Log-periodic asymptotic expansion of the spectral partition functions on self-similar sets

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Abstract: Let $\mathcal{Z}(t)$ be the partition function (the trace of the heat semigroup) of the canonical Laplacian on a post-critically finite self-similar set (with uniform resistance scaling factor and some good symmetry) or on a generalized Sierpiński carpet. It is proved that $\mathcal{Z}(t) = \sum_{k=0}^{n} t^{-d_k/d_\mathrm{w}} G_k(-\log t) + O\bigl(\exp(-ct^{-\frac{1}{d_\mathrm{w}-1}})\bigr)$ as $t \downarrow 0$ for some continuous periodic functions $G_k : \mathbb{R} \to \mathbb{R}$ and $c \in (0,\infty)$. Here $d_\mathrm{w} \in (1,\infty)$ denotes the walk dimension, $n = 1$ for a post-critically finite self-similar set and $n = d$ for a d-dimensional generalized Sierpiński carpet, $\{d_k\}_{k=0}^{n} \subset [0,\infty)$ is strictly decreasing with $d_n = 0$, G_0 is strictly positive and G_1 is either strictly positive or strictly negative depending on the (Neumann or Dirichlet) boundary condition. Furthermore in the setting of a post-critically finite self-similar set, the same result with d_w, d_0, G_0 the same, G_1 strictly negative, $n = 2$ and $d_2 = 0$ is proved also for the partition function of the Laplacian with Dirichlet boundary condition on a general self-similar subset.

Key words. Self-similar Dirichlet form – Laplacian eigenvalues – Partition function – Short time asymptotics – Post-critically finite self-similar sets – Generalized Sierpiński carpets
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1. Introduction

Asymptotic distribution of the eigenvalues is a central topic for analysis of Laplacians and elliptic differential operators. Such an operator is typically given as a non-positive self-adjoint operator A on an L^2-space whose spectrum is discrete, and then asymptotic

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distribution of its eigenvalues referes to asymptotic behavior of the *eigenvalue counting function*

$$\mathcal{N}_A(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^A \le \lambda\} \tag{1.1}$$

as $\lambda \to \infty$, or of the *(spectral) partition function*

$$\mathcal{Z}_A(t) := \operatorname{tr} e^{tA} = \sum_{n\in\mathbb{N}} e^{-\lambda_n^A t} = \int_{\mathbb{R}} e^{-\lambda t} d\mathcal{N}_A(\lambda) \tag{1.2}$$

as $t \downarrow 0$; here $\{\lambda_n^A\}_{n\in\mathbb{N}}$ denotes the non-decreasing enumeration of all the eigenvalues of $-A$ with each eigenvalue repeated according to its multiplicity.

Analysis of the eigenvalue counting functions of Laplacians was initiated by Weyl in [59, 60], where he proved that for the Dirichlet Laplacian Δ_U on a (sufficiently regular) bounded non-empty open subset U of $\mathbb{R}^d$,

$$\mathcal{N}_{\Delta_U}(\lambda) = c_d \operatorname{vol}_d(U)\lambda^{d/2} + o(\lambda^{d/2}) \qquad \text{as } \lambda \to \infty, \tag{1.3}$$

with vol_d the Lebesgue measure on $\mathbb{R}^d$ and $c_d := (2\pi)^{-d} \operatorname{vol}_d\bigl(\{x \in \mathbb{R}^d \mid |x| < 1\}\bigr)$. By Karamata's Tauberian theorem [19, Section XIII.5, Theorem 2], (1.3) is equivalent to

$$\mathcal{Z}_{\Delta_U}(t) = (4\pi)^{-d/2} \operatorname{vol}_d(U)t^{-d/2} + o(t^{-d/2}) \qquad \text{as } t \downarrow 0, \tag{1.4}$$

which can be in fact easily verified for any non-empty open subset U of $\mathbb{R}^d$ with $\operatorname{vol}_d(U) < \infty$ by slight analysis of the Dirichlet heat kernel $p_t^U(x, y)$ on U and the expression

$$\mathcal{Z}_{\Delta_U}(t) = \int_U p_t^U(x, x)dx, \qquad t \in (0, \infty). \tag{1.5}$$

Later in [34] Kac posed his famous question *"Can one hear the shape of a drum?"*, meaning whether the knowledge of the eigenvalues $\{\lambda_n^U\}_{n\in\mathbb{N}}$ of $-\Delta_U$ determines the geometry of the open set U. Although the answer to this original question of Kac is negative as shown by Urakawa [58] for $d \ge 4$ and by Gordon, Webb and Wolpert [26] for $d = 2$, his question motivated numerous works on further detailed asymptotics of $\mathcal{N}_{\Delta_U}(\lambda) - c_d \operatorname{vol}_d(U)\lambda^{d/2}$ and $\mathcal{Z}_{\Delta_U}(t) - (4\pi)^{-d/2} \operatorname{vol}_d(U)t^{-d/2}$. A key observation in this direction due to [14] is that, if the boundary of U is fractal, then the box-counting (Minkowski) dimension of the boundary of U, *not* its Hausdorff dimension, should be involved in the remainder estimate for the asymptotics (1.3) and (1.4). See e.g. [14, 15, 47–50, 52] and references therein for further details in regard to possible refinements of (1.3) and (1.4) in the settings of Euclidean domains and Riemannian manifolds.

The purpose of this paper is to establish similar detailed asymptotic behavior, *beyond the principal order term*, of the partition functions of Laplacians on self-similar sets. Our main results are stated and proved for two large classes of self-similar sets, known as *post-critically finite self-similar sets* (with additional assumption of some geometric symmetry) and *generalized Sierpiński carpets* defined in [7, 8]. In this introduction we illustrate our main results by treating the particular cases of the canonical Laplacians on the Sierpiński gasket and the Sierpiński carpet (see Fig. 1 below), which are among the simplest self-similar fractals and have been intensively studied.

A crucial difference between these two fractals, and also between the two above-mentioned classes of self-similar sets, is that the former are *finitely ramified*, i.e. can be made disconnected by removing a finite subset, while the latter are *infinitely ramified*, i.e. not finitely ramified. For the Sierpiński gasket, the set V_0^{SG} of the three vertices of

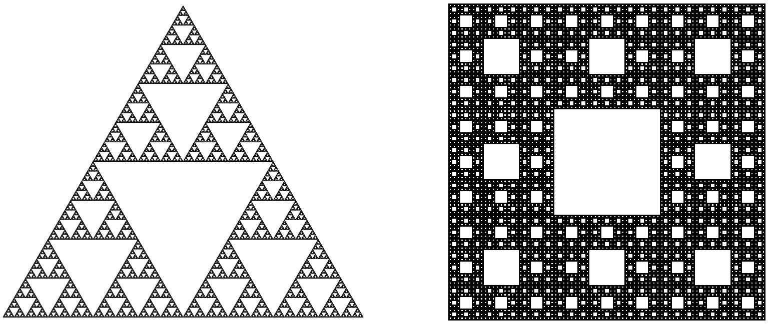

Fig. 1. Sierpiński gasket and Sierpiński carpet

the enclosing equilateral triangle is considered as its *boundary* in the sense that any two distinct copies of itself can intersect only on the copies of V_0^{SG}, whereas the same role is played by the natural boundary V_0^{SC} of the enclosing unit square in the case of the Sierpiński carpet. Then being finitely or infinitely ramified amounts to the finiteness or the infiniteness of the boundary of the self-similar set.

We first recall some basics of the canonical Laplacians on the Sierpiński gasket and the Sierpiński carpet. Let K_{SG} denote the Sierpiński gasket and μ the d_{f}-dimensional Hausdorff measure on K_{SG} with respect to the Euclidean metric $\rho(x, y) := |x - y|$, where $d_{\mathrm{f}} := \log_2 3$ is the Hausdorff dimension of K_{SG} with respect to ρ. The canonical Laplacian on K_{SG} was first constructed by Goldstein [25] and Kusuoka [45] as the generator of a natural μ-symmetric diffusion process on K_{SG}. Their diffusion was obtained as the weak limit of suitably scaled simple random walks on the natural graph approximations and is now called the *Brownian motion on the Sierpiński gasket*. Soon later it turned out to be much easier to construct the Dirichlet form of this μ-symmetric diffusion directly by simple analytic arguments, as described e.g. in [2,40,57], so that the canonical Laplacian and the Brownian motion on K_{SG} can be alternatively obtained by applying the general theory of regular Dirichlet forms presented in [23,16].

A fundamental property of the canonical Laplacian on K_{SG} is the following *sub-Gaussian estimate* of the associated heat kernel due to Barlow and Perkins [10, Theorem 1.5]: the associated heat semigroup has a jointly continuous heat kernel $p_t(x, y)$, and there exist $c_{1.1}, c_{1.2} \in (0, \infty)$ such that for any $(t, x, y) \in (0, 1] \times K_{\mathrm{SG}} \times K_{\mathrm{SG}}$,

$$\frac{c_{1.1}}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}} \exp\biggl(-\Bigl(\frac{\rho(x, y)^{d_{\mathrm{w}}}}{c_{1.1}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr) \leq p_t(x, y) \leq \frac{c_{1.2}}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}} \exp\biggl(-\Bigl(\frac{\rho(x, y)^{d_{\mathrm{w}}}}{c_{1.2}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr), \tag{1.6}$$

where $d_{\mathrm{w}} := \log_2 5$. (1.6) is called *sub*-Gaussian since the exponent d_{w}, the so-called *walk dimension*, satisfies $d_{\mathrm{w}} > 2$, which is known to hold quite in general for Laplacians on fractals.

In the case of the Sierpiński carpet, the infiniteness of its boundary set V_0^{SC}, the natural boundary of the enclosing unit square, makes every step of analysis much more difficult than for post-critically finite self-similar sets. Let K_{SC} denote the Sierpiński carpet and let μ be the d_{f}-dimensional Hausdorff measure on K_{SC} with respect to the Euclidean metric $\rho(x, y) := |x - y|$, where $d_{\mathrm{f}} := \log_3 8$ is the Hausdorff dimension of K_{SC} with respect to ρ. A natural non-degenerate diffusion on K_{SC} was constructed for the first time by M. T. Barlow and R. F. Bass in [3], and later Kusuoka and Zhou [46] also obtained one by constructing a self-similar regular Dirichlet form on $L^2(K_{\mathrm{SC}}, \mu)$.

In fact, it was only very recently that these two diffusions were proved to be the same, as a consequence of the uniqueness result by Barlow, Bass, Kumagai and Teplyaev [8]. By the results of [8], together with slight additional arguments in [33, Proof of Proposition 5.1] and [38, Proposition 5.9], now it is known that there exists a unique non-zero conservative self-similar regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K_{\mathrm{SC}}, \mu)$ that is invariant under the isometries of the enclosing unit square. Thus the canonical Laplacian on K_{SC} is obtained as the non-positive self-adjoint operator on $L^2(K_{\mathrm{SC}}, \mu)$ associated with $(\mathcal{E}, \mathcal{F})$, and furthermore by [6,8], the associated heat semigroup has a jointly continuous heat kernel $p_t(x, y)$ which satisfies (1.6) for a certain specific choice of $d_{\mathrm{w}} \in (2, \infty)$.

Now for $\mathrm{F} \in \{\mathrm{SG}, \mathrm{SC}\}$, let $\mathcal{Z}_{\mathrm{N}}^{\mathrm{F}}$ be the partition function of the canonical Laplacian on K_{F}, which satisfies the Neumann boundary condition on the boundary V_0^{F} of K_{F} in a certain natural sense, and let $\mathcal{Z}_{\mathrm{D}}^{\mathrm{F}}$ be the partition function of the Laplacian with Dirichlet boundary condition on V_0^{F}. In fact, for post-critically finite self-similar sets we will also consider cases with Dirichlet boundary condition on general self-similar subsets, and as one of the simplest examples let L be the line segment at the bottom of K_{SG} and let $\mathcal{Z}_{L,\mathrm{N}}^{\mathrm{SG}}$ and $\mathcal{Z}_{L,\mathrm{D}}^{\mathrm{SG}}$ be the partition functions of the Laplacian on K_{SG} with Dirichlet boundary condition on L and on $L \cup V_0^{\mathrm{SG}}$, respectively. Then our main results for post-critically finite self-similar sets (Theorems 3.4 and 4.2) yield the following asymptotics.

Theorem 1.1. *Set* $d_{\mathrm{f}} := \log_2 3$, $d_{\mathrm{w}} := \log_2 5$, $s_{\mathrm{N}} := 1$ *and* $s_{\mathrm{D}} := -1$. *Then there exist* $c_{1.3} \in (0, \infty)$ *and continuous* $\log 5$*-periodic functions* $G_0, G_1, G_L : \mathbb{R} \to (0, \infty)$ *such that for any* $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, *as* $t \downarrow 0$,

$$\mathcal{Z}_{\mathrm{B}}^{\mathrm{SG}}(t) = t^{-d_{\mathrm{f}}/d_{\mathrm{w}}} G_0(-\log t) + \frac{3}{2} s_{\mathrm{B}} G_1(-\log t) + O\Bigl(\exp\bigl(-c_{1.3} t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr), \quad (1.7)$$

$$\mathcal{Z}_{L,\mathrm{B}}^{\mathrm{SG}}(t) = t^{-d_{\mathrm{f}}/d_{\mathrm{w}}} G_0(-\log t) - t^{-1/d_{\mathrm{w}}} G_L(-\log t) + \frac{1}{2} s_{\mathrm{B}} G_1(-\log t) + O\Bigl(\exp\bigl(-c_{1.3} t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \quad (1.8)$$

On the other hand, our main result for generalized Sierpiński carpets (Theorem 5.11) implies the following asymptotics of $\mathcal{Z}_{\mathrm{B}}^{\mathrm{SC}}$.

Theorem 1.2. *Set* $d_{\mathrm{f}} := \log_3 8$, *let* $d_{\mathrm{w}} \in (2, \infty)$ *be as in* (1.6) *for* K_{SC} *and set* $\tau := 3^{d_{\mathrm{w}}}$. *Then there exist* $c_{1.4} \in (0, \infty)$ *and continuous* $\log \tau$*-periodic functions* $G_{\mathrm{B},k} : \mathbb{R} \to \mathbb{R}$, $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, $k \in \{0, 1, 2\}$, *such that for any* $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, *as* $t \downarrow 0$,

$$\mathcal{Z}_{\mathrm{B}}^{\mathrm{SC}}(t) = t^{-d_{\mathrm{f}}/d_{\mathrm{w}}} G_{\mathrm{B},0}(-\log t) + t^{-1/d_{\mathrm{w}}} G_{\mathrm{B},1}(-\log t) + G_{\mathrm{B},2}(-\log t) + O\Bigl(\exp\bigl(-c_{1.4} t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \quad (1.9)$$

Moreover, $G_{\mathrm{N},0}, G_{\mathrm{N},1}$ *are* $(0, \infty)$*-valued,* $G_{\mathrm{N},0} = G_{\mathrm{D},0}$ *and* $\frac{5}{8} G_{\mathrm{N},1} = -\frac{5}{12} G_{\mathrm{D},1}$.

Note that the factors 5 and τ appearing in the periods of G_0, G_1, G_L and $G_{\mathrm{B},k}$ are the scaling factors for the time variable of the heat kernels; see Lemma 2.21, its proof and Example 4.5 below.

As remarked in the following paragraphs, the principal order terms in (1.7), (1.8) and (1.9) were already known. The new results here are the existence of the other *only finitely many* periodic functions and the *exponentially decaying* remainder estimate. The periodic functions G_1, G_L and $G_{\mathrm{B},k}$, $k \in \{1, 2\}$ derive from, roughly speaking, the self-similarity of each "piece" of the boundary sets V_0^{SG}, L and V_0^{SC}, and the order estimates $t^{-0/d_{\mathrm{w}}}$ and $t^{-1/d_{\mathrm{w}}}$ for these terms result from their Euclidean box-counting (or equivalently, Hausdorff) dimensions. Since the pieces $\{q\}$, $q \in V_0^{\mathrm{SG}}$, of V_0^{SG} do not intersect with each other, in (1.7) we end up with just one extra log-periodic term and an exponentially decaying remainder. On the other hand, L consists of two copies of itself of half length intersecting with each other at the midpoint of L, which has some influence on the constant order term and thereby gives a different coefficient for the constant order term of (1.8). Similarly, V_0^{SC} consists of four edges, whose self-similarity gives rise to $G_{\mathrm{B},1}$ of the second term in (1.9), but they intersect with each other at vertices of the unit square and the three copies of each edge of one third length also intersect with each other, which yields another term $G_{\mathrm{B},2}(-\log t)$ of constant order.

The strict positivity of G_1, G_L in Theorem 1.1 and $G_{\mathrm{N},1}$ in Theorem 1.2, which reflects the influence of the Dirichlet boundary conditions on the eigenvalues and the heat kernels, could be deduced from the author's general result [35, Theorem 7.7], but we provide an alternative proof of this fact based on detailed calculations of the partition functions. On the other hand, $G_{\mathrm{B},2}$ in the third term of (1.9) is a sum of strictly positive and strictly negative $\log \tau$-periodic functions and *the author has no idea which signs* $G_{\mathrm{B},2}$ *actually takes*; see also Theorem 5.15 and its proof below in this connection.

It is very difficult to obtain any further information on the periodic functions G_1, G_L and $G_{\mathrm{B},k}$. In fact, G_0 in Theorem 1.1 is known to be non-constant as explained in the next paragraph, whereas *the author has no idea whether any one of the other periodic functions* G_1, G_L *in Theorem* 1.1 *and* $G_{\mathrm{B},k}$, $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, $k \in \{0, 1, 2\}$ *in Theorem* 1.2 *is non-constant.*

The existence of G_0 in the principal order term for $\mathcal{Z}_{\mathrm{B}}^{\mathrm{SG}}$ in (1.7) is an immediate consequence of the known result for the corresponding eigenvalue counting functions $\mathcal{N}_{\mathrm{B}}^{\mathrm{SG}}$ due to Kigami and Lapidus [43, Theorem 2.4 and Corollary 2.5]: there exists a right-continuous $\log 5$-periodic function $G_{\mathcal{N}} : \mathbb{R} \to \mathbb{R}$ satisfying $0 < \inf_{s \in \mathbb{R}} G_{\mathcal{N}}(s) \leq \sup_{s \in \mathbb{R}} G_{\mathcal{N}}(s) < \infty$ such that for any $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$,

$$\mathcal{N}_{\mathrm{B}}^{\mathrm{SG}}(\lambda) = \lambda^{d_{\mathrm{f}}/d_{\mathrm{w}}} G_{\mathcal{N}}(\log \lambda) + O(1) \qquad \text{as } \lambda \to \infty. \tag{1.10}$$

Moreover, $G_{\mathcal{N}}$ is *discontinuous* and hence non-constant by [9, Theorems 4.4 and 6.6] or [40, Theorem 4.3.4 and Corollary 4.4.11] (see also [24, Theorem 5.2] and [37, Lemma 3.5]), and consequently G_0 in Theorem 1.1 is also non-constant. In view of (1.10), one may expect further detailed asymptotics of $\mathcal{N}_{\mathrm{B}}^{\mathrm{SG}}$ and $\mathcal{N}_{L,\mathrm{B}}^{\mathrm{SG}}$ analogous to (1.7) and (1.8), where $\mathcal{N}_{L,\mathrm{B}}^{\mathrm{SG}}$ is the eigenvalue counting function corresponding to $\mathcal{Z}_{L,\mathrm{B}}^{\mathrm{SG}}$, but at this moment *the author has no idea how* $\mathcal{N}_{\mathrm{B}}^{\mathrm{SG}}(\lambda) - \lambda^{d_{\mathrm{f}}/d_{\mathrm{w}}} G_{\mathcal{N}}(\log \lambda)$ *actually behaves as* $\lambda \to \infty$ *or whether* $\mathcal{N}_{L,\mathrm{B}}^{\mathrm{SG}}(\lambda) - \lambda^{d_{\mathrm{f}}/d_{\mathrm{w}}} G_{\mathcal{N}}(\log \lambda) = o(\lambda^{d_{\mathrm{f}}/d_{\mathrm{w}}})$ *as* $\lambda \to \infty$. In fact, in this particular case of the Sierpiński gasket, some explicit knowledge of the eigenvalues is available in [24,39,56] and there might be some hope of answering those questions by using their results. We leave investigation of this possibility to future studies.

For the Sierpiński carpet, the existence of $G_{\mathrm{D},0}$ in the principal order term for $\mathcal{Z}_{\mathrm{D}}^{\mathrm{SC}}$ in (1.9) was proved by Hambly [30, Theorem 1.1] with the help of the upper inequality in (1.6) and an argument on the associated diffusion, and the same result for $\mathcal{Z}_{\mathrm{N}}^{\mathrm{SC}}$ with $G_{\mathrm{N},0} = G_{\mathrm{D},0}$ can be easily verified by following his argument. (See [35, Theorems 5.2, 5.11, 7.7 and Corollary 7.8] for some generalization and refinement of Hambly's arguments in [30].) On the other hand, for the corresponding eigenvalue counting functions $\mathcal{N}_{\mathrm{B}}^{\mathrm{SC}}$, *even their log-periodic asymptotic behavior similar to* (1.10) *is not known.*

Finally, we also remark that *we have to assume the scaling factors for the self-similar measure and the self-similar Dirichlet form to be uniform among all the cells* (see Definitions 2.4 and 2.6 below), for the sake of the validity of the required renewal theorem (Theorem 2.23) and of simplicity in its actual use. In fact, Fleckinger, Levitin and Vassiliev [21,22] and van den Berg [11,12] obtained short time asymptotics very similar to Theorems 1.1 and 1.2 of the integral of the solution to the heat equation on certain von Koch snowflake domains in $\mathbb{R}^2$ with initial value 0 and boundary value 1. Since van den Berg [11,12] allowed the domains to have different scaling factors for different pieces of the Koch curves constituting the boundary, his method could enable us to relax our assumption of the uniformity of the scaling factors, which we leave to future studies.

The rest of this paper is organized as follows. In Section 2, we introduce our framework of self-similar Dirichlet forms on self-similar sets and collect preliminary facts and lemmas for the proofs of our main results. The key fact is Proposition 2.16, which along with Lemma 2.22 makes it possible to extract lower order terms by formal calculations of the heat kernels with Dirichlet boundary conditions on different subsets. In Section 3, after recalling basics of self-similar Dirichlet forms on post-critically finite self-similar sets, we state and prove the result (Theorem 3.4) for the partition functions of the Laplacians on post-critically finite self-similar sets with Neumann and Dirichlet boundary conditions on the canonical boundary V_0. Section 4 treats the result for the Laplacians on post-critically finite self-similar sets with Dirichlet boundary condition on general self-similar subsets (Theorem 4.2). In Section 5, we first collect important facts concerning generalized Sierpiński carpets and their canonical self-similar Dirichlet form and then state the main result for them (Theorem 5.11). In fact, Theorem 5.11 is essentially a special case of Theorem 5.15 on more detailed information on the lower order terms. Finally, Section 6 is devoted to the proof of Theorems 5.11 and 5.15.

Notation. In this paper, we adopt the following notation and conventions.
(1) $\mathbb{N} = \{1, 2, 3, \dots\}$, i.e. $0 \notin \mathbb{N}$.
(2) The cardinality (the number of elements) of a set A is denoted by $\#A$.
(3) We set $\sup \emptyset := \max \emptyset := 0$, $\inf \emptyset := \min \emptyset := \infty$ and set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ for $a, b \in [-\infty, \infty]$. All functions in this paper are assumed to be $[-\infty, \infty]$-valued.
(4) For $d \in \mathbb{N}$, $\mathbb{R}^d$ is always equipped with the Euclidean norm $|\cdot|$.
(5) Let E be a topological space. The Borel σ-field of E is denoted by $\mathcal{B}(E)$. We set $C(E) := \{u \mid u : E \to \mathbb{R}, u \text{ is continuous}\}$, $\operatorname{supp}_E[u] := \overline{\{x \in E \mid u(x) \neq 0\}}$ and $\|u\|_\infty := \sup_{x \in E} |u(x)|$ for $u \in C(E)$. For $A \subset E$, $\operatorname{int}_E A$ denotes its interior in E.
(6) Let E be a set, $\rho : E \times E \to [0, \infty)$ and $x \in E$. We set $\operatorname{diam}_\rho A := \sup_{y,z \in A} \rho(y, z)$ and $\rho(x, A) := \inf_{y \in A} \rho(x, y)$ for $A \subset E$ and $B_r(x, \rho) := \{y \in E \mid \rho(x, y) < r\}$ for $r \in (0, \infty)$.

2. Preliminaries

In this section, we introduce our framework of a self-similar set and a self-similar Dirichlet form on it and prepare preliminary facts concerning the heat kernels and the eigenvalues of the Laplacians. At the end of this section, we also state and prove a version of the renewal theorem which involves log-periodic reminder terms.

Let us start with standard notions concerning self-similar sets. We refer to [40, Chapter 1] and [41, Section 1.2] for details. Throughout this section, we fix a compact metrizable topological space K with $\#K \geq 2$, a non-empty finite set S and a continuous injective map $F_i : K \to K$ for each $i \in S$. We set $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$.

Definition 2.1. (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_m := S^m = \{w_1 \dots w_m \mid w_i \in S \text{ for } i \in \{1, \dots, m\}\}$ for $m \in \mathbb{N}$ and let $W_* := \bigcup_{m \in \mathbb{N} \cup \{0\}} W_m$. For $w \in W_*$, the unique $m \in \mathbb{N} \cup \{0\}$ satisfying $w \in W_m$ is denoted by $|w|$ and called the *length of* w.
(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3 \dots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology, and define the *shift map* $\sigma : \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3 \dots) := \omega_2\omega_3\omega_4 \dots$. For $i \in S$ we define $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3 \dots) := i\omega_1\omega_2\omega_3 \dots$. For $\omega = \omega_1\omega_2\omega_3 \dots \in \Sigma$ and $m \in \mathbb{N} \cup \{0\}$, we write $[\omega]_m := \omega_1 \dots \omega_m \in W_m$.
(3) For $w = w_1 \dots w_m \in W_*$, we set $F_w := F_{w_1} \circ \dots \circ F_{w_m}$ ($F_\emptyset := \mathrm{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \dots \circ \sigma_{w_m}$ ($\sigma_\emptyset := \mathrm{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$, and if $w \neq \emptyset$ then $w^\infty \in \Sigma$ is defined by $w^\infty := www \dots$ in the natural manner.

Definition 2.2. $\mathcal{L}$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for any $i \in S$. Note that such π, if exists, is unique and satisfies $\{\pi(\omega)\} = \bigcap_{m \in \mathbb{N}} K_{[\omega]_m}$ for any $\omega \in \Sigma$.

In what follows we always assume that $\mathcal{L}$ is a self-similar structure, so that $\#S \geq 2$ by $\#K \geq 2$ and $\pi(\Sigma) = K$. For $A \subset K$, the closure of A in K is denoted by $\overline{A}$.

Definition 2.3. (1) We define the *critical set* $\mathcal{C}$ and the *post-critical set* $\mathcal{P}$ *of* $\mathcal{L}$ by

$$\mathcal{C} := \pi^{-1}\Big(\bigcup_{i,j \in S, i \neq j} K_i \cap K_j\Big) \qquad \text{and} \qquad \mathcal{P} := \bigcup_{n \in \mathbb{N}} \sigma^n(\mathcal{C}). \tag{2.1}$$

$\mathcal{L}$ is called *post-critically finite*, or *p.-c. f.* for short, if and only if $\mathcal{P}$ is a finite set.
(2) We set $V_0 := \pi(\mathcal{P})$, $V_m := \bigcup_{w \in W_m} F_w(V_0)$ for $m \in \mathbb{N}$ and $V_* := \bigcup_{m \in \mathbb{N}} V_m$.
(3) We set $K^I := K \setminus \overline{V_0}$ and $K^I_w := F_w(K^I)$ for $w \in W_*$.

V_0 should be considered as the *"boundary"* of the self-similar set K; recall that $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w \cap \Sigma_v = \emptyset$ by [40, Proposition 1.3.5-(2)]. According to [40, Lemma 1.3.11], $V_{m-1} \subset V_m$ for any $m \in \mathbb{N}$, and if $V_0 \neq \emptyset$ then V_* is dense in K. Furthermore by [35, Lemma 2.11], K^I_w is open in K and $K^I_w \subset K^I$ for any $w \in W_*$.

