THE BRAIDED THOMPSON'S GROUPS ARE OF TYPE F_∞

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Abstract. We prove that the braided Thompson's groups V_{br} and F_{br} are of type F_∞. We do this by considering actions of the groups on certain cube complexes. The key step is proving higher connectivity properties of descending links with respect to a height function. To prove this, we inspect some analogs for matching complexes of graphs, induced on surfaces.

Recall that a group is said to be of *type* F_n if it admits a classifying space with finitely many cells in dimensions up to n. For example, a group is of type F_1 if and only if it is finitely generated, and of type F_2 if and only if it is finitely presented. If a group is of type F_n for all $n \geq 0$ then it is of *type* F_∞.

Some prominent examples of groups of type F_∞ include the well-studied Thompson's groups F, T and V, and the braid groups B_n. A braided variant of Thompson's group V, which we will denote V_{br}, was introduced independently by Brin and Dehornoy [Bri07, Deh06]. This group contains F as a subgroup, along with copies of the braid group B_n for each $n \in \mathbb{N}$, and was shown in [Bri07] to be finitely presented. In [BBCS08], Brady, Burillo, Cleary and Stein introduced another braided Thompson's group, which we denote F_{br}, and which encodes the pure braid groups PB_n in a similar way to how V_{br} encodes B_n. They also proved that F_{br} is finitely presented. The notation used in [Bri07, BBCS08] is BV and BF. As explained in [Bri06], the relationship between V and V_{br} is in many ways reminiscent of the relationship between a Coxeter group and its corresponding Artin group. For example, there is a presentation of V that can be converted to a presentation for V_{br} by dropping the relations that the generators are involutions.

In this paper we prove that the braided Thompson's groups are of type F_∞. In the case of V_{br}, this was conjectured by John Meier already in 2001; see also [FK08, Remark 5.1 (1)] and [FK11, Remark 3.3].

Main Theorem. *The braided Thompson's groups V_{br} and F_{br} are of type F_∞.*

This result can be viewed as part of a general attempt to try and understand how the finiteness properties of a group change when the group is modified in some way, for example by braiding. Another instance of this question concerns the *braided Houghton groups BH_n*. In [Deg00] these are conjectured to be of type F_{n-1} but not of type F_n, and this is proved for $n \leq 3$; see also [Fun07]. A possible approach to proving the complete conjecture is described in [Bux03]. There are also the *braided Ptolemy-Thompson groups $T^\sharp$* and T^* which are shown to be finitely presented in [FK08]. In [FK11] the group T^* is shown to be asynchronously combable, and hence of type F_3 (and conjectured to be of type F_∞).

In the present work, crucial to our analysis of certain descending links is the new notion of a *matching complex on a surface*. This is a melding of the two notions of an arc complex on a surface, and a matching complex on a graph. These complexes

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are of independent topological and combinatorial interest, for example in [Zar13] they are shown to yield examples of highly generating families of subgroups, in the sense of Abels and Holz [AH93], for the pure braid group.

In Section 1 we recall the definitions of V_{br} and F_{br}, using the language from [BBCS08]. We also introduce "spraiges", or "split-braid-merge diagrams", along with the important notion of "dangling". Then in Section 2 we construct a space X on which V_{br} acts, which we call the *Stein space* for V_{br}, and an analogous space for F_{br}. In Section 4 we show that these actions are sufficiently nice as to satisfy the conditions of a well known criterion due to Brown for a group being of type F_∞. Our proof relies on some complexes that we call *matching complexes on surfaces*, which are interesting in their own right; see Section 3 and Corollary 4.4. Finally we prove the main theorem in Section 5.

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1. The braided Thompson's groups

Thompson's groups F and V have been studied at length, and possess many unusual and interesting properties. For example, F is a torsion-free group of type F_∞ with infinite cohomological dimension, and V is an infinite simple group of type F_∞. An introduction to F and V can be found in [CFP96], in which the *paired tree diagrams* approach to the groups is discussed. This is the approach that we will take here as well. We will follow the definitions given in [BBCS08], where in addition the braided Thompson's groups V_{br} and F_{br} are defined in terms of *braided paired tree diagrams*. We will also, less formally, picture elements of these groups in the language of *strand diagrams*, as in [BM07].

1.1. **The group.** We first recall the definition of V. By a *rooted binary tree* we mean a finite tree such that every vertex has degree 3, except the *leaves*, which have degree 1, and the *root*, which has degree 2 (or degree 1 if the root is also a leaf). Usually we draw such trees with the root at the top and the nodes descending from it, down to the leaves. A non-leaf node together with the two nodes directly below it is called a *caret*. If the leaves of a caret in T are leaves of T, we will call the caret *elementary*. Note that a (rooted binary) tree always consists of $(n-1)$ carets and has n leaves, for some $n \in \mathbb{N}$.

By a *paired tree diagram* we mean a triple (T_-, ρ, T_+) consisting of two rooted binary trees T_- and T_+ with the same number of leaves n, and a permutation $\rho \in S_n$. The leaves of T_- are labeled $1, \dots, n$ from left to right, and for each i, the $\rho(i)^{\mathrm{th}}$ leaf of T_+ is labeled i. There is an equivalence relation on paired tree diagrams given by reductions and expansions. By a *reduction* we mean the following: Suppose there is an elementary caret in T_- with left leaf labeled i and right leaf labeled $i+1$, and an elementary caret in T_+ with left leaf labeled i and right leaf labeled $i+1$. Then we can "reduce" the diagram by removing those carets, renumbering the leaves and replacing ρ with the permutation $\rho' \in S_{n-1}$ that sends the new leaf of T_- to the new leaf of T_+, and otherwise behaves like ρ. The resulting paired tree diagram (T'_-, ρ', T'_+) is then said to be obtained by *reducing* (T_-, ρ, T_+). The reverse operation to reduction is called *expansion*, so (T_-, ρ, T_+) is an expansion of (T'_-, ρ', T'_+). A paired tree diagram is called *reduced* if there is no reduction possible. Thus an equivalence class of paired tree diagrams consists of all diagrams having

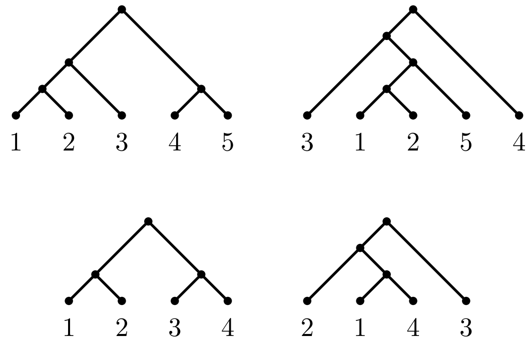

FIGURE 1. Reduction, of the top paired tree diagram to the bottom one.

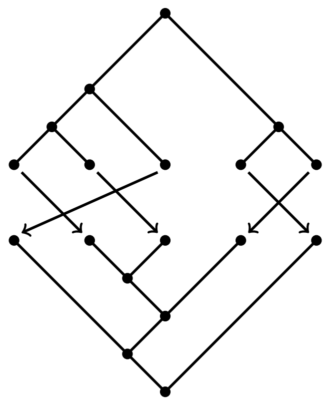

FIGURE 2. An element of V.

a common reduced representative. Such reduced representatives are unique. See Figure 1 for an idea of reduction of paired tree diagrams.

There is a binary operation $*$ on the set of equivalence classes of paired tree diagrams. Let $T = (T_-, \rho, T_+)$ and $S = (S_-, \xi, S_+)$ be reduced paired tree diagrams. By applying repeated expansions to T and S we can find representatives (T'_-, ρ', T'_+) and (S'_-, ξ', S'_+) of the equivalence classes of T and S, respectively, such that $T'_+ = S'_-$. Then we declare $T * S$ to be $(T'_-, \rho'\xi', S'_+)$. This operation is well defined on the equivalence classes, and is a group operation [Bri07, CFP96].

Definition 1.1. Thompson's group V is the group of equivalence classes of paired tree diagrams with the multiplication $*$. Thompson's group F is the subgroup of V consisting of elements where the permutation is the identity.

To deal with braided Thompson's groups it will become convenient to have the following picture in mind for paired tree diagrams. Think of the tree T_+ drawn beneath T_- and upside down, i.e., with the root at the bottom and the leaves at the top. The permutation ρ is then indicated by arrows pointing from the leaves of T_- to the corresponding paired leaf of T_+. See Figure 2 for this visualization of (the unreduced representation of) the element of V in Figure 1.

In the braided version V_{br} of V, the permutations of leaves are replaced by braids between the leaves. Again following [BBCS08] we will first introduce *braided paired tree diagrams* and then copy the construction of V given above to define V_{br}. Then we will mention how things change for F_{br}.

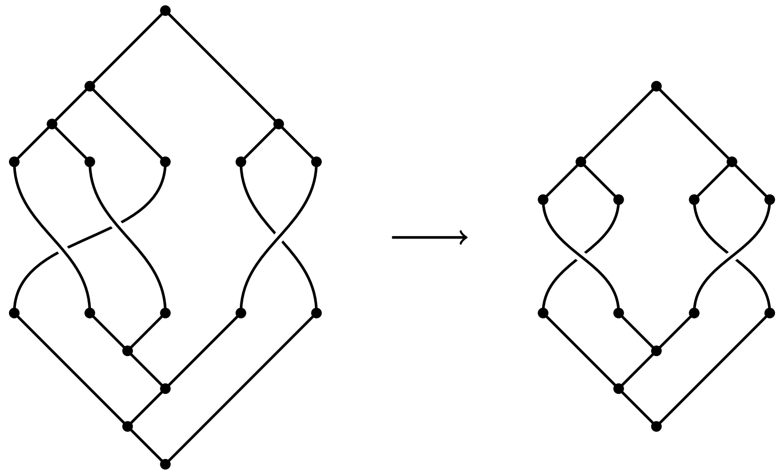

FIGURE 3. Reduction of braided paired tree diagrams.

Definition 1.2. A *braided paired tree diagram* is a triple (T_-, b, T_+) consisting of two rooted binary trees T_- and T_+ with the same number of leaves n and a braid $b \in B_n$.

We draw braided paired tree diagrams with T_+ upside down and below T_-, and the strands of the braid connecting leaves. This is analogous to the visualization of paired tree diagrams in Figure 2, and examples of braided paired tree diagrams can be seen in Figure 3.

As with V, we can define an equivalence relation on the set of braided paired tree diagrams using the notions of reduction and expansion. It is easier to first define expansion and then take reduction as the reverse of expansion. Let $\rho_b \in S_n$ denote the permutation corresponding to the braid $b \in B_n$. Let (T_-, b, T_+) be a braided paired tree diagram. Label the leaves of T_- from 1 to n, left to right, and for each i label the $\rho_b(i)^{\text{th}}$ leaf of T_+ by i. By the i^{th} strand of the braid we will always mean the strand that begins at the i^{th} leaf of T_-, i.e., we count the strands from the top. An *expansion* of (T_-, b, T_+) amounts to the following. For $1 \le i \le n$, replace $T_\pm$ with trees $T'_\pm$ obtained from $T_\pm$ by adding a caret to the leaf labeled i. Then replace b with a braid $b' \in B_{n+1}$, obtained by "doubling" the i^{th} strand of b. The triple (T'_-, b', T'_+) is an *expansion* of (T_-, b, T_+). As with paired tree diagrams, *reduction* is the reverse of expansion, so (T_-, b, T_+) is a reduction of (T'_-, b', T'_+). See Figure 3 for an idea of reduction of braided paired tree diagrams.

Two braided paired tree diagrams are equivalent if one is obtained from the other by a sequence of reductions or expansions. The multiplication operation $*$ on the equivalence classes is defined the same way as for V. It is a well defined group operation [Bri07].

Definition 1.3. The braided Thompson's group V_{br} is the group of equivalence classes of braided paired tree diagrams with the multiplication $*$.

