Cohomology theories of simplicial complexes, algebras, and digraphs

Alexander Grigor'yan [*]
Department of Mathematics
University of Bielefeld
33613 Bielefeld, Germany

Yuri Muranov[†]
Department of Mathematics
Grodno State University
Grodno 230023, Republic of Belarus

Shing-Tung Yau
Department of Mathematics
Harvard University
Cambridge MA 02138, USA

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1 Introduction

Let A be an associative algebra over a field $\mathbb{K}$. Denote by $C^n(A)$ the space of all $\mathbb{K}$-multilinear functions $f : A^n \to A$. Hochschild [5] introduced a differential $D : C^n(A) \to C^{n+1}(A)$ that satisfies $D^2 = 0$ thus making $\{C^n(A)\}_{n=0}^{\infty}$ into a cochain complex. Its cohomologies are called Hochschild cohomologies of A and are denoted by $HH^n(A)$.

In 1983 Gerstenhaber and Schack [3] established the following beautiful relation between the Hochschild and simplicial cohomologies. Let S be a (finite) simplicial complex over $\mathbb{K}$. Denote by V the set of all barycenters of the simplexes of S, and introduce on V a structure of a digraph

(=directed graph) by defining the edges $a \to b$ between two barycenters $a, b \in V$ as follows: if s_a is the simplex whose barycenter is a, and s_b the simplex with the barycenter b, then by definition

$$a \to b \Leftrightarrow s_a \supset s_b. \tag{1.1}$$

In particular, the underlying undirected graph is the 1-skeleton of the barycentric subdivision B_S of S (see Fig. 1)
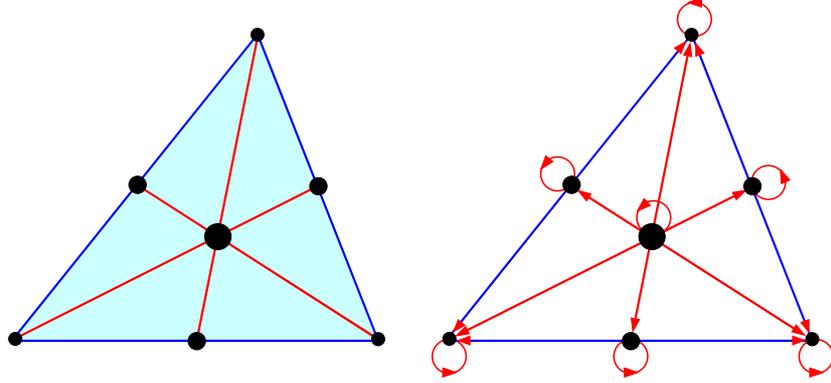

Figure 1: The barycentric subdivision B_S of a 2-simplex (left) and the associated digraph $(V, \to)$ (right).

Denote by A the $\mathbb{K}$-linear space of all formal linear combinations of the edges of V. Define the multiplication of the edges by

$$(a, b)\, (a', b') = \begin{cases} (a, b'), & \text{if } b = a' \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

(cf. Fig. 2) and then extend it to a multiplication in A, thus making A into an associative algebra over $\mathbb{K}$.
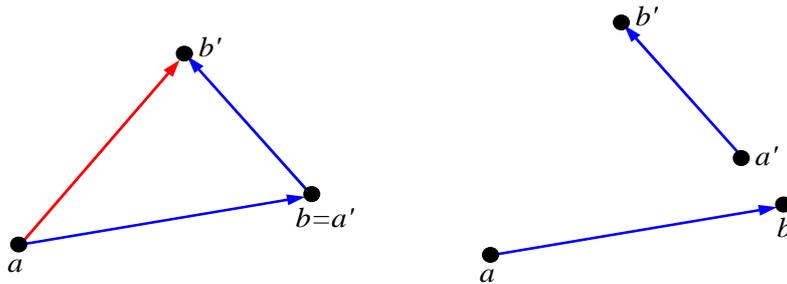

Figure 2: Illustration to (1.2): $(a, b)\,(a', b') = (a, b')$ (left) and $(a, b)\,(a', b') = 0$ (right)

Then Gerstenhaber and Schack proved in [3] that

$$HH^n(A) \cong H^n(S) \tag{1.3}$$

where $H^n(S)$ is the simplicial cohomology of S. They wrote in [3] that although they could define explicitly a cochain map $C^n(S) \to C^n(A)$, they were unable to derive the isomorphism

of cohomologies from this map by a direct computation. They used instead quite advanced tools developed in their previous paper [2].

In this paper we give a direct proof of (1.3) based on the cochain map (Theorem 7.2). The proof consists of a few steps that we describe now. We use the cochain complex $\{\Omega^n\}$ based on the digraph $(V, \rightarrow)$. The elements of Ω^n are $\mathbb{K}$-valued functions on n-paths $i_0 \rightarrow i_1 \rightarrow ... \rightarrow i_n$ of the vertices of V, with appropriately defined exterior differential $d : \Omega^n \rightarrow \Omega^{n+1}$. The notions of exterior differential and exterior forms on general digraphs were developed in [1] and [4]. Here we need a special case of transitive digraphs that can be treated independently as we do in Section 3, but still in the spirit and with notation of [4]. Consider also a regularized cochain complex $\{\Omega^n_{reg}\}$ where the functions are defined only on the paths $i_0 \rightarrow i_1 \rightarrow ... \rightarrow i_n$ with distinct vertices.

Next, consider the product $\widetilde{V} = V \times V$ that also becomes a digraph with the following definition of edges:

$$(i, j) \rightarrow (i', j') \text{ in } \widetilde{V} \Leftrightarrow j \rightarrow i' \text{ in } V$$

(see Section 4). This type of product of graphs is unusual in graph theory, but appropriate for our purpose. On the digraph $(\widetilde{V}, \rightarrow)$ we also introduce a cochain complex $\{\widetilde{\Omega}^n\}$ analogously to $\{\Omega^n\}$ on $(V, \rightarrow)$, with an additional weight in the exterior differential that emphasizes the role of the diagonal diag $\widetilde{V} \cong V$.

Denote by $\overline{A}$ the algebra spanned by all pairs $(a, b) \in V \times V$ with multiplication (1.2), and consider a Hochschild cohomology $HH^n(A, \overline{A})$ that comes from the cochain complex $C^n(A, \overline{A})$ of multilinear functions $f : A^n \rightarrow \overline{A}$.

We prove (1.3) by means of the following chain of relations:

$$HH^n(A) \cong HH^n(A, \overline{A}) \cong H^n(\widetilde{\Omega}) \cong H^n(\Omega) \cong H^n(\Omega_{reg}) \cong H^n(B_S) \cong H^n(S). \qquad (1.4)$$

The corresponding cochain spaces are related as follows:

$$C^n(A) \subset C^n(A, \overline{A}) \cong \widetilde{\Omega}^n \succ \Omega^n \succ \Omega^n_{reg} \cong C^n(B_S) \succ C^n(S),$$

where $\mathcal{A} \succ \mathcal{B}$ means that $\mathcal{B}$ is a quotient space of $\mathcal{A}$. Hence, in the first step we enlarge the cochain space in question, but in each of the following steps we reduce it while preserving cohomologies, until we get to $C^n(S)$. Let us comment on each of the relations in (1.4).

1. The the isomorphism $H^n(\widetilde{\Omega}) \cong H^n(\Omega)$ is a major new result of this paper and is proved in Lemma 4.2. We establish first a map between the corresponding cochain complexes and then prove that the cohomology of the kernel of this map is trivial. This step is crucial in the entire proof as it allows to drastically reduce the cochain space $\widetilde{\Omega}^n$ to Ω^n.

