

AUTOMORPHISMS OF SURFACES OF GENERAL TYPE WITH $q \geq 2$ ACTING TRIVIALY IN COHOMOLOGY

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ABSTRACT. In this note, we prove that, surfaces of general type with irregularity $q \geq 3$ are rationally cohomologically rigidified, and so are minimal surfaces S with $q(S) = 2$ unless $K_S^2 = 8\chi(\mathcal{O}_S)$. Here a compact complex manifold X is said to be rationally cohomologically rigidified if its automorphism group $\text{Aut}(X)$ acts faithfully on the cohomology ring $H^*(X, \mathbb{Q})$.

As examples we give a complete classification of surfaces isogenous to a product with $q(S) = 2$ that are not rationally cohomologically rigidified.

1. INTRODUCTION

A compact complex manifold X is said to be *cohomologically rigidified* if its automorphism group $\text{Aut}(X)$ acts faithfully on the cohomology ring $H^*(X, \mathbb{Z})$, and *rationally cohomologically rigidified* if $\text{Aut}(X)$ acts faithfully on $H^*(X, \mathbb{Q})$; it is said to be *rigidified* if $\text{Aut}(X) \cap \text{Diff}^0(X) = \{\text{id}_X\}$, where $\text{Diff}^0(X)$ is the connected component of the identity of the group of orientation preserving diffeomorphisms of X ([Cat11, Definition 12]).

Note that any element in $\text{Aut}(X) \cap \text{Diff}^0(X)$ acts trivially on the cohomology ring $H^*(X, \mathbb{Z})$. There are obvious implications: rationally cohomologically rigidified \Rightarrow cohomologically rigidified \Rightarrow rigidified.

It is well known that curves of genus ≥ 2 are rationally cohomologically rigidified. There are surfaces of general type with p_g arbitrary large which are not cohomologically rigidified ([Ca07]). An interesting question posed by Catanese ([Cat11, Remark 46]) is whether every surface of general type is rigidified.

The question is closely related to the local moduli problem for X , that is, whether the natural local map $\text{Def}(X) \rightarrow \mathcal{T}(M)_{[X]}$, from the Kuranishi space to the germ of the Teichmüller space at $[X]$, is a homeomorphism or not. Here M is the underlying C^∞ -manifold of X and $[X] \in \mathcal{T}(M)$ is the point corresponding to the complex structure of X ([Cat11, Section 1.4]).

Apart from the local moduli problem, there is also motivation from the global moduli problem, that is, the existence of fine moduli space for polarized manifolds having the same Hilbert polynomial as X together with a so-called level l -structure ([Po77, Lecture 10]). Along this line, many authors studied the action of automorphism groups of compact complex manifolds on their cohomology rings. It is known that K3 surfaces are rationally cohomologically

2000 *Mathematics Subject Classification.* Primary 14J50; Secondary 14J29.

Key words and phrases. surface of general type, automorphism, cohomology.

The first author was supported by the NSFC (No. 11071004). The second author was generously supported by the DFG via Sönke Rollenske's Emmy-Noether Programm and by the Bielefelder Nachwuchsfonds.

rigidified (cf. [BR75], [BHPV04]). For Enriques surfaces S , either S is cohomologically rigidified, or the kernel of $\text{Aut}(S) \rightarrow \text{Aut}(H^*(S, \mathbb{Z}))$ is a cyclic group of order 2 or 4, and the latter case was completely classified ([Mu–Na84], [Mu10]). For elliptic surfaces S , if $\chi(\mathcal{O}_S) > 0$ and $p_g(S) > 0$, then S is rationally cohomologically rigidified ([Pet80]). There are also attempts at the generalization to hyperkähler manifolds ([Be83], [BNS11]); recently Oguiso [O12] proved that generalized Kummer manifolds are cohomologically rigidified.

For surfaces of general type, the problem seems harder, since there is not so many available structures on the cohomology groups as for the special surfaces. At the moment only partial results are known.

Let S be a minimal nonsingular complex projective surface of general type, and $\text{Aut}_0(S)$ the subgroup of automorphisms of X , acting trivially on the cohomology ring $H^*(S, \mathbb{Q})$. Peters [Pet79] showed that, if the canonical linear system $|K_S|$ of S is base-point-free, then S is rationally cohomologically rigidified, with the possible exceptional case where S satisfies either $K_S^2 = 8\chi(\mathcal{O}_S)$ or $K_S^2 = 9\chi(\mathcal{O}_S)$. In [Ca06] and [Ca10], the first author proved that, if either S has a fibration of genus 2 and $\chi(\mathcal{O}_S) \geq 5$, or S is an irregular surface with $K_S^2 \leq 4\chi(\mathcal{O}_S)$ and $\chi(\mathcal{O}_S) > 12$, then either S is rationally cohomologically rigidified, or $\text{Aut}_0(S)$ is of order two and S satisfies $K_S^2 = 4\chi(\mathcal{O}_S)$ and $q(S) = 1$.

In this note, we consider surfaces of general type with $q(S) \geq 2$. Our main theorem is as follows.

Theorem 1.1. *Let S be a minimal nonsingular complex projective surface of general type with $q(S) \geq 2$. Then either S is rationally cohomologically rigidified, or $\text{Aut}_0(S)$ is a group of order two and S satisfies $K_S^2 = 8\chi(\mathcal{O}_S)$, $q(S) = 2$, the Albanese map of S is surjective, and S has a pencil of genus one.*

In particular, if $q(S) \geq 3$, then S is rationally cohomologically rigidified, hence rigidified.

By [Cat11, Theorem 45], we have

Corollary 1.2. *Let S be a minimal surface of general type with $q(S) \geq 3$. If K_S is ample, then the natural map $\text{Def}(S) \rightarrow \mathcal{T}(M)_{[S]}$ is a local homeomorphism between the Kuranishi space and the Teichmüller space. Here M is the underlying differential manifold of S .*

As examples we classify surfaces isogenous to a product that are not rationally cohomologically rigidified.

Theorem 1.3. *Let $S = (C \times D)/G$ be a surface isogenous to a product with $q(S) \geq 2$. Assume that S is not rationally cohomologically rigid. Then S is as in Example 4.6 below; in particular, S is of unmixed type, G is isomorphic to one of the following groups: $\mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}$, $\mathbb{Z}_2 \oplus \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}$ (m, n are arbitrary positive integers), and $g(C/G) = g(D/G) = 1$.*

By a result of Borel-Narasimhan ([BN67]), surfaces in Example 4.6 are rigidified (Proposition 4.8). It is not known whether surfaces S in the latter case of Theorem 1.1 are rigidified. A further step that can be done on these surfaces is to check if the action of $\text{Aut}(S)$ on cohomology with \mathbb{Z} -coefficient or on the fundamental group is faithful.

The paper is organized as follows.

Theorems 1.1 and 1.3 are proved in Sections 2–3 and Section 4, respectively.

In Section 2, we consider projective manifolds (of arbitrary dimension) with maximal Albanese dimension. We show that the group of automorphisms of such varieties behave like that of curves (Theorem 2.6), which is of independent interest.

In Section 3, a combination of the topological and holomorphic Lefschetz formulae for a group action and the Severi inequality helps us pin down the numerical restrictions on the surfaces with non-trivial $\text{Aut}_0(S)$.

In the classification of surfaces isogenous to a product $S = (C \times D)/G$ that are not rationally cohomologically rigidified in Section 4, we first use the theorem proved in Section 2 to exclude the mixed type case; then Broughton's cohomology representation theorem for curves is used to calculate the cohomology of surfaces isogenous to a product of unmixed type; finally we manage to give the classification by finding an appropriate character of the group G through Frobenius' reciprocity theorem.

Notations. A pencil of genus b of a surface S is a fibration $f : S \rightarrow B$, where B is a smooth curve of genus b .

