STRONGLY SOLVABLE SPHERICAL SUBGROUPS AND THEIR COMBINATORIAL INVARIANTS

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Abstract. This is a survey paper devoted to establishing interrelations between the following three combinatorial classifications of strongly solvable spherical subgroups in semisimple complex algebraic groups: Luna's general classification of arbitrary spherical subgroups restricted to the strongly solvable case, Luna's 1993 classification of strongly solvable wonderful subgroups, and the author's 2011 classification of strongly solvable spherical subgroups. We give a detailed presentation of all the three classifications and exhibit interrelations between the corresponding combinatorial invariants, which enables one to pass from one of these classifications to any other.

Introduction

Let G be a connected reductive complex algebraic group and let B be a Borel subgroup of G. A closed subgroup $H \subset G$, as well as the corresponding homogeneous space G/H, is said to be *spherical* if any one of the following equivalent properties (S1)–(S6) is satisfied:

(S1) the group B has an open orbit in G/H;

(S2) there exists an element $g \in G$ such that $\mathfrak{g} = (\operatorname{Ad} g)\mathfrak{h} + \mathfrak{b}$, where $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{h} = \operatorname{Lie} H$, and $\mathfrak{b} = \operatorname{Lie} B$;

(S3) for every homogeneous line bundle L over G/H, the representation of G on the space of regular sections of L is multiplicity free;

(S4) for every simple G-module V and every character χ of H, the subspace $V_\chi^{(H)} \subset V$ of H-semi-invariant vectors of weight χ is at most one-dimensional;

(S5) every normal irreducible G-variety containing G/H as an open orbit has finitely many G-orbits;

(S6) every normal irreducible G-variety containing G/H as an open orbit has finitely many B-orbits.

Property (S2) is just a reformulation of (S1) in terms of tangent algebras. The equivalence (S1)⇔(S3) was proved in [VinK]. The equivalence (S3)⇔(S4) follows from Frobenius reciprocity, see, for instance, [Tim, Corollary 2.13]. The proof of (S3)⇔(S5) is the subject

Date: 6th February 2013.

2010 *Mathematics Subject Classification.* 14M27, 14M17, 05E15.

Key words and phrases. Algebraic group, homogeneous space, representation, spherical variety, wonderful variety, spherical subgroup, solvable subgroup.

Partially supported by Russian Foundation for Basic Research, grant no. 12-01-00704. A considerable part of this paper was written during the author's stay at the University of Bielefeld (Germany) in June and July, 2012 supported by the Alexander von Humboldt Foundation and SFB 701. The work on this paper was completed during the author's stay at the University of Bochum (Germany) in November and December, 2012 supported by DFG priority program 1388 (Representation theory). The hospitality of both universities is gratefully acknowledged.

of the paper [Akh2]. The implication (S1)⇒(S6) was proved in [Vin] and, independently, in [Bri1], and the implication (S6)⇒(S5) is trivial.

We recall that, for an arbitrary homogeneous space G/H, every normal irreducible G-variety containing G/H as an open orbit is said to be an *equivariant embedding* (or simply an *embedding*) of G/H. Equivariant embeddings of spherical homogeneous spaces are said to be *spherical varieties*. Among arbitrary normal irreducible G-varieties, spherical G-varieties are characterized by the existence of an open orbit for the induced action of B.

Spherical varieties form an extremely interesting class of G-varieties. The most famous representatives of this class are toric varieties, symmetric varieties, and flag varieties. Countless papers are devoted to various aspects concerned with spherical varieties or certain subclasses of them. Some of these aspects are reflected in the monograph by Timashev, see [Tim, Chapter 5]. In this context we also mention the recent survey paper [Per] devoted to the geometry of spherical varieties.

One of the important problems in the theory of spherical varieties is the problem of classifying them. The spherical G-varieties with a given open G-orbit, that is, the embeddings of a given spherical homogeneous space, were classified by Luna and Vust in the framework of their general theory of embeddings of arbitrary homogeneous spaces developed in [LunV]. Later, in the particular case of spherical varieties this theory was considerably simplified and restated in a more transparent form by Knop [Kno1]. As a result, the embeddings of a given spherical homogeneous space are classified in terms of certain objects of convex geometry called colored fans, which generalize usual fans used for classifying toric varieties. Thus the classification of spherical varieties reduces to the classification of spherical homogeneous spaces, which amounts to the classification of spherical subgroups of reductive algebraic groups.

In classifying spherical homogeneous spaces G/H (or, equivalently, spherical subgroups $H \subset G$) one may restrict the consideration to the case where G is semisimple. Indeed, let Z denote the center of G. Then, making use of either property (S1) or (S2) it is easy to check that G/H is spherical if and only if the homogeneous space $(G/Z)/(H/(Z \cap H)) = (G/Z)/(ZH/Z) \simeq G/(ZH)$ is such.

For semisimple G, a complete classification of *affine* spherical homogeneous spaces G/H (that is, with reductive H) was obtained by Krämer [Krä] (the case of simple G), Mikityuk [Mik], Brion [Bri2] (both treated independently the case of non-simple semisimple G), and Yakimova [Yak] (small corrections). We note that this classification is essentially given by a list of spaces and does not involve any combinatorial invariants.

At present, there is a complete classification in combinatorial terms of spherical homogeneous spaces G/H, which is a result of joint decade-long efforts of several researchers. The idea of this classification was proposed by Luna in 2001 [Lun4], therefore we shall refer to it as Luna's general classification. This classification is carried out in two steps. The first one is to reduce the classification to the case of so-called wonderful subgroups. A spherical subgroup $H \subset G$ is said to be *wonderful* if G/H admits a wonderful embedding, which is a smooth complete embedding with certain additional properties (see the exact definition in § 2.2). Wonderful embeddings of spherical homogeneous spaces G/H are also known as wonderful G-varieties. The second step of the classification is to describe all the wonderful G-varieties. In his 2001 paper [Lun4], Luna himself performed the first step in the full generality, stated a conjecture for the second step, and proved this conjecture in the case where G is a product of simple groups of type A. Luna's conjecture claimed that wonderful G-varieties are classified by combinatorial objects called *spherical systems*. During the following several years the conjecture was proved in certain other particular cases, see [BraP1], [Bra1], and [BraC]. The uniqueness part of Luna's conjecture was proved by Losev [Los] in 2009 by a general argument. The first general proof of the existence part of this conjecture was obtained by Cupit-Foutou [Cup] via invariant Hilbert schemes. Another proof of the existence part, which follows the lines of the original constructive approach employed by Luna, was recently suggested by Bravi and Pezzini in the series of preprints [BraP2], [BraP3], and [BraP4]. Let us also mention the paper [BraL], which is an introduction to wonderful varieties with an emphasis put on combinatorics of spherical systems.

The combinatorial invariants involved in Luna's general classification are called the *principal combinatorial invariants* in this paper. These invariants come from the Luna–Vust theory mentioned above: the description of all embeddings of a fixed spherical homogeneous space is given in terms of these invariants. A detailed discussion of the principal combinatorial invariants, including definitions and basic properties, can be found in § 1.1.

A subgroup $H \subset G$ is said to be *strongly solvable* if it is contained in a Borel subgroup of G. We note that every connected solvable subgroup of G is automatically strongly solvable. An example of a solvable but not strongly solvable subgroup is given by the normalizer of a maximal torus in SL_2.

Apart from Luna's general classification of arbitrary spherical subgroups, there are two different combinatorial classifications in the case of strongly solvable spherical subgroups. Both of them reduce neither to each other nor to Luna's general classification. The first one was obtained by Luna in his unpublished 1993 preprint [Lun1]. This classification applies only to the case of wonderful strongly solvable subgroups. The approach used here goes back to Demazure's 1970 description of automorphism groups of smooth complete toric varieties [Dem]. In what follows we shall refer to this classification as Luna's 1993 classification. The second classification was recently obtained by the author [Avd2] and deals with connected solvable subgroups. In fact, the connectedness condition turns out to be inessential in this classification, so that it extends almost unchanged to the case of arbitrary strongly solvable subgroups. The main idea employed here is to classify the Lie algebras of subgroups under consideration, which is done in terms of certain root data.

Both Luna's general classification and Luna's 1993 classification have a geometric origin: the invariants involved arise from the geometry of the corresponding homogeneous spaces. As a result, both classifications provide no simple method for relating the invariants with an explicit structure of the corresponding subgroups, which leaves the following two problems unsolved:

(P1) compute the invariants of a given subgroup (specified, for instance, by a Levi subgroup of it together with the Lie algebra of its unipotent radical);

(P2) determine the subgroup corresponding to a given set of invariants.

In contrast to both Luna's classifications, the author's 2011 classification is much more algebraic: the invariants involved encode an explicit structure of Lie algebras of the subgroups under consideration, so that these subgroups can be easily recovered from the corresponding invariants. For this reason, in what follows this classification will be referred to as the explicit classification.

The main goal of this paper is to give a detailed presentation of all the three above-mentioned combinatorial classifications (Luna's general classification of arbitrary spherical subgroups, Luna's 1993 classification of strongly solvable wonderful subgroups, and the explicit classification of strongly solvable spherical subgroups) and reveal interrelations between the combinatorial invariants corresponding to them. Specifically, for every two classifications under consideration we provide either explicit formulas or an effective method for computing the invariants involved in the first one starting from the invariants involved in the second one. Of course, the latter is possible in the situation where both classifications apply, that is, for strongly solvable spherical subgroups or for strongly solvable wonderful subgroups. The only new results of this paper are contained in § 4 and concerned with establishing interrelations between the explicit classification and the other two classifications under consideration. In particular, these results solve problems (P1) and (P2) in the case of strongly solvable spherical subgroups.

A special attention is paid in this paper to Luna's 1993 classification of wonderful strongly solvable subgroups. At the moment, this classification can be found only in Luna's unpublished preprint [Lun1], which is extremely hard to access. Few references to this preprint existing in the literature give no idea on the employed approach and the classification itself. Thus the classification now seems to be almost forgotten. Being sure that it does not deserve such a fate, in this paper we make an attempt to bring this classification back to life. As was already mentioned above, Luna's 1993 classification is based on the description of automorphism groups of smooth complete toric varieties obtained by Demazure in 1970 [Dem]. To present the classification, we use a much more transparent version of this description obtained by Cox in 1995 [Cox] via his realization of toric varieties as quotients of vector spaces by actions of certain diagonalizable groups. In contrast to Luna's preprint, which is written in a rather sketchy style, in our presentation of the classification we tried to give as much details as possible providing complete proofs for all statements.

Along with the invariants involved in the three classifications in question, in this paper we consider one more invariant of arbitrary spherical homogeneous spaces, called the extended weight semigroup (see its definition in § 1.2). The term "extended weight semigroup" was introduced in the recent paper [Avd1] though the semigroup itself appeared implicitly many times in earlier papers of different authors. This semigroup is closely related to the principal combinatorial invariants. Namely, it turns out that, knowing the extended weight semigroup, one can compute all but one principal combinatorial invariants, and in the strongly solvable case this semigroup determines all the principal combinatorial invariants. On the other hand, the extended weight semigroup is recovered from the principal combinatorial invariants. A systematic study of the interrelations between the extended weight semigroup and the principal combinatorial invariants is undertaken in § 1.3.

The significance of the extended weight semigroup in this paper becomes apparent in establishing interrelations between Luna's general classification and the explicit classification. Thanks to the paper [AvdG], extended weight semigroups are computed for all spherical homogeneous spaces with strongly solvable stabilizer in terms of the combinatorial invariants involved in the explicit classification. In view of what we have said in the previous paragraph, this enables one to pass between the explicit classification and Luna's general classification. In this situation, the extended weight semigroup plays the role of an intermediate invariant between the invariants involved in the two classifications.

The paper is organized as follows.

In § 1 we introduce the principal combinatorial invariants and the extended weight semigroup of a spherical homogeneous space and then carry out a detailed study of interrelations between these invariants.

The main goal of § 2 is to present Luna's general classification of spherical homogeneous spaces. To this end, we introduce the important notion of a wonderful G-variety and explain how the classification reduces to that of wonderful G-varieties. Apart from precise description of combinatorial objects involved in Luna's general classification, we also introduce several related notions needed later in this paper. At last, we state and prove a criterion for a wonderful subgroup to be strongly solvable in terms of its combinatorial invariants.

In § 3 we present Luna's 1993 classification of strongly solvable wonderful subgroups. We first show how this classification reduces to that of so-called wonderful B^--varieties, where B^- is the Borel subgroup of G opposite to B with respect to a fixed maximal torus $T \subset B$. Next, we provide a detailed description of automorphism groups of smooth complete toric varieties and then apply it to obtain a classification of wonderful B^--varieties, which also implies a classification of strongly solvable wonderful subgroups. Finally, we find out how the invariants involved in Luna's 1993 classification are related to the invariants involved in Luna's general classification.

The explicit classification of strongly solvable spherical subgroups is presented in § 4. We begin with an outline of main ideas employed in this classification and then state the classification itself. After that, we establish interrelations between the explicit classification and two Luna's classifications. These interrelations solve problems (P1) and (P2) in the case of strongly solvable spherical subgroups.

Section 5 is illustrative. Here, we list all wonderful strongly solvable subgroups in all semisimple groups of rank at most 2 and also in semisimple groups of type A_3. For every such subgroup, we indicate its invariants with respect to all the three classifications under consideration.

In Appendix, we recall the construction and main properties of homogeneous bundles, which play an important role in § 3.

Acknowledgements. The author cordially thanks D. Luna for discussions and private notes that helped the author to learn much about Luna's general classification of spherical homogeneous spaces. Thanks are also due to I. V. Arzhantsev, P. Bravi, S. Cupit-Foutou, D. A. Timashev, E. B. Vinberg, and V. S. Zhgoon for helpful discussions on particular topics.

Some notation and conventions.

In this paper the base field is the field $\mathbb{C}$ of complex numbers. All topological terms relate to the Zarisky topology. All groups are assumed to be algebraic and their subgroups closed. The Lie algebras of groups denoted by capital Latin letters are denoted by the corresponding small Gothic letters.

$\mathbb{Z}^+$ is the set of non-negative integers;

$\mathbb{Q}^+$ is the set of non-negative elements in $\mathbb{Q}$;

$\mathbb{C}^\times$ is the multiplicative group of the field $\mathbb{C}$;

$\langle \cdot\,, \cdot \rangle$ is the natural pairing between $\operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Q})$ and L, where L is a lattice;

e is the identity element of an arbitrary group;
o is the base point of any homogeneous space L/K, $o = eK$
$|E|$ is the cardinality of a finite set E;
V^* is the space of linear functions on a vector space V;
K^0 is the connected component of the identity of a group K;
$K^\sharp$ is the common kernel of all characters of K;
(K, K) is the derived subgroup of a group K;
$\mathfrak{X}(K)$ is the character lattice (in additive notation) of a group K;
$N_L(K)$ is the normalizer of a subgroup K in a group L;
$\mathbb{C}[X]$ is the algebra of regular functions on an algebraic variety X;
$\mathbb{C}(X)$ is the field of rational functions on an algebraic variety X;
$\operatorname{Quot} A$ is the field of quotients of an algebra A with no zero divisors
G is a connected reductive algebraic group;
$B \subset G$ is a fixed Borel subgroup of G;
$T \subset B$ is a fixed maximal torus of G;
$U \subset B$ is the unipotent radical of B;
$B^- \subset G$ is the Borel subgroup opposite to B with respect to T, that is, $B \cap B^- = T$;
$(\cdot\,,\cdot)$ is a fixed inner product on $\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ invariant with respect to the Weyl group $N_G(T)/T$;
$\Delta \subset \mathfrak{X}(T)$ is the root system of G with respect to T;
$\Delta^+ \subset \Delta$ is the subset of positive roots with respect to B;
$\Pi \subset \Delta^+$ is the set of simple roots;
$\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the root subspace corresponding to a root $\alpha \in \Delta$;
$e_\alpha \in \mathfrak{g}_\alpha$ is a fixed non-zero element;
$\alpha^\vee \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z})$ is the dual root corresponding to a root $\alpha \in \Delta$;
$P_{\Pi'} \supset B$ is the parabolic subgroup of G whose Lie algebra is the direct sum of $\mathfrak{b}$ and all root subspaces $\mathfrak{g}_{-\alpha}$ with $\alpha \in \Delta^+$ and $\operatorname{Supp} \alpha \subset \Pi'$;
$P^-_{\Pi'} \supset B^-$ is the parabolic subgroup of G whose Lie algebra is the direct sum of $\mathfrak{b}^-$ and all root subspaces $\mathfrak{g}_\alpha$ with $\alpha \in \Delta^+$ and $\operatorname{Supp} \alpha \subset \Pi'$;
$\mathfrak{X}_+(B) \subset \mathfrak{X}(B)$ is the set of dominant weights of B;
$V(\lambda)$ is the simple G-module with highest weight $\lambda \in \mathfrak{X}_+(B)$;
λ^* is the highest weight of the simple G-module $V(\lambda)^*$, so that $V(\lambda^*) \simeq V(\lambda)^*$;
$\varpi_\alpha \in \mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the fundamental weight associated with a simple root α.

