

# From submodule categories to preprojective algebras.

Claus Michael Ringel and Pu Zhang

Abstract: Let  $S(n)$  be the category of invariant subspaces of nilpotent operators with nilpotency index at most  $n$ . Such submodule categories have been studied already in 1936 by Birkhoff, they have attracted a lot of attention in recent years, for example in connection with some weighted projective lines (Kussin, Lenzing, Meltzer). On the other hand, we consider the preprojective algebra  $\Pi_n$  of type  $\mathbb{A}_n$ ; the preprojective algebras were introduced by Gelfand and Ponomarev, they are now of great interest, for example they form an important tool to study quantum groups (Lusztig) or cluster algebras (Geiss, Leclerc, Schröer). Direct connections between the submodule category  $\mathcal{S}(n)$  and the module category of the preprojective algebra  $\Pi_{n-1}$  have been established quite a long time ago by Auslander and Reiten, and recently also by Li and Zhang, but apparently this remained unnoticed. The aim of this note is to provide details on this relationship. As a byproduct we see that here we deal with ideals  $\mathcal{I}$  in triangulated categories  $\mathcal{T}$  such that  $\mathcal{I}$  is generated by an idempotent and  $\mathcal{T}/\mathcal{I}$  is abelian.

Let  $k$  be a field. Let  $S(n)$  be the category of invariant subspaces of nilpotent operators with nilpotency index at most  $n$ . For a detailed analysis of this category we refer to [RS2]. Let  $k[x]$  be the polynomial ring in one variable  $x$  with coefficients in  $k$  and  $\Lambda_n = k[x]/\langle x^n \rangle$ . We may consider the objects of  $\mathcal{S}(n)$  as pairs  $(X, Y)$  where  $Y$  is a  $\Lambda_n$ -module and  $X$  is a submodule of  $Y$ . We denote the indecomposable  $\Lambda_n$ -module of length  $i$  by  $[i]$ , and  $[0]$  will denote the zero module.

Let  $\Pi_n$  be the preprojective algebra of type  $\mathbb{A}_n$ . The aim of this note is to draw the attention to the following result which is (at least implicitly) in the paper [LZ] by Zhi-Wei Li and Pu Zhang:

**Theorem 1 (Li-Zhang).** *Let  $\mathcal{U}$  be the set of objects of  $\mathcal{S}(n)$  which are of the form  $([i], [j])$  with  $i = j$  or  $j = n$ . Then the factor category  $\mathcal{S}(n)/\mathcal{U}$  is equivalent to  $\text{mod } \Pi_{n-1}$ .*

Addendum. *Under such an equivalence, the objects of  $\mathcal{S}(n)$  of the form  $([0], [j])$  with  $1 \leq j \leq n - 1$  correspond to the indecomposable projective  $\Pi_{n-1}$ -modules.*

Here, given an additive category  $\mathcal{A}$  and an ideal  $\mathcal{I}$  in  $\mathcal{A}$ , we denote by  $\mathcal{A}/\mathcal{I}$  the corresponding factor category (it has the same objects, and the homomorphisms in  $\mathcal{A}/\mathcal{I}$  are the residue classes of the homomorphisms in  $\mathcal{A}$  modulo  $\mathcal{I}$ ). If  $\mathcal{U}$  is a set of objects of the category  $\mathcal{A}$ , we denote by  $\langle \mathcal{U} \rangle$  the ideal of  $\mathcal{A}$  given by all maps which factor through a direct sum of objects in  $\mathcal{U}$ . Instead of writing  $\mathcal{A}/\langle \mathcal{U} \rangle$ , we just will write  $\mathcal{A}/\mathcal{U}$ .

Theorem 1 asserts that the factor category  $\mathcal{S}(n)/\mathcal{U}$  is a module category, thus, in particular, abelian. If we denote by  $\mathcal{W}$  the set of objects of  $\mathcal{S}(n)$  which contains besides

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the objects in  $\mathcal{U}$  also the pairs  $([0], [j])$  with  $1 \leq j \leq n-1$ , then the addendum shows that  $\mathcal{S}(n)/\mathcal{W}$  is equivalent to the stable module category  $\underline{\text{mod}} \Pi_{n-1}$ .

The aim of this paper is to provide some details on such equivalences. In particular, we will show in which way the Auslander-Reiten quiver of  $\text{mod } \Pi_{n-1}$  is obtained from the Auslander-Reiten quiver of  $\mathcal{S}(n)$ , see section 7.

Following [LZ], the equivalence  $\mathcal{S}(n)/\mathcal{U} \rightarrow \text{mod } \Pi_{n-1}$  will be established in two steps, with a category  $\mathcal{F}(n)$  of  $A_n$ -modules as an intermediate category, where  $A_n$  is the Auslander algebra of  $\Lambda_n$ . The functor  $\lambda: \mathcal{S}(n) \rightarrow \mathcal{F}(n)$  to be used was introduced already in [A] (see also [AR1] and [AR2]). The second functor  $\delta: \mathcal{F}(n) \rightarrow \text{mod } \Pi_{n-1}$  is just the projection onto the torsionfree factor module for a torsion pair, it was considered in this context in [DR].

The subcategory  $\mathcal{F}(n)$  is of interest in itself. Proposition 1 provides several characterizations, partly following [DR]: an  $A_n$ -module  $M$  belongs to  $\mathcal{F}(n)$  if and only if  $M$  is torsionless, if and only if  $M$  has projective dimension at most 1, if and only if the injective envelope of  $M$  is projective, if and only if  $M$  has a  $\Delta$ -filtration (note that any Auslander-algebra is quasi-hereditary, and the algebras  $A_n$  are quasi-hereditary for just one ordering of the simple modules, thus there is just one possible choice for the standard modules  $\Delta(i)$ ).

One obtains  $\mathcal{F}(n)$  from  $\mathcal{S}(n)$  by factoring out the ideal generated by the objects of the form  $([i], [i])$ , with  $1 \leq i \leq n$ , and under this equivalence the objects of  $\mathcal{S}(n)$  of the form  $([i], [n])$  with  $0 \leq i \leq n-1$  correspond to the direct summands of the characteristic tilting module of  $A_n$ . Altogether, there is a three step procedure

$$\mathcal{S}(n) \xrightarrow{\lambda} \mathcal{F}(n) \xrightarrow{\delta} \text{mod } \Pi_{n-1} \xrightarrow{\pi} \underline{\text{mod}} \Pi_{n-1},$$

in the first step, we kill the objects of the form  $([i], [i])$  with  $1 \leq i \leq n$ , in the second those of the form  $([i], [n])$  with  $0 \leq i \leq n-1$  and in the third one those of the form  $([0], [j])$  with  $1 \leq j \leq n-1$ . The set  $\mathcal{W}$  of all these objects is closed under the Auslander-Reiten translation  $\tau$  of  $\mathcal{S}(n)$ , it consists of two  $\tau$ -orbits of length 1, namely the objects  $([n], [n])$  and  $([0], [n])$  (they are projective-injective in  $\mathcal{S}(n)$ ), of several orbits of length 6 and at most one orbit of length 3. For  $n$  even, there are  $\frac{n-2}{2}$  orbits of length 6 and one orbit of length 3, whereas, for  $n$  odd, there are just  $\frac{n-1}{2}$  orbits of length 6.

Another way to obtain the category  $\text{mod } \Pi_{n-1}$  as a factor category of  $\mathcal{S}(n)$  was (essentially) presented already in 1974 by Auslander and Reiten [AR1]:

**Theorem 2 (Auslander-Reiten).** *Let  $\mathcal{V}$  be the set of objects of  $\mathcal{S}(n)$  which are of the form  $([i], [j])$  with  $i = j$  or  $i = 0$ . Then the factor category  $\mathcal{S}(n)/\mathcal{V}$  is equivalent to  $\text{mod } \Pi_{n-1}$ .*

*Addendum. Under such an equivalence, the objects of  $\mathcal{S}(n)$  of the form  $([i], [n])$  with  $1 \leq i \leq n-1$  correspond to the indecomposable projective  $\Pi_{n-1}$ -modules.*

The functor  $\rho: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$  used by Auslander and Reiten is given by  $\rho(u) = \text{Cok Hom}(E, q)$ , where  $q: Y \rightarrow Y/X$  is the cokernel map of the embedding  $u: X \rightarrow Y$  and  $W$  is the direct sum  $E$  of all indecomposable  $\Lambda$ -modules, one from each isomorphism class.

