

INTERSECTION FORMS OF SPIN FOUR-MANIFOLDS

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1. INTRODUCTION

The purpose of this note is to prove the following

Theorem 1.1. *Let X be a smooth, closed, connected and simply connected, four-dimensional spin manifold. Then the second Betti number $b^2(X) = \dim H^2(X; \mathbb{R})$ satisfies an estimate*

$$b^2(X) \geq \frac{11}{8} \cdot |\text{signature}(X)|.$$

This theorem applies to the question of which bilinear forms can be realized as intersection forms of simply connected, closed and oriented four-manifolds [22, Problem 4.1]: The case of non-smooth manifolds was completely solved by Freedman [16]. In the smooth setting, Donaldson's results [13] settled the non-spin case.

Corollary 1.2. *The intersection form of a smooth and simply connected spin four-manifold is isometric to the intersection form of a connected sum of copies of $S^2 \times S^2$ and of copies of K3-surfaces, suitably oriented.*

Furuta in [19] had obtained an estimate $b^2(X) \geq 2 + \frac{10}{8} \cdot |\text{signature}(X)|$ for spin manifolds with arbitrary fundamental groups and nonzero signature. The main theorem 2.1 in this note generalizes Furuta's 10/8-theorem to the case of spin four-manifolds bounding disjoint unions of rational homology spheres.

Theorem 1.1 is deduced from this generalized 10/8-theorem. To get an idea, how this comes about, note the extra summand 2 in the statement of the $\frac{10}{8}$ -theorem. Suppose a closed four-manifold is cut into pieces along embedded homology spheres. Then every piece carrying nonzero signature

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contributes an extra such summand. The more and smaller pieces there are, the more we can improve on the factor $\frac{10}{8}$. This argument works best in the case of simply connected manifolds. These can be cut into pieces small and numerous enough such that the extra contributions add up to an extra factor $\frac{1}{8}$.

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2. MAIN THEOREMS

Let X be a smooth, compact, connected and oriented Riemannian spin manifold of dimension 4 with boundary Y a disjoint union of rational homology spheres. We assume the metric on X to be product near the boundary, i. e. the existence of a metric collar

$$X_\varepsilon =]-\varepsilon, 0] \times Y \subset X$$

along the boundary. We will consider differential operators

$$\begin{aligned} D^+ &: \Gamma(S^+) \rightarrow \Gamma(S^-) \\ d^+ &: cE^1(X) \rightarrow \Omega^{2,+}(X) \end{aligned}$$

between spaces of smooth sections of bundles over X : The Dirac operator D^+ maps positive to negative spinors. The operator d^+ maps co-exact 1-forms to the self-dual part of their exterior derivative. On imposing suitable boundary conditions adapted to the gauge theory of the monopole

map (compare section 4), these operators become Fredholm. We set

$$\begin{aligned} s(X) &:= \operatorname{ind}_{\mathbb{H}} D^+ \\ h(X) &:= -\operatorname{ind}_{\mathbb{R}} d^+. \end{aligned}$$

Here we are using the fact that the spinor bundles S^+ and S^- for a spin 4-manifold are quaternionic line bundles and the Dirac operator preserves the quaternionic structure. The group $Pin(2) \subset Sp(1) \subset \mathbb{H}$ is the normalizer of the maximal torus $\mathbb{T} \subset \mathbb{C} \subset \mathbb{H}$ in $Sp(1)$. This subgroup is generated by \mathbb{T} and an additional element $j \in \mathbb{H}$ which satisfies $j^2 = -1$ and $ij + ji = 0$. The operators D^+ and d^+ are $Pin(2)$ -equivariant: On spinors, $Pin(2)$ acts via the quaternionic structure. On forms, the element j acts through multiplication by -1 . The monopole map

$$\mu : \mathcal{A}(X) \rightarrow \mathcal{C}(X),$$

as defined in section 4, is a non-linear $Pin(2)$ -equivariant perturbation of the Fredholm operator $l = D^+ \oplus d^+$. It is a map between Fréchet spaces, elements of which being smooth sections of vector bundles over smooth manifolds.

Recall that a $Pin(2)$ -universe is a real Hilbert space \mathfrak{U} on which $Pin(2)$ acts via isometries in such a way that an irreducible $Pin(2)$ -representation, if contained in \mathfrak{U} , is so with infinite multiplicity. For a finite dimensional real $Pin(2)$ -representation V , let S^V denote the one-point compactification of the underlying real vector space. This is a sphere, equipped with an action of the group $Pin(2)$ and a distinguished base point at infinity. For finite dimensional $Pin(2)$ -representations V and W , we define a stable equivariant homotopy set indexed by \mathfrak{U} as the co-limit

$$\{S^V, S^W\}_{\mathfrak{U}}^{Pin(2)} = \operatorname{colim}_{U \subset \mathfrak{U}} [S^U \wedge S^V, S^U \wedge S^W]^{Pin(2)}$$

over suspensions with finite dimensional sub-representations $U \subset \mathfrak{U}$.