A convenient way to visualize multiplication in V_{br} is via "stacking" braided paired tree diagrams. For $g, h \in V_{\text{br}}$, each pictured as a tree-braid-tree as before, $g*h$ is obtained by attaching the top of h to the bottom of g and then reducing the picture via certain moves. We indicate four of these moves in Figure 4. A "split" followed immediately by a "merge," or a merge followed immediately by a split, is equivalent to doing nothing, as seen in the top two pictures. Also, splits and merges interact with braids in ways indicated by the bottom two pictures. We leave it to the reader to further inspect the details of this visualization of multiplication in V_{br}. This is closely related to the *strand diagram* model for Thompson's groups in [BM07]. See also Section 1.2 in [Bri07] and Figure 2 of [BC09].

From now on we will just refer to the braided paired tree diagrams as being the elements of V_{br}, though one should keep in mind that the elements are actually equivalence classes under the reduction and expansion operations.

We can also define F_{br} as a subgroup of V_{br}. Recall that a braid b is called *pure* if $\rho_b = \mathrm{id}$. The elements of F_{br} are the (equivalence classes of) diagrams where the braid is pure. The fact that V_{br} and F_{br} are finitely presented has been known for some time, and explicit finite presentations are given in [Bri07] and [BBCS08]. Our current goal is to inspect their higher finiteness properties, though first we will need a little bit more language. We now introduce a class of diagrams that will be used throughout the rest of this paper.

1.2. **A general class of diagrams.** To define the spaces we will use, we need a broader class of diagrams that generalizes braided paired tree diagrams, namely we will consider forests instead of trees. We will also continue to informally use the notion of strand diagrams. Here a forest will always mean a union of binary rooted trees. Given a braided paired tree diagram (T_-, b, T_+) we call a caret in T_- a *split*. Similarly a *merge* is a caret in T_+. With this terminology, we can call the picture representing the braided paired tree diagram a *split-braid-merge diagram*, abbreviated *spraige*. That is, we first picture one strand splitting up into n strands in a certain way, representing T_-. Then the n strands braid with each other, representing b, and finally according to T_+ we merge the strands back together. These special kind of diagrams will also be called $(1,1)$-*spraiges*. More generally:

Definition 1.4. An (n,m)-*spraige* is a spraige that begins on n strands, the *heads*, and ends on m strands, the *feet*. As indicated above we can equivalently think of an (n,m)-spraige as a *braided paired forest diagram* (F_-, b, F_+), where F_- has n roots, F_+ has m roots and both have the same number of leaves. By an n-*spraige* we mean an (n,m)-spraige for some m, and by a *spraige* we mean an (n,m)-spraige for some n and m. Let $\mathscr{S}$ denote the set of all spraiges, $\mathscr{S}_{n,m}$ the set of all (n,m)-spraiges, and $\mathscr{S}_n$ the set of all n-spraiges.

Note that an n-spraige has n heads, but can have any number of feet. A function that will be important in what follows is the "number of feet" function, which we define as $f\colon \mathscr{S} \to \mathbb{N}$ given by $f(\sigma) = m$ if $\sigma \in \mathscr{S}_{n,m}$ for some n.

The pictures in Figure 5 are examples of spraiges. One can generalize the notion of reduction and expansion of such diagrams to arbitrary spraiges, and consider equivalence classes under reduction and expansion. Each such class has a unique reduced representative, as was the case for paired tree diagrams and braided paired

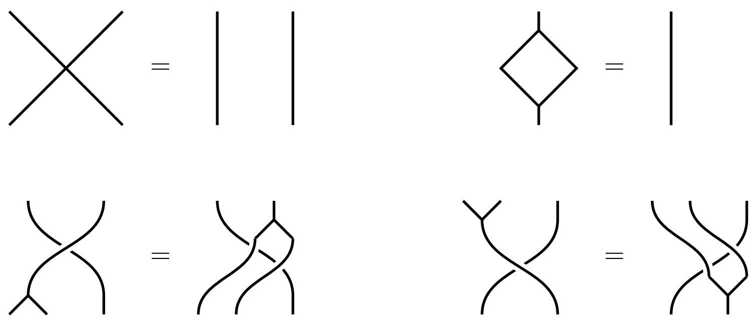

FIGURE 4. Moves to reduce braided paired tree diagrams after stacking.

tree diagrams. We will just call an equivalence class of spraiges a spraige, so in particular the elements of V_{br} are $(1,1)$-spraiges.

The operation $*$ defined for V_{br} can be defined in general for spraiges, via concatenation of diagrams. It is only defined for certain pairs of spraiges, namely we can multiply $\sigma_1 * \sigma_2$ for $\sigma_1 \in \mathscr{S}_{n_1,m_1}$ and $\sigma_2 \in \mathscr{S}_{n_2,m_2}$ if and only if $m_1 = n_2$. In this case we obtain $\sigma_1 * \sigma_2 \in \mathscr{S}_{n_1,m_2}$. The reader may find it helpful to work out why the multiplication in Figure 5 holds. We remark that, in an effort to make the pictures "balanced", for a given forest in a spraige we will usually try to draw all the roots in a row and all the leaves in a row. This becomes impossible if there is a node that is both a root and a leaf, so in this case we will cheat and draw two dots connected by an edge (this can already be seen in the top forest of the first spraige in Figure 5).

Remark 1.5. (i) For every $n \in \mathbb{N}$ there is an identity (n,n)-spraige 1_n with respect to $*$, namely the spraige represented by $(1_n, \mathrm{id}, 1_n)$. Here, by abuse of notation, 1_n also denotes the trivial forest with n roots.

(ii) For every (n,m)-spraige (F_-, b, F_+) there exists an inverse (m,n)-spraige (F_+, b^{-1}, F_-), in the sense that

$$(F_-, b, F_+) * (F_+, b^{-1}, F_-) = 1_n$$

and

$$(F_+, b^{-1}, F_-) * (F_-, b, F_+) = 1_m .$$

(iii) $\mathscr{S}$ is a groupoid.

There are certain forests that will be fundamental to the construction of the Stein space X in Section 2. For $n \in \mathbb{N}$ and $J \subseteq \{1, \dots, n\}$, define $F_J^{(n)}$ to be the forest with n roots and $|J|$ carets, with a caret attached to the i^{th} root for each $i \in J$. These forests are characterized by the property that every caret is elementary, and we will call any such forest *elementary.* Define the spraige $\lambda_J^{(n)}$ to be the $(n, n+|J|)$-spraige $(F_J^{(n)}, \mathrm{id}, 1_{n+|J|})$, and the spraige $\mu_J^{(n)}$ to be its inverse. If $J = \{i\}$ write $F_i^{(n)}$, $\lambda_i^{(n)}$ and $\mu_i^{(n)}$ instead. See Figure 6 for an example of an elementary forest and the corresponding spraiges.

Fix an (n,m)-spraige σ. For any forest F with m roots and l leaves define the *splitting of σ by F* as multiplying σ by the spraige $(F, \mathrm{id}, 1_l)$ from the right. Similarly a *merging of σ by F'* is right multiplication by the spraige $(1_m, \mathrm{id}, F')$, where F' now has l roots and m leaves. In the case where F (respectively F') is an elementary forest, we call this operation *elementary splitting* (respectively *elementary merging).* See Figure 7 for an idea of splitting and Figure 8 for an idea of elementary merging.

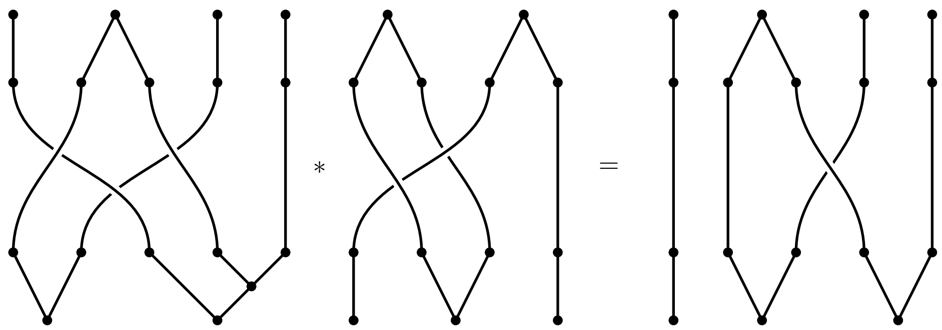

FIGURE 5. Multiplication of spraiges.

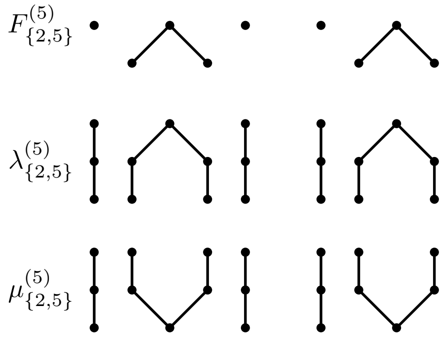

FIGURE 6. The elementary forest $F^{(5)}_{\{2,5\}}$, and the spraiges $\lambda^{(5)}_{\{2,5\}}$ and $\mu^{(5)}_{\{2,5\}}$.

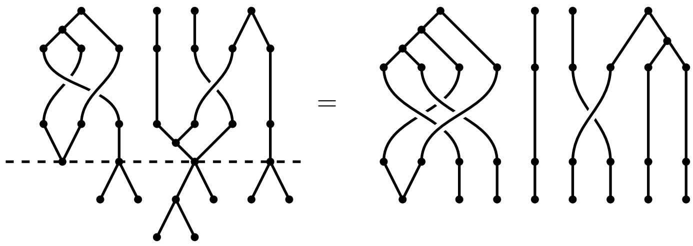

FIGURE 7. A splitting of a spraige.

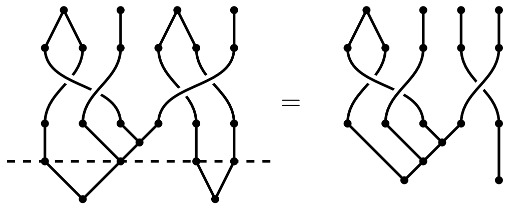

FIGURE 8. An elementary merging of a spraige.

In the special case that $F = F_i^{(n)}$ for $i \in \{1, \dots, n\}$ we can think of a splitting by F as simply attaching a single caret to the i^{th} foot of a spraige, possibly followed by reductions. Similarly a merging by F in this case can be thought of as merging the i^{th} and $(i+1)^{\text{st}}$ feet together. In these cases we will also speak of *adding a split* (respectively *merge*) to the spraige.

The following types of spraiges will prove to be particularly important. First, a *braige* is defined to be a spraige where there are no splits, i.e., a spraige of the form $(1_n, b, F)$ for $b \in B_n$ and F having n leaves. Also, when F is elementary we will call $(1_n, b, F)$ an *elementary braige*. Analogously to spraiges, we define *n-braiges* and *elementary n-braiges*.

To deal with F_{br}, we make the following convention. Whenever we want to only consider pure braids, we will attach the modifier "pure", e.g., we can talk about pure n-spraiges, or elementary pure n-braiges.

1.3. **Dangling spraiges.** We can identify the braid group B_n with a subgroup of $\mathscr{S}_{n,n}$ via $b \mapsto (1_n, b, 1_n)$. In particular for any $n, m \in \mathbb{N}$ there is a right action of the braid group B_m on $\mathscr{S}_{n,m}$, by right multiplication. This action encodes the idea that the feet of a spraige may "dangle". See Figure 9 for an example of the dangling action of B_2 on $\mathscr{S}_{4,2}$.

For $\sigma \in \mathscr{S}_{n,m}$, denote by $[\sigma]$ the orbit of σ under this action, and call $[\sigma]$ a *dangling* (n,m)*-spraige*. We can also refer to a dangling n-spraige or dangling spraige. The action of B_m preserves the property of being a braige or elementary braige, so we can also refer to dangling braiges and dangling elementary braiges.