2. The isomorphism $HH^n(A, \overline{A}) \cong H^n(\widetilde{\Omega})$ follows from the isomorphism of the corresponding cochain complexes $C^n(A, \overline{A})$ and $\widetilde{\Omega}^n$. The proof of the latter contains a trick of changing the tensor notation, which in an instant transforms the elements of $C^n(A, \overline{A})$ to those of $\widetilde{\Omega}^n$ (Lemma 5.1).

3. The isomorphism $HH^n(A) \cong HH^n(A, \overline{A})$ is proved in Lemma 5.2 using an argument from the proof of Lemma 4.2.

4. The isomorphism $H^n(\Omega) \cong H^n(\Omega_{reg})$ is proved in Lemma 6.1, and the proof is also similar to that of Lemma 4.2.

5. The isomorphism $H^n(\Omega_{reg}) \cong H^n(B_S)$ is rather straightforward and follows from the isomorphism of the cochain complexes $\{\Omega^n_{reg}\}$ and $\{C^n(B_S)\}$ (Lemma 7.1).

6. The isomorphism $H^n(B_S) \cong H^n(S)$ is classical.

The structure of the paper is determined by the above sequence of Lemmas. The main Theorem 7.2 is stated and proved at the end of the paper.

2 Hochschild cohomologies

Let $\mathbb{K}$ be a field and A be an associative algebra over $\mathbb{K}$, that is, A is a $\mathbb{K}$-linear space with multiplication $A \times A \ni (a, b) \mapsto ab \in A$ that is an associative bilinear operation.

Let M be $\mathbb{K}$-linear space that has a compatible structure of a A-bimodule (in particular, M can be an algebra over $\mathbb{K}$ containing A as a subalgebra). For any non-negative integer n, denote by $C^n(A, M)$ the set of all $\mathbb{K}$-multilinear functions $f : A^n \to M$. Clearly, $C^n(A, M)$ is a $\mathbb{K}$-linear space. For example, $A^0 = \{0\}$ and, hence, $C^0(A, M) \cong M$.

Consider the Hochschild differential $D : C^n(A, M) \to C^{n+1}(A, M)$ defined on any function $f \in C^n(A, M)$ as follows:

$$
\begin{aligned}
Df(a_1, a_2, ..., a_{n+1}) \;=\;& a_1 f(a_2, ..., a_{n+1}) \\
& + \sum_{k=1}^{n} (-1)^k f(a_1, ..., a_k a_{k+1}, ... a_{n+1}) \\
& + (-1)^{n+1} f(a_1, ..., a_n) a_{n+1},
\end{aligned}
$$

for all $a_i \in A$. In particular, for $n = 0$, the differential of a constant $f = m \in M$ is a function $Df : A \to M$ given by

$$Df(a) = am - ma.$$

It is known [5] that $D^2 = 0$ so that we have a cochain complex

$$0 \to C^0(A, M) \xrightarrow{D} ... \xrightarrow{D} C^n(A, M) \xrightarrow{D} C^{n+1}(A, M) \xrightarrow{D} ...$$

whose cohomologies are called the Hochschild cohomologies and are denoted by $HH^*(A, M)$.

3 Cohomologies of transitive graphs

Let V be a finite set, whose elements will be called vertices. Denote by $\Lambda^n = \Lambda^n(V, \mathbb{K})$ the $\mathbb{K}$-linear space of all functions $f : V^{n+1} \to \mathbb{K}$. We use the following notation

$$f_{i_0...i_n} := f(i_0, ..., i_n)$$

for all $i_k \in V$. Any sequence $i_0...i_n$ of $n + 1$ vertices of V will be called an n-path. Hence, any $f \in \Lambda^n$ is a $\mathbb{K}$-valued function on n-paths.

Fix a function $\delta : V \to \mathbb{K}$ and define the exterior differential $d : \Lambda^n \to \Lambda^{n+1}$ by

$$(df)_{i_0...i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \delta_{i_k} f_{i_0...\widehat{i_k}...i_{n+1}}, \tag{3.1}$$

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where the hat $\widehat{i_k}$ means omission of the index i_k. Strictly speaking, the operator d should have been denoted by d_δ but we omit the index δ as usually this causes no confusion.

Claim. $d^2 = 0$.

Proof. For any $f \in \Lambda^n$, we have

$$
\begin{aligned}
\left(d^2 f\right)_{i_0 \ldots i_{n+2}} &= \sum_{k=0}^{n+2} (-1)^k \, \delta_{i_k} \left(df\right)_{i_0 \ldots \widehat{i_k} \ldots i_{n+2}} \\
&= \sum_{k=0}^{n+2} (-1)^k \, \delta_{i_k} \left(\sum_{l=0}^{k-1} (-1)^l \, \delta_{i_l} f_{i_0 \ldots \widehat{i_l} \ldots \widehat{i_k} \ldots i_{n+2}} + \sum_{l=k+1}^{n+2} (-1)^{l-1} \, \delta_{i_l} f_{i_0 \ldots \widehat{i_k} \ldots \widehat{i_l} \ldots i_{n+2}} \right) \\
&= \sum_{0 \le l < k \le n+2} (-1)^{k+l} \, \delta_{i_k} \delta_{i_l} f_{i_0 \ldots \widehat{i_l} \ldots \widehat{i_k} \ldots i_{n+2}} \\
&\quad - \sum_{0 \le k < l \le n+2} (-1)^{k+l} \, \delta_{i_k} \delta_{i_l} f_{i_0 \ldots \widehat{i_k} \ldots \widehat{i_l} \ldots i_{n+2}}.
\end{aligned}
$$

After switching k and l in the last sum we see that it is equal to the previous one, whence $d^2 f = 0$ follows. ∎

Hence, we obtain a cochain complex

$$
0 \to \Lambda^0 \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^n \xrightarrow{d} \Lambda^{n+1} \xrightarrow{d} \ldots
$$

Next, we assume that V has an additional structure of a digraph, that is, a relation $\to$ is defined on V. Any couple of vertices $i, j \in V$ with $i \to j$ is called a (directed) edge.

Definition 3.1 A n-path $i_0 \ldots i_n$ of vertices of a digraph $(V, \to)$ is called *allowed* if

$$
i_0 \to i_1 \to \ldots \to i_n,
$$

and *non-allowed* otherwise.

On any digraph $(V, \to)$ there is a natural way of defining a cochain complex $\{\Omega^n\}$ with the exterior differential d that takes into account allowed paths (see [4]). However, we need this construction for a specific class of *transitive* digraphs, that can be done easily as below.

Definition 3.2 Let $(V, \to)$ be a digraph and let δ be a function on V. We say that V is a δ-*transitive* if

$$
i \to j \text{ and } j \to k \Rightarrow i \to k, \quad \text{provided } \delta_j \ne 0,
$$

for all $i, j, k \in V$.

Let V be a δ-transitive graph, where δ is the same as in the definition (3.1) of d. Denote by $\Omega^n = \Omega^n(V, \to, \mathbb{K})$ the space of $\mathbb{K}$-valued functions on all allowed n-paths on V. Observe that the formula (3.1) can be used to define d as an operator from Ω^n to Ω^{n+1} as follows. Given $f \in \Omega^n$, first extend f arbitrarily to be a function on all n-paths. Then (3.1) defines df as a function on all $(n+1)$-paths. The restriction of df to the allowed $(n+1)$-paths yields an element of Ω^{n+1}, that will also be denoted by df.