Theorem 2.1. *Suppose the boundary Y of the spin 4-manifold X is a disjoint union of rational homology spheres. Then the monopole map defines a $Pin(2)$ -equivariant stable homotopy element*

$$[\mu] \in \left\{ S^{s(X)\mathbb{H}}, S^{h(X)\mathbb{R}^-} \right\}_{\mathfrak{U}}^{Pin(2)}$$

for any $Pin(2)$ -universe \mathfrak{U} containing a suitable Sobolev completion of $\mathcal{C}(X)$ as a closed subspace. The monopole class $[\mu]$ restricts to the identity on $Pin(2)$ -fixed point spheres.

Indeed, this theorem follows immediately from boundedness (5.1) of the monopole map. Stable equivariant homotopy theory provides obstructions to the existence of such elements, depending on the numbers $s(X)$ and $h(X)$:

Corollary 2.2. *Suppose the boundary Y of the spin 4-manifold X is a disjoint union of rational homology spheres and suppose $s(X) > 0$. Then the following estimates hold:*

$$h(X) \geq 2s(X) + \begin{cases} 1 & \text{if } s(X) \equiv 0 \text{ or } 1 \pmod{4} \\ 2 & \text{if } s(X) \equiv 2 \pmod{4} \\ 3 & \text{if } s(X) \equiv 3 \pmod{4}. \end{cases}$$

If $s = 4$ or 5 , then $h(X) \geq 12$.

For closed manifolds, this corollary is a slight strengthening of Furuta's theorem [19] using [10, 30, 27]. A proof of Furuta's theorem was outlined in [4]. There is but one new ingredient: The definition of a monopole map for manifolds with boundary. It turns out that the monopole map, set up this particular way, enjoys the same features as in the closed manifold case. The rest of the argument is identical to the one given in [4, 9.1].

Theorem 2.3. *Suppose the boundary Y of an oriented 4-manifold X is a disjoint union of rational homology spheres. Then the analytical index $h(X)$ is the sum*

$$h(X) = b^+(X) + b^0(Y)$$

where $b^+(X)$ is the dimension of a maximal linear subspace of $H^2(X, Y; \mathbb{R})$ on which the cup product is positive and $b^0(Y)$ is the number of connected components of Y .

The invariants enjoy an additivity property:

Proposition 2.4. *Suppose $X = X_1 \cup X_2$ and each of X , X_1 , X_2 and $X_1 \cap X_2$ are oriented 4-manifolds bounding rational homology spheres such*

that

$$X_1 \cap X_2 = I \times Y$$

is a (metric) product of an interval times a rational homology sphere. Then $s(X) = s(X_1) + s(X_2)$.

Assuming 2.1-2.4, let's prove the statements from the introduction.

Proof. (1.2 follows from 1.1) Let X be a closed and simply connected spin 4-manifold. Fix the orientation on X such that the signature is not positive. The intersection form on $H^2(X; \mathbb{Z})$ is indefinite by Donaldson's theorem [13]. By Rochlin's theorem [26], the signature of X is divisible by 16. So we set $\sigma(X) = -\text{signature}(X)/16$. According to the classification of unimodular bilinear forms over \mathbb{Z} (compare [29, p 60]), the intersection form splits into an orthogonal sum of hyperbolic planes $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the (negative definite) form E_8

$$Q(X) = b^+(X)U \perp 2\sigma(X)E_8.$$

The hyperbolic plane U is realized as an intersection form by the product $S^2 \times S^2$ of 2-spheres. The K3-surface is the smooth manifold underlying a generic quartic in complex projective 3-space, i. e. the zero-set of a generic homogenous polynomial of order 4. Its intersection form $Q(K3) = 3U \perp 2E_8$ realizes the lower bound in the inequality claimed in 1.1. Intersection forms are additive on connected sums

$$Q(X_0 \# X_1) = Q(X_0) \perp Q(X_1).$$

So if the intersection form of X satisfies the inequality claimed in 1.1, then X has the intersection form of a connected sum of σ copies of K3 and r copies of $S^2 \times S^2$ with

$$2r = b^2 - 22\sigma \geq 0.$$

□

Proof. (of 1.1) Let X be a simply connected and closed spin 4-manifold that does not satisfy the claimed inequality. We may assume the signature to be negative. We may further assume $X \# S^2 \times S^2$ to have the same

intersection form as the connected sum of σ copies of $K3$. Using a theorem of Freedman and Taylor [17], we can find a decomposition

$$X = X_1 \cup X_2 \cup \dots \cup X_\sigma$$

as a union of compact 4-dimensional sub-manifolds such that

- for $i < j$ the intersection $X_i \cap X_j$ is non-empty if and only if $j = i + 1$,
- for $i < \sigma$ the intersection $X_i \cap X_{i+1}$ is a product $I \times Y_i$ of an interval with a homology 3-sphere,
- the integral cohomology rings $H^*(X_1, Y_1)$ and $H^*(K3 \# S^2 \times S^2, D)$ are isomorphic, where D is an embedded 4-disc,
- for $1 < i < \sigma$ the rings $H^*(X_i, Y_{i-1} \sqcup Y_i)$ and $H^*(K3, D \sqcup D)$ are isomorphic,
- the sub-manifold X_σ has one boundary component and intersection form $Q(X_\sigma) = 2E_8 \perp U$.