Let $\mathcal{P}$ denote the set of all dangling spraiges, with $\mathcal{P}_{n,m}$ and $\mathcal{P}_n$ defined in the obvious way. When $m = 1$, B_1 is trivial, so we will identify $\mathcal{P}_{n,1}$ with $\mathscr{S}_{n,1}$ for each n. In particular we identify $\mathcal{P}_{1,1}$ with V_{br}. Note that if $\sigma \in \mathscr{S}_{n,m}$ and $\tau_1, \tau_2 \in \mathscr{S}_{m,\ell}$ with $[\sigma * \tau_1] = [\sigma * \tau_2]$, then $[\tau_1] = [\tau_2]$. We will refer to this fact as *left cancellation*. We remark that while $\mathscr{S}$ has right cancellation, $\mathcal{P}$ does not.

There is also a poset structure on $\mathcal{P}$. For $x, y \in \mathcal{P}$, with $x = [\sigma_x]$, say that $x \le y$ if there exists a forest F with m leaves such that $y = [\sigma_x * (F, \mathrm{id}, 1_m)]$. In other words, $x \le y$ if y is obtained from x via splitting. It is easy to see that this is a partial ordering. Also, if $x \in \mathcal{P}_n$ and $y \in \mathcal{P}$ with $x \le y$ or $y \le x$, then $y \in \mathcal{P}_n$. In other words, two elements are comparable only if they have the same number of heads. Also define a relation $\preceq$ on $\mathcal{P}$ as follows. If $x = [\sigma_x] \in \mathcal{P}$ and $y \in \mathcal{P}$ such that $y = [\sigma_x * \lambda_J^{(n)}]$ for some $n \in \mathbb{N}$ and $J \subseteq \{1, \dots, n\}$, write $x \preceq y$. That is, $x \preceq y$ if y is obtained from x via elementary splitting, and this is a well defined relation with respect to dangling. If $x \preceq y$ and $x \neq y$ then write $x \prec y$. Note that $\preceq$ and $\prec$ are not transitive, though it is true that if $x \preceq z$ and $x \le y \le z$ then $x \preceq y$ and $y \preceq z$. This is all somewhat similar to the corresponding situation for F and V discussed for example in Section 4 of [Bro92].

We remark that a totally analogous construction yields the notion of a *dangling pure spraige*, where the dangling is now via the action of the pure braid group. We also have dangling pure braiges and dangling elementary pure braiges. All of the essential results above still hold.

2. The Stein space

In this section we construct a space X on which V_{br} acts, which we call the *Stein space* for V_{br}. This construction can also be reproduced using pure braids to get a space $X(F_{\mathrm{br}})$ on which F_{br} acts; we will say more about this at the end of the section. Similar spaces, which could be termed the Stein spaces for F and V, were constructed and discussed in [Bro92, Bro06, Ste92]. Also, a Stein space was used in [FSWZ12] to show that the higher dimensional versions sV of V are of type F_∞. In the course of defining our space X, it should be clear how the Stein spaces for F and V would be described with our model. This construction was given in some generality in [Ste92], and was further generalized in [Far03] to get finiteness properties (among other things) for a class of groups called diagram groups, of which F is an example.

Our starting point is the poset $\mathcal{P}_1$ of dangling 1-spraiges, i.e., dangling spraiges with a single head. As with any poset, we have the following terminology. If $x \le z$

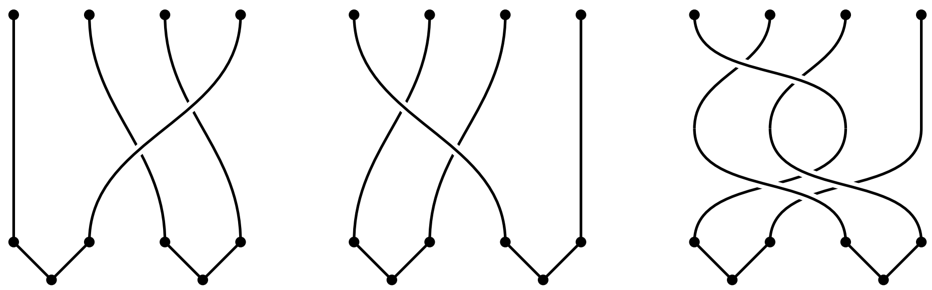

Figure 9. Dangling.

and $y \le z$ call z an *upper bound* of x and y. The minimal elements of the set of upper bounds of x and y are called *minimal upper bounds*. If x and y have a unique minimal upper bound z call z the *least upper bound* of x and y. Similarly define *lower bounds*, *maximal lower bounds* and *greatest lower bounds*.

We remark that while we construct X starting with $\mathcal{P}_1$, the following results remain essentially unchanged if we start instead with $\mathcal{P}_n$ for some other n. Since we want V_{br} to act on X though, $\mathcal{P}_1$ is the right place to start.

Proposition 2.1. *Let $x, y \in \mathcal{P}_1$. Then x and y have a least upper bound. Also, if x and y have a lower bound then they have a greatest lower bound.*

Proof. We first claim that x and y have an upper bound. Represent x by the spraige $\sigma_x = (T, b, F)$, where T is a tree with n leaves, $b \in B_n$ and F is a forest with k roots and n leaves, so x is a dangling $(1, k)$-spraige. Represent y by $\sigma_y = (U, c, G)$, where U is a tree with m leaves, $c \in B_m$, and G is a forest with ℓ roots and m leaves, so y is a dangling $(1, \ell)$-spraige. Now, $[\sigma_x * (F, \mathrm{id}, 1_n)] = [(T, \mathrm{id}, 1_n)]$, so $x \le [(T, \mathrm{id}, 1_n)]$. Similarly $y \le [(U, \mathrm{id}, 1_m)]$. Since T and U are both trees, $[(T, \mathrm{id}, 1_n)]$ and $[(U, \mathrm{id}, 1_m)]$ have an upper bound, and hence so do x and y.

We now claim that there is even a least upper bound. Again take $\sigma_x = (T, b, F)$ and $\sigma_y = (U, c, G)$, and suppose z and w are both minimal upper bounds of $x = [\sigma_x]$ and $y = [\sigma_y]$. Then there is an $(f(x), f(y))$-spraige (H_-, d, H_+) such that $[\sigma_x * (H_-, \mathrm{id}, 1_p)] = z$ and $[\sigma_x * (H_-, d, H_+)] = y$, and there is an $(f(x), f(y))$-spraige (I_-, e, I_+) such that $[\sigma_x * (I_-, \mathrm{id}, 1_q)] = w$ and $[\sigma_x * (I_-, e, I_+)] = y$. Here H_- has p leaves and I_- has q leaves. In particular $[\sigma_x * (H_-, d, H_+)] = [\sigma_x * (I_-, e, I_+)]$, which by left cancellation tells us that $[(H_-, d, H_+)] = [(I_-, e, I_+)]$. Moreover, since z and w are minimal upper bounds of x and y, the spraiges (H_-, d, H_+) and (I_-, e, I_+) are reduced. By uniqueness of reduced representatives, we must have in particular that $H_- = I_-$, and so $z = w$.

Finally suppose x and y have maximal lower bounds z and w. We claim that $w = z$. Of course x and y are upper bounds for z and w, so if v is the least upper bound of z and w then v is a lower bound of x and y. But then since z and w are maximal lower bounds we conclude that $z = v = w$. □

Now consider the geometric realization $|\mathcal{P}_1|$, i.e., the simplicial complex with a k-simplex for every chain $x_0 < \cdots < x_k$ in $\mathcal{P}_1$. We will refer to x_k as the *top* of the simplex and x_0 as the *bottom*. Call such a simplex *elementary* if $x_0 \preceq x_k$.

Definition 2.2. Define the *Stein space* X for V_{br} to be the subcomplex of $|\mathcal{P}_1|$ consisting of all elementary simplices.

Since faces of elementary simplices are elementary, this is a subcomplex. In fact there is a coarser cell decomposition of X, as a cubical complex, which we describe as follows. For $x \le y$ define the closed interval $[x, y] := \{z \mid x \le z \le y\}$. Similarly define the open and half-open intervals (x, y), $(x, y]$ and $[x, y)$. Note that if $x \preceq y$ then the closed interval $[x, y]$ is a Boolean lattice, and so the simplices in its geometric realization piece together into a cube. The *top* of the cube is y and the *bottom* is x. Every elementary simplex is contained in such a cube, and the face of any cube is clearly another cube. Also, the intersection of cubes is either empty or is itself a cube; this is clear since if $[x, y] \cap [z, w] \ne \emptyset$ then y and w have a lower bound, and we get that $[x, y] \cap [z, w] = [\sup(x, z), \inf(y, w)]$. This means that X has the cubical structure of a cubical complex, in the sense of [BH99, p. 112, Definition 7.32]. This is all very similar to the construction of the Stein spaces for F and V given in [Bro92, Ste92].

Lemma 2.3. *For $x < y$ with $x \not\preceq y$, $|(x, y)|$ is contractible.*

Proof. The proof is very similar to the proof of the lemma in Section 4 of [Bro92]. For any $z \in (x, y]$ let z_0 be the largest element of $[x, z]$ such that $x \preceq z_0$. By hypothesis $z_0 \in [x, y)$, and by the definition of $\preceq$ it is clear that $z_0 \in (x, y]$, so in fact $z_0 \in (x, y)$. Also, $z_0 \leq y_0$ for any $z \in (x, y)$. The inequalities $z \geq z_0 \leq y_0$ then imply that $|(x, y)|$ is contractible, by Section 1.5 of [Qui78]. □

Corollary 2.4. *X is contractible.*

Proof. First note that $|\mathcal{P}_1|$ is contractible since $\mathcal{P}_1$ is directed. Similar to the situation in [Bro92], we can build up from X to $|\mathcal{P}_1|$ by attaching new subcomplexes, and we claim that this never changes the homotopy type, so X is contractible. Given a closed interval $[x, y]$, define $r([x, y]) := f(y) - f(x)$. As a remark, if $x \preceq y$ then $r([x, y])$ is the dimension of the cube given by $[x, y]$. We attach the contractible subcomplexes $|[x, y]|$ for $x \not\preceq y$ to X in increasing order of r-value. When we attach $|[x, y]|$ then, we attach it along $|[x, y)| \cup |(x, y]|$. But this is the suspension of $|(x, y)|$, and so is contractible by the previous lemma. We conclude that attaching $|[x, y]|$ does not change the homotopy type, and since $|\mathcal{P}_1|$ is contractible, so is X. □

There is a natural action of V_{br} on the vertices of X. Namely, for $g \in V_{\mathrm{br}}$ and $\sigma \in \mathscr{S}_1$ with $x = [\sigma]$, define $gx := [g * \sigma]$. This action preserves the relations $\leq$ and $\preceq$, and hence extends to an action on the whole space. To prove that V_{br} is of type F_∞, we will apply Brown's Criterion to the action of V_{br} on X.

Brown's Criterion. [Bro87, Corollary 3.3] *Let G be a group and X a contractible G-CW-complex such that the stabilizer of every cell is of type* F_∞. *Let $\{X_j\}_{j\geq 1}$ be a filtration of X such that each X_j is finite* mod G. *Suppose that the connectivity of the pair (X_{j+1}, X_j) tends to ∞ as j tends to ∞. Then G is of type* F_∞.

Extending the definition of f from $\mathscr{S}$ to $\mathcal{P}$, for each $n \in \mathbb{N}$ define $X^{\leq n}$ to be the full subcomplex of X spanned by vertices x with $f(x) \leq n$. Note that the $X^{\leq n}$ are invariant under the action of V_{br}.

Lemma 2.5. *For each $n \geq 1$ the sublevel set $X^{\leq n}$ is finite modulo V_{br}.*

Proof. Note first that for each $k \geq 1$, V_{br} acts transitively on the set of $(1, k)$-spraiges. Thus there exists for each $1 \leq k \leq n$ one orbit of vertices x in $X^{\leq n}$ with $f(x) = k$. Given a vertex x with $f(x) = k$ there exist only finitely many cubes $C_1, \ldots, C_r$ in the sublevel set $X^{\leq n}$ that have x as their bottom. If C is a cube in $X^{\leq n}$ such that its bottom is in the same orbit as x, then the cube C must be in the same orbit as C_i for some $1 \leq i \leq r$. It follows that there can only be finitely many orbits of cubes in the sublevel set $X^{\leq n}$. □

Lemma 2.6. *Let x be a vertex in X with $f(x) = n$. The stabilizer* $\mathrm{Stab}_{V_{\mathrm{br}}}(x)$ *is isomorphic to B_n.*

Proof. As a first step, identify B_n with its image under the inclusion $B_n \hookrightarrow \mathscr{S}_{n,n}$ sending b to $(1_n, b, 1_n)$.

