Paths and animals in unbounded degree graphs with repulsion

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Abstract

A class of countable infinite graphs with unbounded vertex degree is considered. In these graphs, the vertices of large degree 'repel' each other, which means that the path distance between two such vertices cannot be smaller than a certain function of their degrees. Assuming that this function increases sufficiently fast, we prove that the number of finite connected subgraphs (animals) of order N containing a given vertex x is exponentially bounded in N for N belonging to an infinite subset $\mathcal{N}_x \subset \mathbb{N}$. Under a less restrictive condition, the same result is obtained for the number of simple paths originated at a given vertex. These results are then applied to a number of problems, including estimating the growth of the Randić index and of the number of greedy animals.

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1. Introduction

Infinite graphs are used in probabilistic combinatorics, image processing, and many other domains. In particular, they serve as underlying discrete metric spaces for Markov random fields [4, 8, 12, 13, 14, 15, 19, 20]. The structure of such graphs is more accessible for studying if the vertex degrees are globally bounded. However, in many important applications it is essential to employ unbounded degree graphs, see [12, 13, 15]. For these graphs, it is intuitively clear that their global metric properties can be similar to those of bounded degree graphs if the vertices with large degree are 'sparse', see, e.g., the Introduction in [20]. In the present work, we consider two families of 'repulsive graphs', in which vertices with large degree 'repel' each other in the sense of Definition 1 below. For such graphs, we derive exponential upper bounds for the number of connected subgraphs of order N which contain a given vertex, valid for large N. These results allow for obtaining similar estimates also for other metric characteristics, e.g., for the number of vertices in a ball of radius N. In Section 2, we introduce necessary notions and notations and then formulate our main results in Theorems 2, 3 and Corollary 4. In Section 3, we describe some applications of these results. Among them we note an upper estimate for the generalized Randić index (Proposition 7) and an almost sure sublinear growth of weights of greedy graph animals (Proposition 9). In Section 5, we give the proof of the statements just mentioned preceded by the study of the properties of paths and animals in repulsive graphs conducted in Section 4. Here we introduce the notion of a tempered graph by imposing restrictions on the vertex degree growth, see Definition 12. Then in Lemmas 13 and 14, we show that the properties stated in Theorems 2 and 3 hold for such tempered graphs. Thereafter, by a technical result obtained in Lemma 16 we prove Lemma 17 which gives us tools for controlling the vertex degree growth in the repulsive graphs which we study. By means of these tools we prove that our graphs are tempered, which yields the proof of Theorems 2 and 3, as well as of Propositions 7 and 9.

2. Setup and results

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a countably infinite simple graph with no loops. By writing $x \sim y$ we mean that $x, y \in \mathsf{V}$ constitute an edge, $\langle x, y\rangle = \langle y, x\rangle \in \mathsf{E}$. We say that such x and y are adjacent and that they are the endpoints of the edge $\langle x, y\rangle$. For each $x \in \mathsf{V}$, the *degree*

$$n(x) \stackrel{\text{def}}{=} \#\{y \in \mathsf{V} : y \sim x\}$$

is assumed to be finite, whereas

$$n_{\mathsf{G}} \stackrel{\text{def}}{=} \sup_{x \in \mathsf{V}} n(x), \tag{1}$$

can be finite or infinite. A finite connected subgraph, $\mathsf{A} \subset \mathsf{G}$, is called an *animal* (also a *polymer*, cf. [8, 16, 19]). By $\mathsf{V}(\mathsf{A})$ and $\mathsf{E}(\mathsf{A})$ we denote the set of vertices and edges of A, respectively. A *path*, ϑ, is a finite sequence of vertices, $\{x_0, x_1, \ldots, x_n\}$, not necessarily distinct, such that $x_{k+1} \sim x_k$, for all $k = 0, \ldots, n-1$. Then ϑ *originates* at x_0 and *terminates* at x_n. Its *length* $|\vartheta|$ is set to be n. In a *simple path*, all $x_0, x_1, \ldots, x_{n-1}$ are distinct. By G_ϑ we denote the graph generated by ϑ. That is, its vertex set V_ϑ consists of those in ϑ, not counting repeated vertices; the edge set E_ϑ comprises the edges with both endpoints in V_ϑ. Clearly, each G_ϑ is an animal.

By $\vartheta(x, y)$ we denote a path such that $x_0 = x$ and $x_n = y$. The path distance $\rho(x, y)$ is set to be the length of the shortest path $\vartheta(x, y)$. A ball $\mathsf{B}_N(x)$ (resp., a sphere $\mathsf{S}_N(x)$), $N \in \mathbb{N}$ and $x \in \mathsf{V}$, is the set of $y \in \mathsf{V}$ such that $\rho(x, y) \leq N$ (resp., $\rho(x, y) = N$). For $N \in \mathbb{N}$ and $x \in \mathsf{V}$, let $\mathcal{A}_N(x)$ denote the set of all animals such that $x \in \mathsf{V}(\mathsf{A})$ and $|\mathsf{V}(\mathsf{A})| = N$. For such x and N, let also $\Sigma_N(x)$ be the set of all simple paths of length N originated at x. In many applications, see [6, 7, 8, 11, 16, 18, 19], one needs to estimate the growth of the cardinalities of the mentioned sets as $N \to +\infty$. For a graph G with $n_\mathsf{G} < \infty$, there exist positive q_G, $\bar{q}_\mathsf{G}$, and N_G such that the following estimates hold

$$\text{(a)} \quad |\Sigma_N(x)| \leq q_\mathsf{G}^N, \qquad\qquad \text{(b)} \quad |\mathcal{A}_N(x)| \leq \bar{q}_\mathsf{G}^N, \tag{2}$$

for all $N \geq N_\mathsf{G}$. The first estimate can easily be proven to hold with $q_\mathsf{G} = n_\mathsf{G}$ and $N_\mathsf{G} = 1$, cf. (20) below. The second one is not so immediate, see [17, Chapter 2] where a more general estimate was proved. Note that (b) implies (a). By (a) in (2) one readily gets

$$\text{(a)} \quad |\mathsf{S}_N(x)| \leq q_\mathsf{G}^N, \qquad\qquad \text{(b)} \quad |\mathsf{B}_N(x)| \leq \frac{q_\mathsf{G}}{q_\mathsf{G} - 1} q_\mathsf{G}^N, \tag{3}$$

For graphs with $n_\mathsf{G} = +\infty$, the cardinalities in (2) can grow faster than exponentially. Furthermore, if (a) or (b) holds for $N \geq N_*$ with one and the same N_* for all $x \in \mathsf{V}$, then $n_\mathsf{G} < \infty$.