So according to 2.3 we get

$$h(X_i) = \begin{cases} 5 & i < \sigma \\ 2 & i = \sigma \end{cases}$$

and thus by 2.2

$$s(X_i) \leq \begin{cases} 1 & i < \sigma \\ 0 & i = \sigma. \end{cases}$$

Using additivity 2.4 we end up with a strict inequality

$$s(X) = \sum_{i=1}^{\sigma} s(X_i) < \sigma$$

which contradicts the Atiyah-Singer index formula $s(X) = \sigma$ for the index of the Dirac operator. \square

3. HODGE THEORY AND THE SIGNATURE OPERATOR

An outline of Hodge theory for manifolds with boundary can be found in [8]. Here is a short review for which we will assume X to be a smooth, compact, connected, oriented Riemannian manifold of dimension n with

non-empty boundary Y . The co-differential $\delta : \Omega^p(X) \rightarrow \Omega^{p-1}(X)$ on the space of smooth p -forms is related to exterior differentiation via

$$\delta = (-1)^p *^{-1} d*$$

through the Hodge star operator $*$. The space $C^p \subset \Omega^p(X)$ of closed forms is the kernel of d . Because of $d \circ d = 0$, it contains the space of exact forms $E^p = d(\Omega^{p-1}) \subset C^p$. The space $cC^p \subset \Omega^p(X)$ of co-closed forms is the kernel of δ . It contains the space of co-exact forms $cE^p = \delta(\Omega^{p+1})$. Along the boundary, a smooth differential p -form ω decomposes into tangential and normal components

$$\omega(y) = \omega_{tan}(y) + \omega_{norm}(y).$$

At the point $y \in Y$, the form $\omega_{tan}(y)$ agrees with $\omega(y)$ when evaluated on tuples of vectors tangent to Y and vanishes if one of the vectors is orthogonal to Y . The tangential component uniquely determines the pullback $j^*\omega \in \Omega^p(Y)$ along the inclusion. Let N denote a vector field on X which restricts to the outward pointing unit normal vector field along the boundary Y and consider the $(p-1)$ -form $i_N\omega$ obtained by contraction with N . The normal component ω_{norm} of ω uniquely determines the pullback $j^*(i_N\omega) \in \Omega^{p-1}(Y)$. Let

$$\Omega_N^p = \{\omega_{norm} = j^*(i_N\omega) = 0\} \subset \Omega^p(X)$$

denote the space of smooth p -forms satisfying Neumann boundary conditions and let

$$\Omega_D^p = \{\omega_{tan} = j^*\omega = 0\} \subset \Omega^p(X)$$

denote the p -forms satisfying Dirichlet boundary conditions. The exterior differential d preserves Ω_D^* and, likewise, the co-differential δ preserves Ω_N^* . The space C^p of closed forms contains the subspace $C_D^p = C^p \cap \Omega_D^p$ of closed forms satisfying Dirichlet boundary conditions and similarly, we have the subspace $cC_N^p = cC^p \cap \Omega_N^p$ of forms satisfying Neumann boundary condition. Set

$$cE_N^p = \delta(\Omega_N^{p+1}) \quad \text{and} \quad E_D^p = d(\Omega_D^{p-1})$$

and note that boundary conditions are applied before differentiation.

Here is the Hodge Decomposition Theorem, as stated in [8]:

Theorem. (Hodge Decomposition¹) There are direct sum decompositions

$$\begin{aligned}\Omega^p(X) &= cE_N^p \perp (C^p \cap cC^p) \perp E_D^p \\ C^p \cap cC^p &= \begin{cases} (C^p \cap cE^p) \perp (C^p \cap cC^p)_D \\ (E^p \cap cC^p) \perp (C^p \cap cC^p)_N \end{cases}\end{aligned}$$

which are orthogonal with respect to the L^2 inner product.

As orthogonal complement to the co-exact p -forms within the co-closed ones, $(C^p \cap cC^p)_D$ is isomorphic to $H^p(X, Y; \mathbb{R})$. Similarly, $(C^p \cap cC^p)_N$ is isomorphic to $H^p(X; \mathbb{R})$. In case the boundary Y is non-empty, one gets as a consequence of the strong unique continuation theorem ([1, 2])

Proposition 3.1. (cf. [28, 8]) *A smooth differential form which is both closed and co-closed, and which vanishes on the boundary, must be identical zero, i. e.*

$$(C^p \cap cC^p)_D \cap (C^p \cap cC^p)_N = 0.$$

□

To consider the operator $d + \delta$, we use the notation $\Omega(X) = \bigoplus_p \Omega^p(X)$ and according notation for the subspaces defined above.