Claim. *If V is δ-transitive then, for any $f \in \Omega^n$, the differential df is well defined as an element of Ω^{n+1}.*

Proof. We need to show that df on allowed $(n+1)$-paths does not depend on how f was extended. Indeed, let $i_0 \ldots i_{n+1}$ be an allowed path. We need to verify that in the right hand

side of (3.1) the contribution of the terms $\delta_{i_k} f_{i_0 \ldots \widehat{i_k} \ldots i_{n+1}}$ with non-allowed paths $i_0 \ldots \widehat{i_k} \ldots i_{n+1}$ vanishes. Indeed, if $\delta_{i_k} = 0$ then this term does not give any contribution. If $\delta_{i_k} \neq 0$ then $i_{k-1} \to i_k \to i_{k+1}$ imply $i_{k-1} \to i_{k+1}$ so that the n-path

$$i_0 \ldots \widehat{i_k} \ldots i_{n+1} = i_0 \ldots i_{k-1} i_{k+1} \ldots i_{n+1}$$

is allowed, which finishes the proof. ∎

Hence, we obtain a cochain complex

$$0 \to \Omega^0 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n \xrightarrow{d} \Omega^{n+1} \xrightarrow{d} \ldots$$

of δ-transitive graph, whose cohomologies

$$H^n(\Omega) = \ker d|_{\Omega^n} / \operatorname{Im} d|_{\Omega^{n-1}}$$

are of primary interest for us. By construction, $H^n(\Omega)$ is a $\mathbb{K}$-linear space that depends on $V, \mathbb{K}, \to, \delta$.

A digraph $(V, \to)$ is called transitive if

$$i \to j \to k \Rightarrow i \to k.$$

Of course, a transitive digraph is 1-transitive, that is, δ-transitive with $\delta \equiv 1$. For transitive graphs we will use $\delta \equiv 1$ also in the definition (3.1) of d. For example, any poset is a transitive digraph.

4 Self-product of a transitive graph

Let V be a transitive digraph, and $\{\Omega^n\}$ the associated cochain complex.

Set $\widetilde{V} = V \times V$ and define the edges on $\widetilde{V}$ as follows

$$(i, j) \to (i', j') \Leftrightarrow j \to i' \tag{4.1}$$

for all $i, j, i', j' \in V$ (cf. Fig. 3).

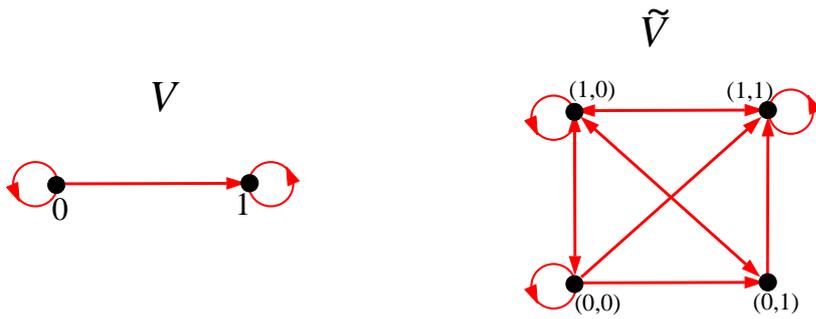Figure 3: An example of digraphs V and $\widetilde{V}$

Besides, define a function $\delta : \widetilde{V} \to \mathbb{K}$ by

$$\delta_{(i,j)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4.2}$$

Claim. *If V is transitive then $\widetilde{V}$ is δ-transitive.*

    Proof. Indeed, if

$$(i,j) \to (i',j') \to (i'',j'')$$

and $\delta_{(i',j')} \neq 0$ then $i' = j'$ and

$$j \to i' = j' \to i''$$

whence $j \to i''$ and, hence, $(i,j) \to (i'',j'')$. $\blacksquare$

Definition 4.1 The δ-transitive graph $\widetilde{V}$ is called a *self-product* of V.

    As before, let $\{\Omega^n\}$ be the cochain complex of the transitive graph V, and denote by $\widetilde{\Omega}^n$ the cochain complex of the δ-transitive graph $\widetilde{V}$.

    We say that a digraph $(V, \to)$ is *reflexive* if $i \to i$ for all $i \in V$. For example, any poset is transitive and reflexive. Now we can state and prove our main technical lemma.

Lemma 4.2 *If V is transitive and reflexive, then we have the isomorphism of cohomologies:*

$$H^n(\widetilde{\Omega}) \cong H^n(\Omega) \tag{4.3}$$

for all non-negative integers n.

    Proof. Let us identify V with diag $\widetilde{V}$. Note that $(i,i) \to (j,j)$ is equivalent to $i \to j$ so that diag $\widetilde{V}$ as a subgraph of $\widetilde{V}$ is isomorphic to a digraph V. For any function f on the allowed n-paths in $\widetilde{V}$, let Φf be a restriction of f to the allowed n-paths on V. In tensor notation, any function $f \in \widetilde{\Omega}^n$ has the components $f_{(i_0,j_0)...(i_n,j_n)}$, and then Φf is defined by

$$(\Phi f)_{i_0...i_n} = f_{(i_0,i_0)...(i_n,i_n)}$$

for all allowed n-paths $i_0...i_n$ on V. Hence, we obtain an epimorphism

$$\Phi : \widetilde{\Omega}^n \to \Omega^n. \tag{4.4}$$

We start with verifying that the following diagram is commutative:

$$
\begin{array}{ccc}
\widetilde{\Omega}^n & \overset{d}{\to} & \widetilde{\Omega}^{n+1} \\
\downarrow^{\Phi} & & \downarrow^{\Phi} \\
\Omega^n & \overset{d}{\to} & \Omega^{n+1}
\end{array}
\tag{4.5}
$$

For any $f \in \widetilde{\Omega}^n$ and any allowed $(n+1)$-path $i_0...i_{n+1}$ on V we have

$$
\begin{aligned}
(d(\Phi f))_{i_0...i_{n+1}} &= \sum_{k=0}^{n+1} (-1)^k (\Phi f)_{i_0...\widehat{i_k}...i_{n+1}} \\
&= \sum_{k=0}^{n+1} (-1)^k \delta_{(i_k,i_k)} f_{(i_0,i_0)...\widehat{(i_k,i_k)}...(i_{n+1},i_{n+1})} \\
&= (df)_{(i_0,i_0)...(i_{n+1},i_{n+1})} \\
&= (\Phi(df))_{i_0...i_{n+1}}
\end{aligned}
$$

whence $d\Phi = \Phi d$ follows.

Denote by K^n the kernel of the mapping (4.4). It follows that d is well-defined on K^n and we have a commutative diagram

$$
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{d} & K^n & \xrightarrow{d} & K^{n+1} & \xrightarrow{d} & \dots \\
& & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{d} & \widetilde{\Omega}^n & \xrightarrow{d} & \widetilde{\Omega}^{n+1} & \xrightarrow{d} & \dots \\
& & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} & & \\
\dots & \xrightarrow{d} & \Omega^n & \xrightarrow{d} & \Omega^{n+1} & \xrightarrow{d} & \dots \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}
$$

where $K^n \to \widetilde{\Omega}^n$ is a natural inclusion and, hence, the columns are short exact sequences. By zigzag Lemma, we obtain an long exact sequence in cohomologies

$$
\dots \to H^n(K) \to H^n(\widetilde{\Omega}) \to H^n(\Omega) \to H^{n+1}(K) \to \dots
$$

Hence, in order to prove the isomorphism (4.3) it suffices to show that, for all non-negative integers n,

$$
H^n(K) \cong \{0\},
$$

that is, if $f \in K^n$ and $df = 0$ then there is $g \in K^{n-1}$ such that $f = dg$.