The lattices $\mathfrak{X}(B)$ and $\mathfrak{X}(T)$ are identified via the restriction of characters from B to T.

If A is an algebra equipped with an action of a group L, then the notation A^L stands for the algebra of L-invariants and, for every $\chi \in \mathfrak{X}(L)$, the notation $A^{(L)}_\chi$ stands for the subspace of L-semi-invariants of weight χ.

The actions of G on itself by left translation ($(g, x) \mapsto gx$) and right translation ($(g, x) \mapsto xg^{-1}$) induce its representations on the space $\mathbb{C}[G]$ of regular functions on G by the formulae $(gf)(x) = f(g^{-1}x)$ and $(gf)(x) = f(xg)$, respectively. For brevity, we refer to these actions as the action *on the left* and *on the right*, respectively. Unless otherwise stated, for every subgroup $L \subset G$ the notation $\mathbb{C}[G]^L$ (resp. $\mathbb{C}[G]^{(L)}_\chi$) stands for the L-invariants (resp. L-semi-invariants of weight χ) with respect to the action of L on the right.

For every element $\gamma = \sum\limits_{\alpha\in\Pi} k_\alpha\alpha \in \mathbb{Q}^+\Pi$, we set $\operatorname{Supp}\gamma = \{\alpha \mid k_\alpha > 0\}$. If moreover $\gamma \in \Delta^+$, then we set $\operatorname{ht}\gamma = \sum\limits_{\alpha\in\Pi} k_\alpha$.

For every weight $\lambda \in \mathfrak{X}_+(B)$, one has $\lambda = \sum\limits_{\alpha\in\Pi} l_\alpha\varpi_\alpha$ for some non-negative integers l_α. We set *supp* $\lambda = \{\varpi_\alpha \mid l_\alpha > 0\}$.

Let K be a group and let K_1, K_2 be subgroups of it. We write $K = K_1 \leftthreetimes K_2$ if K is a semidirect product of K_1, K_2, that is, $K = K_1K_2$, $K_1 \cap K_2 = \{e\}$, and K_2 is a normal subgroup in K.

For connected Dynkin diagrams, the numbering of nodes (that is, simple roots) is the same as in [OniV].

Let Q be a finite-dimensional vector space over $\mathbb{Q}$.

A *cone* in Q is a subset $\mathcal{C} \subset Q$ that is invariant under addition and multiplication by elements in $\mathbb{Q}^+$, that is, $q_1x_1 + q_2x_2 \in \mathcal{C}$ whenever $x_1, x_2 \in \mathcal{C}$ and $q_1, q_2 \in \mathbb{Q}^+$.

A cone $\mathcal{C} \subset Q$ is said to be *finitely generated* if there are finitely many elements $v_1, \ldots, v_s \in Q$ with $\mathcal{C} = \mathbb{Q}^+v_1 + \ldots + \mathbb{Q}^+v_s$. All cones considered in this paper are finitely generated.

A cone $\mathcal{C} \subset Q$ is said to be *strictly convex* if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

The *dimension* of a cone is the dimension of its linear span.

The *dual cone* of a cone $\mathcal{C} \subset Q$ is the cone

$$\mathcal{C}^\vee = \{\xi \in Q^* \mid \xi(v) \geqslant 0 \text{ for all } v \in \mathcal{C}\}.$$

One always has $(\mathcal{C}^\vee)^\vee = \mathcal{C}$.

A *face* of a cone $\mathcal{C} \subset Q$ is a subset of the form $\mathcal{C} \cap \{v \in Q \mid \xi(v) = 0\}$ for some $\xi \in \mathcal{C}^\vee$.

A *facet* of a cone $\mathcal{C} \subset Q$ is a face of codimension 1.

A *fan* in Q is a collection $\mathcal{F}$ of cones in Q satisfying the two axioms below:

(1) if $\mathcal{C} \in \mathcal{F}$, then every face of $\mathcal{C}$ also belongs to $\mathcal{F}$;

(2) if $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{F}$, then $\mathcal{C}_1 \cap \mathcal{C}_2$ is again a cone in $\mathcal{F}$.

A fan $\mathcal{F}$ in Q is said to be *strictly convex* if every cone in $\mathcal{F}$ is strictly convex.

A fan $\mathcal{F}$ in Q is said to be *complete* if $Q = \bigcup\limits_{\mathcal{C}\in\mathcal{F}} \mathcal{C}$.

Let $L \subset Q$ be a fixed sublattice of maximal rank, so that $Q = L \otimes_{\mathbb{Z}} \mathbb{Q}$. For every cone $\mathcal{C} \subset Q$, let $\mathcal{C}^1$ denote the set of primitive elements v of L such that $\mathbb{Q}^+v$ is a face of $\mathcal{C}$. Similarly, for every fan $\mathcal{F}$ in Q, let $\mathcal{F}^1$ denote the set of primitive elements v of L such that $\mathbb{Q}^+v$ is a cone in $\mathcal{F}$. Clearly, $\mathcal{F}^1 = \bigcup\limits_{\mathcal{C}\in\mathcal{F}} \mathcal{C}^1$. In this paper we shall often find ourselves in the situation where $Q = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q})$ and $L = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ for some lattice M.

Under the assumptions of the previous paragraph, a cone $\mathcal{C} \subset Q$ is said to be *regular* if the set $\mathcal{C}^1$ is a part of a basis of L. A fan $\mathcal{F}$ in Q is said to be *regular* if every cone in $\mathcal{F}$ is regular.

1. Combinatorial invariants of spherical homogeneous spaces

In this section, the group G is assumed to be semisimple in §§ 1.2–1.3.

1.1. **Principal combinatorial invariants.** Let G/H be a spherical homogeneous space.

Let $P = P_{G/H}$ be the stabilizer of the open B-orbit in G/H. Evidently, P is a parabolic subgroup of G containing B, so that $P = P_{\Pi^p}$ for some subset $\Pi^p \subset \Pi$. The set $\Pi^p = \Pi^p_{G/H}$ is the first invariant associated with G/H.

The second invariant associated with G/H is the *weight lattice* $\Lambda = \Lambda_{G/H}$. It is defined to be the lattice of B-weights of B-semi-invariant rational functions on G/H:

$$\Lambda = \{\mu \in \mathfrak{X}(T) \mid \mathbb{C}(G/H)^{(B)}_\mu \neq \{0\}\}.$$

We also introduce the rational vector space $\mathcal{Q} = \mathcal{Q}_{G/H} = \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$.

For every $\mu \in \Lambda$, one has $\dim \mathbb{C}(G/H)^{(B)}_\mu = 1$ since G/H is spherical. Fix a non-zero function f_μ in each of the subspaces $\mathbb{C}(G/H)^{(B)}_\mu$.

Let $\mathcal{V} = \mathcal{V}_{G/H}$ denote the set of all discrete $\mathbb{Q}$-valued G-invariant valuations of the field $\mathbb{C}(G/H)$ vanishing on $\mathbb{C}^\times$. We define a map $\phi\colon \mathcal{V} \to \mathcal{Q}$ by the formula

$$\langle \phi(v), \mu\rangle = v(f_\mu),$$

where $v \in \mathcal{V}$, $\mu \in \Lambda$. It is known that the map ϕ is injective (see [LunV, Proposition 7.4] or [Kno1, Corollary 1.8]) and its image is a finitely generated convex cone containing the image in $\mathcal{Q}$ of the antidominant Weyl chamber (see [BriP, Corollary 3.2] or [Kno1, Corollary 5.3]). This cone is called the *valuation cone* of G/H. Later on, we identify it with $\mathcal{V}$. We note that $\mathcal{V}$ spans $\mathcal{Q}$ as a vector space.

Let $\Sigma = \Sigma_{G/H}$ be the set of primitive elements ρ of Λ with the following properties:
(1) $\operatorname{Ker}\rho \subset \mathcal{Q}$ contains a facet of $\mathcal{V}$;
(2) $\langle \rho, \mathcal{V}\rangle \leqslant 0$.
The elements in Σ are called the *spherical roots* of G/H. Brion proved that Σ is the set of simple roots of a root system in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, see [Bri4, § 3]. In particular, the set Σ is linearly independent. We note that the valuation cone $\mathcal{V}$ and the set of spherical roots Σ uniquely determine each other whenever the weight lattice Λ is known. The set Σ is the third invariant associated with G/H.

The fourth invariant associated with G/H is the set $\mathcal{D} = \mathcal{D}_{G/H}$ of B-stable prime divisors in G/H. The elements in $\mathcal{D}$ are called the *colors* of G/H. The set $\mathcal{D}$ is considered together with a map $\varkappa = \varkappa_{G/H}\colon \mathcal{D} \to \mathcal{Q}$ defined in the following way. For a color $D \in \mathcal{D}$, one has $\langle \varkappa(D), \mu\rangle = \operatorname{ord}_D(f_\mu)$ for all $\mu \in \Lambda$, where $\operatorname{ord}_D(f_\mu)$ is the order of f_μ along D.

For every $\alpha \in \Pi$, we regard the corresponding minimal parabolic subgroup $P_{\{\alpha\}} \supset B$ and put $\mathcal{D}(\alpha)$ to be the set of $P_{\{\alpha\}}$-unstable colors.

Proposition 1.1 ([Lun3, §§ 2.7, 3.4])**.** (a) *One has* $\mathcal{D} = \bigcup\limits_{\alpha\in\Pi} \mathcal{D}(\alpha)$.

(b) *Every* $\alpha \in \Pi$ *belong to exactly one of the following types:*
type p: $\mathcal{D}(\alpha) = \varnothing$;
type a: $\alpha \in \Sigma$, $\mathcal{D}(\alpha) = \{D^+, D^-\}$, *and* $\varkappa(D^+) + \varkappa(D^-) = \alpha^\vee|_\Lambda$;
type a′: $2\alpha \in \Sigma$, $\mathcal{D}(\alpha) = \{D\}$, *and* $\varkappa(D) = \frac{1}{2}\,\alpha^\vee|_\Lambda$;
type b: $\mathbb{Q}\alpha \cap \Sigma = \varnothing$, $\mathcal{D}(\alpha) = \{D\}$, *and* $\varkappa(D) = \alpha^\vee|_\Lambda$.

It can be easily seen that the union of all roots $\alpha \in \Pi$ with $\mathcal{D}(\alpha) = \varnothing$ is exactly the set Π^p. In accordance with Proposition 1.1, we denote by $\mathcal{D}^a$ (resp. $\mathcal{D}^{a'}$, $\mathcal{D}^b$) the union of the sets $\mathcal{D}(\alpha)$ over all $\alpha \in \Pi$ such that $\alpha \in \Sigma$ (resp. $2\alpha \in \Sigma$, $\mathbb{Q}\alpha \cap \Sigma = \varnothing$). It turns out (see [Lun3, 2.7] or [Tim, 30.10]) that there is the following disjoint union:

$$\mathcal{D} = \mathcal{D}^a \cup \mathcal{D}^{a'} \cup \mathcal{D}^b. \tag{1.1}$$

For two spherical homogeneous spaces G/H and G/H', we write $\mathcal{D}_{G/H} = \mathcal{D}_{G/H'}$ if there exists a bijection $i\colon \mathcal{D}_{G/H} \to \mathcal{D}_{G/H'}$ such that $\varkappa_{G/H} = \varkappa_{G/H'} \circ i$.

The following theorem is just a reformulation of [Los, Theorem 1].

Theorem 1.2. *The spherical homogeneous space G/H is uniquely determined by the quadruple $(\Lambda, \Pi^p, \Sigma, \mathcal{D})$. In other words, if G/H' is another spherical homogeneous space such that $\Lambda_{G/H} = \Lambda_{G/H'}$, $\Pi^p_{G/H} = \Pi^p_{G/H'}$, $\Sigma_{G/H} = \Sigma_{G/H'}$, and $\mathcal{D}_{G/H} = \mathcal{D}_{G/H'}$, then the subgroups H and H' are conjugate in G.*

A classification of all quadruples $(\Lambda, \Pi^p, \Sigma, \mathcal{D})$ arising from spherical homogeneous spaces is given in § 2.4.

1.2. **Extended weight semigroup.** Basic references for this subsection are [Avd1, §§ 1.2, 1.3] and [AvdG, §§ 1.2, 1.3].

First assume $H \subset G$ to be an arbitrary subgroup. We recall that characters of H are in one-to-one correspondence with homogeneous line bundles (see [Pop, Theorem 4]). Namely, for an arbitrary character $\chi \in \mathfrak{X}(H)$ the corresponding homogeneous line bundle is defined as $L(\chi) = (G \times \mathbb{C}_\chi)/H$, where H acts on G by right translation and on $\mathbb{C}_\chi \simeq \mathbb{C}$ by the character χ. We note that $L(\chi)$ is also denoted by $G *_H \mathbb{C}_\chi$ (see Appendix A). Let $\Gamma(L(\chi))$ denote the space of regular sections of $L(\chi)$. For every $\chi \in \mathfrak{X}(H)$ there is a natural isomorphism

$$\Gamma(L(-\chi)) \simeq \mathbb{C}[G]^{(H)}_\chi. \tag{1.2}$$

For every $\lambda \in \mathfrak{X}_+(B)$, let $\mathbb{C}[G]^{(B\times H)}_{(\lambda,\chi)} \subset \mathbb{C}[G]^{(H)}_\chi$ be the subspace of $(B \times H)$-semi-invariant functions in $\mathbb{C}[G]$ of weight (λ, χ), where the semi-invariants are taken with respect to the action of B on the left and the action of H on the right. Every non-zero function $f \in \mathbb{C}[G]^{(B\times H)}_{(\lambda,\chi)}$ is a highest-weight vector of a simple G-submodule of $\mathbb{C}[G]^{(H)}_\chi$ with highest weight λ, and vice versa. Let $\widehat{\Lambda}^+ = \widehat{\Lambda}^+_{G/H}$ denote the set of all pairs (λ, χ), where $\lambda \in \mathfrak{X}_+(B)$ and $\chi \in \mathfrak{X}(H)$, such that $\dim \mathbb{C}[G]^{(B\times H)}_{(\lambda,\chi)} \ne 0$, that is, the G-module $\mathbb{C}[G]^{(H)}_\chi \simeq \Gamma(L(-\chi))$ contains the simple G-module $V(\lambda)$. The set $\widehat{\Lambda}^+$ is said to be the *extended weight semigroup* of G/H.

