ON QUOTIENTS OF AFFINE SPHERICAL VARIETIES BY UNIPOTENT SUBGROUPS

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Abstract. Let G be a connected complex reductive algebraic group and X be an affine spherical G-variety. Then any Borel subgroup $B \subseteq G$ acts on X with finitely many orbits. Now suppose that $H \subset B$ is a normal unipotent subgroup such that the algebra $\mathbb{C}[G/H]$ is finitely generated. Then $\mathbb{C}[X]^H$ is finitely generated, too, and one may consider the categorical quotient map $\pi : X \to X/\!\!/H := \operatorname{Spec} \mathbb{C}[X]^H$. Clearly, B/H acts on $X/\!\!/H$. Recently Panyushev posed a question whether the number of orbits for this action is always finite. (The point is that π is not always surjective.) He conjectured that this is always the case for $H = [U, U]$, the commutator of the maximal unipotent subgroup $U \subset B$. We discuss an approach to answering the question, give a positive answer in some cases, and provide a counterexample to the above conjecture.

Introduction

The problem which we are discussing in this paper is about pushing down certain geometric properties of an algebraic variety equipped with a group action to the quotient variety. In the geometric invariant theory, there are several concepts of a quotient space, which should morally mean a kind of "moduli space of orbits" for a group action. In the category of affine algebraic varieties an appropriate concept is that of a "categorical quotient", defined as the spectrum of the algebra of invariant polynomials on an initial variety. This makes sense, because functions on the quotient space should indeed lift to invariant functions on a given variety. However, this "categorical quotient space" usually cannot be identified with the set of orbits and even worse, the natural quotient map may be non-surjective for non-reductive group actions. The latter circumstance is a crucial obstruction for pushing down properties which we are interested in here.

Suppose that we are given a connected solvable algebraic group B acting on an affine algebraic variety X with finitely many orbits. This

Date: February 7, 2013.

2010 *Mathematics Subject Classification.* Primary 14L30; Secondary 14M27, 14L24.

Key words and phrases. Spherical variety, categorical quotient, Grosshans subgroup, transfer principle, canonical embedding.

is a nice situation, because the orbital decomposition yields a stratification of X by locally closed "cells" of simple structure, with nice geometric consequences.

If $H \subset B$ is a normal unipotent subgroup, then B/H acts on the categorical quotient $X /\!\!/ H$ (whenever it exists, i.e., the algebra of invariant polynomials on X is finitely generated). (We consider unipotent H, because in this case $X /\!\!/ H$ is quite close to a naïve "space of orbits". Namely H-invariant polynomials separate general H-orbits, so that an open subset of $X /\!\!/ H$ parameterizes orbits in general position.) It is quite natural to ask whether B/H still has finitely many orbits on $X /\!\!/ H$. Of course, it is not a question if the quotient map $X \to X /\!\!/ H$ is surjective. But this is not always the case.

In this paper we consider a very special situation, where B is a Borel subgroup in a bigger reductive group G acting on X, i.e., X is a spherical variety. In this situation, the question about finite number of B/H-orbits on $X /\!\!/ H$ was raised by D. I. Panyushev. In Section 1 we discuss a general approach to this question based on the so-called "Transfer Principle", which allows to reduce the study of quotients for non-reductive group actions to those for reductive group actions, and give some simple examples, where the question is answered affirmatively.

After that, in Section 2 we turn to an interesting case, where H is the commutator of the unipotent radical of B. Panyushev conjectured that in this case B/H has finitely many orbits on $X /\!\!/ H$ for any (affine spherical) X. We disprove this conjecture in Proposition 3 and Theorem 1. In fact, in turns out that for "big" X finiteness is a rare exception, i.e., it holds for few groups G. The proofs are based on a careful study of the structure of the "canonical embedding" $G /\!\!/ H$.

Since a negative answer to the above question is not quite satisfactory, we look for cases, where the finiteness holds for any G and H as in Section 2 or a bigger normal unipotent subgroup of B. In Section 3 we prove that, under some restrictions on X, there are finitely many B/H-orbits on $X /\!\!/ H$ (Theorem 2).

Acknowledgements. A major part of this paper was written during my stay at the University of Bielefeld in summer 2012 in the framework of the SFB 701 Project B1 "Diophantine inequalities, spectral structures, and linear algebraic groups". I would like to express my gratitude to Prof. H. Abels for the invitation and thank the University for warm hospitality and excellent working conditions.

1. Preliminaries

Let a linear algebraic group H act on an affine algebraic variety X. (All algebraic varieties and groups are defined over $\mathbb{C}$.) Suppose that the algebra of invariant polynomials $\mathbb{C}[X]^H$ is finitely generated. Then one can define the *categorical quotient* $X/\!\!/H := \operatorname{Spec}\mathbb{C}[X]^H$ together with the H-invariant quotient map $\pi = \pi_H : X \to X/\!\!/H$ dual to the inclusion of algebras $\mathbb{C}[X]^H \hookrightarrow \mathbb{C}[X]$.

If H is reductive, then, by Hilbert's Endlichkeitssatz, $\mathbb{C}[X]^H$ is finitely generated, so the categorical quotient exists and possesses many nice properties: π is surjective, separates closed orbits (and even disjoint invariant closed subsets), $X/\!\!/H$ carries on the quotient (Zariski) topology with respect to π, etc (see, e.g., [PV, 4.4]). In general, all these properties can be not satisfied.

Sometimes one can guarantee the existence of the categorical quotient for non-reductive group actions. Suppose that H is contained in a bigger group G acting on X. The following assertion is well known (see, e.g., [Gr2]):

Transfer Principle. $\mathbb{C}[X]^H \simeq \mathbb{C}[G/H \times X]^G \simeq (\mathbb{C}[G/H] \otimes \mathbb{C}[X])^G$.

The isomorphism (from the right to the left) is given by restricting functions on $G/H \times X$ to $\{eH\} \times X$. The name is justified by observing that this result allows transfer of information about H-invariants from G-invariants.

Suppose now that G is reductive. If $\mathbb{C}[G/H]$ is finitely generated, then we see from the transfer principle that $\mathbb{C}[X]^H$ is finitely generated, too. Geometrically, if $G/\!\!/H$ exists, then $X/\!\!/H$ exists, too, and $X/\!\!/H \simeq (G/\!\!/H \times X)/\!\!/G$. Recall the following

Definition 1. H is said to be a *Grosshans subgroup* of G if G/H is quasiaffine and $\mathbb{C}[G/H]$ is finitely generated.