Let $g \in \mathrm{Stab}_{V_{\mathrm{br}}}(x)$. Fix $\sigma \in \mathscr{S}_{1,n}$ with $x = [\sigma]$. We have $gx = x$, which means that $[g * \sigma] = [\sigma]$, and so in particular $\sigma^{-1} * g * \sigma \in B_n$.

Define a map

$$\begin{aligned} \psi\colon \mathrm{Stab}_{V_{\mathrm{br}}}(x) &\to B_n \\ g &\mapsto \sigma^{-1} * g * \sigma\,. \end{aligned}$$

This is an isomorphism, with inverse $b \mapsto \sigma * b * \sigma^{-1}$. We remark that ψ depends on the choice of σ, and so is not canonical, but it is uniquely determined up to inner automorphisms of B_n. □

Definition 2.7. Let $J \subseteq \{1, \dots, n\}$. Let $b \in B_n$ and let ρ_b be the corresponding permutation in S_n. If ρ_b stabilizes J set-wise, call b an *J-stabilizing* braid. Let $B_n^J \le B_n$ be the subgroup of J-stabilizing braids.

Corollary 2.8. *Let x be a vertex in X, with $f(x) = n$ and $x = [\sigma]$, and let $F_J^{(n)}$ be an elementary forest. If $y = [\sigma * \lambda_J^{(n)}]$, then the stabilizer in V_{br} of the cube $[x, y]$ is isomorphic to B_n^J. In particular all cell stabilizers are of type F_∞.*

Proof. First observe that $g \in V_{\mathrm{br}}$ stabilizes $[x, y]$ if and only if it stabilizes x and y. For $g \in \mathrm{Stab}_{V_{\mathrm{br}}}(x)$ let $b_g := \sigma^{-1} * g * \sigma \in B_n$, where we identify B_n as a subgroup of $\mathscr{S}_{n,n}$ as in the previous proof. Then g stabilizes y if and only if $[\sigma * b_g * \lambda_J^{(n)}] = [\sigma * \lambda_J^{(n)}]$, which by left cancellation is equivalent to $[b_g * \lambda_J^{(n)}] = [\lambda_J^{(n)}]$. This in turn is equivalent to $b_g \in B_n^J$, and so the cube stabilizer equals $\psi^{-1}(B_n^J)$, where $\psi\colon \mathrm{Stab}_{V_{\mathrm{br}}}(x) \to B_n$ is the map from the previous proof. Since ψ is an isomorphism, the first statement follows.

The second statement follows since braid groups are of type F_∞ [Squ94, Theorem A], and each B_n^J has finite index in B_n. $\square$

The filtration $\{X^{\le n}\}_{n \ge 1}$ of X has so far been shown to satisfy all the conditions of Brown's Criterion save one, namely that the connectivity of the pair $(X^{\le n+1}, X^{\le n})$ tends to ∞ as n tends to ∞. We will prove this in Corollary 4.6, using discrete Morse theory. We now describe the Morse-theoretic tools we will use.

Let Y be a piecewise Euclidean cell complex, and let h be a map from the set of vertices of Y to the integers, such that each cell has a unique vertex maximizing h. Call h a *height function*, and $h(y)$ the *height* of y for vertices y in Y. For $t \in \mathbb{Z}$, define $Y^{\le t}$ to be the full subcomplex of Y spanned by vertices y satisfying $h(y) \le t$. Similarly define $Y^{<t}$, and let $Y^{=t}$ be the set of vertices at height t. The *descending star* $\mathrm{st}{\downarrow}(y)$ of a vertex y is defined to be the open star of y in $Y^{\le y}$. The *descending link* $\mathrm{lk}{\downarrow}(y)$ of y is given by the set of "local directions" starting at y and pointing into $\mathrm{st}{\downarrow}(y)$. More details can be found in [BB97], and the following Morse Lemma is a consequence of [BB97, Corollary 2.6].

Morse Lemma. *With the above setup, the following hold.*

1. *Suppose that for any vertex y with $h(y) = t$, $\mathrm{lk}{\downarrow}(y)$ is $(k-1)$-connected. Then the pair $(Y^{\le t}, Y^{<t})$ is k-connected, that is, the inclusion $Y^{<t} \hookrightarrow Y^{\le t}$ induces an isomorphism in π_j for $j < k$, and an epimorphism in π_k.*
2. *Suppose that for any vertex y with $h(y) \ge t$, $\mathrm{lk}{\downarrow}(y)$ is $(k-1)$-connected. Then $(Y, Y^{<t})$ is k-connected.*

The first part of the Morse Lemma will be applied to the Stein space, and the second part will become convenient in Section 3.2.

Every cell of X has a unique vertex maximizing f, so f is a height function. Hence we can inspect the connectivity of the pair $(X^{\le n}, X^{<n})$ by looking at descending links with respect to f. In the rest of this section, we describe a convenient model for the descending links, and then analyze their connectivity in the following sections.

Recall that we identify $\mathcal{P}_1$ with the vertex set of X, and cubes in X are (geometric realizations of) intervals $[y, x]$ with $x, y \in \mathcal{P}_1$ and $y \preceq x$. For $x \in \mathcal{P}_1$, the descending star $\mathrm{st}{\downarrow}(x)$ of x in X is the set of cubes $[y, x]$ with top x. For such a cube $C = [y, x]$ let $\mathrm{bot}(C) := y$ be the map giving the bottom vertex. This is a bijection from the set of such cubes to the set $D(x) := \{y \in \mathcal{P}_1 \mid y \preceq x\}$. The cube $[y', x]$ is a face of $[y, x]$ if and only if $y' \in [y, x]$, if and only if $y' \ge y$. Hence C' is a face of C if and only if $\mathrm{bot}(C') \ge \mathrm{bot}(C)$, so bot is an order-reversing poset map. By considering cubes $[y, x]$ with $y \ne x$ and restricting to $D(x) \setminus \{x\}$, we obtain a description of $\mathrm{lk}{\downarrow}(x)$. Namely, a simplex in $\mathrm{lk}{\downarrow}(x)$ is a dangling spraige y with $y \prec x$, the rank

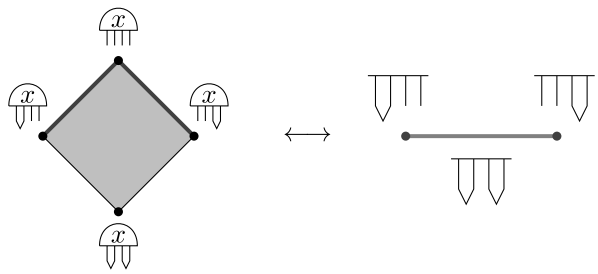

FIGURE 10. The correspondence between $\mathrm{lk}{\downarrow}(x)$ and $\mathcal{EB}_n$.

of the simplex is the number of elementary splits needed to get from y to x (so the number of elementary merges to get from x to y) and the face relation is the reverse of the relation $<$ on $D(x) \setminus \{x\}$. Since X is a cubical complex, $\mathrm{lk}{\downarrow}(x)$ is a simplicial complex.

If $f(x) = n$, then thanks to left cancellation, $\mathrm{lk}{\downarrow}(x)$ is isomorphic to the simplicial complex $\mathcal{EB}_n$ of dangling elementary n-braiges $[(1_n, b, F_J^{(n-|J|)})]$ for $J \neq \emptyset$, with the face relation given by the reverse of the ordering $\leq$ in $\mathcal{P}_n$. See Figure 10 for an idea of the correspondence between $\mathrm{lk}{\downarrow}(x)$ and $\mathcal{EB}_n$. We will usually draw braiges as emerging from a horizontal line, as a visual reminder of this correspondence.

We will prove that $\mathcal{EB}_n$ is highly connected in Corollary 4.4. Our proof relies on a complex that we call the *matching complex on a surface*, which we will define and analyze in the next section. Then we will return to considering dangling elementary braiges in Section 4.

We close this section with some remarks on F_{br}. Restricting to pure braids everywhere in this section does not affect any of the proofs, so we can simply say that $X(F_{\mathrm{br}})$ is the contractible cubical complex of dangling pure 1-spraiges, understood in the same way as X (though now dangling is only via pure braids). We will also denote by f the height function "number of feet" on $X(F_{\mathrm{br}})$. The filtration is still cocompact and the stabilizers are still of type F_∞, being finite index subgroups of braid groups. As for descending links, the descending link of a dangling pure $(1,n)$-spraige in $X(F_{\mathrm{br}})$ is isomorphic to the simplicial complex $P\mathcal{EB}_n$ of dangling elementary pure n-braiges.

3. Matching complexes on surfaces

Throughout this section S denotes a connected surface, with (possibly empty) boundary ∂S, and P denotes a finite set of points in $S \setminus \partial S$. By an *arc*, we mean a simple path in $S \setminus \partial S$ that intersects P precisely at its endpoints, and whose endpoints are distinct. Our standard reference for arc complexes is [Hat91]. Note that our definition of arc is slightly different from the definition given in [Hat91], in that we do not allow the endpoints of a given arc to coincide. Also note that in [Hat91], points in P were allowed to be contained in ∂S, and we will not consider this case here. In Section 4 we will only need the special case where S is a disk, but to prove the results in this section we need to use this degree of generality.

A major theme of this section will be the similarities between certain complexes defined using edges in graphs, and similar complexes defined using arcs on surfaces. Of particular interest is the family of *complete graphs* K_n. The graph K_n is the graph with n nodes and a single edge between any two nodes. Later we will also be interested in the family of subgraphs of *linear graphs* L_n. The linear graph L_n has $n+1$ nodes labeled 1 through $n+1$, and n edges, one connecting the node labeled i to the node labeled $i+1$ for each $1 \leq i \leq n$. Note that when dealing

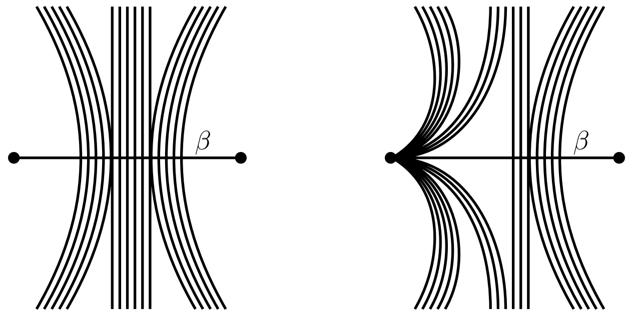

FIGURE 11. The Hatcher flow.

with K_n, n is the number of nodes, but when dealing with L_n, n is the number of edges. This is just for the sake of future ease of notation.

3.1. **The arc complex.** Let $\{\alpha_0, \dots, \alpha_k\}$ be a collection of arcs. If the α_i are all disjoint from each other except possibly at their endpoints, and if no distinct α_i and α_j are homotopic relative P, we call $\{\alpha_0, \dots, \alpha_k\}$ an *arc system.* The homotopy classes, relative P, of arc systems form the simplices of a simplicial complex, with the face relation given by passing to subcollections of arcs.

Definition 3.1. Let Γ be a graph with $|P|$ nodes, and identify P with the set of nodes of Γ. Call an arc *compatible* with Γ if its endpoints are connected by an edge in Γ. Let $\mathcal{HA}(\Gamma)$ be the *arc complex* on (S, P) corresponding to Γ, that is the simplicial complex with a k-simplex for each arc system $\{\alpha_0, \dots, \alpha_k\}$ such that all the α_i are compatible with Γ.