Corollary 3.2. *Let X be connected with non-empty boundary. Then the operator $d + \delta : \Omega(X) \rightarrow \Omega(X)$ is surjective.*

Proof. It suffices to verify $\Omega^p(X) = cE^p + E^p$ for all p . By the Hodge decomposition theorem, the L^2 -orthogonal complement to cE^p in $\Omega^p(X)$ is $(C^p \cap cC^p)_D \perp E_D^p$ and the orthogonal complement to E^p is $cE_N^p \perp (C^p \cap cC^p)_N$. The orthogonal complement to $cE^p + E^p$ thus is contained in the intersection

$$((C^p \cap cC^p)_D \perp E_D^p) \cap (cE_N^p \perp (C^p \cap cC^p)_N) = 0.$$

□

¹In increasing generality: [11, 20, 32, 21, 31, 33, 7, 23, 14, 15, 12, 18, 9, 25]

Corollary 3.3. *Suppose $\dim X = 4k$ and $Y \neq \emptyset$. The Hodge star operator acts as an involution on $\Omega^{2k}(X)$, leading to an L^2 -orthogonal decomposition*

$$\Omega^{2k}(X) = \Omega^{2k,+}(X) \perp \Omega^{2k,-}(X)$$

into ± 1 eigen-spaces. There are further L^2 -orthogonal decompositions

$$\Omega^{2k,\pm}(X) = d^\pm(\Omega_D^{2k-1}) \perp (E^{2k} \cap cE^{2k})^\pm \perp F^\pm$$

with $d^\pm(\Omega_D^{2k-1}) = \delta^\pm(\Omega_N^{2k+1})$ and F^\pm the ± 1 eigen-spaces of the involution $$ interchanging the two summands in the finite dimensional vector space*

$$F := (C^{2k} \cap cC^{2k})_D \oplus (C^{2k} \cap cC^{2k})_N \cong H^{2k}(X, Y; \mathbb{R}) \oplus H^{2k}(X; \mathbb{R}).$$

Proof. The Hodge operator interchanges the two summands cE_N^{2k} and E_D^{2k} , making the sum $*$ -invariant. The orthogonal complement $C^{2k} \cap cC^{2k}$ is $*$ -invariant and orthogonally decomposes into the $*$ -invariant subspaces F and $E^{2k} \cap cE^{2k}$. The claim is immediate. \square

Corollary 3.4. *Suppose $\dim X = 4k$ and $Y \neq \emptyset$. The L^2 -orthogonal projection $p = \frac{1}{2}(\text{id} + *) : \Omega^{2k}(X) \rightarrow \Omega^{2k,+}(X)$, restricted to exact forms, induces a short exact sequence*

$$0 \rightarrow (E^{2k} \cap cE^{2k})^- \rightarrow E^{2k} \rightarrow \Omega^{2k,+}(X) \rightarrow 0.$$

Proof. By Hodge decomposition, the exterior derivative d induces an isomorphism of cE_N^{2k-1} with E^{2k} . The latter space decomposes orthogonally

$$E^{2k} = E_D^{2k} \perp (E^{2k} \cap cE^{2k}) \perp (E^{2k} \cap F).$$

As $E_D^{2k} \cap *(E_D^{2k}) = E_D^{2k} \cap cE_N^{2k} = 0$, the projection $p : E_D^{2k} \rightarrow d^+(\Omega_D^{2k-1})$ is an isomorphism. The second summand is $*$ -invariant, hence decomposes into its ± 1 eigen-spaces. By construction,

$$(E^{2k} \cap F) \cap *(E^{2k} \cap F) = (E^{2k} \cap cE^{2k}) \cap F = 0.$$

Hence the projection $p : (E^{2k} \cap F) \rightarrow F^+$ is injective. It is surjective by dimension reasons. \square

In order to apply boundary conditions, we need to understand the restriction map to the boundary.

Proposition 3.5. *Pullback $j^* : \Omega(X) \rightarrow \Omega(Y)$ along the inclusion factors through an isomorphism*

$$(cE_N \cap d^{-1}(E \cap cC)) \perp (C \cap cE) \rightarrow \Omega(Y).$$

Proof. The map j^* is split surjective: Using a trivialization of a collar at the boundary, any form on Y can be trivially extended over such a collar. Multiplication with a bump function with support along the chosen collar results in a linear splitting $\Omega(Y) \rightarrow \Omega(X)$. The orthogonal decomposition $E = (E \cap cC) \perp E_D$ induces via the isomorphism $d : cE_N \rightarrow E$ a direct sum decomposition

$$\Omega(X) = (cE_N \cap d^{-1}(E \cap cC)) \oplus (C + \Omega_D).$$

Because of the orthogonal decomposition $C = (C \cap cE) \perp C_D$ and $C_D = C \cap \Omega_D$, we have a direct sum decomposition

$$C + \Omega_D = (C \cap cE) \oplus \Omega_D.$$

The claim now follows as $\Omega_D = \ker j^*$. □

The exact forms $E^{2k}(Y)$ on Y admit an L^2 -orthogonal decomposition into eigen-spaces of the self-adjoint operator $d^Y \circ *_Y$. To be precise, we need an orientation convention: Set $dvol_X = dt \wedge dvol_Y$, where dt denotes the dual of the outward pointing unit normal vector field along the boundary.