We recall the following isomorphism of $(G \times G)$-modules (see, for instance, [Tim, Theorem 2.15]):

$$\mathbb{C}[G] \simeq \bigoplus_{\lambda \in \mathfrak{X}_+(B)} V(\lambda^*) \otimes V(\lambda), \tag{1.3}$$

where in the left-hand side $G \times G$ acts on the left and on the right, and in the right-hand side the first (resp. second) factor of $G \times G$ acts on the first (resp. second) tensor factor. For a fixed $\lambda \in \mathfrak{X}_+(B)$, the embedding $V(\lambda^*) \otimes V(\lambda) \hookrightarrow \mathbb{C}[G]$ is defined as follows. For $u \in V(\lambda^*)$ and $v \in V(\lambda)$, $u \otimes v$ maps to the function whose value at a point $g \in G$ is $\langle u, gv\rangle$, where $\langle \cdot\,, \cdot\rangle$ is the natural pairing between $V(\lambda^*)$ and $V(\lambda)$. Under isomorphism (1.3), the subspace $\mathbb{C}[G]^{(B\times H)}_{(\lambda^*,\chi)} \subset \mathbb{C}[G]$ corresponds to the subspace $V(\lambda^*)^{(B)}_{\lambda^*} \otimes V(\lambda)^{(H)}_\chi = \langle v_{\lambda^*}\rangle \otimes V(\lambda)^{(H)}_\chi \subset V(\lambda^*) \otimes V(\lambda)$, where v_{λ^*} is a highest-weight vector in $V(\lambda^*)$. Hence $\dim \mathbb{C}[G]^{(B\times H)}_{(\lambda^*,\chi)} = \dim V(\lambda)^{(H)}_\chi$ and $(\lambda^*, \chi) \in \widehat{\Lambda}^+$ if and only if $V(\lambda)^{(H)}_\chi \ne 0$.

Now assume that H is spherical in G. It is well known (see [VinK, Theorem 1]) that the sphericity of H is equivalent to the property that the representation of G on $\Gamma(L(-\chi)) \simeq \mathbb{C}[G]^{(H)}_{\chi}$ is multiplicity free for every $\chi \in \mathfrak{X}(H)$. Hence $(\lambda^*, \chi) \in \widehat{\Lambda}^+$ if and only if $\dim \mathbb{C}[G]^{(B\times H)}_{(\lambda,\chi)} = \dim V(\lambda^*)^{(H)}_{\chi} = 1$. In case of G simply connected, the semigroup $\widehat{\Lambda}^+$ is free and is isomorphic to the semigroup of effective B-stable divisors in G/H, which is freely generated by colors of G/H (see [AvdG, Theorem 1]). Under this isomorphism, a color D of G/H corresponds to an indecomposable element (λ_D, χ_D) of $\widehat{\Lambda}^+$ such that D is the divisor of a (unique up to proportionality) regular section s_D of $L(-\chi_D)$ that is B-semi-invariant of weight λ_D.

1.3. **Relations between the extended weight semigroup and the principal combinatorial invariants.** For every group K, let $K^\sharp$ denote the subgroup of K defined to be the common kernel of all characters of K. Then $K/K^\sharp$ is a diagonalizable group and there is a natural isomorphism $\mathfrak{X}(K) \simeq \mathfrak{X}(K/K^\sharp)$.

We recall that a subgroup L_0 of a group L is said to be *observable* in L if the homogeneous space L/L_0 is quasi-affine. The following lemma is well known, but for convenience of the reader we provide a proof of it.

Lemma 1.3. *For arbitrary groups $K \subset L$, the subgroup $K^\sharp$ is observable in L.*

Proof. By a theorem of Chevalley (see [Hum, Theorem 11.2]) there exist a finite-dimensional L-module V and a non-zero vector $v \in V$ such that K is the stabilizer of the line $\mathbb{C}v$. Let $K_v \subset K$ be the stabilizer of v. Evidently, $K_v \supset K^\sharp$ and K_v is observable in L. Let $\chi_1, \ldots, \chi_k$ be generators of the group $\mathfrak{X}(K)$. For $i = 1, \ldots, k$ denote by V_i the one-dimensional K-module on which K acts by the character χ_i. In each of the spaces V_i fix a non-zero vector v_i. Then $K^\sharp$ is the stabilizer of the vector $v_1 + \ldots + v_k$ in the K-module $V_1 \oplus \ldots \oplus V_k$, which is also a K_v-module. Hence $K^\sharp$ is observable in K_v. As K_v is observable in L, it follows that $K^\sharp$ is observable in L (see [BHM, § 5] or [Gro, Corollary 2.3]). □

We now turn to the situation where G is a connected reductive group and $H \subset G$ is an arbitrary subgroup.

Lemma 1.4. *The group $H^\sharp$ is the common stabilizer in G of all $H^\sharp$-fixed vectors in all simple G-modules.*

Proof. In view of Lemma 1.3 the group $H^\sharp$ is observable in G, so by [Gro, Theorem 2.1] there exist a finite-dimensional G-module V and a vector $v \in V$ such that $H^\sharp$ is the stabilizer of v in G. Hence $H^\sharp$ is the common stabilizer in G of all $H^\sharp$-fixed points in V. Since G is reductive, the G-module V is completely reducible, which implies that $H^\sharp$ is the common stabilizer in G of all $H^\sharp$-fixed points in all simple G-modules contained in V, whence the required result. □

Corollary 1.5. *The group $H^\sharp$ is the common stabilizer in G of all elements in $\mathbb{C}[G]^{H^\sharp}$ with respect to the action of G on the right.*

Proof. This follows from Lemma 1.4 and isomorphism (1.3). □

We now proceed to establishing relations between the principal combinatorial invariants and the extended weight semigroup of a spherical homogeneous space G/H.

Let $\widehat{\Lambda} = \widehat{\Lambda}_{G/H}$ denote the sublattice in $\mathfrak{X}(B) \oplus \mathfrak{X}(H)$ generated by $\widehat{\Lambda}^+$.

Proposition 1.6. *The map $\mu \mapsto (\mu, 0)$ induces an isomorphism*

$$\Lambda \simeq \widehat{\Lambda} \cap \mathfrak{X}(B) = \{(\lambda, \chi) \in \widehat{\Lambda} \mid \chi = 0\}.$$

Proof. The inclusion $\Lambda \supset \widehat{\Lambda} \cap \mathfrak{X}(B)$ is obvious. To prove the converse inclusion, let $\mu \in \Lambda$ and regard the corresponding element $f_\mu \in \mathbb{C}(G/H)$ as a $(B \times H/H^\sharp)$-semi-invariant rational function on $G/H^\sharp$ of weight $(\mu, 0)$. Lemma 1.3 yields that $G/H^\sharp$ is quasi-affine, therefore $\mathbb{C}(G/H^\sharp) = \operatorname{Quot} \mathbb{C}[G/H^\sharp]$ and by [Tim, Lemma D.7] one has $f_\mu = F_1/F_2$ for some B-semi-invariant functions $F_1, F_2 \in \mathbb{C}[G/H^\sharp]$. As $H/H^\sharp$ is diagonalizable, we may also assume F_1, F_2 to be $H/H^\sharp$-semi-invariant (of the same weight). Let $(\lambda_1, \chi), (\lambda_2, \chi) \in \mathfrak{X}_+(B) \oplus \mathfrak{X}(H/H^\sharp)$ be the $(B \times H/H^\sharp)$-weights of F_1, F_2, respectively. Then $F_1 \in \mathbb{C}[G]^{(B\times H)}_{(\lambda_1,\chi)}$ and $F_2 \in \mathbb{C}[G]^{(B\times H)}_{(\lambda_2,\chi)}$, whence $(\lambda_1, \chi), (\lambda_2, \chi) \in \widehat{\Lambda}^+$ and $(\mu, 0) = (\lambda_1, \chi) - (\lambda_2, \chi)$. □

By Proposition 1.6, for every $\mu \in \Lambda$ there is a unique expression of the form

$$(\mu, 0) = \sum_{D \in \mathcal{D}} c(D, \mu)(\lambda_D, \chi_D),$$

which immediately implies the following result.

Proposition 1.7. *$\langle \varkappa(D), \mu\rangle = c(D, \mu)$ for every $D \in \mathcal{D}$.*

Lemma 1.8. *Suppose that $\alpha \in \Pi$ and $D \in \mathcal{D}$. Then $D \in \mathcal{D}(\alpha)$ if and only if $\varpi_\alpha \in$ supp λ_D.*

Proof. Recall that D is the divisor of zeros of a (unique up to proportionality) regular section $s_D \in \Gamma(-\chi_D)$, which is B-semi-invariant of weight λ_D. Evidently, for every $g \in G$, gD is the divisor zeros of the section gs_D. Hence D is P_α-unstable if and only if s_D is not P_α-semi-invariant. The latter is equivalent to $\varpi_\alpha \in$ *supp* λ_D. □

Lemma 1.8 and Proposition 1.1 imply the proposition below.

Proposition 1.9. *Suppose that $\alpha \in \Pi$. Then:*

(1) $\alpha \in \Pi^p$ *if and only if there is no free generator (λ, χ) of $\widehat{\Lambda}^+$ such that $\varpi_\alpha \in$ supp λ;*
(2) $\alpha \in \Sigma$ *if and only if there are exactly two different free generators (λ_1, χ_1), (λ_2, χ_2) of $\widehat{\Lambda}^+$ such that $\varpi_\alpha \in$ supp λ_1 and $\varpi_\alpha \in$ supp λ_2.*

A combination of Propositions 1.6, 1.7, and 1.9 yields the following result.

Theorem 1.10. *The pair $(\mathfrak{X}(H), \widehat{\Lambda}^+)$ uniquely determines Λ, Π^p, $\mathcal{D}$, and $\Pi \cap \Sigma$.*

Taking into account Theorem 1.2 we get the corollary below.

Corollary 1.11. *The spherical homogeneous space G/H is uniquely determined by the triple $(\mathfrak{X}(H), \widehat{\Lambda}^+, \Sigma)$.*

Example 1.12. Table 1 in [Avd1] contains the free generators of semigroups $\widehat{\Lambda}^+$ for certain affine spherical homogeneous spaces G/H with non-simple G, which by Theorem 1.10 enables one to compute the invariants Λ, Π^p, $\mathcal{D}$, and $\Pi \cap \Sigma$ for these spaces. In particular, for spherical homogeneous spaces $(\mathrm{SL}_n \times \mathrm{SL}_{n+1})/(\mathrm{SL}_n \times \mathbb{C}^\times)$ with $n \geqslant 2$ and $(\mathrm{Spin}_n \times \mathrm{Spin}_{n+1})/\mathrm{Spin}_n$ with $n \geqslant 3$ (see items 1, 2 in [Avd1, Table 1]) it turns out that $\Pi \cap \Sigma = \Pi$, in which case $\Sigma = \Pi$ and the entire quadruple $(\Lambda, \Pi^p, \Sigma, \mathcal{D})$ is uniquely determined by the pair $(\mathfrak{X}(H), \widehat{\Lambda}^+)$.

Example 1.13. A spherical homogeneous space G/H is said to be *model* if it is quasi-affine and there is a G-module isomorphism

$$\mathbb{C}[G/H] \simeq \bigoplus_{\lambda \in \mathfrak{X}_+(B)} V(\lambda),$$

that is, for every $\lambda \in \mathfrak{X}_+(B)$ the G-module $\mathbb{C}[G/H]$ contains a simple G-submodule isomorphic to $V(\lambda)$. An example of such a space is given by G/U. All model spherical homogeneous spaces were classified by Luna in [Lun5]. For such a space G/H, one has $\mathfrak{X}(H) = 0$ and $\widehat{\Lambda}^+ = \{(\lambda, 0) \mid \lambda \in \mathfrak{X}_+(B)\}$. Therefore, G/H is uniquely determined by the set Σ, which may be any subset of a certain finite subset $\Sigma_G^{\mathrm{mod}} \subset \Delta^+$. We note that $\Sigma_G^{\mathrm{mod}} \neq \varnothing$ whenever $G \neq \mathrm{SL}_2$. This example shows that in general the set Σ is not determined by the pair $(\mathfrak{X}(H), \widehat{\Lambda}^+)$.

We now turn to the problem of expressing $\mathfrak{X}(H)$ and $\widehat{\Lambda}^+$ in terms of the quadruple $(\Lambda, \Pi^p, \Sigma, \mathcal{D})$.

Proposition 1.14. *The group $\mathfrak{X}(H)$ is generated by the characters χ_D, $D \in \mathcal{D}$.*

Proof. Let $H_{\mathcal{D}} \subset H$ be the common kernel of all the characters χ_D, $D \in \mathcal{D}$. Clearly, $H^\sharp \subset H_{\mathcal{D}}$. On the other hand, since $\widehat{\Lambda}^+_{G/H}$ is generated by all the elements (λ_D, χ_D), $D \in \mathcal{D}$, one has $\mathbb{C}[G]^{H^\sharp} = \mathbb{C}[G]^{H_{\mathcal{D}}}$. Corollary 1.5 yields $H_{\mathcal{D}} \subset H^\sharp$, whence $H_{\mathcal{D}} = H^\sharp$. □

Let $\mathbb{Z}^{|\mathcal{D}|}$ be the free Abelian group consisting of integer linear combinations of elements in $\mathcal{D}$. Proposition 1.14 yields a surjective homomorphism $\nu \colon \mathbb{Z}^{|\mathcal{D}|} \to \mathfrak{X}(H)$ given by $D \mapsto \chi_D$, hence $\mathfrak{X}(H) \simeq \mathbb{Z}^{|\mathcal{D}|} / \operatorname{Ker} \nu$. Applying Proposition 1.6, we obtain the following result.

Proposition 1.15. *The kernel of ν is generated by elements $\sum\limits_{D \in \mathcal{D}} \langle \varkappa(D), \mu \rangle D$, where μ runs over a basis of Λ.*

For every $D \in \mathcal{D}$, we regard the expression $\lambda_D = \sum\limits_{\alpha \in \Pi} n_{\alpha,D} \varpi_\alpha$.

Proposition 1.16 ([Fos, § 2.2, Theorem 2.2], [Tim, Lemma 30.24])**.** *The numbers $n_{\alpha,D}$ are determined as follows:*

$$n_{\alpha,D} = \begin{cases} 0 & \text{if } D \notin \mathcal{D}(\alpha); \\ 1 & \text{if } D \in \mathcal{D}(\alpha) \text{ and } 2\alpha \notin \Sigma; \\ 2 & \text{if } D \in \mathcal{D}(\alpha) \text{ and } 2\alpha \in \Sigma. \end{cases} \tag{1.4}$$

Corollary 1.17 ([Cup, § 2.1.2], [Tim, Lemma 30.24])**.** *Depending on the type of D, the weight λ_D is determined as follows:*

$$\lambda_D = \begin{cases} \sum\limits_{\alpha \in \Pi \mid D \in \mathcal{D}(\alpha)} \varpi_\alpha & \text{if } D \in \mathcal{D}^a; \\ 2\varpi_\alpha & \text{if } D \in \mathcal{D}^{a'} \text{ and } D \in \mathcal{D}(\alpha); \\ \sum\limits_{\alpha \in \Pi \mid D \in \mathcal{D}(\alpha)} \varpi_\alpha \quad (\leqslant 2 \text{ summands}) & \text{if } D \in \mathcal{D}^b. \end{cases} \tag{1.5}$$

2. Luna's general classification of spherical homogeneous spaces

2.1. **Simple embeddings of spherical homogeneous spaces.** In this subsection we collect some facts from the general theory of spherical embeddings expounded in [Kno1].

Let G/H be a spherical homogeneous space. We retain all the notation introduced in § 1.

An embedding X of G/H is said to be *simple* if it contains exactly one closed G-orbit. Simple embeddings are classified by strictly convex colored cones.

Definition 2.1 ([Kno1, § 3]). A *colored cone* is a pair $(\mathcal{C}, \mathcal{A})$ with $\mathcal{C} \subset \mathcal{Q}$ and $\mathcal{A} \subset \mathcal{D}$ having the following properties:

(CC1) $\mathcal{C}$ is a cone generated by $\varkappa(\mathcal{A})$ and finitely many elements of $\mathcal{V}$;

(CC2) $\mathcal{C}^\circ \cap \mathcal{V} \neq \varnothing$.

A colored cone is said to be *strictly convex* if the following property holds:

(SCC) $\mathcal{C}$ is strictly convex and $0 \notin \varkappa(\mathcal{A})$.

Definition 2.2. A colored cone $(\mathcal{C}, \mathcal{A})$ is said to be a *colored subspace* if $\mathcal{C}$ is a vector subspace of $\mathcal{Q}$.

Let X be a simple embedding of G/H and let Y be its closed G-orbit. We regard all B-stable prime divisors in X containing Y. These can be divided into two parts. The first part, denoted by $\mathcal{G}(X)$, consists of divisors that are G-stable. Divisors in the second part are closures of colors. Let $\mathcal{A}(X)$ denote the set of colors arising in this way.