For a smooth projective variety X , we denote by $p_g(X)$, $q(X)$, $e(X)$, $\chi(\mathcal{O}_X)$, $\chi(\omega_X)$, and K_X the geometric genus, the irregularity, the Euler topological characteristic, the Euler characteristic of the structure sheaf, the Euler characteristic of the canonical sheaf, and a canonical divisor of X , respectively.

We denote by $\text{Aut}_0(X)$ the kernel the natural homomorphism of groups $\text{Aut}(X) \rightarrow \text{Aut}(H^*(X, \mathbb{Q}))$.

We use \equiv , $\equiv_{\mathbb{Q}}$ to denote linear equivalence and \mathbb{Q} -linear equivalence of divisors, respectively.

For a finite group G and an element $g \in G$, we denote by

$|G|$: the order of G ,

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$C_G(g)$: the conjugacy class of g in G ,

$\text{Irr}(G)$: the set of irreducible characters of G ,

$\text{Ker}(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}$, for $\chi \in \text{Irr}(G)$.

For a representation V of G and a character $\chi \in \text{Irr}(G)$, we let V^χ be the sum of irreducible sub- G -modules W of V with $\chi_W = \chi$, where χ_W is the character of G -module W .

Let H be a subgroup of a finite group G , and χ a character of H , we denote by χ^G the induced character from χ . Recall that χ^G is defined by

$$\chi^G(g) = \frac{1}{|H|} \sum_{t \in G} \chi^\circ(tgt^{-1})$$

where for any $g \in G$

$$\chi^\circ(g) = \begin{cases} \chi(g), & \text{if } g \in H, \\ 0, & \text{if } g \notin H. \end{cases}$$

The symbol \mathbb{Z}_n denotes the cyclic group of order n .

Acknowledgements. Part of this note appears in the doctoral thesis of the second author submitted to Universität Bayreuth (2010). The second author

and the third author would like to thank the hospitality of Professor Jin-Xing Cai when they visited School of Mathematical Sciences, Beijing University in August 2012.

2. PROJECTIVE MANIFOLDS WITH MAXIMAL ALBANESE DIMENSION

In this section, we use the generic vanishing theorems of Green and Lazarsfeld ([GL87], [GL91], see also [Ha04]) and the notion of continuous global generation ([PP03], [PP06], [BLNP12]) to show that the groups of automorphisms of projective manifolds of general type with maximal Albanese dimension and with positive generic vanishing index behave like that of curves (Theorem 2.6).

We begin by recalling some notations.

2.1. The generic vanishing index. Let X be a smooth projective variety with $q(X) > 0$, and $a : X \rightarrow \text{Alb}X$ the Albanese map of X . We say that X is of maximal Albanese dimension if a is a generically finite map onto its image.

For $0 \leq i \leq \dim X$, the i -th cohomological support locus of X is defined as

$$V^i(\omega_X) := \{\alpha \in \text{Pic}^0(X) \mid h^i(X, \omega_X \otimes \alpha) > 0\}.$$

Let

$$\text{gv}_i(\omega_X) = \text{codim}_{\text{Pic}^0(X)} V^i(\omega_X) - i, \text{ and } \text{gv}(\omega_X) = \min_{i>0} \{\text{gv}_i(\omega_X)\}.$$

We call $\text{gv}_i(\omega_X)$ and $\text{gv}(\omega_X)$ the i -th generic vanishing index and the generic vanishing index of X , respectively.

2.2. The results of Green and Lazarsfeld. If X is a smooth projective variety of maximal Albanese dimension, by the generic vanishing theorem due to Green and Lazarsfeld (cf. [GL87], [GL91]), one has

(2.2.1) $\text{gv}(\omega_X) \geq 0$. So for a general $\alpha \in \text{Pic}^0(X)$, $h^i(X, \omega_X \otimes \alpha) = 0$ for all $i > 0$, and hence $\chi(\omega_X) = \chi(\omega_X \otimes \alpha) = h^0(X, \omega_X \otimes \alpha)$;

(2.2.2) for each positive dimensional component Z of $V^i(\omega_X)$, Z is a complex sub-torus of $\text{Pic}^0(X)$, and there exists an algebraic variety Y of dimension $\leq \dim X - i$ and a dominant map $f : X \rightarrow Y$ such that $Z \subset \alpha + f^*\text{Pic}^0(Y)$ for some $\alpha \in \text{Pic}^0(X)$.

2.3. A result on varieties with positive generic vanishing indices and its applications.

Theorem 2.4. *Let $f : X \rightarrow Y$ be a generically finite morphism of smooth projective varieties of maximal Albanese dimension. Let $a : X \rightarrow A := \text{Alb}X$ be the Albanese map of X . Assume that the following conditions hold:*

- (i) a factors through f ;
- (ii) $\text{gv}_i(\omega_X) \geq 1$ for all $0 < i < \dim X$;
- (iii) $p_g(X) = p_g(Y)$ if $q(X) = \dim X$.

Then $\chi(\omega_X) \geq \chi(\omega_Y)$ and “=” occurs only when f is birational.

Proof. By the assumption (i), there is a morphism $a' : Y \rightarrow A$, such that $a' \circ f = a$. By the universal property of the Albanese map, we have that a' is just the Albanese map of Y .

Since f is generically finite and Y is smooth, there is an injective morphism of sheaves $f^*\omega_Y \hookrightarrow \omega_X$. Taking f_* and composing with the natural morphism $\omega_Y \rightarrow f_*f^*\omega_Y$, we obtain an inclusion of sheaves $\omega_Y \hookrightarrow f_*\omega_X$.

Taking a'_* , we get an inclusion of sheaves

$$\rho : a'_*\omega_Y \hookrightarrow a'_*(f_*\omega_X) = a_*\omega_X.$$

Hence for every $\alpha \in \text{Pic}^0 A$, we have an inclusion

$$\rho_\alpha : H^0(A, a'_*\omega_Y \otimes \alpha) \hookrightarrow H^0(A, a_*\omega_X \otimes \alpha).$$

By (2.2.1), we have $\chi(\omega_X) \geq \chi(\omega_Y)$ by choosing α to be general.

Now we will show that, if $\chi(\omega_X) = \chi(\omega_Y)$ then $\deg f = 1$.

Again by (2.2.1), the assumption $\chi(\omega_X) = \chi(\omega_Y)$ implies that, for a general $\alpha \in \text{Pic}^0(X)$, $h^0(X, \omega_X \otimes \alpha) = h^0(Y, \omega_Y \otimes \alpha)$. Thus we can find a non-empty Zariski open set $U \subset \text{Pic}^0(A)$ such that for $\alpha \in U$, ρ_α is an isomorphism. Consider the following commutative diagram

$$(2.4.1) \quad \begin{array}{ccc} \bigoplus_{\alpha \in T} H^0(A, a'_*\omega_Y \otimes \alpha) \otimes \alpha^{-1} & \xrightarrow{\text{ev}'_T} & a'_*\omega_Y \\ \bigoplus_{\alpha \in T} \rho_\alpha \otimes \alpha^{-1} \downarrow & & \rho \downarrow \\ \bigoplus_{\alpha \in T} H^0(A, a_*\omega_X \otimes \alpha) \otimes \alpha^{-1} & \xrightarrow{\text{ev}_T} & a_*\omega_X \end{array}$$

where T is a subset of $\text{Pic}^0(A)$ and $\text{ev}_T, \text{ev}'_T$ are evaluation maps.

Case 1. $q(X) > \dim X$. We let $T = U$. Then by the choice of U , $\bigoplus_{\alpha \in T} \rho_\alpha$ is an isomorphism. In this case the assumption (ii) is equivalent to $\text{gv}(\omega_X) \geq 1$. This implies that ev_T is surjective ([PP06, Proposition 5.5], [BLNP12, (a) of Corollary 4.11]). By the commutative diagram (2.4.1), it follows that ρ is surjective.