Note that  $\mathcal{W}$  is the union of  $\mathcal{V}$  and the set of objects of  $\mathcal{S}(n)$  of the form  $([i], [n])$  with  $1 \leq i \leq n-1$ . Thus, the addendum shows again that  $\mathcal{S}(n)/\mathcal{W}$  is equivalent to the stable module category  $\underline{\text{mod}} \Pi_{n-1}$ . It follows that the stabilization of the functor  $\delta\lambda$  of Li-Zhang and of the Auslander-Reiten functor  $\rho$  differ only by an auto-equivalence of  $\underline{\text{mod}} \Pi_{n-1}$ . As we will show, they differ by the syzygy functor  $\Omega: \underline{\text{mod}} \Pi_{n-1} \rightarrow \underline{\text{mod}} \Pi_{n-1}$ . Namely, there is the following assertion:

**Theorem 3.** We have

$$\pi\delta\lambda = \Omega\pi\rho.$$

The paper is organized as follows. In the first section, we study the Auslander algebra  $A_n$  and the category  $\mathcal{F}(n)$  of the torsionless  $A_n$ -modules. The functors  $\lambda$  and  $\rho$  are given by restrictions of the functor  $\alpha$  which sends a morphism  $f$  in  $\text{mod } \Lambda_n$  to  $\text{Cok Hom}(E, f)$ . This functor was introduced by Auslander and studied in several papers by Auslander and Reiten. The functor  $\alpha$  will be discussed in section 2. Sections 3 and 4 are devoted to  $\lambda$  and  $\rho$ , section 5 to the functor  $\delta$ . Section 6 establishes the comparison formula  $\pi\delta\lambda = \Omega\pi\rho$ . In section 7 we provide illustrations concerning the change of the Auslander-Reiten quiver of  $\mathcal{S}(n)$  when we factor out various ideals. The final section 8 draws the attention to the fact that here we deal with ideals in triangulated categories which are generated by an idempotent such that the corresponding factor categories are abelian.

**Bookkeeping.** Going from  $\mathcal{S}(n)$  to  $\mathcal{F}(n)$ , or from  $\mathcal{F}(n)$  to  $\text{mod } \Pi_{n-1}$ , the number of indecomposable objects decreases in both step by  $n$ . Here is the bookkeeping table. We denote the number of isomorphism classes of indecomposable objects in the category  $\mathcal{C}$  by  $\#\text{ind } \mathcal{C}$ .

$n$	$\#\text{ind } \mathcal{S}(n)$	$\#\text{ind } \mathcal{F}(n)$	$\#\text{ind } \text{mod } \Pi_{n-1}$
1	2	1	0
2	3	3	1
3	10	7	4
4	20	16	12
5	50	45	40
6	$\infty$	$\infty$	$\infty$

Going from  $\underline{\mathcal{S}}(n)$  to  $\text{mod } \Pi_{n-1}$  the number of indecomposables decreases by  $2(n-1)$ , going from  $\text{mod } \Pi_{n-1}$  to  $\underline{\text{mod}} \Pi_{n-1}$  the number decreases by  $n-1$ . Here are the actual numbers; for the triangulated categories  $\underline{\mathcal{S}}(n)$  and  $\underline{\text{mod}} \Pi_{n-1}$  we also list the tree type of the corresponding Auslander-Reiten quivers.

$n$	$\#\text{ind } \underline{\mathcal{S}}(n)$	$\#\text{ind } \text{mod } \Pi_{n-1}$	$\#\text{ind } \underline{\text{mod}} \Pi_{n-1}$
1	0	0	0
2	1 $\mathbb{A}_2$	1	0
3	8 $\mathbb{D}_4$	4	2 $\mathbb{A}_1$
4	18 $\mathbb{E}_6$	12	9 $\mathbb{A}_3$
5	48 $\mathbb{E}_8$	40	36 $\mathbb{D}_6$
6	$\infty$	$\infty$	$\infty$

## 1. The Auslander algebra $A_n = A(\Lambda_n)$ .

If  $\Lambda$  is a ring of finite representation type, then  $A(\Lambda)$  denotes the basic Auslander algebra of  $\Lambda$  (the opposite of the endomorphism ring of the direct sum  $E$  of all indecomposable  $\Lambda$ -modules, it is called an *Auslander generator*).

We consider here the special case of  $\Lambda = \Lambda_n = k[x]/\langle x^n \rangle$  and its Auslander algebra  $A_n = (\text{End } E)^{\text{op}}$ . Recall that we denote by  $[i]$  the indecomposable  $\Lambda_n$ -module of length  $i$  and therefore  $E = \bigoplus_{i=1}^n [i]$ .

Let  $P(i) = \text{Hom}(E, [i])$ . This is an indecomposable projective  $A_n$ -module. The inclusions  $[i] \rightarrow [i+1]$  in the category  $\text{mod } \Lambda_n$  yield a chain of inclusions

$$P(1) \subset P(2) \subset \cdots \subset P(n-1) \subset P(n).$$

Let  $\Delta(i) = P(i)/P(i-1)$  (with  $P(0) = 0$ ). Note that  $A_n$  is quasi-hereditary (for this ordering and only for this ordering) and the modules  $\Delta(i)$  are the standard modules (but observe that the labeling of the simple  $A_n$ -modules exhibited here is the opposite of the labeling commonly used (see for example [DR]): in the present paper, it is the module  $\Delta(1)$  which is projective and not  $\Delta(n)$ , and correspondingly, it is  $P(n) = I(n)$  which is projective-injective and not  $P(1)$ ).

Let  $T(i) = P(n)/P(i-1)$  for  $1 \leq i \leq n$  and  $T = \bigoplus_i T(i)$ . This is the characteristic tilting module for the quasi-hereditary algebra  $A_n$ .

**Proposition 1.** *The following conditions are equivalent for an  $A_n$ -module  $M$ .*

- (i)  $M$  is torsionless.
- (ii)  $\text{Ext}^1(M, T) = 0$ .
- (iii)  $M$  has a  $\Delta$ -filtration.
- (iv) The projective dimension of  $M$  is at most 1.
- (v) The injective envelope of  $M$  is projective.

**Remark 1.** We have avoided to refer to the labeling of the simple modules. If we use our labels, so that  $P(n) = I(n)$  is the unique indecomposable module which is both projective and injective, then (v) can be reformulated as saying that (vi) *the injective envelope  $IM$  of  $M$  is a direct sum of copies of  $I(n)$* , or also that (vii) *the socle of  $M$  is a direct sum of copies of  $S(n)$* .

**Remark 2.** It has been shown in [R] that any Auslander algebra is left strongly quasi-hereditary. This means that any module with a  $\Delta$ -filtration has projective dimension at most 1 (the implication (iii)  $\implies$  (iv)).

We denote by  $\mathcal{F}(n)$  the full subcategory of  $\text{mod } A_n$  given by the  $A_n$ -modules which satisfy the equivalent conditions of Proposition 1. It follows directly from (i) or also (v) that  $\mathcal{F}(n)$  is closed under submodules.

Proof of Proposition 1. (i)  $\implies$  (ii). Let  $M$  be torsionless. There is an embedding  $M \rightarrow P$  with  $P$  projective. Since the injective dimension of  $T$  is 1, the canonical map  $\text{Ext}^1(P, T) \rightarrow \text{Ext}^1(M, T)$  is surjective. But  $\text{Ext}^1(P, T) = 0$ , thus  $\text{Ext}^1(M, T) = 0$ .

(ii)  $\implies$  (iii). We have to show that  $\text{Ext}^i(M, T) = 0$  for all  $i \geq 1$ . Since  $T$  has injective dimension 1, we only have to look at  $i = 1$ , but this is assertion (ii).

(iii)  $\implies$  (iv). The modules  $\Delta(i)$  have projective dimension at most 1.