Let us denote the elements of $\widetilde{V}$ by $\mathbf{i}_1, \mathbf{i}_2$ etc. If $\mathbf{i} = (i, j)$ then set

$$
\mathbf{i}' = (j, j) \in \operatorname{diag} \widetilde{V}
$$

so that $\mathbf{i}'$ can be regarded as "projection" of $\mathbf{i}$ onto V. Note that always $\mathbf{i} \to \mathbf{i}'$, which follows from the reflexivity of V, and

$$
\mathbf{i} \to \mathbf{j} \ \Leftrightarrow \ \mathbf{i}' \to \mathbf{j}
$$

for all $\mathbf{i}, \mathbf{j} \in \widetilde{V}$, since $\mathbf{i}$ and $\mathbf{i}'$ have the same second component.

The following claim is a key to the proof.

Claim. *For any $f \in \widetilde{\Omega}^n$ there exists $g \in \widetilde{\Omega}^{n-1}$ such that the following holds:*

(i) *for any allowed $(n-1)$-path $\mathbf{i}_1 \dots \mathbf{i}_n$ that lies on $\operatorname{diag} \widetilde{V}$,*

$$
g_{\mathbf{i}_1 \dots \mathbf{i}_n} = 0; \tag{4.6}
$$

(ii) *for any allowed n-path $\mathbf{i}_1 \dots \mathbf{i}_{n+1}$ on $\widetilde{V}$ that does not lie entirely on $\operatorname{diag} \widetilde{V}$,*

$$
(-1)^k (df)_{\mathbf{i}_1 \dots \mathbf{i}_k \mathbf{i}'_k \dots \mathbf{i}_{n+1}} = f_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}} - (dg)_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}}, \tag{4.7}
$$

where k is the smallest index such that $\mathbf{i}_k \notin \operatorname{diag} \widetilde{V}$.

Let us define g as follows: for any allowed $(n-1)$-paths $\mathbf{i}_1 \dots \mathbf{i}_n$ on $\widetilde{V}$, set $g_{\mathbf{i}_1 \dots \mathbf{i}_n} = 0$ if all $\mathbf{i}_k \in \operatorname{diag} \widetilde{V}$, whereas otherwise set

$$
g_{\mathbf{i}_1 \dots \mathbf{i}_n} = (-1)^k f_{\mathbf{i}_1 \dots \mathbf{i}_{k-1} \mathbf{i}_k \mathbf{i}'_k \mathbf{i}_{k+1} \dots \mathbf{i}_n}, \tag{4.8}
$$

where k is the minimal index such that $\mathbf{i}_k \notin \operatorname{diag} \widetilde{V}$. Observe that the right hand side of (4.8) is well defined, since the n-path $\mathbf{i}_1 \dots \mathbf{i}_{k-1} \mathbf{i}_k \mathbf{i}'_k \mathbf{i}_{k+1} \dots \mathbf{i}_n$ is allowed whenever $\mathbf{i}_1 \dots \mathbf{i}_n$ is allowed, because $\mathbf{i}_k \to \mathbf{i}'_k \to \mathbf{i}_{k+1}$ (see Fig. 4).

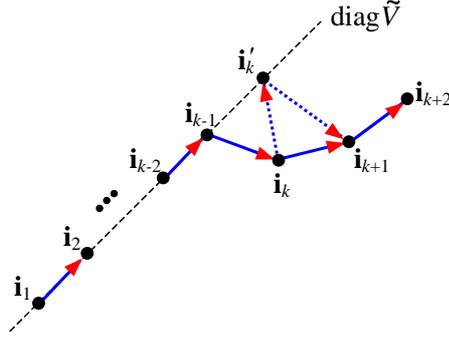Figure 4: The path $\mathbf{i}_1...\mathbf{i}_k\mathbf{i}'_k\mathbf{i}_{k+1}...$ is allowed

We have by (3.1)

$$(dg)_{\mathbf{i}_1...\mathbf{i}_{n+1}} = \sum_{l=1}^{n+1} (-1)^{l-1} \delta_{\mathbf{i}_l} g_{\mathbf{i}_1...\widehat{\mathbf{i}_l}...\mathbf{i}_{n+1}}. \tag{4.9}$$

The term $\delta_{\mathbf{i}_l} g_{\mathbf{i}_1...\widehat{\mathbf{i}_l}...\mathbf{i}_{n+1}}$ in (4.9) vanishes whenever $\delta_{\mathbf{i}_l} = 0$, which, in particular, is the case when $l = k$ as $\mathbf{i}_k \notin \operatorname{diag} \widetilde{V}$. Assume that $\delta_{\mathbf{i}_l} \neq 0$, in particular, $l \neq k$. Then the $(n-1)$-path $\mathbf{i}_1...\widehat{\mathbf{i}_l}...\mathbf{i}_{n+1}$ is allowed. The first term in this path outside $\operatorname{diag} \widetilde{V}$ is again $\mathbf{i}_k$, and it has the ordinal number $k-1$ in this path, if $l < k$, and the ordinal number k if $l > k$. Hence, if $l < k$, then we obtain by (4.8)

$$g_{\mathbf{i}_1...\widehat{\mathbf{i}_l}...\mathbf{i}_{n+1}} = (-1)^{k-1} f_{\mathbf{i}_1...\widehat{\mathbf{i}_l}...\mathbf{i}_k\mathbf{i}'_k...\mathbf{i}_{n+1}},$$

and if $l > k$ then

$$g_{\mathbf{i}_1...\widehat{\mathbf{i}_l}...\mathbf{i}_{n+1}} = (-1)^{k} f_{\mathbf{i}_1...\mathbf{i}_k\mathbf{i}'_k...\widehat{\mathbf{i}_l}...\mathbf{i}_{n+1}}.$$

Combining all the terms together, we obtain

$$(dg)_{\mathbf{i}_1...\mathbf{i}_{n+1}} = \sum_{l=1}^{k-1} (-1)^{k+l} \delta_{\mathbf{i}_l} f_{\mathbf{i}_1...\widehat{\mathbf{i}_l}...\mathbf{i}_k\mathbf{i}'_k...\mathbf{i}_{n+1}} + \sum_{l=k+1}^{n+1} (-1)^{k+l-1} \delta_{\mathbf{i}_l} f_{\mathbf{i}_1...\mathbf{i}_k\mathbf{i}'_k...\widehat{\mathbf{i}_l}...\mathbf{i}_{n+1}}. \tag{4.10}$$

By (3.1) we have also

$$\begin{aligned}
(df)_{\mathbf{i}_1...\mathbf{i}_k\mathbf{i}'_k...\mathbf{i}_{n+1}} = {} & \sum_{l=1}^{k-1} (-1)^{l-1} \delta_{\mathbf{i}_l} f_{\mathbf{i}_1...\widehat{\mathbf{i}_l}...\mathbf{i}_k\mathbf{i}'_k...\mathbf{i}_{n+1}} \\
& + (-1)^{k-1} \delta_{\mathbf{i}_k} f_{\mathbf{i}_1...\widehat{\mathbf{i}_k}\ \mathbf{i}'_k...\mathbf{i}_{n+1}} \\
& + (-1)^{k} \delta_{\mathbf{i}'_k} f_{\mathbf{i}_1...\mathbf{i}_k\widehat{\mathbf{i}'_k}...\mathbf{i}_{n+1}} \\
& + \sum_{l=k+1}^{n+1} (-1)^{l} \delta_{\mathbf{i}_l} f_{\mathbf{i}_1...\mathbf{i}_k\mathbf{i}'_k...\widehat{\mathbf{i}_l}...\mathbf{i}_{n+1}}
\end{aligned}$$

Noticing that $\delta_{\mathbf{i}_k} = 0$, $\delta_{\mathbf{i}'_k} = 1$, $f_{\mathbf{i}_1...\mathbf{i}_k\widehat{\mathbf{i}'_k}...\mathbf{i}_{n+1}} = f_{\mathbf{i}_1...\mathbf{i}_{n+1}}$, substituting here (4.10) and multiplying by $(-1)^k$, we obtain (4.7).