The first condition means that $\mathbb{C}(G/H) = \operatorname{Quot}\mathbb{C}[G/H]$. It is not really essential: one can replace H by a (unique) bigger subgroup $\widehat{H} \subseteq G$ such that $M^H = M^{\widehat{H}}$ for any G-module M [PV, 3.7]; in particular, $\mathbb{C}[G/H] = \mathbb{C}[G/\widehat{H}]$. However, assuming both conditions, we have an open embedding $G/H \hookrightarrow G/\!\!/H$, which is geometrically very natural. The affine closure $G/\!\!/H$ of a quasiaffine homogeneous space G/H is called the *canonical embedding* (of G/H) [AT]. It can be characterized as a (unique) normal affine variety containing G/H as a dense open subset with the complement of codimension ≥ 2. Conversely, if G/H can be embedded into an affine variety as a dense open subset with the complement of codimension ≥ 2, then H is a Grosshans subgroup in G [Gr2, Thm. 4.3].

Example 1. A maximal unipotent subgroup $U \subset G$ is Grosshans; this is due to D. Khadzhiev. More generally, the unipotent radical of any parabolic subgroup of G is a Grosshans subgroup in G [Gr2, Thm. 16.4].

Remark 1. There is a long-standing conjecture, due to K. Pommerening and V. L. Popov, that any unipotent subgroup $H \subseteq U$ normalized by a maximal torus $T \subseteq B = N_G(U)$ is a Grosshans subgroup of G.

The above example fits into this conjecture. The conjecture is by now verified for all G of semisimple rank ≤ 2 and for $G = SL_n(\mathbb{C})$, $n \leq 5$.

Suppose now that X is an affine *spherical* G-variety. This means that a Borel subgroup $B \subseteq G$ has a dense open orbit on X or, equivalently, that $\mathbb{C}[X]$ is a multiplicity free G-module [VK]. In fact, B acts on X with finitely many orbits [Br1], [Vin]. Clearly, $T \simeq B/U$ acts on $X/\!\!/U$ with a dense orbit as well, i.e., $X/\!\!/U$ is an affine toric variety and therefore contains only finitely many T-orbits.

Let us now replace U with an arbitrary subgroup $H \subseteq U$ which is normal in B. Suppose additionally that H is a Grosshans subgroup in G (which is probably always the case). Then, as above, B/H acts on $X/\!\!/H$ with a dense orbit. Recently D. I. Panyushev posed the following

Question. *Does B/H have finitely many orbits on $X/\!\!/H$?*

A delicate point, which prevents us from an "obvious" answer, is that the quotient morphism $\pi : X \to X/\!\!/H$ may be non-surjective, hence not necessarily all B/H-orbits on $X/\!\!/H$ come from B-orbits on X.

An affirmative answer to this question (for some G, H, X) would produce new examples of algebraic varieties acted on by a solvable group with finitely many orbits. Such varieties have a nice stratification (as noted in the Introduction) and therefore, for instance, nice homological properties [FMSS].

Let us consider some easy cases, where the question of Panyushev is answered affirmatively.

Example 2. Let H be the unipotent radical of a parabolic subgroup $P \subseteq G$ containing B. Then B/H is a Borel subgroup in a reductive group P/H acting on $X/\!\!/H$. Thus, $X/\!\!/H$ is a spherical P/H-variety and consequently possesses only finitely many B/H-orbits.

Example 3. Suppose now that X contains a (necessarily unique) G-fixed point 0. Then $\mathbb{C}[X]$ admits a G-invariant positive grading $\mathbb{C}[X] = \mathbb{C} \oplus \mathbb{C}[X]_1 \oplus \mathbb{C}[X]_2 \oplus \cdots$, i.e., there is a multiplicative one-parameter subgroup of G-equivariant automorphisms contracting the whole X to 0 (M. Brion, [Kn, 7.9]). The quotient map π is equidimensional if and only if $\dim \pi^{-1}\pi(0) = \dim Hx$ for a general point $x \in X$. Indeed, the "zero fiber" $\pi^{-1}\pi(0)$ has the biggest possible dimension and generic fibers are just orbits, because generic orbits of a unipotent subgroup on an affine variety are separated by invariant polynomials and closed [PV, 3.2, 1.3].

There is no loss of generality in assuming that X is normal: finite coverings preserve the property of having finitely many orbits. Then $X/\!\!/H$ is normal, too. It follows that π is an open morphism [Che, Ch. V, §5, Prop. 3]. Since the above one-parameter group action descends to $X/\!\!/H$ and contracts it to $\pi(0)$, the map π is surjective, whence all

B/H-orbits on $X/\!\!/H$ come from B-orbits on X and there are finitely many of them.

Generally, to analyze the question of Panyushev, one has to evade somehow the possible non-surjectivity of the quotient map. But this can be done via the Transfer Principle. Indeed, we have the following commutative diagram:

$$\begin{array}{ccc} X & \hookrightarrow & G/\!\!/H \times X \\ \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi_G} \\ X/\!\!/H & = & (G/\!\!/H \times X)/\!\!/G, \end{array}$$

where π_G is already surjective. Note that B/H acts on $G/\!\!/H \times X$ by G-automorphisms via its action on $G/\!\!/H$ "from the right" (coming from $B/H \subseteq N_G(H)/H = \mathrm{Aut}_G(G/H)$) and every B/H-orbit on $X/\!\!/H$ comes from an orbit of $\widehat{G} = G \times B/H$ on $G/\!\!/H \times X$. Thus, a sufficient condition for the affirmative answer to the question is that there are finitely many $\widehat{G}$-orbits on $G/\!\!/H \times X$. (Of course, this condition is not necessary: by the surjectivity of π_G and separation of closed G-orbits, B/H-orbits in $X/\!\!/H$ are in bijection with only those $\widehat{G}$-orbits in $G/\!\!/H \times X$ which intersect fibers of π_G in *closed* G-orbits.) In order to apply this approach, one has to study the structure of the canonical embedding $G/\!\!/H$.

2. Quotients by $[U, U]$

In this section, we consider a particular unipotent subgroup $H = U' := [U, U]$. It is known to be a Grosshans subgroup of G [Gr1, §4]. Our aim is to study the above question in this case.

Conjecture (D. I. Panyushev)**.** *For any affine spherical G-variety X, the group B/U' acts on $X/\!\!/U'$ with finitely many orbits.*

We show that this conjecture fails, with a few "small" exceptions.