Case 2. $q(X) = \dim X$. In this case we let $T = U \cup \{\hat{0}\}$, where $\hat{0}$ is the identity element of $\text{Pic}^0(A)$. By the assumption (iii) and by the choice of U , we have that $\bigoplus_{\alpha \in T} \rho_\alpha \otimes \alpha^{-1}$ is an isomorphism. By [BLNP12, (b) of Corollary 4.11], the assumption (ii) implies ev_T is surjective, and so ρ is surjective.

Since the ranks (at the generic point of A) of $a'_*\omega_Y, a_*\omega_X$ are $\deg a', \deg a$ ($= \deg a' \cdot \deg f$), respectively, the surjection of ρ implies $\deg f = 1$.

This completes the proof of the theorem. \square

Remark 2.5. *After finishing the paper, we are kindly informed by Sofia Tirabassi that, under slightly milder hypothesis, Theorem 2.4 was already independently proved in her thesis ([T11, Proposition 5.2.4]) in the case where $q(X) > \dim X$.*

Theorem 2.6. *Let X be a smooth projective variety of general type and of maximal Albanese dimension. If $\text{gv}_i(\omega_X) \geq 1$ for all $0 < i < \dim X$, then X is rationally cohomologically rigidified.*

Proof. Otherwise, there is a non-trivial automorphism σ of X , which acts trivially on $H^*(X, \mathbb{Q})$. Since X is of general type, σ is of finite order. Replacing σ by a suitable power, we may assume σ is of prime order, say p .

Let

$$\pi : X \rightarrow \bar{X} = X / \langle \sigma \rangle$$

be the quotient map. Since σ acts trivially on $H^i(X, \mathbb{C})$ for all $i \geq 0$, by Hodge theory, we have

$$H^i(\bar{X}, \mathcal{O}_{\bar{X}}) \simeq H^i(X, \pi_* \mathcal{O}_X) = H^i(X, \mathcal{O}_X)^\sigma = H^i(X, \mathcal{O}_X).$$

In particular, we have

$$(2.6.1) \quad h^{\dim X}(\bar{X}, \mathcal{O}_{\bar{X}}) = h^{\dim X}(X, \mathcal{O}_X), \text{ and } \chi(\mathcal{O}_{\bar{X}}) = \chi(\mathcal{O}_X).$$

Let $\rho : Y \rightarrow \bar{X}$ be a resolution of quotient singularities (if any). Then $R^i \rho_* \mathcal{O}_Y = 0$ for $i > 0$ since quotient singularities are rational. Thus

$$(2.6.2) \quad h^{\dim X}(Y, \mathcal{O}_Y) = h^{\dim X}(\bar{X}, \mathcal{O}_{\bar{X}}), \text{ and } \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{\bar{X}}).$$

By (2.6.1) and (2.6.2), using Serre duality, we obtain

$$p_g(Y) = p_g(X), \text{ and } \chi(\omega_Y) = \chi(\omega_X).$$

We claim that $X^\sigma \neq \emptyset$. Otherwise, the map π is étale. This implies $\chi(\omega_X) = p\chi(\omega_{\bar{X}})$. Combining this with (2.6.1), we have $\chi(\omega_X) = 0$. On the other hand, since $\text{gv}_i(\omega_X) \geq 1$ for all $0 < i < \dim X$ by assumption, $\chi(\omega_X) = 0$ is equivalent to X being not of general type ([BLNP12, Proposition 4.10]). So we get a contradiction.

Let $a : X \rightarrow \text{Alb}X$ be the Albanese map of X (the map a is unique up to translations of $\text{Alb}X$ and we fix it once for all). We have that there is an automorphism $\bar{\sigma}$ of $\text{Alb}X$, such that $\bar{\sigma} \circ a = a \circ \sigma$. Since σ induces trivial action on $H^1(X, \mathbb{Q})$, we have that either $\bar{\sigma}$ is a translation or $\bar{\sigma} = \text{id}_{\text{Alb}X}$. If $\bar{\sigma}$ is a translation, then $X^\sigma = \emptyset$ — a contradiction by the claim above. So $\bar{\sigma} = \text{id}_{\text{Alb}X}$, and consequently, a factors through π .

Let $f : X \rightarrow Y$ be the rational map induced by the quotient map π , and $\rho : X' \rightarrow X$ be a birational morphism such that $f \circ \rho$ is a morphism. Since $V^i(\omega_X)$ are birational invariants, using X' instead of X and $a \circ \rho$ instead of a , we may assume that f is a morphism. Then $\deg f$ is divisible by p , which is a contradiction by Theorem 2.4. \square

Corollary 2.7. *Let S be a smooth projective surface of maximal Albanese dimension. Assume that S has no pencils of genus ≥ 2 , and S has no pencils of genus 1 when $q(S) = 2$. Then S is rationally cohomologically rigidified.*

Proof. Note that the assumption of Corollary 2.7 is equivalent to $\text{gv}_1(\omega_S) \geq 1$ by (2.2.2). The corollary follows by Theorem 2.6. \square

Remark 2.8. The assumption on the i -th generic vanishing index (which is slightly weaker than $\text{gv}(\omega_X) \geq 1$) is indispensable for Theorem 2.6. For example, let $X = S \times C$, $\tau = \sigma \times \text{id}_C$, where the pair (S, σ) is as in Example 4.6, and C is a curve of genus ≥ 2 . Then τ is an involution of X , which acts trivially on the cohomology ring $H^*(X, \mathbb{Q})$.

It is interesting to classify smooth projective 3-folds of general type and of maximal Albanese dimension that are not rationally cohomologically rigidified.

3. NUMERICAL CLASSIFICATIONS

Theorem 3.1. *Let S be a minimal nonsingular complex projective surface of general type with $q(S) \geq 2$. Then either $\text{Aut}_0(S)$ is trivial, or $\text{Aut}_0(S)$ is a group of order two and S satisfies $K_S^2 = 8\chi(\mathcal{O}_S)$.*

For the proof of Theorem 3.1, we need the following lemmas.

Lemma 3.2. *Let S be a minimal nonsingular complex projective surface of general type, and $G \subset \text{Aut}(S)$ be a subgroup of automorphisms of S . Assume that the quotient S/G is of general type. Then $K_S^2 \geq |G|K_X^2$, where X is the minimal smooth model of S/G .*

Proof. Let $Y = S/G$, and $\pi : S \rightarrow Y$ be the quotient map. Let $\eta : \tilde{Y} \rightarrow Y$ be the minimal resolution of the quotient singularities of Y . We have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\pi}} & \tilde{Y} \\ \alpha \downarrow & & \downarrow \eta \\ S & \xrightarrow{\pi} & Y \end{array}$$

where \tilde{S} is the minimal resolution of the normalization of the fiber product $S \times_Y \tilde{Y}$, and $\alpha, \tilde{\pi}$ are natural induced maps.