Now assuming that $f \in K^n$ and $df = 0$, let us show that the function g defined by (4.8) belongs to K^{n-1} and satisfies $dg = f$. That $g \in K^{n-1}$ is true by (4.6). Let us show that $dg = f$, that is, for any allowed n-path $\mathbf{i}_1...\mathbf{i}_{n+1}$,

$$(dg)_{\mathbf{i}_1...\mathbf{i}_{n+1}} = f_{\mathbf{i}_1...\mathbf{i}_{n+1}}. \tag{4.11}$$

9

If all $\mathbf{i}_1, ..., \mathbf{i}_{n+1} \in \operatorname{diag} \widetilde{V}$ then we have by (4.6) that $(dg)_{\mathbf{i}_1 ... \mathbf{i}_{n+1}} = 0$, which matches $f_{\mathbf{i}_1 ... \mathbf{i}_{n+1}} = 0$ that is true by $f \in K^n$. Otherwise, let k be the minimal value of the index such that $\mathbf{i}_k \notin \operatorname{diag} \widetilde{V}$. Then by $df = 0$ we have

$$(df)_{\mathbf{i}_1 ... \mathbf{i}_k \mathbf{i}'_k ... \mathbf{i}_{n+1}} = 0,$$

because, as it was observed above, the path $\mathbf{i}_1 ... \mathbf{i}_k \mathbf{i}'_k ... \mathbf{i}_{n+1}$ is allowed. Substituting this into (4.7), we obtain (4.11). ∎

5 Algebra associated with poset

Let V be a finite poset. Denote by $\overline{A}$ the $\mathbb{K}$-linear space of all formal linear combinations of pairs (i, j) of vertices of V. The pair (i, j) as an element of $\overline{A}$ will be denoted by e_{ij} so that any element $a \in \overline{A}$ can be uniquely represented in the form

$$a = \sum_{i,j \in V} a^{ij} e_{ij}$$

where a^{ij} are the coefficients from $\mathbb{K}$. Let A be a subspace of $\overline{A}$ spanned by all e_{ij} with $i \to j$; that is, any $a \in A$ has the form

$$a = \sum_{\{i,j \in V : i \to j\}} a^{ij} e_{ij}.$$

Define in $\overline{A}$ multiplication first on the basis elements

$$e_{ij} \cdot e_{i'j'} = \delta_{ji'} e_{ij'} = \begin{cases} e_{ij'}, & \text{if } i' = j, \\ 0, & \text{otherwise,} \end{cases} \tag{5.1}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \tag{5.2}$$

and then extend multiplication to all the elements of $\overline{A}$ by bilinearity. It follows from (5.1) that, for all $a, b \in \overline{A}$ and $i, j \in V$,

$$(ab)^{ij} = \sum_{l \in V} a^{il} b^{lj}. \tag{5.3}$$

It is easy to see that $\overline{A}$ is an associative algebra over $\mathbb{K}$. Also, $\overline{A}$ has a unity $1_{\overline{A}} = \sum_{i \in V} e_{ii}$.

By a poset property we see that if $e_{ij}, e_{kl} \in A$ then $e_{ij} e_{kl} \in A$ so that A is a subalgebra of $\overline{A}$ (and $1_A = 1_{\overline{A}}$). Hence, the Hochschild cochain complex $\{C^n(A, \overline{A})\}$ is defined as in Section 2.

On the other hand, we have a cochain complex $\{\Omega^n\}$ of a poset V as defined in Section 3, and a cochain complex $\{\widetilde{\Omega}^n\}$ of a δ-transitive graph $\widetilde{V} = V \times V$ as defined in Section 4, where δ is given by (5.2) or, equivalently, by (4.2).

Lemma 5.1 *The cochain complexes $\{C^n(A, \overline{A})\}$ and $\{\widetilde{\Omega}^n\}$ are isomorphic.*

Proof. For any function $f \in C^n(\overline{A}, \overline{A})$ set

$$f_{i_1 j_1 i_2 j_2 ... i_n j_n} = f(e_{i_1 j_1}, ..., e_{i_n j_n})$$

for all $i_k, j_k \in V$. Since f takes values in $\overline{A}$, $f_{i_1 j_1 i_2 j_2 ... i_n j_n}$ is linear combination of the terms e_{ij} as follows:

$$f_{i_1 j_1 i_2 j_2 ... i_n j_n} = \sum_{i,j \in V} f^{ij}_{i_1 j_1 i_2 j_2 ... i_n j_n} e_{ij}$$

with the coefficients $f^{ij}_{i_1j_1i_2j_2...i_nj_n} \in \mathbb{K}$. It follows that, for all $a_1, ..., a_n \in A$,

$$f(a_1, ..., a_n) = \sum f^{ij}_{i_1j_1...i_nj_n} a_1^{i_1j_1}...a_n^{i_nj_n} e_{ij}$$

where the summation is performed over all $i, j, i_k, j_k \in V$.

By definition of the Hochschild differential D on $C^n(\overline{A}, \overline{A})$ we have

$$
\begin{aligned}
(Df)_{i_1j_1...i_{n+1}j_{n+1}} &= (Df)\left(e_{i_1j_1}, ..., e_{i_{n+1}j_{n+1}}\right) \\
&= e_{i_1j_1} f\left(e_{i_2j_2}, ..., e_{i_{n+1}j_{n+1}}\right) \\
&\quad + \sum_{k=1}^{n} (-1)^k f\left(e_{i_1j_1}, ..., e_{i_kj_k} \cdot e_{i_{k+1}j_{k+1}}, ...e_{i_{n+1}j_{n+1}}\right) \\
&\quad + (-1)^{n+1} f\left(e_{i_1j_1}, ..., e_{i_nj_n}\right) e_{i_{n+1}j_{n+1}}.
\end{aligned}
$$

Using (5.1) and (5.3) we obtain

$$
\begin{aligned}
(Df)^{ij}_{i_1j_1...i_{n+1}j_{n+1}} &= \sum_{l \in V} e^{il}_{i_1j_1} f^{lj}_{i_2j_2...i_{n+1}j_{n+1}} \\
&\quad + \sum_{k=1}^{n} (-1)^k \delta_{j_k i_{k+1}} f^{ij}_{i_1j_1...i_{k-1}j_{k-1}i_kj_{k+1}i_{k+2}j_{k+2}...i_{n+1}j_{n+1}} \\
&\quad + (-1)^{n+1} \sum_{l \in V} f^{il}_{i_1j_1...i_nj_n} e^{lj}_{i_{n+1}j_{n+1}}
\end{aligned}
$$

whence

$$
\begin{aligned}
(Df)^{ij}_{i_1j_1...i_{n+1}j_{n+1}} &= \delta_{ii_1} f^{j_1j}_{i_2j_2...i_{n+1}j_{n+1}} \\
&\quad + \sum_{k=1}^{n} (-1)^k \delta_{j_k i_{k+1}} f^{ij}_{i_1j_1...i_{k-1}j_{k-1}i_kj_{k+1}i_{k+2}j_{k+2}...i_{n+1}j_{n+1}} \\
&\quad + (-1)^{n+1} \delta_{j_{n+1}j} f^{ii_{n+1}}_{i_1j_1...i_nj_n}.
\end{aligned}
$$