Let us start with fixing some notation. Capital Roman letters denote algebraic groups and the respective lowercase Gothic letters denote their Lie algebras. There is no loss of generality in assuming G semisimple simply connected. Let $\alpha_1, \dots, \alpha_r$ denote the simple roots of G (with respect to the chosen Borel subgroup B and maximal torus T) and $\omega_1, \dots, \omega_r$ be the respective fundamental weights. Since we use the additive notation for the group law in the weight lattice of T, it will be convenient to denote by t^χ the value of the character $\chi : T \to \mathbb{C}^\times$ at $t \in T$. Let $U_\alpha \subseteq U$ denote the root one-parameter subgroups corresponding to the positive roots α and e_α be the Chevalley generators of their Lie algebras. By $V(\lambda)$ we denote the irreducible G-module of highest weight λ and $v_\mu \in V(\lambda)$ stands for any T-eigenvector of weight μ. (Usually, in our considerations, such vectors will be determined by their eigenweights up to a multiple.) Let us put $Y = G/\!\!/U'$, for brevity. We shall keep other notation and conventions from §1.

Now we describe the canonical embedding Y explicitly. Consider a G-module

$$V = V(\omega_1) \oplus V(\omega_1)' \oplus \cdots \oplus V(\omega_r) \oplus V(\omega_r)' = \bigoplus_{i=1}^{r} V(\omega_i) \otimes \mathbb{C}_i^2.$$

Here $V(\omega_i)' \simeq V(\omega_i)$ is the "second" copy of a fundamental irreducible G-module occurring in the decomposition of V and $\mathbb{C}_i^2 \simeq \mathbb{C}^2$ is the i-th multiplicity space. Take a vector

$$v_0 = (v_{\omega_1}, v'_{\omega_1-\alpha_1}, \dots, v_{\omega_r}, v'_{\omega_r-\alpha_r}) \in V.$$

(Here and below, we mark vectors from $V(\omega_i)'$ by dashes.)

Lemma 1. $Y = \overline{Gv_0}$.

The lemma follows from the fact that $\mathbb{C}[G/U']^U$ is generated by the pullbacks of the U-invariant linear functions on V along the orbit map $G \to V$, $g \mapsto gv_0$. A representation-theoretic proof is given in [Pan, Thm. 2.2]. A simple geometric proof can be found in [Gor]. It is based on an observation that G/U' is the universal torsor over a spherical homogeneous space G/TU' and therefore $\mathbb{C}[G/U']^U$ is freely generated by the canonical sections of the line bundles associated with the B-stable prime divisors on G/TU', which are easy to describe [Ti1], cf. [Br2, 3.2].

Next, we describe the action of B/U' on Y by G-equivariant automorphisms. Since the induced action on $\mathbb{C}[Y]$ preserves the isotypic G-module decomposition, it is easy to see that the desired action comes from a linear action on V commuting with G, i.e., from linear representations of B/U' in the multiplicity spaces $\mathbb{C}_i^2$. The only condition to be satisfied is that the action of B/U' on the base point v_0 must be compatible with the action of B as a subgroup of G. The concluding result is:

Lemma 2. *The G-equivariant B/U'-action on Y is the restriction of a linear representation in V coming from the representations $B \to \mathrm{GL}(\mathbb{C}_i^2)$ which are described as follows. In the standard basis e_{i1}, e_{i2} of $\mathbb{C}_i^2$:*

$$T \ni t \mapsto \begin{pmatrix} t^{-\omega_i} & 0 \\ 0 & t^{-\omega_i+\alpha_i} \end{pmatrix}, \qquad U_{\alpha_i} \ni \exp(u_i e_{\alpha_i}) \mapsto \begin{pmatrix} 1 & 0 \\ -u_i & 1 \end{pmatrix},$$

and all other root unipotent subgroups U_α corresponding to positive roots $\alpha \neq \alpha_i$ act trivially.

Our next aim is to describe the orbital decomposition of Y under the action of $\widehat{G} = G \times B/U'$. Note that, since G/B is complete, $Y = G \cdot \overline{Bv_0}$.

Lemma 3. $\overline{Bv_0} = \overline{Tv_0} + \mathfrak{u} \cdot v_0 = \{(\dots, x_i v_{\omega_i}, y_i v'_{\omega_i - \alpha_i} + z_i v'_{\omega_i}, \dots)\}$, *where* x_i, y_i *are bound by binomial equations which define* $\overline{Tv_0}$ *and* z_i *are arbitrary.*

Proof. We decompose the Borel subgroup as $B = TU_{\alpha_1} \cdots U_{\alpha_r} U'$ and act on v_0 step by step. At the first step, U' just fixes v_0. At the next r steps, each U_{α_i} acts trivially on every summand of the vector already obtained after previous steps, except for $v'_{\omega_i - \alpha_i}$, which transforms as follows: $\exp(u_i e_{\alpha_i}) v'_{\omega_i - \alpha_i} = v'_{\omega_i - \alpha_i} + u_i e_{\alpha_i} v'_{\omega_i - \alpha_i} = v'_{\omega_i - \alpha_i} + u_i v'_{\omega_i}$. (We can normalize the eigenvectors $v'_{\omega_i - \alpha_i}$ and/or the Chevalley generators e_{α_i} in order to obtain the last equality.) Thus, $Uv_0 = v_0 + \langle v'_{\omega_1}, \dots, v'_{\omega_r} \rangle = v_0 + \mathfrak{u} \cdot v_0$. At the last step, since $\mathfrak{u} \cdot v_0$ is already T-stable and v_0 lies in a complementary T-stable subspace, $Bv_0 = Tv_0 + \mathfrak{u} \cdot v_0 \simeq Tv_0 \times \mathfrak{u} \cdot v_0$. It remains to close it up. □

Corollary 1. $\widehat{G}$ *acts on* Y *with finitely many orbits.*

Proof. It suffices to show that $B \times B/U'$ acts on $\overline{Bv_0}$ with finitely many orbits. In fact, even $T \times T \subset B \times B/U'$ has finitely many orbits in $\overline{Bv_0}$, because the latter is a toric $(T \times T)$-variety. Indeed, the biweights of the $(T \times T)$-eigenvectors occurring in the decomposition of each vector in $\overline{Bv_0}$ are: $(\omega_i, -\omega_i)$ of v_{ω_i}, $(\omega_i - \alpha_i, -\omega_i + \alpha_i)$ of $v'_{\omega_i - \alpha_i}$, $(\omega_i, -\omega_i + \alpha_i)$ of v'_{ω_i}. Note that the last r biweights are linearly independent modulo the previous $2r$ biweights. This means that, acting by $T \times T$, we can multiply z_i's (in the notation of Lemma 3) by arbitrary nonzero scalars without change of x_i's and y_i's. As for x_i's and y_i's, we can also reduce their possible values to finitely many possibilities, since $\overline{Tv_0}$ is a toric variety. □

Let us exhibit the orbit representatives explicitly. Given a vector $v = (\dots, x_i v_{\omega_i}, y_i v'_{\omega_i - \alpha_i} + z_i v'_{\omega_i}, \dots) \in \overline{Bv_0}$, we are going to put v into a canonical form by the $(B \times B/U')$-action.