For $x, y \in \mathsf{V}$, we set

$$m_+(x, y) = \max\{n(x); n(y)\}, \qquad m_-(x, y) = \min\{n(x); n(y)\}.$$

Definition 1. *Let $\phi : \mathbb{N} \to (0, +\infty)$ be strictly increasing. By $\mathbb{G}_\pm(\phi)$ we denote the family of graphs, for each of which there exists an integer n_* such that*

$$\rho(x, y) \geq \phi(m_\pm(x, y)), \tag{4}$$

whenever $m_-(x, y) > n_*$. No restrictions are imposed if $m_-(x, y) \leq n_*$.

Note that, for the first time, a condition like (4) appeared in [2]. Clearly,

$$\mathbb{G}_+(\phi) \subset \mathbb{G}_-(\phi).$$

For $\mathsf{G} \in \mathbb{G}_\pm(\phi)$, by (4) vertices of large degrees 'repel' each other. That is why we call these graphs *repulsive*. For such a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, we set

$$\mathsf{V}_* = \{x \in \mathsf{V} \mid n(x) \leq n_*\}, \qquad \mathsf{V}_*^c = \mathsf{V} \setminus \mathsf{V}_*, \tag{5}$$

and consider

$$\mathsf{K}(x) \stackrel{\text{def}}{=} \{y \in \mathsf{V} \mid \rho(y, x) < \phi\,(n(x))\}.$$

Now let G be in $\mathbb{G}_+(\phi)$. For $x \in \mathsf{V}_*^c$, by (4) we have $\mathsf{K}(x) \cap \mathsf{V}_*^c = \{x\}$, i.e., x 'repels' all vertices $y \in \mathsf{V}_*^c$ from $\mathsf{K}(x)$. For the sake of convenience, we shall assume that $\mathsf{K}(x)$ contains the neighborhood of x, which is equivalent to assuming $\phi(4) > 1$. Then for any $\mathsf{G} \in \mathbb{G}_+(\phi)$, we have

$$\phi(n_* + 1) > 1.$$

For $\mathsf{G} \in \mathbb{G}_-(\phi)$, x 'repels' from $\mathsf{K}(x)$ only those $y \in \mathsf{V}_*^c$, for which $n(y) \geq n(x)$.

Our main results are contained in the following two statements.

Theorem 2. *Let $\phi : \mathbb{N} \to (0, +\infty)$ be such that the following holds*

$$\sum_{k=1}^{\infty} \frac{\log t_{k+1}}{\phi(t_k)} < \infty, \tag{6}$$

for some strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Then, for each $\mathsf{G} \in \mathbb{G}_-(\phi)$, there exists $q_{\mathsf{G}} > 1$ such that, for any $x \in \mathsf{V}$, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that the estimate

$$|\Sigma_N(x)| \leq q_{\mathsf{G}}^N \tag{7}$$

holds for all $N = N_k$, $k \in \mathbb{N}$. If $\mathsf{G} \in \mathbb{G}_+(\phi)$, then, for each $x \in \mathsf{V}$, there exists $N_x \in \mathbb{N}$ such that the estimate (7) holds for all $N \geq N_x$.

Theorem 3. *Let $\phi : \mathbb{N} \to (0, +\infty)$ be such that the following holds*

$$\sum_{k=1}^{\infty} \frac{t_{k+1} \log t_{k+1}}{\phi(t_k)} < \infty, \tag{8}$$

for some strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Then, for each $\mathsf{G} \in \mathbb{G}_-(\phi)$, there exists $\bar{q}_\mathsf{G} > 1$ such that, for any $x \in \mathsf{V}$, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that the estimate

$$|\mathcal{A}_N(x)| \leq \bar{q}_\mathsf{G}^N \tag{9}$$

holds for all $N = N_k$, $k \in \mathbb{N}$. If $\mathsf{G} \in \mathbb{G}_+(\phi)$, then, for any $x \in \mathsf{V}$, there exists $\overline{N}_x \in \mathbb{N}$ such that the estimate (9) holds for all $N \geq \overline{N}_x$.

An immediate corollary of Theorem 2 is the following statement.

Corollary 4. *Let G be in $\mathbb{G}_+(\phi)$ with ϕ obeying (6). Let also N_x, $x \in \mathsf{V}$, be as in Theorem 2. Then there exists $B_x > 0$ such that, for all $N > N_x$, the following holds*

$$\text{(a)} \quad |\mathsf{S}_N(x)| \leq q_\mathsf{G}^N, \qquad \text{(b)} \quad |\mathsf{B}_N(x)| \leq B_x q_\mathsf{G}^N. \tag{10}$$

Proof. By the very definition of $\mathsf{S}_N(x)$, we have that $|\mathsf{S}_N(x)| \leq |\Sigma_N(x)|$, which yields (a) in (10), whereas (b) with $B_x = |\mathsf{B}_{N_x}(x)|/q_\mathsf{G}^{N_x}$ follows by (a). □

The optimal choice of $\{t_k\}_{k\in\mathbb{N}}$ in (6) seems to be $t_k = \exp(e^k)$, for big enough k. Then the choice of ϕ can be $\phi(t) = \upsilon \log t (\log\log t)^{1+\epsilon}$, $\epsilon > 0$; cf. [15, Theorem 4].

3. Applications

3.1. Percolation

Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be as above. For $\mathsf{E}' \subset \mathsf{E}$, we set $\mathsf{G}' = (\mathsf{V}, \mathsf{E}')$. Note that G' need not be connected. Let now edges $e \in \mathsf{E}'$ be picked at random, independently and with the same probability p each. This defines a probability measure, μ_p^{b}, on the set of all subsets of E. The corresponding subgraph G' with randomly picked $e \in \mathsf{E}'$ is random as well. The event that it has an infinite connected component (called also *cluster*) occurs with probability either zero or one, dependent on the value of p. This is the Bernoulli bond percolation model, cf. [10, 12].

Proposition 5. *Let ϕ obey (6) and G be in $\mathbb{G}_-(\phi)$, so that (7) holds. Then no cluster appears μ_p^{b}-almost surely whenever $p < 1/q_\mathsf{G}$.*

Proof. Given $x \in \mathsf{V}$, the probability that there exists at least one simple path of length N originated at x does not exceed $p^N|\Sigma_N(x)|$. Then the proof follows by (7) and the Borel-Cantelli lemma. $\square$

Now for $\mathsf{V}' \subset \mathsf{V}$, let $\mathsf{E}' \subset \mathsf{E}$ comprise the edges with both endpoints in V'. Set $\mathsf{G}' = (\mathsf{V}', \mathsf{E}')$. Further, suppose that each vertex of V' is picked at random, independently and with the same probability p each. This defines a probability measure, $\mu_p^{\rm s}$, on the set of all subsets of V. Thereby, the subgraph G' is random. The event that it has a cluster occurs with probability either zero or one, dependent on p. The appearance of a cluster is called the Bernoulli site percolation, see [10, Chapter 3] or [12].