Now we change notation, namely, we rename $f^{ij}_{i_1j_1...i_nj_n}$ to $f_{ii_1j_1...i_nj_nj}$ and then rename in the latter the indices as follows: $f_{i_0j_0i_1j_1...i_nj_n}$. Effectively this means the following change of notation:

$$i \rightsquigarrow i_0, \quad i_k \rightsquigarrow j_{k-1}, \quad j_k \rightsquigarrow i_k, \quad j \rightsquigarrow j_n.$$

In the notation for Df we should use $j \rightsquigarrow j_{n+1}$ instead of $j \rightsquigarrow j_n$, so that the above formula for Df becomes:

$$
\begin{aligned}
(Df)_{i_0j_0...i_{n+1}j_{n+1}} &= \delta_{i_0j_0} f_{i_1j_1...i_{n+1}j_{n+1}} \\
&\quad + \sum_{k=1}^{n} (-1)^k \delta_{i_kj_k} f_{i_0j_0...i_{k-1}j_{k-1}i_{k+1}j_{k+1}...i_{n+1}j_{n+1}} \\
&\quad + (-1)^{n+1} \delta_{i_{n+1}j_{n+1}} f_{i_0j_0...i_nj_n}.
\end{aligned}
$$

All pairs (i_k, j_k) can be regarded as elements of $\widetilde{V} = V \times V$. Using notation

$$\mathbf{i}_k = (i_k, j_k)$$

we see that the components of f can be denoted by $f_{\mathbf{i}_0...\mathbf{i}_n}$. Hence, the above formula for Df can be rewritten as follows

$$(Df)_{\mathbf{i}_0...\mathbf{i}_{n+1}} = \sum_{k=0}^{n+1} (-1)^k \, \delta_{\mathbf{i}_k} f_{\mathbf{i}_0...\widehat{\mathbf{i}_k}...\mathbf{i}_{n+1}} \tag{5.4}$$

for all $\mathbf{i}_k \in \widetilde{V}$. Consequently, any function $f \in C^n(\overline{A}, \overline{A})$ can be identified with a function from $\Lambda^n(\widetilde{V})$, this identification is clearly a linear isomorphism, and it transforms the operator D to d as one can see from comparison of (5.4) and (3.1) (note that we use the same the function δ in (5.4) and (3.1)).

Consider now the space $C^n\left(A, \overline{A}\right)$. The functions from $C^n\left(A, \overline{A}\right)$ can be regarded as restrictions of the functions from $C^n\left(\overline{A}, \overline{A}\right)$ to the domain A^n. In particular, for any $f \in C^n(A, \overline{A})$, the value $f\left(e_{i_1 j_1}, ..., e_{i_n j_n}\right)$ is defined only if $i_k \to j_k$, for all $k = 1, ..., n$. Hence, the component $f^{ij}_{i_1 j_1 ... i_n j_n}$ is defined also under the same condition, that is, for the sequences of indices such that

$$ii_1 \to j_1 i_2 \to j_2 ... i_n \to j_n j.$$

In the notation $f_{i_0 j_0 ... i_n j_n} = f_{\mathbf{i}_0...\mathbf{i}_n}$ this condition becomes $j_{k-1} \to i_k$, that is,

$$i_0 j_0 \to i_1 j_1 \to ... \to i_n j_n,$$

which is equivalent to

$$\mathbf{i}_0 \to \mathbf{i}_1 \to ... \to \mathbf{i}_n,$$

given that the edges in $\widetilde{V}$ are defined by (4.1). Hence, any function $f \in C^n(A, \overline{A})$ is identified with a $\mathbb{K}$-valued function on allowed n-paths on $\widetilde{V}$, and such functions are exactly the elements of $\widetilde{\Omega}^n$, which proves the isomorphism of the cochain complexes $\{C^n(A, \overline{A})\}$ and $\{\widetilde{\Omega}^n\}$. $\blacksquare$

Lemma 5.2 *We have $HH^n(A, A) \cong HH^n\left(A, \overline{A}\right)$.*

Proof. In the previous proof we have identified the space $C^n\left(A, \overline{A}\right)$ with the space $\widetilde{\Omega}^n$ of functions on allowed n-paths on $\widetilde{V}$. The space $C^n\left(A, A\right)$ is a subspace of $C^n\left(A, \overline{A}\right)$, and a function $f \in C^n\left(A, \overline{A}\right)$ belongs to $C^n\left(A, A\right)$ if and only if, for all $i_1 \to j_1, ..., i_n \to j_n$,

$$i \not\to j \Rightarrow f^{ij}_{i_1 j_1 ... i_n j_n} = 0 \tag{5.5}$$

(in other words, the component f^{ij} vanishes if ij is not an edge). Let $\widetilde{\Omega}^n_0$ be the image of $C^n\left(A, A\right)$ in $\widetilde{\Omega}^n$. Then we have the isomorphism of the cochain complexes

$$C^n\left(A, \overline{A}\right) \cong \widetilde{\Omega}^n \quad \text{and} \quad C^n\left(A, A\right) \cong \widetilde{\Omega}^n_0.$$

It suffices to prove that

$$H^n(\widetilde{\Omega}) \cong H^n(\widetilde{\Omega}_0),$$

which is equivalent to

$$H^n(\widetilde{\Omega}/\widetilde{\Omega}_0) = \{0\}. \tag{5.6}$$

Before we can prove (5.6), let us reformulate the property (5.5) in terms of the notation $f_{\mathbf{i}_0...\mathbf{i}_n}$ that we use for the elements of $\widetilde{\Omega}^n$. Let us introduce one more relation $\twoheadrightarrow$ on $\widetilde{V}$:

$$(i, j) \twoheadrightarrow (i', j') \Leftrightarrow i \to j'$$

(observe the difference with the definition (4.1) of $\to$ on $\widetilde{V}$). Then the relation $i \to j$ in the notation $f^{ij}_{i_1 j_1 \dots i_n j_n}$ is equivalent to $\mathbf{i}_0 \twoheadrightarrow \mathbf{i}_n$ in the notation $f_{\mathbf{i}_0 \dots \mathbf{i}_n}$.

We say that an allowed n-path $\mathbf{i}_0 \dots \mathbf{i}_n$ on $\widetilde{V}$ is *proper* if $\mathbf{i}_0 \twoheadrightarrow \mathbf{i}_n$, and *improper* otherwise. Then it follows from (5.5) that a function $f \in \widetilde{\Omega}^n$ belongs to $\widetilde{\Omega}^n_0$ if and only if

$$f_{\mathbf{i}_0 \dots \mathbf{i}_n} = 0 \quad \text{for any allowed improper } n\text{-path } \mathbf{i}_0 \dots \mathbf{i}_n. \tag{5.7}$$

To prove (5.6) we need to show that if $f \in \widetilde{\Omega}^n$ and $df \in \widetilde{\Omega}^{n+1}_0$ then there exists $g \in \widetilde{\Omega}^{n-1}$ such that $dg - f \in \widetilde{\Omega}^n_0$. Define $g \in \widetilde{\Omega}^{n-1}$ as in Claim in the proof of Lemma 4.2. Then we must show that, for any allowed improper n-path $\mathbf{i}_1 \dots \mathbf{i}_{n+1}$ on $\widetilde{V}$,

$$(dg)_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}} = f_{\mathbf{i}_1 \dots \mathbf{i}_{n+1}}. \tag{5.8}$$