Since surface quotient singularities are log-terminal (cf. [Ka84, Corollary 1.9]), we have that there is an effective \mathbb{Q} -divisor D supported on η -exceptional curves, such that

$$(3.2.1) \quad K_{\tilde{Y}} \equiv_{\mathbb{Q}} \eta^* K_Y - D.$$

Let R be the ramification divisor of π , that is, $R = \sum_C (|G_C| - 1)C$ with the sum taken all smooth curves C on S and $G_C = \{g \in G \mid g|_C = \text{id}\}$. We have

$$(3.2.2) \quad K_S \equiv_{\mathbb{Q}} \pi^* K_Y + R.$$

Let $\rho : \tilde{Y} \rightarrow X$ be the contraction of \tilde{Y} onto its uniquely determined minimal model X . Then

$$(3.2.3) \quad K_{\tilde{Y}} = \rho^* K_X + A$$

for some effective exceptional divisor A of ρ . Note that $\pi \circ \alpha = \eta \circ \tilde{\pi}$, and we have

$$(3.2.4) \quad \alpha^* K_S \equiv_{\mathbb{Q}} (\rho \circ \tilde{\pi})^* K_X + \tilde{\pi}^* A + \tilde{\pi}^* D + \alpha^* R$$

by combining (3.2.1), (3.2.2) and (3.2.3). So

$$\begin{aligned} K_S^2 &= \alpha^* K_S ((\rho \circ \tilde{\pi})^* K_X + \tilde{\pi}^* A + \tilde{\pi}^* D + \alpha^* R) \\ &\geq \alpha^* K_S (\rho \circ \tilde{\pi})^* K_X \quad (\text{since } \alpha^* K_S \text{ is nef}) \\ &\geq ((\rho \circ \tilde{\pi})^* K_X)^2 \quad (\text{using (3.2.4), since } (\rho \circ \tilde{\pi})^* K_X \text{ is nef}) \\ &= |G| K_X^2. \end{aligned}$$

□

Lemma 3.3. *Let S be a minimal nonsingular complex projective surface of general type with $q(S) \geq 2$. If S has a pencil of genus larger than one, then $\text{Aut}_0(S)$ is trivial.*

Proof. Let $f: S \rightarrow B$ be such a fibration over a curve B of genus $b \geq 2$. Suppose that there is a non-trivial element $\sigma \in \text{Aut}_0(S)$. Let F be a general fiber of f , and g the genus of F . We have $g \geq 2$. Since σ acts trivially on $\text{NS}(S) \otimes \mathbb{Q} \hookrightarrow H^2(S, \mathbb{Q})$, we have σ^*F is numerically equivalent to F . So f is preserved under the action of σ . Since σ acts trivially on $f^*H^1(B, \mathbb{Q}) \subseteq H^1(S, \mathbb{Q})$, it induces identity action on B , and so $f \circ \sigma = f$. We have $b \leq 1$ by [Ca12, Lemma 2.1] — a contradiction. \square

3.4. Proof of Theorem 3.1. By Lemma 3.3, we may assume that S is of maximal Albanese dimension.

Assume that $G := \text{Aut}_0(S)$ is not trivial. We will show that G is a group of order two and S satisfies $K_S^2 = 8\chi(\mathcal{O}_S)$.

Let X be a minimal smooth model of the quotient S/G . Then $q(X) = q(S)$, $p_g(X) = p_g(S)$, and both the canonical map and the Albanese map of S factorize through the quotient map $S \rightarrow S/G$. In particular, X is of maximal Albanese dimension and the Kodaira dimension of X is at least 1.

If the Kodaira dimension of X is 1, then the canonical map ϕ_X is composed with a pencil of genus $= q(X) - 1$ (cf. [Be82, p. 345, Lemme]). Since ϕ_S factors through the quotient map $S \rightarrow S/G$, we have that ϕ_S is composed with a pencil whose base curve, say C , is of genus $g(C) \geq q(X) - 1 = q(S) - 1 \geq 1$. On the other hand, one has that either $q(S) = g(C) = 1$ or $g(C) = 0$ and $q(S) \leq 2$ by [Xi85]. This is a contradiction.

Now we may assume that X is of general type. By Severi inequality ([Par05]), we have

$$(3.4.1) \quad K_X^2 \geq 4\chi(\mathcal{O}_X) = 4\chi(\mathcal{O}_S).$$

Combining (3.4.1), Lemma 3.2 and the Bogomolov–Miyaoka–Yau inequality $9\chi(\mathcal{O}_S) \geq K_S^2$, we get $|G| = 2$ and

$$(3.4.2) \quad K_S^2 \geq 8\chi(\mathcal{O}_S).$$

Let σ be the generator of G . Let D_i ($1 \leq i \leq u$, $u \geq 0$) be the σ -fixed curves. After suitable re-indexing, we may assume that $D_i^2 \geq 0$ for $i \leq k$ ($0 \leq k \leq u$) and $D_i^2 < 0$ for $i > k$. We may apply the topological and holomorphic Lefschetz formula to σ (cf. [AS, p. 566]):

$$(3.4.3) \quad \begin{aligned} & e(S) + 8(q(S) - \dim_{\mathbb{C}} H^0(S, \Omega_S^1)^{\sigma}) - 2(h^2(S, \mathbb{C}) - \dim_{\mathbb{C}} H^2(S, \mathbb{C})^{\sigma}) \\ &= e(S^{\sigma}) = n + \sum_{i=1}^u e(D_i) \end{aligned}$$

$$\chi(\mathcal{O}_S) + 2(q(S) - \dim_{\mathbb{C}} H^0(S, \Omega_S^1)^{\sigma}) = \frac{n - \sum_{i=1}^u K_S D_i}{4},$$

where n is the number of isolated σ -fixed points. Combining (3.4.3) with Noether's formula, we get

$$(3.4.4) \quad K_S^2 = 8\chi(\mathcal{O}_S) + \sum_{i=1}^u D_i^2 \leq 8\chi(\mathcal{O}_S) + \sum_{i=1}^k D_i^2.$$

Let $\rho: \tilde{S} \rightarrow S$ be the blowup of all isolated fixed points of σ , and $\tilde{\sigma}$ the induced involution on \tilde{S} . Let $\tilde{\pi}: \tilde{S} \rightarrow \tilde{X} := \tilde{S}/\tilde{\sigma}$ be the quotient map. Let

$\eta : \tilde{X} \rightarrow X$ be the map contracting all (-1) -curves on \tilde{X} . We have

$$(3.4.5) \quad \rho^* K_S = (\eta \circ \tilde{\pi})^* K_X + \tilde{\pi}^* A + \sum_{i=1}^u \rho^* D_i$$

for some effective exceptional divisor A of η .

We show that $k = 0$. Otherwise, we have

$$\begin{aligned} K_S^2 &= \rho^* K_S^2 \\ &\geq \rho^* K_S (\eta \circ \tilde{\pi})^* K_X + \sum_{i=1}^k \rho^* K_S \rho^* D_i \quad (\text{using (3.4.5), since } \rho^* K_S \text{ is nef}) \\ &\geq ((\eta \circ \tilde{\pi})^* K_X)^2 + \sum_{i=1}^k \rho^* K_S \rho^* D_i \quad (\text{using (3.4.5), since } (\eta \circ \tilde{\pi})^* K_X \text{ is nef}) \\ &= 2K_X^2 + \sum_{i=1}^k K_S D_i \\ &\geq 8\chi(\mathcal{O}_S) + (K_S - \sum_{i=1}^k D_i) \sum_{i=1}^k D_i + \sum_{i=1}^k D_i^2 \quad (\text{by (3.4.1)}) \\ &\geq 8\chi(\mathcal{O}_S) + 2 + \sum_{i=1}^k D_i^2, \end{aligned}$$

which contradicts (3.4.4), where the last inequality follows since each σ -fixed curve is contained in the fixed part of $|K_S|$ (cf. [Ca04, 1.14]) and $|K_S|$ is 2-connected (cf. [BHPV04, VII, Proposition 6.2]).

So we have $k = 0$ and hence $u = 0$ by combining (3.4.2) with (3.4.4). This finishes the proof of Theorem 3.1. \square

3.5. Proof of Theorem 1.1. By Theorem 3.1, there remains to prove the following claim: if $\text{Aut}_0(S)$ is not trivial, then $q(S) = 2$, the Albanese map of S is surjective, and S has a pencil of genus one.

By Lemma 3.3, we may assume that S is of maximal Albanese dimension, and S has no pencils of genus ≥ 2 . Now the claim follows by Corollary 2.7. \square

4. SURFACES ISOGENOUS TO A PRODUCT

Surfaces isogenous to a product play an important role in studying the geometry and the moduli of surfaces of general type. For examples, they provide simple counterexamples to the DEF=DIFF question whether deformation type and diffeomorphism type coincide for algebraic surfaces ([Cat03]), and they are useful in the construction of Inoue-type manifolds ([BC12, Definition 0.2]). In this section we give an explicit description for surfaces S isogenous to a product with $q(S) \geq 2$ which are not rationally cohomologically rigidified (Examples 4.6 and Theorem 4.9).