Proposition 6. *Let ϕ obey (8) and G be in $\mathbb{G}_-(\phi)$, so that (9) holds. Then no cluster appears $\mu_p^{\rm s}$-almost surely whenever $p < 1/\bar{q}_{\mathsf{G}}$.*

Proof. Given $x \in \mathsf{V}$, the probability that there exists at least one connected subgraph of order N, which contains x, does not exceed $p^N|\mathcal{A}_N(x)|$. Then the proof follows by (9) and the Borel-Cantelli lemma. $\square$

Further applications of the above results to models of dependent percolation, e.g., to the random cluster model, can be developed by means of cluster expansion techniques [7, 8, 16, 17, 19, 20]. Applications in [13, 14, 15] to Gibbs random fields on the graphs considered here are based on the estimate in (7).

3.2. Randić index

For a real θ and an animal, A, we set

$$R^\theta(\mathsf{A}) = \sum_{\langle x,y\rangle \in \mathsf{E}(\mathsf{A})} [n(x)n(y)]^\theta.$$

In mathematical chemistry, large molecules are consideerd as finite trees. For such a tree T, $R^\theta(\mathsf{T})$ is known under the name *generalized Randić* or *connectivity* index, see [5]. It turns out that its value is closely related to chemical properties of the corresponding substance.

For a vertex x and $N \in \mathbb{N}$, we define

$$R^\theta_N(x) = \max_{\mathsf{A} \in \mathcal{A}_N(x)} R^\theta(\mathsf{A}).$$

Proposition 7. *Let $\phi : \mathbb{N} \to (0, +\infty)$ be strictly increasing and such that*

$$\sum_{k=1}^{\infty} \frac{t_{k+1}^{\theta+1}}{\phi(t_k)} < \infty, \qquad (11)$$

for some strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Then, for each $\mathsf{G} \in \mathbb{G}_-(\phi)$, there exists $\tilde{q}_{\mathsf{G}} > 1$ such that, for any $x \in \mathsf{V}$, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$R_N^\theta(x) \leq \tilde{q}_{\mathsf{G}}^N \qquad (12)$$

holds for all $N = N_k$, $k \in \mathbb{N}$. If $\mathsf{G} \in \mathbb{G}_+(\phi)$, then for any $x \in \mathsf{V}$, there exists $\widetilde{N}_x \in \mathbb{N}$ such that (12) holds for all $N \geq \widetilde{N}_x$.

The proof of this statement will be given below.

3.3. Growth of Aut(G)

For a $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, an automorphism, γ, is a bijection $\mathsf{V} \ni x \mapsto x\gamma \in \mathsf{V}$ such that $x \sim y$ implies $x\gamma \sim y\gamma$. The automorphisms constitute a group, denoted by $\mathrm{Aut}(\mathsf{G})$. Assume that V is given the discrete topology, and let $\mathcal{T}$ be the weakest topology on $\mathrm{Aut}(\mathsf{G})$ in which the maps $\mathrm{Aut}(\mathsf{G}) \ni \gamma \mapsto x\gamma \in \mathsf{V}$ are continuous for all $x \in \mathsf{V}$. It is known [1] that $(\mathrm{Aut}(\mathsf{G}), \mathcal{T})$ is a locally compact Polish group. By the local compactness, there exists a right Haar measure on $\mathrm{Aut}(\mathsf{G})$, which we denote by μ. For $x \in \mathsf{V}$, the set

$$\Gamma_x := \{\gamma \in \mathrm{Aut}(\mathsf{G}) : x\gamma = x\}$$

is the *stabilizer* of x. It is compact and open, and thus $0 < \mu(\Gamma_x) < \infty$, for all $x \in \mathsf{V}$. Let Δ stand for a compact neighborhood of the identity of $\mathrm{Aut}(\mathsf{G})$. For $n \in \mathbb{N}$, by Δ^n we denote the set of all products $\gamma_1\gamma_2\cdots\gamma_n$ of the elements of Δ.

Proposition 8. *Let G be in $\mathbb{G}_+(\phi)$ with ϕ obeying (6), and let Δ be as above. Then there exist $C > 0$ and $N_* \in \mathbb{N}$ such that, for all $N \geq N_*$, the following holds*

$$\mu(\Delta^N) \leq C q_{\mathsf{G}}^N, \qquad (13)$$

where q_{G} is the same as in (7).

Proof. By Proposition 3.2 in [1], for each $x \in \mathsf{V}$, there exists $c > 0$ such that, for all $N \in \mathbb{N}$, the following estimate holds

$$\mu(\Delta^N) \leq \mu(\Gamma_x)|\mathsf{B}_{cN}(x)|.$$

We fix x and apply (b) of (10), which yields (13) with $N_* = N_x$ and $C = B_x \mu(\Gamma_x) q_{\mathsf{G}}^c$. □

3.4. Greedy animals

Let $\{Y_x : x \in \mathsf{V}\}$ be a family of independent positive random variables (weights). For $N \in \mathbb{N}$ and $x \in \mathsf{V}$, we define

$$S_N(x) = \max_{\mathsf{A} \in \mathcal{A}_N(x)} \sum_{x \in \mathsf{V}(\mathsf{A})} Y_x. \tag{14}$$

Those $\mathsf{A} \in \mathcal{A}_N(x)$, for which the maximum in (14) is attained are called *greedy animals*, see [6] for motivating examples, applications, and further details.