Let $\mathcal{C}(X)$ be the cone in $\mathcal{Q}$ generated by $\varkappa(\mathcal{A}(X))$ and the images of G-invariant valuations associated with elements in $\mathcal{G}(X)$.

Proposition 2.3 ([LunV, Proposition 8.10], [Kno1, Theorem 3.1]). *The map $X \mapsto (\mathcal{C}(X), \mathcal{A}(X))$ is a bijection between isomorphism classes of simple embeddings and strictly convex colored cones.*

2.2. **Standard completions and wonderful G-varieties.** Let $H \subset G$ be a spherical subgroup.

Definition 2.4. An embedding X of G/H is said to be *toroidal* if no color contains a G-orbit in its closure.

In other words, X is toroidal if every irreducible B-stable closed subvariety containing a closed G-orbit is actually G-stable.

Definition 2.5. A complete simple toroidal embedding of G/H is said to be a *standard completion*[1] of G/H.

Proposition 2.6. *Suppose that X is a simple embedding of G/H. Then X is a standard completion if and only if $\mathcal{A}(X) = \varnothing$ and $\mathcal{C}(X) = \mathcal{V}$. In particular, a standard completion is unique if exists.*

Proof. It follows from the definition that X is toroidal if and only if $\mathcal{A}(X) = \varnothing$ and $\mathcal{C}(X) \subset \mathcal{V}$. By [Kno1, Theorem 4.2], X is complete if and only if $\mathcal{C}(X) \supset \mathcal{V}$. ☐

The subgroup H is said to be *sober* if the group $N_G(H)/H$ is finite.

[1] The term "standard embedding" seems to be more common in this situation, however we avoid using this term since it will appear later in this paper in another context.

Corollary 2.7. *A standard completion of G/H exists if and only if the group H is sober.*

Proof. It follows from Propositions 2.3 and 2.6 that a standard completion exists if and only if the cone $\mathcal{V}$ is strictly convex. By [BriP, 5.3, Corollaire] the latter is equivalent to H being sober. □

Until the end of this subsection we assume that H is sober. Let X be the standard completion of G/H. We put $X_B = X \setminus \bigcup\limits_{D\in\mathcal{D}} \overline{D}$. By [Kno1, Theorem 2.1] the set X_B is B-stable, affine, and open. We call it the *canonical B-chart* of X. We note that X_B is nothing else but the union of B-orbits in X whose closure contains the closed G-orbit. Let $\mathcal{C}_B$ denote the cone in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the weights of B-semi-invariant regular functions on X_B.

Proposition 2.8. *Under the above assumptions, $\mathcal{C}_B^1 = \{-\sigma \mid \sigma \in \Sigma\}$.*

Proof. By [Kno1, Theorem 2.5(a)], $\mathcal{C}_B = \mathcal{C}(X)^\vee = \mathcal{V}^\vee$, hence the required result is implied by the definition of spherical roots. □

A smooth standard completion of G/H is said to be *wonderful*. The subgroup H is said to be *wonderful* if the homogeneous space G/H admits a wonderful completion. Wonderful subgroups $H \subset G$ are characterized by the following property.

Proposition 2.9. *A spherical subgroup $H \subset G$ is wonderful if and only if $\Lambda = \mathbb{Z}\Sigma$.*

This fact seems to be first observed by Knop in [Kno2, Introduction]. For a proof see also [Tim, 30.1].

Definition 2.10. A smooth complete irreducible G-variety X is said to be *wonderful* of rank r if the three conditions below are satisfied:

(1) X contains an open G-orbit whose complement is a union of prime G-stable divisors $D_1, \ldots, D_r$;

(2) $D_1, \ldots, D_r$ are smooth and have a non-empty transversal intersection;

(3) for every two points $x, x' \in X$, $Gx = Gx'$ if and only if $\{i \mid x \in D_i\} = \{j \mid x' \in D_j\}$.

In 1996 Luna [Lun2] proved that every wonderful G-variety is spherical, which implies the following result (for a proof see [Tim, Theorem 30.15]).

Theorem 2.11. *A G-variety is wonderful if and only if it is a wonderful completion of a spherical homogeneous space G/H.*

Remark 2.12. The rank of a wonderful G-variety equals the number of spherical roots of X.

It was noted in [Lun2] that every wonderful G-variety X is projective and the connected center of G acts trivially on X.

2.3. **Spherical closure of a spherical subgroup.** Let $H \subset G$ be a spherical subgroup. We regard the action of $N_G(H)$ on G/H given by $(n, gH) \mapsto gn^{-1}H$ $(n \in N_G(H), g \in G)$. This action commutes with the action of G on G/H by left translation and therefore induces an action of $N_G(H)$ on the set of colors $\mathcal{D}$. The kernel of this action, denoted by $\overline{H}$, is said to be the *spherical closure* of H. The subgroup H is said to be *spherically closed* if $\overline{H} = H$. We note that one always has $\overline{H} \supset N_G(H)^0$.

For every $D \in \mathcal{D}$, let (λ_D, χ_D) be the corresponding indecomposable element of $\widehat{\Lambda}^+_{G/H}$ and let v_D be a non-zero element of the one-dimensional space $V(\lambda_D^*)^{(H)}_\chi \subset V(\lambda_D^*)$.

Proposition 2.13. *The spherical closure $\overline{H}$ of H is the common stabilizer in G of all the lines $\langle v_D \rangle$, where D runs over the set $\mathcal{D}$.*

In the proof of this proposition we shall need the following lemma.

Lemma 2.14. *For a spherical subgroup $H \subset G$, the group $H^\sharp$ is the common stabilizer in G of all the vectors v_D, where D runs over the set $\mathcal{D}$.*

Proof. As the group $H/H^\sharp$ is diagonalizable, it follows from Lemma 1.4 that $H^\sharp$ is the common stabilizer in G of vectors in all the spaces $V(\lambda^*)^{(H)}_\chi$, where (λ, χ) runs over the whole semigroup $\widehat{\Lambda}^+_{G/H}$. In view of isomorphism (1.3), for every $(\lambda_1, \chi_1), (\lambda_2, \chi_2) \in \widehat{\Lambda}^+_{G/H}$ and every $v_1 \in V(\lambda_1^*)^{(H)}_{\chi_1}$, $v_2 \in V(\lambda_2^*)^{(H)}_{\chi_2}$, $v_3 \in V(\lambda_1^* + \lambda_2^*)^{(H)}_{\chi_1+\chi_2}$, the common stabilizer of v_1 and v_2 stabilizes v_3. Consequently, $H^\sharp$ is the common stabilizer in G of all vectors v_D, $D \in \mathcal{D}$. ☐

Proof of Proposition 2.13. Let $\widetilde{H}$ be the common stabilizer in G of all the lines $\langle v_D \rangle$, $D \in \mathcal{D}$. Clearly, $H^\sharp \subset H \subset \widetilde{H}$. We first show that $\overline{H} = \widetilde{H} \cap N_G(H)$. Indeed, the information given in § 1.2 yields that an element $n \in N_G(H)$ fixes a color D if and only if n fixes the line

$$\mathbb{C}[G]^{(B\times H)}_{(\lambda_D,\chi_D)} \subset \mathbb{C}[G]^{(H)}_{\chi_D} \simeq \Gamma(L(-\chi_D)).$$

In view of isomorphism (1.3), the latter holds if and only if n fixes the line $V(\lambda_D^*)^{(H)}_{\chi_D} = \langle v_D \rangle$.

To complete the proof, it suffices to show that $\widetilde{H} \subset N_G(H)$. By Lemma 2.14 the action of the group $\widetilde{H}/H^\sharp$ on the vector space $\bigoplus\limits_{D\in\mathcal{D}} \langle v_D \rangle$ is faithful and diagonalizable, so $\widetilde{H}/H^\sharp$ is commutative. The latter implies that for every $h \in H$ and $\widetilde{h} \in \widetilde{H}$ one has $\widetilde{h}h\widetilde{h}^{-1} \in hH^\sharp \subset H$, therefore $\widetilde{H} \subset N_G(H)$. Now the required result follows from part (a). ☐

Corollary 2.15. *For every spherical subgroup $H \subset G$, the group $\overline{H}$ is spherically closed.*

Theorem 2.16 ([Kno2, §§ 7.6, 7.2])**.** *Let $H \subset G$ be a spherical subgroup. If H is spherically closed, then it is wonderful. In particular, H is wonderful whenever $N_G(H) = H$.*

Remark 2.17. For every spherical subgroup $H \subset G$ one has $\overline{H}^\sharp \subset H \subset \overline{H}$.

2.4. **Classification of spherical homogeneous spaces and wonderful G-varieties.** In this subsection we present Luna's general classification of spherical homogeneous spaces and wonderful G-varieties. The idea of this classification was proposed in the paper [Lun4] and consists in performing two steps. At the first step, one classifies all spherical subgroups $H \subset G$ with a given spherical closure $\overline{H}$. By Theorem 2.16 the classification then reduces to the classification of wonderful G-varieties, which is performed at the second step. As was mentioned in Introduction, Luna himself accomplished the first step, proposed a conjecture for the second step, and managed to prove this conjecture in the case where G is a product of simple groups of type A. The whole history of the proof of Luna's conjecture can be found in Introduction.

In the classification of wonderful G-varieties, an important role is played by wonderful G-varieties of small rank. As follows from Definition 2.10, wonderful G-varieties of rank zero are just complete homogeneous G-varieties, which are well-known to have the form G/P for some parabolic subgroup $P \subset G$. Rank-one wonderful varieties were classified by Akhiezer [Akh1] and, by another method, Brion [Bri3]. Wonderful varieties of rank two were classified by Wasserman [Was].

Elements of $\mathfrak{X}(T)$ appearing as spherical roots of rank-one wonderful G-varieties are said to be the *spherical roots* of G. Let Σ_G denote the set of all spherical roots of G. It is a finite set easily obtained from the classification of rank-one wonderful G-varieties. Spherical roots are non-negative linear combinations of simple roots of G with coefficients in $\frac{1}{2}\mathbb{Z}$. Spherical roots σ that belong to the root lattice of G are listed in Table 1. An element $\mu \in \mathfrak{X}(T) \backslash \mathbb{Z}\Delta$ is a spherical root of G if and only if $\sigma = 2\mu$ appears in Table 1 and its number is marked by an asterisk. (In Table 1, the notation α_i stands for the ith simple root of the set $\operatorname{Supp}\sigma$ whenever the Dynkin diagram of $\operatorname{Supp}\sigma$ is connected. If $\operatorname{Supp}\sigma$ is of type $\mathsf{A}_1 \times \mathsf{A}_1$, then α, α' are the two distinct roots in $\operatorname{Supp}\sigma$.)

A pair (Π^p, σ) with $\Pi^p \subset \Pi$ and $\sigma \in \Sigma_G$ is said to be *compatible* if there exists a rank-one wonderful variety X such that $\Pi^p_X = \Pi^p$ and $\Sigma_X = \{\sigma\}$. Based on the classification of rank-one wonderful G-varieties, the compatibility condition can be reformulated in purely combinatorial terms. Namely, the pair (Π^p, σ) is compatible if and only if

$$\Pi^{pp}(\sigma) \subset \Pi^p \subset \Pi^p(\sigma),$$

where $\Pi^p(\sigma) = \{\alpha \in \Pi \mid (\sigma, \alpha) = 0\}$ and the set $\Pi^{pp}(\sigma) \subset \Pi$ is determined as follows:

$$\Pi^{pp}(\sigma) = \begin{cases} \operatorname{Supp}\sigma \cap \Pi^p(\sigma)\backslash\{\alpha_r\}, & \text{if } \sigma = \alpha_1 + \alpha_2 + \ldots + \alpha_r \text{ with support of type } \mathsf{B}_r; \\ \operatorname{Supp}\sigma \cap \Pi^p(\sigma)\backslash\{\alpha_1\}, & \text{if } \sigma \text{ has support of type } \mathsf{C}_r; \\ \operatorname{Supp}\sigma \cap \Pi^p(\sigma), & \text{otherwise.} \end{cases}$$

For the reader's convenience, in the column "$\Pi^{pp}(\sigma)$" of Table 1 we listed all roots in the set $\Pi^{pp}(\sigma)$ for every spherical root $\sigma \in \mathbb{Z}\Delta$. If $\mu \in \Sigma_G \backslash \mathbb{Z}\Delta$, then $\Pi^{pp}(\mu) = \Pi^{pp}(2\mu)$.

Let $H \subset G$ be a spherical subgroup.

Proposition 2.18. (a) *The quadruple* $(\Pi^p_{G/H}, \Lambda_{G/H}, \Sigma_{G/H}, \mathcal{D}_{G/H})$ *amounts to the quadruple* $(\Pi^p_{G/H}, \Lambda_{G/H}, \Sigma_{G/H}, \mathcal{D}^a_{G/H})$.

(b) *If H is wonderful, then the quadruple* $(\Pi^p_{G/H}, \Lambda_{G/H}, \Sigma_{G/H}, \mathcal{D}_{G/H})$ *amounts to the triple* $(\Pi^p_{G/H}, \Sigma_{G/H}, \mathcal{D}^a_{G/H})$.

Proof. (a) In view of the disjoint union (1.1) we need to show that the sets $\mathcal{D}^{a'}_{G/H}$ and $\mathcal{D}^b_{G/H}$ are uniquely determined by the other combinatorial invariants of G/H. Namely, it follows from Proposition 1.1 that there is a surjective map $\Pi\backslash(\Sigma_{G/H} \cup \Pi^p_{G/H}) \to \mathcal{D}^{a'}_{G/H} \cup \mathcal{D}^b_{G/H}$, which sends α to the unique color in G/H moved by $P_{\{\alpha\}}$. It turns out that two roots $\alpha, \alpha' \in \Pi\backslash(\Sigma_{G/H} \cup \Pi^p_{G/H})$ are taken to the same color if and only if they are orthogonal and $\alpha + \alpha' \in \Sigma_{G/H} \cap 2\Sigma_{G/H}$, see [Lun3, 2.7], or [Lun4, 2.3], or [Tim, 30.10].

Part (b) follows directly from part (a) and Proposition 2.9. □

The set $(\Pi^p_{G/H}, \Lambda_{G/H}, \Sigma_{G/H}, \mathcal{D}^a_{G/H})$ is said to be the *homogeneous spherical datum* of G/H. If H is wonderful, then the set $(\Pi^p_{G/H}, \Sigma_{G/H}, \mathcal{D}^a_{G/H})$ is said to be the *spherical system* of G/H. These two combinatorial objects satisfy certain axioms, which are listed

Table 1. Spherical roots

No.	Type of Supp σ	σ	$\Pi^{pp}(\sigma)$	Note
1	A_1	α_1	$\varnothing$	
2	A_1	$2\alpha_1$	$\varnothing$	
3*	$\mathsf{A}_1 \times \mathsf{A}_1$	$\alpha + \alpha'$	$\varnothing$	
4	A_r	$\alpha_1 + \alpha_2 + \ldots + \alpha_r$	$\alpha_2, \alpha_3, \ldots, \alpha_{r-1}$	$r \geqslant 2$
5*	A_3	$\alpha_1 + 2\alpha_2 + \alpha_3$	α_1, α_3	
6	B_r	$\alpha_1 + \alpha_2 + \ldots + \alpha_r$	$\alpha_2, \alpha_3, \ldots, \alpha_{r-1}$	$r \geqslant 2$
7	B_r	$2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_r$	$\alpha_2, \alpha_3, \ldots, \alpha_r$	$r \geqslant 2$
8*	B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	α_1, α_2	
9	C_r	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{r-1} + \alpha_r$	$\alpha_3, \alpha_4, \ldots, \alpha_r$	$r \geqslant 3$
10*	D_r	$2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$	$\alpha_2, \alpha_3, \ldots, \alpha_r$	$r \geqslant 4$
11	F_4	$2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$	$\alpha_2, \alpha_3, \alpha_4$	
12	G_2	$\alpha_1 + \alpha_2$	$\varnothing$	
13	G_2	$2\alpha_1 + \alpha_2$	α_2	
14	G_2	$4\alpha_1 + 2\alpha_2$	α_2	

in the definition below. These axioms trace back to Proposition 1.1 and to Wasserman's classification of rank-two wonderful varieties [Was].

