DYNAMICS OF PERTURBATIONS OF THE IDENTITY OPERATOR BY MULTIPLES OF THE BACKWARD SHIFT ON $l^\infty(\mathbb{N})$

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ABSTRACT. Let B, I be the unweighted backward shift and the identity operator respectively on $l^\infty(\mathbb{N})$, the space of bounded sequences over the complex numbers endowed with the supremum norm. We prove that $I + \lambda B$ is locally topologically transitive if and only if $|\lambda| > 2$. This, shows that a classical result of Salas, which says that backward shift perturbations of the identity operator are always hypercyclic, or equivalently topologically transitive, on $l^p(\mathbb{N})$, $1 \le p < +\infty$, fails to hold for the notion of local topological transitivity on $l^\infty(\mathbb{N})$. We also obtain further results which complement certain results from [5].

1. INTRODUCTION

We start with some terminology and notation. Throughout this paper the letter X stands for a Banach space, $T : X \to X$ will always be a bounded linear operator and frequently we shall drop the words "bounded linear" for the sake of simplicity. The symbol $L(X)$ stands for the set of all bounded linear operators acting on X. As usual, by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ we denote the sets of positive integers, real numbers and complex numbers respectively. In the following, the symbols $\mathbb{D}$, $\overline{\mathbb{D}}$, $\partial\mathbb{D}$ denote the open unit disk, closed unit disk and the unit circle on the complex plane respectively.

The purpose of this note is to explore the "local" dynamics of certain operators on $l^\infty(\mathbb{N})$, where $l^\infty(\mathbb{N})$ denotes the Banach space of bounded sequences of complex numbers endowed with the usual supremum norm. We will frequently use the subspace $c_0(\mathbb{N})$ of $l^\infty(\mathbb{N})$, which

consists of all the null sequences of complex numbers. Let us proceed with the relevant definitions.

Definition 1.1. Let $T : X \to X$ be an operator. For every $x \in X$ the sets

$$J_T(x) = \{y \in X : \text{ there exist a strictly increasing sequence of positive}$$
$$\text{integers } (k_n) \text{ and a sequence } (x_n) \subset X \text{ such that } x_n \to x \text{ and}$$
$$T^{k_n} x_n \to y\}$$

$$J_T^{mix}(x) = \{y \in X : \text{ there exists a sequence } (x_n) \subset X \text{ such that}$$
$$x_n \to x \text{ and } T^n x_n \to y\}$$

denote the extended (prolongational) limit set and the extended mixing limit set of x under T respectively.

Definition 1.2. Let X be a Banach space and $T : X \to X$ be an operator. Then T is called *topologically transitive* if for every pair of open sets U, V of X there exists a positive integer n such that $T^n U \cap V \neq \emptyset$. The operator T is called *topologically mixing* if for every pair of open sets U, V of X there exists a positive integer k such that $T^n U \cap V \neq \emptyset$ for every $n \geq k$.

Definition 1.3. Let X be a separable Banach space. An operator $T : X \to X$ is called *hypercyclic* if there exists a vector $x \in X$ so that its orbit under T, $Orb(T, x) := \{x, Tx, T^2, \ldots\}$, is dense in X. Then x is called *hypercyclic* for T.

It is easy to see that if X is separable, then an operator T on X is hypercyclic if and only if T is topologically transitive. Clearly the notion of hypercyclicity makes no sense when X is non-separable. However, one can find topologically transitive operators on non-separable Banach spaces, see for instance [4]. On the other hand not every non-separable Banach space supports topologically transitive operators, as for example $l^\infty(\mathbb{N})$ [3]. For a comprehensive treatment on hypercyclicity we refer to the recent books [2], [8].

We now move to "localized analogues" of hypercyclicity and topological transitivity, introduced in [6].

Definition 1.4. An operator $T : X \to X$ will be called *J-class or locally topologically transitive* (*J^{mix}-class or locally topologically mixing*) provided there exists a non-zero vector $x \in X$ so that the extended limit set of x under T is the whole space, i.e. $J_T(x) = X$ (the extended mixing limit set of x under T is the whole space, i.e. $J_T^{mix}(x) = X$). In

this case x will be called a *J-class vector* (J^{mix}-*class vector*) for T and the set of all $y \in X$ with $J_T(y) = X$ ($J_T^{mix}(y) = X$) is denoted by A_T (A_T^{mix}).