Let P_x be the probability measure on $[0, +\infty)$ which is the law of Y_x. Then the law of the family $\{Y_x : x \in \mathsf{V}\}$ is defined as a product measure in a standard way. We assume that, for each $x \in \mathsf{V}$,

$$w_x(t) := \log \mathbb{E} e^{tY_x} < \infty, \tag{15}$$

for a certain $t > 0$. Thus, w_x is analytic in some neighborhood of $t = 0$, and hence $w_x(t)/t \to v_x$ as $t \to 0$, where

$$v_x := \mathbb{E} Y_x. \tag{16}$$

Proposition 9. *Let G be in $\mathbb{G}_-(\phi)$ with ϕ satisfying (8). Suppose also that*

$$v_x \leq Cn(x) \log n(x), \tag{17}$$

for some $C > 0$ and each $x \in \mathsf{V}$. Then there exists $Y > 0$ such that, for each $x \in \mathsf{V}$, there exists an increasing sequence, $\{N_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$, for which

$$\limsup_{k \to +\infty} \frac{1}{N_k} S_{N_k}(x) \leq Y \qquad \text{with probability 1.} \tag{18}$$

The proof of this statement will be given below. Let us now make some comments. The lattice $\mathbb{Z}^d$ can be turned into a graph by setting $x \sim y$ if $|x - y| = 1$. The greedy animals on $\mathbb{Z}^d$ were studied in detail in [6, 9, 11, 18]. In those papers, however, the weights are supposed to be identically distributed with law P_x satisfying less restrictive conditions (as compared to (15)), involving the lattice dimension d, cf. Theorem 1 in [6] or Theorem 3.3 in [18]. In the statement above, we allow the mean value of Y_x to increase in a controlled way (17), which seems to be quite natural for unbounded degree graphs. In a separate work, we shall study greedy animals in such graphs in more detail. In particular, we plan to relax the exponential integrability assumed in (15).

4. Further properties of paths and animals

4.1. Counting paths

We recall that by $\mathsf{G}_\vartheta = (\mathsf{V}_\vartheta, \mathsf{E}_\vartheta)$ we denote the graph generated by path ϑ. For $e \in \mathsf{E}$, we say that ϑ traverses e if $e \in \mathsf{E}_\vartheta$. We say that $\vartheta = \{x_0, \ldots, x_n\}$ leaves x_k towards x_{k+1}, $k = 0, \ldots, n-1$. For $x \in \mathsf{V}_\vartheta$, let $\nu_\vartheta(x)$ be the number of times ϑ leaves x. We also set $\nu_\vartheta(x) = 0$ if x is not in V_ϑ. Then, for a simple path, $\nu_\vartheta(x) \leq 1$. Recall that $\Sigma_N(x)$ denotes the collection of simple paths of length $N \in \mathbb{N}$ originated at a given $x \in \mathsf{V}$. Along with this set we also consider $\Theta_N(x)$ being the collection of paths $\vartheta = \{x, x_1, \ldots, x_N\}$ such that the number of times ϑ leaves each $y \in \mathsf{V}_\vartheta$ towards any $z \in \mathsf{V}_\vartheta$ is at most one. Note that this does not mean $\nu_\vartheta(x) \leq 1$.

Lemma 10. *For any $x \in \mathsf{V}$ and $N \in \mathbb{N}$, it follows that $\Sigma_N(x) \subset \Theta_N(x)$. Each $\vartheta \in \Theta_N(x)$ has the properties: (i) each $e \in \mathsf{E}_\vartheta$ can be traversed by ϑ at most twice; (ii) $\nu_\vartheta(y) \leq n(y)$ for each $y \in \mathsf{V}_\vartheta$.*

Proof: The stated inclusion is immediate, whereas both (i) and (ii) follow from the fact that ϑ leaves each $x \in \mathsf{V}_\vartheta$ towards any $y \sim x$ at most once. □

Lemma 11. *For any $x \in \mathsf{V}$ and $N \in \mathbb{N}$, it follows that*

$$|\Theta_N(x)| \leq \max_{\vartheta \in \Theta_N(x)} \exp\left(\sum_{y \in \mathsf{V}_\vartheta} n(y) \log n(y) \right). \tag{19}$$

Proof: Obviously,

$$|\Theta_N(x)| \leq \sum_{y:\ y\sim x} |\Theta_{N-1}(y)| \leq \sup_{y:\ y\sim x} n(x)\,|\Theta_{N-1}(y)|\,,$$

which by the induction in N yields

$$\begin{aligned} |\Theta_N(x)| &\leq \max_{\vartheta\in\Theta_N(x)} n(x)n(x_1)\cdots n(x_{N-1}) \\ &= \max_{\vartheta\in\Theta_N(x)} \exp\left(\sum_{y\in\mathsf{V}_\vartheta} \nu_\vartheta(y)\log n(y)\right) \\ &\leq \max_{\vartheta\in\Theta_N(x)} \exp\left(\sum_{y\in\mathsf{V}_\vartheta} n(y)\log n(y)\right), \end{aligned}$$

where we have used claim (ii) of Lemma 10. □
Similarly, one proves that

$$|\Sigma_N(x)| \leq \max_{\vartheta\in\Sigma_N(x)} \exp\left(\sum_{y\in\mathsf{V}_\vartheta} \log n(y)\right). \tag{20}$$

4.2. Graphs with tempered growth of vertex degree

For an increasing function $g : \mathbb{N} \to (0, +\infty)$ and an animal A, we set

$$G(\mathsf{A}; g) = \frac{1}{|\mathsf{V}(\mathsf{A})|} \sum_{x\in\mathsf{V}(\mathsf{A})} g\left(n(x)\right).$$

If $n_\mathsf{G} < \infty$, see (1), then $G(\mathsf{A}; g) \leq g(n_\mathsf{G})$ for any animal and any function g. We say that the vertex degree in G is of *tempered* growth if

$$\max_{\mathsf{A}\in\mathcal{A}_N(x)} G(\mathsf{A}; g) \leq \gamma. \tag{21}$$

More precisely, we mean the following.

Definition 12. *The graph G is said to be g-tempered (resp. strongly g-tempered) if there exists a number $\gamma > 0$ such that, for every $x \in \mathsf{V}$, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ (resp. there exists $N_x \in \mathbb{N}$) such (21) holds for all $N = N_k$, $k \in \mathbb{N}$ (resp. for all $N \geq N_x$).*

Lemma 13. *For $g(t) = t \log t$, $t \in \mathbb{N}$, let G be g-tempered. Then there exists $q_{\mathsf{G}} > 1$ such that, for any $x \in \mathsf{V}$, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that the estimate*

$$|\mathcal{A}_N(x)| \leq q_{\mathsf{G}}^N \tag{22}$$

holds for all $N = N_k$, $k \in \mathbb{N}$. If G is strongly g-tempered, then for any $x \in \mathsf{V}$, there exists $N_x \in \mathbb{N}$ such that (22) holds for all $N \geq N_x$.