Note that by [40, Theorem 1.6.2], K is connected if and only if any $i, j \in S$ admit $n \in \mathbb{N}$ and $\{i_k\}_{k=0}^n \subset S$ with $i_0 = i$ and $i_n = j$ such that $K_{i_{k-1}} \cap K_{i_k} \neq \emptyset$ for any $k \in \{1, \dots, n\}$, and if K is connected then it is arcwise connected.

Definition 2.4. A Borel probability measure μ on K is called a *self-similar measure on* $\mathcal{L}$ *with uniform weight* if and only if the following equality (of Borel measures on K) holds:

$$\mu = \frac{1}{\#S} \sum_{i \in S} \mu \circ F_i^{-1}. \tag{2.2}$$

There exists a self-similar measure on $\mathcal{L}$ with uniform weight. Indeed, if ν is the Bernoulli measure on Σ with weight $(1/\#S)_{i \in S}$, then $\nu \circ \pi^{-1}$ is such a self-similar measure on $\mathcal{L}$; see [40, Section 1.4] for details. Moreover by [41, Theorem 1.2.7 and its proof], if $K \neq \overline{V_0}$ and μ is a self-similar measure on $\mathcal{L}$ with uniform weight, then $\mu(K_w) = (1/\#S)^{|w|}$ and $\mu(F_w(\overline{V_0})) = 0$ for any $w \in W_*$. In particular, a self-similar measure on $\mathcal{L}$ with uniform weight is unique if $K \neq \overline{V_0}$.

The following lemma is immediate from the above-mentioned facts.

Lemma 2.5. *Assume $K \neq \overline{V_0}$, let μ be the self-similar measure on $\mathcal{L}$ with uniform weight and let $w \in W_*$. Then $\int_K |u \circ F_w| d\mu = (\#S)^{|w|} \int_{K_w} |u| d\mu$ for any Borel measurable $u : K \to [-\infty, \infty]$. In particular, if we set $F^*_w u := u \circ F_w$ for $u : K \to [-\infty, \infty]$, then F^*_w defines a bounded linear operator $F^*_w : L^2(K, \mu) \to L^2(K, \mu)$.*

Next we define the notion of a self-similar Dirichlet space with uniform weights and state its basic properties. See [23, Section 1.1] for basic notions concerning symmetric regular Dirichlet forms.

Definition 2.6 (Self-similar Dirichlet space with uniform weights)**.** Assume $K \neq \overline{V_0}$, let μ be the self-similar measure on $\mathcal{L}$ with uniform weight and let $r \in (0, \#S)$. $(\mathcal{E}, \mathcal{F})$ is called a *self-similar Dirichlet form on* $L^2(K, \mu)$ *with uniform weight* r if and only if it is a non-zero symmetric regular Dirichlet form on $L^2(K, \mu)$ satisfying the following two conditions:

(SSDF1) $\mathcal{F} \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{F} \text{ for any } i \in S\}$.
(SSDF2) For any $u \in \mathcal{F} \cap C(K)$,

$$\mathcal{E}(u, u) = \frac{1}{r} \sum_{i \in S} \mathcal{E}(u \circ F_i, u \circ F_i). \tag{2.3}$$

If $(\mathcal{E}, \mathcal{F})$ is a self-similar Dirichlet form on $L^2(K, \mu)$ with uniform weight r, then $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F}, r)$ is called a *self-similar Dirichlet space with uniform weights*, and we call $\tau := \#S/r$ its *time scaling factor* and $d_{\mathrm{s}} := 2 \log_\tau \#S$ its *spectral dimension.*

τ naturally appears as the scaling factor for the time variable of the heat kernels; see Lemma 2.21 and its proof below. Note that $\tau > 1$ by $r \in (0, \#S)$.

In the rest of this section, we assume that $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F}, r)$ is a self-similar Dirichlet space with uniform weights. Note that it is contained in the framework of [38, Definition 2.7], which in turn is a special case of [35, Definition 3.3]. In particular, by [35, Lemma 5.5], $u \circ F_w \in \mathcal{F}$ for any $u \in \mathcal{F}$ and any $w \in W_*$, and (2.3) holds for any $u \in \mathcal{F}$. Furthermore by [38, Lemma 2.8], $(\mathcal{E}, \mathcal{F})$ is conservative (i.e. $\mathbf{1} \in \mathcal{F}$ and $\mathcal{E}(\mathbf{1}, \mathbf{1}) = 0$) and strongly local, and $V_0 \neq \emptyset$.

Part of the assertions of our main results could be derived from [35, Theorem 7.7], which requires two conditions (SSDF3$\mathcal{S}$) and (LWTF) stated in [35, Definition 7.6]. Our Dirichlet space $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F}, r)$ obviously satisfies (LWTF) by definition, and it also satisfies (SSDF3$\mathcal{S}$) by virtue of the next proposition, which is an easy consequence of (SSDF1) and requires the following definition.

Definition 2.7. (1) A finite subset Λ of W_* is called a *partition of* Σ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \neq v$ and $\Sigma = \bigcup_{w \in \Lambda} \Sigma_w$.
(2) For $\Gamma \subset W_*$, we set $K(\Gamma) := \bigcup_{w \in \Gamma} K_w$.

Proposition 2.8. *Let Λ_1, Λ_2 be partitions of Σ, let $\Gamma_1 \subset \Lambda_1$ and $\Gamma_2 \subset \Lambda_2$ be non-empty and let $\varphi : \Gamma_1 \to \Gamma_2$ be bijective. Suppose that $F_\varphi : K(\Gamma_1) \to K(\Gamma_2)$ is a homeomorphism and satisfies $F_\varphi|_{K_w} = F_{\varphi(w)} \circ F_w^{-1}$ for any $w \in \Gamma_1$. Then whenever $u \in \mathcal{F} \cap C(K)$ and $u^\varphi \in C(K)$ we have $u^\varphi \in \mathcal{F}$, where $u^\varphi : K \to \mathbb{R}$ is defined by*

$$u^\varphi := \begin{cases} u \circ F_\varphi^{-1} & \text{on } K(\Gamma_2), \\ 0 & \text{on } K \setminus K(\Gamma_2). \end{cases} \tag{2.4}$$

Proof. For a partition Λ of Σ, an induction in $\#\Lambda$ using (SSDF1) easily shows that $\mathcal{F} \cap C(K) = \{u \in C(K) \mid u \circ F_w \in \mathcal{F} \text{ for any } w \in \Lambda\}$. Now let $u \in \mathcal{F} \cap C(K)$ satisfy $u^\varphi \in C(K)$. If $w \in \Gamma_2$ then $u^\varphi \circ F_w = u \circ F_\varphi^{-1} \circ F_w = u \circ F_{\varphi^{-1}(w)} \in \mathcal{F}$. If $w \in \Lambda_2 \setminus \Gamma_2$, then $u^\varphi \circ F_w|_{K \setminus V_0} = 0$, which and $u^\varphi \circ F_w \in C(K)$ imply $u^\varphi \circ F_w = 0 \in \mathcal{F}$ since $K \setminus V_0$ is dense in K by $K \neq \overline{V_0}$ and [40, Theorem 1.3.8]. Thus $u^\varphi \in \mathcal{F}$. $\square$

Definition 2.9. Let U be a non-empty open subset of K. We define $\mu|_U := \mu|_{\mathcal{B}(U)}$,

$$\mathcal{F}_U := \overline{\{u \in \mathcal{F} \cap C(K) \mid \operatorname{supp}_K[u] \subset U\}} \qquad \text{and} \qquad \mathcal{E}^U := \mathcal{E}|_{\mathcal{F}_U \times \mathcal{F}_U}, \tag{2.5}$$

where the closure is taken in the Hilbert space $\mathcal{F}$ with inner product $\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + \int_K uv d\mu$. $(\mathcal{E}^U, \mathcal{F}_U)$ is called the *part of the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *on* U.

Since $u = 0$ μ-a.e. on $K \setminus U$ for any $u \in \mathcal{F}_U$, we can regard $\mathcal{F}_U$ as a linear subspace of $L^2(U, \mu|_U)$ in the natural manner. Under this identification, we have the following lemma.

Lemma 2.10. *Let U be a non-empty open subset of K. Then $(\mathcal{E}^U, \mathcal{F}_U)$ is a strongly local symmetric regular Dirichlet form on $L^2(U, \mu|_U)$.*

Proof. The regularity of $(\mathcal{E}, \mathcal{F})$ easily yields that of $(\mathcal{E}^U, \mathcal{F}_U)$ by (2.5) and [23, Lemma 1.4.2-(ii)], and the strong locality of $(\mathcal{E}, \mathcal{F})$ implies that of $(\mathcal{E}^U, \mathcal{F}_U)$. □

Next we prepare basic facts concerning the heat kernel of $(K, \mu, \mathcal{E}, \mathcal{F})$ and that of $(U, \mu|_U, \mathcal{E}^U, \mathcal{F}_U)$ for non-empty open subsets U of K.

Definition 2.11 (CHK)**.** We say that $(K, \mu, \mathcal{E}, \mathcal{F})$ *satisfies* (CHK), or simply (CHK) *holds*, if and only if the Markovian semigroup $\{T_t\}_{t \in (0,\infty)}$ on $L^2(K, \mu)$ associated with $(\mathcal{E}, \mathcal{F})$ admits a *continuous integral kernel* p, i.e. a continuous function $p = p_t(x, y) : (0, \infty) \times K \times K \to \mathbb{R}$ such that for any $u \in L^2(K, \mu)$ and any $t \in (0, \infty)$,

$$T_t u = \int_K p_t(\cdot, y) u(y) d\mu(y) \qquad \mu\text{-a.e.} \tag{2.6}$$

Such p, if exists, is unique and satisfies $p_t(x, y) = p_t(y, x) \geq 0$ for any $(t, x, y) \in (0, \infty) \times K \times K$ by a standard monotone class argument. p is called the *(continuous) heat kernel of* $(K, \mu, \mathcal{E}, \mathcal{F})$.

Proposition 2.12. *Suppose that* (CHK) *holds and that*

$$\lim_{t \downarrow 0} \int_{K \setminus U} p_t(x, y) d\mu(y) = 0 \qquad \text{for any open subset } U \text{ of } K \text{ and any } x \in U. \tag{2.7}$$

Let U be a non-empty open subset of K.
(1) *The Markovian semigroup $\{T^U_t\}_{t \in (0,\infty)}$ on $L^2(U, \mu|_U)$ associated with $(\mathcal{E}^U, \mathcal{F}_U)$ admits a unique continuous integral kernel $p^U = p^U_t(x, y) : (0, \infty) \times U \times U \to \mathbb{R}$, similarly to* (2.6)*. Moreover, $0 \leq p^U_t(x, y) = p^U_t(y, x) \leq p_t(x, y)$ for any $(t, x, y) \in (0, \infty) \times U \times U$.*
(2) *If U is arcwise connected, then $p^U_t(x, y) > 0$ for any $(t, x, y) \in (0, \infty) \times U \times U$.*

Proof. (1) Set $\mathcal{P}_t(x, A) := \int_A p_t(x, y) d\mu(y)$ for $(t, x) \in (0, \infty) \times K$ and $A \in \mathcal{B}(K)$. Then $\mathcal{P}_t u := \int_K u(y) \mathcal{P}_t(\cdot, dy) \in C(K)$ for any $u \in L^1(K, \mu)$ by the compactness of K and (CHK), and $\mathcal{P}_t u = T_t u$ μ-a.e. for any $u \in L^2(K, \mu)$. In particular, $\mathcal{P}_t(x, K) = 1$ for any $(t, x) \in (0, \infty) \times K$ by the conservativeness of $(\mathcal{E}, \mathcal{F})$, and then (2.7) means that $\lim_{t \downarrow 0} \mathcal{P}_t u(x) = u(x)$ for any $u \in C(K)$, which and [13, Chapter I, Exercise 9.13] imply that $\lim_{t \downarrow 0} \|\mathcal{P}_t u - u\|_\infty = 0$ for any $u \in C(K)$. From these facts, the conclusions of [35, Lemma 7.11] can be verified for our Dirichlet space $(K, \mu, \mathcal{E}, \mathcal{F})$ with exactly the same proof, and hence a continuous integral kernel p^U of $\{T^U_t\}_{t \in (0,\infty)}$ exists. A monotone class argument immediately shows the uniqueness of such p^U, and the last assertion easily follows from [41, (C.2)] and a monotone class argument again.
(2) In view of the validity of the conclusions of [35, Lemma 7.11] (see also Lemma 2.15 below), the assertion is immediate from [35, Proposition A.3-(2)]. □

Throughout the rest of this section, we assume that (CHK) and (2.7) hold.

Definition 2.13. Let U be a non-empty open subset of K. The integral kernel p^U of $\{T_t^U\}_{t\in(0,\infty)}$ as in Proposition 2.12-(1) is called the *Dirichlet heat kernel on* U. We extend p^U to $(0,\infty)\times K\times K$ by setting $p^U|_{(0,\infty)\times(K\times K\setminus U\times U)} := 0$.

We also set $p_t^{\emptyset}(x,y) := 0$ for any $(t,x,y)\in(0,\infty)\times K\times K$.

Note that for $t\in(0,\infty)$, $p_t^U : K\times K\to[0,\infty)$ may not be continuous on $K\times K$.

Lemma 2.14. *Let U, V be open subsets of K with $U\cap V=\emptyset$. Then*

$$p_t^{U\cup V}(x,y) = p_t^U(x,y) + p_t^V(x,y), \qquad (t,x,y)\in(0,\infty)\times K\times K. \tag{2.8}$$

Proof. Since (2.8) is clear if $U=\emptyset$ or $V=\emptyset$, we may assume that U and V are non-empty. Let $u\in\mathcal{F}\cap C(K)$ satisfy $\operatorname{supp}_K[u]\subset U\cup V$. Then $U\cap\operatorname{supp}_K[u]$ is a compact subset of U, and by [23, Exercise 1.4.1] we can choose $v\in\mathcal{F}\cap C(K)$ with $\operatorname{supp}_K[v]\subset U$ so that $v=1$ on $U\cap\operatorname{supp}_K[u]$. Therefore $u\mathbf{1}_U = uv\in\mathcal{F}\cap C(K)$ by [23, Theorem 1.4.2-(ii)], and the (strong) locality of $(\mathcal{E},\mathcal{F})$ together with $\operatorname{supp}_K[u\mathbf{1}_U]\subset U$ and $\operatorname{supp}_K[u\mathbf{1}_V]\subset V$ yields $\mathcal{E}(u,u)=\mathcal{E}(u\mathbf{1}_U,u\mathbf{1}_U)+\mathcal{E}(u\mathbf{1}_V,u\mathbf{1}_V)$, which as well as $u\mathbf{1}_U\in\mathcal{F}_U$ and $u\mathbf{1}_V\in\mathcal{F}_V$ then easily follows for any $u\in\mathcal{F}_{U\cup V}$. Now from this fact and [23, Lemma 1.3.4-(i)], we easily see that $T_t^{U\cup V}u = T_t^U(u|_U)$ on U and $T_t^{U\cup V}u = T_t^V(u|_V)$ on V for any $u\in L^2(U\cup V,\mu|_{U\cup V})$, which immediately imply (2.8) by virtue of the uniqueness of $p^{U\cup V}$, p^U and p^V. □

As mentioned in the above proof of Proposition 2.12, the conclusions of [35, Lemma 7.11] are valid for $(K,\mu,\mathcal{E},\mathcal{F})$ under the present assumption that (CHK) and (2.7) hold. Consequently we have the following lemma, which will be used (only) in the proof of Proposition 2.16 below; see [23, Section A.2] and [16, Section A.1] for details on Hunt processes. For each non-empty open subset U of K, let $U_\partial := U\cup\{\partial_U\}$ denote its one-point compactification.

Lemma 2.15. (1) *There exists a Hunt process $X=\bigl(\Omega,\mathcal{M},\{X_t\}_{t\in[0,\infty]},\{\mathbf{P}_x\}_{x\in K_\partial}\bigr)$ on K such that $\mathbf{P}_x[X_t\in A]=\int_A p_t(x,y)d\mu(y)$ for any $(t,x)\in(0,\infty)\times K$ and any $A\in\mathcal{B}(K)$.*
(2) *For $A\in\mathcal{B}(K_\partial)$, set $\dot\sigma_A(\omega):=\inf\{t\in[0,\infty)\mid X_t(\omega)\in A\}$, $\omega\in\Omega$, and $\tau_A:=\dot\sigma_{K_\partial\setminus A}$. Let U be a non-empty open subset of K. For $t\in[0,\infty]$ and $\omega\in\Omega$, define $X_t^U(\omega):=X_t(\omega)$ if $t<\tau_U(\omega)$ and $X_t^U(\omega):=\partial_U$ if $t\geq\tau_U(\omega)$. Also set $\mathbf{P}_{\partial_U}:=\mathbf{P}_{\partial_K}$. Then $X^U:=\bigl(\Omega,\mathcal{M},\{X_t^U\}_{t\in[0,\infty]},\{\mathbf{P}_x\}_{x\in U_\partial}\bigr)$ is a Hunt process on U such that $\mathbf{P}_x[X_t^U\in A]=\int_A p_t^U(x,y)d\mu(y)$ for any $(t,x)\in(0,\infty)\times U$ and any $A\in\mathcal{B}(U)$.*

In fact, the Hunt processes X and X^U in Lemma 2.15 are diffusions with no killing inside by [23, Theorem 4.5.4-(iii)] and the strong locality of $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E}^U,\mathcal{F}_U)$.

The following proposition is the key to the proof of our main theorems.

Proposition 2.16. *Let U be an open subset of K, let J_0 be a finite set and let U_j be an open subset of K for each $j\in J_0$. Set $U_J:=\bigcap_{j\in J}U_j$ for $J\subset J_0$ ($U_\emptyset:=K$) and define*

$$p_t^U(x,y|\{U_j\}_{j\in J_0}) := \sum_{J\subset J_0}(-1)^{\#J}p_t^{U\cap U_J}(x,y) \tag{2.9}$$

for $(t,x,y)\in(0,\infty)\times K\times K$. Then we have the following statements.
(1) $0\leq p_t^U(x,y|\{U_j\}_{j\in J_0})\leq p_t^U(x,y)$ *for any* $(t,x,y)\in(0,\infty)\times K\times K$.

(2) *Let $k \in J_0$. Then for any $(t,x,y) \in (0,\infty) \times \overline{U_k} \times \overline{U_k}$,*

$$p_t^U(x,y|\{U_j\}_{j\in J_0}) \le \sup_{s\in[t/2,t]} \sup_{z\in\overline{U_k}\setminus U_k} p_s(x,z) + \sup_{s\in[t/2,t]} \sup_{z\in\overline{U_k}\setminus U_k} p_s(z,y). \quad (2.10)$$

Proof. Let $(t,x,y) \in (0,\infty)\times K\times K$. If $x \notin U$ or $y \notin U$ then $p_t^U(x,y|\{U_j\}_{j\in J_0}) = 0$. Suppose $x, y \in U$ and set $J_{x,y} := \{j \in J_0 \mid x, y \in U_j\}$, so that $p_t^{U\cap U_J}(x,y) = 0$ for $J \subset J_0$ with $J \not\subset J_{x,y}$. If $J_{x,y} = \emptyset$ then $p_t^U(x,y|\{U_j\}_{j\in J_0}) = p_t^U(x,y)$. Now suppose $J_{x,y} \neq \emptyset$. Then $p_t^U(x,y|\{U_j\}_{j\in J_0}) = p_t^U(x,y|\{U_j\}_{j\in J_{x,y}})$, and for $J \subset J_{x,y}$, $x, y \in U \cap U_J$ and hence $p_t^{U\cap U_J}(x,\cdot)$ is continuous at y. Therefore choosing a metric ρ on K compatible with the original topology of K and following the notation in Lemma 2.15, we obtain

$$\begin{aligned}
&p_t^U(x,y|\{U_j\}_{j\in J_0}) = p_t^U(x,y|\{U_j\}_{j\in J_{x,y}})\\
&= \lim_{s\downarrow 0} \frac{1}{\mu(B_s(y,\rho))} \sum_{J\subset J_{x,y}} (-1)^{\#J} \int_{B_s(y,\rho)} p_t^{U\cap U_J}(x,z)\,d\mu(z)\\
&= \lim_{s\downarrow 0} \frac{1}{\mu(B_s(y,\rho))} \sum_{J\subset J_{x,y}} (-1)^{\#J} \mathbf{P}_x[X_t \in B_s(y,\rho),\ t < \tau_{U\cap U_J}]\\
&= \lim_{s\downarrow 0} \frac{1}{\mu(B_s(y,\rho))} \sum_{J\subset J_{x,y}} (-1)^{\#J} \mathbf{P}_x[X_t \in B_s(y,\rho),\ t < \tau_U \wedge \min_{j\in J} \tau_{U_j}]\\
&= \lim_{s\downarrow 0} \frac{\mathbf{P}_x[X_t \in B_s(y,\rho),\ t < \tau_U] - \mathbf{P}_x[X_t \in B_s(y,\rho),\ t < \tau_U \wedge \max_{j\in J_{x,y}} \tau_{U_j}]}{\mu(B_s(y,\rho))}\\
&= \lim_{s\downarrow 0} \frac{\mathbf{P}_x[X_t \in B_s(y,\rho),\ \max_{j\in J_{x,y}} \tau_{U_j} \le t < \tau_U]}{\mu(B_s(y,\rho))} \qquad (2.11)\\
&\le \lim_{s\downarrow 0} \frac{\mathbf{P}_x[X_t \in B_s(y,\rho),\ t < \tau_U]}{\mu(B_s(y,\rho))} = p_t^U(x,y),
\end{aligned}$$

where we used the inclusion-exclusion formula for the equality in the fifth line. The expression (2.11) also shows that $p_t^U(x,y|\{U_j\}_{j\in J_0}) \ge 0$, proving (1).

For (2), let $k \in J_0$ and suppose $x, y \in \overline{U_k}$. Since $p_t^U(x,y|\{U_j\}_{j\in J_0}) = 0$ if $x \notin U$ or $y \notin U$, we may assume $x, y \in U$. If $x \in \overline{U_k} \setminus U_k$ or $y \in \overline{U_k} \setminus U_k$, then (2.10) holds since $p_t^U(x,y|\{U_j\}_{j\in J_0}) \le p_t^U(x,y) \le p_t(x,y)$ and $p_t(x,y)$ is obviously bounded by the right-hand side of (2.10). Now suppose $x, y \in U_k$. Then $k \in J_{x,y}$ and hence

$$\begin{aligned}
p_t^U(x,y|\{U_j\}_{j\in J_0}) &\le \lim_{s\downarrow 0} \frac{\mathbf{P}_x[X_t \in B_s(y,\rho),\ \tau_{U_k} \le t]}{\mu(B_s(y,\rho))} = p_t(x,y) - p_t^{U_k}(x,y)\\
&\le \sup_{s\in[t/2,t]} \sup_{z\in\overline{U_k}\setminus U_k} p_s(x,z) + \sup_{s\in[t/2,t]} \sup_{z\in\overline{U_k}\setminus U_k} p_s(z,y)
\end{aligned}$$

by virtue of (2.11), where the inequality in the second line follows from [28, Theorem 5.1] (or [27, Theorem 10.4]), the continuity of p on $(0,\infty) \times K \times K$, that of p^{U_k} on $(0,\infty) \times U_k \times U_k$ and the compactness of $\overline{U_k}$. □

Definition 2.17. For a non-empty open subset U of K, the non-positive self-adjoint operator on $L^2(U,\mu|_U)$ associated with $(\mathcal{E}^U, \mathcal{F}_U)$ (the generator of $\{T_t^U\}_{t\in(0,\infty)}$; see [23, Section 1.3]) is denoted by Δ_U, and its domain is written as $\mathcal{D}[\Delta_U]$.

Recall that $\mathcal{D}[\Delta_U] \subset \mathcal{F}_U$ and that for $u \in \mathcal{F}_U$ and $f \in L^2(U, \mu|_U)$,

$$u \in \mathcal{D}[\Delta_U] \text{ and } -\Delta_U u = f \quad \text{if and only if} \quad \mathcal{E}(u, v) = \int_U f v \, d\mu \text{ for any } v \in \mathcal{F}_U. \tag{2.12}$$

Our main interest is in short time asymptotic behavior of the partition function, which is defined as follows.

Definition 2.18. Let U be a non-empty open subset of K. Noting that Δ_U has discrete spectrum and that $\operatorname{tr} T_t^U < \infty$ for $t \in (0, \infty)$ by [17, Theorem 2.1.4], let $\{\lambda_n^U\}_{n \in \mathbb{N}}$ be the non-decreasing enumeration of all the eigenvalues of $-\Delta_U$, where each eigenvalue is repeated according to its multiplicity. The *eigenvalue counting function* $\mathcal{N}_U$ and the *partition function* $\mathcal{Z}_U$ *on* U (or *of the Dirichlet space* $(U, \mu|_U, \mathcal{E}^U, \mathcal{F}_U)$) are defined respectively by, for $\lambda \in \mathbb{R}$ and $t \in (0, \infty)$,

$$\mathcal{N}_U(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^U \leq \lambda\}, \tag{2.13}$$

$$\mathcal{Z}_U(t) := \operatorname{tr} T_t^U = \sum_{n \in \mathbb{N}} e^{-\lambda_n^U t} = \int_{\mathbb{R}} e^{-\lambda t} d\mathcal{N}_U(\lambda) = \int_U p_t^U(x, x) d\mu(x). \tag{2.14}$$

Furthermore we define $\mathcal{N}_{\mathrm{N}} := \mathcal{N}_K$, $\mathcal{Z}_{\mathrm{N}} := \mathcal{Z}_K$, $\mathcal{N}_{\mathrm{D}} := \mathcal{N}_{K^I}$ and $\mathcal{Z}_{\mathrm{D}} := \mathcal{Z}_{K^I}$, and we also set $\mathcal{N}_\emptyset(\lambda) := 0$ for $\lambda \in \mathbb{R}$ and $\mathcal{Z}_\emptyset(t) := 0$ for $t \in (0, \infty)$.

The subscripts $_{\mathrm{N}}$ and $_{\mathrm{D}}$ stand for the Neumann and Dirichlet boundary conditions on $\overline{V_0}$, respectively. In the situation of Definition 2.18, $\mathcal{N}_U(\lambda) < \infty$ for any $\lambda \in \mathbb{R}$ by $\lim_{n \to \infty} \lambda_n^U = \infty$, and $\mathcal{Z}_U$ is $(0, \infty)$-valued and continuous. Moreover, we have the following basic facts for $\{\lambda_n^U\}_{n \in \mathbb{N}}$ and $\mathcal{Z}_U$. Recall that by [18, Theorems 4.5.1 and 4.5.3], the smallest eigenvalue λ_1^U of $-\Delta_U$ is given by $\lambda_1^U = \inf_{u \in \mathcal{F}_U \setminus \{0\}} \mathcal{E}(u, u) / \int_U u^2 d\mu$, which easily implies that λ_1^U is non-increasing in U and that for $u \in \mathcal{F}_U$,

$$u \in \mathcal{D}[\Delta_U] \text{ and } -\Delta_U u = \lambda_1^U u \quad \text{if and only if} \quad \mathcal{E}(u, u) \leq \lambda_1^U \int_U u^2 d\mu. \tag{2.15}$$

Lemma 2.19. *Let U be an arcwise connected non-empty open subset of K and let φ_1^U be an eigenfunction of $-\Delta_U$ with eigenvalue λ_1^U. Then $\varphi_1^U|_U \in C(U)$, $\varphi_1^U|_U$ is either $(0, \infty)$-valued or $(-\infty, 0)$-valued, and $\{u \in \mathcal{D}[\Delta_U] \mid -\Delta_U u = \lambda_1^U u\} = \mathbb{R}\varphi_1^U$.*

Proof. This lemma is well-known, but here we provide a complete proof for the reader's convenience. We have $T_1^U \varphi_1^U = e^{-\lambda_1^U} \varphi_1^U$, hence $\varphi_1^U = e^{\lambda_1^U} \int_U p_1^U(\cdot, y) \varphi_1^U(y) d\mu(y)$ μ-a.e., where the right-hand side is continuous on U by Proposition 2.12-(1), and therefore $\varphi_1^U|_U \in C(U)$ with the same equality valid at any point of U. Moreover, if $\varphi_1^U \geq 0$ on U then $\varphi_1^U = e^{\lambda_1^U} \int_U p_1^U(\cdot, y) \varphi_1^U(y) d\mu(y) > 0$ on U by Proposition 2.12-(2).