We begin by recalling some notations of surfaces isogenous to a product; we refer to [Cat00] for properties of these surfaces.

Definition 4.1. ([Cat00, Definition 3.1]) A smooth projective surface S is isogenous to a (higher) product if it is a quotient $S = (C \times D)/G$, where C, D are curves of genus at least two, and G is a finite group acting freely on $C \times D$.

Let $S = (C \times D)/G$ be a surface isogenous to a product. Let G° be the intersection of G and $\text{Aut}(C) \times \text{Aut}(D)$ in $\text{Aut}(C \times D)$. Then G° acts on the two factors C, D and acts on $C \times D$ via the diagonal action. If G° acts faithfully on both C and D , we say $(C \times D)/G$ is a minimal realization of S . By [Cat00, Proposition 3.13], a minimal realization exists and is unique. In the following we always assume $S = (C \times D)/G$ is the minimal realization.

We say that S is of unmixed type if $G = G^\circ$; otherwise S is of mixed type.

Proposition 4.2. *If $S = (C \times C)/G$ is a surface isogenous to a product of mixed type with $q(S) \geq 2$, then S is rationally cohomologically rigidified.*

Proof. Let $\sigma \in G \setminus G^\circ$. Up to coordinate change in both factors of $C \times C$, we can assume $\sigma(x, y) = (y, \tau x)$ for some $\tau \in G^\circ$ (cf. [Cat00, Proposition 3.16]). Then $X := (C \times C)/\sigma$ is smooth, and the natural maps $C \times C \rightarrow X$ and $\pi: X \rightarrow S$ are both étale coverings.

Note that

$$(4.2.1) \quad \begin{aligned} \text{Pic}^0(X) &\cong \text{Pic}^0(C \times C)^\sigma \cong (\text{Pic}^0(C) \times \text{Pic}^0(C))^\sigma \\ &= \{(\alpha, \beta) \in \text{Pic}^0(C) \times \text{Pic}^0(C) \mid \alpha = \tau^* \beta, \beta = \alpha\} \end{aligned}$$

Hence we can identify $\text{Pic}^0(X)$ with the set

$$\{(\alpha, \alpha) \in \text{Pic}^0(C) \times \text{Pic}^0(C) \mid \tau^* \alpha = \alpha\}.$$

We have

$$(4.2.2) \quad \begin{aligned} &H^1(X, (\alpha, \alpha)) \\ &\cong H^1(C \times C, (\alpha, \alpha))^\sigma \\ &\cong (H^1(C, \alpha) \otimes_{\mathbb{C}} H^0(C, \alpha) \oplus H^0(C, \alpha) \otimes_{\mathbb{C}} H^1(C, \alpha))^\sigma \end{aligned}$$

which is zero unless $\alpha = \hat{0}$, the identity element of $\text{Pic}^0(C)$. Using Serre duality we have $V^1(\omega_X) = \{\hat{0}\}$.

Note that $\pi: X \rightarrow S$ is an étale covering. We have for any $\gamma \in \text{Pic}^0(S)$

$$(4.2.3) \quad H^i(X, \omega_X \otimes \pi^* \gamma) = H^i(S, \omega_S \otimes \gamma \otimes \pi_* \mathcal{O}_X)$$

by the projection formula and the Leray spectral sequence. The left hand side of (4.2.3) is zero unless $\pi^* \gamma = \hat{0}$, while the right hand side contains a direct summand of $H^i(S, \omega_S \otimes \gamma)$. In other words, $H^i(S, \omega_S \otimes \gamma) = 0$ unless $\pi^* \gamma = \hat{0}$. Since $\pi^*: \text{Pic}^0(S) \rightarrow \text{Pic}^0(X)$ is a finite map onto its image, we conclude that $V^1(\omega_S)$ is a finite set. In particular, we have $\text{gv}_1(\omega_S) \geq 1$. By Theorem 2.6, we have that S is rationally cohomologically rigidified. \square

Contrary to the case of surfaces isogenous to a product of mixed type, there are surfaces isogenous to a product of unmixed type which are not rationally cohomologically rigidified. Before giving such examples, we insert here two facts on curves as well as an expression for the second cohomology of surfaces isogenous to a product of unmixed type that will be used in the sequel.

4.3. Riemann's existence theorem. Let $m_1, \dots, m_r \geq 2$ be r integers, and G a finite group. Let B be a curve of genus b , and let $p_1, \dots, p_r \in B$ be r different points.

Assume that there are $2b + r$ elements of G (not necessarily different), $\alpha_j, \beta_j, \gamma_i$ ($1 \leq j \leq b$, $1 \leq i \leq r$), such that these elements generate G , and satisfy

$$(4.3.1) \quad \prod_{j=1}^b [\alpha_j \beta_j] \prod_{i=1}^r \gamma_i = 1, \text{ and } |\gamma_i| = m_i.$$

If the Riemann–Hurwitz equation

$$2g - 2 = |G|(2b - 2 + \sum_{i=1}^r (1 - \frac{1}{m_i}))$$

is satisfied for some non-negative integer g , then there exists a curve C of genus g with a faithful G -action, such that the quotient map $C \rightarrow C/G \simeq B$ branched exactly at p_1, \dots, p_r , and m_i is the ramification numbers of q over p_i .

In what follows we call a $2b+r$ -tuple $(\alpha_1, \dots, \alpha_b, \beta_1, \dots, \beta_b, \gamma_1, \dots, \gamma_r)$ of elements of G a generating vector of type $(b; m_1, \dots, m_r)$, if these $2b+r$ elements generate G and satisfy (4.3.1).

4.4. The cohomology representation of the group of automorphisms of a curve. Let C be a smooth curve of genus $g(C) \geq 2$ and G a group of automorphisms of C . Let r be the number of branch points of the quotient map $C \rightarrow C/G$, and $C_G(\sigma_1), \dots, C_G(\sigma_r)$ the conjugacy classes whose elements generate the stabilizers over the branch points.

For $\sigma \in G$ and $\chi \in \text{Irr}(G)$, we denote by $l_\sigma(\chi)$ the multiplicity of the trivial character in the restriction of χ to $\langle \sigma \rangle$. Clearly, $l_\sigma(\chi) \leq \chi(1)$, and the equality holds if and only if $\sigma \in \text{Ker}(\chi)$.

By [Br87, Proposition 2], for any nontrivial irreducible character χ of G ,

$$h^1(C, \mathbb{C})^\chi = \chi(1)(2g(C/G) - 2 + r) - \sum_{j=1}^r l_{\sigma_j}(\chi),$$

where $h^1(C, \mathbb{C})^\chi = \dim H^1(C, \mathbb{C})^\chi$.