Proof: Let A be an animal such that $x \in \mathsf{V}(\mathsf{A})$. Consider the multi-graph $\widetilde{\mathsf{A}}$ which has the same vertices as A but doubled edges. This means that the edge set of $\widetilde{\mathsf{A}}$ consists of the pairs $e, \tilde{e}$, both connecting the same vertices, such that $e \in \mathsf{E}(\mathsf{A})$. Then the graph $\widetilde{\mathsf{A}}$ is Eulerian, see, e.g., [3, page 51], and hence there exists a path in $\widetilde{\mathsf{A}}$, which originates and terminates at x, enters each $y \in \mathsf{V}(\mathsf{A})$, and traverses each edge of $\widetilde{\mathsf{A}}$ exactly once. Therefore, $\mathsf{A} = \mathsf{G}_\vartheta$ for some path $\vartheta(x,x) \in \Theta_M(x)$ with $M = 2|\mathsf{E}(\mathsf{A})|$. Thus, by (19) we have

$$\begin{aligned} |\mathcal{A}_N(x)| \leq |\Theta_M(x)| &\leq \max_{\vartheta \in \Theta_M(x)} \exp\left(\sum_{y \in \mathsf{V}_\theta} g(n(y)) \right) \\ &= \max_{\mathsf{A} \in \mathcal{A}_N(x)} \exp\left(\sum_{y \in \mathsf{V}(\mathsf{A})} g(n(y)) \right) \\ &\leq \max_{\mathsf{A} \in \mathcal{A}_N(x)} \exp\left(N G(\mathsf{A}; g) \right). \end{aligned}$$

Then we set $q_{\mathsf{G}} = e^\gamma$ and obtain (22) from (21). □

In the same way, by means of (20) one proves the following

Lemma 14. *For $g(t) = \log t$, $t \in \mathbb{N}$, let G be g-tempered. Then there exists $\bar{q}_{\mathsf{G}} > 1$ such that, for any $x \in \mathsf{V}$, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that the estimate*

$$|\Sigma_N(x)| \leq \bar{q}_{\mathsf{G}}^N \tag{23}$$

holds for all $N = N_k$, $k \in \mathbb{N}$. If G is strongly g-tempered, then for any $x \in \mathsf{V}$, there exists $\overline{N}_x \in \mathbb{N}$ such that the estimate (23) holds for all $N \geq \overline{N}_x$.

4.3. Capacity of animals

For an animal, $\mathsf{A} \subset \mathsf{G}$, by $\rho_{\mathsf{A}}(x,y)$, we denote the length of the shortest path $\vartheta(x,y)$ in A, i.e., such that $\mathsf{G}_\vartheta \subset \mathsf{A}$. We shall be interested in estimating the number of vertices in subsets $\mathsf{B} \subset \mathsf{V}(\mathsf{A})$, which have the following property.

Definition 15. *Given $\lambda > 1$, a set, $\mathsf{B} \subset \mathsf{V}(\mathsf{A})$, is said to be λ-admissible in A if $\rho_{\mathsf{A}}(x,y) \geq \lambda$ for any distinct $x, y \in \mathsf{B}$. The quantity*

$$C(\mathsf{A};\lambda) = \max\{|\mathsf{B}| : \mathsf{B} \text{ is } \lambda-\text{admissible in } \mathsf{A}\}$$

is called the λ-capacity of A.

Hence, if $\mathsf{A}' \subset \mathsf{A}$ is a connected spanning subgraph, then

$$C(\mathsf{A};\lambda) \leq C(\mathsf{A}';\lambda). \tag{24}$$

If ϑ is a simple path of length N, then

$$C(\mathsf{G}_\vartheta;\lambda) \leq 1 + N/\lambda. \tag{25}$$

Lemma 16. *Let A be an animal of size N. Then, for any $\lambda > 0$,*

$$C(\mathsf{A};\lambda) \leq \max\{1; 2N/\lambda\}. \tag{26}$$

Proof: Suppose first that $N \leq \lambda$. As any simple path in A cannot be longer than $N-1$, one has $\rho(x,y) \leq N-1$ for any $x, y \in \mathsf{V}(\mathsf{A})$. Thus, any λ-admissible set can contain at most one element, and hence (26) holds. For $N > \lambda$, we use the induction in N assuming that (26) holds for all $N' < N$.

Let $\mathsf{T} \subset \mathsf{A}$ be a spanning tree for A. We are going to estimate its capacity and then to use (24). Let $\mathsf{B} \subset \mathsf{V}(\mathsf{T})$ be such that

$$\min_{x,y \in \mathsf{B},\ x \neq y} \rho_{\mathsf{T}}(x,y) \geq \lambda.$$

To estimate $|\mathsf{B}|$, we pick $z, z' \in \mathsf{V}(\mathsf{A})$ such that the simple path $\vartheta(z,z')$ in T is the longest one among such paths (backbone); that is, $\rho_{\mathsf{T}}(z,z')$ is the diameter of T, see Fig. 1. Let $\mathsf{T}_0 \subset \mathsf{T}$ be the graph generated by this path. Then we split

$$\mathsf{E}(\mathsf{T}) = \mathsf{E}(\mathsf{T}_0) \cup \mathsf{E}' \cup \mathsf{E}'',$$

where E' (resp. E'') consists of those edges of T which have exactly one endpoint (resp. no endpoints) in $\mathsf{V}(\mathsf{T}_0)$. Note that E' and E'' may be void. Then the graph $(\mathsf{V}(\mathsf{T}), \mathsf{E}(\mathsf{T}_0) \cup \mathsf{E}'')$ is disconnected and falls into $r+1$ connected components $\mathsf{T}_0, \mathsf{T}_1, \ldots, \mathsf{T}_r$, $r \geq 0$. As T is a tree, $r = |\mathsf{E}'|$; that is, $\mathsf{E}' = \{\langle z_1, z_1'\rangle, \ldots \langle z_r, z_r'\rangle\}$ with $z_s \in \mathsf{V}(\mathsf{T}_0)$, $s = 1, \ldots, r$. We call z_s the *root* of T_s in T_0. As $\rho_{\mathsf{T}}(z, z')$ is the diameter of T, both z and z' cannot be among the roots. Note also that some of the trees T_s may have common roots.

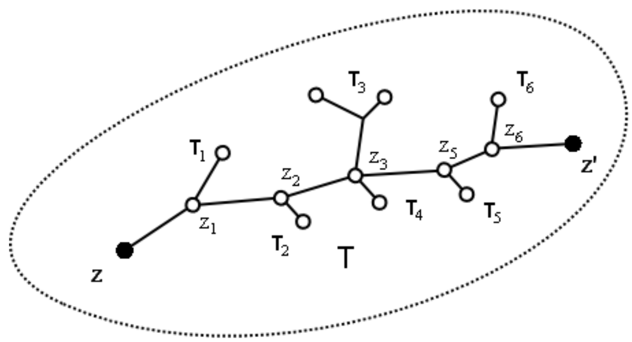

Figure 1: Trees $\mathsf{T}, \mathsf{T}_0, \mathsf{T}_1, \ldots, \mathsf{T}_6$.