We remark that $\mathcal{HA}(K_n)$ is a proper subcomplex of the space $\mathcal{A}(S, P)$ in [Hat91], since we only consider arcs with two distinct endpoints.

Proposition 3.2. *For any $n \geq 2$ the complex $\mathcal{HA}(K_n)$ is contractible.*

The proof here is essentially the same as the proof of the theorem in [Hat91], so we will not be overly precise. Indeed there is only one extra step, which we will point out when it comes.

Proof. Fix an arc β, i.e., a vertex in $\mathcal{HA}(K_n)$. We will retract $\mathcal{HA}(K_n)$ to the star of β. We use the "Hatcher flow" introduced in [Hat91]. Let $\sigma = \{\alpha_0, \dots, \alpha_k\}$ be a simplex in $\mathcal{HA}(K_n)$ and let p be a point in σ, expressed in terms of barycentric coordinates $p = \sum_{i=0}^{k} c_i \alpha_i$, with $c_i \geq 0$ and $\sum_{i=0}^{k} c_i = 1$. Interpret p geometrically by saying that each α_i is thickened to a "band" of thickness c_i. Wherever the bands cross β, pinch them into a single band of thickness θ. Now the Hatcher flow is as follows. At time $t \in [0, 1]$, push p to the point p_t obtained by leaving $(1-t)\theta$ worth of the band in place and pushing the remaining θ-thick part of the band all the way to one end of β; see Figure 11. The additional consideration we have to make is, if at any point we create a new arc whose endpoints coincide, discard this from p_t. This is allowed, since if none of the α_i are loops then there will always exist at least one non-loop arc used in p_t. One checks that this flow is continuous and respects the face relation, and at time $t = 1$ we have deformed $\mathcal{HA}(K_n)$ into the star of β, so we conclude that $\mathcal{HA}(K_n)$ is contractible. □

As a remark, note that the above proof yields contractibility for more general $\mathcal{HA}(\Gamma)$; the only requirement is that there exists a node of Γ that shares an edge with every other node.

We now want to consider a subspace of $\mathcal{HA}(K_n)$ that is related to the matching complex of a complete graph, which we call the matching complex on a surface. In the next subsection, we will first cover some background on matching complexes of graphs and establish some results, and then we will inspect the surface version.

3.2. **Matching complexes.** Matching complexes of graphs are defined as follows.

Definition 3.3. Let Γ be a graph. The *matching complex* $\mathcal{M}(\Gamma)$ of Γ is the simplicial complex with a k-simplex for every collection $\{e_0, \dots, e_k\}$ of $k+1$ pairwise disjoint edges, with the face relation given by passing to subcollections.

Observe that $\mathcal{M}(K_n)$ is non-empty if and only if $n \geq 2$, and as an exercise one can verify that it is connected for $n \geq 5$. As we will see in Proposition 3.5, $\mathcal{M}(K_n)$ is $(\nu(n)-1)$-connected, where we define $\nu(m) := \lfloor \frac{m+1}{3} \rfloor - 1$ for any $m \in \mathbb{Z}$.

Before proving this, we need a technical lemma.

Lemma 3.4. *For $m_1, \dots, m_\ell \in \mathbb{Z}$, we have*
$$\sum_{i=1}^{\ell} \nu(m_i) \geq \nu\left(\sum_{i=1}^{\ell} m_i - 4(\ell-1)\right).$$

Proof. We induct on ℓ. The case $\ell = 1$ is trivially true. In order to prove the case $\ell = 2$ we need the following observation:
$$\left\lfloor \frac{m_1}{3} \right\rfloor + \left\lfloor \frac{m_2}{3} \right\rfloor \geq \left\lfloor \frac{m_1+m_2+1}{3} \right\rfloor - 1 \qquad (*)$$
for any $m_1, m_2 \in \mathbb{Z}$. It suffices to consider the cases where m_1 and m_2 are 0, 1 or 2, and these cases are readily checked. Thus we obtain
$$\begin{aligned}
\nu(m_1) + \nu(m_2) &= \left\lfloor \frac{m_1+1}{3} \right\rfloor + \left\lfloor \frac{m_2+1}{3} \right\rfloor - 2 \\
&\geq \left\lfloor \frac{m_1+m_2+3}{3} \right\rfloor - 3 && \text{(using the inequality $(*)$)} \\
&= \left\lfloor \frac{m_1+m_2-3}{3} \right\rfloor - 1 \\
&= \nu(m_1+m_2-4)
\end{aligned}$$
and this finishes the case $\ell = 2$. Finally, suppose that $\ell > 2$. Then
$$\begin{aligned}
\sum_{i=1}^{\ell} \nu(m_i) &= \sum_{i=1}^{\ell-1} \nu(m_i) + \nu(m_\ell) \\
&\geq \nu\left(\sum_{i=1}^{\ell-1} m_i - 4(\ell-2)\right) + \nu(m_\ell) && \text{(by induction)} \\
&\geq \nu\left(\sum_{i=1}^{\ell-1} m_i - 4(\ell-2) + m_\ell - 4\right) && \text{(by the $\ell = 2$ case)} \\
&= \nu\left(\sum_{i=1}^{\ell} m_i - 4(\ell-1)\right).
\end{aligned}$$
$\square$

To prove that $\mathcal{M}(K_n)$ is $(\nu(n)-1)$-connected it will be convenient to embed it into a contractible space, so we can use the Morse Lemma. Let $\mathcal{H}(K_n)$ be the simplicial complex with a simplex for every subgraph of K_n that has the same vertex set as K_n and has at least one edge, with face relation given by inclusion. Hence a 0-simplex in $\mathcal{H}(K_n)$ is a subgraph with a single edge, a 1-simplex has two edges, and so forth.

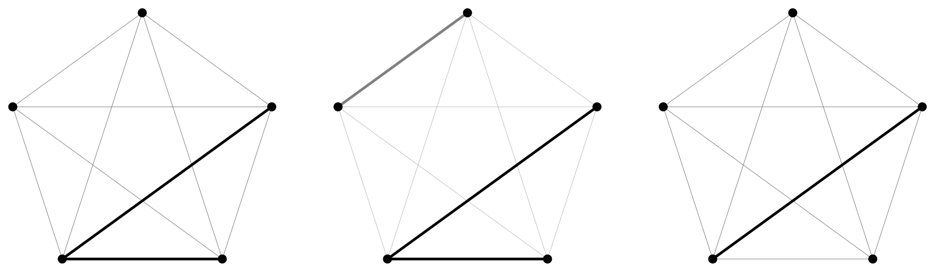

FIGURE 12. Three vertices in $\mathcal{H}(K_5)$. From left to right: a graph Γ with defect 1, a graph in the up-link of Γ and a graph in the down-link of Γ.

In fact, $\mathcal{H}(K_n)$ is isomorphic to an $\left(\binom{n}{2} - 1\right)$-simplex, and so is contractible. Think of $\mathcal{M}(K_n)$ as a subcomplex of the contractible complex $\mathcal{H}(K_n)$. Consider a simplex Γ in $\mathcal{H}(K_n)$. Let $e(\Gamma)$ be the number of edges of Γ and let $r(\Gamma)$ be the number of non-isolated nodes of Γ. Define the *defect* of Γ to be the number $d(\Gamma) := 2e(\Gamma) - r(\Gamma)$. This measures the failure of Γ to be in $\mathcal{M}(K_n)$, in that $\mathcal{M}(K_n)$ is precisely the set of simplices of $\mathcal{H}(K_n)$ with defect 0. Note that $\mathcal{M}(K_n)$ already contains every 0-simplex of $\mathcal{H}(K_n)$. Also note that a subgraph Γ' of a graph Γ cannot have higher defect than Γ. Now define a function $h(\Gamma) := (d(\Gamma), -e(\Gamma))$, and consider its values ordered lexicographically. Think of h as a function on the vertex set of $\mathcal{H}(K_n)'$, the barycentric subdivision of $\mathcal{H}(K_n)$. Note that adjacent vertices have distinct e-values, and hence distinct h-values, so this is a height function on the vertex set of $\mathcal{H}(K_n)'$.

Consider the descending link $\mathrm{lk}{\downarrow}(\Gamma)$ of Γ in $\mathcal{H}(K_n)'$. There are two types of vertices in $\mathrm{lk}{\downarrow}(\Gamma)$, namely graphs $\widetilde{\Gamma} > \Gamma$ with $h(\widetilde{\Gamma}) < h(\Gamma)$ and graphs $\Gamma' < \Gamma$ with $h(\Gamma') < h(\Gamma)$. Define the *up-link* (respectively *down-link*) to the the full subcomplex of $\mathrm{lk}{\downarrow}(\Gamma)$ spanned by vertices of the first type (respectively second type). Observe that $h(\widetilde{\Gamma}) < h(\Gamma)$ is equivalent to $d(\widetilde{\Gamma}) = d(\Gamma)$, and $h(\Gamma') < h(\Gamma)$ is equivalent to $d(\Gamma') < d(\Gamma)$. Any graph in the down-link is a subgraph of any graph in the up-link, so $\mathrm{lk}{\downarrow}(\Gamma)$ is the join of the up-link and down-link.

Proposition 3.5. *The complex $\mathcal{M}(K_n)$ is $(\nu(n) - 1)$-connected.*

This result is well-known, see for example [Ath04, BLVŽ94]. We will prove it here using a method that we will use later to prove the main result of this section, Theorem 3.7.

Proof. As a base case, $\mathcal{M}(K_n)$ is non-empty for $n \geq 2$. Now suppose $n \geq 5$. Since $\mathcal{H}(K_n)$ is contractible and its 0-skeleton is already in $\mathcal{M}(K_n)$, to show that $\mathcal{M}(K_n)$ is $(\nu(n) - 1)$-connected it suffices by the Morse Lemma to show that for any Γ with $e(\Gamma) \geq 2$ and $d(\Gamma) \geq 1$, the descending link $\mathrm{lk}{\downarrow}(\Gamma)$ is $(\nu(n) - 1)$-connected. First consider the down-link. A subgraph of $\Gamma' < \Gamma$ fails to be in the down-link precisely if each edge in $\Gamma \setminus \Gamma'$ is disjoint from every other edge of Γ, since then and only then do Γ and Γ' have the same defect. Let Γ_0 be the subgraph of Γ consisting precisely of all such edges, if any exist. The space of all proper subgraphs of Γ is an $(e(\Gamma) - 2)$-sphere, and the complement in this space of the down-link is either empty, or is contractible with cone point Γ_0. Hence the down-link is either an $(e(\Gamma) - 2)$-sphere or is contractible. Now consider the up-link. The graphs in the up-link are given by adding edges to Γ that are all disjoint from each other and from the edges of Γ. Hence the up-link is isomorphic to $\mathcal{M}(K_{n-r(\Gamma)})$, and so is $(\nu(n - r(\Gamma)) - 1)$-connected by induction. Since the down-link is $(e(\Gamma)-3)$-connected, this tells us that $\mathrm{lk}{\downarrow}(\Gamma)$ is $(e(\Gamma)+\nu(n-r(\Gamma))-2)$-connected. Since $e(\Gamma) \geq 2$ and $d(\Gamma) \geq 1$, we have

$$\begin{aligned} e(\Gamma)+\nu(n-r(\Gamma))-2 &= \nu(n+3e(\Gamma)-r(\Gamma)-3)-1 \\ &= \nu(n+d(\Gamma)+e(\Gamma)-3)-1 \\ &\geq \nu(n)-1 \end{aligned}$$

and so we conclude that $\mathrm{lk}{\downarrow}(\Gamma)$ is indeed $(\nu(n)-1)$-connected. □

We now define the notion of the matching complex on a surface. We retain all the notation and definitions from Section 3.1, including the surface S with points P. Fix a labeling of the points in P by the numbers $1,\dots,n$, and identify P with the set of nodes of K_n, as above.