Definition 2.19 ([Lun4, § 2])**.** Suppose that Λ is a sublattice in $\mathfrak{X}(T)$, Π^p is a subset of Π, $\Sigma \subset \Sigma_G \cap \Lambda$ is a linearly independent set consisting of indivisible elements in Λ, and $\mathcal{D}^a$ is a finite set equipped with a map $\varkappa\colon \mathcal{D}^a \to \Lambda^\vee$. For every $\alpha \in \Pi \cap \Sigma$, put $\mathcal{D}(\alpha) = \{D \in \mathcal{D}^a \mid \langle \varkappa(D), \alpha\rangle = 1\}$.

The quadruple $(\Lambda, \Pi^p, \Sigma, \mathcal{D}^a)$ is said to be a *homogeneous spherical datum* if it satisfies the following axioms:

(A1) $\langle \varkappa(D), \sigma\rangle \leqslant 1$ for all $D \in \mathcal{D}^a$ and $\sigma \in \Sigma$, and the equality is attained if and only if $\sigma = \alpha \in \Pi \cap \Sigma$ and $D \in \mathcal{D}(\alpha)$;

(A2) for every $\alpha \in \Pi \cap \Sigma$, the set $\mathcal{D}(\alpha)$ contains exactly two elements D_α^+ and D_α^- such that $\langle \varkappa(D_\alpha^+), \lambda\rangle + \langle \varkappa(D_\alpha^-), \lambda\rangle = \langle \alpha^\vee, \lambda\rangle$ for all $\lambda \in \Lambda$;

(A3) the set $\mathcal{D}^a$ is the union of the sets $\mathcal{D}(\alpha)$ over all $\alpha \in \Pi \cap \Sigma$;

(Σ1) if $\alpha \in \Pi \cap \frac{1}{2}\Sigma$, then $\langle \alpha^\vee, \Lambda\rangle \subset 2\mathbb{Z}$ and $\langle \alpha^\vee, \sigma\rangle \leqslant 0$ for all $\sigma \in \Sigma \backslash \{2\alpha\}$;

(Σ2) if $\alpha, \beta \in \Pi$, $\alpha \perp \beta$, and $\alpha + \beta \in \Sigma \cup 2\Sigma$, then $\langle \alpha^\vee, \lambda\rangle = \langle \beta^\vee, \lambda\rangle$ for all $\lambda \in \Lambda$;

(S) $\langle \alpha^\vee, \lambda\rangle = 0$ for all $\alpha \in \Pi^p$ and $\lambda \in \Lambda$, and for every $\sigma \in \Sigma$ the pair (Π^p, σ) is compatible.

The triple $(\Pi^p, \Sigma, \mathcal{D}^a)$ is said to be a *spherical system* if it satisfies the above axioms with $\Lambda = \mathbb{Z}\Sigma$.

Remark 2.20. For every homogeneous spherical datum $(\Pi^p, \Lambda, \Sigma, \mathcal{D}^a)$, the triple $(\Pi^p, \Sigma, \mathcal{D}^a)$ with $\varkappa$ restricted to $\mathbb{Z}\Sigma$ is a spherical system.

Luna's general classification is given by the following theorem.

Theorem 2.21. (a) *The map $X \mapsto (\Pi^p_X, \Sigma_X, \mathcal{D}^a_X)$ is a bijection between wonderful G-varieties and spherical systems for G.*

(b) *The map $G/H \mapsto (\Lambda_{G/H}, \Pi^p_{G/H}, \Sigma_{G/H}, \mathcal{D}^a_{G/H})$ is a bijection between spherical homogeneous spaces and homogeneous spherical data for G.*

Example 2.22. Homogeneous wonderful G-varieties (that is, rank zero wonderful G-varieties) are characterized by the condition $\Sigma = \varnothing$, which immediately implies $\mathcal{D}^a = \varnothing$. It is well known that, for every subset $\Pi' \subset \Pi$, the stabilizer of the open B-orbit in $X = G/P_{\Pi'}$ coincides with $P_{\Pi'}$, whence the spherical system of X is $(\Pi', \varnothing, \varnothing)$.

Let $H \subset G$ be a spherical subgroup and let $\overline{H}$ be its spherical closure. For every $\gamma \in \Sigma_{G/H}$, we introduce the spherical root $\overline{\gamma} \in \Sigma_G$ in the following way.

$$\overline{\gamma} = \begin{cases} 2\gamma & \text{if } \gamma \notin \Pi, 2\gamma \in \Sigma_G, \text{ and the pair } (\Pi^p_{G/H}, 2\gamma) \text{ is compatible;} \\ \gamma & \text{otherwise.} \end{cases}$$

We set $\overline{\Sigma}_{G/H} = \{\overline{\gamma} \mid \gamma \in \Sigma_{G/H}\}$.

Proposition 2.23 ([Lun4, Lemma 7.1])**.** *Let $H \subset G$ be a spherical subgroup. The spherical system of $G/\overline{H}$ is determined as follows:*

(1) $\Pi^p_{G/\overline{H}} = \Pi^p_{G/H}$;

(2) $\Sigma_{G/\overline{H}} = \overline{\Sigma}_{G/H}$;

(3) *the natural map $G/H \to G/\overline{H}$ induces a bijection $\mathcal{D}_{G/H} \to \mathcal{D}_{G/\overline{H}}$, and the map $\varkappa_{G/\overline{H}}$ is the restriction of the map $\varkappa_{G/H}$ to $\Lambda_{G/\overline{H}}$.*

2.5. **Distinguished subsets of colors and quotient systems.** Let X be a wonderful variety with spherical system $(\Pi^p, \Sigma, \mathcal{D}^a)$.

For every subset $\mathcal{D}' \subset \mathcal{D}$ let $\mathcal{C}_{\mathcal{D}'}$ denote the convex cone in $\mathcal{Q}$ generated by the set $\varkappa(\mathcal{D}')$.

Definition 2.24. A subset $\mathcal{D}' \subset \mathcal{D}$ is said to be *distinguished* if either of the two equivalent conditions below holds:

- the set $\mathcal{C}^\circ_{\mathcal{D}'}$ meets $-\mathcal{V}$;
- there exists an element $\delta = \sum\limits_{D \in \mathcal{D}'} n_D \varkappa(D)$ with $n_D > 0$ such that $\langle \delta, \sigma \rangle \geqslant 0$ for all $\sigma \in \Sigma$.

Let $\mathcal{D}' \subset \mathcal{D}$ be a distinguished subset of colors. We put $\Sigma_{\mathcal{D}'} \subset \Sigma$ to be the set of spherical roots σ such that $\langle \delta, \sigma \rangle = 0$ for all $\delta \in \mathcal{C}^\circ_{\mathcal{D}'} \cap (-\mathcal{V})$. Let $\mathcal{V}_{\mathcal{D}'}$ be the largest face of $\mathcal{V}$ such that $\mathcal{C}^\circ_{\mathcal{D}'} \cap (-\mathcal{V}^\circ_{\mathcal{D}'}) \neq \varnothing$. It is defined by the vanishing of all the elements in $\Sigma_{\mathcal{D}'}$. Let $V_{\mathcal{D}'}$ be the vector subspace of $\mathcal{Q}$ generated by the set $\varkappa(\mathcal{D}') \cup \mathcal{V}_{\mathcal{D}'}$. Clearly, $(V_{\mathcal{D}'}, \mathcal{D}')$ is a colored subspace (see Definition 2.2) and $V_{\mathcal{D}'} \cap \mathcal{V} = \mathcal{V}_{\mathcal{D}'}$.

Given a distinguished subset of colors $\mathcal{D}' \subset \mathcal{D}$, one defines the *quotient system* $(\Pi^p, \Sigma, \mathcal{D}^a)/\mathcal{D}' = (\Pi^p/\mathcal{D}', \Sigma/\mathcal{D}', \mathcal{D}^a/\mathcal{D}')$. To do that, we first set

$$\begin{aligned} \Lambda/\mathcal{D}' = \{\lambda \in \Lambda \mid \langle v, \lambda \rangle = 0 \text{ for all } v \in V_{\mathcal{D}'}\} = \\ \{\lambda \in \Lambda \mid \langle \varkappa(D), \lambda \rangle = 0 \text{ for all } D \in \mathcal{D}' \text{ and } \langle \alpha^\vee, \lambda \rangle = 0 \text{ for all } \alpha \in \Sigma \backslash \Sigma_{\mathcal{D}'}\} = \\ \{\lambda \in \mathbb{Z}\Sigma_{\mathcal{D}'} \mid \langle \varkappa(D), \lambda \rangle = 0 \text{ for all } D \in \mathcal{D}'\}. \end{aligned}$$

Then the elements of the quotient system are defined as follows:

- $\Pi^p/\mathcal{D}' = \{\alpha \in \Pi \mid \mathcal{D}(\alpha) \subset \mathcal{D}'\}$;
- $\Sigma/\mathcal{D}'$ is the set of indecomposable elements of the semigroup $\mathbb{Z}^+\Sigma \cap \Lambda/\mathcal{D}'$;
- $\mathcal{D}^a/\mathcal{D}'$ is the union of the sets $\mathcal{D}(\alpha)$ over all $\alpha \in \Pi \cap (\Sigma/\mathcal{D}')$, and the map $\varkappa/\mathcal{D}'$ is the restriction of $\varkappa$ to $\mathcal{D}^a/\mathcal{D}'$ followed by the projection $\mathcal{Q} \to \operatorname{Hom}_{\mathbb{Z}}(\Lambda/\mathcal{D}', \mathbb{Q})$.

Remark 2.25. Using the condition $\mathcal{C}^\circ_{\mathcal{D}'} \cap (-\mathcal{V}^\circ_{\mathcal{D}'}) \ne \varnothing$, one can show that the semigroup $\mathbb{Z}^+\Sigma \cap \Lambda/\mathcal{D}'$ can be expressed in the following way:

$$\mathbb{Z}^+\Sigma \cap \Lambda/\mathcal{D}' = \{\lambda \in \mathbb{Z}^+\Sigma \mid \langle \varkappa(D), \lambda\rangle = 0 \text{ for all } D \in \mathcal{D}'\}.$$

Recently Bravi proved that the semigroup $\mathbb{Z}^+\Sigma \cap \Lambda/\mathcal{D}'$ is always free, see [Bra2, Theorem 3.1]. So $\Sigma/\mathcal{D}'$ is the set of free generators of this semigroup.

A G-equivariant morphism $X \to X'$ between two wonderful varieties is said to be *wonderful* if it is dominant (and thereby surjective) and has connected fibers. We note that, in case where $X' = G/P$ for a parabolic subgroup $P \subset G$, every G-equivariant morphism $X \to G/P$ is wonderful.

Let $\phi\colon X \to X'$ be a wonderful morphism between two wonderful varieties. Let $\mathcal{D}(\phi) \subset \mathcal{D}_X$ denote the set of colors that map dominantly (and thereby surjectively) onto X'. Bravi's result together with [Lun4, Proposition 3.3.2] imply the following result.

Proposition 2.26. (a) *The map $\phi \mapsto \mathcal{D}(\phi)$ is a bijection between wonderful morphisms $\phi\colon X \to X'$ and distinguished subsets of $\mathcal{D}$.*

(b) *For the wonderful morphism $X \to X'$ corresponding to a distinguished subset $\mathcal{D}' \subset \mathcal{D}$, the spherical system of X' is given by $(\Pi^p_X, \Sigma_X, \mathcal{D}^a_X)/\mathcal{D}'$. In particular, $(\Pi^p_X, \Sigma_X, \mathcal{D}^a_X)/\mathcal{D}'$ is a spherical system.*

2.6. **Characterization of strongly solvable wonderful subgroups.** The goal of this subsection is to obtain a characterization of strongly solvable wonderful subgroups in G in terms of their spherical systems.

Definition 2.27. A spherical system is said to be *strongly solvable* if the corresponding wonderful subgroup of G is strongly solvable.

Proposition 2.28 (see [Lun1, § 1])**.** *A spherical system $\mathscr{S} = (\Pi^p, \Sigma, \mathcal{D}^a)$ is strongly solvable if and only if there exists a subset $\mathcal{D}' \subset \mathcal{D}^a$ having the following properties:*

(1) *the convex cone generated by the set $\mathcal{V} \cap \varkappa(\mathcal{D}')$ coincides with $\mathcal{Q}$ or, equivalently, the set $\mathcal{C}^\circ_{\mathcal{D}'}$ meets $-\mathcal{V}^\circ$ or, equivalently, there exists an element $\delta = \sum\limits_{D \in \mathcal{D}'} n_D \varkappa(D)$ with $n_D > 0$ such that $\langle \delta, \sigma\rangle > 0$ for all $\sigma \in \Sigma$;*

(2) $|\mathcal{D}\backslash\mathcal{D}'| = |\Pi|$, *where $\mathcal{D}$ is the set of colors associated with $\mathscr{S}$.*

Remark 2.29. Condition (1) guarantees that the set $\mathcal{D}'$ is distinguished.

Proof of Proposition 2.28. Let $H \subset G$ be a wonderful subgroup corresponding to $\mathscr{S}$ and let X be the wonderful embedding of G/H. Clearly, $\mathcal{D}$ is naturally identified with the set of colors of X. Regard a wonderful morphism $X \to X'$ and denote by $\mathcal{D}'$ the corresponding distinguished subset of colors. By the definition of quotient system, condition (1) holds for $\mathcal{D}'$ if and only if $\Sigma/\mathcal{D}' = \varnothing$, which in turn is equivalent to X' being homogeneous (see Example 2.22), that is, $X' = G/P$ for a parabolic subgroup $P \subset G$. Now assume X' to be homogeneous so that condition (1) holds for $\mathcal{D}'$. Then the colors in $\mathcal{D}\backslash\mathcal{D}'$ are exactly the preimages of colors in X'. So condition (2) holds for $\mathcal{D}'$ if and only if X' contains exactly $|\Pi|$ colors. It is well known that the latter is equivalent to $X' = G/B$. In this case, for every $\alpha \in \Pi$ one has $|(\mathcal{D}\backslash\mathcal{D}) \cap \mathcal{D}(\alpha)| = 1$, whence $\mathcal{D}' \subset \mathcal{D}^a$ by Proposition 1.1.

Now let us prove the assertion. The subgroup H is strongly solvable if and only if there exists a G-equivariant morphism $G/H \to G/B$. By [Kno1, Theorem 4.1], such a morphism always extends to a G-equivariant morphism $X \to G/B$, and we may apply the above reasoning. □

3. Luna's 1993 approach for classifying strongly solvable wonderful subgroups

3.1. **Spherical and wonderful B^--varieties.** In this subsection we introduce the notions of a spherical and wonderful B^--variety and show the role played by them in the classification of strongly solvable wonderful subgroups of G.

Definition 3.1. A B^--variety Z is said to be *spherical* if T has an open orbit in Z.

Definition 3.2. A spherical B^--variety Z is said to be *wonderful* if it possesses the following properties:
(1) Z is smooth and complete;
(2) there is exactly one closed B^--orbit in Z (which is necessarily a fixed point z_0);
(3) every irreducible T-stable closed subvariety $Z' \subset Z$ containing z_0 is actually B^--stable.

Proposition 3.3. *Let Z be a B^--variety and regard the G-variety $X = G *_{B^-} Z$.*
(a) *Z is a spherical B^--variety if and only if X is a spherical G-variety;*
(b) *Z is a wonderful B^--variety if and only if X is a wonderful G-variety.*

To prove this proposition, we need an additional consideration. Namely, we regard the natural G-equivariant morphism $\phi\colon X \to G/B^-$. Note that Z is naturally identified with the subset $\phi^{-1}(o) \subset X$. Since the subset $Bo = Uo \subset G/B^-$ is open, the subset $X_0 = \phi^{-1}(Bo) \subset X$ is open and B-stable. Applying Proposition A.2 to the B-equivariant morphism $X_0 \to Bo \simeq B/T$, we get $X_0 \simeq B *_T Z \simeq U \times Z$, where the latter variety is acted on by B by the formula

$$tv \cdot (u, z) = (tvut^{-1}, tz) \quad (t \in T, v, u \in U, z \in Z). \tag{3.1}$$

Proof of Proposition 3.3. (a) Evidently, X is spherical if and only if B has an open orbit in X_0. Formula (3.1) shows that the latter holds if and only if T has an open orbit in Z.