It is very easy to see that $T \in L(X)$ is topologically transitive if and only if $J_T(x) = X$ for every $x \in X$ and that T is topologically mixing if and only if $J_T^{mix}(x) = X$ for every $x \in X$. This observation justifies the term "local topological transitive". For examples of operators which are locally topologically transitive (locally topologically mixing) but not topologically transitive (topologically mixing) as well as properties of J-class operators we refer to [6], [5]. For recent additional work on local topological transitivity see [1], [9].

By $l^p(\mathbb{N})$ $1 \leq p < +\infty$ we denote the space of p-summable sequences of complex numbers which becomes a Banach space under the usual p-norm. Recall that, given a sequence of positive and bounded weights $w = (w_n)$, the operator $T_w \in L(l^p(\mathbb{N}), 1 \leq p < +\infty$ defined by $T_w(x_1, x_2, \ldots) = (w_1 x_2, w_2 x_3, \ldots)$ is called a backward unilateral weighted shift. A beautiful result of Salas [10] asserts that if $T_w : l^p(\mathbb{N}) \to l^p(\mathbb{N})$ is a backward unilateral weighted shift then $I + T_w$ is hypercyclic on $l^p(\mathbb{N})$, $1 \leq p < +\infty$; actually it is even mixing [7]. Therefore, it is natural to seek a "localized analogue" of Salas result, if it exists, in the setting of weighted shifts acting on $l^\infty(\mathbb{N})$. Of course, when we say "localized analogue" we mean that, one should replace the term "hypercyclicity" by "local topological transitivity", since $l^\infty(\mathbb{N})$ is non-separable. So, in view of the above, the following question arises naturally.

Question. *Let* $T_w : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ *be a backward unilateral weighted shift. Is it true that* $I + T_w$ *is locally topologically transitive on* $l^\infty(\mathbb{N})$?

The purpose of the present note is to give a negative answer to this question by proving the following

Theorem 1.5. *Let* $B : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ *be the unweighted backward unilateral shift, i.e.*

$$B(x_1, x_2, \ldots) := (x_2, x_3, \ldots), \text{ for } x = (x_1, x_2, \ldots) \in l^\infty(\mathbb{N}).$$

(i) *If* $|\lambda| \leq 2$ *the set* $J_{I+\lambda B}(x)$ *has empty interior for every* $x \in X$ *and, in particular, the operator* $I + \lambda B$ *is not locally topologically transitive.*

(ii) *If* $|\lambda| > 2$ *then* $I + \lambda B$ *is locally topologically mixing and* $A_{I+\lambda B}^{mix} = c_0(\mathbb{N})$.

Part (i) of Theorem 1.5 gives a negative answer to the above question. Actually, Theorem 1.5 provides a characterization of J^{mix}-class operators of the form $I + \lambda B$, $\lambda \in \mathbb{C}$ and in addition describes the set $A^{mix}_{I+\lambda B}$. Our paper is organized as follows. In section 2 we give the proof of Theorem 1.5. In the final section, Section 3, we establish a variant of Theorem 1.5 and we also exhibit some information on the structure of the set A^{mix}_T for certain $T \in L(l^\infty(\mathbb{N}))$.

2. Proof of the main result

2.1. **Preparatory lemmas.** We start with a lemma which can be found in [6, Corollary 3.4].

Lemma 2.1. *Let $T : X \to X$ be an operator. Suppose there exists a vector $x \in X$ such that the set $J_T(x)$ has non-empty interior. Then for every $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$ the operator $T - \lambda I$ has dense range.*

The next lemma also appears in [6] and gives information on the spectrum of a J-class operator.