(iv)  $\implies$  (v). Let  $0 \rightarrow P_1 \xrightarrow{u} P_0 \rightarrow M \rightarrow 0$  be a projective resolution. We have to show that  $M$  embeds into a module which is both injective and projective. Consider the injective envelopes of  $v_i: P_i \rightarrow I(P_i)$  for  $i = 0, 1$ , this yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{u} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & v_1 \downarrow & & \downarrow v_0 & & \downarrow f & & \\ 0 & \longrightarrow & I(P_1) & \xrightarrow{u'} & I(P_0) & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

Since  $v_0$  is injective, the snake lemma yields an embedding of the kernel  $K$  of  $f$  into the cokernel of  $v_1$ . For any projective module  $P$  with injective envelope  $IP$ , the cokernel  $IP/P$  embeds into a projective-injective module (since the dominant dimension of  $\Lambda_n$  is at least 2).

On the other hand,  $u'$  is a monomorphism and  $I(P_1)$  is injective, thus  $u'$  is a split monomorphism. Thus  $M'$  is a direct summand of  $I(P_0)$  and this module is projective-injective. The exact sequence  $0 \rightarrow K \rightarrow M \rightarrow M'$  shows that  $M$  embeds into  $IK \oplus M'$  which is projective-injective.

(v)  $\implies$  (i). If the injective envelope of  $M$  is projective, then  $M$  embeds into a projective module, thus  $M$  is torsionless.

**Proposition 2.** *If  $M$  is an  $A_n$ -module of projective dimension at most 1 and generated by  $P(n)$ , then  $M$  belongs to  $\text{add } T$ .*

Proof: Write  $M = P/U$  where  $P$  is a direct sum of copies of  $P(n)$  and  $U$  is a submodule of  $P$ . Since the projective dimension of  $M$  is at most 1, the submodule  $U$  is projective. Now  $P$  is injective, thus we may embed an injective envelope  $IU$  of  $U$  into  $P$ , this is a direct summand of  $P$ , say  $P = IU \oplus C$  for some submodule  $C$ . Since  $P$  is a direct sum of copies of  $P(n)$ , also  $C$  is a direct sum of copies of  $P(n) = T(1)$ . On the other hand, the exact sequences  $0 \rightarrow P(i) \rightarrow P(1) \rightarrow T(i+1) \rightarrow 0$  for  $1 \leq i \leq n$  (with  $T(n+1) = 0$ ) show that for any indecomposable projective module  $P'$ , the module  $I(P')/P'$  belongs to  $\text{add } T$ , thus  $IU/I$  belongs to  $\text{add } T$ . Altogether we see that  $P/U = IU/U \oplus C$  belongs to  $\text{add } T$ .

**Remark.** The modules generated by  $P(n) = T(1)$  are the modules with a  $\nabla$ -filtration (see [DR]), thus proposition 2 just asserts that modules which have both a  $\Delta$ -filtration and a  $\nabla$ -filtration belong to  $\text{add } T$ .

## 2. The functor $\alpha: \text{mod } T_2(\Lambda) \rightarrow \text{mod } A(\Lambda)$ .

Let  $\Lambda$  be a representation-finite algebra with Auslander generator  $E$  and  $A(\Lambda) = (\text{End } E)^{\text{op}}$  the corresponding Auslander algebra.

We denote by  $T_2(\Lambda)$  the ring of all upper triangular  $(2 \times 2)$ -matrices with coefficients in  $\Lambda$ . The category of  $T_2(\Lambda)$ -modules may be seen as the category of all morphisms  $X \rightarrow Y$  in  $\text{mod } \Lambda$ .

**Proposition 3 (Auslander-Reiten).** *Let  $\mathcal{X}$  consist of the two objects  $(1: E \rightarrow E)$  and  $(E \rightarrow 0)$  in  $\text{mod } T_2(\Lambda)$ . Then the functor  $\alpha: \text{mod } T_2(\Lambda) \rightarrow \text{mod } A(\Lambda)$  defined by*

$\alpha(f) = \text{Cok Hom}(E, f)$  for a morphism  $f$  in  $\text{mod } \Lambda$  yields an equivalence  $(\text{mod } T_2(\Lambda))/\mathcal{X} \rightarrow \text{mod } A(\Lambda)$ .

For a slightly weaker statement we refer to Theorem 1.1 in [AR2]. Let us recall and complete the proof. Given a morphism  $f: X \rightarrow Y$  in  $\text{mod } \Lambda$ , we obtain an exact sequence

$$\text{Hom}(E, X) \xrightarrow{\text{Hom}(E, f)} \text{Hom}(E, Y) \rightarrow \alpha(f) \rightarrow 0.$$

Since the  $A(\Lambda)$ -modules  $\text{Hom}(E, X)$  and  $\text{Hom}(E, Y)$  are projective, we obtain in this way a projective presentation of  $\alpha(f)$ . Conversely, given an  $A(\Lambda)$ -module  $M$ , take a projective resolution  $P_1 \xrightarrow{p} P_0 \rightarrow M \rightarrow 0$ . Now, the category of projective  $A(\Lambda)$ -modules is equivalent to the category  $\text{mod } \Lambda$ , thus we can assume that there are  $\Lambda$ -modules  $X$  and  $Y$  such that  $P_1 = \text{Hom}(E, X)$  and  $P_0 = \text{Hom}(E, Y)$  and a  $\Lambda$ -homomorphism  $f: X \rightarrow Y$  such that  $\text{Hom}(E, f) = p$ . In this way, we see that the functor is dense. Similarly, starting with an  $A(\Lambda)$ -homomorphism  $M \rightarrow M'$ , we can lift it to projective presentations of  $M$  and  $M'$  and using again the fact that the category of projective  $A(\Lambda)$ -modules is equivalent to the category  $\text{mod } \Lambda$ , we see that  $M \rightarrow M'$  is in the image of the functor  $\alpha$ . Thus, it remains to calculate the kernel of  $\alpha$ .

Of course, under this functor  $\alpha$  the two objects in  $\mathcal{X}$  are sent to zero. Thus the ideal  $\langle \mathcal{X} \rangle$  is contained in the kernel of  $\alpha$ . Conversely, assume that there is given a map

$$(g_1, g_0): (f: X_1 \rightarrow X_0) \longrightarrow (f': (X'_1 \rightarrow X'_0))$$

(thus  $g_0 f = f' g_1$ ), such that  $\alpha(g_1, g_0) = 0$ . Thus the following diagram commutes and its rows are projective presentations:

$$\begin{array}{ccccccc} \text{Hom}(E, X_1) & \xrightarrow{\text{Hom}(E, f)} & \text{Hom}(E, X_0) & \xrightarrow{e} & \alpha(f) & \longrightarrow & 0 \\ \text{Hom}(E, g_1) \downarrow & & \downarrow \text{Hom}(E, g_0) & & \downarrow \alpha(g_1, g_0)=0 & & \\ \text{Hom}(E, X'_1) & \xrightarrow{\text{Hom}(E, f')} & \text{Hom}(E, X'_0) & \xrightarrow{e'} & \alpha(f') & \longrightarrow & 0. \end{array}$$

Let us show that the map  $(g_1, g_0)$  factors through  $[1 \ 0]: X_0 \oplus X_1 \rightarrow X_0$ . Now  $e' \text{Hom}(E, g_0) = 0$ , thus there is a map  $\tilde{h}: \text{Hom}(E, X_0) \rightarrow \text{Hom}(E, X'_1)$  such that  $\text{Hom}(E, f')\tilde{h} = \text{Hom}(E, g_0)$ . Again using the equivalence of  $\text{mod } \Lambda_n$  and the category of projective  $A_n$ -modules, we get a map  $h: X_0 \rightarrow X'_1$  with  $\tilde{h} = \text{Hom}(E, h)$  and  $g_0 = f'h$ . Therefore  $f'hf = g_0f = f'g_1$ , and thus  $f'(g_1 - hf) = 0$ . As a consequence, the following diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_0 \\ \left[ \begin{array}{c} f \\ 1 \end{array} \right] \downarrow & & \downarrow 1 \\ X_0 \oplus X_1 & \xrightarrow{[1 \ 0]} & X_0 \\ \left[ \begin{array}{c} h \ g_1 - hf \end{array} \right] \downarrow & & \downarrow g_0 \\ X'_1 & \xrightarrow{f'} & X'_0 \end{array}$$

and the composition of the vertical maps is just  $(g_1, g_0)$ . Since  $E$  is an Auslander generator, all  $\Lambda_n$ -modules belong to add  $E$ , thus  $[1 \ 0]: X_0 \oplus X_1 \rightarrow X_0$  belongs to add  $\mathcal{X}$ . This shows that the map  $(g_1, g_0)$  belongs to  $\langle \mathcal{X} \rangle$ .

