

τ -rigid modules for algebras with radical square zero ^{*†}Xiaojin Zhang[‡]*School of Mathematics and Statistics, Nanjing University of Information Science and Technology,
Nanjing 210044, P. R. China***Abstract**

In this paper, we give some methods to find τ -rigid modules for a basic indecomposable Artin algebra Λ with radical square zero. More precisely, we show that for an algebra Λ with radical square zero and an indecomposable Λ -module M such that τM is τ -rigid, then M is τ -rigid if and only if the first two projective terms of a minimal projective resolution of M have no non-zero direct summands in common. Moreover, if Λ is a Gorenstein algebra of finite type, then M is τ -rigid if and only if the first and the second projective terms in a minimal projective resolution of M have no non-zero direct summands in common.

1 Introduction

In October of 2012, Adachi, Iyama and Reiten introduced the notion of τ -tilting modules, which is a generalization of the classical tilting modules [APR, BB, HR]. For an Artin algebra Λ , a Λ -module M is called τ -rigid if $\text{Hom}(M, \tau M) = 0$, moreover, M is called τ -tilting if it is τ -rigid and $|M| = |\Lambda|$, where τ denotes the Auslander-Reiten translation and $|M|$ denotes the number of non-isomorphic direct summands. This important class of modules which admit very similar properties to the classical tilting modules are very close to silting objects in [AiI] and the cluster tilting objects in 2-Calabi-Yau triangulated categories [IY]. It is not difficult to show that τ -tilting modules coincide with tilting modules and τ -rigid modules coincide with the rigid modules in case that Λ is hereditary.

On the other hand, algebras with radical square zero have been studied by Auslander, Reiten and Smalø in [AuRS], which play an important role in classifying Nakayama algebras. Recently Chen, Ringel and Xiong successfully classified this kind of algebras in terms of Gorenstein projective modules in [C] and [RX], respectively.

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Notice that in [AIR] Adachi, Iyama and Reiten give examples to show the main results. All algebras there are of radical square zero. So a natural question is: Whether can we find all τ -rigid modules over algebras with radical square zero?

In this paper, we give a partial answer to this question. The paper is organized as follows:

In Section 2, we will recall some preliminaries on algebras with radical square zero. In section 3, we will give some methods to determine the τ -rigid modules for algebras with radical square zero and prove the following:

Theorem 1 *Let Λ be a Nakayama algebra with $r^2 = 0$ which is not self-injective local and let n be the number of non-isomorphic simple modules. Then*

- (1) *Every indecomposable module M is τ -rigid.*
- (2) *Every τ -tilting module T is of the form $S_1 \oplus S_2 \oplus \cdots \oplus S_t \oplus (\Lambda/P_0(\tau(S_1 \oplus \cdots \oplus S_t)))$, where S_j is simple for $1 \leq j \leq t$, t is an integer such that $0 \leq t \leq \text{int}(n/2)$ and $\text{int}(m)$ denotes the largest integer less than or equal to m for any real number m .*

Theorem 2 *Let Λ be a Gorenstein algebra with $r^2 = 0$.*

- (1) *If Λ is self-injective local, then every indecomposable τ -rigid module is projective.*
- (2) *If Λ is self-injective but not local, then every indecomposable module M is τ -rigid.*
- (3) *If Λ is representation finite of finite global dimension, then an indecomposable module M is τ -rigid if and only if there is no non-zero direct summand of $P_0(M)$ and $P_1(M)$ in common, where $P_0(M)$ and $P_1(M)$ are the first and second projective terms of a minimal projective resolution of M , respectively.*

Theorem 3 *Let Λ be an algebra with $r^2 = 0$ and let M be an indecomposable Λ -module such that τM is τ -rigid. Then M is τ -rigid if and only if there is no non-zero direct summand of $P_0(M)$ and $P_1(M)$ in common, where $P_0(M)$ and $P_1(M)$ are the first and second projective terms of a minimal projective resolution of M , respectively.*

In Section 4, we will give examples to show our results.

Throughout this paper, all algebras are basic indecomposable non-semi-simple Artin algebras over a commutative Artin ring R , $\mathbb{D} = \text{Hom}_R(-, I^0(R/J))$ is the ordinary dual, where J is the Jacobson radical of R and $I^0(R/J)$ is the injective envelope of R/J . All modules are finitely generated left Λ -modules if not claimed.

2 Properties for algebras with radical square zero

In this section we will recall some properties for algebras with radical square zero. Denote by r the Jacobson radical of an algebra Λ . Λ is called radical square zero if $r^2 = 0$. Let Γ be

another algebra. We say that Λ is stable equivalent to Γ if there is an equivalence functor $F : \underline{\text{mod}}\Lambda \rightarrow \underline{\text{mod}}\Gamma$, where $\underline{\text{mod}}\Lambda$ and $\underline{\text{mod}}\Gamma$ denote the associate module categories modulo the projective modules, respectively.