An improper path cannot lie entirely on diag $\widetilde{V}$ as otherwise $\mathbf{i}_1 \to \dots \to \mathbf{i}_{n+1}$ implies $\mathbf{i}_1 \to \mathbf{i}_{n+1}$ and $\mathbf{i}_1 \twoheadrightarrow \mathbf{i}_{n+1}$. Let k be the smallest index with $\mathbf{i}_k \notin$ diag $\widetilde{V}$. Then g satisfies (4.7), and the identity (5.8) will follow from (4.7) if we show that

$$(df)_{\mathbf{i}_1 \dots \mathbf{i}_k \mathbf{i}'_k \dots \mathbf{i}_{n+1}} = 0. \tag{5.9}$$

The path $\mathbf{i}_1 \dots \mathbf{i}_k \mathbf{i}'_k \dots \mathbf{i}_{n+1}$ is allowed, so that we obtain (5.9) from $df \in \widetilde{\Omega}^{n+1}_0$ if we show that this path is improper. If $k < n+1$ then it is improper because its first and last vertices are the same as those of the improper path $\mathbf{i}_1 \dots \mathbf{i}_{n+1}$. In the case $k = n+1$ we need to verify that the path $\mathbf{i}_1 \dots \mathbf{i}_{n+1} \mathbf{i}'_{n+1}$ is improper, that is, $\mathbf{i}_1 \not\twoheadrightarrow \mathbf{i}'_{n+1}$. Indeed, since the last components of $\mathbf{i}_{n+1}$ and $\mathbf{i}'_{n+1}$ are the same, the condition $\mathbf{i}_1 \not\twoheadrightarrow \mathbf{i}'_{n+1}$ follows from $\mathbf{i}_1 \not\twoheadrightarrow \mathbf{i}_{n+1}$. $\blacksquare$

6 Regularized cohomologies of posets

A n-path $i_0 \dots i_n$ of the vertices of a digraph $(V, \to)$ is called *regular* if $i_{k-1} \neq i_k$ for all $k = 1, \dots, n$, and irregular otherwise. Let from now on $(V, \to)$ be a poset. Any the regular allowed n-path $i_0 \dots i_n$ on a poset has an additional property that all the vertices i_k in this path are distinct. Indeed, if $i_l = i_k$ for some $l < k - 1$ then $i_l \to i_{l+1} \to \dots i_{k-1} \to i_k$ implies by transitivity that $i_l \to i_{k-1} \to i_k$ whence by a poset property $i_{k-1} = i_k$, which contradicts the regularity.

Alongside the space Ω^n of $\mathbb{K}$-valued functions on all allowed n-paths on V, consider the space Ω^n_{reg} of $\mathbb{K}$-valued functions on all regular allowed n-paths. Observe that $d\Omega^n_{reg} \subset \Omega^{n+1}_{reg}$ in the following sense: any function $f \in \Omega^n_{reg}$ can be extended to a function from Ω^n, also denoted by f, then df on the regular allowed $(n+1)$-paths does not depend on the choice of extension. Indeed, if $i_0 \dots i_{n+1}$ is a regular allowed $(n+1)$-path then

$$(df)_{i_0 \dots i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k f_{i_0 \dots \widehat{i_k} \dots i_{n+1}}, \tag{6.1}$$

and each path $i_0 \dots \widehat{i_k} \dots i_{n+1}$ is regular and allowed. Hence, $\{\Omega^n_{reg}\}$ is a cochain complex, that will be referred to as the regularized cochain complex of the poset V.

Lemma 6.1 *If V is a poset then*

$$H^n(\Omega) \cong H^n(\Omega_{reg}). \tag{6.2}$$

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Proof. Consider the following diagram

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{d} & K^n & \xrightarrow{d} & K^{n+1} & \xrightarrow{d} & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{d} & \Omega^n & \xrightarrow{d} & \Omega^{n+1} & \xrightarrow{d} & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{d} & \Omega^n_{reg} & \xrightarrow{d} & \Omega^{n+1}_{reg} & \xrightarrow{d} & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

where the mappings $\Omega^n \to \Omega^n_{reg}$ are just restrictions of the functions from Ω^n to regular allowed paths, K^n is the kernel of this restriction, and $K^n \to \Omega^n$ is a natural inclusion. This diagram is clearly commutative and the columns are short exact sequences. Hence, the isomorphism (6.2) will follow if we prove that

$$H^n(K) \cong \{0\},$$

that is, for any $f \in K^n$ with $df = 0$ there is $g \in K^{n-1}$ such that $dg = f$.

If $n = 0$ then $K^0 = \{0\}$ so the claim is trivial. For $n = 1$, a function $f \in K^1$ may have non-zero components only of the form f_{ii}. Since $(df)_{iii} = f_{ii}$, it follows from $df = 0$ that also $f_{ii} = 0$ so that $f = 0$. Hence, $g = 0$ is a solution.

Assume $n \geq 2$. We will construct explicitly a solution $g \in K^{n-1}$ of the equation $df = g$. Let us first illustrate this construction in the case $n = 2$. For any allowed 1-path $i \to j$, set

$$
g_{ij} = \left\{
\begin{array}{ll}
0, & i \neq j \\
f_{iii}, & i = j.
\end{array}
\right.
$$

Then $g \in K^1$ and let us show that $dg = f$, that is,

$$(dg)_{ijk} = f_{ijk} \tag{6.3}$$

for all allowed paths ijk. If the path ijk is regular then all i, j, k are distinct, and

$$(dg)_{ijk} = g_{jk} - g_{ik} + g_{ij} = 0 = f_{ijk}.$$

If $i = j$ then

$$(dg)_{iik} = g_{ii} = f_{iii}.$$

Since $df = 0$, we have

$$0 = (df)_{iiik} = f_{iik} - f_{iii}$$

whence

$$(dg)_{iik} = f_{iik}.$$

Similarly one treats the case $j = k$, which settles (6.3).

Now consider the general case $n \geq 2$. If $i_0...i_{n-1}$ is a regular allowed path then set

$$g_{i_0...i_{n-1}} = 0.$$

If $i_0...i_{n-1}$ is an irregular allowed path then let k be the smallest index with the property that $i_k = i_{k+1}$. Denoting by j the common value of i_k and i_{k+1}, we can write

$$i_0...i_{n-1} = i_0...i_{k-1}jji_{k+2}...i_{n-1}.$$

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Then set

$$g_{i_0...i_{n-1}} = (-1)^k f_{i_0...i_{k-1}jjji_{k+2}...i_{n-1}}. \tag{6.4}$$

Note that the n-path $i_0...i_{k-1}jjji_{k+2}...i_{n-1}$ on the right hand side is obtained by inserting one more vertex j between already existing in $i_0...i_{n-1}$ two vertices j at positions k and $k+1$ (see Fig. 5).
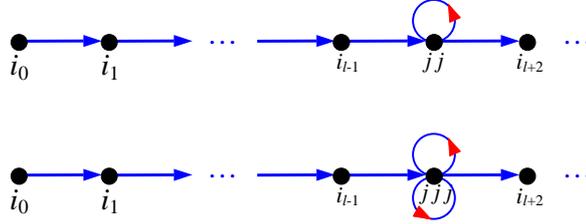

Figure 5: Paths $i_0...jj...i_{n-1}$ and $i_0...jjj...i_{n-1}$

By definition, we have $g \in K^{n-1}$. Let us show that $dg = f$ in Ω^n, that is,

$$(dg)_{i_0...i_n} = f_{i_0...i_n} \tag{6.5}$$

for any allowed path $i_0...i_n$. If the path $i_0...i_n$ is regular then

$$(dg)_{i_0...i_n} = \sum_{l=0}^{n} (-1)^l g_{i_0...\widehat{i_l}...i_n} = 0,$$

because all the paths $i_0...\widehat{i_l}...i_n$ are also regular. Since also $f_{i_0...i_n} = 0$ by $f \in K^n$, we obtain the identity (6.5) in this case.