### 3. The functor $\lambda: \mathcal{S}(n) \rightarrow \mathcal{F}(n)$ .

It may be appropriate to focus first the attention to the category  $\mathcal{S}(n)$  itself. An object  $(X, Y)$  of  $\mathcal{S}(n)$  with inclusion map  $u: X \rightarrow Y$  will also be denoted by  $u$ . This stresses the fact that objects of  $\mathcal{S}(n)$  are given by maps in the category  $\text{mod } \Lambda_n$ , thus one may consider  $\mathcal{S}(n)$  as a full subcategory of the category of  $T_2(\Lambda_n)$ -modules. Note that  $\mathcal{S}(n)$  turns out to be just the category of all Gorenstein-projective  $T_2(\Lambda_n)$ -modules.

We take as functor  $\lambda: \mathcal{S}(n) \rightarrow \mathcal{F}(n)$  the restriction of  $\alpha$  to the full subcategory  $\mathcal{S}(n)$  of  $\text{mod } T_2(\Lambda_n)$ , thus

$$\lambda(u) = \text{Cok Hom}(E, u).$$

for  $(u: X \rightarrow Y)$  in  $\mathcal{S}(n)$ . Since  $u$  is a monomorphism, also  $\text{Hom}(E, u)$  is a monomorphism, thus there is the following exact sequence

$$0 \rightarrow \text{Hom}(E, X) \rightarrow \text{Hom}(E, Y) \rightarrow \lambda(u) \rightarrow 0.$$

Since both  $\text{Hom}(E, X)$  and  $\text{Hom}(E, Y)$  are projective  $A_n$ -modules, we see that the projective dimension of  $\lambda(u)$  is at most 1, thus  $\lambda(u)$  belongs to  $\mathcal{F}(n)$ .

Under the functor  $\lambda$ , we have

$$\begin{aligned} \lambda([i], [i]) &= 0 & 1 \leq i \leq n, \\ \lambda([i], [n]) &= P(n)/P(i) = T(i+1) & \text{for } 0 \leq i \leq n-1, \\ \lambda([0], [j]) &= P(j) & 1 \leq j \leq n-1. \end{aligned}$$

**Proposition 4.** *Let  $\mathcal{U}_1$  be the set of objects of  $\mathcal{S}(n)$  of the form  $([i], [i])$  with  $1 \leq i \leq n$ . The functor  $\lambda$  yields an equivalence between the factor category of  $\mathcal{S}(n)/\mathcal{U}_1$  and the category  $\mathcal{F}(n)$ . It maps the pairs  $([i], [n])$  with  $0 \leq i \leq n-1$  to  $T(i+1)$  and the pairs  $([0], [j])$  with  $1 \leq j \leq n-1$  to  $P(j)$ .*

*Proof.* By definition,  $\lambda$  is the restriction of the functor  $\alpha$  to the subcategory  $\mathcal{S}(n)$  of  $\text{mod } T_2(\Lambda_n)$ . We have noted already that the image of  $\lambda$  consists of modules of projective dimension at most 1. Also, conversely, if  $M$  is a  $A_n$ -module of projective dimension at most 1, say with a projective presentation

$$0 \rightarrow P_1 \xrightarrow{p} P_0 \rightarrow M \rightarrow 0,$$

then we can write  $p = \text{Hom}(E, f)$  for some map  $f: X_1 \rightarrow X_0$  in  $\text{mod } \Lambda_n$ . Clearly,  $f$  has to be a monomorphism, since  $E$  is an Auslander generator. Thus we can assume that  $f$  belongs to  $\mathcal{S}(n)$  and we have  $\lambda(f) = M$ .

According to Proposition 3 the kernel of the functor  $\lambda$  is given by all morphisms in  $\mathcal{S}(n)$  which factor through a  $T_2(\Lambda_n)$ -module of the form  $[1 \ 0]: V \oplus V' \rightarrow V$ . Thus assume we have the following commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{u} & Y \\
 \downarrow [g' ] & & \downarrow g'' \\
 V \oplus V' & \xrightarrow{[1 \ 0]} & V \\
 \downarrow [h \ h' ] & & \downarrow h'' \\
 X' & \xrightarrow{u'} & Y'.
 \end{array}$$

(\*)

The commutativity of the lower square means that we have

$$[h'', 0] = h''[1, 0] = u'[h, h'] = [u'h, u'h'],$$

thus  $u'h' = 0$ . Since  $u'$  is a monomorphism, it follows that  $h' = 0$ . But then also the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{u} & Y \\
 g \downarrow & & \downarrow g'' \\
 V & \xrightarrow{1} & V \\
 h \downarrow & & \downarrow h'' \\
 X' & \xrightarrow{u'} & Y'.
 \end{array}$$

commutes and the composition of the vertical maps is the same as the composition of the vertical maps in (\*). This shows that the kernel of the functor  $\lambda$  consists of all the morphisms in  $\mathcal{S}(n)$  which factor through objects of the form  $(1: V \rightarrow V)$ , but this is just the ideal  $\mathcal{U}_1$ .

#### 4. The functor $\rho: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$ .

As we have mentioned in the introduction, the functor  $\rho: \mathcal{S}(n) \rightarrow \text{mod } \Pi_{n-1}$  is defined by

$$\rho(X, Y) = \text{Cok Hom}(E, Y \rightarrow Y/X),$$

this means: we form the cokernel map  $q: Y \rightarrow Y/X$  of the inclusion map  $X \rightarrow Y$ , apply the functor  $\text{Hom}(E, -)$  so that we obtain a map

$$\text{Hom}(E, q): \text{Hom}(E, Y) \rightarrow \text{Hom}(E, Y/X)$$

and take its cokernel  $\text{Cok Hom}(E, q)$ .

We may phrase the definition of  $\rho$  also differently: Let  $\mathcal{Q}(n)$  be the category of all epimorphisms  $(Y \rightarrow Z)$  in  $\text{mod } \Lambda_n$ , this is again a subcategory of  $\text{mod } T_2(\Lambda_n)$  and we

may look at the restriction of the functor  $\alpha$  to  $\mathcal{Q}(n)$ . Of course, there is an obvious categorical equivalence  $\iota: \mathcal{S}(n) \rightarrow \mathcal{Q}(n)$ , it sends an object  $(u: X \rightarrow Y)$  to the canonical map  $(Y \rightarrow Y/X)$ . Clearly,  $\rho = \alpha\iota$ .

**Remark.** Instead of looking at the categories  $\mathcal{S}(n)$  and  $\mathcal{Q}(n)$ , we may consider the category  $\mathcal{E}(n)$  of all short exact sequences in the category  $\text{mod } \Lambda_n$ . There are forgetful functors  $\mathcal{E}(n) \rightarrow \mathcal{S}(n)$  and  $\mathcal{E}(n) \rightarrow \mathcal{Q}(n)$  which send an exact sequence  $0 \rightarrow X \xrightarrow{u} Y \xrightarrow{q} Z \rightarrow 0$  to  $u$  or  $q$ , respectively. Obviously, both functors are categorical equivalences.

Under the functor  $\rho$ , we have

$$\begin{aligned} \rho([i], [i]) &= 0 & 1 \leq i \leq n, \\ \rho([i], [n]) &= P_{\Pi}(n-i) & \text{for } 1 \leq i \leq n-1, \\ \rho([0], [j]) &= 0 & 1 \leq j \leq n. \end{aligned}$$

Proof of Theorem 2. The essential observation is the following assertion: *A map  $f: Y \rightarrow Z$  in  $\text{mod } \Lambda_n$  is an epimorphism if and only if  $\text{Cok Hom}(P(n), f) = 0$ , thus if and only if  $\alpha(f)$  is a  $\Pi_{n-1}$ -module, this is Proposition 4.1 in [AR1]. Indeed,  $f$  is an epimorphism, if and only if  $\text{Hom}([n], f)$  is surjective.*

Proposition 3 asserts that  $\alpha$  is a full and dense functor from  $\mathcal{Q}(n)$  onto the category  $\text{mod } \Pi_{n-1}$  and that its kernel is the set of morphisms which factor through an object of the form  $([1, 0]: V \oplus V' \rightarrow V)$ . Under the equivalence  $\iota: \mathcal{S}(n) \rightarrow \mathcal{Q}(n)$ , the objects of the form  $([1, 0]: V \oplus V' \rightarrow V)$  in  $\mathcal{Q}(n)$  correspond to the objects of the form  $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]: V' \rightarrow V \oplus V'$  in  $\mathcal{S}(n)$ , but these are precisely the objects in  $\text{add } \mathcal{V}$ . This shows that  $\rho = \alpha\iota$  yields an equivalence  $\mathcal{S}(n)/\mathcal{V} \rightarrow \text{mod } \Pi_{n-1}$ .