Proposition 3.6. *Suppose $\dim X = 4k$. The $*$ -invariant bilinear form on $C^{2k} \cap cC^{2k}$*

$$q : (a_0, a_1) \mapsto \int_X a_0 \wedge a_1$$

is symmetric. It is positive definite on $(C^{2k} \cap cC^{2k})^+$ and negative definite on $(C^{2k} \cap cC^{2k})^-$. The inverse $g : E^{2k}(Y) \rightarrow cE^{2k-1}(Y)$ to the exterior differential d^Y induces a bilinear form

$$(b_0, b_1) \mapsto \int_Y b_0 \wedge g(b_1)$$

which is symmetric and non-degenerate on $E^{2k}(Y)$. Pullback of forms along the inclusion of the boundary

$$j^* : E^{2k} \cap cC^{2k} \rightarrow E^{2k}(Y)$$

is an isometry with respect to these bilinear forms.

Proof. The first part is immediate from the defining equation $a_0 \wedge *a_1 = \langle a_0, a_1 \rangle d\text{vol}_X$ for the Hodge star operator. Given exact forms $b_i = d^Y \beta_i$, exterior differentiation gives

$$d^Y(\beta_0 \wedge \beta_1) = (d^Y \beta_0) \wedge \beta_1 - \beta_0 \wedge (d^Y \beta_1).$$

Together with Stokes' theorem this implies symmetry of the bilinear form on $E^{2k}(Y)$. To show non-degeneracy, consider an eigen-vector b of $d^Y \circ *_Y$ with eigen-value λ . Then

$$\int_Y b \wedge g(b) = \frac{1}{\lambda} \|b\|_{L^2}^2.$$

Suppose, $a_i = d\alpha_i$ and $j^*a_i = b_i$, then Stokes' theorem with the given orientation convention gives:

$$\int_X a_0 \wedge a_1 = \int_X a_0 \wedge d\alpha_1 = \int_X d(a_0 \wedge \alpha_1) = \int_Y b_0 \wedge g(b_1).$$

This shows that j^* is an isometry. Injectivity follows from Hodge decomposition. \square

We now turn to the case of a manifold X of dimension 4 bounding a 3-manifold Y . Let $cE^{1,\pm}(Y) \subset cE^1(Y)$ denote the closure in the C^∞ -topology of the union of eigen-spaces with positive and negative eigen-values, respectively, of the self-adjoint operator $*d$. The decomposition

$$cE^1(Y) = cE^{1,+}(Y) \perp cE^{1,-}(Y)$$

is L^2 -orthogonal.

Theorem 3.7. *Let X be a compact, connected, oriented Riemannian 4-manifold bounding a disjoint union of rational homology 3-spheres. Consider the linear operator*

$$d^+ + p : cE_N^1(X) \rightarrow \Omega^{2,+}(X) \oplus cE^{1,-}(Y),$$

where p is the pullback $j^* : cE_N^1(X) \rightarrow \Omega^1(Y)$, followed by L^2 -orthogonal projection $\Omega^1(Y) \rightarrow cE^{1,-}(Y)$. The operator $d^+ + p$ is injective and has co-kernel of dimension $b^+(X)$.

Proof. Because the cohomology of the boundary vanishes in dimensions 1 and 2, the intersection form on X is non-degenerate. The intersection form on de Rham cohomology is represented by the bilinear form q , restricted to the subspace $(C \cap cC)_N$. This form is by definition of type (b^+, b^-) . On the L^2 -orthogonal complement

$$(C \cap cC)_N^\perp \subset F$$

in the finite dimensional space $F \subset (C \cap cC)$, the bilinear form q has to be of type (b^-, b^+) as the signature of q on F is zero. According to the Hodge decomposition theorem, there is an L^2 -orthogonal decomposition

$$(E \cap cC) = (E \cap cE) \perp (C \cap cC)_N^\perp.$$

As the first summand is $*$ -invariant, this decomposition is also orthogonal with respect to the bilinear form q . In particular, the bilinear form q is non-degenerate on $(E \cap cC)$. Being an isometry (cf. 3.6), the pullback $j^* : E \cap cC \rightarrow E^2(Y)$ is injective. As pullback commutes with exterior derivation d , the map

$$p : cE_N^1 \cap d^{-1}(E \cap cC) \rightarrow cE^1(Y)$$

is injective as well. It is surjective because of 3.5 and the fact that $j^*(C^1 \cap cE^1) \subset C^1(Y)$ as j^* commutes with d . Note that the isomorphism

$$d : cE^1(Y) \rightarrow E^2(Y)$$

does respect the decomposition into positive and negative subspaces with respect to the self-adjoint operators $*d$ and $d*$, respectively. In this way, we have established that the pullback map

$$j^* : (E \cap cC) \rightarrow E^2(Y)$$

is an invertible isometry with respect to the bilinear forms specified in (3.6).