Since $|\varphi_1^U| \in \mathcal{F}_U$ and $\mathcal{E}(|\varphi_1^U|, |\varphi_1^U|) \leq \mathcal{E}(\varphi_1^U, \varphi_1^U)$ by [16, Theorem 1.1.3], (2.15) implies that $|\varphi_1^U|$ belongs to $\{u \in \mathcal{D}[\Delta_U] \mid -\Delta_U u = \lambda_1^U u\}$, and hence so do $(\varphi_1^U)^+ := (|\varphi_1^U| + \varphi_1^U)/2$ and $(\varphi_1^U)^- := (|\varphi_1^U| - \varphi_1^U)/2$. If $\varphi_1^U(x) > 0$ for some $x \in U$, then the last argument in the previous paragraph shows that $(\varphi_1^U)^+ > 0$ on U and hence $\varphi_1^U > 0$ on U. Otherwise $\varphi_1^U(x) < 0$ for some $x \in U$ and then similarly we get $\varphi_1^U < 0$ on U.

Finally, let u be an eigenfunction of $-\Delta_U$ with eigenvalue λ_1^U. If the function $v := u - \left(\int_U u \varphi_1^U d\mu / \int_U (\varphi_1^U)^2 d\mu\right) \varphi_1^U$ were not identically 0 on U, then v would be an eigenfunction of $-\Delta_U$ with eigenvalue λ_1^U and hence either $v > 0$ on U or $v < 0$ on U, contradicting $\int_U v \varphi_1^U d\mu = 0$. Thus $v = 0$ and hence $u \in \mathbb{R}\varphi_1^U$. $\square$

Lemma 2.20. *Let U be a non-empty open subset of K. Then*

$$e^{-\lambda_1^U t} \leq \mathcal{Z}_U(t) \leq \mathcal{Z}_U(1)e^{\lambda_1^U} e^{-\lambda_1^U t}, \qquad t \in [1,\infty). \tag{2.16}$$

Proof. Let $\{\lambda_n^U\}_{n\in\mathbb{N}}$ be as in Definition 2.18. Then for $t \in [1,\infty)$, $1 \leq e^{\lambda_1^U t}\mathcal{Z}_U(t) = \sum_{n\in\mathbb{N}} e^{(\lambda_1^U-\lambda_n^U)t} \leq \sum_{n\in\mathbb{N}} e^{\lambda_1^U-\lambda_n^U} = \mathcal{Z}_U(1)e^{\lambda_1^U}$ since $\lambda_1^U - \lambda_n^U \leq 0$ for $n \in \mathbb{N}$. □

Lemma 2.21. *Let U be a non-empty open subset of K^I and let $w \in W_*$. Then*

$$\mathcal{Z}_U(\tau^{|w|}t) = \mathcal{Z}_{F_w(U)}(t), \qquad t \in (0,\infty). \tag{2.17}$$

Proof. Note that $F_w(U)$ is open in K since K_w^I is open in K and $F_w|_{K^I} : K^I \to K_w^I$ is a homeomorphism. Similarly to [35, Lemma 5.5], F_w^* defines a bijection $F_w^* : L^2(F_w(U), \mu|_{F_w(U)}) \to L^2(U, \mu|_U)$ such that $\int_U (F_w^* u)^2 d\mu = (\#S)^{|w|} \int_{F_w(U)} u^2 d\mu$ for any $u \in L^2(F_w(U), \mu|_{F_w(U)})$ by Lemma 2.5, $F_w^*(\mathcal{F}_{F_w(U)}) = \mathcal{F}_U$ by (SSDF1) and $\mathcal{E}(F_w^* u, F_w^* u) = r^{|w|}\mathcal{E}(u,u)$ for any $u \in \mathcal{F}_{F_w(U)}$ by (SSDF2). It easily follows from these facts and [23, Lemma 1.3.4-(i)] that $F_w^* T_t^{F_w(U)} = T_{\tau^{|w|}t}^U F_w^*$ for any $t \in (0,\infty)$, which and the uniqueness of the continuous heat kernels $p^{F_w(U)}$ and p^U imply

$$p_{\tau^{|w|}t}^U(x,y) = \frac{1}{(\#S)^{|w|}} p_t^{F_w(U)}\bigl(F_w(x), F_w(y)\bigr), \qquad (t,x,y) \in (0,\infty)\times K\times K. \tag{2.18}$$

Now (2.17) is immediate from (2.14), (2.18) and Lemma 2.5. □

At the last of this section, we present two fundamental tools for the proofs of our main results. The first lemma is used to relate differences of heat kernels with different boundary conditions to alternating sums of the form (2.9). The second tool is a version of the renewal theorem which yields log-periodic asymptotic behavior of functions.

Lemma 2.22. *Let J_0 be a finite set and let $a_J \in \mathbb{R}$ for each $J \subset J_0$. Then*

$$a_{J_0} = \sum_{J\subset J_0} \sum_{A\subset J} (-1)^{\#A} a_{J\setminus A}. \tag{2.19}$$

Proof. For each $J \subset J_0$, the coefficient of the term a_J in the summation in (2.19) is $\sum_{A\subset J_0\setminus J} (-1)^{\#(J\cup A)}$, which is 1 for $J = J_0$ and 0 for $J \subsetneq J_0$, proving the lemma. □

Theorem 2.23. *Let $\alpha_0 \in \mathbb{R}$, $\gamma \in (1,\infty)$ and $n \in \mathbb{N}$. For each $k \in \{1,\ldots,n\}$, let $\alpha_k \in (-\infty,\alpha_0)$ and let $G_k : \mathbb{R} \to \mathbb{R}$ be bounded and $\log\gamma$-periodic. Assume that $\mathcal{Z} : (0,\infty) \to \mathbb{R}$ and $\mathcal{R} : (0,1] \to [0,\infty)$ satisfy*

$$\left| \mathcal{Z}(t) - \gamma^{\alpha_0}\mathcal{Z}(\gamma t) - \sum_{k=1}^{n} t^{-\alpha_k} G_k(-\log t) \right| \leq \mathcal{R}(t) \leq c t^{-\alpha_0+\varepsilon}, \qquad t \in (0,1] \tag{2.20}$$

and $|\mathcal{Z}(t)| \leq c t^{-\alpha_0-\varepsilon}$ for any $t \in [1,\infty)$ for some $c,\varepsilon \in (0,\infty)$. Then there exists a unique $\log\gamma$-periodic function $G_0 : \mathbb{R} \to \mathbb{R}$ such that for any $t \in (0,\gamma]$,

$$\left| \mathcal{Z}(t) - t^{-\alpha_0} G_0(-\log t) + \sum_{k=1}^{n} \frac{t^{-\alpha_k} G_k(-\log t)}{\gamma^{\alpha_0-\alpha_k} - 1} \right| \leq \sum_{j\in\mathbb{N}} \frac{\mathcal{R}(\gamma^{-j}t)}{\gamma^{\alpha_0 j}} =: \mathcal{R}_{\alpha_0,\gamma}(t). \tag{2.21}$$

Moreover, G_0 is bounded, and if $\mathcal{Z}$ is continuous then so is G_0.

Proof. We follow [40, Proofs of Theorems 4.1.5 and B.4.3]. Define $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(s) := e^{-\alpha_0 s}\mathcal{Z}(e^{-s}) \qquad \text{and} \qquad g(s) := e^{-\alpha_0 s}\bigl(\mathcal{Z}(e^{-s}) - \gamma^{\alpha_0}\mathcal{Z}(\gamma e^{-s})\bigr),$$

so that $f(s) - f(s - \log\gamma) = g(s)$ for any $s \in \mathbb{R}$. Then (2.20) means that

$$\left| g(s) - \sum_{k=1}^{n} e^{-(\alpha_0-\alpha_k)s} G_k(s) \right| \le e^{-\alpha_0 s}\mathcal{R}(e^{-s}) \le ce^{-\varepsilon s}, \qquad s \in [0,\infty), \tag{2.22}$$

whereas the latter assumption means that for $s \in (-\infty, 0]$, $|f(s)| \le ce^{\varepsilon s}$ and hence $|g(s)| \le 2ce^{\varepsilon s}$. Therefore the series $G_0 := \sum_{j\in\mathbb{Z}} g(\cdot - j\log\gamma)$ is uniformly absolutely convergent on any compact subset of $\mathbb{R}$ and defines a bounded $\log\gamma$-periodic function, which is continuous if $\mathcal{Z}$ is continuous. Since $\lim_{s\to-\infty} f(s) = 0$, by using $f = f(\cdot - \log\gamma) + g$ repeatedly we see that for any $s \in \mathbb{R}$ and any $J \in \mathbb{N}$,

$$f(s) = f(s - J\log\gamma) + \sum_{j=0}^{J-1} g(s - j\log\gamma) = \sum_{j\in\mathbb{N}\cup\{0\}} g(s - j\log\gamma).$$

Hence $G_0 - f = \sum_{j\in\mathbb{N}} g(\cdot + j\log\gamma)$, and then by the $\log\gamma$-periodicity of G_k for $k \in \{1,\ldots,n\}$ and (2.22), for any $s \in [-\log\gamma, \infty)$ we have

$$\begin{aligned}
&\left| f(s) - G_0(s) + \sum_{k=1}^{n} \frac{e^{-(\alpha_0-\alpha_k)s} G_k(s)}{\gamma^{\alpha_0-\alpha_k} - 1} \right| \\
&\le \sum_{j\in\mathbb{N}} \left| g(s + j\log\gamma) - \sum_{k=1}^{n} e^{-(\alpha_0-\alpha_k)(s+j\log\gamma)} G_k(s + j\log\gamma) \right| \\
&\le \sum_{j\in\mathbb{N}} e^{-\alpha_0(s+j\log\gamma)}\mathcal{R}(e^{-s-j\log\gamma}) = e^{-\alpha_0 s}\mathcal{R}_{\alpha_0,\gamma}(e^{-s}),
\end{aligned}$$

from which (2.21) is immediate.

Finally for the uniqueness of G_0, let $G : \mathbb{R} \to \mathbb{R}$ be $\log\gamma$-periodic and satisfy (2.21) for $t \in (0, \gamma]$. Then since $\mathcal{R}_{\alpha_0,\gamma}(t) \le c(\gamma^{\varepsilon} - 1)^{-1} t^{-\alpha_0+\varepsilon}$ for any $t \in (0, \gamma]$ by the upper inequality of (2.20), we easily see from (2.21) for G and G_0 that $\lim_{t\downarrow 0} |G(-\log t) - G_0(-\log t)| = 0$, which and the $\log\gamma$-periodicity of $G - G_0$ yield $G = G_0$. □

3. Post-critically finite self-similar fractals I: canonical boundary

Now we start to discuss log-periodic asymptotic behavior of the partition functions. First we consider the case of post-critically finite self-similar fractals in this and the next sections. This section is devoted to the case of $\mathcal{Z}_{\mathrm{N}}$ and $\mathcal{Z}_{\mathrm{D}}$, and the next section treats the asymptotics of $\mathcal{Z}_U$ for non-empty open subsets U of K with self-similar boundaries. The case of generalized Sierpiński carpets is studied in Sections 5 and 6.

Throughout this section, we assume that $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a post-critically finite self-similar structure with K connected and $\#K \ge 2$. In particular, $2 \le \#V_0 < \infty$ and V_* is countably infinite and dense in K, so that $K \ne \overline{V_0} = V_0$.

We first describe briefly the construction of a self-similar Dirichlet space over K; see [40, Chapter 3] for details. Let $D = (D_{xy})_{x,y\in V_0}$ be a real symmetric matrix of size $\#V_0$ (which we also regard as a linear operator on $\mathbb{R}^{V_0}$) such that

(D1) $\{u \in \mathbb{R}^{V_0} \mid Du = 0\} = \mathbb{R}\mathbf{1}_{V_0}$,
(D2) $D_{xy} \geq 0$ for any $x, y \in V_0$ with $x \neq y$.

We define $\mathcal{E}^{(0)}(u,v) := -\sum_{x,y \in V_0} D_{xy}u(y)v(x)$ for $u, v \in \mathbb{R}^{V_0}$, so that $(\mathcal{E}^{(0)}, \mathbb{R}^{V_0})$ is a Dirichlet form on $L^2(V_0, \#)$. Furthermore let $r \in (0,\infty)$ and define

$$\mathcal{E}^{(m)}(u,v) := \frac{1}{r^m} \sum_{w \in W_m} \mathcal{E}^{(0)}(u \circ F_w|_{V_0}, v \circ F_w|_{V_0}), \qquad u, v \in \mathbb{R}^{V_m} \tag{3.1}$$

for each $m \in \mathbb{N}$. We assume that (D, r) is a *harmonic structure on* $\mathcal{L}$, i.e. (D, r) satisfies

$$\mathcal{E}^{(0)}(u,u) = \inf\{\mathcal{E}^{(1)}(v,v) \mid v \in \mathbb{R}^{V_1}, v|_{V_0} = u\} \qquad \text{for any } u \in \mathbb{R}^{V_0}.$$

Then $\mathcal{E}^{(m)}(u,u) = \min_{v \in \mathbb{R}^{V_n}, v|_{V_m} = u} \mathcal{E}^{(n)}(v,v)$ for any $m, n \in \mathbb{N} \cup \{0\}$ with $m \leq n$ and any $u \in \mathbb{R}^{V_m}$ by [40, Proposition 3.1.3], and $r \in (0,1)$ by [40, Proposition 3.1.8]. In particular, $\{\mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m})\}_{m \in \mathbb{N} \cup \{0\}}$ is non-decreasing and hence has the limit in $[0,\infty]$ for any $u \in C(K)$, and we define

$$\begin{aligned} \mathcal{F} &:= \{u \in C(K) \mid \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) < \infty\}, \\ \mathcal{E}(u,v) &:= \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, v|_{V_m}) \in \mathbb{R}, \qquad u, v \in \mathcal{F}. \end{aligned} \tag{3.2}$$

$(\mathcal{E}, \mathcal{F})$ is easily seen to satisfy (SSDF1) and (SSDF2) of Definition 2.6. By [40, Theorem 3.3.4], $(\mathcal{E}, \mathcal{F})$ is a resistance form on K whose resistance metric $R : K \times K \to [0,\infty)$ is compatible with the original topology of K, and then [42, Corollary 6.4 and Theorem 9.4] imply that $(\mathcal{E}, \mathcal{F})$ is a non-zero symmetric regular Dirichlet form on $L^2(K, \mu)$, where μ denotes the self-similar measure on $\mathcal{L}$ with uniform weight. (See [40, Definition 2.3.1] or [42, Definition 3.1] for the definition of resistance forms and their resistance metrics.) Thus $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F}, r)$ is a self-similar Dirichlet space with uniform weight. It easily follows from [42, (3.1)] and [23, Theorem 1.4.2-(iv)] that $\mathcal{F}_U = \{u \in \mathcal{F} \mid u|_{K \setminus U} = 0\}$. Moreover, (CHK) and (2.7) hold by [42, Theorem 10.4] (or by [40, Section 5.1 and Proposition 5.2.6-(2)]), and we have the following estimate. Recall that we set $\tau := \#S/r$ and $d_{\mathrm{s}} := 2\log_\tau \#S$.

Proposition 3.1. *Let* $d_{\mathrm{w}}^R := \log_{1/r} \tau$ (note that $d_{\mathrm{w}}^R > 1$). *Then there exist* $c_{3.1}, c_{3.2} \in (0,\infty)$ *such that for any* $(t,x,y) \in (0,1] \times K \times K$,

$$p_t(x,y) \leq c_{3.1} t^{-d_{\mathrm{s}}/2} \exp\biggl(-c_{3.2}\Bigl(\frac{R(x,y)^{d_{\mathrm{w}}^R}}{t}\Bigr)^{\frac{1}{d_{\mathrm{w}}^R - 1}}\biggr). \tag{3.3}$$

Proof. We easily see from [40, Lemmas 3.3.5 and 4.2.4] that

$$c_{3.3} s^{d_{\mathrm{w}}^R - 1} \leq \mu(B_s(x,R)) \leq c_{3.4} s^{d_{\mathrm{w}}^R - 1}, \qquad (s,x) \in (0, 1 + \mathrm{diam}_R K] \times K \tag{3.4}$$

for some $c_{3.3}, c_{3.4} \in (0,\infty)$. (3.4) means that the condition [42, Theorem 15.10-(b)] is satisfied with $d = R$ and $g(t) = t^{d_{\mathrm{w}}^R}$, and hence (3.3) follows by [42, Theorem 15.10] and the strong locality of $(\mathcal{E}, \mathcal{F})$. $\square$

In fact, *the power* $1/(d_{\mathrm{w}}^{R}-1)$ *in the exponential in the right-hand side of* (3.4) *is not best possible in general*; see [31] for a sharp two-sided estimate of $p_t(x,y)$ in the present setting and see [29, Section 6] for a generalization of such an estimate in the framework of a metric measure space. To state and prove our main results with the best exponent in the reminder estimates, we introduce the following condition, *which is always satisfied in the present setting* on account of Proposition 3.1 but with a possibly worse exponent $d_{\mathrm{w}}=d_{\mathrm{w}}^{R}$ (and with $\rho=R$).

Definition 3.2. Let $d_{\mathrm{w}}\in(1,\infty)$. We say that $(K,\mu,\mathcal{E},\mathcal{F})$ *satisfies* $(\mathrm{UHK})_{d_{\mathrm{w}}}$, or simply $(\mathrm{UHK})_{d_{\mathrm{w}}}$ *holds*, if and only if there exist a metric ρ on K compatible with the original topology of K and $c_{3.5},c_{3.6}\in(0,\infty)$ such that for any $(t,x,y)\in(0,1]\times K\times K$,

$$p_t(x,y)\leq c_{3.5}t^{-d_{\mathrm{s}}/2}\exp\biggl(-c_{3.6}\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr). \tag{3.5}$$

For example, the d-dimensional level-l Sierpiński gasket (see the first line of Fig. 2 below) equipped with its canonical harmonic structure satisfies $(\mathrm{UHK})_{d_{\mathrm{w}}}$ and a lower bound of the same form as (3.5) with $d_{\mathrm{w}}=\log_l\tau$ and ρ the Euclidean metric. Such a two-sided heat kernel estimate, called *sub-Gaussian*, was proved first by Barlow and Perkins [10] for the d-dimensional *standard* (i.e. level-2) Sierpinski gasket and extended later to nested fractals by Kumagai [44] and to affine nested fractals by Fitzsimmons, Hambly and Kumagai [20]. Their results, however, were stated and proved for unbounded fractals and their proofs were rather involved. Now such results can be easily deduced from general results like [42, Theorem 15.10] on the basis of certain necessary geometric considerations; see Remark 3.8-(2) below in this regard, and see Example 4.5 for a detailed argument in the case of the d-dimensional level-l Sierpiński gasket.

We need the following definition for the main theorems of this and the next sections.

Definition 3.3. We define the *symmetry group* $\mathcal{G}$ *of* $(\mathcal{L},(D,r),\mu)$ by

$$\mathcal{G}:=\left\{g\;\middle|\;\begin{array}{l}g\text{ is a homeomorphism from }K\text{ to itself, }g(V_0)=V_0,\ \mu\circ g=\mu,\\ u\circ g,u\circ g^{-1}\in\mathcal{F}\text{ and }\mathcal{E}(u\circ g,u\circ g)=\mathcal{E}(u,u)\text{ for any }u\in\mathcal{F}\end{array}\right\}, \tag{3.6}$$

which clearly forms a subgroup of the group of homeomorphisms of K.

The following is the main theorem of this section.

Theorem 3.4. *Let* $q\in V_0$, $d_{\mathrm{w}}\in(1,\infty)$ *and suppose* $\{g(q)\mid g\in\mathcal{G}\}=V_0$ *and that* $(\mathrm{UHK})_{d_{\mathrm{w}}}$ *holds. Set* $n_{\mathrm{N}}:=\frac{\#S\#V_0-\#V_1}{\#S-1}$ *and* $n_{\mathrm{D}}:=-\frac{\#(V_1\setminus V_0)}{\#S-1}$ (note that $n_{\mathrm{N}}>0>n_{\mathrm{D}}$). *Then there exist* $c_{3.7}\in(0,\infty)$ *and continuous* $\log\tau$*-periodic functions* $G_0,G_1:\mathbb{R}\to(0,\infty)$ *such that for any* $\mathrm{B}\in\{\mathrm{N},\mathrm{D}\}$, *as* $t\downarrow 0$,

$$\mathcal{Z}_{\mathrm{B}}(t)=t^{-d_{\mathrm{s}}/2}G_0(-\log t)+n_{\mathrm{B}}G_1(-\log t)+O\Bigl(\exp\bigl(-c_{3.7}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \tag{3.7}$$

Remark 3.5. (1) G_0 in Theorem 3.4 was essentially obtained in [43]. Indeed, according to [43, Theorem 2.4 and Corollary 2.5], there exists a right-continuous $\log\tau$-periodic function $G_{\mathcal{N}}:\mathbb{R}\to\mathbb{R}$ with $0<\inf_{s\in\mathbb{R}}G_{\mathcal{N}}(s)\leq\sup_{s\in\mathbb{R}}G_{\mathcal{N}}(s)<\infty$ such that for any $\mathrm{B}\in\{\mathrm{N},\mathrm{D}\}$,

$$\mathcal{N}_{\mathrm{B}}(\lambda)=\lambda^{d_{\mathrm{s}}/2}G_{\mathcal{N}}(\log\lambda)+O(1)\qquad\text{as }\lambda\to\infty, \tag{3.8}$$

where the remainder estimate $O(1)$ follows from [40, Lemma 4.1.9] and Theorem 2.23 with $\mathcal{Z}(t)=\mathcal{N}_{\mathrm{D}}(t^{-1})$. Then a direct calculation using (3.8) (see [1, Proof of Theorem

2.1]) easily shows that $\mathcal{Z}_{\mathrm{B}}(t) = t^{-d_{\mathrm{s}}/2}G_0(-\log t) + O(1)$ as $t \downarrow 0$ for any $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, where $G_0 := \int_0^\infty e^{-\lambda}\lambda^{d_{\mathrm{s}}/2}G_{\mathcal{N}}(\cdot + \log\lambda)d\lambda$.

In view of (3.7) and (3.8), one may expect a more detailed description of the asymptotic behavior of $\mathcal{N}_{\mathrm{B}}$ similar to (3.7), but at this moment *the author has no idea how* $\mathcal{N}_{\mathrm{B}}(\lambda) - \lambda^{d_{\mathrm{s}}/2}G_{\mathcal{N}}(\log\lambda)$ *actually behaves as* $\lambda \to \infty$.

(2) By [9, Theorem 4.4 and Section 5] or [40, Theorem 4.3.4 and Section 4.4] (see also [37, Lemma 3.5]), under certain mild conditions on $\mathcal{G}$ the periodic function $G_{\mathcal{N}}$ in (3.8) is *discontinuous* and hence non-constant, which and Karamata's Tauberian theorem [19, Section XIII.5, Theorem 2] imply that G_0 in Theorem 3.4 is also non-constant. On the other hand, *the author does not know any example where it can be proved that* G_1 *in Theorem* 3.4 *is non-constant.*

(3) The strict positivity of G_1 in Theorem 3.4 could be derived from the author's general result [35, Theorem 7.7], but we provide an alternative simpler proof of this fact; see the proof of Proposition 3.12 below.

Before we give the proof of Theorem 3.4, we briefly recall the definition of (affine) nested fractals and see that Theorem 3.4 is applicable to them. Recall that $f : \mathbb{R}^d \to \mathbb{R}^d$ is called a *contractive similitude on* $\mathbb{R}^d$ if and only if there exist $\alpha \in (0,1)$, $b \in \mathbb{R}^d$ and a d-dimensional real orthogonal matrix U such that $f(x) = \alpha Ux + b$ for any $x \in \mathbb{R}^d$. Then such α is called the *contraction ratio of* f. For $x, y \in \mathbb{R}^d$ with $x \neq y$, let $g_{xy} : \mathbb{R}^d \to \mathbb{R}^d$ denote the reflection in the hyperplane $\{z \in \mathbb{R}^d \mid |z-x| = |z-y|\}$.

Definition 3.6. (1) A homeomorphism $g : K \to K$ is called a *symmetry of* $\mathcal{L}$ if and only if, for any $m \in \mathbb{N} \cup \{0\}$, there exists an injective map $g^{(m)} : W_m \to W_m$ such that $g(F_w(V_0)) = F_{g^{(m)}(w)}(V_0)$ for any $w \in W_m$.
(2) Assume the following situation:

$$\begin{aligned}&d \in \mathbb{N},\ K \text{ is a compact subset of } \mathbb{R}^d, \text{ and for each } i \in S,\ F_i = f_i|_K\\ &\text{for some contractive similitude } f_i \text{ on } \mathbb{R}^d \text{ with contraction ratio } \alpha_i.\end{aligned} \tag{3.9}$$

$\mathcal{L}$ is called an *affine nested fractal* if and only if it is post-critically finite, K is connected and $g_{xy}|_K$ is a symmetry of $\mathcal{L}$ for any $x, y \in V_0$ with $x \neq y$. An affine nested fractal $\mathcal{L}$ is called a *nested fractal* if and only if $\alpha_i = \alpha$ for any $i \in S$ for some $\alpha \in (0,1)$.

Now Theorem 3.4 yields the following corollary for affine nested fractals.

Corollary 3.7. *Assume that* $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ *is an affine nested fractal and that* $D_{xy} = D_{x'y'}$ *for any* $x, y, x', y' \in V_0$ *with* $|x-y| = |x'-y'|$. *Further assume that*

$$\#\bigl(F_i(V_0) \cap F_j(V_0)\bigr) \leq 1 \qquad \text{for any } i, j \in S \text{ with } i \neq j. \tag{3.10}$$

Then $\{g(q) \mid g \in \mathcal{G}\} = V_0$ *for* $q \in V_0$. *In particular, if* $d_{\mathrm{w}} \in (1,\infty)$ *and* $(\mathrm{UHK})_{d_{\mathrm{w}}}$ *holds, then there exist* $c_{3.7} \in (0,\infty)$ *and continuous* $\log\tau$*-periodic functions* $G_0, G_1 : \mathbb{R} \to (0,\infty)$ *such that* (3.7) *holds as* $t \downarrow 0$ *for any* $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, *where* $n_{\mathrm{N}} := \frac{\#S\#V_0 - \#V_1}{\#S-1}$ *and* $n_{\mathrm{D}} := -\frac{\#(V_1\setminus V_0)}{\#S-1}$.

Proof. For any $x, y \in V_0$ with $x \neq y$, $g_{xy}|_K \in \mathcal{G}$ by [37, Proof of Theorem 4.5] and [40, Corollary 3.8.21], and $g_{xy}(x) = y$. Thus $\{g(q) \mid g \in \mathcal{G}\} = V_0$ for $q \in V_0$ and hence Theorem 3.4 applies. □

Remark 3.8. (1) As a special case of Remark 3.5-(2), by [40, Theorem 4.3.4 and Corollary 4.4.11] (see also [9, Theorems 4.4 and 6.6]), *if* $\#V_0 \geq 3$ *in the situation of Corollary*

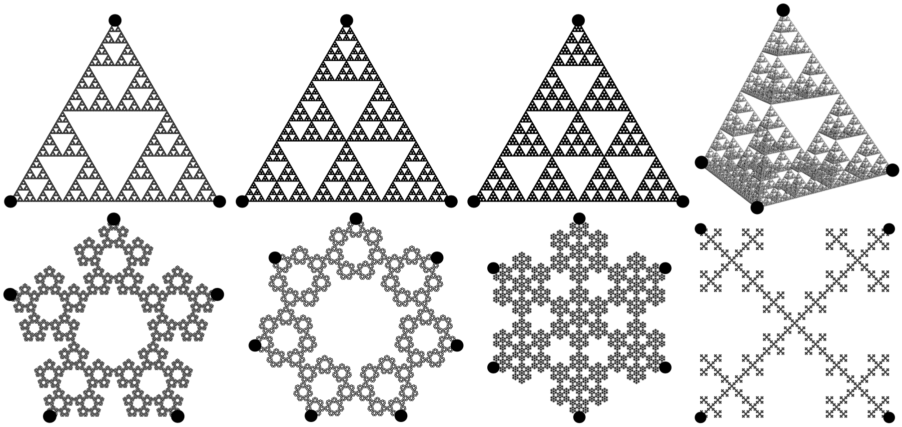

Fig. 2. Examples of nested fractals. From the *upper left*, two-dimensional level-l Sierpiński gasket ($l = 2, 3, 4$), three-dimensional standard (level-2) Sierpiński gasket, pentagasket (5-polygasket), heptagasket (7-polygasket), snowflake and the Vicsek set. In each fractal, the set V_0 of its boundary points is marked by solid circles.