Now we can recall the following result for algebras with radical square zero from [AuRS, X, Theorem 2.4, Lemma 2.1].

Lemma 2.1 *Let Λ be an algebra with $r^2 = 0$. Denote by Γ the triangular matrix algebra*

$$\begin{pmatrix} \Lambda/r & 0 \\ r & \Lambda/r \end{pmatrix} \quad (1)$$

and denote by $F : \underline{\text{mod}}\Lambda \rightarrow \underline{\text{mod}}\Gamma$ the functor via $F(M) = (M/rM, rM, f)$ and $F(g) = (g_1, g_2)$ for any $M, N, L \in \text{mod } \Lambda$ and $g : N \rightarrow L$, where $f : r \otimes_{\Lambda/r} M/rM \rightarrow rM$ is an epimorphism, $g_1 : N/rN \rightarrow L/rL$ and $g_2 : rN \rightarrow rL$ are induced by g . Then

- (1) F is an equivalence and hence Λ is stable equivalent to Γ .
- (2) $F(M)$ is indecomposable if and only if M is indecomposable.
- (3) $F(M)$ is projective if and only if M is projective.

Recall that a morphism $h : E \rightarrow M$ is called right minimal if for any $l : E \rightarrow E$ $h = hl$ implies that l is an isomorphism. h is right almost split if h is not a split epimorphism and for any $m : N \rightarrow E$ which is not a split epimorphism there exists a $t : N \rightarrow E$ such that $m = ht$. Dually, one can define left minimal morphisms and left almost split sequences. An exact sequence $0 \rightarrow A \xrightarrow{g} B \xrightarrow{h} C \rightarrow 0$ is called almost split if g is left almost split and h is right almost split. Now we are ready to recall the following properties of almost split sequences for algebras with $r^2 = 0$ from [AuRS, V, Proposition 3.5, X, Proposition 2.5].

Lemma 2.2 *Let Λ be an algebra with $r^2 = 0$ and let $0 \rightarrow A \xrightarrow{g} B \xrightarrow{h} C \rightarrow 0$ be an almost split sequence. Then*

- (1) B is projective if and only if A is non-injective simple. If A is simple non-injective, then $h : B \rightarrow C$ is a projective cover.
- (2) B is injective if and only if C is non-projective simple. If C is simple non-projective, then $g : A \rightarrow B$ is an injective envelope.

Lemma 2.3 *Let Λ , F and Γ be as in Lemma 2.1 and let $0 \rightarrow A \xrightarrow{g} B \xrightarrow{h} C \rightarrow 0$ (*) be an exact sequence such that A and C are indecomposable and A is not simple. Then*

- (1) The sequence (*) is almost split in $\text{mod } \Lambda$ if and only if $0 \rightarrow F(A) \xrightarrow{F(g)} F(B) \xrightarrow{F(h)} F(C) \rightarrow 0$ is almost split in $\text{mod } \Gamma$.
- (2) If (*) is almost split, then $F(A) = F(\tau_\Lambda C) = \tau_\Gamma F(C)$.

Proof. (2) follows from (1). □

The following result which gives a connection between morphisms in $\text{mod } \Lambda$ and $\text{mod } \Gamma$ is very important to the proof of the main results.

Lemma 2.4 *Let Λ , F and Γ be as in Lemma 2.1. Then*

(1) *For any $M, N \in \text{mod } \Lambda$ we have the following exact sequence of Abelian groups:*

$$0 \rightarrow \text{Hom}_\Lambda(M, rN) \rightarrow \text{Hom}_\Lambda(M, N) \rightarrow \text{Hom}_\Gamma(F(M), F(N)) \rightarrow 0$$

(2) $\underline{\text{Hom}}_\Lambda(M, N) \simeq \text{Hom}_\Gamma(F(M), F(N))$ *if both M and N have no projective direct summands.*

Proof. (1) follows from [AuRS, X, Lemma 2.1] and (2) follows from [AuRS, X, Lemma 2.3] and (1). \square

In order to show the main result on Nakayama algebra with $r^2 = 0$, we need the following:

Lemma 2.5 *Let Λ be a Nakayama algebra with $r^2 = 0$. Then every indecomposable module $M \in \text{mod } \Lambda$ is either simple or projective.*

Proof. By using $r^2 = 0$ and [AsSS, V, Theorem 4.1]. \square

3 Finding τ -rigid modules for algebras with radical square zero

In this section, we will determine the τ -rigid modules in term of almost split sequences and simple modules. Firstly, we recall the notions of τ -tilting modules and τ -rigid modules in [AIR].