Let now the path $i_0...i_n$ be irregular. Let k be the smallest index with the property $i_k = i_{k+1}$, and denote by j the common value of i_k and i_{k+1}, that is,

$$i_0...i_n = i_0...i_{k-1}jji_{k+2}...i_n.$$

Observe that the k-path $i_0...i_k$ is regular and allowed, which implies that all the vertices in this path are distinct. We have by (6.1)

$$
\begin{aligned}
(dg)_{i_0...i_n} &= \sum_{l=0}^{k-1} (-1)^l g_{i_0...\widehat{i_l}...jj...i_n} \tag{6.6} \\
&+ (-1)^k g_{i_0...\widehat{jj}...i_n} + (-1)^{k+1} g_{i_0...j\widehat{j}...i_n} \tag{6.7} \\
&+ \sum_{l=k+2}^{n} (-1)^l g_{i_0...jj...\widehat{i_l}...i_n} \tag{6.8}
\end{aligned}
$$

Clearly, the two terms in (6.7) cancel out. Observe that in the paths $i_0...\widehat{i_l}...jj...i_n$ and $i_0...jj...\widehat{i_l}...i_n$ the first couple of equal consecutive vertices is jj, the same as in the full paths $i_0...jj...i_n$. By (6.4) we obtain

$$g_{i_0...\widehat{i_l}...jj...i_n} = (-1)^{k-1} f_{i_0...\widehat{i_l}...jjj...i_n}$$

and

$$g_{i_0...jj...\widehat{i_l}...i_n} = (-1)^k f_{i_0...jjj...\widehat{i_l}...i_n}$$

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whence

$$(dg)_{i_0...i_n} = \sum_{l=0}^{k-1} (-1)^{l+k-1} f_{i_0...\widehat{i_l}...jjj...i_n} + \sum_{l=k+2}^{n} (-1)^{l+k} f_{i_0...jjj...\widehat{i_l}...i_n}. \qquad (6.9)$$

On the other hand, since the $(n+1)$-path $i_0...i_{k-1}jjji_{k+2}...i_n$ is allowed, we have

$$(df)_{i_0...i_{k-1}jjji_{k+2}...i_n} = 0. \qquad (6.10)$$

Computing this component by (6.1), we obtain

$$
\begin{aligned}
(df)_{i_0...i_{k-1}jjji_{k+2}...i_n} &= \sum_{l=0}^{k-1} (-1)^l f_{i_0...\widehat{i_l}...jjj...i_n} && (6.11) \\
&\quad + (-1)^k f_{i_0...\widehat{j}jj...i_n} + (-1)^{k+1} f_{i_0...j\widehat{j}j...i_n} + (-1)^{k+2} f_{i_0...jj\widehat{j}...i_n} && (6.12) \\
&\quad + \sum_{l=k+2}^{n} (-1)^{l+1} f_{i_0...jjj...\widehat{i_l}...i_n} && (6.13)
\end{aligned}
$$

The three terms in (6.12) sum up to $(-1)^k f_{i_0...jj...i_n} = (-1)^k f_{i_0...i_n}$. Combining (6.11)-(6.13) with (6.9), we obtain

$$(df)_{i_0...i_{k-1}jjji_{k+2}...i_n} = (-1)^{k-1} (dg)_{i_0...i_n} + (-1)^k f_{i_0...i_n}.$$

Substituting here (6.10), we obtain (6.5). ∎

7 Poset associated with a simplicial complex

Let S be a finite simplicial complex. By definition, S can be regarded as a family of subsets of a finite set M such that if $s \in S$ then any non-empty subset t of s also belongs to S (we do not allow the empty set to be in S). Then we obtain immediately a poset structure on S as follows:

$$s \to t \quad \Leftrightarrow \quad s \supset t.$$

Let B_S be the barycentric subdivision of S, and let V be the set of vertices of B_S, that is, V is the set of the barycenters of all simplexes of S. For any vertex $i \in V$ denote by s_i the simplex from S whose barycenter is i. Clearly, the set V has the same poset structure as S: if i and i are two vertices from V then we set

$$i \to j \quad \Leftrightarrow \quad s_i \supset s_j$$

(cf. Fig. 1). As in Section 6, let $\left\{\Omega_{reg}^n\right\}$ be a regularized cochain complex of the poset V.

Lemma 7.1 *We have isomorphism*

$$H^n(\Omega_{reg}) \cong H^n(S).$$

Proof. We will use the chain complexes $\{\Omega_n^{reg}\}$ and $\{C_n(S)\}$ that determine the homologies $H_n(\Omega^{reg})$ and $H_n(S)$ that are dual to the cohomologies $H^n(\Omega_{reg})$ and $H^n(S)$, respectively. Hence, it suffices to prove that

$$H_n(\Omega^{reg}) \cong H_n(S). \qquad (7.1)$$

The space Ω_n^{reg} consists of all formal $\mathbb{K}$-linear combination of the regular allowed n-paths $i_0...i_n$ on V. The n-path $i_0...i_n$ as an element of Ω_n^{reg} will be denoted by $e_{i_0...i_n}$. The boundary operator $\partial : \Omega_n^{reg} \to \Omega_{n-1}^{reg}$ is defined by

$$\partial e_{i_0...i_n} = \sum_{k=0}^{n} (-1)^k e_{i_0...\widehat{i_k}...i_n} \tag{7.2}$$

(observe that the $(n-1)$-path $i_0...\widehat{i_k}...i_n$ is regular allowed). It is clear that ∂ is dual to $d :$ $\Omega_{reg}^{n-1} \to \Omega_{reg}^n$ (cf. [4]).

By construction of the barycentric subdivision B_S of S, each regular allowed n-path $i_0...i_n$ in V gives rise to a n-simplex of B_S with the vertices $i_0...i_n$, and vice versa (see Fig. 6).
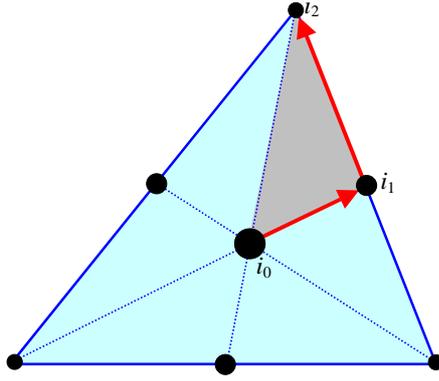

Figure 6: A regular allowed path $i_0 i_1 i_2$ determines a 2-simplex $[i_0, i_1, i_2]$ of B_S.

Since the boundary operator on simplexes is defined by

$$\partial [i_0, ..., i_n] = \sum_{k=0}^{n} (-1)^k [i_0, ..., \widehat{i_k}, ..., i_n]$$

that matches (7.2), we see that the chain complex Ω_n^{reg} is isomorphic to the chain complex $C_n(B_S)$ of the simplicial complex B_S. In particular, it follows that

$$H_n(\Omega^{reg}) \cong H_n(B_S).$$

Since $H_n(B_S) \cong H_n(S)$, we obtain (7.1). ∎

As in Section 2, define the algebras A and $\overline{A}$ based on a poset V and the associated Hochschild cohomologies $HH^n(A, A)$. Now we can state and prove the main result.

Theorem 7.2 *We have the following isomorphism of cohomologies*

$$HH^n(A, A) \cong H^n(S),$$

where $H^n(S)$ is the simplicial cohomologies of the simplicial complex S.

Proof. This follows from Lemmas 4.2, 5.1, 5.2, 6.1, 7.1 as in (1.4). ∎

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