3.7*, then* $G_{\mathbb{N}}$ *in* (3.8) *is discontinuous and hence non-constant, so that* G_0 *in* (3.7) *is also non-constant.*
(2) *If* $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ *is an affine nested fractal satisfying* (3.10)*, then a harmonic structure* (D, r) *on* $\mathcal{L}$ *as in Corollary* 3.7 *exists and is unique (up to constant multiples of* D*).* Here the existence part is essentially due to Lindstrøm [51]; see [40, Section 3.8] and references therein for further details, and see [32,53–55] for more recent results on existence of harmonic structures.

Also in this situation (to be precise, with slight changes in the definition of affine nested fractals), it was proved in [20, Theorems 5.7 and 6.1] that *there exist* $d_{\mathrm{w}} \in (1, \infty)$ *and a metric* ρ *on* K *compatible with the original topology of* K *such that (UHK)*$_{d_{\mathrm{w}}}$ *and a lower bound of the same form as* (3.5) *hold.* Their heat kernel estimates were stated for unbounded affine nested fractals, but the same results for bounded ones can be easily deduced from a recent general result [42, Theorem 15.10].

Indeed, from the construction and the properties of the metric ρ given in [20, Section 3], where ρ is denoted as d, we easily see that $\mathrm{diam}_\rho K_w \leq c_{\rho.1}\tau^{-|w|/d_{\mathrm{w}}}$ for any $w \in W_*$ and $\rho(x, y) \geq c_{\rho.2}\tau^{-|w|/d_{\mathrm{w}}}$ for any $m \in \mathbb{N}$, any $w, v \in W_m$ with $K_w \cap K_v = \emptyset$ and any $(x, y) \in K_w \times K_v$ for some $c_{\rho.1}, c_{\rho.2} \in (0, \infty)$. Also $\mathrm{diam}_R K_w \leq r^{|w|} \mathrm{diam}_R K$ for any $w \in W_*$ by [35, Lemma 3.3.5] and $R(x, y) \geq c_R r^m$ for any $m \in \mathbb{N}$, any $w, v \in W_m$ with $K_w \cap K_v = \emptyset$ and any $(x, y) \in K_w \times K_v$ for some $c_R \in (0, \infty)$ by [40, Proof of Lemma 4.2.4]. It follows from these facts that $c_{R,\rho} R^{d_{\mathrm{w}}^R} \leq \rho^{d_{\mathrm{w}}} \leq c_{\rho,R} R^{d_{\mathrm{w}}^R}$ for some $c_{R,\rho}, c_{\rho,R} \in (0, \infty)$. Now in view of (3.4), we can apply [42, Theorem 15.10] to conclude that $(\mathrm{UHK})_{d_{\mathrm{w}}}$ and a lower bound of the same form as (3.5) hold; note that the chain condition in [42, Theorem 15.10, Case 2] is valid for ρ by its construction.
(3) For the same reason as [37, Theorem 4.5] (see [37, Remark 4.6-(2)]), it is unclear whether the (technical) assumption (3.10) can be removed from Corollary 3.7.

The rest of this section is devoted to the proof of Theorem 3.4, which requires a few intermediate steps. We start with the following easy lemma. Recall (see Definition 2.18) that λ_1^U denotes the smallest eigenvalue of $-\Delta_U$ for a non-empty open subset U of K.

Lemma 3.9. *Let U, V be non-empty open subsets of K. If V is connected and $U \subsetneqq V$, then $\lambda_1^U > \lambda_1^V$. In particular, if $U \neq K$, then $\lambda_1^U > 0$.*

Proof. The latter assertion follows from the former since K is connected and $\lambda_1^K = 0$. Suppose V is connected and $U \subsetneqq V$. Note that then V is arcwise connected since K is locally arcwise connected by [40, Proposition 1.3.6] and the connectedness of K. Suppose $\lambda_1^U \leq \lambda_1^V$ and let φ_1^U be an eigenfunction of $-\Delta_U$ with eigenvalue λ_1^U. Then $\varphi_1^U \in \mathcal{F}_V$ and $\mathcal{E}(\varphi_1^U, \varphi_1^U) \leq \lambda_1^V \int_V (\varphi_1^U)^2 d\mu$ by $U \subset V$, so that φ_1^U would be an eigenfunction of $-\Delta_V$ with eigenvalue λ_1^V on account of (2.15). Now Lemma 2.19 would imply that either $\varphi_1^U > 0$ on V or $\varphi_1^U < 0$ on V, which would contradict $\varphi_1^U \in \mathcal{F}_U = \{u \in \mathcal{F} \mid u|_{K\setminus U} = 0\}$ since $U \subsetneqq V$. Thus $\lambda_1^U > \lambda_1^V$. $\square$

Definition 3.10. For $m \in \mathbb{N} \cup \{0\}$ and $x \in V_m$, we define

$$W_{m,x} := \{w \in W_m \mid x \in K_w\}, \quad n_{x,m} := \# W_{m,x} \quad \text{and} \quad U_m^x := \{x\} \cup \bigcup_{w \in W_{m,x}} K_w^I;$$

note that U_m^x is open in K. We also set $U^q := U_0^q = K^I \cup \{q\}$ for $q \in V_0$.

Proposition 3.11. *Let $q \in V_0$ and suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$. Let $m \in \mathbb{N} \cup \{0\}$ and $x \in V_m$. Then for any $t \in (0, \infty)$,*

$$\mathcal{Z}_{U_m^x}(t) = \mathcal{Z}_{U^q}(\tau^m t) + (n_{x,m} - 1)\mathcal{Z}_{\mathrm{D}}(\tau^m t). \tag{3.11}$$

Proof. For each $U \in \{U^q, K^I\}$, let $\{\varphi_n^U\}_{n\in\mathbb{N}}$ be a complete orthonormal system of $L^2(U, \mu|_U)$ consisting of eigenfunctions of $-\Delta_U$ with eigenvalues $\{\lambda_n^U\}_{n\in\mathbb{N}}$, which we use below to write down all the eigenfunctions of $-\Delta_{U_m^x}$. Let $\{a_k\}_{k=1}^{n_{x,m}} \subset \mathbb{R}^{W_{m,x}}$, $a_k = (a_{k,w})_{w \in W_{m,x}}$, be an orthonormal basis of $\mathbb{R}^{W_{m,x}}$ with $a_1 = n_{x,m}^{-1/2} \mathbf{1}_{W_{m,x}}$. For each $w \in W_{m,x}$, $x \in K_w \cap V_m = F_w(V_0)$, and hence by $\{g(q) \mid g \in \mathcal{G}\} = V_0$ we can choose $g_w \in \mathcal{G}$ so that $x = F_w(g_w(q))$. Now for $n \in \mathbb{N}$, we define $\varphi_{n,k} : K \to \mathbb{R}$, $k \in \{1, \ldots, n_{x,m}\}$, by $\varphi_{n,k}|_{K\setminus U_m^x} := 0$ and

$$\varphi_{n,k}|_{K_w^I \cup \{x\}} := \begin{cases} n_{x,m}^{-1/2} (\# S)^{m/2} \varphi_n^{U^q} \circ g_w^{-1} \circ F_w^{-1}|_{K_w^I \cup \{x\}} & \text{if } k = 1 \\ a_{k,w} (\# S)^{m/2} \varphi_n^{K^I} \circ g_w^{-1} \circ F_w^{-1}|_{K_w^I \cup \{x\}} & \text{if } k \geq 2 \end{cases} \tag{3.12}$$

for each $w \in W_{m,x}$; note that the value at x of the right-hand side of (3.12) is independent of $w \in W_{m,x}$ by $\varphi_n^{K^I}(q) = 0$. Then the expression (3.12) of $\varphi_{n,k}$ extends to K_w for each $w \in W_{m,x}$, hence $\varphi_{n,k} \in C(K)$, and it follows from $\{g_w\}_{w \in W_{m,x}} \subset \mathcal{G}$, (SSDF1) and $\varphi_{n,k}|_{K\setminus U_m^x} = 0$ that $\varphi_{n,k} \in \mathcal{F}_{U_m^x}$.

Next we prove that $\{\varphi_{n,k}\}_{n\in\mathbb{N}, k\in\{1,\ldots,n_{x,m}\}}$ is a complete orthonormal system of $L^2(U_m^x, \mu|_{U_m^x})$. Indeed, it is easily seen to be orthonormal in $L^2(U_m^x, \mu|_{U_m^x})$ by a direct calculation using Lemma 2.5 and $\{g_w\}_{w \in W_{m,x}} \subset \mathcal{G}$. Let $u \in L^2(U_m^x, \mu|_{U_m^x})$ and suppose that $\int_{U_m^x} u \varphi_{n,k} d\mu = 0$ for any $(n, k) \in \mathbb{N} \times \{1, \ldots, n_{x,m}\}$. Then for any $n \in \mathbb{N}$, $0 = n_{x,m}^{1/2} (\# S)^{m/2} \int_{U_m^x} u \varphi_{n,1} d\mu = \int_{U^q} \varphi_n^{U^q} \bigl(\sum_{w \in W_{m,x}} u \circ F_w \circ g_w\bigr) d\mu$, and hence $\sum_{w \in W_{m,x}} u \circ F_w \circ g_w = 0$ μ-a.e. by the completeness of $\{\varphi_n^{U^q}\}_{n\in\mathbb{N}}$. On the other hand, for each $n \in \mathbb{N}$, $\int_{U_m^x} u \varphi_{n,k} d\mu = 0$, $k \in \{2, \ldots, n_{x,m}\}$, implies that $(\# S)^m \int_{K_w} (\varphi_n^{K^I} \circ g_w^{-1} \circ F_w^{-1}) u \, d\mu = \int_{K^I} (u \circ F_w \circ g_w) \varphi_n^{K^I} d\mu$ is independent of $w \in W_{m,x}$, so that $u \circ F_w \circ g_w = u \circ F_v \circ g_v$ μ-a.e. for any $w, v \in W_{m,x}$ by the completeness of $\{\varphi_n^{K^I}\}_{n\in\mathbb{N}}$. Thus $0 = n_{x,m}^{-1}\sum_{v\in W_{m,x}} u\circ F_v\circ g_v = u\circ F_w\circ g_w$ μ-a.e. for any $w\in W_{m,x}$ and hence $u=0$ μ-a.e. on U_m^x.

Finally, we show that for $n\in\mathbb{N}$, $\varphi_{n,k}$ is an eigenfunction of $-\Delta_{U_m^x}$ with eigenvalue $\tau^m\lambda_n^{U^q}$ for $k=1$ and $\tau^m\lambda_n^{K^I}$ for $k\in\{2,\dots,n_{x,m}\}$. Let $n\in\mathbb{N}$. By using (SSDF2), $\{g_w\}_{w\in W_{m,x}}\subset\mathcal{G}$ and the fact that $\varphi_n^{U^q}\in\mathcal{D}[\Delta_{U^q}]$ and $-\Delta_{U^q}\varphi_n^{U^q}=\lambda_n^{U^q}\varphi_n^{U^q}$ (recall (2.12)), we easily see that $\mathcal{E}(\varphi_{n,1},u)=\tau^m\lambda_1^{U^q}\int_{U_m^x}\varphi_{n,1}u\,d\mu$ for any $u\in\mathcal{F}_{U_m^x}$. For the proof for $k\in\{2,\dots,n_{x,m}\}$, for each $u:U_m^x\to\mathbb{R}$ we define $Pu:U_m^x\to\mathbb{R}$ by

$$Pu|_{K_w^I\cup\{x\}} := n_{x,m}^{-1}\sum_{v\in W_{m,x}} u\circ F_v\circ g_v\circ g_w^{-1}\circ F_w^{-1}|_{K_w^I\cup\{x\}},\quad w\in W_{m,x},\quad (3.13)$$

so that $Pu(x):=u(x)$ regardless of choices of $w\in W_{m,x}$ in (3.13). Identifying each $u\in\mathcal{F}_{U_m^x}$ with $u|_{U_m^x}\in C(U_m^x)$, we have $P(\mathcal{F}_{U_m^x})\subset\mathcal{F}_{U_m^x}$ by virtue of (SSDF1) and $\{g_w\}_{w\in W_{m,x}}\subset\mathcal{G}$. Now let $k\in\{2,\dots,n_{x,m}\}$ and $u\in\mathcal{F}_{U_m^x}$. (SSDF2) and $\{g_w\}_{w\in W_{m,x}}\subset\mathcal{G}$ yield $\mathcal{E}(\varphi_{n,k},Pu)=0=\int_{U_m^x}\varphi_{n,k}Pu\,d\mu$, hence $\mathcal{E}(\varphi_{n,k},u)=\mathcal{E}(\varphi_{n,k},u-Pu)$ and $\int_{U_m^x}\varphi_{n,k}u\,d\mu=\int_{U_m^x}\varphi_{n,k}(u-Pu)\,d\mu$, which and $\varphi_n^{K^I}\in\mathcal{D}[\Delta_{K^I}]$, $-\Delta_{K^I}\varphi_n^{K^I}=\lambda_n^{K^I}\varphi_n^{K^I}$, together easily imply $\mathcal{E}(\varphi_{n,k},u)=\tau^m\lambda_n^{K^I}\int_{U_m^x}\varphi_{n,k}u\,d\mu$ since $(u-Pu)\circ F_w\in\mathcal{F}_{K^I}$ for any $w\in W_{m,x}$ by $u-Pu\in\mathcal{F}_{U_m^x}$ and $(u-Pu)(x)=0$.

Thus it follows that $\{\lambda_{n,k}\}_{n\in\mathbb{N},k\in\{1,\dots,n_{x,m}\}}$, $\lambda_{n,1}:=\tau^m\lambda_n^{U^q}$ and $\lambda_{n,k}:=\tau^m\lambda_n^{K^I}$, $k\in\{2,\dots,n_{x,m}\}$, gives an enumeration of all the eigenvalues of $-\Delta_{U_m^x}$ with each eigenvalue repeated according to its multiplicity, and hence (3.11) follows. □

Proposition 3.12. *Let $q\in V_0$, $d_{\mathrm{w}}\in(1,\infty)$ and suppose $\{g(q)\mid g\in\mathcal{G}\}=V_0$ and that* $(\mathrm{UHK})_{d_{\mathrm{w}}}$ *holds. Then there exist $c_{3.8}\in(0,\infty)$ and a continuous $\log\tau$-periodic function $G_1:\mathbb{R}\to(0,\infty)$ such that, as $t\downarrow 0$,*

$$\mathcal{Z}_{U^q}(t)-\mathcal{Z}_{\mathrm{D}}(t)=G_1(-\log t)+O\Bigl(\exp\bigl(-c_{3.8}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr).\qquad(3.14)$$

Remark 3.13. *The periodic function G_1 in the conclusion of Theorem* 3.4 *is nothing but G_1 given by Proposition* 3.12 *as above*; see the end of the proof of Theorem 3.4 below.

Proof. Let us verify the assumptions of Theorem 2.23 with $\alpha_0=0$ and $\gamma=\tau$ for $\mathcal{Z}_{U^q}-\mathcal{Z}_{\mathrm{D}}$. Since $n_{q,1}=1$ by [37, Remark 6.4], $W_{1,q}=\{i\}$ and $U_1^q=K_i^I\cup\{q\}$ for some $i\in S$, and $q\in K_i\cap V_1=F_i(V_0)$. Let $t\in(0,\infty)$. Recalling (2.9), by Proposition 3.11, Lemma 2.21, $U_1^q\subset U^q$ and Proposition 2.16-(1) we have

$$\begin{aligned}\mathcal{Z}_{U^q}(t)-\mathcal{Z}_{\mathrm{D}}(t)-\bigl(\mathcal{Z}_{U^q}(\tau t)-\mathcal{Z}_{\mathrm{D}}(\tau t)\bigr)&=\mathcal{Z}_{U^q}(t)-\mathcal{Z}_{\mathrm{D}}(t)-\mathcal{Z}_{U_1^q}(t)+\mathcal{Z}_{K_i^I}(t)\\&=\int_K p_t^{U^q}\bigl(x,x\bigm|\{K\setminus\{q\},U_1^q\}\bigr)d\mu(x)\geq 0.\end{aligned}\qquad(3.15)$$

Let ρ be as in $(\mathrm{UHK})_{d_{\mathrm{w}}}$ and $x\in K$. Since $\rho(x,q)\vee\rho(x,F_i(V_0)\setminus\{q\})\geq\rho(q,F_i(V_0)\setminus\{q\})/2$ and $(K\setminus K_i)\cup(F_i(V_0)\setminus\{q\})\subset K\setminus U_1^q$, Proposition 2.16, (3.5) and $d_{\mathrm{s}}<2$ imply that for $t\in(0,1]$,

$$\begin{aligned}0&\leq p_t^{U^q}\bigl(x,x\bigm|\{K\setminus\{q\},U_1^q\}\bigr)\qquad(3.16)\\&\leq\begin{cases}4c_{3.5}t^{-d_{\mathrm{s}}/2}\exp\Bigl(-c_{3.6}\bigl(\frac{\rho(q,F_i(V_0)\setminus\{q\})^{d_{\mathrm{w}}}}{2^{d_{\mathrm{w}}}t}\bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\Bigr)&\text{if } x\in K_i\\4c_{3.5}t^{-d_{\mathrm{s}}/2}\exp\Bigl(-c_{3.6}\bigl(\frac{\rho(q,x)^{d_{\mathrm{w}}}}{t}\bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\Bigr)&\text{if } x\in K\setminus K_i\end{cases}\end{aligned}$$

$$\leq 4c_{3.5}t^{-d_{\mathrm{s}}/2}\exp\bigl(-c_{3.8}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr),$$

where $c_{3.8} := c_{3.6}\bigl(\rho(q, K \setminus U_1^q)/2\bigr)^{\frac{d_{\mathrm{w}}}{d_{\mathrm{w}}-1}}$. Now (3.15) and (3.16) together imply that (2.20) holds for $\mathcal{Z}_{U^q} - \mathcal{Z}_{\mathrm{D}}$ with $\alpha_0 = 0$, $\gamma = \tau$, $n = 1$, $\alpha_1 = -1$, $G_1 = 0$ and $\mathcal{R}(t) := 4c_{3.5}t^{-d_{\mathrm{s}}/2}\exp\bigl(-c_{3.8}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$. Moreover $\sum_{j\in\mathbb{N}}\mathcal{R}(\tau^{-j}t) \leq c_{3.9}\exp\bigl(-c_{3.8}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$ for any $t \in (0, \tau]$ for some $c_{3.9} \in (0, \infty)$, $0 \leq \mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) \leq \mathcal{Z}_{U^q}(1)e^{\lambda_1^{U^q}}e^{-\lambda_1^{U^q}t}$ for any $t \in [1, \infty)$ by $p^{K^I} \leq p^{U^q}$ and Lemma 2.20, and $\lambda_1^{U^q} > 0$ by Lemma 3.9. Hence Theorem 2.23 applies to $\mathcal{Z}_{U^q} - \mathcal{Z}_{\mathrm{D}}$ to yield a continuous $\log\tau$-periodic function $G_1 : \mathbb{R} \to \mathbb{R}$ satisfying (3.14).

It remains to show that G_1 is $(0, \infty)$-valued. Note that U^q is connected since K^I is connected by $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and [40, Proposition 1.6.9]. Therefore $\lambda_1^{K^I} > \lambda_1^{U^q}$ by Lemma 3.9, which and Lemma 2.20 imply that $\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) > 0$ for any $t \in [T, \infty)$ for some $T \in [1, \infty)$. Now since $\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) \geq \mathcal{Z}_{U^q}(\tau t) - \mathcal{Z}_{\mathrm{D}}(\tau t)$ for any $t \in (0, \infty)$ by (3.15), $\inf_{t\in(0,\tau T]}\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) = \min_{t\in[T,\tau T]}\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) > 0$, which together with (3.14) shows that G_1 is $(0, \infty)$-valued. □

Proposition 3.14. *Let $q \in V_0$, $d_{\mathrm{w}} \in (1, \infty)$ and suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and that* (UHK)$_{d_{\mathrm{w}}}$ *holds. Then there exists $c_{3.10} \in (0, \infty)$ such that, as $t \downarrow 0$,*

$$\mathcal{Z}_{\mathrm{N}}(t) - \mathcal{Z}_{\mathrm{D}}(t) = (\#V_0)\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) + O\Bigl(\exp\bigl(-c_{3.10}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr), \tag{3.17}$$

$$\mathcal{Z}_{\mathrm{D}}(t) - (\#S)\mathcal{Z}_{\mathrm{D}}(\tau t) = \#(V_1 \setminus V_0)\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) + O\Bigl(\exp\bigl(-c_{3.10}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \tag{3.18}$$

Proof. Let ρ be as in (UHK)$_{d_{\mathrm{w}}}$ and set $\delta := \min_{x\in V_1}\rho(x, K \setminus U_1^x)/2 > 0$, so that $\rho(z, x) \vee \rho(z, y) \geq \delta$ for any $z \in K$ and any $x, y \in V_1$ with $x \neq y$. Let $t \in (0, \infty)$. Since $\mathcal{Z}_{U^x} = \mathcal{Z}_{U^q}$ for any $x \in V_0$ by Proposition 3.11, we see from Lemma 2.22, Proposition 2.16 and (3.5) that

$$\begin{aligned}
&\mathcal{Z}_{\mathrm{N}}(t) - \mathcal{Z}_{\mathrm{D}}(t) - (\#V_0)\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) \qquad (3.19)\\
&= \int_K\Bigl(p_t(y, y) - p_t^{K^I}(y, y) - \sum_{x\in V_0}\bigl(p_t^{U^x}(y, y) - p_t^{K^I}(y, y)\bigr)\Bigr)d\mu(y)\\
&= \int_K\Bigl(\sum_{\emptyset\neq V\subset V_0}\sum_{A\subset V}(-1)^{\#A}p_t^{K^I\cup(V\setminus A)}(y, y) - \sum_{x\in V_0}p_t^{U^x}(y, y|\{K \setminus \{x\}\})\Bigr)d\mu(y)\\
&= \int_K\Bigl(\sum_{\emptyset\neq V\subset V_0}p_t^{K^I\cup V}(y, y|\{K \setminus \{x\}\}_{x\in V}) - \sum_{x\in V_0}p_t^{U^x}(y, y|\{K \setminus \{x\}\})\Bigr)d\mu(y)\\
&= \sum\nolimits_{V\subset V_0,\#V\geq 2}\int_K p_t^{K^I\cup V}(y, y|\{K \setminus \{x\}\}_{x\in V})d\mu(y)\ (\geq 0)\\
&\leq \sum\nolimits_{V\subset V_0,\#V\geq 2}\int_K 2\min_{x\in V}\Bigl\{\sup_{s\in[t/2,t]}p_s(x, y)\Bigr\}d\mu(y)\\
&\leq 2^{\#V_0+2}c_{3.5}t^{-d_{\mathrm{s}}/2}\exp\bigl(-c_{3.6}(\delta^{d_{\mathrm{w}}}/t)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr)
\end{aligned}$$

where the inequality in the last line of (3.19) is valid only for $t \in (0, 1]$.

On the other hand, $(\#S)\mathcal{Z}_{\mathrm{D}}(\tau t) = \sum_{i\in S}\mathcal{Z}_{K_i^I}(t) = \mathcal{Z}_{K\setminus V_1}(t)$ and $n_{x,1}\mathcal{Z}_{\mathrm{D}}(\tau t) = \sum_{i\in W_{1,x}}\mathcal{Z}_{K_i^I}(t) = \mathcal{Z}_{U_1^x\setminus\{x\}}(t)$ for any $x\in V_1$ by Lemmas 2.21 and 2.14, and hence $\mathcal{Z}_{U^q}(\tau t)-\mathcal{Z}_{\mathrm{D}}(\tau t) = \mathcal{Z}_{U_1^x}(t)-\mathcal{Z}_{U_1^x\setminus\{x\}}(t)$ for any $x\in V_1$ by Proposition 3.11. Therefore similarly to (3.19), setting $K_1^V := (K\setminus V_1)\cup V$ for each $V\subset V_1\setminus V_0$, from Lemmas 2.22, 2.14, Proposition 2.16 and (3.5) we obtain

$$
\begin{aligned}
&\mathcal{Z}_{\mathrm{D}}(t)-(\#S)\mathcal{Z}_{\mathrm{D}}(\tau t)-\#(V_1\setminus V_0)\bigl(\mathcal{Z}_{U^q}(\tau t)-\mathcal{Z}_{\mathrm{D}}(\tau t)\bigr) \qquad (3.20)\\
&=\int_K\biggl(p_t^{K^I}(y,y)-p_t^{K\setminus V_1}(y,y)-\sum_{x\in V_1\setminus V_0}\bigl(p_t^{U_1^x}(y,y)-p_t^{U_1^x\setminus\{x\}}(y,y)\bigr)\biggr)d\mu(y)\\
&=\int_K\biggl(\sum_{\emptyset\neq V\subset V_1\setminus V_0}\sum_{A\subset V}(-1)^{\#A}p_t^{K_1^{V\setminus A}}(y,y)-\sum_{x\in V_1\setminus V_0}p_t^{U_1^x}(y,y|K\setminus\{x\})\biggr)d\mu(y)\\
&=\int_K\biggl(\sum_{\emptyset\neq V\subset V_1\setminus V_0}p_t^{K_1^V}(y,y|\{K\setminus\{x\}\}_{x\in V})-\sum_{x\in V_1\setminus V_0}p_t^{U_1^x}(y,y|K\setminus\{x\})\biggr)d\mu(y)\\
&=\sum\nolimits_{V\subset V_1\setminus V_0,\#V\geq 2}\int_K p_t^{K_1^V}(y,y|\{K\setminus\{x\}\}_{x\in V})d\mu(y)\ (\geq 0)\\
&\leq\sum\nolimits_{V\subset V_1\setminus V_0,\#V\geq 2}\int_K 2\min_{x\in V}\Bigl\{\sup_{s\in[t/2,t]}p_s(x,y)\Bigr\}d\mu(y)\\
&\leq 2^{\#(V_1\setminus V_0)+2}c_{3.5}t^{-d_{\mathrm{s}}/2}\exp\bigl(-c_{3.6}(\delta^{d_{\mathrm{w}}}/t)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr)
\end{aligned}
$$

where again the inequality in the last line of (3.20) follows only for $t\in(0,1]$.