Set $N_s = |\mathsf{V}(\mathsf{T}_s)|$, $s = 0, 1, \ldots, r$. As $\rho_{\mathsf{T}}(z, z')$ is the diameter of T, we have $N_s \leq N_0$ for all $s = 1, \ldots, r$. If $N_0 < \lambda$, the diameter of A is less than λ and hence $|\mathsf{B}| \leq 1$, which yields (26). Thus, we assume in the sequel that $N_0 \geq \lambda$. If $r = 0$, then $N_0 = N$ and $\mathsf{B} \subset \mathsf{V}(\mathsf{T}_0)$. In this case, by (25) we have

$$|\mathsf{B}| \leq 1 + (N-1)/\lambda < 2N/\lambda.$$

For $N_0 < N$, we number the trees T_s in such a way that, for some $k \in \{0, 1, \ldots r\}$, $N_s \geq \lambda/2$ for $s = 0, 1, \ldots, k$, and $N_s < \lambda/2$ for $s = k+1, \ldots, r$. Then by the inductive assumption, we have

$$|\mathsf{B} \cap \mathsf{V}(\mathsf{T}_s)| \leq 2N_s/\lambda, \quad s = 0, \ldots, k. \tag{27}$$

Since $N = \sum_{s=0}^{r} N_s$, for $k = r$, we have from (27) that $|\mathsf{B}| \leq 2N/\lambda$ and hence (26) holds. Suppose now that $k < r$. Then for any $y \in \mathsf{V}(\mathsf{T}_s)$ with $s = k+1, \ldots, r$, we have that $\rho_{\mathsf{T}}(y, z_s) \leq N_s < \lambda/2$. Therefore, each such a tree can contain at most one element of B, and the trees with the same root can contain at most one such element in common. If none of them contain elements of B, we have

$$|\mathsf{B}| = \sum_{s=0}^{k} |\mathsf{B} \cap \mathsf{V}(\mathsf{T}_s)| \leq \frac{2}{\lambda} \sum_{s=0}^{k} N_s \leq 2N/\lambda,$$

which again yields (26). Let us forget about those trees which do not contain elements of B and suppose that, for some $n \in \{k+1, \ldots, r\}$, each of $\mathsf{T}_{k+1}, \ldots, \mathsf{T}_n$ contains a single element $\tilde{y}_s \in \mathsf{B}$, $s = k+1, \ldots, n$. Then all their roots $z_{k+1}, \ldots, z_n$ are distinct, and, for all such s,

$$\rho_{\mathsf{T}}(\tilde{y}_s, z_s) \leq N_s. \tag{28}$$

The total number of $\tilde{y}_s$'s is $n-k$. Let us now estimate the maximum possible number of elements of B in the tree T_0. To this end we consider

$$\mathsf{D}_s = \{y \in \mathsf{V}(\mathsf{T}_0) : \rho_{\mathsf{T}}(y, z_s) < \lambda - N_s\}, \quad s = k+1, \ldots, n. \tag{29}$$

Each D_s is in fact an interval of the path $\vartheta(z, z')$, centered at z_s. As for any $y \in \mathsf{D}_s$, we have $\rho_{\mathsf{T}}(y, \tilde{y}_s) < \lambda$; hence, none of D_s can contain elements of B. Some of D_s can overlap, which reduces the part of T_0 free of elements of B. If D_s and $\mathsf{D}_{s'}$ overlap, then

$$\lambda \leq \rho_{\mathsf{T}}(\tilde{y}_s, \tilde{y}_{s'}) \leq N_s + N_{s'} + \rho_{\mathsf{T}}(z_s, z_{s'}) \tag{30}$$

which gives the lower bound for $\rho_{\mathsf{T}}(z_s, z_{s'})$.

Suppose now that the roots $z_{k+1}, \ldots, z_n$, and hence the corresponding intervals (29), are distributed among q groups, $q \geq 1$, consisting of $l_1, \ldots, l_q$ elements, $l_1 + \cdots + l_q = n - k$. We also suppose that consecutive intervals in each group overlap (if the corresponding $l_j > 1$), whereas the intervals belonging to distinct groups do not overlap. The roots are numbered in such a way that, for $j = 0, 1, \ldots, q-1$, the j-th group is

$$Z_j = \{z_{t_j+1}, \ldots, z_{t_{j+1}}\}, \quad t_j = k + l_1 + \cdots + l_j, \quad l_0 = 0.$$

For such a group, let y_j^* (resp. z_{j+1}^*) be the closest to z_{t_j+1} (resp. to $z_{t_{j+1}}$) element of $\mathsf{B} \cap \mathsf{V}(\mathsf{T}_0)$. Then $\rho_{\mathsf{T}}(y_j^*, z_{t_j+1}) + \rho_{\mathsf{T}}(\tilde{y}_{t_j+1}, z_{t_j+1}) \geq \lambda$ and $\rho_{\mathsf{T}}(z_{j+1}^*, z_{t_{j+1}}) + \rho_{\mathsf{T}}(\tilde{y}_{t_{j+1}+1}, z_{t_{j+1}}) \geq \lambda$, which yields, see (28),

$$\rho_{\mathsf{T}}(y_j^*, z_{t_j+1}) \geq \lambda - N_{t_j+1}, \quad \rho_{\mathsf{T}}(z_{j+1}^*, z_{t_{j+1}}) \geq \lambda - N_{t_{j+1}}.$$

In what follows, the elements of $\mathsf{B} \cap \mathsf{V}(\mathsf{T}_0)$ are contained in the paths $\vartheta(z, y_0^*)$, $\vartheta(y_j^*, z_j^*)$, $j = 1, \ldots, q-1$, and $\vartheta(z_q^*, z')$. The number of elements of B in each such a path can be estimated by (25); thus, we have to estimate the total length of such paths. The latter quantity is equal to the length of $\vartheta(z, z')$ minus the total length of the intervals (29); that is,

$$
\begin{aligned}
L &\stackrel{\text{def}}{=} |\vartheta(z, y_0^*)| + |\vartheta(z_q^*, z')| + \sum_{j=1}^{q-1} |\vartheta(y_j^*, z_j^*)| \qquad (31)\\
&\leq N_0 - 1 - \sum_{j=0}^{q-1} |\vartheta(z_{t_j+1}, z_{t_{j+1}})|.
\end{aligned}
$$