In particular, if $g(C/G) = 1$, then we have that,

$$(4.4.1) \quad h^1(C, \mathbb{C})^\chi \neq 0 \text{ if and only if } \chi(\sigma_j) \neq \chi(1) \text{ for some } j.$$

4.5. The second cohomology of surfaces isogenous to a product of unmixed type. Let $S = (C \times D)/G$ be a surface isogenous to a product of unmixed type. Then the second cohomology of S is

$$(4.5.1) \quad \begin{aligned} H^2(S, \mathbb{C}) &= H^2(C \times D, \mathbb{C})^{\Delta_G} \\ &= W \bigoplus \left(\bigoplus_{\chi_1, \chi_2 \in \text{Irr}(\Delta_G)} H^1(C, \mathbb{C})^{\chi_1} \otimes_{\mathbb{C}} H^1(D, \mathbb{C})^{\chi_2} \right)^{\Delta_G}, \end{aligned}$$

where $W = H^2(C, \mathbb{C}) \otimes_{\mathbb{C}} H^0(D, \mathbb{C}) \bigoplus H^0(C, \mathbb{C}) \otimes_{\mathbb{C}} H^2(D, \mathbb{C})$ and Δ_G is the diagonal subgroup of $G \times G$. As a representation of Δ_G , the irreducible constituents

of

$$\bigoplus_{\chi_1, \chi_2 \in \text{Irr}(G)} \mathbb{H}^1(C, \mathbb{C})^{\chi_1} \otimes_{\mathbb{C}} \mathbb{H}^1(D, \mathbb{C})^{\chi_2}$$

all has the same character $\chi_1 \chi_2$. Hence the multiplicity of the trivial representation $\mathbf{1}_{\Delta_G}$ in such a irreducible constituent is

$$\langle \chi_1 \chi_2, \mathbf{1}_{\Delta_G} \rangle_c = \langle \chi_1, \bar{\chi}_2 \rangle_c = \begin{cases} 1, & \text{if } \chi_2 = \bar{\chi}_1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\langle \cdot \rangle_c$ is the inner product on the vector space of class functions on G . Therefore

(4.5.1) becomes

$$(4.5.2) \quad \mathbb{H}^2(S, \mathbb{C}) = W \bigoplus \left(\bigoplus_{\chi \in \text{Irr}(G)} \mathbb{H}^1(C, \mathbb{C})^\chi \otimes_{\mathbb{C}} \mathbb{H}^1(D, \mathbb{C})^{\bar{\chi}} \right)^{\Delta_G}.$$

Example 4.6. Surfaces S isogenous to a product of unmixed type with $\text{Aut}_0(S) \simeq \mathbb{Z}_2$. Let m, n, k, l be positive integers. Let G be one of the following groups:

$$\mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}, \quad \mathbb{Z}_2 \oplus \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}.$$

Let \bar{C}, \bar{D} be elliptic curves. Let $\mathcal{V} := (\alpha_1, \beta_1, \overbrace{\gamma, \dots, \gamma}^{2k})$, $\mathcal{V}' := (\alpha'_1, \beta'_1, \overbrace{\gamma', \dots, \gamma'}^{2l})$ be generating vectors of G of type $(1; 2, \dots, 2)$ with $\gamma \neq \gamma'$, and

$$\rho : C \rightarrow \bar{C}, \quad \rho' : D \rightarrow \bar{D}$$

the G -coverings of smooth curves corresponding to $\mathcal{V}, \mathcal{V}'$, respectively (cf. 4.3).

For example, if $G = \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}$, we may take $\mathcal{V} = (\alpha, \beta, \alpha^m, \dots, \alpha^m)$, $\mathcal{V}' = (\alpha, \beta, \beta^{mn}, \dots, \beta^{mn})$, where $\alpha := (1, 0), \beta := (0, 1) \in G$; if $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}$, we may take $\mathcal{V} = (\mu, \nu, \lambda, \dots, \lambda)$, $\mathcal{V}' = (\mu, \nu, \lambda \mu^m, \dots, \lambda \mu^m)$, where $\lambda := (1, 0, 0), \mu := (0, 1, 0), \nu := (0, 0, 1) \in G$.

Let G act diagonally on $C \times D$. Note that the stabilizer of each point lying over any branch point of ρ is $\langle \gamma \rangle$, and that of ρ' is $\langle \gamma' \rangle$ (cf. 4.3). Since $\langle \gamma \rangle \cap \langle \gamma' \rangle = 1$ by assumption, we have that G acts freely on $C \times D$, and hence $S := (C \times D)/G$ is a surface isogenous to a product of curves.

By Hurwitz formula, we have $g(C) = 2\delta m^2 nk + 1$ and $g(D) = 2\delta m^2 nl + 1$, where $\delta = 1$ or 4 depending on $G = \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}$ or not. So the numerical invariants of S are as below:

$$p_g(S) = \delta m^2 nkl + 1, \quad q(S) = 2 \text{ and } K_S^2 = 8\delta m^2 nkl.$$

Let $I = \{\chi \in \text{Irr}(G) \mid \chi(\gamma) \neq 1 \text{ and } \bar{\chi}(\gamma') \neq 1\}$. By (4.4.1) and (4.5.2), we have

$$(4.6.1) \quad \mathbb{H}^2(S, \mathbb{C}) = W \bigoplus_{\chi \in I} \mathbb{H}^1(C, \mathbb{C})^\chi \otimes_{\mathbb{C}} \mathbb{H}^1(D, \mathbb{C})^{\bar{\chi}},$$

with $W = \mathbb{H}^2(C, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{H}^0(D, \mathbb{C}) \oplus \mathbb{H}^0(C, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{H}^2(D, \mathbb{C})$.

Since γ (resp. γ') is of order two, it induces $-\text{id}$ on $\mathbb{H}^1(C, \mathbb{C})^\chi$ (resp. $\mathbb{H}^1(D, \mathbb{C})^{\bar{\chi}}$) for all $\chi \in I$. So (γ, γ') induces identity on the right-side hand of (4.6.1).

Let σ be the automorphism of S induced by $(\gamma, \gamma') \in \text{Aut}(C) \times \text{Aut}(D) \subseteq \text{Aut}(C \times D)$. Then σ is an involution of S and it acts trivially on $H^2(S, \mathbb{Q})$ and hence on $H^*(S, \mathbb{Q})$.

Remark 4.7. Surfaces in Example 4.6 are rigidified by Proposition 4.8 below. It is not known whether they are cohomologically rigidified.

Proposition 4.8. *Let S be a smooth projective surface. Assume that the universal cover of S is a bounded domain in \mathbb{C}^2 . Then S is rigidified.*

Proof. Otherwise, there is an automorphism $\sigma \in \text{Aut}(X) \cap \text{Diff}^0(X)$, of prime order. The assumption implies S is of general type; in particular $\chi(\omega_S) > 0$. So we have $S^\sigma \neq \emptyset$ by the proof of Theorem 2.6. Thus σ and id_S are homotopic automorphisms which agree at S^σ . Since the universal cover of S is a product of two unit disks in \mathbb{C} , a bounded domain, it follows that $\sigma = \text{id}_S$ by [BN67, Theorem 3.6] — a contradiction. \square

Theorem 4.9. *Let $S = (C \times D)/G$ be a surface isogenous to a product of unmixed type with $q(S) \geq 2$. If S is rationally cohomologically rigidified, then S is as in Example 4.6.*

Before proving Theorem 4.9, we show the following preparatory results.

Proposition 4.10. *Let $S = (C \times D)/G$ be a surface isogenous to a product (with minimal realization). Denote by Δ_G the diagonal of $G \times G$. Then*

$$\text{Aut}(S) = N(\Delta_G)/\Delta_G,$$

where $N(\Delta_G)$ is the normalizer of Δ_G in $\text{Aut}(C \times D)$.

Proof. For each $\sigma \in \text{Aut}(S)$, there is an automorphism $\tilde{\sigma} \in \text{Aut}(C \times D)$ such that

$$\begin{array}{ccc} C \times D & \xrightarrow{\tilde{\sigma}} & C \times D \\ \pi \downarrow & & \downarrow \pi \\ S & \xrightarrow{\sigma} & S \end{array}$$

is commutative, where π is the quotient map. The existence of such a lift $\tilde{\sigma}$ of σ follows simply from the uniqueness of minimal realization of S .