Now with $c_{3.10} := c_{3.6}\delta^{\frac{d_{\mathrm{w}}}{d_{\mathrm{w}}-1}}/2$, (3.17) is immediate from (3.19), whereas (3.18) follows from (3.20), (3.15) and (3.16) since $c_{3.8}\geq c_{3.6}\delta^{\frac{d_{\mathrm{w}}}{d_{\mathrm{w}}-1}}$ for $c_{3.8}$ as in (3.16). □

Proof of Theorem 3.4. (3.18) and Proposition 3.12 together yields (2.20) for $\mathcal{Z}_{\mathrm{D}}$ with $\alpha_0 = d_{\mathrm{s}}/2$, $\gamma=\tau$, $n=1$, $\alpha_1=0$, $-n_{\mathrm{D}}(\#S-1)G_1$ in place of G_1, and $\mathcal{R}(t) := c_{3.11}\exp\bigl(-(c_{3.8}\wedge c_{3.10})t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$ for some $c_{3.11}\in(0,\infty)$. Moreover, $\lambda_1^{K^I}>0$ by Lemma 3.9 and $0\leq\mathcal{Z}_D(t)\leq\mathcal{Z}_D(1)e^{\lambda_1^{K^I}}e^{-\lambda_1^{K^I}t}$ for any $t\in[1,\infty)$ by Lemma 2.20. Hence we can apply Theorem 2.23 to $\mathcal{Z}_{\mathrm{D}}$ to conclude that there exists a continuous $\log\tau$-periodic function $G_0:\mathbb{R}\to\mathbb{R}$ such that (3.7) holds for $\mathrm{B}=\mathrm{D}$ with G_1 as in Proposition 3.12 and $c_{3.7} := c_{3.8}\wedge c_{3.10}$. Then (3.7) for $\mathrm{B}=\mathrm{N}$ follows from that for $\mathrm{B}=\mathrm{D}$, (3.17) and Proposition 3.12. Finally, since $\mathcal{Z}_{U^q}-\mathcal{Z}_{\mathrm{D}}\geq 0$ by Proposition 2.16-(1), $\mathcal{Z}_{\mathrm{D}}(t)\geq(\#S)\mathcal{Z}_{\mathrm{D}}(\tau t)$ and hence $t^{d_{\mathrm{s}}/2}\mathcal{Z}_{\mathrm{D}}(t)\geq(\tau t)^{d_{\mathrm{s}}/2}\mathcal{Z}_{\mathrm{D}}(\tau t)$ for any $t\in(0,\infty)$ by (3.20) and therefore $\inf_{t\in(0,1]}t^{d_{\mathrm{s}}/2}\mathcal{Z}_{\mathrm{D}}(t) = \min_{t\in[\tau^{-1},1]}t^{d_{\mathrm{s}}/2}\mathcal{Z}_{\mathrm{D}}(t)>0$, which together with (3.7) shows that G_0 is $(0,\infty)$-valued. □

4. Post-critically finite self-similar fractals II: general self-similar boundary

Throughout this section, we follow the framework of the last section. Namely, $\mathcal{L} = (K,S,\{F_i\}_{i\in S})$ is a post-critically finite self-similar structure with K connected and $\#K\geq 2$, (D,r) is a harmonic structure on $\mathcal{L}$, $(\mathcal{E},\mathcal{F})$ is the resistance form on K resulting from (D,r) defined by (3.2), R is the resistance metric of $(\mathcal{E},\mathcal{F})$ and μ is the self-similar measure on $\mathcal{L}$ with uniform weight. Recall that then $(\mathcal{L},\mu,\mathcal{E},\mathcal{F},r)$ is a self-similar Dirichlet space with uniform weight, that we set $\tau := \#S/r$ and $d_{\mathrm{s}} := 2\log_\tau\#S$ and that the symmetry group $\mathcal{G}$ of $(\mathcal{L},(D,r),\mu)$ is defined by (3.6).

Definition 4.1. Let $m \in \mathbb{N}$ and let $X \subset W_m$ be non-empty. We set $d_X := \frac{2}{m} \log_\tau \#X$,

$$\Sigma[X] := \{\omega \in \Sigma \mid [\sigma^{mn}(\omega)]_m \in X \text{ for any } n \in \mathbb{N} \cup \{0\}\} \quad \text{and} \quad K[X] := \pi(\Sigma[X]).$$

We also define $\Sigma_w[X] := \sigma_w(\Sigma[X])$ and $K_w[X] := \pi(\Sigma_w[X])$ for $w \in W_*$ and set $\mathcal{N}_{X,\mathrm{N}} := \mathcal{N}_{K \setminus K[X]}$, $\mathcal{Z}_{X,\mathrm{N}} := \mathcal{Z}_{K \setminus K[X]}$, $\mathcal{N}_{X,\mathrm{D}} := \mathcal{N}_{K^I \setminus K[X]}$ and $\mathcal{Z}_{X,\mathrm{D}} := \mathcal{Z}_{K^I \setminus K[X]}$.

The purpose of this section is to prove the following theorem.

Theorem 4.2. *Let $q \in V_0$, $d_\mathrm{w} \in (1,\infty)$, suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and that* (UHK)$_{d_\mathrm{w}}$ *holds, and let $n_\mathrm{N}, n_\mathrm{D}, G_0, G_1$ be as in Theorem* 3.4. *Let $m \in \mathbb{N}$, let $X \subsetneqq W_m$ be non-empty and assume $K[X] \not\subset V_0$ if $\#X = 1$. Set $n_{X,\mathrm{N}} := n_{X,\mathrm{D}} := 0$ if $\#X = 1$, and $n_{X,\mathrm{N}} := \frac{\#X\#(K[X]\cap V_0) - \#(K[X]\cap V_m)}{\#X-1}$ and $n_{X,\mathrm{D}} := -\frac{\#(K[X]\cap(V_m\setminus V_0))}{\#X-1}$ if $\#X \geq 2$* (note that $n_{X,\mathrm{N}} \geq 0 \geq n_{X,\mathrm{D}}$). *Then there exist $c_{4.1} \in (0,\infty)$ which is independent of m and X and a continuous $m\log\tau$-periodic function $G_X : \mathbb{R} \to (0,\infty)$ such that for any* $\mathrm{B} \in \{\mathrm{N},\mathrm{D}\}$, *as $t \downarrow 0$,*

$$\begin{aligned} \mathcal{Z}_{X,\mathrm{B}}(t) &= t^{-d_\mathrm{s}/2} G_0(-\log t) - t^{-d_X/2} G_X(-\log t) \\ &\quad + (n_\mathrm{B} - n_{X,\mathrm{B}}) G_1(-\log t) + O\Bigl(\exp\bigl(-c_{4.1}(c_{m,X}/t)^{\frac{1}{d_\mathrm{w}-1}}\bigr)\Bigr), \end{aligned} \tag{4.1}$$

where $c_{m,X} := \tau^{-m} \wedge \bigl(\min_{x\in V_m\setminus K[X]} \rho(x, K[X]) \wedge \min_{x \in V_m} \rho(x, V_m \setminus \{x\})\bigr)^{d_\mathrm{w}}$ with ρ the metric on K given in (UHK)$_{d_\mathrm{w}}$.

Remark 4.3. (1) Only very little is known for $\mathcal{N}_{X,\mathrm{B}}$. In fact, *the author has no idea even whether* $\mathcal{N}_{X,\mathrm{B}}(\lambda) - \lambda^{d_\mathrm{s}/2} G_\mathcal{N}(\log\lambda) = o(\lambda^{d_\mathrm{s}/2})$ *as* $\lambda \to \infty$, where $G_\mathcal{N}$ is as in (3.8).
(2) Similarly to Remark 3.5-(2), *the author does not know any example where it can be proved that G_X in Theorem* 4.2 *is non-constant.*
(3) Similarly to Remark 3.5-(3), the strict positivity of G_X in Theorem 4.2 could be derived from the author's general result [35, Theorem 7.7], but we give an alternative simpler proof of this fact; see the proof of Theorem 4.2 below.

The following proposition is the core of the proof of Theorem 4.2.

Proposition 4.4. *Let $q \in V_0$ and suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$. Let $m \in \mathbb{N}$ and let $X \subsetneqq W_m$ be non-empty. Then for any $t \in (0,\infty)$,*

$$\begin{aligned} &\mathcal{Z}_\mathrm{N}(t) - \mathcal{Z}_{K\setminus K[X]}(t) - \bigl(\mathcal{Z}_\mathrm{D}(t) - \mathcal{Z}_{K^I\setminus K[X]}(t)\bigr) - \#(K[X]\cap V_0)\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_\mathrm{D}(t)\bigr) \\ &= \sum\nolimits_{\substack{V\subset V_0 \\ \#V\geq 2 \text{ or } V\not\subset K[X]}} \int_K p_t^{K^I\cup V}(y,y|\{K\setminus K[X]\}\cup\{K\setminus\{x\}\}_{x\in V})\,d\mu(y), \end{aligned} \tag{4.2}$$

$$\begin{aligned} &\mathcal{Z}_\mathrm{D}(t) - \mathcal{Z}_{K^I\setminus K[X]}(t) - (\#X)\bigl(\mathcal{Z}_\mathrm{D}(\tau^m t) - \mathcal{Z}_{K^I\setminus K[X]}(\tau^m t)\bigr) \\ &\qquad\qquad - \#\bigl(K[X]\cap(V_m\setminus V_0)\bigr)\bigl(\mathcal{Z}_{U^q}(\tau^m t) - \mathcal{Z}_\mathrm{D}(\tau^m t)\bigr) \\ &= \sum\nolimits_{\substack{V\subset V_m\setminus V_0 \\ \#V\geq 2 \text{ or } V\not\subset K[X]}} \int_K p_t^{(K\setminus V_m)\cup V}(y,y|\{K\setminus K[X]\}\cup\{K\setminus\{x\}\}_{x\in V})\,d\mu(y). \end{aligned} \tag{4.3}$$

Proof. Let $t \in (0,\infty)$. We see from Lemma 2.22 and (2.9) that

$$\begin{aligned} &\mathcal{Z}_\mathrm{N}(t) - \mathcal{Z}_{K\setminus K[X]}(t) - \bigl(\mathcal{Z}_\mathrm{D}(t) - \mathcal{Z}_{K^I\setminus K[X]}(t)\bigr) \\ &= \int_K \Bigl(p_t(y,y) - p_t^{K\setminus K[X]}(y,y) - p_t^{K^I}(y,y) + p_t^{K^I\setminus K[X]}(y,y)\Bigr)d\mu(y) \end{aligned} \tag{4.4}$$

$$= \int_K \sum_{\emptyset \neq V \subset V_0} \sum_{A \subset V} (-1)^{\#A} \Big(p_t^{K^I \cup (V \setminus A)}(y,y) - p_t^{(K^I \cup (V \setminus A)) \setminus K[X]}(y,y) \Big) d\mu(y)$$
$$= \sum_{\emptyset \neq V \subset V_0} \int_K p_t^{K^I \cup V}(y, y | \{K \setminus K[X]\} \cup \{K \setminus \{x\}\}_{x \in V}) d\mu(y)$$
$$= \sum_{x \in K[X] \cap V_0} \int_K p_t^{U^x}(y, y | \{K \setminus K[X], K \setminus \{x\}\}) d\mu(y)$$
$$+ \sum_{\substack{V \subset V_0 \\ \#V \geq 2 \text{ or } V \not\subset K[X]}} \int_K p_t^{K^I \cup V}(y, y | \{K \setminus K[X]\} \cup \{K \setminus \{x\}\}_{x \in V}) d\mu(y).$$

Now (4.2) follows from (4.4) since, for $x \in K[X] \cap V_0$, Proposition 3.11 yields

$$\int_K p_t^{U^x}(y, y | \{K \setminus K[X], K \setminus \{x\}\}) d\mu(y)$$
$$= \int_K \Big(p_t^{U^x}(y,y) - p_t^{K^I}(y,y) \Big) d\mu(y) = \mathcal{Z}_{U^x}(t) - \mathcal{Z}_{\mathrm{D}}(t) = \mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t).$$

For (4.3), since $K_w^I \cap K[X] = \emptyset$ for $w \in W_m \setminus X$, setting $K_m^V := (K \setminus V_m) \cup V$ for each $V \subset V_m \setminus V_0$, we see from Lemmas 2.21, 2.14, 2.22 and (2.9) that

$$\mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{K^I \setminus K[X]}(t) - (\#X)\big(\mathcal{Z}_{\mathrm{D}}(\tau^m t) - \mathcal{Z}_{K^I \setminus K[X]}(\tau^m t)\big) \tag{4.5}$$
$$= \mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{K^I \setminus K[X]}(t) - \sum_{w \in X} \big(\mathcal{Z}_{K_w^I}(t) - \mathcal{Z}_{K_w^I \setminus K_w[X]}(t) \big)$$
$$= \mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{K^I \setminus K[X]}(t) - \sum_{w \in W_m} \big(\mathcal{Z}_{K_w^I}(t) - \mathcal{Z}_{K_w^I \setminus K[X]}(t) \big)$$
$$= \mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{K^I \setminus K[X]}(t) - \int_K \sum_{w \in W_m} \Big(p_t^{K_w^I}(y,y) - p_t^{K_w^I \setminus K[X]}(y,y) \Big) d\mu(y)$$
$$= \int_K \Big(p_t^{K^I}(y,y) - p_t^{K^I \setminus K[X]}(y,y) - p_t^{K \setminus V_m}(y,y) + p_t^{(K \setminus V_m) \setminus K[X]}(y,y) \Big) d\mu(y)$$
$$= \int_K \sum_{\emptyset \neq V \subset V_m \setminus V_0} \sum_{A \subset V} (-1)^{\#A} \Big(p_t^{K_m^{V \setminus A}}(y,y) - p_t^{K_m^{V \setminus A} \setminus K[X]}(y,y) \Big) d\mu(y)$$
$$= \sum_{\emptyset \neq V \subset V_m \setminus V_0} \int_K p_t^{K_m^V}(y, y | \{K \setminus K[X]\} \cup \{K \setminus \{x\}\}_{x \in V}) d\mu(y)$$
$$= \sum_{x \in K[X] \cap (V_m \setminus V_0)} \int_K p_t^{(K \setminus V_m) \cup \{x\}}(y, y | \{K \setminus K[X], K \setminus \{x\}\}) d\mu(y)$$
$$+ \sum_{\substack{V \subset V_m \setminus V_0 \\ \#V \geq 2 \text{ or } V \not\subset K[X]}} \int_K p_t^{(K \setminus V_m) \cup V}(y, y | \{K \setminus K[X]\} \cup \{K \setminus \{x\}\}_{x \in V}) d\mu(y).$$

For $x \in K[X] \cap (V_m \setminus V_0)$, Lemmas 2.14, 2.21 and Proposition 3.11 imply that

$$\int_K p_t^{(K \setminus V_m) \cup \{x\}}(y, y | \{K \setminus K[X], K \setminus \{x\}\}) d\mu(y)$$
$$= \int_K \Big(p_t^{(K \setminus V_m) \cup \{x\}}(y,y) - p_t^{K \setminus V_m}(y,y) \Big) d\mu(y)$$

$$
\begin{aligned}
&= \int_K \Bigl(p_t^{U_m^x}(y,y) - p_t^{U_m^x \setminus \{x\}}(y,y) \Bigr) d\mu(y) \\
&= \mathcal{Z}_{U_m^x}(t) - n_{x,m} \mathcal{Z}_{\mathrm{D}}(\tau^m t) = \mathcal{Z}_{U^q}(\tau^m t) - \mathcal{Z}_{\mathrm{D}}(\tau^m t),
\end{aligned}
$$

which and (4.5) yield (4.3). □

Proof of Theorem 4.2. Recall that ρ denotes the metric on K given in $(\mathrm{UHK})_{d_\mathrm{w}}$. Set $\delta := \frac{1}{2}\bigl(\min_{x \in V_m \setminus K[X]} \rho(x, K[X]) \wedge \min_{x \in V_m} \rho(x, V_m \setminus \{x\})\bigr)$, let $t \in (0,\infty)$ and let $V \subset V_m$ satisfy $\#V \geq 2$ or $V \not\subset K[X]$. Then $\rho(y, K[X]) \vee \max_{x \in V} \rho(x,y) \geq \delta$ for any $y \in K$, and since $\mathrm{int}_K K[X] = \emptyset$ by $X \subsetneqq W_m$, we see from Proposition 2.16 and $(\mathrm{UHK})_{d_\mathrm{w}}$ that for any open subset U of K,

$$
\begin{aligned}
0 &\leq \int_K p_t^U(y,y \mid \{K \setminus K[X]\} \cup \{K \setminus \{x\}\}_{x \in V}) d\mu(y) \qquad (4.6)\\
&\leq 2 \int_K \Bigl(\sup_{s \in [t/2,t]} \sup_{x \in K[X]} p_s(x,y) \Bigr) \wedge \min_{x \in V} \Bigl\{ \sup_{s \in [t/2,t]} p_s(x,y) \Bigr\} d\mu(y) \\
&\leq 4 c_{3.5} t^{-d_\mathrm{s}/2} \int_K \exp\biggl(-c_{3.6} \Bigl(\frac{\rho(y,K[X]) \vee \max_{x \in V} \rho(x,y)}{t^{1/d_\mathrm{w}}} \Bigr)^{\frac{d_\mathrm{w}}{d_\mathrm{w}-1}} \biggr) d\mu(y) \\
&\leq 4 c_{3.5} t^{-d_\mathrm{s}/2} \exp\bigl(-c_{3.6} (\delta^{d_\mathrm{w}}/t)^{\frac{1}{d_\mathrm{w}-1}} \bigr)
\end{aligned}
$$

where the inequality in the third line of (4.6) is valid only for $t \in (0,1]$.

If $\#X = 1$, then it easily follows from $K[X] \not\subset V_0$ that $K[X] \cap V_* = \emptyset$, and therefore (4.3) and (4.6) imply (2.20) for $\mathcal{Z}_{\mathrm{D}} - \mathcal{Z}_{K^I \setminus K[X]}$ with $\alpha_0 = 0$, $\gamma = \tau^m$, $n = 1$, $\alpha_1 = -1$, 0 in place of G_1, and $\mathcal{R}(t) := 2^{\#V_m - \#V_0 + 2} c_{3.5} t^{-d_\mathrm{s}/2} \exp\bigl(-c_{3.6}(\delta^{d_\mathrm{w}}/t)^{\frac{1}{d_\mathrm{w}-1}}\bigr)$. If $\#X \geq 2$, then (4.3), (4.6) and Proposition 3.12 together imply that $\mathcal{Z}_{\mathrm{D}} - \mathcal{Z}_{K^I \setminus K[X]}$ satisfies (2.20) with $\alpha_0 = d_X/2$, $\gamma = \tau^m$, $n = 1$, $\alpha_1 = 0$, $-n_{X,\mathrm{D}}(\#X - 1)G_1$ in place of G_1, and $\mathcal{R}(t) := c_{4.2} t^{-d_\mathrm{s}/2} \exp\bigl(-c_{4.3} t^{-\frac{1}{d_\mathrm{w}-1}}\bigr)$ for some $c_{4.2} \in (0,\infty)$, where $c_{4.3} := \bigl(c_{3.6} \delta^{\frac{d_\mathrm{w}}{d_\mathrm{w}-1}}\bigr) \wedge \bigl(c_{3.8} \tau^{-\frac{m}{d_\mathrm{w}-1}}\bigr)$. Also $\sum_{j \in \mathbb{N}} (\#X)^{-j} \mathcal{R}(\tau^{-mj} t) \leq c_{4.4} \exp\bigl(-c_{4.3} t^{-\frac{1}{d_\mathrm{w}-1}}\bigr)$ for any $t \in (0,\tau^m]$ for some $c_{4.4} \in (0,\infty)$ in both cases, $0 \leq \mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{K^I \setminus K[X]}(t) \leq \mathcal{Z}_{\mathrm{D}}(1) e^{\lambda_1^{K^I}} e^{-\lambda_1^{K^I} t}$ for any $t \in [1,\infty)$ by $p^{K^I \setminus K[X]} \leq p^{K^I}$ and Lemma 2.20, and $\lambda_1^{K^I} > 0$ by Lemma 3.9. Hence Theorem 2.23 applies to $\mathcal{Z}_{\mathrm{D}} - \mathcal{Z}_{K^I \setminus K[X]}$ to yield a continuous $m \log \tau$-periodic function $G_X : \mathbb{R} \to \mathbb{R}$ such that for any $t \in (0,\tau^m]$,

$$
\begin{aligned}
&\Bigl| \mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{K^I \setminus K[X]}(t) - t^{-d_X/2} G_X(-\log t) - n_{X,\mathrm{D}} G_1(-\log t) \Bigr| \\
&\qquad\qquad \leq c_{4.4} \exp\bigl(-c_{4.3} t^{-\frac{1}{d_\mathrm{w}-1}}\bigr). \qquad (4.7)
\end{aligned}
$$

Now since $n_{X,\mathrm{N}} = n_{X,\mathrm{D}} + \#(K[X] \cap V_0)$, (4.1) follows from Theorem 3.4, Proposition 3.12, (4.2), (4.6) and (4.7).

Finally, we prove that G_X is $(0,\infty)$-valued. Since $K^I \setminus K[X] \subsetneqq K^I$ by $K[X] \not\subset V_0$ and K^I is connected by $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and [40, Proposition 1.6.9], Lemma 3.9 yields $\lambda_1^{K^I \setminus K[X]} > \lambda_1^{K^I}$, and therefore by Lemma 2.20 there exists $T \in [1,\infty)$ such that $\mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{K^I \setminus K[X]}(t) > 0$ for any $t \in [T,\infty)$. On the other hand, (4.3) and Proposition 2.16-(1) imply that $\mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{K^I \setminus K[X]}(t) \geq (\#X)\bigl(\mathcal{Z}_{\mathrm{D}}(\tau^m t) - \mathcal{Z}_{K^I \setminus K[X]}(\tau^m t)\bigr)$

and hence $t^{d_X/2}\bigl(\mathcal{Z}_{\mathrm{D}}(t)-\mathcal{Z}_{K^I\setminus K[X]}(t)\bigr)\geq(\tau^m t)^{d_X/2}\bigl(\mathcal{Z}_{\mathrm{D}}(\tau^m t)-\mathcal{Z}_{K^I\setminus K[X]}(\tau^m t)\bigr)$ for any $t\in(0,\infty)$. Now it follows from these facts that

$$\inf_{t\in(0,\tau^m T]} t^{d_X/2}\bigl(\mathcal{Z}_{\mathrm{D}}(t)-\mathcal{Z}_{K^I\setminus K[X]}(t)\bigr)=\min_{t\in[T,\tau^m T]} t^{d_X/2}\bigl(\mathcal{Z}_{\mathrm{D}}(t)-\mathcal{Z}_{K^I\setminus K[X]}(t)\bigr)>0,$$

which and (4.7) together show that G_X is $(0,\infty)$-valued. □

Example 4.5 (Sierpinski gaskets)**.** Let $d,l\in\mathbb{N}$, $d\geq 2$, $l\geq 2$, and let $\{q_k\}_{k=0}^d\subset\mathbb{R}^d$ be the set of the vertices of a regular d-dimensional simplex. Further let $S:=\{(i_k)_{k=1}^d\in(\mathbb{N}\cup\{0\})^d \mid \sum_{k=1}^d i_k\leq l-1\}$, and for each $i=(i_k)_{k=1}^d\in S$ we set $q_i:=q_0+\sum_{k=1}^d(i_k/l)(q_k-q_0)$ and define $f_i:\mathbb{R}^d\to\mathbb{R}^d$ by $f_i(x):=q_i+l^{-1}(x-q_0)$. Let K be the *self-similar set associated with* $\{f_i\}_{i\in S}$, i.e. the unique non-empty compact subset of $\mathbb{R}^d$ such that $K=\bigcup_{i\in S}f_i(K)$, and set $F_i:=f_i|_K$ for $i\in S$. Then $\mathcal{L}=(K,S,\{F_i\}_{i\in S})$ is a self-similar structure, which is called the *d-dimensional level-l Sierpinski gasket* (see the first line of Fig. 2 above). This is a nested fractal satisfying (3.10), and we have $\mathcal{P}=\{\boldsymbol{i}_k^\infty \mid k\in\{0,1,\ldots,d\}\}$ and $V_0=\{q_k\mid k\in\{0,1,\ldots,d\}\}$, where $\boldsymbol{i}_k:=((l-1)\mathbf{1}_{\{k\}}(j))_{j=1}^d\in S$.

Define $D=(D_{xy})_{x,y\in V_0}$ by $D_{xx}:=-d$ and $D_{xy}:=1$ for $x,y\in V_0$, $x\neq y$. Note that any real symmetric matrix satisfying (D1), (D2) and the condition in the first sentence of Corollary 3.7 is a constant multiple of D. By the symmetry of $\mathcal{L}$ and D, there exists a unique $r\in(0,\infty)$ such that (D,r) is a harmonic structure on $\mathcal{L}$. In this situation, it is not difficult to verify that $l^{-1}<r<l^{-2}\#S$.

Let R be the resistance metric of the resistance form $(\mathcal{E},\mathcal{F})$ resulting from (D,r) and define metrics $\rho,\hat{\rho}:K\times K\to[0,\infty)$ on K by $\rho(x,y):=|x-y|$ and $\hat{\rho}(x,y):=\inf\{\ell_{\mathbb{R}^d}(\gamma)\mid\gamma:[0,1]\to K,\gamma$ is continuous, $\gamma(0)=x,\gamma(1)=y\}$ where $\ell_{\mathbb{R}^d}$ denotes the length with respect to $|\cdot|$. Then clearly $\operatorname{diam}_{\hat{\rho}}K_w=l^{-|w|}$ for any $w\in W_*$ and $\rho(x,y)\geq c_\rho l^{-m}$ for any $m\in\mathbb{N}$, any $w,v\in W_m$ with $K_w\cap K_v=\emptyset$ and any $(x,y)\in K_w\times K_v$ for some $c_\rho\in(0,\infty)$. Therefore similarly to Remark 3.8-(2), we easily have $c_{R,\rho}R^{\log_{1/r}l}\leq\rho\leq\hat{\rho}\leq c_{\hat{\rho},R}R^{\log_{1/r}l}$ for some $c_{R,\rho},c_{\hat{\rho},R}\in(0,\infty)$, and in view of (3.4) we can apply [42, Theorem 15.10] to conclude that $(\mathrm{UHK})_{d_{\mathrm{w}}}$ and a lower bound of the same form as (3.5) hold with the Euclidean metric ρ and $d_{\mathrm{w}}:=\log_l\tau$; note that $2<d_{\mathrm{w}}<d_{\mathrm{w}}^R$ by $l^{-1}<r<l^{-2}\#S$. Note also that the chain condition in [42, Theorem 15.10, Case 2] holds for $\hat{\rho}$ by its definition and hence for ρ as well. Thus Corollary 3.7 and Theorem 4.2 apply to this example with the reminder estimates given in terms of this exponent d_{w}, which is the smallest in order for $(\mathrm{UHK})_{d_{\mathrm{w}}}$ to be satisfied.

In the case of the d-dimensional standard (i.e. level-2) Sierpiński gasket, $\#S=d+1$, $n_{\mathrm{N}}=\frac{d+1}{2}$, $n_{\mathrm{D}}=-\frac{d+1}{2}$ and we can easily verify that $r=\frac{d+1}{d+3}$, so that $\tau=d+3$, $d_{\mathrm{s}}=2\log_{d+3}(d+1)$ and $d_{\mathrm{w}}=\log_2(d+3)$. Furthermore if we consider $X:=S\setminus\{0_{\mathbb{R}^d}\}\subset W_1$, then $K[X]$ is the $(d-1)$-dimensional standard Sierpiński gasket with vertices $\{q_k\}_{k=1}^d$ and it holds that $d_X=2\log_{d+3}d$, $n_{X,\mathrm{N}}=d/2$ and $n_{X,\mathrm{D}}=-d/2$. Consequently we obtain (3.7) and (4.1) with $n_{\mathrm{N}}-n_{X,\mathrm{N}}=1/2$ and $n_{\mathrm{D}}-n_{X,\mathrm{D}}=-1/2$, which include Theorem 1.1 as a special case.

5. Sierpiński carpets

Our main concern in the rest of this paper is the case of generalized Sierpiński carpets, which are among the most typical examples of *infinitely ramified* self-similar fractals and have been intensively studied e.g. in [3–8,46,41,30,35,33].

In this section, we first collect fundamental facts concerning generalized Sierpiński carpets and their canonical self-similar Dirichlet form and then state our main theorems (Theorems 5.11 and 5.15) of asymptotic expansion of the partition functions for them. The proof of Theorems 5.11 and 5.15 is given in the next section.

We fix the following setting throughout this and the next sections.