Lemma 2.2. *Let X be a Banach space and $T \in L(X)$. If T is J-class then the spectrum of T, $\sigma(T)$, intersects the unit circle $\partial \mathbb{D}$.*

Lemma 2.3. *Let T be an operator on a Banach space X. If $0 \in J^{mix}_T(x)$ for some $x \in X$ then $J^{mix}_T(x) = J^{mix}_T(0)$.*

Proof. Assume that $0 \in J^{mix}_T(x)$. Then there exists a sequence $(x_n) \subset X$ such that $x_n \to x$ and $T^n x_n \to 0$. Let us show first that $J^{mix}_T(0) \subset J^{mix}_T(x)$. Indeed, take $y \in J^{mix}_T(0)$. There exists a sequence $(y_n) \subset X$ such that $y_n \to 0$ and $T^n y_n \to y$. Hence $T^n(x_n + y_n) \to y$ and $x_n + y_n \to x$. Therefore $y \in J^{mix}_T(x)$. For the converse inclusion let $y \in J^{mix}_T(x)$. There exists a sequence $(v_n) \subset X$ such that $v_n \to x$ and $T^n v_n \to y$. Hence $T^n(v_n - x_n) \to y$, $v_n - x_n \to 0$ and we conclude that $y \in J^{mix}_T(0)$. This completes the proof. $\square$

Lemma 2.4. *Let T be an operator on a Banach space X. If $J^{mix}_T(0) = X$ then $J^{mix}_T(x) = X$ for every $x \in Ker T$.*

Proof. Observe that $0 \in J^{mix}_T(x)$ for every $x \in Ker T$ and we conclude by Lemma 2.3. $\square$

In the next elementary, but very useful in what follows, lemma we calculate the primage of a vector $y = (y_n) \in \mathbb{C}^{\mathbb{N}}$ under $I + \lambda B$, where $\lambda \in \mathbb{C} \setminus \{0\}$.

Lemma 2.5. *Let $B : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ be the unweighted backward unilateral shift and let $x = (x_n)$, $y = (y_n)$ be vectors in $\mathbb{C}^{\mathbb{N}}$ such that $(I + \lambda B)x = y$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then,*

$$x_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k + \frac{x_1}{(-\lambda)^{n-1}}$$

for every $n = 2, 3, \ldots$.

Proof. Notice that $x_n + \lambda x_{n+1} = y_n$, for every $n \in \mathbb{N}$. It is easy to show that the kernel of $I + \lambda B$ consists of all vectors $w = (w_n)$ such that $w_n = \frac{w_1}{(-\lambda)^{n-1}}$, for every $n \in \mathbb{N}$. Define the vector $z = (z_n)$ by $z_n := \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k$ for $n = 2, 3, \ldots$ and $z_1 := 0$. Straightforward calculations give $(I + \lambda B)z = y$. Thus, $(I + \lambda B)(x - z) = 0$ and from the above description of the kernel of $(I + \lambda B)$ we conclude that $x_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k + \frac{x_1}{(-\lambda)^{n-1}}$ for every $n = 2, 3, \ldots$. $\qquad\square$

Lemma 2.6. *Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then the operator $I + \lambda B : l^{\infty}(\mathbb{N}) \to l^{\infty}(\mathbb{N})$ does not have dense range.*

Proof. Consider the vector $y = (y_n) \in l^{\infty}(\mathbb{N})$ with $y_n = (-\lambda)^{-n}$, for every $n \in \mathbb{N}$. We will show that the open ball $B(y, 1/2)$ centered at y with radius $\frac{1}{2}$ does not intersect the range of $I + \lambda B$. Assume the contrary, i.e. there is a vector $w = (w_n) \in B(y, 1/2)$ and a vector $x \in l^{\infty}(\mathbb{N})$ such that $(I + \lambda B)x = w$. By Lemma 2.5 we deduce that

$$(2.1) \qquad x_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k w_k + \frac{x_1}{(-\lambda)^{n-1}},$$

for every $n = 2, 3, \ldots$. Since $w \in B(y, 1/2)$ we have $w_k = y_k + \varepsilon_k$ for some $\varepsilon_k \in \mathbb{C}$ with $|\varepsilon_k| < \frac{1}{2}$, for every $k \in \mathbb{N}$. Substituting w_k in (2.1) we get

$$\left| \sum_{k=1}^{n-1} (-\lambda)^k (y_k + \varepsilon_k) - \lambda x_1 \right| \leq \|x\|_{\infty}$$

for every $n \in \mathbb{N}$, which in turn implies

$$\left| n - 1 + \sum_{k=1}^{n-1} (-\lambda)^k \varepsilon_k \right| \leq \|x\|_{\infty} + |x_1| \leq 2 \|x\|_{\infty},$$