An element $\exp(u_i e_{\alpha_i}) \in U_{\alpha_i} \subset B$ acts on v by a transform $z_i \mapsto z_i + u_i y_i$, while being considered as an element of B/U', it acts by a transform $z_i \mapsto z_i - u_i x_i$. U' leaves v fixed and the $(T \times T)$-action was described above.

It follows that for each $i \in \{1, \dots, r\}$ we can make the coordinates x_i, y_i, z_i equal to the following specified values, thus attributing the index i (or the simple root α_i, or the fundamental weight ω_i, or the respective node of the Dynkin diagram) to one of the five types (depending on v, of course):

$$\begin{cases} x_i = y_i = 1,\ z_i = 0 & \text{(type XY)} \\ x_i = 1,\ y_i = z_i = 0 & \text{(type X)} \\ y_i = 1,\ x_i = z_i = 0 & \text{(type Y)} \\ x_i = y_i = 0,\ z_i = 1 & \text{(type Z)} \\ x_i = y_i = z_i = 0 & \text{(type O)} \end{cases}$$

What are the possible decompositions of $\{1,\dots,r\}$ into types?

In order to determine which of x_i, y_i can be nonzero, we have to describe the faces of the convex cone spanned by the weights ω_1, $\omega_1-\alpha_1$, $\dots$, ω_r, $\omega_r-\alpha_r$. Namely the T-orbits in $\overline{Tv_0}$ are in bijection with these faces, and $x_i \neq 0$ (resp. $y_i \neq 0$) on an orbit if and only if ω_i (resp. $\omega_i-\alpha_i$) belongs to the corresponding face [Ful].

The faces are determined by supporting linear functions on the weight lattice, i.e., by elements h in the dual space such that $\langle h,\omega_i\rangle \geq 0$, $\langle h,\omega_i-\alpha_i\rangle \geq 0$, $i=1,\dots,r$. This means that $h = c_1\alpha_1^\vee+\dots+c_r\alpha_r^\vee$ and $\sum_{j\neq i} c_j|\langle\alpha_i,\alpha_j^\vee\rangle| \geq c_i \geq 0$, $i=1,\dots,r$. The weight ω_i, resp. $\omega_i-\alpha_i$ lies on the face determined by h if and only if the second, resp. first, one of the latter inequalities turns into an equality.

We readily see that: i is of type XY if and only if $c_i=0$ and there is no neighboring vertex j of the Dynkin diagram with $c_j>0$; i is of type X if and only if $c_i=0$ and there exists at least one neighbor j of i with $c_j>0$; i is of type Y, Z, or O if and only if $c_i>0$.

We claim that the vertices of types Y, Z, O may constitute any subdiagram without isolated vertices (for appropriate h). (Clearly, an isolated vertex i cannot occur in such a subdiagram, because this would contradict the first inequality for i.) Indeed, take any subdiagram without isolated vertices and, for every component of it, put

$$c_i = \begin{cases} 1, & i \text{ non-extreme, or extreme with the arrow pointing outside,} \\ \delta, & i \text{ extreme, with no arrow at } i \text{ or the arrow pointing to } i, \end{cases}$$

where $1/2<\delta<1$. Also put $c_i=0$ for all i outside the subdiagram. It is easy to see that the above inequalities will be satisfied.

Further, observe that both vertices in each A_2-component of the above subdiagram are of type Y and the vertices of type Y in any other component are isolated from each other. Indeed, for any two adjacent vertices i_1, i_2 of type Y, the equalities $c_{i_k} = \sum_{j\neq i_k} c_j|\langle\alpha_{i_k},\alpha_j^\vee\rangle|$ imply that $c_{i_1}\geq c_{i_2}$, $c_{i_2}\geq c_{i_1}$, and one of these inequalities is strict if there is at least one more vertex in the component or i_1 and i_2 are connected by a multiple edge, a contradiction. If i,j constitute a component of type A_2, then we also have $c_i\geq c_j$ and $c_j\geq c_i$, whence the equality, i.e., both i and j are of type Y. We attribute the isolated vertices of type Y to the subtype Y1 and the non-isolated ones to the subtype Y2.

Conversely, if we choose an arbitrary subset of pairwise isolated vertices in each component of the subdiagram whose type is not A_2, then we can find h such that these chosen vertices together with the vertices of the A_2-components are precisely the vertices of type Y. Indeed, we can start from h with the coefficients c_i given by the above displayed formula; then the only vertices of type Y will be those in the components of type A_2. After that, for each i from the chosen subset, we increase c_i in order to turn the inequality $c_i < \sum_{j\neq i} c_j|\langle\alpha_i,\alpha_j^\vee\rangle|$ into an equality. Since these i's are isolated from each other, this procedure can be performed simultaneously and clearly it will not break the other inequalities or turn them into equalities.

Finally, the remaining vertices may be distributed between the types Z and O in an arbitrary way: this corresponds to choosing orbits of $\operatorname{diag} T \subset T \times T$ acting on the linear span of the respective vectors v'_{ω_i} with linearly independent eigenweights α_i.