On the other hand, given $\tilde{\sigma} \in \text{Aut}(C \times D)$, $\tilde{\sigma}$ descends to an automorphism $\sigma \in \text{Aut}(S)$ if and only if it is in the normalizer $N(\Delta_G)$ of Δ_G in $\text{Aut}(C \times D)$. Hence we have a surjective homomorphism of groups $N(\Delta_G) \rightarrow \text{Aut}(S)$ and its kernel is easily seen to be Δ_G . So $\text{Aut}(S) = N(\Delta_G)/\Delta_G$. \square

Proposition 4.11. *Let S be as in Proposition 4.10. If S is of unmixed type, then*

$$\text{Aut}_0(S) \subseteq (G \times G) \cap N(\Delta_G)/\Delta_G.$$

Proof. For each $\sigma \in \text{Aut}_0(S)$, let $\tilde{\sigma} \in \text{Aut}(C \times D)$ be its lift as in the proof of Proposition 4.10. By the proof of Lemma 3.3, σ preserves the two induced fibrations $\pi_1: S \rightarrow C/G$ and $\pi_2: S \rightarrow D/G$, and it induces identity on their bases C/G and D/G . Hence $\tilde{\sigma}$ does not interchange the factors of $C \times D$. By [Cat00, Rigidity Lemma 3.8]), there are automorphisms σ_C and σ_D , of C and D , respectively, such that $\tilde{\sigma} = (\sigma_C, \sigma_D)$. Since $\tilde{\sigma}$ induces identity on bases of π_1 and π_2 , we have $\sigma_C, \sigma_D \in G$.

On the other hand, By Proposition 4.10, we have $\tilde{\sigma} \in N(\Delta_G)$. So $\tilde{\sigma} \in (G \times G) \cap N(\Delta_G)$, and $\sigma = \tilde{\sigma} \bmod \Delta_G \in (G \times G) \cap N(\Delta_G)/\Delta_G$. \square

Remark 4.12. Let Z_G be the center of G . Then $(G \times G) \cap N(\Delta_G)$ is generated by $Z_G \times \{1\}$ and Δ_G , and the map

$$Z_G \rightarrow (G \times G) \cap N(\Delta_G)/\Delta_G, \sigma \mapsto (\sigma, 1) \bmod \Delta_G$$

is an isomorphism of groups. In what follows, we regard Z_G as a subgroup of $\text{Aut}(S)$ under such an isomorphism.

So by Proposition 4.11, $\text{Aut}_0(S)$ is (isomorphic to) a subgroup of Z_G ; in particular, if G is centerless, then $\text{Aut}_0(S)$ is trivial.

Lemma 4.13. *Let S be as in Proposition 4.10. If S is of unmixed type, then for each $\sigma \in Z_G \subseteq \text{Aut}(S)$, we have that, $\sigma \notin \text{Aut}_0(S)$ if and only if there is an irreducible $\chi \in \text{Irr}(G)$ such that $\sigma \notin \text{Ker}(\chi)$, $H^1(C, \mathbb{C})^\chi \neq 0$, and $H^1(D, \mathbb{C})^{\bar{\chi}} \neq 0$.*

Proof. Since for each $\sigma \in Z_G \subseteq \text{Aut}(S)$, $(\sigma, 1) \in \text{Aut}(C \times D)$ is a lift of σ (cf. Remark 4.12), the lemma follows from the fact that, for each $\sigma \in \text{Aut}(S)$, the quotient map $C \times D \rightarrow S$ induces an isomorphism between the action of σ on $H^2(S, \mathbb{C})$ and that of its lift $\tilde{\sigma}$ on the right-hand side of (4.5.2). \square

Lemma 4.14. *Let H be a subgroup of a finite group G , and χ an irreducible character of H . Let $H' \subseteq H$ be a subset such that $H' \cap \text{Ker}(\chi) = \emptyset$. Then*

- (i) *for any irreducible constituent φ of χ^G , $H' \cap \text{Ker}(\varphi) = \emptyset$;*
- (ii) *if moreover $\chi^G(g) = 0$ for some $g \in G$, then there is an irreducible constituent φ' of χ^G such that $(\{g\} \cup H') \cap \text{Ker}(\varphi') = \emptyset$.*

Proof. For any irreducible constituent φ of χ^G , Frobenius reciprocity theorem gives $(\varphi, \chi^G) = (\varphi|_H, \chi)$. Hence the multiplicity of χ in $\varphi|_H$ is the same as that of φ in χ^G . In particular χ is a constituent of $\varphi|_H$ and $\text{Ker}(\varphi) \cap H \subseteq \text{Ker}(\chi)$. So (i) follows.

If $\chi^G(g) = 0$ for some $g \in G$, then there exists an irreducible constituent φ' such that $g \notin \text{Ker}(\varphi')$. Combining this with (i), we obtain (ii). \square

4.15. Proof of Theorem 4.9. Suppose that $\text{Aut}_0(S)$ is not trivial. By Proposition 4.2, we may assume that S is of unmixed type. Consider the induced fibrations $S \rightarrow C/G$ and $S \rightarrow D/G$. By Lemma 3.3, note that $q(S) = g(C/G) + g(D/G)$, we have that $g(C/G) = g(D/G) = 1$.

Let $\mathcal{U} := (a, b, \sigma_1, \dots, \sigma_r)$ (resp. $\mathcal{U}' := (c, d, \tau_1, \dots, \tau_s)$) be the generating vector of G for the branch covering $C \rightarrow C/G$ (resp. $D \rightarrow D/G$) (cf. 4.3). Denote by m_i, n_j the order of σ_i, τ_j , respectively.

Let $\Sigma_1 = \cup_{g \in G} \cup_{1 \leq i \leq r} \langle g\sigma_i g^{-1} \rangle$ (resp. $\Sigma_2 = \cup_{g \in G} \cup_{1 \leq j \leq s} \langle g\tau_j g^{-1} \rangle$). Since the action of G on $C \times D$ is free, we have $\Sigma_1 \cap \Sigma_2 = \{1\}$.

By Theorem 1.1, $\text{Aut}_0(S)$ is of order two. Let $\sigma \in Z_G$ such that σ is the generator of $\text{Aut}_0(S)$ (cf. Remark 4.12). By Lemma 4.13 and (4.4.1), we have that

$$(4.15.1) \quad \text{for any } \chi \in I, \sigma \in \text{Ker}(\chi),$$

where I is the set of irreducible characters χ of G such that $\sigma_i, \tau_j \notin \text{Ker}(\chi)$ for some i, j .

4.15.1. *Claim: G is abelian.*

4.15.2. *Proof of Claim.* If G is not abelian, we will get a contradiction by finding an irreducible character $\chi \in I$ such that $\sigma \notin \text{Ker}(\chi)$. By Lemma 4.14, it is enough to find a subgroup H of G and an irreducible character χ of H , such that $\sigma, \sigma_i, \tau_j \in H$ and $\sigma, \sigma_i, \tau_j \notin \text{Ker}(\chi)$ for some i, j .

For each $1 \leq i \leq r$ and $1 \leq j \leq s$, let G_{ij} be the subgroup of G generated by σ_i and τ_j , φ_i the linear character of the cyclic group $\langle \sigma_i \rangle$ such that $\varphi_i(\sigma_i) = \xi$, where ξ is a primitive m_i -th root, and $\varphi_i^{G_{ij}}$ the induced character from φ_i . Since $\Sigma_1 \cap \Sigma_2 = \emptyset$, we have

$$\varphi_i^{G_{ij}}(\tau_j) = 0$$

for all $1 \leq i \leq r$ and $1 \leq j \leq s$. By Lemma 4.14, there is an irreducible character χ_{ij} of $\varphi_i^{G_{ij}}$ such that

$$(4.15.2) \quad \sigma_i, \tau_j \notin \text{Ker}(\chi_{ij}).$$

Similarly, starting with a primitive linear character of $\langle \tau_j \rangle$, we can construct a character χ'_{ji} of G_{ij} such that

$$\sigma_i, \tau_j \notin \text{Ker}(\chi'_{ji}).$$

Step 1. First we assume $\sigma \notin G_{ij}$ for some i, j . Since σ is of order two and $\sigma \in Z_G$, we have $\langle \sigma \rangle \cap G_{ij} = \{1\}$, and for any $g \in G$, $g\sigma g^{-1} = \sigma \notin G_{ij}$. So by the definition of induced character, we have $\chi_{ij}^{(G_{ij}, \sigma)}(\sigma) = 0$. By Lemma 4.14, there is an irreducible character $\tilde{\chi}_{ij}$ of $\langle G_{ij}, \sigma \rangle$ such that $\sigma, \sigma_i, \tau_j \notin \text{Ker}(\tilde{\chi}_{ij})$.

