$

On the other hand, (2.10) and (7.3) imply

$$\begin{aligned} \int_{D \setminus D(x)} G_D(y, z) dz &\leq \mathbb{E}_y[\tau_D] = \mathbb{E}_y[\tau_{D(y)}] + \mathbb{E}_y[\mathbb{E}_{X_{\tau_{D(y)}}}[\tau_D]] \\ &\leq \frac{c_3}{\sqrt{\phi(\delta_D(y)^{-2})}} + \mathbb{E}_y \left[\int_0^{\tau_{D(y)}} \int_{D \setminus D(y)} j(|X_s - z|) \mathbb{E}_z[\tau_D] dz ds \right] \\ &\leq \frac{c_3}{\sqrt{\phi(\delta_D(y)^{-2})}} + c_4 j(R) \mathbb{E}_y[\tau_{D(y)}] \leq \frac{c_5}{\sqrt{\phi(\delta_D(y)^{-2})}}. \end{aligned}$$

We conclude from the last three displays and (7.2) that there is a constant $c_6 \geq 1$ such that

$$\frac{c_6^{-1}}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \leq G_D(x, y) \leq \frac{c_6}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}. \quad (7.4)$$

Noting that

$$R \leq |x - y| \leq \text{diam}(D)$$

when x and y are in different components of D , we obtain (1.7).

Now we assume that x, y are in the same component U of D . Applying (1.7) to U we get

$$\begin{aligned} G_D(x, y) &\geq G_U(x, y) \geq c_7 \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_U(x)^{-2})\phi(\delta_U(y)^{-2})}} \right) \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \\ &= c_7 \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right) \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}. \end{aligned}$$

For the upper bound, we use the strong Markov property, (2.10) and (7.3)–(7.4) to get

$$\begin{aligned}
& G_D(x, y) \\
&= G_U(x, y) + \mathbb{E}_x [G_D(X_{\tau_U}, y)] \\
&\leq c_8 \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right) \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} + \mathbb{E}_x \left[\int_0^{\tau_U} \int_{D \setminus U} j(|X_s - z|) G_D(z, y) dz ds \right] \\
&\leq c_8 \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right) \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} + j(R) \mathbb{E}_x[\tau_U] \int_{D \setminus U} G_D(y, z) dz \\
&\leq c_8 \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right) \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} + \frac{c_9 \int_{D \setminus U} \frac{dz}{\sqrt{\phi(\delta_D(z)^{-2})}}}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}. \tag{7.5}
\end{aligned}$$

Since D is bounded, we get

$$\begin{aligned}
\frac{1}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \int_{D \setminus U} \frac{dz}{\sqrt{\phi(\delta_D(z)^{-2})}} &\leq \frac{|D|}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})\phi(\text{diam}(D)^{-2})}} \\
&\leq c_{10} \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right),
\end{aligned}$$

which together with (7.5) gives

$$G_D(x, y) \leq c_{11} \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}} \right).$$

□

Before we prove Corollary 1.3, we record a simple fact.

Lemma 7.2. *If $\delta_* \in (0, 1)$ and ψ is a Bernstein function satisfying*

$$\frac{\psi(\lambda x)}{\psi(\lambda)} \geq \sigma x^{1-\delta_*} \quad \text{for all } x \geq 1 \text{ and } \lambda \geq \lambda_0, \tag{7.6}$$

for some $\sigma > 0$. Then there exists a constant $c > 0$ such that $\psi(\lambda) \leq c\lambda\psi'(\lambda)$ for all $\lambda \geq \lambda_0$.

Proof. Let $a_1 = 2 \vee (\frac{2}{\sigma})^{\frac{1}{1-\delta_*}}$. Since ψ' is decreasing,

$$(a_1 - 1)\lambda\psi'(\lambda) \geq \int_{\lambda}^{a_1\lambda} \psi'(t) dt = \psi(a_1\lambda) - \psi(\lambda). \tag{7.7}$$

Since $\psi(a_1\lambda) \geq \sigma a_1^{1-\delta_*}\psi(\lambda)$ by (7.6), from (7.7), we get

$$(a_1 - 1)\lambda\psi'(\lambda) \geq (\sigma a_1^{1-\delta_*} - 1)\psi(\lambda) \geq \psi(\lambda).$$

□

Proof of Corollary 1.3. It follows from the assumptions of the corollary that we can apply Lemma 7.2. Thus by (2.2) and Lemma 7.2 we obtain $\lambda\phi'(\lambda) \leq \phi(\lambda) \leq c\lambda\phi'(\lambda)$ for all $\lambda \geq \lambda_0$. Therefore (A-1)–(A-5) hold and (1.7) is equivalent to (1.10). \square

Proof of Theorem 1.4. When $d = 1$, the theorem follows from Proposition 2.3, Theorem 3.1 and Theorem 5.6 (i).

Note that the result in [CKSV12, Lemma 4.2] is true in our case too. By this result, Theorem 2.9, Theorem 5.6 (i) and Theorem 1.2, the proof of Theorem 1.4 is the same as the proof of [KSV12b, Theorem 1.3] when $d \geq 3$.

Note that if D is a $C^{1,1}$ open set in \mathbb{R}^{d-1} with characteristics (R, Λ) , then $D \times \mathbb{R}$ is clearly $C^{1,1}$ open set in \mathbb{R}^d with the same characteristics (R, Λ) . Thus the case $d = 2$ can be handled in the same way as in the proof of Theorem 5.6 (i). \square

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