We sum up in the following

Lemma 4. *The orbits of $B \times B/U'$ on $\overline{Bv_0}$ are labeled by* type markings *of the Dynkin diagram of G, which assign a type XY, X, Y, Z, or O to each vertex in such a way that:*

1. *the subdiagram consisting of the vertices marked by Y, Z, O (the* YZO-subdiagram*, for brevity) has no components of type A_1, each component of type A_2 has both vertices marked by Y, and the remaining vertices of type Y are isolated from each other;*
2. *each vertex outside the YZO-subdiagram has type X if it is connected with a vertex from the YZO-subdiagram and type XY otherwise.*

For any type marking Γ, let

$$v_\Gamma = (\dots, v_{\omega_i}, v'_{\omega_i - \alpha_i}, \dots, v_{\omega_j}, \dots, v'_{\omega_k - \alpha_k}, \dots, v'_{\omega_l}, \dots)$$

denote the respective orbit representative (where i, j, k, l are of types XY, X, Y, Z, respectively). In order to compute the stabilizers in $\widehat{G}$, it will be convenient to change v_Γ slightly. Namely, let $w_{\mathrm{Y1}} = \prod_m s_m \in W = N_G(T)/T$ be the product of simple reflections over all vertices m of subtype Y1 and put

$$\begin{aligned} v_{\Gamma'} &= w_{\mathrm{Y1}} v_\Gamma = \\ &= (\dots, v_{\omega_i}, v'_{\omega_i - \alpha_i}, \dots, v_{\omega_j}, \dots, v'_{\omega_k - \alpha_k}, v'_{\omega_{k'} - \alpha_{k'}}, \dots, v'_{\omega_m}, \dots, v'_{\omega_l}, \dots), \end{aligned}$$

where k, k' are connected vertices of subtype Y2 and m is of type Y1. (To be more precise, we choose a representative of each s_m in $N_G(T)$ mapping $v'_{\omega_m - \alpha_m}$ to v'_{ω_m}.) In other words, each $\widehat{G}$-orbit contains a representative corresponding to a *reduced type marking* of the Dynkin diagram, by which we mean a type marking with no vertices of subtype Y1.

The set of simple roots of types Y2 and O determines a standard parabolic subgroup $P = P_\Gamma \supseteq B$, which has a Levi decomposition $P = P_{\mathrm{u}} \leftthreetimes L$ with the Levi subgroup $L \supseteq T$ having the roots of types Y2 and O as the simple roots. We decompose the semisimple part of L in a direct product $L' = L'_{\mathrm{Y2}} \times L'_{\mathrm{O}}$, where the factors correspond to the subsets of simple roots of types Y2 and O, respectively.

Proposition 1. *The vectors v_Γ over all reduced type markings Γ constitute a complete set of $\widehat{G}$-orbit representatives on Y.*

Put $H = G_{v_\Gamma}$ and let N be the projection of $\widehat{G}_{v_\Gamma}$ to G. Then there are Levi decompositions $H = H_{\mathrm{u}} \rightthreetimes K$ and $N = P_{\mathrm{u}} \rightthreetimes F$, where $H_{\mathrm{u}} \subseteq P_{\mathrm{u}}$ is normalized by T and its roots are obtained from the set of roots of P_{u} by removing the simple roots of type XY.

The Levi subgroups K and F are also normalized by T. K is semisimple with the root set consisting of all roots of L'_{O} together with $\pm(\alpha_k + \alpha_{k'})$, where k, k' are connected vertices of type Y2. We have $F = K \cdot T_0$, $T_0 \subseteq T$ being the common kernel of all weights of the form $\alpha_1 + \alpha_2 - \alpha_4 - \alpha_5 + \alpha_7 + \alpha_8 - \dots$, where α_i are the simple roots of the components of types A_{3p+2} ($p > 0$) and E_6 in the YZO-subdiagram such that α_i is of type Z unless $3|i$, in the numbering of [OV].

Remark 2. The coefficients in the above linear combination of simple roots are nothing but the coefficients of an (essentially unique) linear dependence between the rows of the matrix $E - A$, where $A = (\langle \alpha_i, \alpha_j^\vee \rangle)_{i,j}$ is the Cartan matrix of a considered component of the YZO-subdiagram and E is the identity matrix. In fact, $E - A$ is degenerate (of corank 1) only for Dynkin diagrams of types A_{3p+2} and E_6.

Proof. The computation of H is quite easy: we just have to intersect the stabilizers of all components of v_Γ, which are maximal quasi-parabolic subgroups in G normalized by T. Note that $N_G(H_{\mathrm{u}})$ is the standard parabolic subgroup of G corresponding to the set of simple roots of types Y2, O, and of type XY which are isolated from other roots of type XY.

The subgroup $N \subseteq N_G(H) \subseteq N_G(H_{\mathrm{u}})$ consists of g such that there exists $\bar{b} \in B/U'$ such that $gv_\Gamma = \bar{b}v_\Gamma$. Recalling the description of the B/U'-action in Lemma 2, it is now easy to see that N is a product of $H \cdot \prod_i U_{\alpha_i}$ (over all i of type XY) $= P_{\mathrm{u}} \cdot K$ and of a certain subgroup in T.

This subgroup consists of $t \in T$ for which there exists $\bar{t} \in T$ such that the weights ω_i, $\omega_i - \alpha_i$, ω_j, $\omega_k - \alpha_k$ have the same values on t and $\bar{t}$ for all i, j, k of types XY, X, Y2, respectively, and $t^{\omega_l} = (\bar{t})^{\omega_l - \alpha_l}$ for all l of type Z. Putting $\bar{t} = ts$, we rewrite these conditions in the following form: $s^{\omega_i} = s^{\omega_i - \alpha_i} = s^{\omega_j} = s^{\omega_k - \alpha_k} = 1$, $s^{\omega_l - \alpha_l} = t^{\alpha_l}$. In other words, s must belong to the standard maximal torus of the standard semisimple Levi subgroup corresponding to the YZO-subdiagram, the roots α_l of type Z must take on t the same values as the weights $\omega_l - \alpha_l$ take on s, and the additional condition $s^{\omega_k - \alpha_k} = 1$ (for k of type Y2) does not affect the previous ones, because the roots of type Y2 are isolated from the roots of types Z and O. This means that the values t^{α_l} must satisfy the same multiplicative integral linear dependencies as $s^{\omega_l - \alpha_l}$ do, i.e., t is in the common kernel of all linear combinations of α_l's with coefficients taken from the linear relations between $\omega_l - \alpha_l$. Hence, by Remark 2, $T = T_0$.

Finally, the vectors v_Γ corresponding to different reduced type markings are not $\widehat{G}$-equivalent, because their stabilizers are not conjugated, which is readily seen from the above description of N and H. □

Proposition 2. *In the notation of Proposition 1, $G/\!\!/H \simeq \overline{Gv_\Gamma} \subseteq Y$.*

Proof. It suffices to prove that $\mathbb{C}[G/H]^U$ is generated by the pullbacks of the U-invariant linear functions on V along the orbit map $G \to V$, $g \mapsto gv_\Gamma$. To this end, we exploit the same geometric idea as for $H = U'$ above.