(b) We shall use the interpretaion of wonderful G-varieties as wonderful completions of spherical homogeneous spaces, see Theorem 2.11.

First suppose that X is a wonderful G-variety and let $X^c \subset X$ be the closed G-orbit. Since X is smooth, Z is also smooth by Proposition A.3. Clearly, Z is complete. Next, for every G-orbit $\mathcal{O} \subset X$, the intersection $\mathcal{O} \cap Z$ is a B^--orbit, and $\mathcal{O}$ is closed if and only if $\mathcal{O} \cap Z$ is closed. This proves that $X^c \cap Z$ is a unique closed B^--orbit in Z, which is necessarily a fixed point z_0. Let $Z' \subset Z$ be an irreducible T-stable closed subvariety containing z_0. Then BZ' is a closed subvariety in X_0 containing $X^c \cap X_0$. Let X' be the closure in X of BZ'. Clearly, X' is an irreducible B-stable closed subvariety containing X^c. As X is toroidal, we obtain that X' is G-stable, whence $Z' = X' \cap Z$ is B^--stable, so that Z is a wonderful B^--variety.

Now suppose that Z is a wonderful B^--variety. Then X is smooth and complete by Propositions A.3 and A.4. If z_0 is the (unique) point in Z fixed by B^-, then $X^c = Gz_0$ is a unique closed G-orbit in X. Let $X' \subset X$ be an irreducible B-stable closed subvariety containing X^c. Then $Z' = X' \cap Z$ is an irreducible T-stable closed subvariety containing z_0, whence Z' is B^--stable. The latter implies that $X' = GZ'$. Thus X is a wonderful G-variety. □

Theorem 3.4. *There is a bijection between wonderful B^--varieties and conjugacy classes in B^- of strongly solvable wonderful subgroups of G contained in B^-.*

Proof. Let Z be a wonderful B^--variety and let $H \subset B^-$ be the stabilizer of a point of the open B^--orbit in Z. Proposition 3.3(b) yields that $X = G *_{B^-} Z$ is a wonderful G-variety, and its open G-orbit is isomorphic to G/H, so that H is wonderful in G.

Conversely, let $H \subset G$ be a wonderful subgroup contained in B^- and let X be the wonderful embedding of the homogeneous space G/H. By [Kno1, Theorem 4.1] the natural morphism $G/H \to G/B^-$ extends to a G-equivariant morphism $\phi\colon X \to G/B^-$. Therefore $X = G *_{B^-} Z$, where $Z = \phi^{-1}(o)$ (see Proposition A.2). Proposition 3.3(b) implies that Z is a wonderful B^--variety. □

3.2. **Automorphism groups of smooth complete toric varieties.** In this subsection we present the description of the automorphism group of a smooth complete toric variety, which goes back to Demazure [Dem] (see also [Oda, § 3.4]). In our exposition we follow a more modern viewpoint on this description, which is due to Cox [Cox]. Naturally, from Cox's paper we extract only results that are necessary for our subsequent considerations. For instance, we shall only need a description of the connected component of the automorphism group of a smooth complete toric variety.

In this paper, we adopt the following definition of a toric variety.

Definition 3.5. A T-variety Z is said to be *toric* if it possesses an open T-orbit.

Suppose Z to be a toric T-variety and let T_0 denote the quotient of T by the kernel of its action on Z. Let M be the lattice of weights of T-semi-invariant rational functions on Z and put $N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Clearly, $M \simeq \mathfrak{X}(T_0)$. Put also $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q})$. It is well known that Z determines a strictly convex fan $\mathcal{F}$ in $N_{\mathbb{Q}}$ and the map $Z \to (M, \mathcal{F})$ is a bijection between toric T-varieties and all pairs $(M, \mathcal{F})$ with M a sublattice of $\mathfrak{X}(T)$ and $\mathcal{F}$ a strictly convex fan in $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q})$.

Let Z be a toric variety and let $\mathcal{F}$ be the corresponding fan in $N_{\mathbb{Q}}$. Then:
(1) Z is complete if and only if $\mathcal{F}$ is complete;
(2) Z is smooth if and only if $\mathcal{F}$ is regular.

Let Z be a smooth complete toric variety and let $\mathcal{F}$ be the corresponding fan in $N_{\mathbb{Q}}$. We recall that the set $\mathcal{F}^1$ is in bijection with the set of T-stable prime divisors in Z. For every $\varrho \in \mathcal{F}^1$, let D_ϱ denote the corresponding T-stable prime divisor in Z.

To each $\varrho \in \mathcal{F}^1$ we assign a variable x_ϱ. Let $\mathrm{CR} = \mathrm{CR}(Z)$ denote the polynomial ring in variables x_ϱ, where ϱ runs over the set $\mathcal{F}^1$. Put $f = |\mathcal{F}^1|$.

Definition 3.6. The ring CR is said to be the *Cox ring* of the toric variety Z.

The ring CR is naturally acted on by a torus $\mathbf{T} \simeq (\mathbb{C}^\times)^f$. For every $\varrho \in \mathcal{F}^1$, let χ_ϱ be the character by which $\mathbf{T}$ acts on x_ϱ. Clearly, the map $\chi_\varrho \mapsto D_\varrho$ identifies $\mathfrak{X}(\mathbf{T})$ with the group of T-stable Weil divisors in Z. We denote this subgroup by $\mathbb{Z}^f$.

Every monomial $\prod\limits_{\varrho \in \mathcal{F}^1} x_\varrho^{a_\varrho}$ determines a T-stable Weil divisor $D = \sum\limits_{\varrho \in \mathcal{F}^1} a_\varrho D_\varrho$. We shall also write this monomial as x^D. The ring CR has a natural grading by the divisor class group $\operatorname{Cl} Z$. By definition, the degree of a monomial x^D is $[D] \in \operatorname{Cl} Z$. For every $c \in \operatorname{Cl} Z$, let CR_c denote the linear span of all monomials x^D with $[D] = c$, so that $\mathrm{CR} = \bigoplus_{c \in \mathrm{Cl}\,Z} \mathrm{CR}_c$. We note that for complete Z each of the subspaces CR_c is finite-dimensional (see [Cox, Corollary 1.2(i)]).

It is known that the group $\mathrm{Cl}\,Z$ is finitely generated, whence it may be identified with the character group of a uniquely determined quasitorus $\mathbf{S}$. Fix an isomorphism $\mathrm{Cl}\,Z \to \mathfrak{X}(\mathbf{S})$, $c \mapsto \chi_c$. The group $\mathbf{S}$ acts naturally on CR preserving the grading: each component CR_c is multiplied by the character χ_c.

Set $\widetilde{Z} = \operatorname{Spec} \mathrm{CR}$.

For every cone $\mathcal{C} \in \mathcal{F}$, we define the monomial $r(\mathcal{C}) = \prod_{\varrho \in \mathcal{F}^1 \setminus \mathcal{C}} x_\varrho$ and let I be the ideal of CR generated by all the monomials $r(\mathcal{C})$, $\mathcal{C} \in \mathcal{F}$. Let E be the subvariety of $\widetilde{Z}$ defined by the vanishing of all polynomials in I. As $I \subset \mathrm{CR}$ is $\mathbf{S}$-stable, $E \subset \widetilde{Z}$ is $\mathbf{S}$-stable as well. Besides, by [Cox, Lemma 1.4] the set E has codimension at least two in $\widetilde{Z}$.

Theorem 3.7 (see [Cox, Theorem 2.1(iii)])**.** *The variety Z is naturally isomorphic to the geometric quotient*[2] *of $\widetilde{Z} \backslash E$ by the action of* $\mathbf{S}$.

We note that for every $\varrho \in \mathcal{F}^1$ the preimage of the divisor D_ϱ under the morphism $\widetilde{Z} \backslash E \to Z$ is defined by the equation $x_\varrho = 0$.

For every $\alpha \in M$, let $f_\alpha \in \mathbb{C}(Z)$ be a T-semi-invariant rational function of weight α, which is unique up to proportionality. Its Weil divisor is given by $\operatorname{div} f_\alpha = \sum_{\varrho \in \mathcal{F}^1} \langle \varrho, \alpha \rangle D_\varrho$. By [Ful, § 3.4], the map $M \to \mathbb{Z}^f$ defined by by $\alpha \mapsto \operatorname{div} f_\alpha$ is included into the following exact sequence:

$$0 \to M \to \mathbb{Z}^f \to \mathrm{Cl}\,Z \to 0$$

The identifications $M \simeq \mathfrak{X}(T_0)$, $\mathbb{Z}^f \simeq \mathfrak{X}(\mathbf{T})$, and $\mathrm{Cl}\,Z \simeq \mathfrak{X}(\mathbf{S})$ yield the exact sequence

$$0 \to \mathfrak{X}(T_0) \to \mathfrak{X}(\mathbf{T}) \to \mathfrak{X}(\mathbf{S}) \to 0,$$

where the map $\mathfrak{X}(T_0) \to \mathfrak{X}(\mathbf{T})$ is defined by $\alpha \mapsto \sum_{\varrho \in \mathcal{F}^1} \langle \varrho, \alpha \rangle \chi_\varrho$. In particular, $T_0 \simeq \mathbf{T}/\mathbf{S}$.

We now turn to the problem of determining the automorphism group of Z.

Let $\mathrm{Aut}_g(\mathrm{CR})$ be the group of graded $\mathbb{C}$-algebra automorphisms of CR. By [Cox, Proposition 4.3] the group $\mathrm{Aut}_g(\mathrm{CR})$ is a connected affine algebraic group. Clearly, $\mathbf{T} \subset \mathrm{Aut}_g(\mathrm{CR})$ and the subgroup $\mathbf{S}$ is identified with a central subgroup of $\mathrm{Aut}_g(\mathrm{CR})$. Therefore every element of $\mathrm{Aut}_g(\mathrm{CR})$, regarded as an automorphism of $\widetilde{Z}$, preserves $\mathbf{S}$-orbits and by Theorem 3.7 descends to an automorphism of Z, so that there is a homomorphism

$$d\colon \mathrm{Aut}_g(\mathrm{CR}) \to \mathrm{Aut}(Z).$$

Hence we have the following sequence of homomorphisms:

$$1 \to \mathbf{S} \to \mathrm{Aut}_g(\mathrm{CR}) \xrightarrow{d} \mathrm{Aut}(Z)^0 \to 1. \tag{3.2}$$

Definition 3.8 (see [Dem, 4.5])**.** An element $\alpha \in M$ is said to be a *root* of the fan $\mathcal{F}$ if there exists an element $\varrho(\alpha) \in \mathcal{F}^1$ with $\langle \varrho(\alpha), \alpha \rangle = 1$ and $\langle \varrho, \alpha \rangle \leqslant 0$ for $\varrho \in \mathcal{F}^1 \backslash \{\varrho(\alpha)\}$.

[2] See the definition of a geometric quotient, for instance, in [PopV, 4.2].

We note that the element $\varrho(\alpha)$ is unique if exists.

Let $R(\mathcal{F})$ denote the set of roots of the fan $\mathcal{F}$. Since $\mathcal{F}$ is complete, it can be easily shown that the set $R(\mathcal{F})$ is finite.

For a root $\alpha \in R(\mathcal{F})$, let $y_{-\alpha}$ be the derivation of the ring CR defined on the generators as follows:

$$y_{-\alpha}(x_{\varrho(\alpha)}) = \prod_{\varrho \in \mathcal{F}^1 \setminus \{\varrho(\alpha)\}} x_\varrho^{-\langle \varrho, \alpha \rangle}; \qquad y_{-\alpha}(x_\varrho) = 0 \text{ for } \varrho \in \mathcal{F}^1 \setminus \{\varrho(\alpha)\}.$$

For every $\zeta \in \mathbb{C}$, we put $Y_{-\alpha}(\zeta) = \exp(\zeta y_{-\alpha})$ and introduce the one-parameter subgroup

$$Y_{-\alpha} = \{Y_{-\alpha}(\zeta) \mid \zeta \in \mathbb{C}\} \subset \operatorname{Aut}(\mathrm{CR}).$$

The action of $Y_{-\alpha}$ on CR is described as follows:

(3.3)

$$[Y_{-\alpha}(\zeta)]x_{\varrho(\alpha)} = x_{\varrho(\alpha)} + \zeta \prod_{\varrho \in \mathcal{F}^1 \setminus \{\varrho(\alpha)\}} x_\varrho^{-\langle \varrho, \alpha \rangle}; \qquad [Y_{-\alpha}(\zeta)]x_\varrho = x_\varrho \text{ for } \varrho \in \mathcal{F}^1 \setminus \{\varrho(\alpha)\}.$$

Clearly, $y_{-\alpha}$ preserves the grading, and so $Y_{-\alpha} \subset \operatorname{Aut}_g(\mathrm{CR})$. Moreover, $Y_{-\alpha}$ maps isomorphically onto $d(Y_{-\alpha})$.

The minus sign in the notation $y_{-\alpha}$ and $Y_{-\alpha}$ is justified by the following proposition.

Proposition 3.9. *The subgroup $d(Y_{-\alpha}) \subset \operatorname{Aut}(Z)$ is normalized by T with weight $-\alpha$.*

Proof. Let $t \in \mathbf{T}$ and $\zeta \in \mathbb{C}$. We have

$$(3.4)\quad [t \cdot Y_{-\alpha}(\zeta) \cdot t^{-1}]x_{\varrho(\alpha)} = [t \cdot Y_{-\alpha}(\zeta)]\chi_{\varrho(\alpha)}(t)^{-1}x_{\varrho(\alpha)} = \chi_{\varrho(\alpha)}(t)^{-1}x_{\varrho(\alpha)} + \chi_{\varrho(\alpha)}(t)^{-1}\zeta \prod_{\varrho \in \mathcal{F}^1 \setminus \{\varrho(\alpha)\}} x_\varrho^{-\langle \varrho, \alpha \rangle} = x_{\varrho(\alpha)} + \zeta \prod_{\varrho \in \mathcal{F}^1} \chi_\varrho(t)^{-\langle \varrho, \alpha \rangle} \prod_{\varrho \in \mathcal{F}^1 \setminus \{\varrho(\alpha)\}} x_\varrho^{-\langle \varrho, \alpha \rangle}.$$

We have obtained the equality $t \cdot Y_{-\alpha}(\zeta) \cdot t^{-1} = Y_{-\alpha}(\chi(t)\zeta)$ in the group $\operatorname{Aut}_g(\mathrm{CR})$, where $\chi = -\sum_{\varrho \in \mathcal{F}^1} \langle \varrho, \alpha \rangle \chi_\varrho$. Since χ is nothing else but the image of $-\alpha$ in $\mathfrak{X}(\mathbf{T})$, the group $d(Y_{-\alpha})$ is normalized by $d(\mathbf{T}) \simeq T_0$ with weight $-\alpha$. □

Theorem 3.10 (see [Cox, Corollary 4.7])**.** (a) *The group* $\operatorname{Aut}(Z)$ *is an affine algebraic group, and the group $T_0 = d(\mathbf{T})$ is a maximal torus of* $\operatorname{Aut}(Z)$.

(b) *The group* $\operatorname{Aut}(Z)^0$ *is generated by T_0 and the groups $d(Y_{-\alpha})$ for all $\alpha \in R(\mathcal{F})$.*

(c) *Sequence* (3.2) *is exact; in particular,* $\operatorname{Aut}(Z)^0 \simeq \operatorname{Aut}_g(\mathrm{CR})/\mathbf{S}$.

In the remaining part of this subsection we state and prove several lemmas that will be needed in the following subsections.