The latter summand can be estimated by means of (30), which, for $j = 0, 1, \ldots, q-1$, yields

$$
|\vartheta(z_{t_j+1}, z_{t_{j+1}})| \geq \lambda(l_{j+1} - 1) - \sum_{s=t_j+1}^{t_{j+1}-1} (N_s + N_{s+1}).
$$

Applying this estimate in (31) and taking into account that the total number of elements of B in $\mathsf{T}_1, \ldots, \mathsf{T}_k$ was estimated in (27), we arrive at

$$
\begin{aligned}
|\mathsf{B}| &\leq q + 1 + L/\lambda + (n-k) + \frac{2}{\lambda}\sum_{s=1}^{k} N_s \\
&\leq q + 1 + (n-k) + \frac{2}{\lambda}\sum_{s=1}^{k} N_s + (N_0 - 1)/\lambda - (n-k) - q \\
&- \frac{1}{\lambda}\sum_{j=0}^{q-1}(N_{t_j+1} + N_{t_{j+1}}) + \frac{2}{\lambda}\sum_{s=k+1}^{n} N_s \\
&= \frac{2}{\lambda}\sum_{s=0}^{n} N_s + 1 - \frac{1}{\lambda}\left(N_0 + \sum_{j=0}^{q-1}(N_{t_j+1} + N_{t_{j+1}})\right) \\
&\leq \frac{2}{\lambda}\sum_{s=0}^{n} N_s \leq 2N/\lambda,
\end{aligned}
$$

where we have taken into account that $N_0 \geq \lambda$ and $N = N_0 + \cdots + N_n + \cdots + N_r$. □

It is worthwhile to note that the estimate in (26) is optimal, that is, for each $\varepsilon > 0$, one can pick A of size N and $\lambda > 0$ such that $C(\mathsf{A}, \lambda) > 2N/\lambda - \varepsilon$.

An instance can be G_ϑ, ϑ being a simple path, cf. (25). In this case,

$$C(\mathsf{G}_\vartheta, |\vartheta|) = 2 > \frac{2(|\vartheta|+1)}{|\vartheta|} - \varepsilon,$$

for sufficiently big $|\vartheta|$.

4.4. Balls in repulsive graphs

We recall that $\mathsf{B}_N(x) = \{y \in \mathsf{V} : \rho(x,y) \leq N\}$, $N \in \mathbb{N}$, denotes the ball in $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ of radius N centered at x, cf. (3) and Corollary 4. Further properties of such sets are described in the following statement.

Lemma 17. *Let G be in $\mathbb{G}_-(\phi)$. Then for each $x \in \mathsf{V}$, there exists a strictly increasing sequence $\{N_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, such that, for all $k \in \mathbb{N}$,*

$$\max_{y \in \mathsf{B}_{N_k}(x)} n(y) \leq \phi^{-1}(2N_k + 1). \tag{32}$$

If $\mathsf{G} \in \mathbb{G}_+(\phi)$, then, for every $x \in \mathsf{V}$, there exists $N_x \in \mathbb{N}$ such that the estimate

$$\max_{y \in \mathsf{B}_N(x)} n(y) \leq \phi^{-1}(2N) \tag{33}$$

holds for all $N \geq N_x$.

Proof: First we consider the case of $\mathsf{G} \in \mathbb{G}_-(\phi)$. Let $x_1 \in \mathsf{V}_*^c$ be the closest vertex to x such that $n(x_1) > n(x)$. If there are several such vertices, we take the one with the biggest degree. In the same way, we pick x_2, being the closest vertex to x such that $n(x_2) > n(x_1)$. Then we set $N_1 = \rho(x, x_2) - 1$, which yields $n(x_1) = \max_{y \in \mathsf{B}_{N_1}(x)} n(y)$. By (4) $\rho(x_1, x_2) \geq \phi(n(x_1))$; hence, $2N_1 + 1 \geq \rho(x, x_2) + \rho(x, x_1) \geq \phi(n(x_1))$. Thus, (32) holds for $N = N_1$. Next we take x_3 such that $n(x_3) > n(x_2)$ and set $N_2 = \rho(x, x_3) - 1$. In this way, we construct the whole sequence $\{N_k\}_{k\in\mathbb{N}}$ for which (32) holds.

Let now G be in $\mathbb{G}_+(\phi)$. Given x, let $\tilde{x} \in \mathsf{V}_*^c$ be the closest vertex to x, see (5). If there are several such vertices, we take the one with the biggest degree. Consider the following cases: (i) $\rho(x, \tilde{x}) > \phi(n(\tilde{x}))/2$; (ii) $\rho(x, \tilde{x}) \leq \phi(n(\tilde{x}))/2$. The latter one includes the case $\tilde{x} = x$, i.e., x itself is in V_*^c. In case (i), we set N_x to be the smallest integer number such that $N_x > \phi(n_*)/2$. Then for $N \geq N_x$, we have the following possibilities: (a) $N < \rho(x, \tilde{x})$; (b) $N \geq \rho(x, \tilde{x})$. If (a) holds, then the ball $\mathsf{B}_N(x)$ contains only elements of V_* and hence (33) holds true. In case (b), we have : (c) $\max_{y \in \mathsf{B}_N(x)} n(y) = n(\tilde{x})$;

(d) there exists $z \neq \tilde{x}$ such that $\max_{y \in \mathsf{B}_N(x)} n(y) = n(z)$. If (c) holds, we again obtain (33) since $N \geq \rho(x, \tilde{x}) > \phi(n(\tilde{x}))/2$, by (i). If (d) holds, by (4) we have $\rho(z, \tilde{x}) \geq \phi(n(z))$. On the other hand, by the triangle inequality $\rho(z, \tilde{x}) \leq \rho(x, \tilde{x}) + \rho(z, x) \leq 2N$, which again yields (33). If (ii) holds, let x_1 be the closest vertex to x such that $n(x_1) > n(\tilde{x})$, again we take the one with the maximum degree among such vertices. Then we set $N_x = \rho(x, x_1)$. For $N \geq N_x$, let z be such that $n(z) = \max_{y \in \mathsf{B}_N(x)} n(y)$. Then by (4) $\rho(\tilde{x}, z) \geq \phi(n(z))$. By the triangle inequality this yields