since $y_k = (-\lambda)^{-k}$. Now, the estimate $|\sum_{k=1}^{n-1} (-\lambda)^k \varepsilon_k| \leq \frac{n-1}{2}$ and the triangle inequality yield

$$\frac{n-1}{2} = n - 1 - \frac{(n-1)}{2} \leq \left| n - 1 + \sum_{k=1}^{n-1} (-\lambda)^k \varepsilon_k \right| \leq 2 \|x\|_\infty$$

for every $n \in \mathbb{N}$, which is a contradiction. $\square$

For the next two lemmas we need to introduce some terminology and notation. Let X be a Banach space and let $T : X \to X$ be a bounded linear operator. For a closed and T-invariant subspace M of X we define the induced operator

$$\widehat{T} : X/M \to X/M \ \text{by} \ \widehat{T}[x]_M := [Tx]_M.$$

$\hat{T}$ is well defined, linear and continuous in the induced quotient topology. In addition we have $\|\widehat{T}\| \leq \|T\|$. In the following we will just write $[x]$ instead of $[x]_M$.

Lemma 2.7. *Let X be a Banach space and $T \in L(X)$. Assume that T is J^{mix}-class and let M be a closed and T-invariant subspace of X such that $M \subset A_T^{mix}$ and $A_T^{mix} \setminus M \neq \emptyset$. Then the induced operator $\widehat{T} : X/M \to X/M$ is J^{mix}-class.*

Proof. Let $x \in A_T^{mix} \setminus M$. Then $[x] \neq 0$. Consider any $[y] \in X/M$. There exists a sequence (x_n) in X such that $x_n \to x$ and $T^n x_n \to y$. From the last we conclude that $[x_n] \to [x]$ and $\widehat{T}^n[x_n] = [T^n x_n] \to [y]$. This shows that $\widehat{T}$ is J^{mix}-class. $\square$

Lemma 2.8. *Let B be the backward shift on $l^\infty(\mathbb{N})$ and consider the induced operator $\hat{B} : l^\infty(\mathbb{N})/c_0(\mathbb{N}) \to l^\infty(\mathbb{N})/c_0(\mathbb{N})$. Then $\sigma(\hat{B}) = \partial\mathbb{D}$.*

Proof. Choose any $\mu := e^{i\theta} \in \partial\mathbb{D}$, $\theta \in \mathbb{R}$. It is straightforward to check that the vector $[(x_n)] \in l^\infty(\mathbb{N})/c_0(\mathbb{N})$ with $x_n = e^{i\theta(n-1)}$ for $n = 1, 2, \ldots$ is an eigenvector for $\hat{B}$ corresponding to μ. Hence, $\partial\mathbb{D} \subset \sigma(\hat{B})$. It remains to show the converse inclusion. To this end, let $\mu \in \mathbb{C}$ with $|\mu| < 1$. Then $\hat{B} - \mu I$ is injective. Indeed, if not, there exists $x = (x_n) \notin c_0(\mathbb{N})$ such that $(\hat{B} - \mu I)[x] = 0$ and therefore $(x_{n+1} - \mu x_n) \in c_0(\mathbb{N})$. Set $\delta := \limsup_n |x_n|$ and let (n_k) be a strictly increasing sequence of positive integers such that $|x_{n_k+1}| \to \delta$. Defining $\epsilon_n := x_{n+1} - \mu x_n$, then $|x_{n_k+1}| \leq |\mu||x_{n_k}| + |\epsilon_{n_k}|$ and we deduce that

$$\delta = \lim_{k \to +\infty} |x_{n_k+1}| \leq |\mu| \limsup_k |x_{n_k}|.$$

Since $|\mu| < 1$ and $\delta > 0$, the above inequality gives a contradiction. It is now easy to show that $\hat{B} - \mu I$ is surjective. Let $[y] \in l^\infty(\mathbb{N})/c_0(\mathbb{N})$

with $y = (y_n)$. Then we have $(\hat{B} - \mu I)[x] = [y]$, where $x = (x_n)$ belongs to $l^\infty(\mathbb{N})$ and it is defined by $x_1 = 0$ and $x_{n+1} := \sum_{k=0}^{n-1} y_{n-k}\mu^k$, $n = 1, 2, \ldots$. Therefore $\hat{B} - \mu I$ is invertible and since $\|\hat{B}\| \le \|B\| = 1$ the conclusion follows. $\qquad\square$

2.2. **Proof of Theorem 1.5.** If $\lambda = 0$ then, clearly, the J-sets of the identity operator are singletons. So, from now on, we may assume that $\lambda \neq 0$.