Definition 3.6. Let $\mathcal{MA}(K_n)$ be the subcomplex of $\mathcal{HA}(K_n)$ whose simplices are given by arc systems whose arcs are pairwise disjoint including at their endpoints. For a subgraph Γ of K_n let $\mathcal{MA}(\Gamma)$ be the preimage of $\mathcal{M}(\Gamma)$ under the map $\mathcal{MA}(K_n) \to \mathcal{M}(K_n)$ that sends an arc with endpoints labeled i and j to the edge of K_n with endpoints i and j. We call $\mathcal{MA}(\Gamma)$ the *matching complex* on (S,P) corresponding to Γ.

Our goal now is to show that $\mathcal{MA}(K_n)$ is $(\nu(n)-1)$-connected, just like $\mathcal{M}(K_n)$. Our proof mimics the proof of Proposition 3.5. As a remark, in the next section we will analyze the connectivity of $\mathcal{MA}(L_n)$ using different methods, in Corollary 3.10, and also give an alternate proof that $\mathcal{MA}(K_n)$ is $(\nu(n)-1)$-connected. The proof that we give in this section for $\mathcal{MA}(K_n)$ is more in line with the analogous situation for graphs, but does not generalize readily to $\mathcal{MA}(L_n)$.

We begin by defining a height function h on the vertex set of the barycentric subdivision $\mathcal{HA}(K_n)'$ of $\mathcal{HA}(K_n)$, similar to the height function on the vertex set of $\mathcal{H}(K_n)'$. In fact we will reuse much of the notion from the $\mathcal{M}(K_n)$ case. Let σ be a k-simplex in $\mathcal{HA}(K_n)$, represented by the arcs $\alpha_0,\dots,\alpha_k$. Choose these arcs to be pairwise disjoint, except possibly at their endpoints. Let $A(\sigma) := \alpha_0 \cup \cdots \cup \alpha_k$ as a subspace of S. Also let $a(\sigma) := k+1$ be the number of arcs, let $r(\sigma) := |A \cap P|$ and define the *defect* $d(\sigma) := 2a(\sigma) - r(\sigma)$. For example if every arc in σ has the same two points as endpoints then $d(\sigma) = 2a(\sigma)-2$, the maximum possible. The defect is zero if and only if $\sigma \in \mathcal{MA}(K_n)$, so in particular $\mathcal{MA}(K_n)$ is a sublevel set. Now define $h(\sigma) := (d(\sigma), -a(\sigma))$ with the lexicographic ordering. In particular, adjacent vertices in $\mathcal{HA}(K_n)'$ have distinct h-values. As in the case of $\mathcal{M}(K_n)$, $\mathrm{lk}{\downarrow}(\sigma)$ decomposes as the join of an up-link and a down-link. The up-link is given by simplices $\widetilde{\sigma} > \sigma$ with $d(\widetilde{\sigma}) = d(\sigma)$, and the down-link is given by faces $\sigma' < \sigma$ with $d(\sigma') < d(\sigma)$; see Figure 13.

Theorem 3.7. *The complex $\mathcal{MA}(K_n)$ is $(\nu(n)-1)$-connected.*

Proof. The proof runs very similarly to the proof of Proposition 3.5. Dealing with the up-link is the biggest difference.

As a base case, $\mathcal{MA}(K_n)$ is non-empty for $n \geq 2$. Now suppose $n \geq 5$. Since $\mathcal{HA}(K_n)$ is contractible by Proposition 3.2, and the 0-skeleton of $\mathcal{HA}(K_n)$ is already in $\mathcal{MA}(K_n)$, to show that $\mathcal{MA}(K_n)$ is $(\nu(n)-1)$-connected it suffices by the Morse Lemma to show that for any k-simplex σ in $\mathcal{HA}(K_n)$ with $k \geq 1$ and $d(\sigma) \geq 1$, the descending link $\mathrm{lk}{\downarrow}(\sigma)$ is $(\nu(n)-1)$-connected. First consider the down-link. A face σ' of σ fails to be in the down-link precisely when each arc in $\sigma \setminus \sigma'$ is disjoint from every other arc of σ, since then and only then do σ and σ' have the same defect. Let σ_0 be the face of σ consisting precisely of all such arcs, if any exist. The boundary of σ is an $(a(\sigma)-2)$-sphere. The complement in this

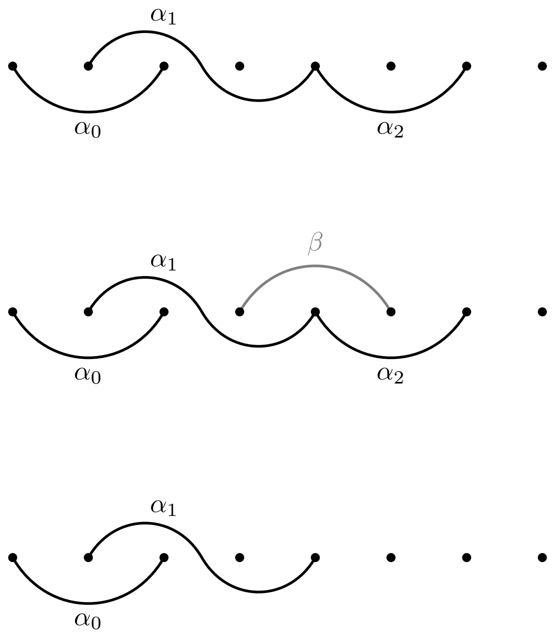

FIGURE 13. From top to bottom: an arc system σ with defect 1, a simplex in the up-link of σ and a simplex in the down-link of σ.

space of the down-link is either empty, or is a cone with cone point σ_0. Hence the down-link is either an $(a(\sigma)-2)$-sphere or is contractible.

Now consider the up-link. The simplices in the up-link are given by adding arcs to σ that are all disjoint from each other and from the arcs in σ. Consider the surface $S \setminus A$, obtained by cutting out the arcs α_i. Denote the connected components of $S \setminus A$ by $C_1, \dots, C_\ell$, and let $n_i := |C_i \cap P|$ for each $1 \le i \le \ell$. Note that $\sum_{i=1}^{\ell} n_i + r(\sigma) = n$. The up-link is isomorphic to the join $*_{i=1}^{\ell} \mathcal{MA}(K_{n_i})$, which by induction is $\big(\sum_{i=1}^{\ell} \nu(n_i) + \ell - 2\big)$-connected. Since the down-link is $(a(\sigma)-3)$-connected, this tells us that $\mathrm{lk}{\downarrow}(\sigma)$ is $\big(a(\sigma) - 2 + \sum_{i=1}^{\ell} \nu(n_i) + \ell - 1\big)$-connected. By Lemma 3.4,

$$a(\sigma) - 2 + \sum_{i=1}^{\ell} \nu(n_i) + \ell - 1 \ge a(\sigma) - 2 + \nu\Bigg(\sum_{i=1}^{\ell} n_i - 4(\ell-1)\Bigg) + \ell - 1$$

which equals

$$\begin{aligned} \nu\Bigg(\sum_{i=1}^{\ell} n_i - 4(\ell-1) + 3a(\sigma) + 3\ell - 6\Bigg) - 1 &= \nu(n - r(\sigma) - \ell + 3a(\sigma) - 2) - 1 \\ &= \nu(n + d(\sigma) + a(\sigma) - \ell - 2) - 1. \end{aligned}$$

Here the first equality follows because $\sum_{i=1}^{\ell} n_i = n - r(\sigma)$, and the second equality follows from the definition of defect, namely $d(\sigma) = 2a(\sigma) - r(\sigma)$. Since $a(\sigma) \ge 2$, this last quantity is at least $\nu(n + d(\sigma) - \ell) - 1$.

To show that $\mathrm{lk}{\downarrow}(\sigma)$ is $(\nu(n)-1)$-connected, it now suffices to show that $d(\sigma) \ge \ell$. By an Euler characteristic argument, we know that $r(\sigma) - a(\sigma) + \ell \le 1 + z$, where z is defined to be the number of connected components of $A(\sigma) = \bigcup_{i=0}^{k} \alpha_i$. Clearly each connected component of $A(\sigma)$ contains at least one arc, and since $d(\sigma) \ge 1$ at least one component must have more than one arc. Hence $1 + z \le a(\sigma)$, which implies that $r(\sigma) + \ell \le 2a(\sigma)$, and so indeed $\ell \le d(\sigma)$. $\square$

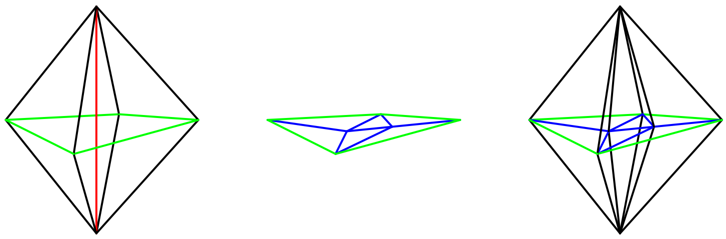

FIGURE 14. Illustration of the proof of Lemma 3.8. The red edge is the simplex σ, that is, both of its vertices are mapped to the same vertex under ψ. The green circle is the link of σ. The link of $\psi(\sigma)$ is simply connected by assumption, so ψ can be extended to a filling disk B (blue).

3.3. **Connectivity of $\mathcal{MA}(L_n)$.** Recall the family of linear graphs L_n from Section 3.2. In this section we analyze the connectivity of $\mathcal{MA}(L_n)$, by following the line of approach used in the proof of Proposition 5.2 in [Put12]. This approach can also be adapted to recover the connectivity of $\mathcal{MA}(K_n)$.

We first need a lemma that allows us to make certain assumptions about maps from spheres to $\mathcal{MA}(\Gamma)$. To state it we need to recall some definitions. By a *combinatorial k-sphere (respectively k-disk)* we mean a simplicial complex that can be subdivided to be isomorphic to a subdivision of the boundary of a $(k+1)$-simplex (respectively to a subdivision of a k-simplex). An m-dimensional *combinatorial manifold* is an m-dimensional simplicial complex in which the link of every simplex σ of dimension k is a combinatorial $(m-k-1)$-sphere. In an m-dimensional *combinatorial manifold with boundary* the link of a k-simplex σ is allowed to be homeomorphic to a combinatorial $(m-k-1)$-disk; its *boundary* consists of all the simplices whose link is indeed a disk.

A simplicial map is called *simplexwise injective* if its restriction to any simplex is injective.

Lemma 3.8. *Let Y be an m-dimensional combinatorial manifold. Let X be a simplicial complex and assume that the link of every k-simplex in X is $(m-2k-2)$-connected. Let $\psi\colon Y \to X$ be a simplicial map whose restriction to ∂Y is simplexwise injective. Upon changing the simplicial structure of Y, ψ is homotopic relative ∂Y to a simplexwise injective map.*

Compare the statement of the lemma to the statement of the claim in the proof of Proposition 5.2 in [Put12].

Proof. The proof is by induction on m and the statement is trivial for $m = 0$.

If ψ is not simplexwise injective, there exists a simplex whose vertices do not map to pairwise distinct points. In particular we can choose a simplex $\sigma \subseteq Y$ of maximal dimension $k > 0$ such that for every vertex x of σ there is another vertex y of σ with $\psi(x) = \psi(y)$. By assumption, σ is not contained in ∂Y. Maximality of the dimension of σ implies that the restriction of ψ to the $(m-k-1)$-sphere $\mathrm{lk}_Y(\sigma)$ is simplexwise injective. It also implies that $\psi(\mathrm{lk}_Y(\sigma)) \subseteq \mathrm{lk}_X(\psi(\sigma))$. Note further that $\psi(\sigma)$ has dimension at most $(k-1)/2$. Therefore its link in X is $(m-k-1)$-connected by assumption. Hence there is an $(m-k)$-disk B with $\partial B = \mathrm{lk}_Y(\sigma)$ and a map $\varphi\colon B \to \mathrm{lk}_X(\psi(\sigma))$ such that $\varphi|_{\partial B}$ coincides with $\psi|_{\mathrm{lk}_Y(\sigma)}$. Inductively applying the lemma, we may assume that φ is simplexwise injective.