$$N \geq \rho(x, z) \geq \rho(\tilde{x}, z) - \rho(x, \tilde{x}) \geq \phi(n(z)) - \phi(n(\tilde{x}))/2 \geq \phi(n(z))/2,$$

which yields (33) and hence completes the proof. □

5. The proof of Theorems 2 and 3 and Propositions 7 and 9

The proof of the statements in question relies upon showing that the graphs $\mathsf{G} \in \mathbb{G}_-(\phi)$ (resp. $\mathsf{G} \in \mathbb{G}_+(\phi)$) are g-tempered (resp. strongly g-tempered) if g and ϕ satisfy a certain condition. Then we apply Lemmas 13 and 14 and obtain the result we want. To realize this we introduce one more notion. Set

$$n_{\mathsf{A}} = \max_{x \in \mathsf{V}(\mathsf{A})} n(x).$$

Definition 18. *Given $\mathsf{G} \in \mathbb{G}_\pm(\phi)$, an $\mathsf{A} \subset \mathsf{G}$ is said to be a good animal if*

$$|\mathsf{V}(\mathsf{A})| \geq \phi(n_{\mathsf{A}})/2. \tag{34}$$

By $\mathcal{A}_{\rm good}$ we denote the set of all good animals, cf. (32) and (33).

Lemma 19. *Let the functions g and ϕ be such that the following holds*

$$\sum_{k=1}^{\infty} \frac{g(t_{k+1})}{\phi(t_k)} < \infty, \tag{35}$$

for some strictly increasing sequence $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$. Then any $\mathsf{G} \in \mathbb{G}_-(\phi)$ (resp. any $\mathsf{G} \in \mathbb{G}_+(\phi)$) is g-tempered (resp. is strongly g-tempered).

Proof: First we consider the case $\mathsf{G} \in \mathbb{G}_-(\phi)$. Let the sequence in (35) be such that $t_1 = n_*$. The proof will be done by showing that: (a) the upper bound as in (21) holds for any good animal; (b) for each $x \in \mathsf{V}$, one can pick $\{N_k\}_{k \in \mathbb{N}}$ such that each $\mathsf{A} \in \mathcal{A}_{N_k}(x)$, $k \in \mathbb{N}$, is a good animal.

For $\mathsf{A} \in \mathcal{A}_{\rm good}$, we set

$$\begin{aligned} \mathsf{M}_k(\mathsf{A}) &= \{x \in \mathsf{V}(\mathsf{A}) : n(x) \in (t_k, t_{k+1}]\}, \quad k = 1, \ldots, l, \\ m_k(\mathsf{A}) &= |\mathsf{M}_k(\mathsf{A})|, \end{aligned} \tag{36}$$

where $l \in \mathbb{N}$ is the smallest number for which $n_{\mathsf{A}} \leq t_{l+1}$, see (34). By (4) we then get $\rho(x, y) \geq \phi(t_k)$ for each $x, y \in \mathsf{M}_k(\mathsf{A})$. Hence, by Lemma 16 we have

$$m_k(\mathsf{A}) \leq C(\mathsf{A}, \phi(t_k)) \leq 2|\mathsf{V}(\mathsf{A})|/\phi(t_k),$$

which leads to the following estimate, cf. (21),

$$G(\mathsf{A}; g) \leq \frac{1}{|\mathsf{V}(\mathsf{A})|} \sum_{k=1}^{l} g(t_{k+1}) m_k(\mathsf{A}) \leq 2 \sum_{k=1}^{\infty} \frac{g(t_{k+1})}{\phi(t_k)} \stackrel{\rm def}{=} \gamma(g, \phi). \tag{37}$$

Let x be an arbitrary vertex. For this x, let $\{N_k\}_{k\in\mathbb{N}}$ be the sequence as in Lemma 17. Then, for any A such that $x \in \mathsf{V}(\mathsf{A})$ and $|\mathsf{V}(\mathsf{A})| = N_1$, we have $\mathsf{V}(\mathsf{A}) \subset \mathsf{B}_{N_1-1}(x)$. Then by (32)

$$2N_1 > 1 + 2(N_1 - 1) \geq \phi\left(\max_{y\in\mathsf{V}(\mathsf{A})} n(y)\right),$$

which yields $\mathsf{A} \in \mathcal{A}_{\rm good}$. Hence, (37) holds for any $\mathsf{A} \in \mathcal{A}_{N_1}(x)$. Then we repeat the same procedure with N_2, N_3, and so on. For $\mathsf{G} \in \mathbb{G}_+(\phi)$, the proof follows along the same line of arguments, with the only difference that by (33) we show that $\mathcal{A}_N(x) \subset \mathcal{A}_{\rm good}$ whenever $N \geq N_x$. $\square$

The proof of Theorem 2 readily follows from Lemmas 13 and 19 with $g(t) = t \log t$. In the same way, by taking $g(t) = \log t$ (resp. $g(t) = t^{\theta+1}$, cf. (11)) we prove Theorem 3 (resp. Proposition 7), see Lemma 14.

To prove Proposition 9 we proceed as follows. Set

$$S(\mathsf{A}) = \sum_{x\in\mathsf{V}(\mathsf{A})} Y_x,$$

cf. (14). Then, for $Y > 0$ and $t > 0$, we have, cf. (15),

$$\begin{aligned}\mathbb{P}\Big(S(\mathsf{A}) \geq Y|\mathsf{V}(\mathsf{A})|\Big) &\leq \exp(-tY|\mathsf{V}(\mathsf{A})|)\mathbb{E}\exp\left(t\sum_{x\in\mathsf{V}(\mathsf{A})} Y_x\right) \\ &= \exp\left(-t\sum_{x\in\mathsf{V}(\mathsf{A})}(Y - w_x(t)/t)\right) \\ &\leq \exp\left(-tY|\mathsf{V}(\mathsf{A})| + tC\sum_{x\in\mathsf{V}(\mathsf{A})} n(x)\log n(x)\right),\end{aligned}$$

which holds for small enough $t > 0$, see (16) and (17). For ϕ satisfying (8), the graph in question is g-tempered with $g(t) = t\log t$, see Lemma 19. Given $x \in \mathsf{V}$, let $\{N_k\}_{k\in\mathbb{N}}$ be the sequence as in Definition 12. Then, for $\mathsf{A} \in \mathcal{A}_{N_k}(x)$, by (21) and the latter estimate we obtain

$$\mathbb{P}\Big(S_{N_k}(x) \geq YN_k\Big) \leq \exp\left(-tN_k(Y - \gamma C)\right).$$

Now we take $Y > \gamma C$ and obtain (18) by applying the Borel-Cantelli lemma.

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