Step 2. Next, suppose $\sigma \in G_{ij}$ for any $1 \leq i \leq r$ and $1 \leq j \leq s$.

If $\sigma \in \langle \sigma_i \rangle$ for some $1 \leq i \leq r$, then $\varphi_i(\sigma) \neq 1$. By Lemma 4.14 and (4.15.2), $\sigma, \sigma_i, \tau_j \notin \text{Ker}(\chi_{ij})$.

Similarly, If $\sigma \in \langle \tau_j \rangle$ for some $1 \leq j \leq s$, then $\sigma, \sigma_i, \tau_j \notin \text{Ker}(\chi'_{ji})$.

Step 3. Now we can assume that, (*) $\sigma \in G_{ij}$ but $\sigma \notin \langle \sigma_i \rangle$ and $\sigma \notin \langle \tau_j \rangle$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$.

Since σ is in the center of G , we have $\langle \sigma, \sigma_i \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{m_i}$. Let ψ_i be the linear character of $\langle \sigma, \sigma_i \rangle$ such that

$$\psi_i(\sigma) = -1, \quad \psi_i(\sigma_i) = \xi,$$

where ξ is a primitive m_i -th root. Let ${}_j\psi_i$ be the induced character of ψ_i on the group $\langle \sigma, \sigma_i, \tau_j \rangle$.

Step 3.1. If $C_G(\tau_j) \cap \langle \sigma, \sigma_i \rangle = \emptyset$ for some $1 \leq i \leq r$ and $1 \leq j \leq s$, where $C_G(\tau_j)$ is the conjugate class of τ_j in G , then by the definition of induced character, ${}_j\psi_i(\tau_j) = 0$. By Lemma 4.14, there is a constituent ψ of ${}_j\psi_i$ such that $\sigma, \sigma_i, \tau_j \notin \text{Ker}(\psi)$.

Step 3.2. Next, we assume additionally $C_G(\tau_j) \cap \langle \sigma, \sigma_i \rangle \neq \emptyset$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. Then for each $1 \leq i \leq r$ and $1 \leq j \leq s$, there is an element $g_{ij} \in G$ such that $g_{ij}\tau_j g_{ij}^{-1} \in \langle \sigma, \sigma_i \rangle$.

If $m_i \geq 3$ for some $1 \leq i \leq r$, then it is easy to find a linear character χ of $\langle \sigma, \sigma_i \rangle$ such that $\sigma, \sigma_i, g_{ij}\tau_j g_{ij}^{-1} \notin \text{Ker}(\chi)$ and hence $\sigma, \sigma_i, \tau_j \notin \text{Ker}(\chi)$. (Since $H_i := \langle \sigma, \sigma_i \rangle$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{m_i}$, we can find characters $\phi, \phi' \in H_i$ such that

$$\phi(\sigma) = 1, \quad \phi(\sigma_i) = \xi_i, \quad \phi'(\sigma_i) = -1, \quad \phi'(\sigma_i) = 1,$$

where ξ_i is a root of unity of order m_i . Write $\tau'_j := g_{ij}\tau_j g_{ij}^{-1} = \sigma^a \sigma_i^b$ for some $a, b \geq 0$. Since $\tau'_j \neq \sigma$, we have $b \neq 0$. Let $\phi_k = \phi^k \phi'$ for $k = 1, 2$. Then

$\phi_k(\tau'_j) = (-1)^a \xi_i^{bk}$. So among ϕ_1 and ϕ_2 , there is at least one character, say ϕ_1 , such that $\phi_1(\tau'_j) \neq 1$.

Similarly, if $n_j \geq 3$ for some $1 \leq j \leq s$, then we can find a linear character χ of $\langle \sigma, \tau_j \rangle$ such that $\sigma, \sigma_i, \tau_j \notin \text{Ker}(\chi)$.

Step 3.3. Finally we may assume further $m_i = n_j = 2$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. This implies that $G_{ij} = D_{2k_{ij}}$ for some $k_{ij} \geq 2$ since any finite group generated by two elements of order two is dihedral.

If $k_{ij} > 2$ for some i and j , then it is well known that there is a faithful (2-dimensional) representation ρ of G_{ij} . Let χ be the corresponding character of ρ . Since ρ is faithful, $\sigma, \sigma_i, \tau_j \notin \text{Ker}(\chi)$.

If $k_{ij} = 2$ for all i and j , the assumption (*) above implies that $\sigma = \sigma_i \tau_j$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. So $\sigma_1 = \cdots = \sigma_r$, $\tau_1 = \cdots = \tau_s$ and $\sigma_1 \neq \tau_1$.

Step 4. We show that σ_1, τ_1 are in the center of G .

Note that G_{ij} in the proof above can be replaced by $G'_{ij} = \langle \sigma'_i, \tau'_j \rangle$ for any $\sigma'_i \in C_G(\sigma_i), \tau'_j \in C_G(\tau_j)$, since the characters of G do not distinguish conjugate elements. So we can assume much more, namely, $\langle \sigma'_i, \tau'_j \rangle \cong \mathbb{Z}^{\oplus 2}$, $\sigma \in \langle \sigma'_i, \tau'_j \rangle$, $\sigma \notin \langle \sigma'_i \rangle$ or $\langle \tau'_j \rangle$ for any i, j , $\sigma'_i \in C_G(\sigma_i)$ and $\tau'_j \in C_G(\tau_j)$. Under these assumptions, we have

$$\sigma \in A := \langle \sigma'_1, \tau_1 \rangle \cong \mathbb{Z}_2^{\oplus 2}$$

for any $\sigma'_1 \in C_G(\sigma_1)$. This implies that A is generated by σ and τ_1 , and hence it is generated by σ_1 and τ_1 since $\sigma = \sigma_1 \tau_1$. So we have $\sigma'_1 = \sigma_1$ and hence $C_G(\sigma_1) = \langle \sigma_1 \rangle$.

Since $\sigma \in Z_G$ and $\sigma = \sigma_1 \tau_1$, we have that τ_1 is in the center of G .

Now by the definition of a generator vector, G is generated by a, b and τ_1 , and $aba^{-1}b^{-1}\sigma_1^r = 1$.

This implies that the commutator subgroup G' is contained in $\langle \sigma_1 \rangle$. Similarly G' is a subgroup of $\langle \tau_1 \rangle$. Since $\langle \sigma_i \rangle \cap \langle \tau_j \rangle = \{1\}$, G' is trivial and G is abelian — a contradiction. This finishes the proof of the claim. \square

Now we may assume that G is abelian. By the proof of the claim, we have that $\sigma_1 = \cdots = \sigma_r$, $\tau_1 = \cdots = \tau_s$ and $\sigma = \sigma_1 \tau_1$. Thus G can be generated by three elements, namely a, b and σ_1 (or c, d and τ_1). By the structure theorem of finitely generated abelian groups, we may write $G = \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \mathbb{Z}_{d_3}$ with $d_1 \mid d_2 \mid d_3$. Since G has at least two elements of order 2, both d_2 and d_3 are even. If $d_1 = 1$, then $G \cong \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}$ for some positive integers m, n .

If $d_1 \geq 2$, then G needs three generators, one of which is σ_1 or τ_1 . Since σ_1 and τ_1 have order 2, we see that $d_1 = 2$. Hence in this case $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2m} \oplus \mathbb{Z}_{2mn}$ for some positive integers m, n .

Since $aba^{-1}b^{-1}\sigma_1^r = 1$ and $cdc^{-1}d^{-1}\tau_1^s = 1$ in G , we have that both r and s are even. So S is as in Example 4.6 with $\mathcal{V} = \mathcal{U}$ and $\mathcal{V}' = \mathcal{U}'$.

This completes the proof of Theorem 4.9. \square

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