Framework 5.1. Let $d, l \in \mathbb{N}$, $d \geq 2$, $l \geq 2$ and set $Q_0 := [0,1]^d$. Let $S \subset \{0,1,\ldots,l-1\}^d$ be non-empty, and for each $i \in S$ define $f_i : \mathbb{R}^d \to \mathbb{R}^d$ by $f_i(x) := l^{-1}i + l^{-1}x$. Set $Q_1 := \bigcup_{i\in S} f_i(Q_0)$, which satisfies $Q_1 \subset Q_0$. Let K be the self-similar set associated with $\{f_i\}_{i\in S}$, i.e. the unique non-empty compact subset of $\mathbb{R}^d$ such that $K = \bigcup_{i\in S} f_i(K)$, and set $F_i := f_i|_K$ for $i \in S$, so that $\mathrm{GSC}(d,l,S) := (K, S, \{F_i\}_{i\in S})$ is a self-similar structure. Also let $\rho : K \times K \to [0,\infty)$ be the Euclidean metric on K given by $\rho(x,y) := |x-y|$, set $d_{\mathrm{f}} := \log_l \#S$ and let μ be the self-similar measure on $\mathcal{L}$ with uniform weight.

Recall that d_{f} is the Hausdorff dimension of (K,ρ) and that μ is a constant multiple of the d_{f}-dimensional Hausdorff measure on (K,ρ); see e.g. [40, Theorem 1.5.7 and Proposition 1.5.8].

The following definition is essentially due to M. T. Barlow and R. F. Bass [7].

Definition 5.2 (Generalized Sierpiński carpets)**.** $\mathrm{GSC}(d,l,S)$ is called a *generalized Sierpiński carpet* if and only if S satisfies the following four conditions:

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of $\mathbb{R}^d$ with $f(Q_0) = Q_0$.

(GSC2) (Connectedness) Q_1 is connected.

(GSC3) (Non-diagonality) $\mathrm{int}_{\mathbb{R}^d}\bigl(Q_1 \cap \prod_{k=1}^d [(i_k-\varepsilon_k)l^{-1}, (i_k+1)l^{-1}]\bigr)$ is either empty or connected for any $(i_k)_{k=1}^d \in \mathbb{Z}^d$ and any $(\varepsilon_k)_{k=1}^d \in \{0,1\}^d$.

(GSC4) (Borders included) $\{(x_1,0,\ldots,0) \in \mathbb{R}^d \mid x_1 \in [0,1]\} \subset Q_1$.

As special cases of Definition 5.2, $\mathrm{GSC}(2,3,S_{\mathrm{SC}})$ and $\mathrm{GSC}(3,3,S_{\mathrm{MS}})$ are called the *Sierpiński carpet* and the *Menger sponge*, respectively, where $S_{\mathrm{SC}} := \{0,1,2\}^2 \setminus \{(1,1)\}$ and $S_{\mathrm{MS}} := \bigl\{(i_1,i_2,i_3) \in \{0,1,2\}^3 \bigm| \sum_{k=1}^3 \mathbf{1}_{\{1\}}(i_k) \leq 1\bigr\}$ (see Fig. 3).

We remark that there are several equivalent ways of stating the non-diagonality condition, as in the following proposition.

Proposition 5.3 ([36, §2])**.** *Set* $|x|_1 := \sum_{k=1}^d |x_k|$ *for* $x = (x_k)_{k=1}^d \in \mathbb{R}^d$*. Then* (GSC3) *is equivalent to any one of the following three conditions:*

$(\mathrm{ND})_{\mathbb{N}}$ $\mathrm{int}_{\mathbb{R}^d}\bigl(Q_1 \cap \prod_{k=1}^d [(i_k-1)l^{-m}, (i_k+1)l^{-m}]\bigr)$ *is either empty or connected for any* $m \in \mathbb{N}$ *and any* $(i_k)_{k=1}^d \in \{1,\ldots,l^m-1\}^d$.

$(\mathrm{ND})_2$ *The case of* $m = 2$ *of* $(\mathrm{ND})_{\mathbb{N}}$ *holds.*

(NDF) *For any* $i,j \in S$ *with* $f_i(Q_0) \cap f_j(Q_0) \neq \emptyset$ *there exists* $\{n(k)\}_{k=0}^{|i-j|_1} \subset S$ *such that* $n(0) = i$, $n(|i-j|_1) = j$ *and* $|n(k) - n(k+1)|_1 = 1$ *for any* $k \in \{0,\ldots,|i-j|_1 - 1\}$.

Remark 5.4. Only the case of $m = 1$ of $(\mathrm{ND})_{\mathbb{N}}$ was assumed in the original definition of generalized Sierpiński carpets in [7, Section 2], but Barlow, Bass, Kumagai and Teplyaev [8] have recently realized that it is too weak for [7, Proof of Theorem 3.19] and has to be replaced by $(\mathrm{ND})_{\mathbb{N}}$ (or equivalently, by (GSC3)).

Now in view of (NDF) in Proposition 5.3, (GSC2) and (GSC3) together imply that $\mathrm{int}_{\mathbb{R}^d} Q_1$ is connected, so that Definition 5.2 turns out to be equivalent to the definition of generalized Sierpiński carpets in [8, Subsection 2.2].

In the rest of this section, we assume that $\mathcal{L} := \mathrm{GSC}(d,l,S) = (K, S, \{F_i\}_{i\in S})$ is a generalized Sierpiński carpet. Then we easily see the following proposition.

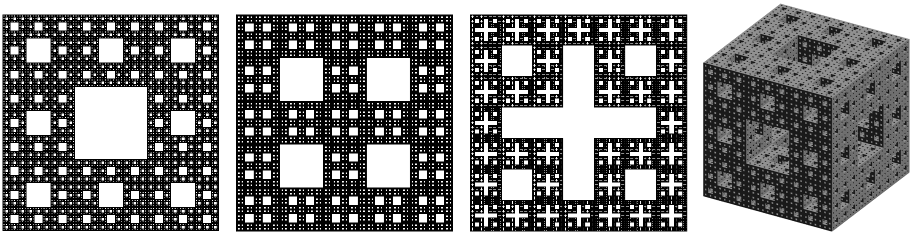

Fig. 3. Sierpiński carpet, some other generalized Sierpiński carpets with $d = 2$ and Menger sponge

Proposition 5.5. *Set* $S_{k,\varepsilon} := \{(i_n)_{n=1}^d \in S \mid i_k = (l-1)\varepsilon\}$ *for* $k \in \{1, 2, \ldots, d\}$ *and* $\varepsilon \in \{0, 1\}$. *Then* $\mathcal{P} = \bigcup_{k=1}^d (S_{k,0}^{\mathbb{N}} \cup S_{k,1}^{\mathbb{N}})$ *and* $\overline{V_0} = V_0 = K \setminus (0,1)^d \neq K$.

Analysis on generalized Sierpiński carpets was initiated by M. T. Barlow and R. F. Bass in [3]: they obtained a non-degenerate μ-symmetric diffusion X on K in the case of $d = 2$ by taking a certain scaling limit of (a suitable subsequence of) the reflecting Brownian motions $X^{(m)}$ on $Q_m := \bigcup_{w \in W_m} f_w(Q_0)$, where $f_w := f_{w_1} \circ \cdots \circ f_{w_m}$ ($f_\emptyset := \mathrm{id}_{\mathbb{R}^d}$) for $w = w_1 \ldots w_m \in W_*$. Then they studied the diffusion X intensively in a series of papers [4–6] and extended their results to the case of $d \geq 3$ in [7]. On the other hand, Kusuoka and Zhou [46] also obtained a non-degenerate diffusion on K in the case of $d = 2$ by constructing a self-similar Dirichlet form on $L^2(K, \mu)$ (with uniform weight r for some $r \in (0,1)$) via a discrete approximation of K. It had been a long-standing problem to prove that the constructions in [3,7] and in [46] give rise to the same diffusion on K, until Barlow, Bass, Kumagai and Teplyaev [8] finally solved it by proving the uniqueness of a non-zero conservative symmetric regular Dirichlet form on $L^2(K, \mu)$ possessing certain local symmetry properties. The following is the statement of their uniqueness theorem [8, Theorem 1.2].

Definition 5.6. (1) We define

$$\mathcal{G}_0 := \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^d \text{ with } f(Q_0) = Q_0\}, \tag{5.1}$$

which forms a subgroup of the group of homeomorphisms of K by virtue of (GSC1).
(2) Define $\psi : \mathbb{R}^d \to Q_0$ by $\psi\big((x_k)_{k=1}^d\big) := \big(\min_{n \in \mathbb{Z}} |x_k - 2n|\big)_{k=1}^d$. For each $w \in W_*$, we set $q^w := F_w(0)$ and define the *folding map* $\varphi_w : K \to K_w$ *into* K_w by

$$\varphi_w(x) := q^w + l^{-|w|}\psi\big(l^{|w|}(x - q^w)\big), \tag{5.2}$$

so that $\varphi_w|_{K_w} = \mathrm{id}_{K_w}$ and $\varphi_w \circ \varphi_v = \varphi_w$ for any $w, v \in W_*$ with $|w| = |v|$.

Note that $\mu \circ g = \mu$ for any $g \in \mathcal{G}_0$. We set $\mu|_A := \mu|_{\mathcal{B}(A)}$ for $A \in \mathcal{B}(K)$. For each $w \in W_*$, if $u : K_w \to [-\infty, \infty]$ is Borel measurable then $\int_K |u \circ \varphi_w| d\mu = (\#S)^{|w|} \int_{K_w} |u| d\mu$, so that $\varphi_w^* u := u \circ \varphi_w$ defines a bounded linear operator $\varphi_w^* : L^2(K_w, \mu|_{K_w}) \to L^2(K, \mu)$, which is called the *unfolding operator from* K_w.

Theorem 5.7 ([8, Theorem 1.2]). *There exists a unique (up to constant multiples of* $\mathcal{E}$*) non-zero conservative symmetric regular Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(K, \mu)$ *satisfying the following conditions:*

(BBKT1) $u \circ \varphi_w \in \mathcal{F}$ *for any* $u \in \mathcal{F}$ *and any* $w \in W_*$.

(BBKT2) *For any* $m \in \mathbb{N}$ *and any* $u \in \mathcal{F}$,

$$\mathcal{E}(u,u) = \frac{1}{(\#S)^m} \sum_{w \in W_m} \mathcal{E}(u \circ \varphi_w, u \circ \varphi_w). \tag{5.3}$$

(BBKT3) *Let* $w, v \in W_*$, $|w| = |v|$ *and* $g \in \mathcal{G}_0$. *If* $u \in L^2(K_v, \mu|_{K_v})$ *and* $u \circ \varphi_v \in \mathcal{F}$, *then* $u^g_{w,v} := u \circ F_v \circ g \circ F_w^{-1} \circ \varphi_w \in \mathcal{F}$ *and* $\mathcal{E}(u^g_{w,v}, u^g_{w,v}) = \mathcal{E}(u \circ \varphi_v, u \circ \varphi_v)$.

In fact, we can also deduce from Theorem 5.7 and [8, Theorem 4.32] the following simpler, though more restrictive, characterization of $(\mathcal{E}, \mathcal{F})$, by virtue of [33, Proof of Proposition 5.1] which for $f \in C(K)$ is based only on [8, Theorem 4.32] and (NDF); see [38, Section 5] for details. Recall Definition 2.6 for (SSDF1) and (SSDF2).

Proposition 5.8 ([38, Proposition 5.9]). $(\mathcal{E}, \mathcal{F})$ *is the unique (up to constant multiples of* $\mathcal{E}$*) non-zero conservative symmetric regular Dirichlet form on* $L^2(K, \mu)$ *satisfying* (SSDF1), (SSDF2) *for some* $r \in (0, \infty)$ *and the following condition:*

(GSCDF) *If* $u \in \mathcal{F} \cap C(K)$ *and* $g \in \mathcal{G}_0$ *then* $u \circ g \in \mathcal{F}$ *and* $\mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u)$.

Moreover, $r \in (0, \infty)$ *for which* (SSDF2) *holds is unique and satisfies* $r \leq l^{-2}\#S$.

It follows from Proposition 5.8 that $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F}, r)$ is a self-similar Dirichlet space with uniform weights, and (GSCDF) and the regularity of $(\mathcal{E}, \mathcal{F})$ easily yield $u \circ g \in \mathcal{F}$ and $\mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u)$ for any $u \in \mathcal{F}$ and any $g \in \mathcal{G}_0$. Moreover, it satisfies (CHK) and (2.7) by the following theorem. Recall that we set $\tau := \#S/r$ and $d_{\mathrm{s}} := 2 \log_\tau \#S$.

Theorem 5.9 ([7, Theorem 1.3], [8, Theorem 4.30 and Remark 4.33]). $(K, \mu, \mathcal{E}, \mathcal{F})$ *satisfies* (CHK) *and there exist* $c_{5.1}, c_{5.2} \in (0, \infty)$ *such that, with* $d_{\mathrm{w}} := \log_l \tau$ (note that $d_{\mathrm{w}} \geq 2$ and that $d_{\mathrm{f}}/d_{\mathrm{w}} = d_{\mathrm{s}}/2$)*, for any* $(t, x, y) \in (0, 1] \times K \times K$,

$$\frac{c_{5.1}}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}} \exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{c_{5.1}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr) \leq p_t(x,y) \leq \frac{c_{5.2}}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}} \exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{c_{5.2}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr). \tag{5.4}$$

Remark 5.10. *The strict inequality* $d_{\mathrm{w}} > 2$ *holds if* $\#S < l^d$. In the case of $d = 2$, this estimate follows from [5, Proof of Proposition 5.2] (see also [6, (2.5)]), whereas for $d \geq 3$ this fact is only stated in [7, Remarks 5.4-1.] without proof.

Now we are in the stage of stating our main theorems of asymptotic expansion of the partition function on generalized Sierpiński carpets. Recall that $\mathcal{L} = \mathrm{GSC}(d, l, S) = (K, S, \{F_i\}_{i \in S})$ is a generalized Sierpiński carpet, that μ is the self-similar measure on $\mathcal{L}$ with uniform weight and that $(\mathcal{E}, \mathcal{F})$ is the self-similar Dirichlet form on $L^2(K, \mu)$ with uniform weight r given by Theorem 5.7 and Proposition 5.8.

Theorem 5.11. *Set* $d_k := \log_l \#\bigl(S \cap (\mathbb{Z}^{d-k} \times \{0\}^k)\bigr)$ *for* $k \in \{0, 1, \ldots, d\}$ $(d_0 = d_{\mathrm{f}})$. *Then there exist* $c_{5.3} \in (0, \infty)$ *and continuous* $\log\tau$*-periodic functions* $G_{\mathrm{B},k} : \mathbb{R} \to \mathbb{R}$, $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, $k \in \{0, \ldots, d\}$*, such that for any* $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$*, as* $t \downarrow 0$,

$$\mathcal{Z}_{\mathrm{B}}(t) = \sum_{k=0}^{d} t^{-d_k/d_{\mathrm{w}}} G_{\mathrm{B},k}(-\log t) + O\Bigl(\exp\bigl(-c_{5.3}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \tag{5.5}$$

Moreover, $G_{\mathrm{N},0}, G_{\mathrm{N},1}$ *are* $(0, \infty)$*-valued,* $G_{\mathrm{N},0} = G_{\mathrm{D},0}$ *and* $n_{\mathrm{N}}^{-1} G_{\mathrm{N},1} = n_{\mathrm{D}}^{-1} G_{\mathrm{D},1}$*, where* $n_{\mathrm{N}} := \frac{\#\{(i,j) \in S \times S \mid |i-j|=1\}}{2\#(S \setminus (\mathbb{Z}^{d-1} \times \{0\}))}$ *and* $n_{\mathrm{D}} := n_{\mathrm{N}} - 2d$ (note that $n_{\mathrm{N}} > 0 > n_{\mathrm{D}}$).

Note that $d_{d-1} = 1$ and $d_d = 0$ by (GSC4) and that for each $k \in \{0, 1, \dots, d\}$, d_k is the box-counting and Hausdorff dimensions of $K \cap (\mathbb{R}^{d-k} \times \{0\}^k)$ with respect to ρ by [35, Propositions 2.24 and 6.7] and [40, Corollary 1.5.9] in view of $K \cap (\mathbb{R}^{d-k} \times \{0\}^k) = K[S \cap (\mathbb{Z}^{d-k} \times \{0\}^k)]$ with the notation of Definition 4.1. Note also that $d_k > d_{k+1}$ for any $k \in \{0, \dots, d-1\}$ since $\bigl((l-1)\mathbf{1}_{\{1,\dots,d-k\}}(j)\bigr)_{j=1}^d \in S \cap (\mathbb{Z}^{d-k} \times \{0\}^k) \setminus (\mathbb{Z}^{d-k-1} \times \{0\}^{k+1})$ by (GSC4) and (GSC1).

Remark 5.12. The same remarks as in the introduction apply to Theorem 5.11. Namely: (1) $G_{\mathrm{D},0}$ in Theorem 5.11 was obtained in [30, Theorem 4.1] where it was proved that

$$\mathcal{Z}_{\mathrm{D}}(t) = t^{-d_{\mathrm{f}}/d_{\mathrm{w}}} G_{\mathrm{D},0}(-\log t) + O(t^{-d_1/d_{\mathrm{w}}}) \qquad \text{as } t \downarrow 0. \tag{5.6}$$

The same result for $\mathcal{Z}_{\mathrm{N}}$ can be easily verified since a slight modification of [30, Proof of Theorem 4.1] shows that $\mathcal{Z}_{\mathrm{N}}(t) - \mathcal{Z}_{\mathrm{D}}(t) = O(t^{-d_1/d_{\mathrm{w}}})$ as $t \downarrow 0$ (see also [35, Theorem 5.11]). Unfortunately, however, *the corresponding log-periodic asymptotic behavior of* $\mathcal{N}_{\mathrm{N}}$ *and* $\mathcal{N}_{\mathrm{D}}$ *similar to* (3.8) *is not known when* $\#S < l^d$.
(2) *The author has no idea whether any one of the periodic functions* $G_{\mathrm{B},k}$, $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, $k \in \{0, \dots, d\}$ *in Theorem* 5.11 *is non-constant when* $\#S < l^d$.
(3) The strict positivity of $G_{\mathrm{N},1}$ (the strict negativity of $G_{\mathrm{D},1}$) in Theorem 5.11 follows from [35, Theorem 7.7] (see [35, Theorems 8.4 and 8.5]), but we give an alternative proof of this fact in the next section as a special case of Theorem 5.15 below. On the other hand, $G_{\mathrm{B},k}$ for $k \geq 2$ in (5.5) are given as sums of strictly positive and strictly negative $\log \tau$-periodic functions and *the author has no idea which signs* $G_{\mathrm{B},k}$ *actually takes for* $k \in \{2, \dots, d\}$; see also Theorem 5.15 and its proof below in this connection.

Remark 5.13. It should be possible to prove asymptotics analogous to (4.1) of Theorem 4.2 for generalized Sierpiński carpets, but the statement and the proof would get much more complicated than in Section 4 because of the possible complexity of the boundary set $K[X]$ which does not arise in the setting of a post-critically finite self-similar fractal. Since the simplest case of the Neumann and Dirichlet boundary conditions on V_0 treated in Theorem 5.11 is already quite involved, we content ourselves with just this case.

In fact, the assertion of Theorem 5.11 for $\mathcal{Z}_{\mathrm{D}}$ is a special case of the following more general theorem, which requires the following definition.

Definition 5.14. Let $\varepsilon = (\varepsilon_k)_{k=1}^d \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$ and $i = (i_k)_{k=1}^d \in \mathbb{R}^d$.
(1) We set $|\varepsilon| := \sum_{k=1}^d \varepsilon_k$, $\varepsilon^{-1}(j) := \{k \in \{1, \dots, d\} \mid \varepsilon_k = j\}$ for $j \in \{0, 1\}$, $S_\varepsilon := S \cap (\mathbb{Z}^{\varepsilon^{-1}(0)} \times \{0\}^{\varepsilon^{-1}(1)})$, $\mathcal{I}_m^{i,\varepsilon} := \{(k, i_k + j l^{-m-1}) \mid k \in \varepsilon^{-1}(0), j \in \{1, \dots, l-1\}\}$, $\mathcal{I}_i(J) := \{(k, i_k) \mid k \in J\}$ for $J \subset \{1, \dots, d\}$, and $\mathcal{I}_{i,\varepsilon} := \mathcal{I}_i(\varepsilon^{-1}(1))$.
(2) We define $R_m^{i,\varepsilon} := \prod_{k=1}^d [i_k - \varepsilon_k l^{-m}, i_k + l^{-m}]$, $U_m^{i,\varepsilon} := K \cap \mathrm{int}_{\mathbb{R}^d} R_m^{i,\varepsilon}$, $W_m^{i,\varepsilon} := \{w \in W_m \mid K_w \subset R_m^{i,\varepsilon}\}$, $\hat{R}_m^{i,\varepsilon} := \mathbb{R}^{\varepsilon^{-1}(0)} \times \prod_{k \in \varepsilon^{-1}(1)} [i_k - l^{-m-1}, i_k + l^{-m-1}]$ and $\hat{U}_m^{i,\varepsilon} := K \cap \mathrm{int}_{\mathbb{R}^d}(R_m^{i,\varepsilon} \cap \hat{R}_m^{i,\varepsilon})$. Note that $U_m^{i,\varepsilon} \neq \emptyset$ if and only if $W_m^{i,\varepsilon} \neq \emptyset$.
(3) We set $H_{k,s} := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_k = s\}$ for $k \in \{1, \dots, d\}$ and $s \in \mathbb{R}$ and $H_{\mathcal{I}} := \bigcup_{(k,s) \in \mathcal{I}} H_{k,s}$ for $\mathcal{I} \subset \{1, \dots, d\} \times \mathbb{R}$. Recalling (2.9), for $t \in (0, \infty)$ we define

$$\mathcal{Z}_m^{i,\varepsilon}(t) := \sum_{\mathcal{I} \subset \mathcal{I}_{i,\varepsilon}} (-1)^{\#\mathcal{I}} \mathcal{Z}_{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}}(t) = \int_{U_m^{i,\varepsilon}} p_t^{U_m^{i,\varepsilon}}(x, x | \{K \setminus H_{k,s}\}_{(k,s) \in \mathcal{I}_{i,\varepsilon}}) d\mu(x). \tag{5.7}$$

Theorem 5.15. *Let $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and assume $W_m^{i,\varepsilon} \neq \emptyset$. Then there exist $c_{5.4}, c_{5.5} \in (0,\infty)$ which are independent of ε, m, i, and continuous* $\log\tau$*-periodic functions $G_m^{i,\varepsilon,k} : \mathbb{R} \to \mathbb{R}$, $k \in \{|\varepsilon|, \ldots, d\}$, such that for any $t \in (0,\tau]$,*

$$\left| \mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}t) - \sum_{k=|\varepsilon|}^{d} t^{-d_k/d_{\mathrm{w}}} G_m^{i,\varepsilon,k}(-\log t) \right| \leq c_{5.4} \exp\bigl(-c_{5.5} t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr). \tag{5.8}$$

Moreover, $G_m^{i,\varepsilon,|\varepsilon|}$ is $(0,\infty)$-valued, and $G_m^{i,\varepsilon,|\varepsilon|+1}$ is $(-\infty,0)$-valued if $|\varepsilon| < d$.

Remark 5.16. (1) Similarly to Remark 5.12-(2), *the author has no idea whether any one of the periodic functions $G_m^{i,\varepsilon,k}$ for $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ with $W_m^{i,\varepsilon} \neq \emptyset$ and $k \in \{|\varepsilon|, \ldots, d\}$ in Theorem* 5.15 *is non-constant when $\#S < l^d$.*
(2) Similarly to Remark 5.12-(3), $G_m^{i,\varepsilon,k}$ for $k \geq |\varepsilon| + 2$ in Theorem 5.15 are given as sums of strictly positive and strictly negative $\log\tau$-periodic functions and *the author has no idea which signs $G_m^{i,\varepsilon,k}$ actually takes for $k \geq |\varepsilon| + 2$.*
(3) According to Lemma 6.2 and Proposition 6.3 below, for each $\varepsilon \in \{0,1\}^d$ there exists $\Xi_\varepsilon \subset \bigcup_{m\in\mathbb{N}\cup\{0\}}(l^{-m}\mathbb{Z}^d \times \{m\})$ with $\#\Xi_\varepsilon \leq 2^{2^{|\varepsilon|}}$ such that for any $m \in \mathbb{N} \cup \{0\}$ and any $i \in l^{-m}\mathbb{Z}^d$ we can choose $(j,n) \in \Xi_\varepsilon$ so that $\mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}t) = \mathcal{Z}_n^{j,\varepsilon}(\tau^{-n}t)$ for any $t \in (0,\infty)$. In particular, the set of all continuous $\log\tau$-periodic functions appearing in Theorem 5.15 is given by $\bigl\{G_m^{i,\varepsilon,k} \bigm| \varepsilon \in \{0,1\}^d, (i,m) \in \Xi_\varepsilon, k \in \{|\varepsilon|, \ldots, d\}\bigr\}$, *which has at most $(d+1)2^{d+2^d}$ elements.*
(4) In fact, $\mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}t)$ *is independent of $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$ and $i \in l^{-m}\mathbb{Z}^d$ as long as $|\varepsilon| = 1$ and $W_m^{i,\varepsilon} \neq \emptyset$, and $G_{\mathrm{N},1} = n_{\mathrm{N}} G_m^{i,\varepsilon,1}$ and $G_{\mathrm{D},1} = n_{\mathrm{D}} G_m^{i,\varepsilon,1}$ for any such ε, m, i*; see Proposition 6.7 and (6.15) below.

The proof of Theorems 5.11 and 5.15 is given in the next section.

6. Proof of Theorems 5.11 and 5.15

Throughout this section, we fix the setting of Framework 5.1 and assume that $\mathcal{L} := \mathrm{GSC}(d,l,S) = (K, S, \{F_i\}_{i\in S})$ is a generalized Sierpiński carpet and that $(\mathcal{E},\mathcal{F})$ is the self-similar Dirichlet form on $L^2(K,\mu)$ with uniform weight r given by Theorem 5.7 and Proposition 5.8. As in the previous section, we set $\tau := \#S/r$, $d_{\mathrm{w}} := \log_l \tau$ and $d_k := \log_l \#\bigl(S \cap (\mathbb{Z}^{d-k} \times \{0\}^k)\bigr)$ for $k \in \{0,1,\ldots,d\}$ ($d_0 = d_{\mathrm{f}}$), and we also follow the notation introduced in Definitions 5.6 and 5.14.

Similarly to the proofs of Theorems 3.4 and 4.2, we need several intermediate steps to prove Theorems 5.11 and 5.15. We start with some discussion on the scaling property between open subsets of K of the form $U_m^{i,\varepsilon}$.

Definition 6.1. Let $\varepsilon \in \{0,1\}^d$, $m, n \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and $j \in l^{-n}\mathbb{Z}^d$. Define $F_{m,n}^{i,j} : \mathbb{R}^d \to \mathbb{R}^d$ by $F_{m,n}^{i,j}(x) := j + l^{m-n}(x-i)$, so that $F_{m,n}^{i,j}(R_m^{i,\varepsilon}) = R_n^{j,\varepsilon}$. We say that (i,m) is *ε-equivalent* to (j,n), and write $(i,m) \overset{\varepsilon}{\sim} (j,n)$, if and only if there exists a bijection $\varphi : W_m^{i,\varepsilon} \to W_n^{j,\varepsilon}$ such that $F_{m,n}^{i,j}|_{K_w} = F_{\varphi(w)} \circ F_w^{-1}$ for any $w \in W_m^{i,\varepsilon}$.

Clearly, $\overset{\varepsilon}{\sim}$ is an equivalence relation on $\bigcup_{m\in\mathbb{N}\cup\{0\}}(l^{-m}\mathbb{Z}^d \times \{m\})$.