Put $\widetilde{H} = TH$. We start by noting that $B \times \widetilde{H}$ acts on $\mathbb{C}[G]$ by left and right translation of an argument, and $\mathbb{C}[G/H]^U = \mathbb{C}[G]^{U\times H}$ is preserved by this action and spanned by $(B\times\widetilde{H})$-eigenfunctions. Since G is assumed simply connected, $\mathbb{C}[G]$ is a unique factorization domain, whence $\mathbb{C}[G/H]^U$ is generated, as an algebra, by multiplicatively indecomposable $(B \times \widetilde{H})$-semi-invariant polynomials. Indeed, $B \times \widetilde{H}$ being connected, it cannot permute the prime factors of a semi-invariant polynomial, but preserves them, up to a scalar multiple.

These indecomposable eigenpolynomials are nothing but the equations in $\mathbb{C}[G]$ of the pullbacks of the B-stable prime divisors in $G/\widetilde{H}$ along the orbit map. In a more sophisticated language, these polynomials are identified with the canonical sections of the line bundles over $G/\widetilde{H}$ corresponding to the B-stable prime divisors, see e.g. [Ti2, Rem. 13.4].

But $G/\widetilde{H}$ is a spherical homogeneous space. Hence there are finitely many B-stable prime divisors in $G/\widetilde{H}$ and they are in fact easy to describe. For technical reasons, we rather describe B^--stable prime divisors, where B^- is the opposite Borel subgroup, but this is the same thing, because all Borel subgroups are conjugated.

First of all, since $\widetilde{H} \subset P$, there are divisors pulled back from the Schubert divisors in G/P. The respective polynomials are pullbacks of lowest weight linear functions on $V(\omega_j)$ for all j of types X and XY, and on $V(\omega_l)'$ for all l of type Z. The remaining B-stable prime divisors project dominantly to G/P and intersect the fiber $P/\widetilde{H}$ of $G/\widetilde{H} \to G/P$ in $(B^- \cap L)$-stable prime divisors. The latter divisors either are pullbacks of the $(B^- \cap L)$-stable divisors on $L/\widetilde{K}$ along the natural map $P/\widetilde{H} \to L/\widetilde{K}$, where $\widetilde{K} = TK$, or project dominantly to $L/\widetilde{K}$ and intersect the fiber over a general point, which is isomorphic to $P_{\mathrm{u}}/H_{\mathrm{u}}$, in T-stable prime divisors.

The space $L/\widetilde{K}$ is a product of several copies of $\mathrm{SL}_3/\mathrm{GL}_2$ (over all Y2-components of the YZO-subdiagram). The $(B^- \cap L)$-stable prime divisors on it and their pullbacks are given by the equations pulled back from the lowest weight linear functions on $V(\omega_k)'$ for k of type Y2, cf. [Ti2, Ex. 17.7]. The T-stable prime divisors on $P_{\mathrm{u}}/H_{\mathrm{u}} \simeq \mathfrak{p}_{\mathrm{u}}/\mathfrak{h}_{\mathrm{u}} \simeq \langle e_{\alpha_i} \mid i \text{ of type XY}\rangle$ are just coordinate hyperplanes and the respective polynomials are pullbacks of lowest weight linear functions on $V(\omega_i)'$ for all i of type XY.

We conclude that the equations of all B-stable prime divisors are pulled back from B-semi-invariant linear functions on V, whence the claim. $\square$

Now we are ready to examine the conjecture of Panyushev. Let us observe that the orbits of $\widehat{G}$ in $\widehat{G}v_\Gamma \times X$ are in bijection with the orbits of N in X, in the notation of Proposition 1. Consider first the rank 2 case.

Proposition 3. *If the semisimple part of G is of type B_2 or G_2, then B/U' acts on $X/\!\!/U'$ with finitely many orbits, for any affine spherical G-variety X. If the semisimple part of G is of type A_2, then there exists such an X with infinitely many B/U'-orbits on $X/\!\!/U'$.*

Proof. In the B_2 or G_2 case, it is easily deduced from Proposition 1 that $N \supseteq B$ for any reduced type marking Γ. Therefore N has finitely many orbits in X and $\widehat{G}$ has finitely many orbits in $Y \times X$, whence the claim.

In the A_2 case, there are exactly two $\widehat{G}$-orbits in Y, which are also G-orbits: the open one Gv_0, with the type marking XY for both vertices, and the closed one Gv, with the type marking Y2 for both vertices. For the closed orbit, $H \simeq \mathrm{SL}_2$ is the 3-dimensional simple subgroup corresponding to the highest root and $N \simeq \mathrm{GL}_2$ is its normalizer. It follows by the Transfer Principle that $X/\!\!/U' \simeq (Y \times X)/\!\!/G$ contains a closed subset $(Gv \times X)/\!\!/G \simeq X/\!\!/H$ and the B/U'-action on $X/\!\!/U'$ restricts to the N/H-action on $X/\!\!/H$.

Now suppose that X is factorial and of rank 2 (where the *rank* of X is by definition the rank of the lattice spanned by the highest weights of the irreducible G-modules occurring in $\mathbb{C}[X]$, cf. [Ti2, 5.1, 5.3]). Examples include: $X = \mathrm{SL}_3/\!\!/U$, $\mathrm{SL}_3/\mathrm{SL}_2$, $\mathrm{SL}_3/\mathrm{SO}_3$. It follows from the local structure theorem of Brion–Luna–Vust and Grosshans [Ti2, Thm. 4.7] that $\dim X = \dim B = 5$. By factoriality and since H has no characters, $\operatorname{Quot}\mathbb{C}[X]^H = \mathbb{C}(X)^H$ [PV, Thm. 3.3], whence by the Rosenlicht theorem $\mathbb{C}[X]^H$ separates generic H-orbits and $\dim X/\!\!/H \geq \dim X - \dim H = 2$ [PV, Prop. 3.4]. Clearly, in this case the one-dimensional torus N/H acts on $X/\!\!/H$ with infinitely many orbits. $\square$

The general case is settled by similar arguments.

Theorem 1. *There are finitely many B/U'-orbits in $X/\!\!/U'$ for any affine spherical G-variety X if and only if all simple factors of G are of types $\mathsf{A}_1, \mathsf{B}_2, \mathsf{G}_2$.*

Proof. If all simple factors of G are of types A_1, B_2, or G_2, then, by Proposition 1, $N \supseteq B$ for all reduced type markings of the Dynkin diagram of G. It follows that N has finitely many orbits in X (since B

does) and $\widehat{G} = G \times B/U'$ has finitely many orbits in $Y \times X$ (because there are finitely many $\widehat{G}$-orbits in $Y = G/\!\!/U'$ and their stabilizers have finitely many orbits in X, by the above). Hence $B/\!\!/U'$ has finitely many orbits in $(Y \times X)/\!\!/G \simeq X/\!\!/U'$.