Definition 3.1 For an algebra Λ , a Λ -module M is called τ -rigid if $\text{Hom}(M, \tau M) = 0$, where τ denotes the Auslander-Reiten translation. A module N is τ -tilting if it is τ -rigid and $|N| = |\Lambda|$, where $|N|$ denotes the number of non-isomorphic direct summands of N . Any τ -rigid module is a direct summand of a τ -tilting module. We also note that if Λ is hereditary then τ -tilting modules and τ -rigid modules coincide with tilting modules and rigid modules, respectively.

For any indecomposable M in $\text{mod } \Lambda$, if M is projective, then it is τ -rigid. So we can assume that M is not projective. Denote by $\cdots \rightarrow P_t(M) \rightarrow \cdots \rightarrow P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0$ be a minimal projective resolution of M , where t is a non-negative integer. And denote by $\Omega^i M$ the i -th syzygy of M for any $i \geq 0$. Considering the almost split sequence $0 \rightarrow \tau M \rightarrow E \rightarrow M \rightarrow 0$, we have the following:

Proposition 3.2 *Let Λ be an algebra with $r^2 = 0$. Then Λ is self-injective local if and only if there is an almost split sequence $0 \rightarrow S \rightarrow P \rightarrow S \rightarrow 0$, where S is a simple Λ -module.*

Proof. \Rightarrow Since Λ is a basic indecomposable local algebra, one can get the following exact sequence: $0 \rightarrow r \rightarrow \Lambda \rightarrow S \rightarrow 0$, where S is simple and r is the radical of Λ . Notice that $r^2 = 0$ then r is semi-simple. Because Λ is self-injective, we get r is simple by [HuZ, Lemma 2.6], and hence $r \simeq S$. Then the sequence is almost split by [AsSS, IV, Proposition 3.11].

\Leftarrow By Lemma 2.2, one gets that P is projective and injective. It is enough to prove that Λ has a unique simple module S up to isomorphism. On the contrary, Suppose that there is another simple $S' \not\simeq S$. We claim that $\text{Hom}_\Lambda(P_0(S'), P_0(S)) = \text{Hom}_\Lambda(P_0(S), P_0(S')) = 0$, where $P_0(M)$ is the projective cover of M .

$$(1) \text{Hom}_\Lambda(P_0(S'), P_0(S)) = 0.$$

Suppose that there is an $f \in \text{Hom}_\Lambda(P_0(S'), P_0(S))$ such that $f \neq 0$, then f is not epic since $P_0(S)$ is projective and $S' \not\simeq S$. Denote by $\text{Im}f$ the image of f , then $\text{Im}f \subseteq rP_0(S)$. Notice that $0 \rightarrow S \rightarrow P \rightarrow S \rightarrow 0$ is almost split, then $P_0(S) \simeq P$ and $rP_0(S) \simeq S$ by Lemma 2.2, and hence $\text{Im}f = S$, then $f : P_0(S') \rightarrow S$ is epic, and hence $P_0(S') \simeq P_0(S)$. One gets a contradiction since $S \not\simeq S'$.

$$(2) \text{Hom}_\Lambda(P_0(S), P_0(S')) = 0.$$

Suppose that there is a $g \in \text{Hom}_\Lambda(P_0(S), P_0(S'))$ such that $g \neq 0$, then g is not epic and $\text{Im}g \subseteq rP_0(S')$ by a similar argument in (1). Notice that $r^2 = 0$, then $rP_0(S')$ is semi-simple. So we get $\text{Im}g = S$, and hence $j : S \hookrightarrow P_0(S')$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} & & P_0(S') & & & & \\ & & \uparrow j & \swarrow h & & & \\ 0 & \longrightarrow & S & \xrightarrow{i} & P & \longrightarrow & S \longrightarrow 0 \end{array}$$

By Lemma 2.2 we get that $P_0(S) \simeq P$, P is injective and $i : S \rightarrow P$ is an injective envelope. Then i is an essential monomorphism implies that $h : P \rightarrow P_0(S')$ is monic, and hence $P \simeq P_0(S')$, a contradiction.

Since Λ is indecomposable, one gets the assertion by the claim and [AsSS, II, Lemma 1.6]. \square

Now we can give a class of τ -rigid modules over algebras with $r^2 = 0$ which are not local-self-injective.