(i) Let $0 < |\lambda| \le 2$. We can rewrite the previous inequality as $||\lambda| - 1| \le 1$. Assume that $J(x)$ has non-empty interior for some $x \in X$. Then, by Lemma 2.1, the operator $(I + \lambda B) - (1 - |\lambda|)I$ has dense range. Writing $(I + \lambda B) - (1 - |\lambda|)I = |\lambda|(I + \frac{\lambda}{|\lambda|}B)$ we conclude that the operator $I + \frac{\lambda}{|\lambda|}B$ has dense range, which contradicts Lemma 2.6.

(ii) Let $\lambda \in \mathbb{C}$ with $|\lambda| > 2$. Fix a vector $y = (y_n) \in l^\infty(\mathbb{N})$ and let $x = (x_n) \in \mathbb{C}^\mathbb{N}$ be such that $(I + \lambda B)x = y$. Then, by Lemma 2.5,

$$x_n = \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k + \frac{x_1}{(-\lambda)^{n-1}},$$

for every $n = 2, 3, \ldots$. If we take $x_1 = 0$ then it readily follows that $|x_n| \le \frac{\|y\|_\infty}{|\lambda|-1}$ for every $n = 2, 3, \ldots$. Therefore, defining $w^{(1)} = (w_n^{(1)}) \in \mathbb{C}^\mathbb{N}$ by

$$w_1^{(1)} := 0, \quad w_n^{(1)} := \frac{-1}{(-\lambda)^n} \sum_{k=1}^{n-1} (-\lambda)^k y_k + \frac{x_1}{(-\lambda)^{n-1}} \quad n = 2, 3, \ldots$$

we have $\|w^{(1)}\|_\infty \le \frac{\|y\|_\infty}{|\lambda|-1}$, hence $w^{(1)} \in l^\infty(\mathbb{N})$, and $(I + \lambda B)w^{(1)} = y$. If we repeat the same argument for $w^{(1)}$ in place of y we find a vector $w^{(2)} \in l^\infty(\mathbb{N})$ such that $(I + \lambda B)w^{(2)} = w^{(1)}$ and $\|w^{(2)}\|_\infty \le \frac{\|w^{(1)}\|_\infty}{|\lambda|-1} \le \frac{\|y\|_\infty}{(|\lambda|-1)^2}$. Proceeding inductively, for every positive integer n we find a vector $w^{(n)} \in l^\infty(\mathbb{N})$ such that $(I + \lambda B)^n w^{(n)} = y$ and $\|w^{(n)}\|_\infty \le \frac{\|y\|_\infty}{(|\lambda|-1)^n}$. Since $|\lambda| > 2$, $w^{(n)} \to 0$. Thus, $J_{I+\lambda B}^{mix}(0) = l^\infty(\mathbb{N})$. Noticing that the kernel of $I + \lambda B$ in $l^\infty(\mathbb{N})$ is non-trivial and taking any non-zero vector w in this kernel, it follows that $J_{I+\lambda B}^{mix}(w) = l^\infty(\mathbb{N})$ by Lemma 2.4. Thus, $I + \lambda B$ is locally topologically mixing. It remains to show that $A_{I+\lambda B}^{mix} = c_0(\mathbb{N})$. Take any $x \in c_0(\mathbb{N})$. The restricted operator $(I + \lambda B)|_{c_0(\mathbb{N})} : c_0(\mathbb{N}) \to c_0(\mathbb{N})$ is topologically mixing, see [7]. This, implies that $c_0(\mathbb{N}) \subset J_{I+\lambda B}^{mix}(x)$. In particular, $0 \in J_{I+\lambda B}^{mix}(x)$ and by Lemma 2.3 we get $J_{I+\lambda B}^{mix}(x) = J_{I+\lambda B}^{mix}(0)$. Since $J_{I+\lambda B}^{mix}(0) = l^\infty(\mathbb{N})$ we conclude that $c_0(\mathbb{N}) \subset A_{I+\lambda B}^{mix}$. To show the converse inclusion we