Lemma 6.2. *Let $\varepsilon \in \{0,1\}^d$. Then there exists $\Xi_\varepsilon \subset \bigcup_{m\in\mathbb{N}\cup\{0\}}(l^{-m}\mathbb{Z}^d \times \{m\})$ with $\#\Xi_\varepsilon \leq 2^{2^{|\varepsilon|}}$ such that whenever $m \in \mathbb{N} \cup \{0\}$ and $i \in l^{-m}\mathbb{Z}^d$, $(i,m) \overset{\varepsilon}{\sim} (j,n)$ for some $(j,n) \in \Xi_\varepsilon$.*

Proof. For $m \in \mathbb{N} \cup \{0\}$ and $i = (i_k)_{k=1}^d \in l^{-m}\mathbb{Z}^d$, set

$$A_m^{i,\varepsilon} := \left\{ (\delta_k)_{k=1}^d \in \{0\}^{\varepsilon^{-1}(0)} \times \{0,1\}^{\varepsilon^{-1}(1)} \;\middle|\; \prod_{k=1}^d \left[i_k - \frac{\delta_k}{l^m}, i_k + \frac{1-\delta_k}{l^m} \right] \subset Q_m \right\}, \tag{6.1}$$

where $Q_m := \bigcup_{w_1 \ldots w_m \in W_m} f_{w_1} \circ \cdots \circ f_{w_m}(Q_0)$. Then it is easy to see that for $m, n \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and $j \in l^{-n}\mathbb{Z}^d$, $(i,m) \overset{\varepsilon}{\sim} (j,n)$ if and only if $A_m^{i,\varepsilon} = A_n^{j,\varepsilon}$. Thus the set $\bigl(\bigcup_{m \in \mathbb{N} \cup \{0\}} (l^{-m}\mathbb{Z}^d \times \{m\})\bigr) / \overset{\varepsilon}{\sim}$ has at most $2^{2^{|\varepsilon|}}$ elements and hence the assertion follows by choosing a representative from each of the equivalence classes under $\overset{\varepsilon}{\sim}$. □

Proposition 6.3. *Let $\varepsilon \in \{0,1\}^d$, $m, n \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and $j \in l^{-n}\mathbb{Z}^d$. If $(i,m) \overset{\varepsilon}{\sim} (j,n)$, then $\mathcal{Z}_U(\tau^{n-m}t) = \mathcal{Z}_{F_{m,n}^{i,j}(U)}(t)$ for any $t \in (0,\infty)$ and any non-empty open subset U of $U_m^{i,\varepsilon}$, and in particular $\mathcal{Z}_m^{i,\varepsilon}(\tau^{n-m}t) = \mathcal{Z}_n^{j,\varepsilon}(t)$ for any $t \in (0,\infty)$.*

Proof. Note that $F_{m,n}^{i,j}(H_{k,i_k}) = H_{k,j_k}$ for any $k \in \{1,\ldots,d\}$, where $i = (i_k)_{k=1}^d$ and $j = (j_k)_{k=1}^d$. By $(i,m) \overset{\varepsilon}{\sim} (j,n)$ there exists a bijection $\varphi : W_m^{i,\varepsilon} \to W_n^{j,\varepsilon}$ such that $F_{m,n}^{i,j}|_{K_w} = F_{\varphi(w)} \circ F_w^{-1}$ for any $w \in W_m^{i,\varepsilon}$, then $F_{m,n}^{i,j}\bigl(K(W_m^{i,\varepsilon})\bigr) = K\bigl(W_n^{j,\varepsilon}\bigr)$ (recall Definition 2.7-(2)) and hence $F_{m,n}^{i,j}(U_m^{i,\varepsilon}) = U_n^{j,\varepsilon}$. If $W_m^{i,\varepsilon} = \emptyset$, then $U_m^{i,\varepsilon} = U_n^{j,\varepsilon} = \emptyset$ and thus $\mathcal{Z}_m^{i,\varepsilon}(\tau^{n-m}t) = \mathcal{Z}_n^{j,\varepsilon}(t) = 0$ for any $t \in (0,\infty)$. Assume $W_m^{i,\varepsilon} \neq \emptyset$, let U be a non-empty open subset of $U_m^{i,\varepsilon}$ and set $(F_{m,n}^{i,j})_*^U u := u \circ (F_{m,n}^{i,j})^{-1}|_{F_{m,n}^{i,j}(U)}$ for $u : U \to [-\infty,\infty]$, so that $\int_U |u| d\mu = (\#S)^{n-m} \int_{F_{m,n}^{i,j}(U)} \bigl|(F_{m,n}^{i,j})_*^U u\bigr| d\mu$ if u is Borel measurable. Then $(F_{m,n}^{i,j})_*^U$ defines a bijective linear map $(F_{m,n}^{i,j})_*^U : L^2(U,\mu|_U) \to L^2\bigl(F_{m,n}^{i,j}(U), \mu|_{F_{m,n}^{i,j}(U)}\bigr)$. Moreover, regarding $\{u \in \mathcal{F} \cap C(K) \mid \operatorname{supp}_K[u] \subset V\} =: \mathcal{C}_V$ and $\mathcal{F}_V$ as subsets of $L^2(V,\mu|_V)$ for each non-empty open subset V of K, we have $(F_{m,n}^{i,j})_*^U(\mathcal{F}_U) = \mathcal{F}_{F_{m,n}^{i,j}(U)}$ and $\mathcal{E}(u,u) = r^{n-m}\mathcal{E}\bigl((F_{m,n}^{i,j})_*^U u, (F_{m,n}^{i,j})_*^U u\bigr)$ for any $u \in \mathcal{F}_U$, since the same are valid with $\mathcal{C}_U, \mathcal{C}_{F_{m,n}^{i,j}(U)}$ in place of $\mathcal{F}_U, \mathcal{F}_{F_{m,n}^{i,j}(U)}$ by (SSDF1), Proposition 2.8 and (SSDF2) (recall Proposition 5.8). Now it follows from the above facts and [23, Lemma 1.3.4-(i)] that $(F_{m,n}^{i,j})_*^U T_{\tau^{n-m}t}^U = T_t^{F_{m,n}^{i,j}(U)} (F_{m,n}^{i,j})_*^U$ for any $t \in (0,\infty)$, which together with the uniqueness of p^U implies that $p_{\tau^{n-m}t}^U(x,y) = (\#S)^{m-n} p_t^{F_{m,n}^{i,j}(U)}\bigl(F_{m,n}^{i,j}(x), F_{m,n}^{i,j}(y)\bigr)$ for any $(t,x,y) \in (0,\infty) \times U \times U$. Thus for any $t \in (0,\infty)$, $\mathcal{Z}_U(\tau^{n-m}t) = \mathcal{Z}_{F_{m,n}^{i,j}(U)}(t)$, which with $U = U_m^{i,\varepsilon} \setminus H_{\mathcal{I}_i(J)}$, $J \subset \varepsilon^{-1}(1)$, yields $\mathcal{Z}_m^{i,\varepsilon}(\tau^{n-m}t) = \mathcal{Z}_n^{j,\varepsilon}(t)$ by virtue of $F_{m,n}^{i,j}(H_{k,i_k}) = H_{k,j_k}$, $k \in \varepsilon^{-1}(1)$. □

Lemma 6.4. *Let $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$ and $i \in l^{-m}\mathbb{Z}^d$. Then for any $j \in S_\varepsilon$, $(i,m) \overset{\varepsilon}{\sim} (i + l^{-m-1}j, m+1)$. Moreover,*

$$\hat{U}_m^{i,\varepsilon} \setminus H_{\mathcal{I}_m^{i,\varepsilon}} = \bigcup_{j \in S_\varepsilon} U_{m+1}^{i+l^{-m-1}j,\varepsilon} \qquad \textit{(disjoint)}. \tag{6.2}$$

Proof. These assertions are immediate from Definition 5.14 and (GSC1). □

Recall (see Definition 2.18) that λ_1^U denotes the smallest eigenvalue of $-\Delta_U$ for a non-empty open subset U of K.

Lemma 6.5. *Let $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and assume $W_m^{i,\varepsilon} \neq \emptyset$. Then $\lambda_1^{U_m^{i,\varepsilon}} > 0$, and $0 \leq \mathcal{Z}_m^{i,\varepsilon}(t) \leq \mathcal{Z}_{U_m^{i,\varepsilon}}(t) \leq 2^d c_{5.2}$ for any $t \in [\tau^{-m}, \infty)$. Moreover, there exists $T_m^{i,\varepsilon} \in [1,\infty)$ such that $\mathcal{Z}_m^{i,\varepsilon}(t) > 0$ for any $t \in [T_m^{i,\varepsilon}, \infty)$.*

Proof. Since $0_{\mathbb{R}^d} \in S_\varepsilon$ by (GSC4), we have $(i,m) \overset{\varepsilon}{\sim} (i, m+2)$ by Lemma 6.4, then $\mathcal{Z}_{U_m^{i,\varepsilon}}(\tau^2 t) = \mathcal{Z}_{U_{m+2}^{i,\varepsilon}}(t)$ for $t \in (0,\infty)$ by Proposition 6.3 and hence $\tau^2 \lambda_1^{U_m^{i,\varepsilon}} = \lambda_1^{U_{m+2}^{i,\varepsilon}}$ by Lemma 2.20. On the other hand, Theorem 5.9 easily shows that $(\mathcal{E},\mathcal{F})$ is irreducible, so that $\{u \in \mathcal{F} \mid \mathcal{E}(u,u) = 0\} = \mathbb{R}\mathbf{1}_K$ by [16, Theorem 2.1.11], and $\mathbf{1}_K \notin \mathcal{F}_{U_{m+2}^{i,\varepsilon}}$ since $\mu(K \setminus U_{m+2}^{i,\varepsilon}) > 0$ by (GSC4). Thus $\lambda_1^{U_{m+2}^{i,\varepsilon}} > 0$ and $\lambda_1^{U_m^{i,\varepsilon}} = \tau^{-2}\lambda_1^{U_{m+2}^{i,\varepsilon}} > 0$.

Next let $t \in [\tau^{-m}, \infty)$. Since $p_{(\cdot)}(x,x)$ is non-increasing for any $x \in K$ by [17, (2.1.4)], it follows from Proposition 2.16-(1), Theorem 5.9 and $\#W_m^{i,\varepsilon} \leq 2^d$ that

$$\begin{aligned} 0 \leq \mathcal{Z}_m^{i,\varepsilon}(t) \leq \mathcal{Z}_{U_m^{i,\varepsilon}}(t) &\leq \int_{U_m^{i,\varepsilon}} p_t(x,x)\,d\mu(x) \leq \int_{U_m^{i,\varepsilon}} p_{\tau^{-m}}(x,x)\,d\mu(x) \\ &\leq c_{5.2}(\tau^{-m})^{-d_{\mathrm{f}}/d_{\mathrm{w}}}\mu(U_m^{i,\varepsilon}) = (\#W_m^{i,\varepsilon})c_{5.2} \leq 2^d c_{5.2}. \end{aligned}$$

For the last assertion, by $\mathcal{Z}_m^{i,\varepsilon} = \sum_{\mathcal{I} \subset \mathcal{I}_{i,\varepsilon}} (-1)^{\#\mathcal{I}} \mathcal{Z}_{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}}$ and Lemma 2.20 it suffices to show that $\lambda_1^{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}} > \lambda_1^{U_m^{i,\varepsilon}}$ for $\mathcal{I} \subset \mathcal{I}_{i,\varepsilon}$ with $\mathcal{I} \neq \emptyset$. Suppose $\lambda_1^{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}} \leq \lambda_1^{U_m^{i,\varepsilon}}$ and let $\varphi_1^{\mathcal{I}}$ be an eigenfunction of $-\Delta_{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}}$ with eigenvalue $\lambda_1^{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}}$. Then $\varphi_1^{\mathcal{I}} \in \mathcal{F}_{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}} \subset \mathcal{F}_{U_m^{i,\varepsilon}}$ and $\mathcal{E}(\varphi_1^{\mathcal{I}}, \varphi_1^{\mathcal{I}}) \leq \lambda_1^{U_m^{i,\varepsilon}} \int_{U_m^{i,\varepsilon}} (\varphi_1^{\mathcal{I}})^2 d\mu$, so that $\varphi_1^{\mathcal{I}}$ would be an eigenfunction of $-\Delta_{U_m^{i,\varepsilon}}$ with eigenvalue $\lambda_1^{U_m^{i,\varepsilon}}$ in view of (2.15). Furthermore from (GSC1), (GSC2) and (GSC4) we can easily verify that $U_m^{i,\varepsilon}$ is arcwise connected, and hence Lemma 2.19 would imply that there would exist a μ-version of $\varphi_1^{\mathcal{I}}$ satisfying $\varphi_1^{\mathcal{I}}|_{U_m^{i,\varepsilon}} \in C(U_m^{i,\varepsilon})$ and that then either $\varphi_1^{\mathcal{I}} > 0$ on $U_m^{i,\varepsilon}$ or $\varphi_1^{\mathcal{I}} < 0$ on $U_m^{i,\varepsilon}$. On the other hand, let $\mathrm{Cap}_{\mathcal{E}}$ denote the 1-capacity associated with $(K,\mu,\mathcal{E},\mathcal{F})$ defined by $\mathrm{Cap}_{\mathcal{E}}(A) := \inf_{U \subset K \text{ open in } K,\, A \subset U} \inf_{u \in \mathcal{F},\, u \geq 1\ \mu\text{-a.e. on } U} \mathcal{E}_1(u,u)$ for each $A \subset K$ (see [23, Section 2.1]). Then $\mathrm{Cap}_{\mathcal{E}}(U_m^{i,\varepsilon} \cap H_{\mathcal{I}}) > 0$ by [35, Lemma 7.14 and Theorem 7.18], but $\mathrm{Cap}_{\mathcal{E}}(U_m^{i,\varepsilon} \cap H_{\mathcal{I}}) = \mathrm{Cap}_{\mathcal{E}}\bigl(\{x \in U_m^{i,\varepsilon} \cap H_{\mathcal{I}} \mid \varphi_1^{\mathcal{I}}(x) \neq 0\}\bigr) = 0$ by $\varphi_1^{\mathcal{I}} \neq 0$ on $U_m^{i,\varepsilon}$, $\varphi_1^{\mathcal{I}} \in \mathcal{F}_{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}}$ and [23, Corollary 2.3.1], a contradiction. Thus $\lambda_1^{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}} > \lambda_1^{U_m^{i,\varepsilon}}$. □

The first key step for the proof of Theorems 5.11 and 5.15 is the following proposition, which together with Lemma 6.5 allows us to apply Theorem 2.23 successively to conclude (5.8) and then to verify the strict positivity of $G_m^{i,\varepsilon,|\varepsilon|}$ and $-G_m^{i,\varepsilon,|\varepsilon|+1}$.

Proposition 6.6. *Let $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$ and $i \in l^{-m}\mathbb{Z}^d$. Then there exist $c_{6.1}, c_{6.2} \in (0,\infty)$ which are independent of ε, m, i, such that for any $t \in (0,\infty)$,*

$$\begin{aligned} 0 &\leq \mathcal{Z}_m^{i,\varepsilon}(t) - \tau^{d_{|\varepsilon|}/d_{\mathrm{w}}} \mathcal{Z}_m^{i,\varepsilon}(\tau t) - \sum_{\delta \in \{0,1\}^d,\, \varepsilon^{-1}(1) \subsetneqq \delta^{-1}(1)} \sum_{j \in J_{\varepsilon,\delta}} \mathcal{Z}_{m+1}^{i+l^{-m-1}j,\delta}(t) \\ &\leq c_{6.1} \exp\bigl(-c_{6.2}(\tau^m t)^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr), \end{aligned} \tag{6.3}$$

where $J_{\varepsilon,\delta} := \{0,\ldots,l-1\}^{\delta^{-1}(0)} \times \{1,\ldots,l-1\}^{\varepsilon^{-1}(0) \cap \delta^{-1}(1)} \times \{0\}^{\varepsilon^{-1}(1)}$.

Proof. Set $\mathcal{I}_0 := \mathcal{I}_m^{i,\varepsilon}$ and let $t \in (0,\infty)$. Noting that $\tau^{d_{|\varepsilon|}/d_{\mathrm{w}}} = \#S_\varepsilon$ by (GSC1), we see from Proposition 6.3, Lemmas 6.4 and 2.14 that

$$\mathcal{Z}_m^{i,\varepsilon}(t) - \tau^{d_{|\varepsilon|}/d_{\mathrm{w}}}\mathcal{Z}_m^{i,\varepsilon}(\tau t) = \mathcal{Z}_m^{i,\varepsilon}(t) - \sum_{j\in S_\varepsilon} \mathcal{Z}_{m+1}^{i+l^{-m-1}j,\varepsilon}(t) \tag{6.4}$$
$$= \mathcal{Z}_m^{i,\varepsilon}(t) - \sum_{j\in S_\varepsilon}\int_K p_t^{U_{m+1}^{i+l^{-m-1}j,\varepsilon}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})d\mu(x)$$
$$= \mathcal{Z}_m^{i,\varepsilon}(t) - \int_K p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})d\mu(x)$$
$$= \int_{U_m^{i,\varepsilon}}\Bigl(p_t^{U_m^{i,\varepsilon}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}_m^{i,\varepsilon}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})$$
$$+ p_t^{\hat{U}_m^{i,\varepsilon}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})\Bigr)d\mu(x).$$

Let $x \in K$. Since $\hat{U}_m^{i,\varepsilon} = U_m^{i,\varepsilon}\cap \mathrm{int}_{\mathbb{R}^d}\ \hat{R}_m^{i,\varepsilon}$, Proposition 2.16 and Theorem 5.9 yield

$$p_t^{U_m^{i,\varepsilon}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}_m^{i,\varepsilon}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) \tag{6.5}$$
$$= p_t^{U_m^{i,\varepsilon}}(x,x|\{K\cap \mathrm{int}_{\mathbb{R}^d}\ \hat{R}_m^{i,\varepsilon}\}\cup\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})\ (\geq 0)$$
$$\leq \frac{2c_{5.2}}{(t/2)^{d_{\mathrm{f}}/d_{\mathrm{w}}}}\exp\biggl(-\Bigl(\frac{\rho(x,K\setminus \mathrm{int}_{\mathbb{R}^d}\ \hat{R}_m^{i,\varepsilon})\vee \max_{(k,s)\in\mathcal{I}_{i,\varepsilon}}\rho(x,K\cap H_{k,s})}{(c_{5.2}t)^{1/d_{\mathrm{w}}}}\Bigr)^{\frac{d_{\mathrm{w}}}{d_{\mathrm{w}-1}}}\biggr)$$
$$\leq 2^{1+d_{\mathrm{f}}/d_{\mathrm{w}}}c_{5.2}t^{-d_{\mathrm{f}}/d_{\mathrm{w}}}\exp\bigl(-(2^{d_{\mathrm{w}}}c_{5.2}\tau^{m+1}t)^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$$

where we have the inequality in the third line of (6.5) only for $t \in (0,1]$.

On the other hand, setting $J_\delta^0 := \{0,\ldots,l-1\}^{\delta^{-1}(0)}\times\{0\}^{\delta^{-1}(1)}$ and $J_{\varepsilon,\delta}^1 := \{0\}^{\delta^{-1}(0)}\times\{1,\ldots,l-1\}^{\varepsilon^{-1}(0)\cap\delta^{-1}(1)}\times\{0\}^{\varepsilon^{-1}(1)}$ for $\delta\in\{0,1\}^d$ with $\varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)$, by using Lemmas 2.22 and 2.14 we get

$$p_t^{\hat{U}_m^{i,\varepsilon}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) \tag{6.6}$$
$$= \sum_{J\subset\varepsilon^{-1}(1)}(-1)^{\#J}\Bigl(p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_i(J)}}(x,x) - p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0\cup\mathcal{I}_i(J)}}(x,x)\Bigr)$$
$$= \sum_{J\subset\varepsilon^{-1}(1)}(-1)^{\#J}\sum_{\emptyset\neq\mathcal{I}\subset\mathcal{I}_0}\sum_{A\subset\mathcal{I}}(-1)^{\#A}p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{(\mathcal{I}_0\setminus\mathcal{I})\cup A\cup\mathcal{I}_i(J)}}(x,x)$$
$$= \sum_{\emptyset\neq\mathcal{I}\subset\mathcal{I}_0}\sum_{A\subset\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}}(-1)^{\#A}p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{(\mathcal{I}_0\setminus\mathcal{I})\cup A}}(x,x)$$
$$= \sum_{\emptyset\neq\mathcal{I}\subset\mathcal{I}_0}p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}})$$
$$= \sum\nolimits_{\mathcal{I}\subset\mathcal{I}_0,\,\#(\mathcal{I}\cap(\{k\}\times\mathbb{R}))\geq 2\text{ for some }k\in\varepsilon^{-1}(0)}p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}})$$
$$+ \sum\nolimits_{\substack{\delta\in\{0,1\}^d\\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}}\sum\nolimits_{j\in l^{-m-1}J_{\varepsilon,\delta}^1}p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0\setminus\mathcal{I}_{i+j,\delta}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+j,\delta}})$$

$$= \sum_{\mathcal{I}\subset\mathcal{I}_0,\,\#(\mathcal{I}\cap(\{k\}\times\mathbb{R}))\geq 2 \text{ for some } k\in\varepsilon^{-1}(0)} p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}})$$
$$+ \sum_{\substack{\delta\in\{0,1\}^d \\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}} \sum_{j\in l^{-m-1}J^1_{\varepsilon,\delta}} \biggl(p_t^{(\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0\setminus\mathcal{I}_{i+j,\delta}})\setminus\hat{R}_m^{i+j,\delta}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+j,\delta}})$$
$$+ \sum_{j_0\in l^{-m-1}J^0_\delta} p_t^{U_{m+1}^{i+j+j_0,\delta}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+j+j_0,\delta}})\biggr).$$

Then from (6.6), Proposition 2.16-(2) and Theorem 5.9 we conclude that

$$p_t^{\hat{U}_m^{i,\varepsilon}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) \tag{6.7}$$
$$-\sum_{\substack{\delta\in\{0,1\}^d \\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}} \sum_{j\in J_{\varepsilon,\delta}} p_t^{U_{m+1}^{i+l^{-m-1}j,\delta}}\bigl(x,x\bigl|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+l^{-m-1}j,\delta}}\bigr)$$
$$= \sum_{\mathcal{I}\subset\mathcal{I}_0,\,\#(\mathcal{I}\cap(\{k\}\times\mathbb{R}))\geq 2 \text{ for some } k\in\varepsilon^{-1}(0)} p_t^{\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}})$$
$$+ \sum_{\substack{\delta\in\{0,1\}^d \\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}} \sum_{j\in l^{-m-1}J^1_{\varepsilon,\delta}} p_t^{(\hat{U}_m^{i,\varepsilon}\setminus H_{\mathcal{I}_0\setminus\mathcal{I}_{i+j,\delta}})\setminus\hat{R}_m^{i+j,\delta}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+j,\delta}})$$
$$\leq 2\sum_{\mathcal{I}\subset\mathcal{I}_0,\,\#(\mathcal{I}\cap(\{k\}\times\mathbb{R}))\geq 2 \text{ for some } k\in\varepsilon^{-1}(0)} \min_{(k,u)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}} \Bigl\{ \sup_{s\in[t/2,t]} \sup_{z\in K\cap H_{k,u}} p_s(x,z)\Bigr\}$$
$$+ 2 \sum_{\substack{\delta\in\{0,1\}^d \\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}} \sum_{j\in l^{-m-1}J^1_{\varepsilon,\delta}} \mathbf{1}_{K\setminus\hat{R}_m^{i+j,\delta}}(x) \min_{(k,u)\in\mathcal{I}_{i+j,\delta}} \Bigl\{ \sup_{s\in[t/2,t]} \sup_{z\in K\cap H_{k,u}} p_s(x,z)\Bigr\}$$
$$\leq 2^{d(l-1)+1+d_{\mathrm{f}}/d_{\mathrm{w}}} c_{5.2} t^{-d_{\mathrm{f}}/d_{\mathrm{w}}} \exp\bigl(-(2^{d_{\mathrm{w}}} c_{5.2}\tau^{m+1}t)^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$$

where the inequality in the last line of (6.7) is valid only for $t \in (0,1]$.

Now (6.4), (6.5), (6.7) and Proposition 2.16-(1) immediately show the lower inequality in (6.3) and also the upper inequality in (6.3) with $c_{6.2} := 4^{-1}(2c_{5.2}\tau)^{-\frac{1}{d_{\mathrm{w}}-1}}$ for $t \in (0,1]$ by virtue of $\mu(U_m^{i,\varepsilon}) = (\#S)^{-m}\#W_m^{i,\varepsilon} \leq 2^d\tau^{-md_{\mathrm{f}}/d_{\mathrm{w}}}$. Finally, the second assertion of Lemma 6.5 yields the upper inequality in (6.3) for $t \in [1,\infty)$. □

Proof of Theorem 5.15. The proof is by induction in $d - |\varepsilon|$. Suppose first $|\varepsilon| = d$. Then $0 \leq \mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}t) - \mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}\tau t) \leq c_{6.1}\exp\bigl(-c_{6.2}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$ for any $t \in (0,\infty)$ by Proposition 6.6, which together with Lemmas 6.5 and 2.20 implies that Theorem 2.23 applies to $\mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}(\cdot))$ with $\alpha_0 = 0$, $\gamma = \tau$, $n = 1$, $\alpha_1 = -1$, $G_1 = 0$ and $\mathcal{R}(t) := c_{6.1}\exp\bigl(-c_{6.2}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$. Therefore there exists a continuous $\log\tau$-periodic function $G_m^{i,\varepsilon,d} : \mathbb{R} \to \mathbb{R}$ such that $\bigl|\mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}t) - G_m^{i,\varepsilon,d}(-\log t)\bigr| \leq \sum_{j\in\mathbb{N}}\mathcal{R}(\tau^{-j}t) \leq c_{6.1}c_{6.3}\exp\bigl(-c_{6.2}(\tau^{-1}t)^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$ for any $t \in (0,\tau]$, where $c_{6.3} \in (1,\infty)$ is explicit in terms of $\tau, d_{\mathrm{w}}, c_{6.2}$, proving (5.8) with $c_{5.4} = c_d := c_{6.1}c_{6.3}$ and $c_{5.5} := c_{6.2}\tau^{\frac{1}{d_{\mathrm{w}}-1}}$.

Next assume $|\varepsilon| < d$ and suppose that (5.8) for $t \in (0,\tau]$ holds with δ, n, j in place of ε, m, i and with $c_{5.4} = c_{|\varepsilon|+1}$ and $c_{5.5} = c_{6.2}\tau^{\frac{1}{d_{\mathrm{w}}-1}}$ for some $c_{|\varepsilon|+1} \in (0,\infty)$ whenever $\delta \in \{0,1\}^d$, $n \in \mathbb{N}\cup\{0\}$ and $j \in l^{-n}\mathbb{Z}^d$ satisfy $|\delta| \geq |\varepsilon| + 1$ and $W_n^{j,\delta} \neq \emptyset$.

For $\delta \in \{0,1\}^d$, $n \in \mathbb{N} \cup \{0\}$ and $j \in l^{-n}\mathbb{Z}^d$ with $|\delta| \geq |\varepsilon| + 1$ and $W_n^{j,\delta} = \emptyset$, we set $G_n^{j,\delta,k} := 0$, $k \in \{|\delta|, \ldots, d\}$, so that (5.8) trivially holds. Then Proposition 6.6 implies that (2.20) holds for $\mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}(\cdot))$ with $\alpha_0 = d_{|\varepsilon|}/d_{\mathrm{w}}$, $\gamma = \tau$, $n = d - |\varepsilon|$, $\alpha_{k-|\varepsilon|} = d_k/d_{\mathrm{w}}$ and

$$G_{k-|\varepsilon|} := \tau^{-d_k/d_{\mathrm{w}}} \sum\nolimits_{\delta \in \{0,1\}^d,\, \varepsilon^{-1}(1) \subsetneqq \delta^{-1}(1),\, |\delta| \leq k} \sum\nolimits_{j \in J_{\varepsilon,\delta}} G_{m+1}^{i+l^{-m-1}j,\delta,k} \tag{6.8}$$

for $k \in \{|\varepsilon| + 1, \ldots, d\}$, and $\mathcal{R}(t) := (c_{6.1} + (2l)^d c_{|\varepsilon|+1}) \exp\bigl(-c_{6.2} t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$. Now from this fact and Theorem 2.23 together with Lemmas 6.5 and 2.20, we conclude that there exists a continuous $\log\tau$-periodic function $G_m^{i,\varepsilon,|\varepsilon|} : \mathbb{R} \to \mathbb{R}$ such that (5.8) for $t \in (0,\tau]$ holds with $G_m^{i,\varepsilon,k} := -\bigl(\tau^{(d_{|\varepsilon|}-d_k)/d_{\mathrm{w}}} - 1\bigr)^{-1} G_{k-|\varepsilon|}$ for $k \in \{|\varepsilon| + 1, \ldots, d\}$, $c_{5.4} = c_{|\varepsilon|} := c_{6.3}(c_{6.1} + (2l)^d c_{|\varepsilon|+1})$ and $c_{5.5} = c_{6.2}\tau^{\frac{1}{d_{\mathrm{w}}-1}}$, where $G_{k-|\varepsilon|}$ is as in (6.8). Thus the induction procedure in $d - |\varepsilon|$ is completed and (5.8) is proved.