If we choose a maximal negative definite subspace of $A^- \subset (C \cap cC)_N^\perp$, then the closed subspace $(E \cap cE)^- \perp A^-$ is a maximal negative definite

subspace of $(E \cap cC)$ with respect to q . The composition of j^* with the projection of $E^2(Y) = E^{2,+}(Y) \oplus E^{2,-}(Y)$ onto the second factor thus is an isomorphism.

According to 3.4, the linear operator $d^+ : cE_N^1 \rightarrow \Omega^{2,+}(X)$ is surjective. So it suffices to consider the map

$$p : \ker d^+ = (E_N^1 \cap d^{-1}(E \cap cE)^-) \rightarrow cE^{1,-}(Y)$$

or, equivalently, the composition of $j^* : (E \cap cE)^- \rightarrow E^2(Y)$ with the projection onto $E^{2,-}(Y)$. This map is injective and has co-kernel of dimension $\dim A^- = b^+(X)$. \square

4. THE MONOPOLE MAP

The Seiberg-Witten equations [34] for a four-dimensional manifold X describe the zero-set of the monopole map. This non-linear map is between spaces of sections of certain vector bundles over X . In the case of a closed four-manifold, the topology of this map is known to carry subtle information on the differentiable structure (compare [6, 5, 4]). In this section a variant of the monopole map will be defined which is adapted to the situation at hand: The 4-dimensional spin manifold X shall be smooth, compact, connected, oriented and equipped with a Riemannian metric which is product along a collar at its boundary Y . The connected components of the boundary are assumed to be rational homology 3-spheres.

The spin structure on X is defined by a principal bundle $P \rightarrow X$ which is a $Spin(4)$ -reduction of the orthonormal frame bundle. The vector bundles we are interested in are associated to the principal bundle P and arise through representations of the group

$$Spin^c(4) \cong Sp(1)^+ \times U(1) \times Sp(1)^- / \pm 1,$$

restricted to the subgroup $Spin(4) \cong Sp(1)^+ \times \{1\} \times Sp(1)^-$. The representations $\rho, \sigma^\pm : Spin^c(4) \rightarrow Aut_{\mathbb{R}}(\mathbb{H})$ are defined by the formulae

$$\begin{aligned} \rho((g^+, u, g^-), h) &= g^+ \cdot h \cdot \overline{g^-} \\ \sigma^\pm((g^+, u, g^-), h) &= g^\pm \cdot h \cdot \bar{u} \end{aligned}$$

The representations $\lambda^\pm : Spin^c(4) \rightarrow Aut_{\mathbb{R}}(Im(\mathbb{H}))$ are defined via

$$\lambda^\pm((g^+, u, g^-), h) = g^\pm \cdot h \cdot \overline{g^\pm}.$$

The equivariant bilinear map $cl : \rho \otimes \sigma^+ \rightarrow \sigma^-, v \otimes h \mapsto \bar{v} \cdot h$ describes Clifford multiplication

$$T^*X \otimes S^+ = P \times_{\rho \otimes \sigma^+} \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \rightarrow P \times_{\sigma^-} \mathbb{H} = S^-.$$

Given 1-forms $a \in \Omega^1(X)$ and $\phi \in \Gamma(S^+)$, we will use the shorthand $ia\phi$ to stand for $cl(a \otimes \phi) \cdot \bar{i}$.

Conjugation $q : \sigma^+ \rightarrow \lambda^+, q(h) = h \cdot i \cdot \bar{h}$ defines the quadratic map

$$q : S^+ = P \times_{\sigma^+} \mathbb{H} \rightarrow P \times_{\lambda^+} Im(\mathbb{H}) = \Lambda^{2,+}T^*X$$

from the positive spinor bundles to the bundle of self-dual 2-forms.

The monopole map μ_X will depend on further data, not explicitly reflected in the notation: We fix base points $x \in X \setminus Y$ and $y_l \in Y_l$ in each connected component of the boundary. Furthermore, we choose for each component Y_l a smooth path γ_l from x to y_l . We also choose a monotone non-increasing smooth function $g : \mathbb{R} \rightarrow \mathbb{R}$ which is identical 1 for r near 0 and equals the function $r \mapsto r^{-1}$ for $r \geq 2$.

The source of the monopole map is the space of pairs

$$\mathcal{A}(X) = \Gamma(S^+) \times cE_N^1(X)$$

consisting of smooth spinors and co-exact 1-forms satisfying Neumann boundary conditions. The target of the monopole map has five factors

$$\mathcal{C}(X) = \Gamma(S^-) \times \Omega^{2,+}(X) \times H^0(Y; \mathbb{R}) \times \Gamma^-(S) \times cE^{1,-}(Y).$$

The first three factors consist of smooth negative spinors, self-dual 2-forms and locally constant functions on Y . The factor $cE^{1,-}(Y)$ was discussed in theorem 3.7. The remaining factor needs some explanation: The Levi-Civita connection on X uniquely defines connections on all vector bundles associated to the principal bundle P and thus, using Clifford multiplication, a Dirac operator $D : \Gamma(S^+) \rightarrow \Gamma(S^-)$. Along the collar the connection induces a product structure

$$S^+ \upharpoonright_{X_\varepsilon} \cong \text{pr}_Y^* S$$

on the spinor bundle with $S = S^+ \upharpoonright_Y$. Clifford multiplication with idt identifies the Dirac operator $D : \Gamma(S^+) \rightarrow \Gamma(S^-)$ along the collar with the operator $\frac{\partial}{\partial t} + D_Y$, where $D_Y : \Gamma(S) \rightarrow \Gamma(S)$ denotes the restricted Dirac operator on Y . The subspace $\Gamma^-(S) \subset \Gamma(S)$ is the closure in the \mathcal{C}^∞ -topology of the union of eigen-spaces of the self-adjoint operator D_Y with negative eigen-values.