argue by contradiction, so assume that $A^{mix}_{I+\lambda B} \setminus c_0(\mathbb{N}) \neq \emptyset$. Now, Lemma 2.7 implies that $\widehat{I+\lambda B}$ is J^{mix}-class and in view of Lemma 2.2 the spectrum of $\widehat{I+\lambda B}$ should intersect the unit circle $\partial \mathbb{D}$. However, by Lemma 2.8 and the spectral mapping theorem it follows that $\sigma(\widehat{I+\lambda B}) = \{1 + \lambda e^{i\theta} : \theta \in \mathbb{R}\}$. Hence, $\sigma(\widehat{I+\lambda B})$ does not intersect the unit circle since $|\lambda| > 2$, which is a contradiction. This finishes the proof of the theorem. $\square$

3. FURTHER RESULTS

3.1. **A variation of Theorem 1.5.** The proof of the next lemma is similar to the proof of item (i) of Proposition 5.9 in [6] and it is left to the interested reader.

Lemma 3.1. *Let $T : X \to X$ be an operator acting on a Banach space X. Then $J^{mix}_T(0) = J^{mix}_{T^n}(0)$ for every positive integer n.*

For $r > 0$ the symbol $D(0, r)$ stands for the open disk in the complex plane with center 0 and radius r.

Theorem 3.2. *Let f be a holomorphic function on $D(0, r)$ for $r > 1$ such that*
$$\overline{\mathbb{D}} \subseteq f(\overline{\mathbb{D}}).$$
Then there exists a positive number R_0 such that for all $R > R_0$ the operator $Rf(B) : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ is J^{mix}-class.

Proof. Since f is holomorphic in $D(0, r)$ with $r > 1$, f has a finite number of zeros on $\overline{\mathbb{D}}$, say z_k, $k = 1, \ldots, n$ for some $n \in \mathbb{N}$. The assumption $\overline{\mathbb{D}} \subseteq f(\overline{\mathbb{D}})$ implies that $\emptyset = \mathbb{D} \cap \partial f(\overline{\mathbb{D}}) = \mathbb{D} \cap f(\partial \mathbb{D})$ from which it follows that $|z_k| < 1$ for every $k = 1, \ldots, n$. Then there exists a holomorphic function g on $D(0, r')$ for some $r' \in (1, r]$ such that $f(z) = g(z)(z - z_1) \cdot \ldots \cdot (z - z_n)$ and $g(z) \neq 0$ for all $z \in D(0, r')$. It follows that $g(\overline{\mathbb{D}}) \subseteq \mathbb{C} \setminus D(0, \delta)$ for some $\delta > 0$ and hence we can choose an $R_1 > 1$ such that $R_1 g(\overline{\mathbb{D}}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ and $R_1 \prod_{k=1}^n ||z_k| - 1| > 1$. Define now
$$h(z) := R_1^2 f(z) = \tilde{h}(z) \cdot p(z),$$
where $p(z) := R_1(z - z_1) \cdot \ldots \cdot (z - z_n)$ and $\tilde{h}(z) = R_1 g(z)$.

Claim. Let $y \in l^\infty(\mathbb{N})$. There exists $x \in l^\infty(\mathbb{N})$ such that:
$$p(B)x = y \text{ and } \|x\| \leq \frac{1}{R_1 \prod_{k=1}^n ||z_k| - 1|} \|y\|.$$

Proof of Claim. We have $p(B) = R_1(B - z_1 I) \cdots (B - z_n I)$. Suppose that $z_j \neq 0$ for every $j = 1, \ldots, n$. Now we can write $B - z_n I =$

$-z_n(\frac{1}{(-z_n)}B + I)$ and then applying Lemma 2.5 and following the proof of item (ii) in Theorem 1.5 we get

$$\left(\frac{1}{(-z_n)}B + I\right)x^{(n)} = \frac{1}{R_1(-z_n)}y \ \text{ and } \ \|x^{(n)}\| \le \frac{\|y\|}{R_1|z_n|\left(\frac{1}{|z_n|} - 1\right)}$$

for some $x^{(n)} \in l^\infty(\mathbb{N})$, or equivalently

$$R_1(B - z_n I)x^{(n)} = y \ \text{ and } \ \|x^{(n)}\| \le \frac{\|y\|}{R_1(1 - |z_n|)}.$$