Otherwise the Dynkin diagram of G contains a fragment of type A_2. Consider a type marking Γ in which the two vertices of this fragment are of type Y2 and the other vertices are of types XY or X, according to the rules of Lemma 4. Let us consider the respective subgroups $H \subset N \subset G$. By Proposition 2, $G/\!\!/H$ is embedded in Y as $Z = \overline{Gv_\Gamma}$.

It follows from the Transfer Principle that $X/\!\!/U' \simeq (Y \times X)/\!\!/G$ contains $(Z \times X)/\!\!/G \simeq X/\!\!/H$ as a closed subset and the B/U'-action on $X/\!\!/U'$ restricts to the N/H-action on $X/\!\!/H$.

Now if X is factorial and of rank r (e.g., $X = Y$), then, as in the proof of Proposition 3, $\dim X = \dim B$ and $\mathbb{C}[X]^H$ separates generic H-orbits, so that $\dim X/\!\!/H \geq \dim B - \dim H > \dim N - \dim H$, because $\dim N = \dim B - 1$ in this case. Hence N/H acts on $X/\!\!/H$ with infinitely many orbits and so does B/U' on $X/\!\!/U'$. □

3. Positive results

The above negative answer to Panyushev's conjecture is somewhat disappointing, because interesting applications would rather come from a positive answer. Therefore we are interested in a positive solution of Panyushev's question in particular cases, under some restrictions on X. On the other side, we shall replace U' by a slightly bigger natural class of unipotent subgroups.

Namely, suppose that $H_0 \subseteq U$ is normalized by B, as above, and $\widetilde{H}_0 = TH_0$ is a *spherical* subgroup of G, i.e., $G/\widetilde{H}_0$ contains an open B-orbit. Connected solvable spherical subgroups were explicitly described by R. S. Avdeev [Avd]. In this particular situation, $\widetilde{H}_0$ is spherical if and only if $\mathfrak{h}_0$ is spanned by the e_α corresponding to all positive roots α except for a certain subset of simple roots.

Indeed, B has an open orbit in $G/\widetilde{H}_0$ if and only if $\widetilde{H}_0$ has an open orbit in G/B or, equally, in the big Schubert cell of G/B (where B acts transitively with stabilizer T). The latter holds if and only if T has an open orbit in $U/H_0 \simeq \mathfrak{u}/\mathfrak{h}_0$ under the adjoint action, i.e., $\mathfrak{h}_0$ is spanned by the e_α corresponding to all positive roots α except for a certain linearly independent subset. We shall attribute the roots from this subset to the type A (*active roots*) and the remaining positive roots to the type I (*inactive roots*). Since H_0 is normal in U, the sum of an inactive root with any positive root is still inactive, whenever it is a root. As active roots are linearly independent, the sum of any two active roots is inactive as well. This means that all active roots are simple.

A large amount of what was said above about G/U' extends to G/H_0 as well, with slight modifications in formulations and proofs. Let us be more precise here.

The canonical embedding $G/\!\!/H_0$ is constructed as $\overline{Gv_0} \subseteq V$, where

$$V = \bigoplus_i \bigl(V(\omega_i) \oplus V(\omega_i)'\bigr) \oplus \bigoplus_j V(\omega_j),$$
$$v_0 = (\dots, v_{\omega_i}, v'_{\omega_i - \alpha_i}, \dots, v_{\omega_j}, \dots),$$

and i, resp. j, runs over all numbers of simple roots of type A, resp. I. The action of B/H_0 on $G/\!\!/H_0$ comes from an action of B on V given by the same formulæ as in Lemma 2 on $V(\omega_i) \oplus V(\omega_i)'$ (i of type A), while on $V(\omega_j)$ (j of type I) U acts trivially and T acts by the weight $-\omega_j$. We have $\overline{Gv_0} = G \cdot \overline{Bv_0}$ and $\overline{Bv_0} = \overline{Tv_0} + \mathfrak{u} \cdot v_0$ is the set of all vectors of the form

$$(\dots, x_i v_{\omega_i}, y_i v'_{\omega_i - \alpha_i} + z_i v'_{\omega_i}, \dots, x_j v_{\omega_j}, \dots),$$

where x_k, y_k are bound by binomial equations which define $\overline{Tv_0}$ and z_i are arbitrary. The $\widehat{G}$-orbit representatives v_Γ are given by reduced type markings Γ satisfying the following conditions (cf. Lemma 4):

(1) all vertices of types XY, Y, and Z belong to the type A at the same time;
(2) in the YZO-subdiagram, each isolated vertex belongs to the type I, each component of type A_2 with both vertices of type A has them marked by Y, and there are no other vertices of type Y;
(3) each vertex outside the YZO-subdiagram has type X if it is either of type I, or of type A and connected with a vertex from the YZO-subdiagram, and type XY otherwise.

The description of stabilizers is exactly the same as in Proposition 1. Proposition 2 holds true in this more general situation.

Now we are going to give an affirmative answer to Panyushev's question under some restrictions on X. The restrictions will be imposed on the G-module structure of $\mathbb{C}[X]$. For any dominant weight λ, we consider the subdiagram of the Dynkin diagram whose nodes correspond to the fundamental weights occurring in the decomposition of λ with positive coefficients, and call it the *support* of λ.

Theorem 2. *Suppose that the supports of all highest weights of the irreducible G-modules occurring in $\mathbb{C}[X]$ do not contain subdiagrams of type A_2 with both nodes of type A. Then B/H_0 acts on $X/\!\!/H_0$ with finitely many orbits.*

Proof. Recall that $X/\!\!/H_0 \simeq (G/\!\!/H_0 \times X)/\!\!/G$ is covered by finitely many π_G-images of $\widehat{G}v_\Gamma \times X$ (over all reduced type markings Γ satisfying the conditions (1)–(3) above). Each $\widehat{G}v_\Gamma \times X$ contains a subvariety $\{v_\Gamma\} \times X$ acted on by $\widehat{H} = \widehat{G}_{v_\Gamma}$ via its projection to G, which we denote by N.