The monopole map μ associates to a pair $(\phi, a) \in \mathcal{A}(X)$

$$\mu(\phi, a) = \left(D\phi + ia\phi, d^+a - g(|\phi|)q(\phi), \prod_l \int_{\gamma_l} a, p(\phi, a) \right)$$

The linear map p was defined in (3.7) on 1-forms. On spinors p is the restriction to Y , followed by L^2 -orthogonal projection $\Gamma(S) \rightarrow \Gamma^-(S)$.

A few explanations are due. The factor $g(|\phi|)$ is a purely technical device. In the estimates below it allows to lift the a priori estimate over the L^2_1 -threshold for the bootstrapping argument. In the case of a closed manifold, this ‘‘regularized’’ monopole map is homotopic to the usual monopole map and thus leads to the same stable cohomotopy invariants. Atiyah-Patodi-Singer boundary conditions on the linearized monopole map are incorporated through the projection map p . The integrals over the curves γ_l are needed to synchronize the residual $Pin(2)$ -action of the gauge group after dividing out the pointed gauge group at x with the residual $Pin(2)$ -actions of the gauge group on the individual boundary components after dividing out the gauge groups pointed at the respective base points y_l . Note that because of Hodge decomposition the space $\mathcal{A}(X)$ is a slice of the pointed gauge group action on $\Gamma(S^+) \times \Omega^1(X)$ in case the first Betti number of X vanishes.

5. THE BASIC ESTIMATE

This section contains the analytical heart of the present paper. The main result is an analogue to the boundedness property of the monopole map in the case of a manifold with boundary. According to [4, thm 2.1], theorem 2.1 is a consequence of the following boundedness result:

Proposition 5.1. *Suppose $k \gg 2$ and assume there are L^2_{k-1} -bounds on $D_a\phi := D\phi + ia\phi$ and $d^+a - Q(\phi) := d^+a - g(|\phi|)q(\phi)$ and furthermore*

$L^2_{k-\frac{1}{2}}$ -bounds on $p(\phi, a)$ and bounds on the absolute value of $\int_{\gamma_l} a$ for every l . Then there are L^2_k -bounds on ϕ and a .

Proof. The Weitzenböck formula for the Dirac operator D_a reads

$$D_a^* D_a = \nabla_a^* \nabla_a + \frac{scal}{4} + \frac{i}{2} d^+ a,$$

with $scal$ denoting the scalar curvature of X . The form

$$\left(\langle D_a^* D_a \phi, \phi \rangle - |D_a \phi|^2 \right) * 1 - \left(\langle \nabla_a^* \nabla_a \phi, \phi \rangle - |\nabla_a \phi|^2 \right) * 1$$

is an exterior differential $d\omega$ for a suitably chosen 3-form $\omega = \omega(\phi, a)$. Here we denote by $\langle \cdot, \cdot \rangle$ the real part of a Hermitian product. Combining these equations, we obtain:

$$\frac{1}{2} \langle i d^+ a \cdot \phi, \phi \rangle = |D_a \phi|^2 - |\nabla_a \phi|^2 - \frac{scal}{4} |\phi|^2 + *d\omega.$$

We replace the term $d^+ a$ by $Q(\phi) + (d^+ a - Q(\phi))$ to get

$$\frac{1}{2} g(|\phi|) |\phi|^4 = |D_a \phi|^2 - |\nabla_a \phi|^2 - \frac{1}{2} \left(\langle i (d^+ a - Q(\phi)) \phi, \phi \rangle + \frac{scal}{2} |\phi|^2 \right) + *d\omega.$$

As we will see, this formula leads to an a priori estimate on ϕ as the left hand side grows with order 3 in ϕ , whereas the right hand side grows with order 2. This will be made precise momentarily. To keep notation under control in the estimates to follow, let's adopt the convention to write $x \leq O(y^n)$ if an inequality

$$x \leq c_n y^n + \dots + c_1 y + c_0$$

holds for some constants c_0, \dots, c_n .