Proposition 3.3 *Let Λ be an algebra with $r^2 = 0$ which is not self-injective local and let M be an indecomposable Λ -module. If M satisfies (1) τM is simple projective, or (2) both τM and M are simple, then M is τ -rigid.*

Proof. Suppose that there is a non-zero $f \in \text{Hom}_\Lambda(M, \tau M)$. For the both cases, we have f is epic, and hence $M \simeq \tau M$. For the first case, we get that M is projective, a contradiction. For the second one, by Lemma 2.2 there is an almost split sequence $0 \rightarrow S \rightarrow P \rightarrow S \rightarrow 0$ with S a simple module. By Proposition 3.2 Λ is self-injective local, a contradiction. \square

Remark For any algebra Σ and an indecomposable Σ -module N with τN simple projective, one can show that N is τ -rigid by formulating the proof of Proposition 3.3 (1).

Now we are in a position to state the τ -tilting and τ -rigid modules for Nakayama algebras with $r^2 = 0$.

Theorem 3.4 *Let Λ be a Nakayama algebra with $r^2 = 0$ which is not self-injective local and let n be the number of non-isomorphic simple modules. Then*

(1) *Every indecomposable module M is τ -rigid.*

(2) *Every τ -tilting module T is of the form $S_1 \oplus S_2 \oplus \cdots \oplus S_t \oplus (\Lambda/P_0(\tau(S_1 \oplus \cdots \oplus S_t)))$, where S_j is simple for $1 \leq j \leq t$, t is an integer such that $0 \leq t \leq \text{int}(n/2)$ and $\text{int}(m)$ denotes the largest integer less than or equal to m for any real number m .*

Proof. (1) By Lemma 2.5, M is simple or projective for any indecomposable $M \in \text{mod } \Lambda$. If M is projective, there is nothing to prove. If M is simple non-projective, then τM is simple, by Proposition 3.3, M is τ -rigid.

(2) For any τ -tilting module T , we claim that there is at least one indecomposable P as a direct summand of T .

On the contrary, suppose that $T = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ with all S_j simple non-projective for $1 \leq j \leq n$. Without loss of generality, we can assume that $\tau S_1 = S_2$ by Lemma 2.5 or (1). Then $0 \neq \text{Hom}_\Lambda(S_2, S_2) \subseteq \text{Hom}_\Lambda(T, \tau T) = 0$, a contradiction.

Next we will show if S is a direct summand of T , then $P_0(S)$ is a direct summand of T .

If S is projective, then the assertion holds true. We can assume that S is not projective. Then we get the following almost split sequence: $0 \rightarrow S' \rightarrow P \rightarrow S \rightarrow 0$ with S' simple by Lemma 2.2 and Lemma 2.5. Again by Lemma 2.2, P is indecomposable projective and $P \simeq P_0(S)$. Because T is τ -tilting and S is direct summand of T , it is not difficult to show that S' is not a direct summand of T . Similarly, if S is not injective, then a simple module S^* with $\tau S^* \simeq S$ is not a direct summand of T . Since τT is semi-simple, then $\text{Hom}_\Lambda(P_0(S), \tau T) = 0$, that is, $P_0(S)$ is in $\text{Fac}(T)$ by [AIR, Theorem 2.10], where $\text{Fac}(T)$ denotes the category consisting of factor modules of T . So $P_0(S)$ is a direct summand of T . Notice that $|T| = |\Lambda| = n$, so the number of simple direct summands of T has to be at most $\text{int}(n/2)$. \square

The conditions in Proposition 3.3 are not easy to be satisfied. In the following we will generalize it into a general framework. Denote by $\text{gl.dim } \Lambda$ the global dimension of Λ and denote by $\text{pd}_\Lambda M$ the projective dimension of M . We have:

Lemma 3.5 *Let Λ be an algebra with $r^2 = 0$. If S is a simple module with $\text{pd}_\Lambda S = m < \infty$. Then S is τ -rigid. Moreover, if $\text{gl.dim } \Lambda = m < \infty$, then every simple module S is τ -rigid.*

Proof. We only have to the first one since the last follows from the first. By [AIR, Proposition 1.2] a simple module S is τ -rigid if and only if it is rigid, that is, $\text{Ext}_\Lambda^1(S, S) = 0$. If S is projective, there is nothing to prove. So we can assume that $m \geq 1$. Take the following part of a minimal projective resolution of S : $0 \rightarrow \Omega^1 S \rightarrow P_0(S) \rightarrow S \rightarrow 0$, where $\Omega^1 S$ denotes the first syzygy of S . One gets that $\text{pd}_\Lambda \Omega^1 S = m - 1$ since $\text{pd}_\Lambda S = m < \infty$. Since $r^2 = 0$, we have $\Omega^1 S \simeq rP_0(S)$ is semi-simple and any direct summand of it is of projective dimension at most $m - 1$. So it is not difficult to show $\text{Ext}_\Lambda^1(S, S) \simeq \text{Hom}_\Lambda(\Omega^1 S, S) = 0$. \square

Denote by $\overline{\text{mod}}\Lambda$ the associate modules category modulo injective modules and denote by $\overline{\text{Hom}}_\Lambda(L, N)$ and $\underline{\text{Hom}}_\Lambda(L, N)$ classes of morphisms from L to N in $\overline{\text{mod}}\Lambda$ and $\underline{\text{mod}}\Lambda$, respectively. Now we are in a position to state another main result on judging the τ -rigid properties by simple modules.