We now replace Y by Y', the space obtained by replacing the closed star of σ by $B * \partial\sigma$. The map $\psi'\colon Y' \to X$ is the map that coincides with ψ outside the open star of σ, coincides with φ on B and is affine on simplices. It is clearly homotopic to ψ, since the image of B under φ is contained in $\mathrm{lk}_X(\psi(\sigma))$. Since the restriction of ψ' to B is simplexwise injective, the restriction to any k-simplex of $B * \partial\sigma$ is injective. Carrying on in this way, we eventually obtain a map that is simplexwise injective. □

We now describe the general procedure we will use to analyze $\mathcal{MA}(\Gamma)$ for a graph Γ, after which we will look at the specific graphs K_n and (subgraphs of) L_n.

Pick an edge e of Γ, say with endpoints v and w. Identify the vertex set of Γ with the set P of distinguished points in the surface S. Let

$$q\colon \mathcal{MA}(\Gamma)^{(0)} \to \{0,1,2,3\}$$

be the function that sends an arc to 0 if it has neither v nor w as endpoints, 1 if it has only v, 2 if it has only w, and 3 if it has both. For any arc α, say with endpoints v_1 and v_2, the link of α in $\mathcal{MA}(\Gamma)$ is isomorphic to $\mathcal{MA}(\Gamma')$, where Γ' is the graph obtained from Γ by removing the stars of v_1 and v_2. Here the surface on which the matching complex is being considered is not S, but rather S with a new boundary component obtained by "slicing" S along α. The idea therefore is to build up from $\mathcal{MA}(\Gamma)^{q=0}$ to $\mathcal{MA}(\Gamma)$ by gluing in missing vertices (arcs) along their relative links, in increasing order of q-value. Since Γ' has fewer vertices and edges than Γ, these links will be highly connected by induction. By the Morse Lemma, to prove that $\mathcal{MA}(\Gamma)$ is highly connected it suffices to prove that the inclusion $\iota\colon \mathcal{MA}(\Gamma)^{q=0} \to \mathcal{MA}(\Gamma)$ induces the trivial map in homotopy π_k for k up to some bound. This last step is accomplished as follows. Fix an arc β with endpoints v and w, and let $\overline{\psi}\colon S^m \to \mathcal{MA}(\Gamma)^{q=0}$ be a simplicial map. The goal is to show that $\psi := \iota \circ \overline{\psi}$ is homotopy equivalent to the constant map sending S^m to β, if m is sufficiently large. A variant of the Hatcher flow becomes useful here; we look at arcs crossing β, choose one closest to w, say α, and "push" it over w and off of β, to the arc α'. See Figure 15 for a visualization. We can homotope ψ to the map ψ' using α in lieu of α', assuming the mutual link $\mathrm{lk}(\alpha) \cap \mathrm{lk}(\alpha')$ is sufficiently highly connected, which we can engineer to be true by induction if the structure of Γ is sufficiently easy to control. A key step to making this rigorous is being able to use Lemma 3.8.

We first carry out this program for the family of subgraphs of linear graphs. A key observation in this setting is that any node has degree at most 2, and so removing the stars of two adjacent nodes results in removing at most 3 edges.

It will be convenient to define $\eta(\ell) := \lfloor \frac{\ell-1}{4} \rfloor$ for $\ell \in \mathbb{Z}$.

Theorem 3.9. *Let Γ_n be any subgraph of a linear graph, with Γ_n having n edges. Then $\mathcal{MA}(\Gamma_n)$ is $(\eta(n)-1)$-connected.*

Proof. We induct on n, with the base case being that $\mathcal{MA}(\Gamma_n)$ is non-empty for $n \geq 1$, which is clear. Now assume $n \geq 5$. Choose an edge e in Γ_n with at least one endpoint of degree 1. Let v and w be the endpoints of e, say w has degree 1. Let q be the function defined above. For an arc α with $q(\alpha) = 1$, the descending link of α with respect to q is isomorphic to $\mathcal{MA}(\Gamma_{n'})$, where $\Gamma_{n'}$ is a subgraph of Γ_n with n' edges. Since every vertex has degree at most 2, $n' \geq n-3$, so by induction $\mathcal{MA}(\Gamma_{n'})$ is $(\eta(n)-2)$-connected. Similarly if $q(\alpha) = 3$ then the descending link of α is isomorphic to $\mathcal{MA}(\Gamma_{n'})$, now with $n' \geq n-2$, so again induction tells us that $\mathcal{MA}(\Gamma_{n'})$ is $(\eta(n)-2)$-connected. (Note that $q = 2$ cannot happen in the present situation.) It now suffices to show that the inclusion $\iota\colon \mathcal{MA}(\Gamma_n)^{q=0} \to \mathcal{MA}(\Gamma_n)$ induces the trivial map in π_m for $m < \eta(n)$, since then the Morse Lemma will tell us that $\pi_m(\mathcal{MA}(\Gamma_n)) = 0$ for $m < \eta(n)$.

First we check a hypothesis on $\mathcal{MA}(\Gamma_n)$ that allows us to apply Lemma 3.8, namely that the link of a k-simplex should be $(m-2k-2)$-connected. A k-simplex σ

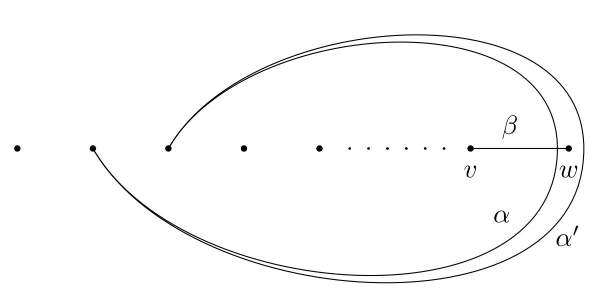

FIGURE 15. Pushing the arc α over the vertex w to obtain the arc α', as described in the proof of Theorem 3.9.

is determined by $k+1$ disjoint arcs. Hence, the link of σ is isomorphic to $\mathcal{MA}(\Gamma_{n'})$ where n' is at least $n-(3k+3)$. By induction, this is $(\eta(n-3k-3)-1)$-connected. Moreover,

$$\begin{aligned}\eta(n-3k-3)-1 &= \left\lfloor \frac{n-3k-4}{4} \right\rfloor - 1\\ &\geq \frac{n-3k-4}{4} - 2\\ &\geq \eta(n) - 2k - 3 \geq m - 2k - 2\,.\end{aligned}$$

We conclude that the hypothesis of Lemma 3.8 is satisfied.

Let S^m be an m-sphere with simplicial decomposition. Let $\overline{\psi}\colon S^m \to \mathcal{MA}(\Gamma_n)^{q=0}$ be a simplicial map and let $\phi := \iota \circ \overline{\psi}$. It suffices by simplicial approximation [Spa66, Theorem 3.5.8] to homotope ψ to a constant map. By Lemma 3.8 we may assume ψ is simplexwise injective. Fix an arc β with endpoints v and w. We claim that ψ can be homotoped in $\mathcal{MA}(\Gamma_n)$ to land in the star of β, which will finish the proof. We will proceed in a similar way to the Hatcher flow used in the proof of Proposition 3.2. None of the arcs in the image of ψ use v or w as vertices, but among the finitely many such arcs, some might cross β. Pick the one, say α, intersecting β at a point closest along β to w, and let x be a vertex of S^m mapping to α. By simplexwise injectivity, none of the vertices in $\operatorname{lk}_{S^m}(x)$ map to α. Let α' be the arc with the same endpoints as α such that together α and α' bound a disk whose interior contains no boundary components, punctures or points of P other than w. See Figure 15 for an example. Note that α and α' do not share a simplex, so none of the vertices in $\operatorname{lk}_{S^m}(x)$ map to α'.

Define a simplicial map $\psi'\colon S^m \to \mathcal{MA}(\Gamma_n)$ that sends the vertex x to α' and sends all other vertices y to $\psi(y)$. This map exists by our choice of α. We claim that we can homotope ψ to ψ'. Once we do this, we will have reduced the number of crossings with β, and so continuing this procedure we will have homotoped our map so as to land in the star of β, finishing the proof.

The mutual link $\operatorname{lk}(\alpha) \cap \operatorname{lk}(\alpha')$ is isomorphic to $\mathcal{MA}(\Gamma_{n'})$, where $\Gamma_{n'}$ now is the graph obtained from Γ_n by removing e, and removing any edge sharing an endpoint with an endpoint of α. Here n' is the number of edges of the resulting graph. Since Γ_n is a subgraph of a linear graph, we have thrown out at most 4 edges, and so $n' \geq n-4$. Hence by induction $\operatorname{lk}(\alpha) \cap \operatorname{lk}(\alpha')$ is $(\eta(n)-2)$-connected, and in particular $(m-1)$-connected. Since $\operatorname{lk}_{S^m}(x)$ is an $(m-1)$-sphere, this tells us that there exists an m-disk B with $\partial B = \operatorname{lk}_{S^m}(x)$ and a simplicial map $\varphi\colon B \to \operatorname{lk}(\alpha) \cap \operatorname{lk}(\alpha')$ so that φ restricted to ∂B coincides with ψ restricted to $\mathrm{lk}_{S^m}(x)$. Since the image of B under φ is contained in $\mathrm{lk}(\alpha)$, we can homotope ψ, replacing $\psi|_{\mathrm{st}_{S^m}(x)}$ with φ. Since the image of B under φ is contained in $\mathrm{lk}(\alpha')$, we can similarly homotope ψ', replacing $\psi'|_{\mathrm{st}_{S^m}(x)}$ with φ. These both yield the same map, so we are finished. □

Corollary 3.10. *$\mathcal{MA}(L_n)$ is $(\eta(n)-1)$-connected.* □

As a remark, we expect that a better connectivity bound should be possible. Indeed, one can check that $\mathcal{MA}(L_n)$ is already connected for $n \geq 4$, and that $\mathcal{M}(L_n)$ is $(\nu(n)-1)$-connected, which for large n is better than $(\eta(n)-1)$-connected. For now however, we will content ourselves with this bound.

Using these techniques, we can also recover the connectivity of $\mathcal{MA}(K_n)$.

Alternate proof of Theorem 3.7. The base case is that $\mathcal{MA}(K_n) \neq \emptyset$ for $n \geq 2$, which is clear. Let $n \geq 5$. Choose any edge e, with endpoints v and w. Let q be as above. For an arc α with $q(\alpha) = 1$, the descending link of α is isomorphic to $\mathcal{MA}(K_{n-3})$. If $q(\alpha) = 2$ or 3, the descending link is isomorphic to $\mathcal{MA}(K_{n-2})$. In any case, by induction all descending links are $(\nu(n)-2)$-connected. Hence we need only check that $\iota\colon \mathcal{MA}(K_n)^{q=0} \to \mathcal{MA}(K_n)$ induces the trivial map in π_m for $m < \nu(n)$.

First we check the hypothesis of Lemma 3.8. The link of a k-simplex is a copy of $\mathcal{MA}(K_{n-2k-2})$, which by induction is $(\nu(n-2k-2)-1)$-connected. We need this to be bounded below by $m-2k-2$. Indeed,

$$\nu(n-2k-2)-1 \geq \frac{n-2k-4}{3} - 2 \geq \nu(n) - 2k - 3 \geq m - 2k - 2\,.$$

Now we consider a simplicial map $\overline{\psi}\colon S^m \to \mathcal{MA}(K_n)^{q=0}$, with $\psi := \iota \circ \overline{\psi}$. We claim that we can homotope ψ to a constant map. By the same argument as in the proof of Theorem 3.9, the problem reduces to inspecting the mutual link $\mathrm{lk}(\alpha) \cap \mathrm{lk}(\alpha')$, where α and α' are again as in Figure 15. This mutual link is isomorphic to $\mathcal{MA}(K_{n-3})$, since compatible arcs may use any endpoints other than the endpoints of α, or the point w. Hence by induction $\mathrm{lk}(\alpha) \cap \mathrm{lk}(\alpha')$ is $(\nu(n)-2)$-connected, and by the same argument as in the proof of Theorem 3.9, we can eventually homotope ψ to land in the star of β, so we are done. □

3.4. **Cyclic graphs.** Let C_n be the cyclic graph with n nodes, labeled 1 through n in sequence. If α is an arc in $\mathcal{MA}(C_n)$ with endpoints 1 and n, the relative link of α is a copy of $\mathcal{MA}(L_{n-3})$. Gluing in these arcs, in any order, we build up from $\mathcal{MA}(L_{n-1})$ to $\mathcal{MA}(C_n)$. Hence it is immediate from Corollary 3.10 and the Morse Lemma that $\mathcal{MA}(C_n)$ is $(\eta(n-1)-1)$-connected. The upshot of this is that the methods of the present article could be used to prove that T_{br}, "braided T", is of type F_∞, once the group has been defined.