The addendum relies on the fact that the set of projective objects of an abelian category is uniquely determined by the categorical structure. Since  $\rho([i], [n]) = P_{\Pi}(n-i)$ , for  $1 \leq i \leq n-1$ , we know that the objects  $([1], [n]), \dots, ([n-1], [n])$  in  $\mathcal{S}(n)/\mathcal{V}$  are the indecomposable projective objects in  $\mathcal{S}(n)/\mathcal{V}$ . Thus any equivalence between  $\mathcal{S}(n)/\mathcal{V}$  and  $\text{mod } \Pi_{n-1}$  sends these objects to the indecomposable projective  $\Pi_{n-1}$ -modules.

## 5. The functor $\delta: \mathcal{F}(n) \rightarrow \text{mod } \Pi_{n-1}$ .

Let  $\Pi_n$  be the preprojective algebra of type  $\mathbb{A}_n$ ; note that  $A_n/\langle e(n) \rangle = \Pi_{n-1}$ .

We consider in  $\text{mod } A_n$  the following torsion pair: the torsion modules are the modules generated by  $P(n)$ , thus the modules with top being a direct sum of copies of  $S(n)$ , the torsionfree modules are the modules which do not have  $S(n)$  as a composition factor (thus the torsionfree modules form a Serre subcategory). Note that the torsionfree modules are just the  $A_n$ -modules  $M$  with  $e(n)M = 0$ , thus these are the  $\Pi_{n-1}$ -modules.

Given an  $A_n$ -module  $M$ , let  $tM$  be its torsion submodule, and

$$\delta M = M/tM.$$

Recall that the indecomposable projective  $A_n$ -modules are  $P(i) = \text{Hom}(E, [i])$  with  $1 \leq i \leq n$ , if necessary, we will denote them by  $P_A(i)$ . The indecomposable projective  $\Pi_{n-1}$ -modules are factor modules of the modules  $P_A(i)$  with  $1 \leq i \leq n-1$ , namely  $P_\Pi(i) = P_A(i)/tP_A(i)$ .

Since we want to keep track of the selected objects  $([i], [j])$  in  $\mathcal{W}$ , let us repeat what happens under the functor  $\delta$ :

$$\begin{aligned} \delta(T(i)) &= 0 & \text{for } & 1 \leq i \leq n, \\ \delta(P_A(i)) &= P_\Pi(i) & & 1 \leq j \leq n-1. \end{aligned}$$

**Proposition 5.** *The functor  $\delta$  yields an equivalence between the factor category  $\mathcal{F}(n)/T$  and the category  $\text{mod } \Pi_{n-1}$ . It maps  $P_A(j)$  with  $1 \leq j \leq n-1$  to  $P_\Pi(j)$ .*

*Proof.* First, let us show that the functor  $\delta$  is dense. Thus, let  $N$  be a  $\Pi_{n-1}$ -module, but consider it as an  $A_n$ -module. Actually, all the modules to be considered now are  $A_n$ -modules; in particular  $I(i)$  will denote (as before) the indecomposable injective  $A_n$ -module corresponding to the vertex  $i$ .

Let  $u: N \rightarrow IN$  be an injective envelope of  $N$  (as an  $A_n$ -module!) and  $p: PIN \rightarrow IN$  a projective cover of  $IN$ . Let  $T'$  be the kernel of  $p$ . Now  $IN$  is a direct sum of modules of the form  $I(i)$  with  $1 \leq i \leq n-1$  and the exact sequence  $0 \rightarrow T(i+1) \rightarrow P(n) \rightarrow I(i) \rightarrow 0$  shows that the kernel of the projective cover  $P(n) \rightarrow I(i)$  is  $T(i+1)$ . Thus we see that  $T'$  is a direct sum of copies of modules of the form  $T(j)$  with  $2 \leq j \leq n$ .

Forming the induced exact sequence with respect to  $u$ , we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T' & \xrightarrow{v'} & \tilde{N} & \xrightarrow{p'} & N & \longrightarrow & 0 \\ & & \parallel & & \downarrow u' & & \downarrow u & & \\ 0 & \longrightarrow & T' & \xrightarrow{v} & PIN & \xrightarrow{p} & IN & \longrightarrow & 0 \end{array}$$

The monomorphism  $u'$  shows that  $\tilde{N}$  is torsionless, thus in  $\mathcal{F}(n)$ . Clearly,  $tT' = T'$  and  $tN = 0$ , thus  $t\tilde{N}$  is the image of  $v'$  and therefore  $\delta(\tilde{N}) = N$ . This shows that  $\delta$  is dense. In the same way, we see that  $\delta$  is also full. Namely, given a morphism  $f: N_1 \rightarrow N_2$  of  $\Pi_{n-1}$ -modules, we can extend  $f$  to a morphism  $f': I(N_1) \rightarrow I(N_2)$  and then lift it to a morphism  $f'': PI(N_1) \rightarrow PI(N_2)$ . Using the pullback property of the commutative square

$$\begin{array}{ccc} \tilde{N}_2 & \xrightarrow{p'} & N_2 \\ \downarrow u' & & \downarrow u \\ PI(N_2) & \xrightarrow{p} & I(N_2) \end{array}$$

we finally obtain a map  $\tilde{f}: \tilde{N}_1 \rightarrow \tilde{N}_2$  such that  $\delta(\tilde{f}) = f$ .

It remains to determine the kernel of  $\delta$ . Since the modules  $T(i)$  are generated by  $T(1)$ , they are torsion modules, thus  $\delta(T(i)) = 0$  for all  $1 \leq i \leq n$ . Conversely, let  $M_1, M_2$

belong to  $\mathcal{F}(n)$  and let  $f: M_1 \rightarrow M_2$  be a homomorphism such that  $\delta(f) = 0$ . This means that the image of  $f$  is contained in the torsion submodule  $tM_2$ . Now  $tM_2$  is a submodule of  $M_2 \in \mathcal{F}(n)$  and  $\mathcal{F}(n)$  is closed under submodules, thus  $tM_2$  has projective dimension at most 1. On the other hand,  $tM$  is generated by  $P(n)$ . Thus Proposition 2 asserts that  $tM_2$  is in  $\text{add } T$ . It follows that  $f$  belongs to the ideal  $\langle T \rangle$ .

**Remark 1.** In order to show the density of the functor  $\delta$ , the paper [DR] started with a  $\Pi_{n-1}$ -module  $N$ , considered it as an  $A_n$ -module and used a universal extension

$$0 \rightarrow T' \rightarrow \tilde{N} \rightarrow N \rightarrow 0$$

of  $N$  by a module  $T'$  in  $\text{add } T$ . Of course, the pullback recipe given in the proof above provides such a universal extension.

**Remark 2.** Looking at the exact sequence

$$0 \rightarrow T' \rightarrow \tilde{N} \rightarrow N \rightarrow 0$$

constructed in the proof of Proposition 5, one may decompose  $T' = \bigoplus_i T(i)^{n(i)}$ . Then  $n(1) = 0$  and  $n(i+1)$  is precisely the multiplicity of  $S(i)$  in the socle of  $N$ .

Combining Propositions 4 and 5 we obtain Theorem 1. The addendum relies again on the fact that the set of projective objects of an abelian category is uniquely determined.