Arguing as before, there exists $x^{(j-1)} \in l^\infty(\mathbb{N})$ such that

$$(B - z_{j-1}I)x^{(j-1)} = x^{(j)} \ \text{ and } \ \|x^{(j-1)}\| \le \frac{\|x^{(j)}\|}{1 - |z_{j-1}|},$$

for every $j = n, n-1, \dots, 2$. Setting $x := x^{(1)}$, we conclude that

$$p(B)x = y \ \text{ and } \ \|x\| \le \frac{1}{R_1 \prod_{k=1}^n ||z_k| - 1|}\|y\|.$$

Of course if $z_j = 0$ for some $j \in \{1, \dots, n\}$ a similar argument can be applied without any difficulty. This completes the proof of the claim.

By the spectral theorem we get

$$\sigma(\tilde{h}(B)) = \tilde{h}(\sigma(B)) = R_1 g(\sigma(B)) = R_1 g(\overline{\mathbb{D}}) \subset \mathbb{C}\backslash\overline{\mathbb{D}}.$$

It follows that the inverse operator $\tilde{h}(B)^{-1}$ exists and $\sigma(\tilde{h}(B)^{-1}) \subset \mathbb{D}$. Therefore, for a given $0 < a < 1$ there exists $n_0 \in \mathbb{N}$ such that

$$\left\|\tilde{h}(B)^{-n_0}\right\| \le a.$$

Consider now any $y \in l^\infty$. The equation $\tilde{h}(B)^{n_0}w^{(1)} = y$ has a unique solution $w^{(1)} \in l^\infty(\mathbb{N})$ and

$$\left\|w^{(1)}\right\| = \left\|\tilde{h}(B)^{-n_0}y\right\| \le a\|y\|.$$

Define

$$b := \frac{1}{(R_1 \prod_{k=1}^n ||z_k| - 1|)^{n_0}} < 1.$$

Applying the Claim n_0 times, there exists a vector $v^{(1)} \in l^\infty(\mathbb{N})$ such that $\left\|v^{(1)}\right\| \le b\left\|w^{(1)}\right\|$ and $p(B)^{n_0}v^{(1)} = w^{(1)}$. Altogether we get

$$h(B)^{n_0}v^{(1)} = y \ \text{ with } \ \left\|v^{(1)}\right\| \le b\left\|w^{(1)}\right\| \le ab\|y\|.$$

The equation $\tilde{h}(B)^{n_0}w^{(2)} = v^{(1)}$ has a unique solution $w^{(2)} \in l^\infty(\mathbb{N})$ and

$$\left\|w^{(2)}\right\| = \left\|\tilde{h}(B)^{-n_0}v^{(1)}\right\| \le a\left\|v^{(1)}\right\|.$$

Again by the Claim there exists $v^{(2)} \in l^\infty(\mathbb{N})$ such that $\left\| v^{(2)} \right\| \leq b \left\| w^{(2)} \right\|$ and $p(B)^{n_0} v^{(2)} = w^{(2)}$. From the above we have

$$h(B)^{2n_0} v^{(2)} = y \text{ and } \left\| v^{(2)} \right\| \leq b \left\| w^{(2)} \right\| \leq (ab)^2 \left\| y \right\|.$$

Proceeding inductively, we construct a sequence $(v^{(m)})$ in $l^\infty(\mathbb{N})$ such that

$$h(B)^{mn_0} v^{(m)} = y \text{ and } \left\| v^{(m)} \right\| \leq (ab)^m \left\| y \right\|$$

for every positive integer m. Since $ab < 1$, $v^{(m)} \to 0$ and we conclude that $J_{h(B)^{n_0}}^{mix}(0) = l^\infty(\mathbb{N})$. Now, in view of Lemma 3.1, $J_{h(B)}^{mix}(0) = l^\infty(\mathbb{N})$. Observe that $p(B)$ has non-trivial kernel, which in turn implies that $h(B)$ has non-trivial kernel. Hence, for every non-zero vector $x \in Ker(h(B))$, $J_{h(B)}^{mix}(x) = l^\infty(\mathbb{N})$. Clearly, for every $R \geq R_0$, where $R_0 := R_1^2$, the operator $Rf(B)$ is locally topologically mixing in $l^\infty(\mathbb{N})$. This completes the proof of the theorem. $\qquad\square$