It remains to prove the strict positivity of $G_m^{i,\varepsilon,|\varepsilon|}$ and $-G_m^{i,\varepsilon,|\varepsilon|+1}$. By Proposition 6.6 and Proposition 2.16-(1), $t^{d_{|\varepsilon|}/d_{\mathrm{w}}} \mathcal{Z}_m^{i,\varepsilon}(t) \geq (\tau t)^{d_{|\varepsilon|}/d_{\mathrm{w}}} \mathcal{Z}_m^{i,\varepsilon}(\tau t)$ for any $t \in (0,\infty)$, and hence

$$\inf_{t \in (0,\tau T_m^{i,\varepsilon}]} t^{d_{|\varepsilon|}/d_{\mathrm{w}}} \mathcal{Z}_m^{i,\varepsilon}(t) = \min_{t \in [T_m^{i,\varepsilon}, \tau T_m^{i,\varepsilon}]} t^{d_{|\varepsilon|}/d_{\mathrm{w}}} \mathcal{Z}_m^{i,\varepsilon}(t) > 0, \tag{6.9}$$

where $T_m^{i,\varepsilon} \in [1,\infty)$ is as in Lemma 6.5. Now (6.9) and (5.8) together imply that $G_m^{i,\varepsilon,|\varepsilon|}$ is $(0,\infty)$-valued. Moreover, suppose $|\varepsilon| < d$, choose $\delta \in \{0,1\}^d$ so that $\varepsilon^{-1}(1) \subsetneqq \delta^{-1}(1)$ and $|\delta| = |\varepsilon| + 1$, and let $j := \bigl(\mathbf{1}_{\varepsilon^{-1}(0) \cap \delta^{-1}(1)}(k)\bigr)_{k=1}^d$. Then $j \in J_{\varepsilon,\delta}$ with $J_{\varepsilon,\delta}$ as in Proposition 6.6, and we easily see from $W_m^{i,\varepsilon} \neq \emptyset$, (GSC1) and (GSC4) that $W_{m+1}^{i+l^{-m-1}j,\delta} \neq \emptyset$. Hence Proposition 6.6, Proposition 2.16-(1) and (6.9) together yield

$$\inf_{t \in (0,1]} t^{d_{|\varepsilon|+1}/d_{\mathrm{w}}} \bigl(\mathcal{Z}_m^{i,\varepsilon}(t) - \tau^{d_{|\varepsilon|}/d_{\mathrm{w}}} \mathcal{Z}_m^{i,\varepsilon}(\tau t)\bigr) \geq \inf_{t \in (0,1]} t^{d_{|\delta|}/d_{\mathrm{w}}} \mathcal{Z}_{m+1}^{i+l^{-m-1}j,\delta}(t) > 0,$$

which implies the strict negativity of $G_m^{i,\varepsilon,|\varepsilon|+1}$ since we have by (5.8), as $t \downarrow 0$,

$$t^{d_{|\varepsilon|+1}/d_{\mathrm{w}}} \bigl(\mathcal{Z}_m^{i,\varepsilon}(t) - \tau^{d_{|\varepsilon|}/d_{\mathrm{w}}} \mathcal{Z}_m^{i,\varepsilon}(\tau t)\bigr) = -c_{|\varepsilon|,m} G_m^{i,\varepsilon,|\varepsilon|+1}(-\log t) + o(1),$$

where $c_{|\varepsilon|,m} := \bigl(\tau^{(d_{|\varepsilon|}-d_{|\varepsilon|+1})/d_{\mathrm{w}}} - 1\bigr)\tau^{-m d_{|\varepsilon|+1}/d_{\mathrm{w}}}$. Thus Theorem 5.15 is proved. $\square$

Next we prove Theorem 5.11. (5.5) for $\mathcal{Z}_{\mathrm{D}}$ with strictly positive $G_{\mathrm{D},0}, -G_{\mathrm{D},1}$ is just a special case of Theorem 5.15 with $i = \varepsilon = 0_{\mathbb{R}^d}$ and $m = 0$. To prove (5.5) for $\mathcal{Z}_{\mathrm{N}}$, $G_{\mathrm{N},0} = G_{\mathrm{D},0}$ and $n_{\mathrm{N}}^{-1} G_{\mathrm{N},1} = n_{\mathrm{D}}^{-1} G_{\mathrm{D},1}$, we need the following two propositions.

Proposition 6.7. *Let $\delta, \varepsilon \in \{0,1\}^d$ satisfy $|\delta| = |\varepsilon| = 1$, let $j \in \{0,1\}$, $m \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and suppose $W_m^{i,\varepsilon} \neq \emptyset$. Then for any $t \in (0,\infty)$,*

$$\mathcal{Z}_{U_m^{i,\varepsilon}}(t) = \mathcal{Z}_{U_0^{j\delta,\delta}}(\tau^m t) + (\#W_m^{i,\varepsilon} - 1)\mathcal{Z}_{\mathrm{D}}(\tau^m t) \quad \text{and} \quad \mathcal{Z}_m^{i,\varepsilon}(t) = \mathcal{Z}_0^{j\delta,\delta}(\tau^m t). \tag{6.10}$$

Proof. Note that $\#W_m^{i,\varepsilon} \in \{1,2\}$. Since $\mathcal{Z}_m^{i,\varepsilon}(t) = \mathcal{Z}_{U_m^{i,\varepsilon}}(t) - (\#W_m^{i,\varepsilon})\mathcal{Z}_{\mathrm{D}}(\tau^m t)$ for any $t \in (0,\infty)$ by Lemmas 2.14 and 2.21, the two equalities in (6.10) are equivalent, so that it suffices to prove the former equality. We set $U^{j,\delta} := U_0^{j\delta,\delta}$.

We first assume $\#W_m^{i,\varepsilon} = 1$ with $W_m^{i,\varepsilon} = \{w\}$. Then $U_m^{i,\varepsilon} \subset K_w$, and we can choose $g_w \in \mathcal{G}_0$ (recall (5.1)) so that $U_m^{i,\varepsilon} = F_w(g_w(U^{j,\delta}))$. Set $(F_w \circ g_w)_* u := u \circ g_w^{-1} \circ F_w^{-1}|_{U_m^{i,\varepsilon}}$ for $u : U^{j,\delta} \to [-\infty, \infty]$, so that $(\#S)^m \int_{U_m^{i,\varepsilon}} |(F_w \circ g_w)_* u| d\mu = \int_{U^{j,\delta}} |u| d\mu$ if u is Borel measurable. Then $(F_w \circ g_w)_*$ defines a bijective linear map $(F_w \circ g_w)_* : L^2(U^{j,\delta}, \mu|_{U^{j,\delta}}) \to L^2\big(U_m^{i,\varepsilon}, \mu|_{U_m^{i,\varepsilon}}\big)$, and it follows from (SSDF1), (SSDF2) and (GSCDF) (recall Proposition 5.8) that $(F_w \circ g_w)_*(\mathcal{F}_{U^{j,\delta}}) = \mathcal{F}_{U_m^{i,\varepsilon}}$ and that $r^m \mathcal{E}\big((F_w \circ g_w)_* u, (F_w \circ g_w)_* u\big) = \mathcal{E}(u, u)$ for any $u \in \mathcal{F}_{U^{j,\delta}}$, similarly to the proof of Proposition 6.3. Now these facts and [23, Lemma 1.3.4-(i)] yield $(F_w \circ g_w)_* T_{\tau^m t}^{U^{j,\delta}} = T_t^{U_m^{i,\varepsilon}} (F_w \circ g_w)_*$ for $t \in (0, \infty)$, which together with the uniqueness of $p^{U^{j,\delta}}$ implies that $p_{\tau^m t}^{U^{j,\delta}}(x, y) = (\#S)^{-m} p_t^{U_m^{i,\varepsilon}}\big(F_w \circ g_w(x), F_w \circ g_w(y)\big)$ for $(t, x, y) \in (0, \infty) \times U^{j,\delta} \times U^{j,\delta}$ and hence that $\mathcal{Z}_{U^{j,\delta}}(\tau^m t) = \mathcal{Z}_{U_m^{i,\varepsilon}}(t)$ for $t \in (0, \infty)$.

Next we assume $\#W_m^{i,\varepsilon} = 2$ with $W_m^{i,\varepsilon} = \{w, v\}$. Then $U_m^{i,\varepsilon} \subset K_w \cup K_v$, and we can choose $g_w, g_v \in \mathcal{G}_0$ so that $K_w \cap U_m^{i,\varepsilon} = F_w(g_w(U^{j,\delta}))$, $K_v \cap U_m^{i,\varepsilon} = F_v(g_v(U^{j,\delta}))$ and $g_v \circ g_w^{-1} = g_{0\varepsilon}|_K$, where $g_{0\varepsilon} : \mathbb{R}^d \to \mathbb{R}^d$ denotes the reflection in the hyperplane $\{z \in \mathbb{R}^d \mid |z| = |z - \varepsilon|\}$. For $u : U^{j,\delta} \to [-\infty, \infty]$, we define $\iota u : U_m^{i,\varepsilon} \to [-\infty, \infty]$ by $\iota u|_{K_w \cap U_m^{i,\varepsilon}} := u \circ g_w^{-1} \circ F_w^{-1}|_{K_w \cap U_m^{i,\varepsilon}}$ and $\iota u|_{K_v \cap U_m^{i,\varepsilon}} := u \circ g_v^{-1} \circ F_v^{-1}|_{K_v \cap U_m^{i,\varepsilon}}$, where the two definitions are consistent on $K_w \cap K_v \cap U_m^{i,\varepsilon}$ by virtue of $g_v \circ g_w^{-1} = g_{0\varepsilon}|_K$. Then $2 \int_{U^{j,\delta}} |u| d\mu = (\#S)^m \int_{U_m^{i,\varepsilon}} |\iota u| d\mu$ if u is Borel measurable, ι defines an injective linear map $\iota : L^2(U^{j,\delta}, \mu|_{U^{j,\delta}}) \to L^2\big(U_m^{i,\varepsilon}, \mu|_{U_m^{i,\varepsilon}}\big)$, and similarly to the proof of Proposition 6.3 we easily see from (SSDF1), (SSDF2) and (GSCDF) that $\iota(\mathcal{F}_{U^{j,\delta}}) \subset \mathcal{F}_{U_m^{i,\varepsilon}}$ and that $2\mathcal{E}(u, u) = r^m \mathcal{E}(\iota u, \iota u)$ for any $u \in \mathcal{F}_{U^{j,\delta}}$.

Now as in the proof of Proposition 3.11, for $U \in \{U^{j,\delta}, K^I\}$ let $\{\varphi_n^U\}_{n \in \mathbb{N}}$ be a complete orthonormal system of $L^2(U, \mu|_U)$ consisting of eigenfunctions of $-\Delta_U$ with eigenvalues $\{\lambda_n^U\}_{n \in \mathbb{N}}$, which we use below to write down all the eigenfunctions of $-\Delta_{U_m^{i,\varepsilon}}$. Let $n \in \mathbb{N}$. Noting that $\varphi_n^{U^{j,\delta}} \in \mathcal{F}_{U^{j,\delta}}$, we define $\varphi_{n,1} \in \mathcal{F}_{U_m^{i,\varepsilon}}$ by

$$\varphi_{n,1} := 2^{-1/2} (\#S)^{m/2} \iota \varphi_n^{U^{j,\delta}}. \tag{6.11}$$

On the other hand, for any $g \in \mathcal{G}_0$, (GSCDF) and $g(K^I) = K^I$ easily imply that $\{u \circ g \mid u \in \mathcal{F}_{K^I}\} = \mathcal{F}_{K^I}$. Then since $\varphi_n^{K^I} \in \mathcal{F}_{K^I}$ and $K_w^I \cup K_v^I \subset U_m^{i,\varepsilon}$, the function $\varphi_{n,2} : K_w^I \cup K_v^I \to \mathbb{R}$ defined by

$$\begin{aligned} \varphi_{n,2}|_{K_w^I} &:= 2^{-1/2} (\#S)^{m/2} \varphi_n^{K^I} \circ g_w^{-1} \circ F_w^{-1}|_{K_w^I}, \\ \varphi_{n,2}|_{K_v^I} &:= -2^{-1/2} (\#S)^{m/2} \varphi_n^{K^I} \circ g_v^{-1} \circ F_v^{-1}|_{K_v^I}, \end{aligned} \tag{6.12}$$

satisfies $\varphi_{n,2} \in \mathcal{F}_{K_w^I \cup K_v^I} \subset \mathcal{F}_{U_m^{i,\varepsilon}}$ by [35, Lemma 5.5].

We claim that the system $\{\varphi_{n,k}\}_{n \in \mathbb{N}, k \in \{1,2\}}$ forms a complete orthonormal system of $L^2\big(U_m^{i,\varepsilon}, \mu|_{U_m^{i,\varepsilon}}\big)$. Indeed, it is easily seen to be orthonormal in $L^2\big(U_m^{i,\varepsilon}, \mu|_{U_m^{i,\varepsilon}}\big)$ by a direct calculation. Let $u \in L^2\big(U_m^{i,\varepsilon}, \mu|_{U_m^{i,\varepsilon}}\big)$ and suppose that $\int_{U_m^{i,\varepsilon}} u \varphi_{n,k} d\mu = 0$ for any $(n, k) \in \mathbb{N} \times \{1, 2\}$. Then for any $n \in \mathbb{N}$, $0 = 2^{1/2} (\#S)^{m/2} \int_{U_m^{i,\varepsilon}} u \varphi_{n,1} d\mu = \int_{U^{j,\delta}} \varphi_n^{U^{j,\delta}} (u \circ F_w \circ g_w + u \circ F_v \circ g_v) d\mu$, and hence $u \circ F_w \circ g_w + u \circ F_v \circ g_v = 0$ μ-a.e. by the completeness of $\{\varphi_n^{U^{j,\delta}}\}_{n \in \mathbb{N}}$. On the other hand, for each $n \in \mathbb{N}$, $\int_{U_m^{i,\varepsilon}} u \varphi_{n,2} d\mu = 0$

implies $\int_{K^I}(u\circ F_w\circ g_w)\varphi_n^{K^I}\,d\mu = \int_{K^I}(u\circ F_v\circ g_v)\varphi_n^{K^I}\,d\mu$, and hence $u\circ F_w\circ g_w = u\circ F_v\circ g_v$ μ-a.e. by the completeness of $\{\varphi_n^{K^I}\}_{n\in\mathbb{N}}$. Thus $u\circ F_w\circ g_w = u\circ F_v\circ g_v = (u\circ F_w\circ g_w + u\circ F_v\circ g_v)/2 = 0$ μ-a.e. and hence $u=0$ μ-a.e. on $U_m^{i,\varepsilon}$, proving the completeness of the orthonormal system $\{\varphi_{n,k}\}_{n\in\mathbb{N},\,k\in\{1,2\}}$ in $L^2\big(U_m^{i,\varepsilon},\mu|_{U_m^{i,\varepsilon}}\big)$.

Finally, we show that for $n\in\mathbb{N}$, $\varphi_{n,k}$ is an eigenfunction of $-\Delta_{U_m^{i,\varepsilon}}$ with eigenvalue $\tau^m\lambda_n^{U^{j,\delta}}$ for $k=1$ and $\tau^m\lambda_n^{K^I}$ for $k=2$. Let $n\in\mathbb{N}$. By using (SSDF1), (SSDF2), (GSCDF) and the fact that $\varphi_n^{U^{j,\delta}}\in\mathcal{D}[\Delta_{U^{j,\delta}}]$ and $-\Delta_{U^{j,\delta}}\varphi_n^{U^{j,\delta}} = \lambda_n^{U^{j,\delta}}\varphi_n^{U^{j,\delta}}$ (recall (2.12)), we easily see that $\mathcal{E}(\varphi_{n,1},u) = \tau^m\lambda_1^{U^{j,\delta}}\int_{U_m^{i,\varepsilon}}\varphi_{n,1}u\,d\mu$ for $u\in\mathcal{F}\cap C(K)$ with $\mathrm{supp}_K[u]\subset U_m^{i,\varepsilon}$ and hence for any $u\in\mathcal{F}_{U_m^{i,\varepsilon}}$. For the proof for $\varphi_{n,2}$, for $u: U_m^{i,\varepsilon}\to\mathbb{R}$ we define $Pu: K\to\mathbb{R}$ by $Pu|_{K\setminus U_m^{i,\varepsilon}} := 0$, $Pu|_{K_w\cap U_m^{i,\varepsilon}} := (u + u\circ F_v\circ g_{0\varepsilon}\circ F_w^{-1})/2|_{K_w\cap U_m^{i,\varepsilon}}$ and $Pu|_{K_v\cap U_m^{i,\varepsilon}} := (u + u\circ F_w\circ g_{0\varepsilon}\circ F_v^{-1})/2|_{K_v\cap U_m^{i,\varepsilon}}$ (recall $g_{0\varepsilon}|_K = g_v\circ g_w^{-1} = g_w\circ g_v^{-1}$), where the two definitions are again consistent on $K_w\cap K_v\cap U_m^{i,\varepsilon}$ with $Pu|_{K_w\cap K_v\cap U_m^{i,\varepsilon}} = u|_{K_w\cap K_v\cap U_m^{i,\varepsilon}}$. Now let $u\in\mathcal{F}\cap C(K)$ satisfy $\mathrm{supp}_K[u]\subset U_m^{i,\varepsilon}$. Then $Pu\in\mathcal{F}\cap C(K)$ by (SSDF1) and (GSCDF), $\mathcal{E}(\varphi_{n,2},Pu) = 0 = \int_{U_m^{i,\varepsilon}}\varphi_{n,2}Pu\,d\mu$ by (SSDF2) and (GSCDF), hence $\mathcal{E}(\varphi_{n,2},u) = \mathcal{E}(\varphi_{n,2},u-Pu)$ and $\int_{U_m^{i,\varepsilon}}\varphi_{n,2}u\,d\mu = \int_{U_m^{i,\varepsilon}}\varphi_{n,2}(u-Pu)\,d\mu$. Now since $u-Pu\in\mathcal{F}\cap C(K)$ and $(u-Pu)|_{K\setminus(K_w^I\cup K_v^I)} = 0$, we easily see from [23, Theorem 1.4.2-(iv)] and (SSDF1) that $(u-Pu)\circ F_w\circ g_w, (u-Pu)\circ F_v\circ g_v\in\mathcal{F}_{K^I}$, which together with (SSDF2), (GSCDF) and $\varphi_n^{K^I}\in\mathcal{D}[\Delta_{K^I}]$, $-\Delta_{K^I}\varphi_n^{K^I} = \lambda_n^{K^I}\varphi_n^{K^I}$, yield $\mathcal{E}(\varphi_{n,2},u-Pu) = \tau^m\lambda_n^{K^I}\int_{U_m^{i,\varepsilon}}\varphi_{n,2}(u-Pu)\,d\mu$. Thus $\mathcal{E}(\varphi_{n,2},u) = \tau^m\lambda_n^{K^I}\int_{U_m^{i,\varepsilon}}\varphi_{n,2}u\,d\mu$ for any $u\in\mathcal{F}\cap C(K)$ with $\mathrm{supp}_K[u]\subset U_m^{i,\varepsilon}$ and hence for any $u\in\mathcal{F}_{U_m^{i,\varepsilon}}$.

It follows that the sequence $\{\lambda_{n,k}\}_{n\in\mathbb{N},\,k\in\{1,2\}}$, $\lambda_{n,1} := \tau^m\lambda_n^{U^{j,\delta}}$, $\lambda_{n,2} := \tau^m\lambda_n^{K^I}$, gives an enumeration of all the eigenvalues of $-\Delta_{U_m^{i,\varepsilon}}$ with each eigenvalue repeated according to its multiplicity, and hence (6.10) with $\#W_m^{i,\varepsilon} = 2$ follows. $\square$

Proposition 6.8. *There exist $c_{6.4}, c_{6.5}\in(0,\infty)$ such that for any $t\in(0,\infty)$,*

$$0 \le \mathcal{Z}_{\mathrm{N}}(t) - \mathcal{Z}_{\mathrm{D}}(t) - \sum_{\substack{\delta,j\in\{0,1\}^d,\,|\delta|\ge 1\\ \delta^{-1}(0)\subset j^{-1}(0)}} \mathcal{Z}_0^{j,\delta}(t) \le c_{6.4}\exp\big(-c_{6.5}t^{-\frac{1}{d_{\mathrm{w}}-1}}\big). \tag{6.13}$$

Proof. Set $\mathcal{I}_0 := \{1,\dots,d\}\times\{0,1\}$ and let $t\in(0,\infty)$. We see from Lemma 2.22 that

$$\begin{aligned}
&\mathcal{Z}_{\mathrm{N}}(t) - \mathcal{Z}_{\mathrm{D}}(t) = \int_K \Big(p_t(x,x) - p_t^{K\setminus H_{\mathcal{I}_0}}(x,x)\Big)d\mu(x) \qquad (6.14)\\
&= \int_K \sum_{\emptyset\ne\mathcal{I}\subset\mathcal{I}_0}\sum_{A\subset\mathcal{I}}(-1)^{\#A} p_t^{K\setminus H_{(\mathcal{I}_0\setminus\mathcal{I})\cup A}}(x,x)\,d\mu(x)\\
&= \sum_{\emptyset\ne\mathcal{I}\subset\mathcal{I}_0}\int_K p_t^{K\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}})\,d\mu(x)\\
&= \sum\nolimits_{\mathcal{I}\subset\mathcal{I}_0,\,(k,0),(k,1)\in\mathcal{I}\text{ for some }k\in\{1,\dots,d\}}\int_K p_t^{K\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}})\,d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&+\sum\nolimits_{\delta,j\in\{0,1\}^d,\,|\delta|\geq 1,\,\delta^{-1}(0)\subset j^{-1}(0)}\int_K p_t^{U_0^{j,\delta}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{j,\delta}})d\mu(x)\\
=&\sum\nolimits_{\mathcal{I}\subset\mathcal{I}_0,\,(k,0),(k,1)\in\mathcal{I}\text{ for some }k\in\{1,\ldots,d\}}\int_K p_t^{K\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}})d\mu(x)\\
&+\sum\nolimits_{\delta,j\in\{0,1\}^d,\,|\delta|\geq 1,\,\delta^{-1}(0)\subset j^{-1}(0)}\mathcal{Z}_0^{j,\delta}(t),
\end{aligned}
$$

which and Proposition 2.16-(1) yield the lower inequality in (6.13). Furthermore for $t\in(0,1]$, (6.14), Proposition 2.16-(2) and Theorem 5.9 together imply that

$$
\begin{aligned}
&\mathcal{Z}_{\mathrm{N}}(t)-\mathcal{Z}_{\mathrm{D}}(t)-\sum\nolimits_{\delta,j\in\{0,1\}^d,\,|\delta|\geq 1,\,\delta^{-1}(0)\subset j^{-1}(0)}\mathcal{Z}_0^{j,\delta}(t)\\
&\leq\sum\nolimits_{\substack{\mathcal{I}\subset\mathcal{I}_0,\,(k,0),(k,1)\in\mathcal{I}\\ \text{for some }k\in\{1,\ldots,d\}}}\int_K 2\min_{(k,u)\in\mathcal{I}}\Bigl\{\sup_{s\in[t/2,t]}\sup_{z\in K\cap H_{k,u}}p_s(x,z)\Bigr\}d\mu(x)\\
&\leq 2^{2d+1+d_{\mathrm{f}}/d_{\mathrm{w}}}c_{5.2}t^{-d_{\mathrm{f}}/d_{\mathrm{w}}}\exp\bigl(-(2^{d_{\mathrm{w}}}c_{5.2}t)^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\leq c_{6.6}\exp\bigl(-c_{6.5}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)
\end{aligned}
$$

for an explicit constant $c_{6.6}\in(0,\infty)$, where $c_{6.5}:=4^{-1}(2c_{5.2})^{-\frac{1}{d_{\mathrm{w}}-1}}$. Finally for $t\in[1,\infty)$, $\mathcal{Z}_{\mathrm{N}}(t)\leq\mathcal{Z}_{\mathrm{N}}(1)=\int_K p_1(x,x)d\mu(x)\leq c_{5.2}\leq c_{5.2}e^{c_{6.5}}\exp\bigl(-c_{6.5}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$ by Theorem 5.9, which and Proposition 2.16-(1) show the upper inequality in (6.13). □

Proof of Theorem 5.11. As mentioned before Proposition 6.7, (5.5) for $\mathcal{Z}_{\mathrm{D}}$ with strictly positive $G_{\mathrm{D},0},-G_{\mathrm{D},1}$ is just a special case of Theorem 5.15 with $i=\varepsilon=0_{\mathbb{R}^d}$ and $m=0$. Theorem 5.15 and Proposition 6.8 together immediately imply (5.5) for $\mathcal{Z}_{\mathrm{N}}$, $\lim_{t\downarrow 0}t^{d_0/d_{\mathrm{w}}}\bigl(\mathcal{Z}_{\mathrm{N}}(t)-\mathcal{Z}_{\mathrm{D}}(t)\bigr)=0$ and hence $G_{\mathrm{N},0}=G_{\mathrm{D},0}$.

To prove $n_{\mathrm{N}}^{-1}G_{\mathrm{N},1}=n_{\mathrm{D}}^{-1}G_{\mathrm{D},1}$, let $\delta\in\{0,1\}^d$ satisfy $|\delta|=1$ and let $j\in\{0,1\}$. Then in view of (5.5) (with $G_{\mathrm{N},0}=G_{\mathrm{D},0}$) and Theorem 5.15, we easily see from Propositions 6.7 and 6.8 that $G_{\mathrm{N},1}=G_{\mathrm{D},1}+2dG_0^{j\delta,\delta,1}$, and from Proposition 6.6 with $i=\varepsilon=0_{\mathbb{R}^d}$ and $m=0$ and Proposition 6.7 that $\mathcal{Z}_{\mathrm{D}}(t)-\tau^{d_0/d_{\mathrm{w}}}\mathcal{Z}_{\mathrm{D}}(\tau t)-n_{1,0}\mathcal{Z}_0^{j\delta,\delta}(\tau t)=o(t^{-d_1/d_{\mathrm{w}}})$ as $t\downarrow 0$, whence $(\tau^{d_1/d_{\mathrm{w}}}-\tau^{d_0/d_{\mathrm{w}}})G_{\mathrm{D},1}=n_{1,0}G_0^{j\delta,\delta,1}$; here $n_{1,0}:=\#\{F_i(K\cap H_{k,s})\mid i\in S,k\in\{1,\ldots,d\},s\in\{0,1\},F_i(K\cap H_{k,s})\not\subset V_0\}$ which is easily seen to be equal to $-n_{\mathrm{D}}\#\bigl(S\setminus(\mathbb{Z}^{d-1}\times\{0\})\bigr)$. Thus we obtain

$$
G_{\mathrm{D},1}=n_{\mathrm{D}}G_0^{j\delta,\delta,1}\qquad\text{and}\qquad G_{\mathrm{N},1}=G_{\mathrm{D},1}+2dG_0^{j\delta,\delta,1}=n_{\mathrm{N}}G_0^{j\delta,\delta,1},\tag{6.15}
$$

so that $n_{\mathrm{N}}^{-1}G_{\mathrm{N},1}=G_0^{j\delta,\delta,1}=n_{\mathrm{D}}^{-1}G_{\mathrm{D},1}$, completing the proof of Theorem 5.11. □

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