4. Descending links in the Stein space

We now return to the Stein space X from Section 2 and inspect the descending links of vertices with respect to the height function f. As explained before, the descending link of a vertex x with $f(x) = n$ is isomorphic to the complex $\mathcal{EB}_n$ of dangling elementary n-braiges $[(1_n, b, F_J^{(n-|J|)})]$ with $J \neq \emptyset$. The idea now is to construct a projection $\mathcal{EB}_n \to \mathcal{MA}(K_n)$ and then, having calculated the connectivity of $\mathcal{MA}(K_n)$ in the previous section, use tools of Quillen [Qui78] to obtain the connectivity of $\mathcal{EB}_n$. As usual we will wait until the end of the section to mention the "pure" case.

The first key observation is that a matching on a linear graph encodes the same information as an elementary forest. Recall that L_{n-1} is the linear graph

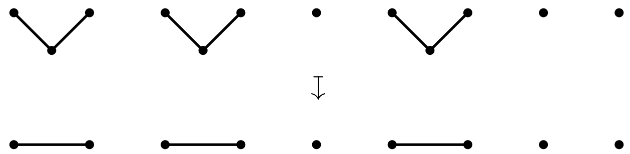

FIGURE 16. An example of the bijective correspondence between elementary forests with 9 leaves and simplices of $\mathcal{M}(L_8)$.

with n vertices, labeled 1 through n, and $n-1$ edges, one connecting i to $i+1$ for each $1 \le i < n$. Let $\mathcal{M}(L_{n-1})$ be the matching complex of L_{n-1}.

Observation 4.1. Elementary forests with n leaves correspond bijectively to simplices of $\mathcal{M}(L_{n-1})$. Under the identification, carets correspond to edges. See Figure 16 for an example.

In light of the observation, we can denote an elementary n-braige by (b, Γ), where $b \in B_n$ and Γ is a simplex in $\mathcal{M}(L_{n-1})$. As usual, the equivalence class under dangling will be denoted $[(b, \Gamma)]$.

Let S be the unit disk, and fix an embedding $L_{n-1} \hookrightarrow S$ of the linear graph with $n-1$ edges into S. Let P be the image of the vertex set, so P is a set of n points in S, labeled 1 through n. With these data in place we can consider $\mathcal{MA}(K_n)$, the matching complex on (S, P), and we have an induced embedding of simplicial complexes $\mathcal{M}(L_{n-1}) \hookrightarrow \mathcal{MA}(K_n)$. The braid group B_n on n strands is isomorphic to the mapping class group of the n-punctured disc D_n, cf. [Bir74]. Since $S \setminus P = D_n$, we have an action of B_n on $\mathcal{MA}(K_n)$. In what follows it will be convenient to consider this as a right action (much as dangling is a right action on braiges), so for $b \in B_n$ and $\sigma \in \mathcal{MA}(K_n)$ we will write $(\sigma)b$ to denote the image of σ under b.

Define a map π from $\mathcal{EB}_n$ to $\mathcal{MA}(K_n)$ as follows. We view $\mathcal{M}(L_{n-1})$ as a subcomplex of $\mathcal{MA}(K_n)$, so we can associate to any elementary n-braige (b, Γ) the arc complex $(\Gamma)b^{-1}$ in $\mathcal{MA}(K_n)$. By construction, the map $(b, \Gamma) \mapsto (\Gamma)b^{-1}$ is well defined on equivalence classes under dangling, so we obtain a simplicial map

$$\begin{aligned} \pi \colon EB_n &\to MA(K_n) \\ [(b, \Gamma)] &\mapsto (\Gamma)b^{-1} \ . \end{aligned}$$

Note that π is surjective, but not injective.

One can visualize this map by considering the merges as arcs, then "combing straight" the braid and seeing where the arcs are taken, as in Figure 17. Note that the resulting simplex $(\Gamma)b^{-1}$ of $\mathcal{MA}(K_n)$ has the same dimension as the simplex $[(b, \Gamma)]$ of $\mathcal{EB}_n$, namely one less than the number of edges in Γ.

The next lemma and proposition are concerned with the fibers of π.

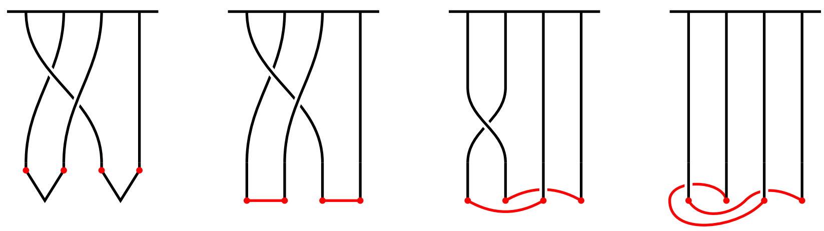

FIGURE 17. From braiges to arc systems. From left to right the pictures show the process of "combing straight" the braid.

Lemma 4.2. *Let E and Γ be simplices in $\mathcal{M}(L_{n-1})$, such that E has one edge and Γ has $e(\Gamma)$ edges. Let $[(b, E)]$ and $[(c, \Gamma)]$ be dangling elementary n-braiges. Suppose that their images under the map π are contained in a simplex of $\mathcal{MA}(K_n)$. Then there exists a simplex in $\mathcal{EB}_n$ that contains $[(b, E)]$ and $[(c, \Gamma)]$.*

Proof. There is an action of B_n on $\mathcal{EB}_n$ ("from above"), given by $b'[(c', \Gamma')] = [(b'c', \Gamma')]$. One can check that for each $k \geq 0$, this action is transitive on the k-simplices of $\mathcal{EB}_n$. We can therefore assume without loss of generality that $c = \mathrm{id}$, and Γ is the subgraph of L_{n-1} whose edges are precisely those connecting j to $j+1$, for $j \in \{1, 3, \dots, 2e(\Gamma) - 1\}$.

Choose an arc α representing $\pi([(b, E)])$ and an arc system σ representing $\pi([(\mathrm{id}, \Gamma)])$. If this can be done so that α is an arc in σ, then we are done, so assume this is not the case. Then we may take σ to be Γ, and choose α disjoint from Γ. This disjointness ensures that, after dangling, we can assume the following condition on b: for each edge of Γ, say with endpoints j and $j + 1$, b can be represented as a braid in such a way that the j^{th} and $(j+1)^{\text{st}}$ strands of b run straight down, parallel to each other, and no strands cross between them. In particular $[(b, \Gamma)] = [(\mathrm{id}, \Gamma)]$, so $[(b, \Gamma \cup E)]$ is a simplex in $\mathcal{EB}_n$ with $[(b, E)]$ and $[(\mathrm{id}, \Gamma)]$ as faces. $\square$

Proposition 4.3. *Let σ be a k-simplex in $\mathcal{MA}(K_n)$ with vertices $v_0, \dots, v_k$. Then*

$$\pi^{-1}(\sigma) = \mathop{\ast}_{j=0}^{k} \pi^{-1}(v_j).$$

In particular $\pi^{-1}(\sigma)$ is k-spherical.

Proof. "$\subseteq$": Suppose that $[(b, \Gamma)] \in \pi^{-1}(\sigma)$, so $\sigma = (\Gamma)b^{-1}$. The vertices of $[(b, \Gamma)]$ are of the form $[(b, E)]$ for E a one-edge subgraph of Γ, and $[(b, E)]$ maps under π to $(E)b^{-1}$. The arcs of Γ are precisely these E, so for each such E, $(E)b^{-1} = v_j$ for some j. Since π is a simplicial map, we conclude that $[(b, \Gamma)] \in \ast_{j=0}^{k} \pi^{-1}(v_j)$.

"$\supseteq$": The 0-skeleton of $\ast_{j=0}^{k} \pi^{-1}(v_j)$ is automatically contained in $\pi^{-1}(\sigma)$. Now assume that the same is true of the r-skeleton, for $r > 0$. Let τ be an $(r + 1)$-simplex in $\ast_{j=0}^{k} \pi^{-1}(v_j)$, and decompose τ as the join of a vertex $[(b, E)]$ and an r-simplex $[(c, \Gamma)]$. By induction, these are both in $\pi^{-1}(\sigma)$, and by Lemma 4.2 they share a simplex in $\mathcal{EB}_n$. The minimal dimensional such simplex maps to σ under π, so we are done. $\square$

Recall that in Sections 3.2 and 3.3 we defined the integers $\nu(n) = \lfloor \frac{n+1}{3} \rfloor - 1$ and $\eta(n) = \lfloor \frac{n-1}{4} \rfloor$.

Corollary 4.4. *$\mathcal{EB}_n$ is $(\nu(n) - 1)$-connected. Hence for any vertex x in X with $f(x) = n$, $\mathrm{lk}{\downarrow}(x)$ is $(\nu(n) - 1)$-connected.*

Proof. We know that $\mathcal{MA}(K_n)$ is $(\nu(n) - 1)$-connected. For any k-simplex σ in $\mathcal{MA}(K_n)$, $\pi^{-1}(\sigma)$ is $(k - 1)$-connected. Also, $\mathrm{lk}(\sigma)$ is isomorphic to $\mathcal{MA}(K_{n-2k-2})$, which is $(\nu(n - 2k - 2) - 1)$-connected and hence $(\nu(n) - k - 2)$-connected. It follows from [Qui78, Theorem 9.1] that $\mathcal{EB}_n$ is $(\nu(n) - 1)$-connected. $\square$

In the pure case, we consider descending links of vertices in $X(F_{\mathrm{br}})$. For a vertex x with $n + 1$ feet, $\mathrm{lk}{\downarrow}(x)$ is isomorphic to the poset $P\mathcal{EB}_{n+1}$ of dangling elementary pure $(n + 1)$-braiges. This projects onto the complex $\mathcal{MA}(L_n)$, using an analogous projection as from $\mathcal{EB}_n$ to $\mathcal{MA}(K_n)$. (Recall that L_n is indexed by the number of edges, not nodes.) By the same argument as in the previous proof, we can get the connectivity of $P\mathcal{EB}_{n+1}$ from that of $\mathcal{MA}(L_n)$.

Corollary 4.5. *$P\mathcal{EB}_{n+1}$ is $(\eta(n) - 1)$-connected.* $\square$

From the above corollaries and the Morse Lemma, we conclude the following

Corollary 4.6. *For each $n \geq 1$, the pair $(X^{\leq n}, X^{<n})$ is $\nu(n)$-connected and the pair $(X(F_{\mathrm{br}})^{\leq n}, X(F_{\mathrm{br}})^{<n})$ is $\eta(n-1)$-connected.* □

5. Proof of the Main Theorem

We are now in a position to prove our main theorem.

Proof of Main Theorem. Consider the action of V_{br} on the complex X, which is contractible by Corollary 2.4. We want to apply Brown's Criterion. By Corollary 2.8, all cell stabilizers are of type F_∞. By Lemma 2.5, each $X^{\leq n}$ is finite modulo V_{br}, and by Corollary 4.6 the connectivity of the pair $(X^{\leq n}, X^{<n})$ tends to ∞ as n tends to ∞. Hence Brown's Criterion tells us that V_{br} is of type F_∞. A parallel argument applies to F_{br} acting on $X(F_{\mathrm{br}})$, and so F_{br} is of type F_∞. □

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