Corollary 3.3. *Let f be a holomorphic function on $D(0, r)$ for some $r > 1$ such that 0 belongs to the interior of $f(\overline{\mathbb{D}})$. Then there exists $R_0 > 0$ such that $Rf(B) : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ is J^{mix}-class for every $R > R_0$.*

3.2. **Properties of the set A_T^{mix}.** Our last result in this section concerns the structure of the set A_T^{mix} of a J^{mix}-class operator on $l^\infty(\mathbb{N})$. Observe that if T is J^{mix}-class on a Banach space X then A_T^{mix} is a closed subspace. So far, in all examples of J^{mix}-class operators T on $l^\infty(\mathbb{N})$, the closed subspace A_T^{mix} is separable. For instance, if $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ then $\lambda B : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ is J^{mix}-class and $A_{\lambda B}^{mix} = c_0(\mathbb{N})$, see [6]. A similar result holds for the operator $I + \lambda B : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ whenever $|\lambda| > 2$, as Theorem 1.5 shows. In the following proposition we exhibit an operator T on $l^\infty(\mathbb{N})$ such that A_T^{mix} is non-separable.

Proposition 3.4. *There exists a J^{mix}-class operator T on $l^\infty(\mathbb{N})$ such that A_T^{mix} is non-separable.*

Proof. Fix an isomorphism $S : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N}) \oplus l^\infty(\mathbb{N})$ and consider the projection $P : l^\infty(\mathbb{N}) \oplus l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ defined by $P(x \oplus y) := x$. Now choose $|\lambda| > \|S^{-1}\|$ and define the operator $T : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$ by $T := \lambda B \circ P \circ S$. Then T is J^{mix}-class with $M := S^{-1}(\{0\} \oplus l^\infty(\mathbb{N})) \subset A_T^{mix}$. To see this take any $y = (y_n) \in l^\infty(\mathbb{N})$. Define $x^{(1)} := (0, \lambda^{-1} y_1, \lambda^{-1} y_2, \ldots)$. Then $\lambda B x^{(1)} = y$ and $\left\| x^{(1)} \right\| = \frac{1}{|\lambda|} \|y\|$. Note that $P(x^{(1)} \oplus 0) = x^{(1)}$ with $\left\| x^{(1)} \oplus 0 \right\| = \left\| x^{(1)} \right\|$. Choose now $z^{(1)} \in l^\infty(\mathbb{N})$ with $Sz^{(1)} = x^{(1)} \oplus 0$. Then $Tz^{(1)} = y$ and

$$\left\| z^{(1)} \right\| \leq \left\| S^{-1} \right\| \left\| x^{(1)} \oplus 0 \right\| = \left\| S^{-1} \right\| \cdot \left\| x^{(1)} \right\| = \frac{\left\| S^{-1} \right\|}{|\lambda|} \|y\| = q \|y\|,$$

where $q := \|S^{-1}\| \, |\lambda|^{-1} < 1$. Proceeding in this way, we find $z^{(2)} \in l^{\infty}(\mathbb{N})$ with $Tz^{(2)} = z^{(1)}$ and $\|z^{(2)}\| \leq q \|z^{(1)}\| \leq q^2 \|y\|$. Inductively, we get a sequence $(z^{(n)})$ in $l^{\infty}(\mathbb{N})$ such that $T^n z^{(n)} = y$ and $z^{(n)} \to 0$. Since y is arbitrary we conclude that $J_T^{mix}(0) = l^{\infty}(\mathbb{N})$. It is easy to see that $M := S^{-1}(\{0\} \oplus l^{\infty}(\mathbb{N}))$ is non separable and $M \subset KerT$, where $KerT$ is the kernel of T. Since $J_T^{mix}(0) = l^{\infty}(\mathbb{N})$ we have $KerT \subset A_T^{mix}$, and the conclusion follows. $\qquad \square$

As we observed above for any $T \in L(X)$ the set A_T^{mix} is a closed subspace. However for the set A_T the situation is less clear and so we ask the following

Question. Does there exist a J-class operator T acting on some Banach space X such that A_T is not a vector space?

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