Theorem 3.6 *Let Λ be an algebra with $r^2 = 0$ and let M be indecomposable with τM simple. We have (1) M is τ -rigid if and only if τM is τ -rigid. (2) If $\text{pd}_\Lambda M < \infty$, then M is τ -rigid. Moreover, if $\text{gl.dim } \Lambda < \infty$, then M is τ -rigid.*

Proof. Since (2) is a straight result of (1), Lemma 2.2 and Lemma 3.5, we only show (1).

\Leftarrow By the remark of Proposition 3.3, it is enough to show the case of τM is not projective.

On the contrary, suppose that M is not τ -rigid, that is, $\text{Hom}(M, \tau M) \neq 0$. We get that f is epic for any $0 \neq f \in \text{Hom}(M, \tau M)$ since τM is simple. By Lemma 2.2, one gets the following almost split sequence: $0 \rightarrow \tau M \xrightarrow{i} P_0(M) \rightarrow M \rightarrow 0$. So $P_0(\tau M)$ is a direct summand of $P_0(M)$. Notice that i is left minimal, then by [AuRS, I, Theorem 2.4] τM can be embedded into $P_0(\tau M)$, and hence a direct summand of $rP_0(\tau M)$ since $r^2 = 0$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & rP_0(\tau M) & \longrightarrow & P_0(\tau M) & \xrightarrow{\pi} & \tau M & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \parallel & & \\ 0 & \longrightarrow & \tau M & \longrightarrow & E & \longrightarrow & \tau M & \longrightarrow & 0 \end{array}$$

with α an epimorphism. By the snake lemma, β is also an epimorphism. Since $\text{pd}_\Lambda M < \infty$, one can show that $\text{pd}_\Lambda \tau M$ is of finite projective dimension by Lemma 2.2. Then by the

assumption τM is τ -rigid and hence the bottom row in the commutative diagram is split. So one gets an epimorphism $P_0(\tau M) \rightarrow \tau M \oplus \tau M$, a contradiction.

\Rightarrow Since M is τ -rigid, one gets $\text{Hom}_\Lambda(M, \tau M) = 0$ which implies that $\underline{\text{Hom}}_\Lambda(M, \tau M) = 0$. Notice that $\tau : \underline{\text{mod}}\Lambda \rightarrow \overline{\text{mod}}\Lambda$ is an equivalence, one can get $\overline{\text{Hom}}_\Lambda(\tau M, \tau^2 M) = \underline{\text{Hom}}_\Lambda(M, \tau M) = 0$. By AR-formula one gets $\text{Ext}_\Lambda^1(\tau M, \tau M) \simeq \mathbb{D}\overline{\text{Hom}}_\Lambda(\tau M, \tau^2 M) = 0$. Then by [AIR, Proposition 1.2] τM is τ -rigid since τM is simple. \square

To find more τ -rigid modules for algebras with $r^2 = 0$. We will focus on hereditary algebras with $r^2 = 0$. Moreover, we will show that all indecomposable modules over a hereditary algebra of finite type are τ -rigid, and hence rigid. Firstly we have:

Proposition 3.7 *Let Λ be a hereditary algebra and let M be an indecomposable non-projective module. If τM is projective, then M is τ -rigid.*

Proof. Suppose that $\text{Hom}_\Lambda(M, \tau M) \neq 0$. Then there is a non-zero morphism $f : M \rightarrow \tau M$. Since Λ is hereditary and τM is projective, one gets that $\text{Im}f$ is projective and hence M is projective, a contradiction. \square

The following property between τM and M is very important to the proof of the main result on hereditary algebras.

Proposition 3.8 *Let Λ be a hereditary algebra and let M be an indecomposable non-projective module. If τM is τ -rigid, then M is τ -rigid.*

Proof. If τM is projective, the assertion holds from Proposition 3.7. We only show the case τM is not projective. By [AsSS, IV, Corollary 2.15 (b)], one gets $\text{Hom}_\Lambda(M, \tau M) \simeq \text{Hom}_\Lambda(\tau M, \tau^2 M) = 0$. \square

Recall that an indecomposable module M over a hereditary algebra is preprojective if there is a non-negative integer j such that $\tau^j M$ is a non-zero projective module. Now we can state another main result of this section.