## 6. Comparison of $\delta\lambda$ and $\rho$ .

We want to compare the functor  $\delta\lambda$  and  $\rho$ . In order to prove the equality  $\pi\delta\lambda = \Omega\pi\rho$ , we start with an object  $u: X \rightarrow Y$  in  $\mathcal{S}(n)$ , thus with an exact sequence

$$0 \rightarrow X \xrightarrow{u} Y \xrightarrow{q} Z \rightarrow 0.$$

We apply the functor  $\text{Hom}(E, -)$  and obtain the exact sequence

$$0 \rightarrow \text{Hom}(E, X) \xrightarrow{\text{Hom}(E, u)} \text{Hom}(E, Y) \xrightarrow{\text{Hom}(E, q)} \text{Hom}(E, Z)$$

Now, the cokernel of  $\text{Hom}(E, q)$  is  $\rho(u)$ . The cokernel of  $\text{Hom}(E, u)$  and thus the image of  $\text{Hom}(E, q)$  is  $\lambda(u)$ , thus there is the following exact sequence

$$0 \rightarrow \lambda(u) \rightarrow \text{Hom}(E, Z) \rightarrow \rho(u) \rightarrow 0$$

and we can assume that the map  $\lambda(u) \rightarrow \text{Hom}(E, Z)$  is an inclusion map. Since  $\rho(u)$  is a  $\Pi_{n-1}$ -module, we have  $t\rho(u) = 0$ , therefore  $t\text{Hom}(E, Z) \subseteq \lambda(u)$ , and therefore  $t\text{Hom}(E, Z) = t\lambda(u)$ . Thus we have the following exact sequence:

$$0 \rightarrow \lambda(u)/t\lambda(u) \rightarrow \text{Hom}(E, Z)/t\text{Hom}(E, Z) \rightarrow \rho(u) \rightarrow 0.$$

Note that  $\lambda(u)/t\lambda(u) = \delta\lambda(u)$  and that  $\text{Hom}(E, Z)/t\text{Hom}(E, Z)$  is a projective  $\Pi_{n-1}$ -module. It follows that  $\delta\lambda(u)$  coincides in  $\underline{\text{mod}} \Pi_{n-1}$  with  $\Omega\rho(u)$ .

**Remark.** It should be stressed that for  $n \geq 2$ , there cannot exist an endofunctor  $\phi$  of  $\text{mod} \Pi_{n-1}$  such that  $\delta\lambda = \phi\rho$  or  $\phi\delta\lambda = \rho$ . For example, if we would have  $\delta\lambda = \phi\rho$ , then the set of objects of  $\mathcal{S}(n)$  killed by  $\rho$  would be contained in the set of objects killed by  $\delta\lambda$ . However  $([0], [1])$  is killed by  $\rho$ , but not by  $\delta\lambda$ . Similarly,  $([1], [n])$  is killed by  $\delta\lambda$ , but not by  $\rho$ .

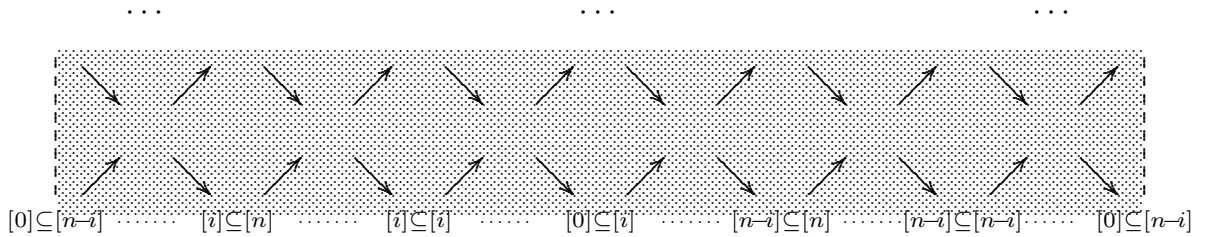
## 7. Auslander-Reiten orbits.

**Some Auslander-Reiten orbits in the category  $\mathcal{S}(n)$ .** Let  $\tau$  be the Auslander-Reiten translation in  $\mathcal{S}(n)$ . The paper [RS1] describes in detail how to obtain for the pair  $(X, Y)$  in  $\mathcal{S}(n)$  the pair  $\tau(X, Y)$ .

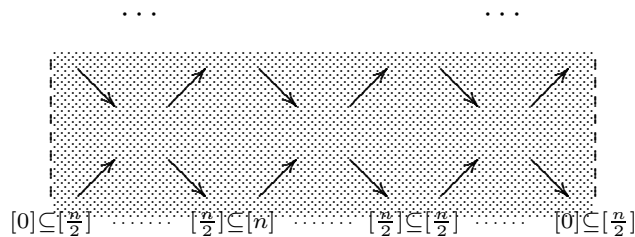
We are interested in some of the objects of the form  $([i], [j])$  (with  $0 \leq i \leq j \leq n$ ). The objects  $([n], [n])$  and  $([0], [n])$  are projective-injective, thus they are sent to zero by  $\tau$ . The following assertions for  $1 \leq i \leq n-1$

$$\begin{aligned}\tau([0], [i]) &= ([i], [i]) \\ \tau([i], [i]) &= ([i], [n]) \\ \tau([i], [n]) &= ([0], [n-i])\end{aligned}$$

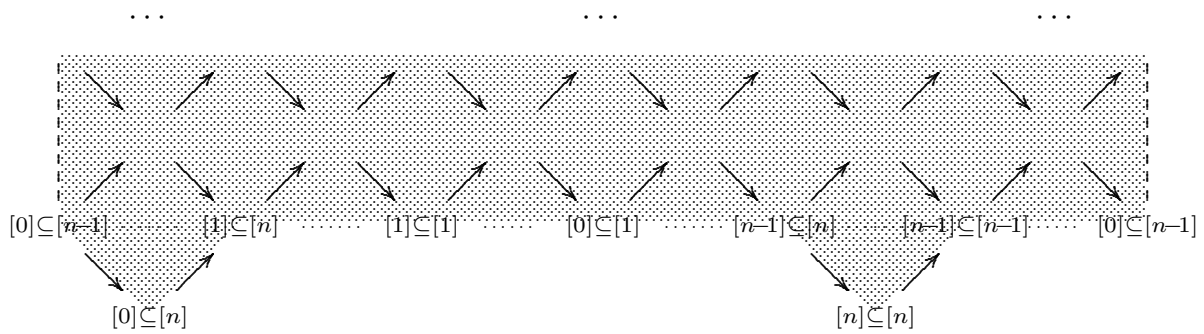
are easily verified. Of course,  $1 \leq i \leq n-1$  implies that also  $1 \leq n-i \leq n-1$ , thus we see that the set of objects of the form  $([0], [i]), ([i], [i]), ([i], [n])$  with  $1 \leq i \leq n-1$  is closed under  $\tau$ . Let us present the corresponding parts of Auslander-Reiten components. First of all, for  $2 \leq i < \frac{n}{2}$ , we deal with a  $\tau$ -orbit of length 6:



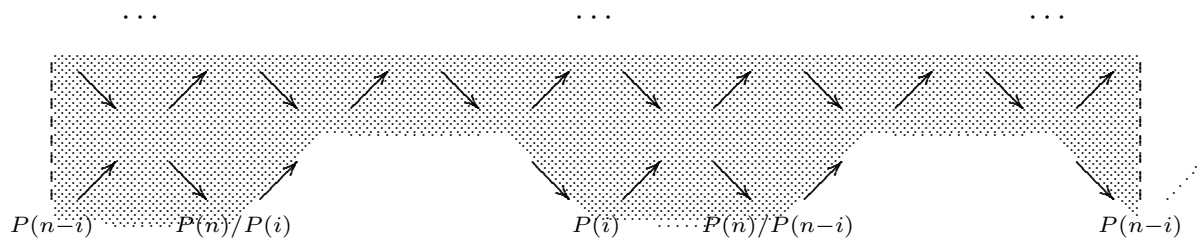
If  $n \geq 4$  is even and  $i = \frac{n}{2}$ , there is the following orbit of length 3:



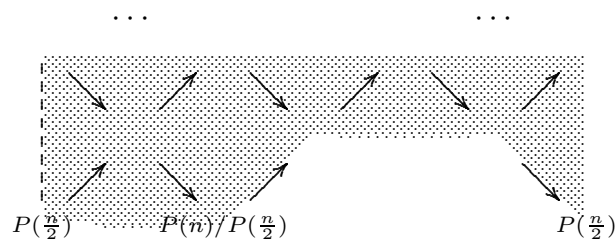
Finally, for  $i = 1$  we get:



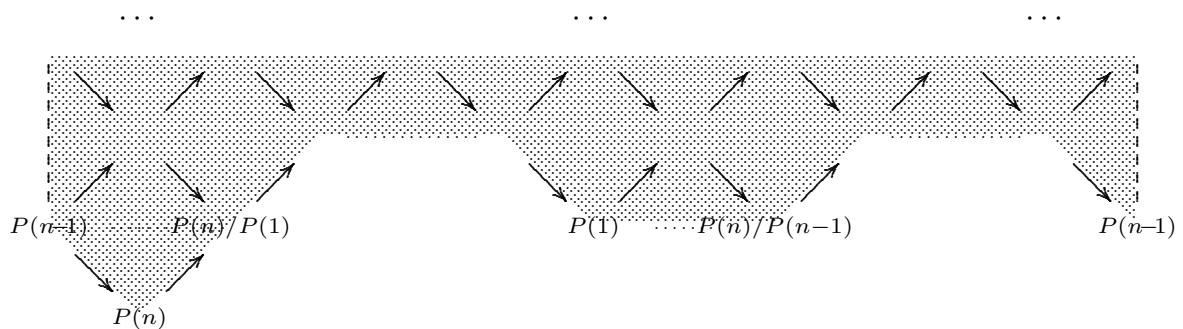
The corresponding  $\tau$ -orbits in the category  $\mathcal{F}(n)$ . First, those for  $2 \leq i < \frac{n}{2}$ :



Second, for  $n \geq 4$  even and  $i = \frac{n}{2}$ :



And finally, for  $i = 1$  we get:

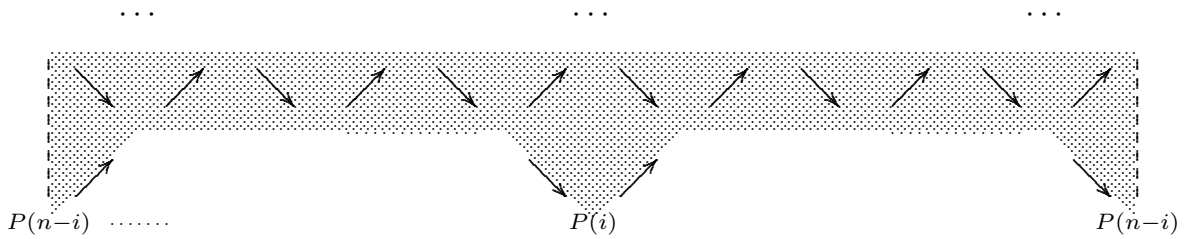


**Remark.** The paper [RS2] describes in detail the Auslander-Reiten quivers of the categories  $\mathcal{S}(n)$  with  $1 \leq n \leq 6$ . Similarly, in [DR] the Auslander-Reiten quivers of the categories  $\mathcal{F}(n)$  with  $2 \leq n \leq 5$  are presented. As we see here, the Auslander-Reiten quiver

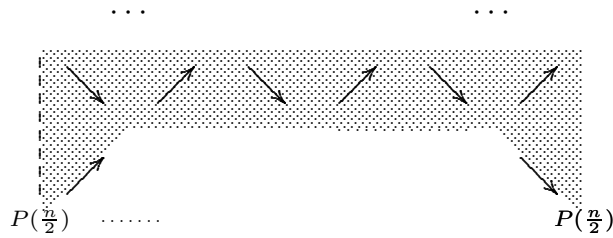
of  $\mathcal{F}(n)$  can be obtained from that of  $\mathcal{S}(n)$  by just deleting some vertices, thus it is easy to obtain the illustrations presented in [DR] from those in [RS2]. The deletion process explains also some of the features of the shape of the Auslander-Reiten quiver of  $\mathcal{F}(n)$ : Of course, there is precisely one projective-injective vertex, namely the module  $P(n) = I(n) = T(1)$ . The remaining transjective orbits contain precisely two vertices, namely  $T(n+1-i)$  and  $P(i) = \tau_{\mathcal{F}(n)}T(n+1-i)$ , here  $1 \leq i \leq n-1$ . As we now know, this concerns the  $\tau$ -orbit of  $\mathcal{S}(n)$  which contains the pairs  $([i], [i])$  and  $([n-i], [n-i])$ : both pairs are killed by the functor  $\lambda$ , but in-between the Auslander-Reiten sequence starting with  $([0], [i])$  and ending in  $([n-i], [n])$  is not touched and it yields under  $\lambda$  the Auslander-Reiten sequence starting with  $P(i)$  and ending in  $T(n+1-i)$ .

**The corresponding  $\tau$ -orbits of the category  $\mathcal{S}(n)/\mathcal{U} \simeq \text{mod } \Pi_{n-1}$ .**

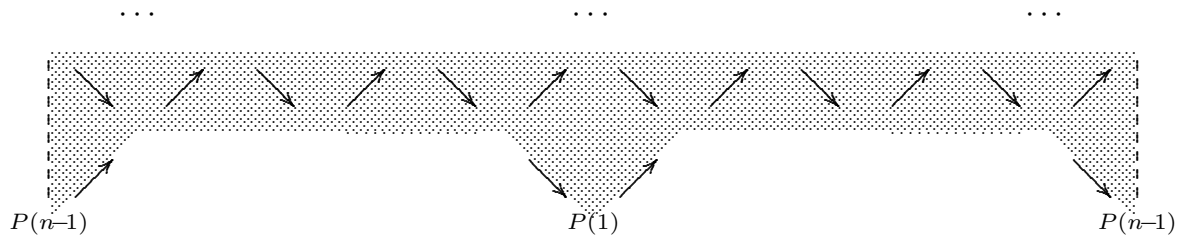
For  $2 \leq i < \frac{n}{2}$ :



For  $n \geq 4$  even and  $i = \frac{n}{2}$ :



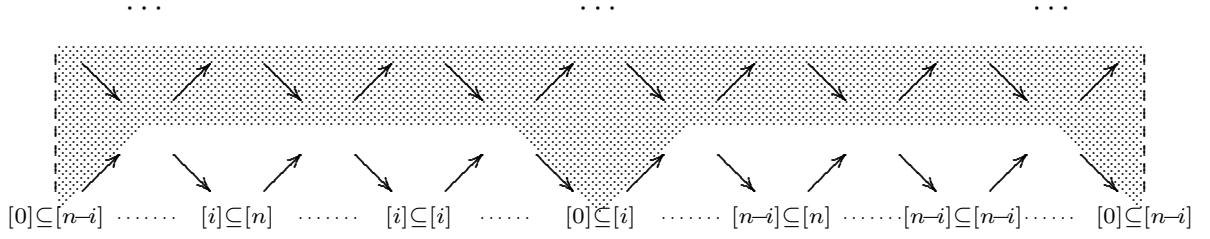
And finally, for  $i = 1$  we get:



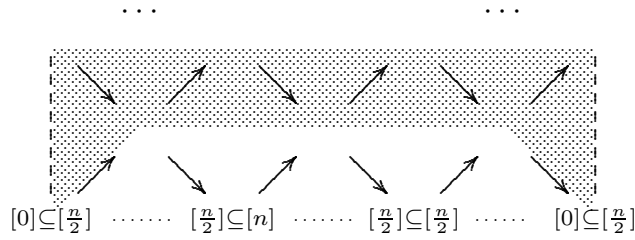
**A simultaneous view.** Let us draw again the relevant components of  $\mathcal{S}(n)$  and use the shading in order to illustrate what remains when we delete the objects in  $\mathcal{U}$ . Note that under the functor  $\delta\lambda$ , we have

$$\begin{aligned} \delta\lambda([i], [i]) &= 0 & 1 \leq i \leq n, \\ \delta\lambda([i], [n]) &= 0 & \text{for } 0 \leq i \leq n-1, \\ \delta\lambda([0], [j]) &= P_{\Pi}(j) & 1 \leq j \leq n-1. \end{aligned}$$