Lemma 3.11. *Let $\alpha, \beta \in R(\mathcal{F})$.*

(a) *if $\langle \varrho(\beta), \alpha \rangle < 0$ and $\langle \varrho(\alpha), \beta \rangle < 0$, then $\alpha + \beta = 0$;*

(b) *if $\langle \varrho(\beta), \alpha \rangle = 0$ and $\langle \varrho(\alpha), \beta \rangle = -p < 0$, then $\alpha + \beta \in R(\mathcal{F})$, $\varrho(\alpha + \beta) = \varrho(\beta)$, and $\langle \varrho(\alpha), \alpha + \beta \rangle = -p + 1$.*

Proof. (a) The hypothesis implies that $\langle \varrho, \alpha + \beta \rangle \leqslant 0$ for all $\varrho \in \mathcal{F}^1$. Since the fan $\mathcal{F}$ is complete, it follows that $\alpha + \beta = 0$.

(b) Obvious. □

The following proposition is obtained by a direct check.

Lemma 3.12. *Suppose that $\alpha, \beta \in R(\mathcal{F})$, $\langle \varrho(\beta), \alpha\rangle = 0$, and $\langle \varrho(\alpha), \beta\rangle = -p \leqslant 0$.*
(a) $[y_\alpha, y_\beta] = -py_{\alpha+\beta}$ *for* $p > 0$ *and* $[y_\alpha, y_\beta] = 0$ *for* $p = 0$*;*
(b) $(\operatorname{ad} y_\alpha)^q y_\beta \neq 0$ *for* $0 \leqslant q \leqslant p$ *and* $(\operatorname{ad} y_\alpha)^q y_\beta = 0$ *for* $q \geqslant p+1$.

Proof. Part (a) is obtained by a direct computation, part (b) is a consequence of (a) and Lemma 3.11(b). □

Fix a root $\alpha \in \mathcal{R}(\mathcal{F})$ and an element $\varrho \in \mathcal{F}^1$.

Lemma 3.13. *The divisor D_ϱ is $d(Y_{-\alpha})$-unstable if and only if $\varrho = \varrho(\alpha)$.*

Proof. The divisor D_ϱ is $d(Y_{-\alpha})$-unstable if and only if the element $x_\varrho \in \mathrm{CR}$ is not $Y_{-\alpha}$-invariant. By (3.3) the latter holds if and only if $\varrho = \varrho(\alpha)$. □

Fix a root $\alpha \in \mathcal{R}(\mathcal{F})$ and a maximal cone $\mathcal{C} \in \mathcal{F}$. Let $z \in Z$ be the T-fixed point corresponding to the cone $\mathcal{C}$.

Lemma 3.14. *The point z is $Y_{-\alpha}$-unstable if and only if $\varrho(\alpha) \in \mathcal{C}^1$ and $\langle \varrho, \alpha\rangle = 0$ for all $\varrho \in \mathcal{C}^1 \backslash \{\varrho(\alpha)\}$.*

Proof. Since $\{z\} = \bigcap\limits_{\varrho \in \mathcal{F}^1} D_\varrho$, the preimage of z under the morphism $\widetilde{Z}\backslash E \to Z$ is the set of zeros of the ideal $I_z \subset \mathrm{CR}$ generated by all variables x_ϱ with $\varrho \in \mathcal{F}^1$. So z is $Y_{-\alpha}$-unstable if and only if I_z is $Y_{-\alpha}$-unstable. By Lemma 3.13, the condition $\varrho(\alpha) \in \mathcal{C}^1$ is necessary for z to be $Y_{-\alpha}$-stable. Under this condition, from (3.3) we see that I_z is $Y_{-\alpha}$-unstable if and only if $\prod\limits_{\varrho \in \mathcal{F}^1 \backslash \{\varrho(\alpha)\}} x_\varrho^{-\langle \varrho, \alpha\rangle} \notin I_z$. Evidently, the latter holds if and only if $\langle \varrho, \alpha\rangle = 0$ for all $\varrho \in \mathcal{C}^1 \backslash \{\varrho(\alpha)\} = 0$. □

3.3. **Smooth complete spherical B^--varieties.** In this subsection we apply the description of connected automorphism groups of smooth complete spherical varieties, presented in the previous section, to the problem of classifying smooth complete spherical B^--varieties.

Smooth complete spherical B^--varieties are classified by combinatorial objects called Enriques' B^--systems.

Definition 3.15 (see [Lun1, § 2])**.** An *Enriques' B^--system* is a triple $(\mathfrak{X}, \mathcal{F}, \rho)$ consisting of the following elements:
(1) $\mathfrak{X}$ is a sublattice of $\mathfrak{X}(T)$;
(2) $\mathcal{F}$ is a regular complete strictly convex fan in $Q = \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{X}, \mathbb{Q})$;
(3) $\rho \colon \Pi \to \mathcal{F}^1 \cup \{0\}$ is a map satisfying the following conditions:
  (a) if $\rho(\alpha) \neq 0$, then $\alpha \in \mathfrak{X}$, $\langle \rho(\alpha), \alpha\rangle = 1$, and $\langle \varrho, \alpha\rangle \leqslant 0$ for every $\varrho \in \mathcal{F}^1 \backslash \{\rho(\alpha)\}$;
  (b) if $\langle \rho(\alpha), \beta\rangle < 0$, then $\langle \rho(\beta), \alpha\rangle = 0$;
  (c) $\langle \rho(\alpha), \beta\rangle \geqslant \langle \alpha^\vee, \beta\rangle$ for every $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $\rho(\beta) \neq 0$.

Let Z be a smooth complete spherical B^--variety and let $\theta \colon B^- \to \operatorname{Aut} Z$ be the natural homomorphism. By definition of a spherical B^--variety, Z is a toric T-variety. In this subsection we shall use all the notation associated with Z in the previous subsection. Let $\mathfrak{X}_Z$ be the weight lattice of T-semi-invariant rational functions on Z. Note that $\mathfrak{X}_Z$ is naturally identified with the character lattice of the torus $\theta(T)$. We put $Q_Z =$

$\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}_Z, \mathbb{Q})$. The structure of a toric T-variety on Z determines a regular complete fan $\mathcal{F}_Z$ in Q_Z. For every $\alpha \in \Pi$, let $U_{-\alpha}$ denote the corresponding one-dimensional unipotent subgroup of U^-. If $U_{-\alpha} \in \mathrm{Ker}\,\theta$, then we put $\rho_Z(\alpha) = 0$. Otherwise the group $\theta(U_{-\alpha})$ is a unipotent subgroup of $\mathrm{Aut}\,Z$ normalized by $\theta(T)$. Theorem 3.10(b) implies that $\alpha \in R(\mathcal{F}_Z)$ and $\theta(U_{-\alpha})$ coincides with $Y_{-\alpha}$. We take $\rho_Z(\alpha) \in Q_Z$ to be the element $\rho(\alpha)$ associated with α as a root of the fan $\mathcal{F}_Z$ (see Definition 3.8).

Proposition 3.16. *For every smooth complete spherical B-variety Z, the triple $(\mathfrak{X}_Z, \mathcal{F}_Z, \rho_Z)$ is an Enriques' B^--system. Conversely, for every Enriques' B^--system $(\mathfrak{X}, \mathcal{F}, \rho)$, up to an isomorphism, there exists a unique smooth complete spherical B^--variety Z with $\mathfrak{X}_Z = \mathfrak{X}$, $\mathcal{F}_Z = \mathcal{F}$, and $\rho_Z = \rho$.*

Proof. Let Z be a smooth complete spherical B^--variety. To show that $(\mathfrak{X}_Z, \mathcal{F}_Z, \rho_Z)$ is an Enriques' B^--system, it remains to establish properties (a), (b), (c) of Definition 3.15 for the map ρ_Z. Property (a) holds by construction and Definition 3.8. Next, suppose that $\langle \rho_Z(\alpha), \beta \rangle < 0$. Then $\rho_Z(\alpha) \neq \rho_Z(\beta)$, whence $\langle \rho_Z(\beta), \alpha \rangle \leqslant 0$. By Lemma 3.11(a) the inequality $\langle \rho_Z(\beta), \alpha \rangle < 0$ would imply $\alpha + \beta = 0$, which is not the case. Hence $\langle \rho_Z(\beta), \alpha \rangle = 0$ and we have (b). At last, let us show (c). Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$ and $\rho_Z(\beta) \neq 0$. Since $\langle \alpha^\vee, \beta \rangle \leqslant 0$, the required inequality holds automatically whenever $\langle \rho_Z(\alpha), \beta \rangle \geqslant 0$. Therefore in what follows we assume $\langle \rho_Z(\alpha), \beta \rangle < 0$ (in particular, $\rho_Z(\alpha) \neq 0$). Then we have $\langle \rho_Z(\beta), \alpha \rangle = 0$ by (b). Clearly, the homomorphism

$$\theta|_{U^-} : U^- \to \mathrm{Aut}\,Z$$

lifts to a unique homomorphism

$$\widetilde{\theta} \colon U \to \mathrm{Aut}_g(\mathrm{CR}(Z)),$$

and one has $\widetilde{\theta}(U_{-\alpha}) = Y_{-\alpha}$ and $\widetilde{\theta}(U_{-\beta}) = Y_{-\beta}$. Regard the corresponding Lie algebra homomorphism

$$d\widetilde{\theta} \colon \mathfrak{u}^- \to \mathrm{Der}\,\mathrm{CR}\,.$$

It sends $e_{-\alpha}$ to a multiple of $y_{-\alpha}$ and $e_{-\beta}$ to a multiple of $y_{-\beta}$. One of the Serre relations (see [Ser, Chapter VI, Theorem 6(c)] says that

$$(\mathrm{ad}\, e_{-\alpha})^{1 - \langle \alpha^\vee, \beta \rangle} e_{-\beta} = 0,$$

hence $(\mathrm{ad}\, y_{-\alpha})^{1 - \langle \alpha^\vee, \beta \rangle} y_{-\beta} = 0$. By Lemma 3.12(b) we have $1 - \langle \alpha^\vee, \beta \rangle \geqslant 1 - \langle \rho(\alpha), \beta \rangle$, which implies $\langle \rho(\alpha), \beta \rangle \geqslant \langle \alpha^\vee, \beta \rangle$.

Conversely, let $(\mathfrak{X}, \mathcal{F}, \rho)$ be an Enriques' B^--system and let Z be the toric T-variety associated with the pair $(\mathfrak{X}, \mathcal{F})$. We set

$$\Pi_0 = \{\alpha \in \Pi \mid \rho(\alpha) \neq 0\}. \tag{3.5}$$

Property (a) of the map ρ implies $\Pi_0 \subset \mathfrak{X}$ and $\Pi_0 \subset R(\mathcal{F})$. We recall (see [Ser, Chapter VI, Theorem 7(i)]) that the Lie algebra $\mathfrak{u}$ is the quotient of the free Lie algebra $\widehat{\mathfrak{u}}^-$ generated by the set $\{e_{-\alpha} \mid \alpha \in \Pi\}$ by the ideal J generated by the set

$$\{(\mathrm{ad}\, e_{-\alpha})^{1 - \langle \alpha^\vee, \beta \rangle} e_{-\beta} \mid \alpha, \beta \in \Pi, \alpha \neq \beta\} \qquad \text{(Serre relations)}.$$

Regard the homomorphism $\iota \colon \widehat{\mathfrak{u}} \to \mathrm{Der}\,\mathrm{CR}(Z)$ given by

$$\iota(e_{-\alpha}) = \begin{cases} y_{-\alpha} & \text{if } \alpha \in \Pi_0; \\ 0 & \text{if } \alpha \notin \Pi_0. \end{cases}$$

Property (c) combined with Lemma 3.12(b) implies that the ideal J is contained in $\operatorname{Ker}\iota$, which gives rise to a homomorphism $\mathfrak{u}^- \to \operatorname{Der}\operatorname{CR}(Z)$. By abuse of notation, we denote this homomorphism by the same letter ι. Let $\mathfrak{u}_0^-$ denote the subalgebra in $\mathfrak{u}^-$ generated by the set $\{e_{-\alpha} \mid \alpha \in \Pi_0\}$ and let $\mathfrak{u}_r^-$ denote the subspace in $\mathfrak{u}^-$ generated by the set $\{e_{-\gamma} \mid \gamma \in \Delta^+, e_{-\gamma} \notin \mathfrak{u}_0^-\}$. Clearly, $\mathfrak{u}^- = \mathfrak{u}_0^- \oplus \mathfrak{u}_r^-$, $\mathfrak{u}_r^-$ is an ideal of $\mathfrak{u}^-$, and $\mathfrak{u}_r^- \subset \operatorname{Ker}\iota$. Let U_0^- (resp. U_r^-) be the subgroup of U^- whose Lie algebra is $\mathfrak{u}_0^-$ (resp. $\mathfrak{u}_r^-$). Since $\Pi_0 \subset \mathfrak{X}$, one has a natural action of T on U_0^- given by $t \cdot \exp(\zeta e_{-\gamma}) = \exp(\gamma(t)^{-1}\zeta e_{-\gamma})$, where $t \in T$, $\gamma \in \Delta^+ \cap \langle \Pi_0 \rangle$, $\zeta \in \mathbb{C}$. The latter enables us to consider the group $A = T \leftthreetimes U_0^-$. The homomorphism ι induces a homomorphism $U^- \to \operatorname{Aut}_g(\operatorname{CR}(Z)) \to \operatorname{Aut} Z$, hence one has an action of U^- on Z whose kernel contains U_r^-. Moreover, the actions on Z of T and U_0^- naturally extend to an action of A. Regard the natural epimorphism $U^- = U_0^- \leftthreetimes U_r^- \to U_0^-$. It extends to a homomorphism $B^- = T \leftthreetimes U^- \to T \leftthreetimes U_0^- = A$, and the chain $B^- \to A \to \operatorname{Aut} Z$ provides a structure of a B^--variety on Z. The equalities $\mathfrak{X}_Z = \mathfrak{X}$, $\mathcal{F}_Z = \mathcal{F}$, and $\rho_Z = \rho$ hold by construction. ☐

Let $(\mathfrak{X}, \mathcal{F}, \rho)$ be an Enriques' B^--system and let Z be the corresponding smooth complete spherical B-variety.

Lemma 3.17. *Let $D \subset Z$ be a T-stable prime divisor and let $\varrho \in \mathcal{F}^1$ be the corresponding element. The divisor D is B^--stable if and only if $\langle \varrho, \alpha \rangle \leqslant 0$ for all $\alpha \in \Pi$ with $\rho(\alpha) \neq 0$.*

Proof. The divisor D is B^--stable if and only if D is $U_{-\alpha}$-stable for all $\alpha \in \Pi$ with $\rho(\alpha) \neq 0$. Since $\theta(U_{-\alpha}) = d(Y_{-\alpha})$, the assertion follows from Lemma 3.13. ☐

Lemma 3.18. *Let $z \in Z$ be a T-fixed point and let $\mathcal{C} \in \mathcal{F}$ be the corresponding maximal cone. The point z is B^--unstable if and only if there exists a root $\alpha \in \Pi$ such that $\rho(\alpha) \in \mathcal{C}^1$ and $\langle \varrho, \alpha \rangle = 0$ for all $\varrho \in \mathcal{C}^1 \backslash \{\rho(\alpha)\}$.*

Proof. Clearly, z is B^--unstable if and only if there exists a root $\alpha \in \Pi$ such that z is $U_{-\alpha}$-unstable. Since $\theta(U_{-\alpha}) = d(Y_{-\alpha})$ for all $\alpha \in \Pi$ with $\rho(\alpha) \neq 0$, the assertion follows from Lemma 3.14. ☐

3.4. **Classification of wonderful B^--varieties.** In this subsection we obtain a classification of wonderful B^--varieties based on the classification of smooth complete spherical varieties obtained in the previous subsection.

Wonderful B^--varieties are classified by admissible maps.

Definition 3.19 (see [Lun1, § 2]). A map $\eta\colon \Pi \times \Pi \to \{-3, -2, -1, 0, 1\}$ is said to be *admissible* if it satisfies the following five conditions:

(AM1) $\eta(\alpha, \alpha) \in \{0, 1\}$ for every $\alpha \in \Pi$;
(AM2) if $\eta(\alpha, \alpha) = 0$, then $\eta(\alpha, \beta) = \eta(\beta, \alpha) = 0$ for every $\beta \in \Pi$;
(AM3) if $\eta(\alpha, \beta) = 1$, then $\eta(\alpha, \gamma) = \eta(\beta, \gamma)$ for every $\gamma \in \Pi$;
(AM4) if $\eta(\alpha, \beta) < 0$, then $\eta(\beta, \alpha) = 0$;
(AM5) $\eta(\alpha, \beta) \geqslant \langle \alpha^\vee, \beta \rangle$ for every $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$.