Theorem 3.9 *Let Λ be a hereditary algebra. Then (1) Every preprojective module M is τ -rigid. (2) If Λ is of finite type, then every indecomposable module is τ -rigid, and hence rigid.*

Proof. (1) We can assume that $\tau^j M$ is projective for some non-negative integer j . By induction on j and Proposition 3.8, one gets the assertion. Then by [AuRS, VIII, Proposition 1.13] and (1), one can show (2). \square

In the following we try to find more τ -rigid modules for Gorenstein algebras with $r^2 = 0$

by using the stable equivalence given in Section 2. Denote by $\text{id}_\Lambda M$ (*resp.* $\text{id}_{\Lambda^\circ} M$) the injective dimension of M for an M in $\text{mod } \Lambda$ (*resp.* $\text{mod } \Lambda^\circ$). Recall that an algebra Λ is called Gorenstein if $\text{id}_\Lambda \Lambda = \text{id}_{\Lambda^\circ} \Lambda = n$ for some integer $n \geq 0$. We have the following:

Lemma 3.10 *Let Λ be a Gorenstein algebra with $r^2 = 0$. Then Λ is either self-injective or of finite global dimension.*

Proof. By [C], we can get that every algebra with $r^2 = 0$ is either self-injective or CM-free. Recall that an algebra is called CM-free if every finitely generated Gorenstein projective module is projective. We will show $\text{gl.dim } \Lambda = n$ if $\text{id}_\Lambda \Lambda = \text{id}_{\Lambda^\circ} \Lambda = n$ for some $n > 0$. By [AuR, Proposition 3.1] one can show that $\Omega^n M$ is Gorenstein projective and hence projective for any M in $\text{mod } \Lambda$ since Λ is CM-free. The assertion holds true. \square

Notice that a self-injective algebra with $r^2 = 0$ is Nakayama by [AuRS, IV, Proposition 2.16]. Since the Nakayama case is completely classified in Proposition 3.2 and Theorem 3.4, we only have to find the τ -rigid modules for algebras of finite global dimension with $r^2 = 0$.

Denote by $\text{Soc}M$ the socle of M . We have the following easy observation:

Lemma 3.11 *Let Λ be an algebra and let M be an indecomposable and non-projective Λ -module. Then (1) $\text{Soc}\tau M \simeq \Omega^1 M/r\Omega^1 M$. (2) $\text{Soc}\tau M \simeq \Omega^1 M$ if $r^2 = 0$.*

Proof. (1) Taking the following part of a minimal projective resolution of M : $P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0$ and then applying the functor $\text{Hom}_\Lambda(-, \Lambda) = (-)^*$, one gets the following part of minimal projective resolution of $\text{Tr}M$: $P_0(M)^* \rightarrow P_1(M)^* \rightarrow \text{Tr}M \rightarrow 0$, where Tr is the Auslander-Bridger transpose. Then applying the functor \mathbb{D} , one has a minimal injective resolution of τM : $0 \rightarrow \tau M \rightarrow \mathbb{D}P_1(M)^* \rightarrow \mathbb{D}P_0(M)^*$. Then $\text{Soc}\tau M \simeq \text{Soc}\mathbb{D}P_1(M)^* \simeq \Omega^1 M/r\Omega^1 M$.

(2) Taking the following part of a minimal projective resolution of M : $0 \rightarrow \Omega^1 M \rightarrow P_0(M) \rightarrow M \rightarrow 0$, one can get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1 M & \longrightarrow & P_0(M) & \longrightarrow & M & \longrightarrow & 0, \\ & & \downarrow g & & \parallel & & \downarrow f & & \\ 0 & \longrightarrow & rP_0(M) & \longrightarrow & P_0(M) & \longrightarrow & M/rM & \longrightarrow & 0 \end{array}$$

where f is epic. Then by the snake lemma, one gets g is a monomorphism. Since $r^2 = 0$, we have that $rP_0(M)$ is semi-simple and hence $\Omega^1 M$ is semi-simple. We are done. \square

Now we are ready to determine all the τ -rigid modules for a Gorenstein algebra of finite type with $r^2 = 0$. Combined with Proposition 3.2, Lemma 3.11 and Theorem 3.4, one can show:

Theorem 3.12 *Let Λ be a Gorenstein algebra of finite type with $r^2 = 0$.*

- (1) *If Λ is self-injective local, then every indecomposable τ -rigid module is projective.*
- (2) *If Λ is self-injective but not local, then every indecomposable module M is τ -rigid*
- (3) *If Λ is of finite global dimension, then an indecomposable module M is τ -rigid if and only if there is no non-zero direct summand of $P_0(M)$ and $P_1(M)$ in common.*

Proof. (1) is clear and (2) is showed in Theorem 3.4.