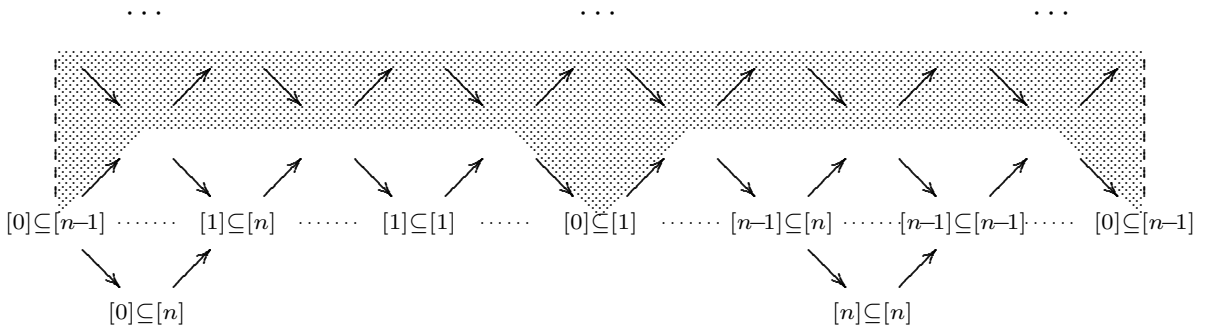
First of all, for  $2 \leq i < \frac{n}{2}$ , two objects of the  $\tau_{\mathcal{S}}$ -orbit of  $([i], [i])$  survive:



If  $n \geq 4$  is even and  $i = \frac{n}{2}$ , the pair  $([0], [i])$  is the only object in the  $\tau_{\mathcal{S}}$ -orbit of  $([i], [i])$  which survives:

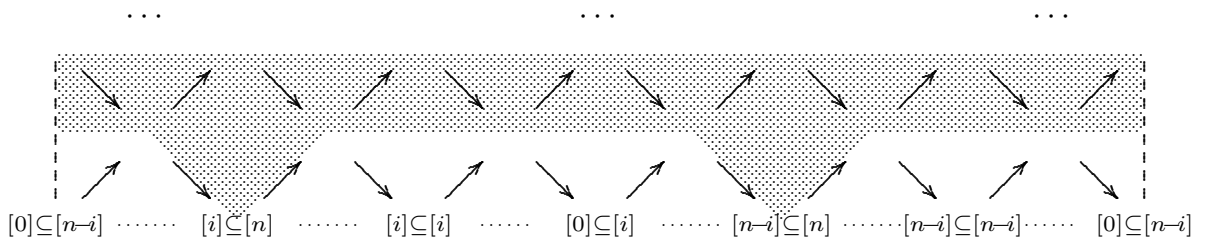


Finally, for  $i = 1$ , again two objects in the  $\tau_{\mathcal{S}}$ -orbit of  $([1], [1])$  survive:

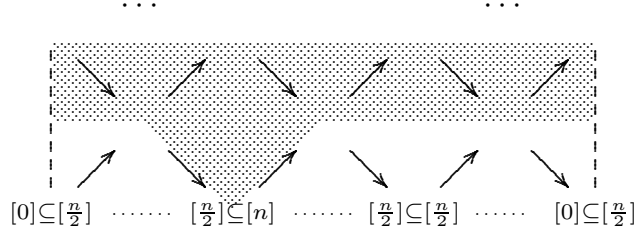


**The relevant components of  $\mathcal{S}(n)/\mathcal{V}$ .** In the same way as we have presented components of  $\mathcal{S}(n)$  shading the parts which remain after deleting  $\mathcal{U}$ , we now show what remains from these components when we remove  $\mathcal{V}$ .

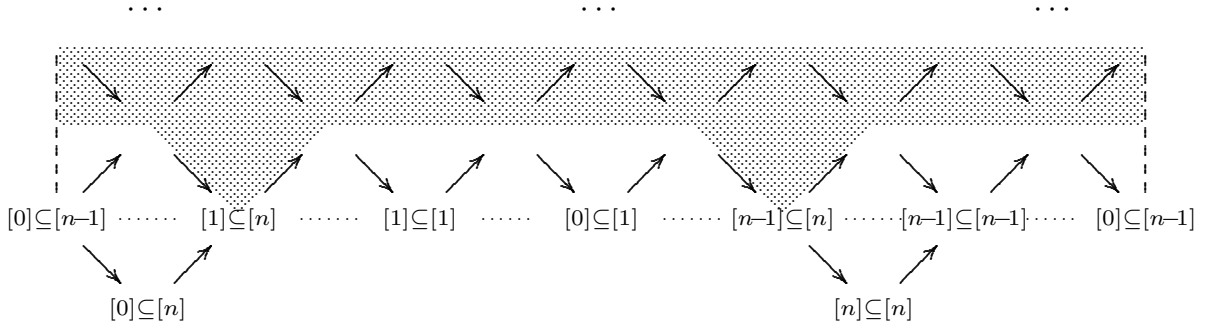
First of all, for  $2 \leq i < \frac{n}{2}$ , two objects in the  $\tau_{\mathcal{S}}$ -orbit of  $([i], [i])$  survive:



Next, for  $n \geq 4$  even and  $i = \frac{n}{2}$ , the pair  $([i], [n])$  is the only object in the  $\tau_{\mathcal{S}}$ -orbit of  $([i], [i])$  which survives:



Finally, for  $i = 1$ , again two objects in the  $\tau_{\mathcal{S}}$ -orbit of  $([i], [i])$  survive:



## 8. Abelian factor categories of triangulated categories

As a byproduct of our consideration, we see that here we deal with examples of triangulated categories  $\mathcal{T}$  with an ideal  $\mathcal{I}$  generated by an idempotent such that  $\mathcal{T}/\mathcal{I}$  is abelian.

Namely, let  $\mathcal{T} = \underline{\mathcal{S}}(n)$  be the stable category of  $\mathcal{S}(n)$ , it is obtained from  $\mathcal{S}(n)$  by factoring out the ideal generated by the objects  $([0], [n])$  and  $([n], [n])$ . Now  $\mathcal{S}(n)$  is in a natural way (see [RS2]) a Frobenius category such that  $([0], [n])$  and  $([n], [n])$  are the only indecomposable objects which are both projective and injective, thus the category  $\mathcal{T}$  is a triangulated category.

Let  $\mathcal{I}$  be the ideal in  $\mathcal{T}$  generated by the objects  $([i], [j])$  with  $1 \leq i \leq n-1$  and either  $i = j$  or  $j = n$ . Then, as additive categories, we have equivalences

$$\mathcal{T}/\mathcal{I} \simeq \mathcal{S}(n)/\mathcal{U} \simeq \text{mod } \Pi_{n-1},$$

thus  $\mathcal{T}/\mathcal{I}$  is an abelian category.

Similarly, let  $\mathcal{J}$  be the ideal in  $\mathcal{T}$  generated by the objects  $([i], [j])$  with  $1 \leq j \leq n-1$  and either  $i = j$  or  $i = 0$ . Then, as additive categories, we have equivalences

$$\mathcal{T}/\mathcal{J} \simeq \mathcal{S}(n)/\mathcal{V} \simeq \text{mod } \Pi_{n-1},$$

thus  $\mathcal{T}/\mathcal{J}$  is again an abelian category.

Here, the ideals  $\mathcal{I}, \mathcal{J}$  of  $\mathcal{T}$  each are generated by  $2n-2$  indecomposable objects, whereas the rank of the Grothendieck group  $K_0(\mathcal{T}/\mathcal{I})$  is  $n-1$ . One may compare this

with corresponding examples which involve cluster categories and cluster tilted algebras, see [BMR] and [K]. Let  $T$  be a cluster tilting object in a cluster category  $\mathcal{T}$  say of type  $Q$ , where  $Q$  is a directed quiver with  $n$  vertices. Then the factor category  $\mathcal{T}/T$  is an abelian category and the rank of the Grothendieck group  $K_0(\mathcal{T}/T)$  is equal to  $n$ , and this is also the number of isomorphism classes of indecomposable direct summands of  $T$ .

## 9. References.

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C. M. Ringel  
 Fakultät für Mathematik,  
 Universität Bielefeld, PO Box 100 131  
 D-33501 Bielefeld, Germany  
 e-mail: ringel@math.uni-bielefeld.de

P. Zhang  
 Department of Mathematics,  
 Shanghai Jiao Tong University  
 Shanghai 200240, P. R. China,  
 e-mail: pzhang@sjtu.edu.cn