We have a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi_H} & X/\!\!/H \\ \| & & \| \\ \{v_\Gamma\} \times X \subseteq \overline{Gv_\Gamma} \times X \simeq G/\!\!/H \times X & \xrightarrow{\pi_G} & (G/\!\!/H \times X)/\!\!/G, \end{array}$$

where $H = G_{v_\Gamma}$. Thus $X/\!\!/H$ is embedded into $X/\!\!/H_0$ as a closed subset. The B/H_0-orbits in $\pi_G(\widehat{G}v_\Gamma \times X)$ intersect $\pi_G(\{v_\Gamma\} \times X)$ in N/H-orbits. Therefore it suffices to prove that there are finitely many N/H-orbits in $\pi_H(X) \subseteq X/\!\!/H$.

We prove the latter by a reduction argument. Retain the notation introduced before Proposition 1 and consider the standard parabolic subgroup $Q \supseteq P$ corresponding to the set of simple roots of types XY, Y2, and O (for the chosen type marking Γ). We have a Levi decomposition $Q = Q_{\mathrm{u}} \rtimes M$, where $M \supseteq L$ has the semisimple part decomposed as $M' = M'_{\mathrm{XY}} \times L'$ with the first factor corresponding to the subset of simple roots of type XY.

Put $Q_0 = Q_{\mathrm{u}} \rtimes L'_{\mathrm{O}}$. It is a subgroup of H normalized by Q and *a fortiori* by N. We have a commutative triangle of N-equivariant maps:

$$\begin{array}{ccc} X & \longrightarrow & X/\!\!/H \\ & \searrow \quad \nearrow & \\ & \widetilde{X} = X/\!\!/Q_0. & \end{array}$$

It suffices to prove that the number of N-orbits in $\widetilde{X}$ is finite.

The variety $\widetilde{X} = X/\!\!/Q_0$ is spherical with respect to the action of $\widetilde{G} = Q/Q_0$ (cf. Example 2). We shall denote the images in $\widetilde{G}$ of various subgroups of Q by putting tildes in the notation. Note that $\widetilde{G} = \widetilde{G}_{\mathrm{XY}} \cdot \widetilde{G}_{\mathrm{Y2}} \cdot \widetilde{T}$, $\widetilde{N} = \widetilde{B}_{\mathrm{XY}} \cdot \widetilde{N}_{\mathrm{Y2}} \cdot \widetilde{T}_0$, and $\widetilde{H} = \widetilde{U}'_{\mathrm{XY}} \cdot \widetilde{H}_{\mathrm{Y2}}$, where $\widetilde{G}_{\mathrm{XY}}, \widetilde{G}_{\mathrm{Y2}}$ are the standard semisimple Levi subgroups corresponding to the subsets of simple roots of types XY, resp. Y2, $\widetilde{B}_{\mathrm{XY}} = \widetilde{B} \cap \widetilde{G}_{\mathrm{XY}}$, $\widetilde{U}'_{\mathrm{XY}} = \widetilde{U}' \cap \widetilde{G}_{\mathrm{XY}}$, $\widetilde{N}_{\mathrm{Y2}}$ is the product of GL_2's in all SL_3-factors of $\widetilde{G}_{\mathrm{Y2}}$, and $\widetilde{H}_{\mathrm{Y2}}$ is the commutator subgroup of $\widetilde{N}_{\mathrm{Y2}}$. In fact, if we disregard the central tori, then $\widetilde{H}, \widetilde{N}$ play the same role for $\widetilde{G}/\!\!/\widetilde{U}'$ as H, N do for $G/\!\!/H_0$. So, in a sense, we reduce the problem to the situation, where all simple roots of G are of types XY and Y2.

However, under our restrictions on the G-module structure of $\mathbb{C}[X]$, $\widetilde{X}$ is spherical even for the action of $\widetilde{S} = \widetilde{G}_{\mathrm{XY}} \cdot \widetilde{N}_{\mathrm{Y2}} \cdot \widetilde{T}_0$. Indeed, each irreducible $\widetilde{G}$-module $\widetilde{V}(\lambda)$ occurring in $\mathbb{C}[\widetilde{X}]$ decomposes in a tensor product

$$\widetilde{V}(\lambda) = \widetilde{V}(\lambda_{\mathrm{XY}}) \otimes \widetilde{V}(\lambda_1) \otimes \cdots \otimes \widetilde{V}(\lambda_s) \otimes \mathbb{C}(\lambda_0),$$

where $\widetilde{V}(\lambda_{\mathrm{XY}})$ is an irreducible $\widetilde{G}_{\mathrm{XY}}$-module, $\widetilde{V}(\lambda_i)$'s are irreducible modules for the simple factors of $\widetilde{G}_{\mathrm{Y2}}$, and $\mathbb{C}(\lambda_0)$ is a one-dimensional representation space of the connected center of $\widetilde{G}$, where it acts via a character λ_0. Under our restrictions on highest weights, the restriction from T to T_0 is injective on the set of λ's and all λ_i are multiples of fundamental weights of SL_3. It is now easy to see that the restriction of the representation of SL_3 in $\widetilde{V}(\lambda_i)$ to GL_2 is multiplicity free and furthermore, the irreducible GL_2-submodules of distinct $\widetilde{V}(\lambda_i)$'s are pairwise non-isomorphic. We conclude that the representation of $\widetilde{S}$ in $\mathbb{C}[\widetilde{X}]$ is multiplicity free, i.e., $\widetilde{X}$ is spherical.

But $\widetilde{N} \cap \widetilde{B}$ is a Borel subgroup in $\widetilde{S}$. Hence even $\widetilde{N} \cap \widetilde{B}$ acts on $\widetilde{X}$ with finitely many orbits, and we are done. □

Example 4. Let $X = \mathrm{SL}_{2n}(\mathbb{C})/\mathrm{Sp}_{2n}(\mathbb{C})$ be the moduli space of unimodular symplectic structures on $\mathbb{C}^{2n}$. It is a symmetric (hence spherical) homogeneous space and it is well known that the highest weights occurring in $\mathbb{C}[X]$ are precisely the non-negative linear combinations of ω_{2i}, $i = 1, \dots, n-1$, see e.g. [Ti2, Prop. 26.22, 26.24, and Table 26.3]. Hence Theorem 2 applies.

Remark 3. One can see from the proof of Theorem 2 that the restrictions on the action of G on X coming from representation theory are not sharp and can be weakened. But I do not see at the moment any sufficiently general formulation for an improved version of the theorem.

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