The assumed L^2_{k-1} -bounds imply via the Sobolev embedding theorems point-wise estimates on $D_a \phi$ and $d^+ a - Q(\phi)$ and hence a point-wise estimate

$$g(|\phi|) |\phi|^4 \leq O(|\phi|^2) + 2 * d\omega.$$

Integration over X gives

$$\begin{aligned} \|\phi\|_{L^3}^3 &\leq \|\phi\|_{L^3(X_2)}^3 + 8Vol(X) \leq \|g(|\phi|)|\phi|^4\|_{L^1} + 8Vol(X) \\ &\leq O(\|\phi\|_{L^3}^2) + 2 \int_X d\omega, \end{aligned}$$

with $X_2 = \{x \in X \mid |\phi| \geq 2\}$. Stokes' theorem converts the last term into a boundary integral of the form ω . This boundary integral is computed in 5.2:

$$\int_X d\omega = \int_Y \omega \leq O\left(\|p(\phi)\|_{L^2_{\frac{1}{2}}}\right).$$

Here we used that for $\phi \in \Gamma^\pm(S)$ the square root of the integral

$$\pm \int_Y \langle D_Y \phi, \phi \rangle * 1$$

is a norm equivalent to the $L^2_{\frac{1}{2}}$ -norm. As this term is bounded by assumption, we get an estimate

$$\|\phi\|_{L^3}^3 \leq O(\|\phi\|_{L^3}^2)$$

and thus an a-priori bound for $\|\phi\|_{L^3}$. Using Sobolev and elliptic estimates, we obtain

$$\begin{aligned} \|a\|_{L^{12}} &\leq O\left(\|a\|_{L^3_1}\right) \leq O\left(\|d^+ a\|_{L^3} + \|p(a)\|_{L^3_{2/3}}\right) \\ &\leq O\left(\|d^+ a - Q(\phi)\|_{L^3} + \|Q(\phi)\|_{L^3}\right) \\ &\leq O(\|\phi\|_{L^3}). \end{aligned}$$

Given these a-priori estimates on (ϕ, a) , we obtain the claimed estimates via standard bootstrapping. \square

Proposition 5.2. *Let X be a Riemannian spin 4-manifold which is a product in a collar at its boundary Y and let D_a denote the Dirac operator twisted by a 1-form $ia \in \Omega^1_N(X; i\mathbb{R})$ satisfying Neumann boundary conditions. Then the following equality holds:*

$$\int_X (\langle D_a^* D_a \phi, \phi \rangle - |D_a \phi|^2 - \langle \nabla_a^* \nabla_a \phi, \phi \rangle + |\nabla_a \phi|^2) * 1 = - \int_Y \langle D_Y \phi, \phi \rangle * 1.$$

Proof. Let ξ be the vector field given by $\beta(\xi) = \langle D_a\phi, \beta \cdot \phi \rangle$ for $\beta \in \Omega^1(X)$. The equation (cf [24, p 115])

$$\langle D_a^* D_a \phi, \phi \rangle - |D_a \phi|^2 = -\operatorname{div}(\xi).$$

describes how D_a^* fails to be adjoint. The covariant derivative satisfies

$$\langle \nabla_a^* \nabla_a \phi, \phi \rangle - |\nabla_a \phi|^2 = \frac{1}{2} \Delta |\phi|^2.$$

Along the collar, D is of the form $\tau \left(\frac{\partial}{\partial t} + D_Y \right)$, where τ is multiplication with idt . Hence the relevant term $\operatorname{div}(\xi) + \frac{1}{2} \Delta |\phi|^2$ takes the form

$$\begin{aligned} \frac{\partial}{\partial t} \langle \tau \left(\frac{\partial}{\partial t} + D_Y \right) \phi + ia\phi, \tau(\phi) \rangle + \operatorname{div}_Y(\xi_Y) - \frac{1}{2} \left(\frac{\partial}{\partial t} \right)^2 |\phi|^2 + \frac{1}{2} \Delta_Y |\phi|^2 \\ = \frac{\partial}{\partial t} (\langle D_Y \phi, \phi \rangle + \langle ia\phi, \tau(\phi) \rangle) + \operatorname{div}_Y(\xi_Y) + \frac{1}{2} \Delta_Y |\phi|^2. \end{aligned}$$

According to Stokes' theorem, the integral in question then is of the form

$$\left(\int_Y \langle D_Y \phi, \phi \rangle * 1 \right) + \left(\int_Y \langle ia\phi, \tau(\phi) \rangle * 1 \right).$$

Now the Neumann boundary condition comes into play: The restriction of a to Y is orthogonal to dt . So the second summand vanishes. \square

6. CONCLUDING ARGUMENTS

Proof. (of 2.3) The claim is immediate from the construction of the monopole map in section 4, combined with theorem 3.7. \square

Proof. (of 2.4) This follows from the Atiyah-Patodi-Singer index theorem [3] as the boundary contributions cancel. Alternatively, one can use the snake lemma for the Dirac operator as map from a short exact sequence

$$0 \rightarrow \Gamma(X_1 \cup X_2; S^+) \rightarrow \Gamma\left(X_1 \amalg X_2; S^+\right) \rightarrow \Gamma(X_1 \cap X_2; S^+) \rightarrow 0$$

to the analogous sequence with negative spinors plus boundary terms, together with the fact that the index vanishes on tubes $[0, 1] \times Y$. \square

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