(3) \Rightarrow It is a straight result of [AIR, Proposition 2.4].

\Leftarrow Without loss of generality, we can assume that τM is not zero. If τM is simple, then the assertion holds true by Theorem 3.6. Now we can assume that τM is not simple. Let Γ and F be as in Lemma 2.1. By Lemma 2.1 and [AuRS, X, Proposition 1.1], we get that Γ is hereditary of finite type. Then by Theorem 3.9(2), $F(M) \in \text{mod } \Gamma$ is τ -rigid. So $\text{Hom}_\Gamma(F(M), F(\tau M)) \simeq \text{Hom}_\Gamma(F(M), \tau F(M)) = 0$ by Lemma 2.3. Then by Lemma 2.4 M is τ -rigid if and only if $\text{Hom}_\Lambda(M, r\tau M) = 0$.

We show that $\text{Hom}_\Lambda(M, r\tau M) = 0$. Since $r^2 = 0$, we get that $r\tau M$ is semi-simple and hence a direct summand of $\text{Soc } \tau M \simeq \Omega^1 M$ by Lemma 3.11. Notice that there is no common direct summand of $P_0(M)$ and $P_1(M)$, one can show $\text{Hom}_\Lambda(M, \text{Soc } \tau M) = 0$ which implies that $\text{Hom}_\Lambda(M, r\tau M) = 0$. Then M is τ -rigid by Lemma 2.4. \square

In general, for algebras mentioned in Theorem 3.12 (3) we don't know whether there is a common direct summand in $P_0(M)$ and $P_1(M)$ for an indecomposable M (see Example 4.3). However, we get the following:

Proposition 3.13 *Let Λ be an algebra of finite global dimension with $r^2 = 0$ and let M be an indecomposable module. If M/rM is simple, then $P_0(M)$ and $P_1(M)$ have no non-zero direct summands in common.*

Proof. Denote by $S = M/rM$. It is enough to show that $\text{Hom}_\Lambda(\Omega^1 M, S) = 0$. On the contrary, suppose that $\text{Hom}_\Lambda(\Omega^1 M, S) \neq 0$. Then S is a direct summand of $\Omega^1 M$ since $r^2 = 0$ and S is simple. Moreover, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega^1 M & \longrightarrow & P_0(M) & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow f & & \parallel & & \downarrow g \\
 0 & \longrightarrow & \Omega^1 S & \longrightarrow & P_0(S) & \longrightarrow & S \longrightarrow 0
 \end{array}$$

Since g is epic, one can show that $f : \Omega^1 M \rightarrow \Omega^1 S$ is a monomorphism, and hence $\Omega^1 M$ is a direct summand of $\Omega^1 S$ because $r^2 = 0$. So S is a direct summand of $\Omega^1 S$, that is

$\text{pd}_\Lambda S \leq \text{pd}_\Lambda S - 1 < \infty$, a contradiction. \square

Denote by $l(M)$ the length of M . As a result of Proposition 3.13, we can get:

Corollary 3.14 *Let Λ be an algebra of finite global dimension with $r^2 = 0$ and M be an indecomposable module. if $l(M) \leq 2$, then $P_0(M)$ and $P_1(M)$ have no non-zero direct summands in common.*

Proof. By Proposition 3.13, it suffice to prove that M/rM is simple. If $l(M) = 1$ then M is simple, the assertion holds true. If $l(M) = 2$ then $rM = \text{Soc}M$ is simple, and hence M/rM is simple. \square

At the end of this section we will give a method to find more τ -rigid modules for any algebra with $r^2 = 0$. As we know all the projective Λ -modules are τ -rigid. In case that the algebra Λ is not self-injective, there must be some indecomposable M such that τM is projective. It is interesting to know whether M is τ -rigid. A more general question is: Whether is M τ -rigid if τM is τ -rigid? To answer this question, we have

Theorem 3.15 *Let Λ be an algebra with $r^2 = 0$ and let M be an indecomposable Λ -module such that τM is τ -rigid. Then M is τ -rigid if and only if there is no non-zero direct summand of $P_0(M)$ and $P_1(M)$ in common.*

Proof. \Rightarrow By [AIR, Proposition 2.4 (d)].

\Leftarrow If τM is simple, then by Theorem 3.6 (1) M is τ -rigid of and only if τM is τ -rigid. Then the assertion holds by the assumption τM is τ -rigid.

Now we can assume that τM is not simple. Let Γ and F be as in Lemma 2.1. We claim that $\text{Hom}_\Gamma(F(M), F(\tau M)) = 0$.

Since τM is τ -rigid, we get $\text{Hom}_\Lambda(\tau M, \tau^2 M) = 0$ and hence $\overline{\text{Hom}}_\Lambda(\tau M, \tau^2 M) = 0$. Notice that $\tau : \underline{\text{mod}} \Lambda \rightarrow \overline{\text{mod}} \Lambda$ is an equivalence, we get that $\underline{\text{Hom}}_\Lambda(M, \tau M) = 0$. If τM is not projective, then we get $\text{Hom}_\Gamma(F(M), F(\tau M)) = 0$ by Lemma 2.4. If τM is projective, then $\text{Hom}_\Gamma(F(M), F(\tau M)) = 0$ since $F(\tau M)$ is projective and Γ is hereditary. Otherwise, one can get a non-zero $g : F(M) \rightarrow F(\tau M)$. So $\text{Im} g$ is projective, and hence $F(M)$ is projective, that is, M is projective by Lemma 2.1, a contradiction.

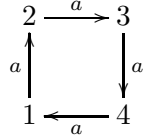
By Lemma 2.4, we only have to show $\text{Hom}_\Lambda(M, r\tau M) = 0$. One can get the assertion by a similar argument in Theorem 3.12. \square

For a non-Nakayama algebra Λ with $r^2 = 0$, by Theorem 3.15 one can find the τ -rigid modules one by one from the projective vertices of the AR -quiver of Λ since here Λ is not self-injective. For the Nakayama case with $r^2 = 0$, we refer to Theorem 3.4 or Theorem 3.12.

4 Examples

In this section we give examples to show our results. Let \mathbf{Q} be a quiver. Denote by $P(i)$, $I(i)$ and $S(i)$ the indecomposable projective module, indecomposable injective module and the simple module according to the vertex $i \in \mathbf{Q}$, respectively. The following example is a Nakayama algebra with $r^2 = 0$.

Example 4.1 Let Λ be given by the quiver:

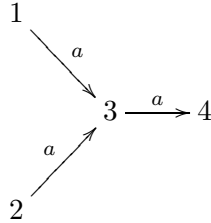


with relations $a^2 = 0$. By Theorem 3.4, every indecomposable module is τ -rigid. The τ -tilting modules are of the following forms:

- (1) 0-simple module. $P(1) \oplus P(2) \oplus P(3) \oplus P(4)$
- (2) 1-simple module. $P(1) \oplus S(1) \oplus P(3) \oplus P(4), P(1) \oplus P(2) \oplus S(2) \oplus P(4)$
 $P(1) \oplus P(2) \oplus P(3) \oplus S(3), S(4) \oplus P(2) \oplus P(3) \oplus P(4)$
- (3) 2-simple modules. $P(1) \oplus S(1) \oplus P(3) \oplus S(3), S(4) \oplus P(2) \oplus S(2) \oplus P(4)$

In the following we give an example to show that there does exist an algebra of finite global dimension with $r^2 = 0$ which is of finite type but not Nakayama.

Example 4.2 Let Λ is given by the quiver:

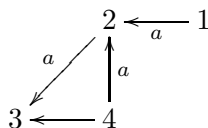


with relations $a^2 = 0$. Then

- (1) Λ is a representation finite algebra of global dimension 2 with $r^2 = 0$.
- (2) $\tau S(3) \simeq S(4)$ and $\tau I(3) \simeq S(3)$. So $S(3)$ and $I(3)$ are τ -rigid by Theorem 3.6.
- (3) By Theorem 3.12 or Theorem 3.15, Corollary 3.14 and (2), every indecomposable Λ -module is τ -rigid.

To show Theorem 3.12 and Theorem 3.15, in the following we will construct an algebra Λ and an indecomposable Λ -module M such that Λ is of finite type and finite global dimension with $r^2 = 0$ and there is no non-zero direct summand of $P_0(M)$ and $P_1(M)$ in common.

Example 4.3 Let Λ is given by the quiver:



with relations $a^2 = 0$. Then

(1) Λ is a representation finite algebra of global dimension 2 with $r^2 = 0$.

(2) The injective module $I(3)$ has a minimal projective resolution:

$$P(2) \oplus P(3) \rightarrow P(2) \oplus P(4) \rightarrow I(3) \rightarrow 0.$$

By Theorem 3.12 (3), $I(3)$ is not τ -rigid.
 (3) $\tau^2 I(3) \simeq S(2)$, then by Theorem 3.6 (2) $\tau I(3)$ is τ -rigid. So τM is τ -rigid can not imply that M is τ -rigid in general.

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