Let η be an admissible map. Our immediate goal is to associate an Enriques' B^--system with η. First, we set $\Pi_\eta = \{\alpha \in \Pi \mid \eta(\alpha, \alpha) = 1\}$ and let $\mathfrak{X}_\eta$ denote the sublattice in $\mathfrak{X}(T)$ generated by Π_η. Second, we set $Q_\eta = \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{X}_\eta, \mathbb{Q})$ and let $\{\breve{\alpha} \mid \alpha \in \Pi_\eta\}$ be the basis of

Q_η dual to Π_η. We introduce the map $\rho_\eta\colon \Pi \to Q_\eta$ by the formula

$$\rho_\eta(\alpha) = \sum_{\gamma \in \Pi_\eta} \eta(\alpha, \gamma)\breve{\gamma}. \tag{3.6}$$

Next, let $\widetilde{\Pi}_\eta$ denote the collection of subsets $\Pi' \subset \Pi_\eta$ such that the restriction of ρ_η to Π' is injective. For every $\Pi' \in \widetilde{\Pi}_\eta$, let $\mathcal{C}_{\Pi'}$ be the convex cone generated by the set

$$S(\Pi') = \{\rho_\eta(\alpha) \mid \alpha \in \Pi'\} \cup \{-\breve{\alpha} \mid \alpha \in \Pi_\eta \backslash \Pi'\}.$$

At last, let $\mathcal{F}_\eta$ denote the set formed by all the cones $\mathcal{C}_{\Pi'}$ ($\Pi' \in \widetilde{\Pi}_\eta$) and their faces.

Lemma 3.20. *Let $\Pi' \in \widetilde{\Pi}_\eta$ and let $\alpha, \beta \in \Pi'$ be different roots. The following properties hold:*

(a) $\eta(\alpha, \beta) \leqslant 0$*;*

(b) *if $(\alpha, \beta) = 0$, then $\eta(\alpha, \beta) = \eta(\beta, \alpha) = 0$.*

Proof. To prove part (a), we note that by (AM3) the condition $\eta(\alpha, \beta) = 1$ would imply $\rho_\eta(\alpha) = \rho_\eta(\beta)$ contradicting the definition of $\widetilde{\Pi}_\eta$. Part (b) is a direct consequence of (a) and (AM5). □

Lemma 3.21. *Let $\Pi' \in \widetilde{\Pi}_\eta$.*

(a) *The set $S(\Pi')$ is linearly independent. In particular, $S(\Pi') = \mathcal{C}^1_{\Pi'}$ and the cone $\mathcal{C}_{\Pi'}$ is simplicial.*

(b) *The cone $\mathcal{C}_{\Pi'}$ is regular.*

Proof. To prove (a) it suffices to show that the set

$$\{\sum_{\gamma \in \Pi'} \eta(\alpha, \gamma)\breve{\gamma} \mid \alpha \in \Pi'\}$$

is linearly independent. (Note that the sum is taken over the set Π' instead of Π_η.) Let $\alpha' \in \Pi'$ be such that the corresponding node of the Dynkin diagram of Π' is incident to at most one edge. By Lemma 3.20(b) this means that there is at most one root $\gamma' \in \Pi'$ with $\eta(\alpha', \gamma') \neq 0$ (which implies $\eta(\alpha', \gamma') < 0$ by Lemma 3.20(a)). Applying (AM4) we obtain that either $\eta(\alpha', \gamma) = 0$ for all $\gamma \in \Pi' \backslash \{\alpha'\}$ or $\eta(\gamma, \alpha') = 0$ for all $\gamma \in \Pi' \backslash \{\alpha'\}$. In any case, the problem reduces to the linear independence of the set

$$\{\sum_{\gamma \in \Pi' \backslash \{\alpha'\}} \eta(\alpha, \gamma)\breve{\gamma} \mid \alpha \in \Pi' \backslash \{\alpha'\}\}.$$

The proof of (a) is completed by induction. From the above argument it also follows that the determinant of the transformation matrix from Π' to $S(\Pi')$ equals ± 1, which implies (b). □

Lemma 3.22. *For every subset $\Pi' \subset \Pi_\eta$, there is a root $\beta \in \Pi'$ such that $\eta(\beta, \gamma) \geqslant 0$ for all $\gamma \in \Pi'$.*

Proof. Assume the converse. Then there exists an infinite sequence $\beta_1, \beta_2, \beta_3, \ldots$ of roots in Π' with $\eta(\beta_i, \beta_{i+1}) < 0$ for all i. Applying (AM5) we obtain $(\beta_i, \beta_{i+1}) < 0$. Next, condition (AM4) implies $\eta(\beta_{i+1}, \beta_i) = 0$, whence $\beta_{i+2} \neq \beta_i$ for all i. Consequently, the Dynkin diagram of Π' contains a cycle, a contradiction. □

According to Lemma 3.22, we choose successively subsets $\Pi_1, \ldots, \Pi_s \subset \Pi_\eta$, where $s = s(\eta) = |\rho_\eta(\Pi_\eta)|$, in the following way. At first, we take a root $\beta_1 \in \Pi_\eta$ such that $\eta(\beta_1, \gamma) \geqslant 0$ for all $\gamma \in \Pi_\eta$ and set $\Pi_1 = \Pi_1(\eta) = \{\gamma \in \Pi_\eta \mid \eta(\beta, \gamma) = 1\}$. Next, assume that $i \in \{2, \ldots, s\}$ and the sets $\Pi_1, \ldots, \Pi_{i-1}$ are chosen. We put $\overline{\Pi}_i = \Pi_\eta \backslash (\Pi_1 \cup \ldots \cup \Pi_{i-1})$ and choose a root $\beta_i \in \overline{\Pi}_i$ such that $\eta(\beta_i, \gamma) \geqslant 0$ for all $\gamma \in \overline{\Pi}_i$. We set

$$\Pi_i = \Pi_i(\eta) = \{\gamma \in \overline{\Pi}_i \mid \eta(\beta_i, \gamma) = 1\}.$$

By (AM3), for every $i = 1, \ldots, s$ the set $\rho_\eta(\Pi_i)$ contains exactly one element; we denote it by $\rho_i = \rho_i(\eta)$.

Remark 3.23. For every $\Pi' \in \widetilde{\Pi}_\eta$ and every $i = 1, \ldots, s$, there is exactly one element σ_i in the set $\{-\breve{\alpha} \mid \alpha \in \Pi_i\} \cup \{\rho_i\}$ that is not contained in $S(\Pi')$. Moreover, all possible sets $(\sigma_1, \ldots, \sigma_s)$ are obtained in this way, so that

$$|\widetilde{\Pi}_\eta| = (|\Pi_1| + 1) \cdot \ldots \cdot (|\Pi_s| + 1).$$

For every $i = 1, \ldots, s$, let Q_i be the subspace of Q_η generated by the set

$$\{\breve{\beta} \mid \beta \in \Pi_1 \cup \ldots \cup \Pi_s\}.$$

Then by construction one has $\rho_i \in Q_i$ for every $i = 1, \ldots, s$.

For a fixed $i \in \{1, \ldots, s\}$, let $q \mapsto q^\diamond$ be the natural epimorphism from Q_i to Q_i/Q_{i-1}. Then $\rho_i^\diamond = -\sum\limits_{\gamma \in \Pi_i} \breve{\gamma}^\diamond$. A key observation is that the cones generated by all proper subsets of the set $\{\breve{\gamma}^\diamond \mid \gamma \in \Pi_i\} \cup \{\rho_i^\diamond\}$ form a complete fan in Q_i/Q_{i-1}; we denote this fan by $\mathcal{F}_i$.

Lemma 3.24. *The set $\mathcal{F}_\eta$ is a complete regular fan in $\mathfrak{X}_\eta$, and*

$$\mathcal{F}_\eta^1 = \rho_\eta(\Pi_\eta) \cup \{-\breve{\alpha} \mid \alpha \in \Pi_\eta\}.$$

Proof. The second assertion becomes obvious as soon as the first one has been proved.

We first show that $\mathcal{F}_\eta$ is a fan. To this end, it suffices to check that the intersection of any two cones in $\mathcal{F}_\eta$ is a face of each. Let $\mathcal{C}', \mathcal{C}'' \in \mathcal{F}_\eta$. Choose subsets $\Pi', \Pi'' \in \widetilde{\Pi}_\eta$ such that $\mathcal{C}'$ (resp. $\mathcal{C}''$) is a face of the cone $\mathcal{C}_{\Pi'}$ (resp. $\mathcal{C}_{\Pi''}$). For every $x \in \mathcal{C}' \cap \mathcal{C}''$, one has

$$x = \sum_{\alpha \in \Pi'} a_\alpha \rho_\eta(\alpha) - \sum_{\alpha \in \Pi_\eta \backslash \Pi'} b_\alpha \breve{\alpha} = \sum_{\alpha \in \Pi''} k_\alpha \rho_\eta(\alpha) - \sum_{\alpha \in \Pi_\eta \backslash \Pi''} l_\alpha \breve{\alpha},$$

where $a_\alpha, b_\alpha \in \mathbb{Q}^+$. We note that $a_\alpha = 0$ for all $\alpha \in \Pi'$ with $\rho_\eta(\alpha) \notin \mathcal{C}'$ and $b_\alpha = 0$ for all $\alpha \in \Pi_\eta \backslash \Pi'$ with $-\breve{\alpha} \notin \mathcal{C}'$; similarly, $k_\alpha = 0$ for all $\alpha \in \Pi''$ with $\rho_\eta(\alpha) \notin \mathcal{C}''$ and $l_\alpha = 0$ for all $\alpha \in \Pi_\eta \backslash \Pi''$ with $-\breve{\alpha} \notin \mathcal{C}''$.

We rewrite the two expressions for x in a slightly different form as follows:

$$x = \sum_{i=1}^{s} a_i \rho_i - \sum_{\alpha \in \Pi_\eta} b_\alpha \breve{\alpha} = \sum_{i=1}^{s} k_i \rho_i - \sum_{\alpha \in \Pi_\eta} l_\alpha \breve{\alpha}. \tag{3.7}$$

Here the coefficients $a_i, b_\alpha, k_i, l_\alpha \in \mathbb{Q}^+$ are determined by the following rule:

- if the set $\Pi_i \cap \Pi'$ is empty, then $a_i = 0$; otherwise this set contains exactly one root γ, in which situation $a_i = a_\gamma$;
- the coefficients b_α are already defined for $\alpha \in \Pi_\eta \backslash \Pi'$, all the others are zero;
- if the set $\Pi_i \cap \Pi''$ is empty, then $k_i = 0$; otherwise this set contains exactly one root γ, in which situation $k_i = k_\gamma$;

- the coefficients l_α are already defined for $\alpha \in \Pi_\eta \backslash \Pi''$, all the others are zero.

Regard the images in Q_s/Q_{s-1} of all parts of equality (3.7):

$$x^\diamond = a_s\rho_s^\diamond - \sum_{\alpha\in\Pi_s} b_\alpha\breve{\alpha}^\diamond = k_s\rho_s^\diamond - \sum_{\alpha\in\Pi_s} l_\alpha\breve{\alpha}^\diamond.$$

Clearly, for each of the cones $\mathcal{C}', \mathcal{C}''$ its image in Q_s/Q_{s-1} is a cone of the fan $\mathcal{F}_s$, and so the intersection of these images is again a cone in $\mathcal{F}_\eta$. This immediately implies $a_s = k_s$ and $b_\alpha = l_\alpha$ for all $\alpha \in \Pi_s$. Therefore,

$$\sum_{i=1}^{s-1} a_i\rho_i - \sum_{\alpha\in\Pi_\eta\backslash\Pi_s} b_\alpha\breve{\alpha} = \sum_{i=1}^{s-1} k_i\rho_i - \sum_{\alpha\in\Pi_\eta\backslash\Pi_s} l_\alpha\breve{\alpha}.$$

Applying induction, we obtain $a_i = k_i$ for all $i = 1, \ldots, s$ and $b_\alpha = l_\alpha$ for all $\alpha \in \Pi_\eta$. In particular, $a_i = 0$ for all $i = 1, \ldots, s$ with $\rho_i \notin \mathcal{C}''$ and $b_\alpha = 0$ for all $\alpha \in \Pi_\eta$ with $-\breve{\alpha} \notin \mathcal{C}''$; similarly, $k_i = 0$ for all $i = 1, \ldots, s$ with $\rho_i \notin \mathcal{C}'$ and $l_\alpha = 0$ for all $\alpha \in \Pi_\eta$ with $-\breve{\alpha} \notin \mathcal{C}'$. Hence $\mathcal{C}' \cap \mathcal{C}''$ is a common face of $\mathcal{C}', \mathcal{C}''$.

Now let us show that the fan $\mathcal{F}_\eta$ is complete. Let $x \in Q_\eta$ be an arbitrary element. Clearly, the element $x^\diamond \in Q_s/Q_{s-1}$ turns out to lie in a cone of the fan $\mathcal{F}_s$, and so there is a unique expression $x^\diamond = a_s\rho_s^\diamond - \sum\limits_{\alpha\in\Pi_s} b_\alpha\breve{\alpha}^\diamond$ with $a_s, b_\alpha \in \mathbb{Q}^+$ and at least one of the coefficients a_s, b_α being zero. Next, the element $x - a_s\rho_s + \sum\limits_{\alpha\in\Pi_s} b_\alpha\breve{\alpha}$ lies in the space Q_{s-1}, and by induction we get an expression $x = \sum\limits_{i=1}^{s} a_i\rho_i - \sum\limits_{\alpha\in\Pi_\eta} b_\alpha\breve{\alpha}$, where $a_i, b_\alpha \in \mathbb{Q}^+$ and for every $i = 1, \ldots, s$ at least one of the coefficients a_i, b_α ($\alpha \in \Pi_i$) is zero. For each $i = 1, \ldots, s$, we regard the coefficient a_i. If it is non-zero, then we choose any root $\alpha \in \Pi_i$ with $b_\alpha = 0$ and set $\Pi_i(x) = \{\alpha\}$. Otherwise we set $\Pi_i(x) = \varnothing$. At last, we set $\Pi(x) = \Pi_1(x) \cup \ldots \cup \Pi_s(x)$. By construction, $\Pi(x) \in \widetilde{\Pi}_\eta$ and $x \in \mathcal{C}_{\Pi(x)}$. ☐

Proposition 3.25 (see [Lun1, § 2])**.** *The triple $(\mathfrak{X}_\eta, \mathcal{F}_\eta, \rho_\eta)$ is an Enriques' B^--system.*

Proof. By construction and Lemma 3.24, it suffices to check conditions (a), (b), (c) of Definition 3.15. But these follow directly from properties (AM1)–(AM5) since for every $\alpha, \beta \in \Pi$ one has $\langle\rho_\eta(\alpha), \beta\rangle = \eta(\alpha, \beta)$ by (3.6). ☐

For an admissible map η, we denote by Z_η the smooth complete spherical B-variety corresponding to the Enriques' B^--system $(\mathfrak{X}_\eta, \mathcal{F}_\eta, \rho_\eta)$.

Given a wonderful B-variety Z, we let $(\mathfrak{X}_Z, \mathcal{F}_Z, \rho_Z)$ be the corresponding Enriques' B^--system and introduce the map $\eta_Z \colon \Pi \times \Pi \to \mathbb{Z}$ as follows:

$$\eta_Z(\alpha, \beta) = \begin{cases} \langle\rho_Z(\alpha), \beta\rangle & \text{if } \beta \in \mathfrak{X}_Z; \\ 0 & \textit{otherwise}. \end{cases} \tag{3.8}$$

Proposition 3.26 ([Lun1, Proposition 1])**.** *If Z is a wonderful B^--variety, then η_Z is an admissible map and $Z = Z_{\eta_Z}$. Conversely, if $\eta \colon \Pi \times \Pi \to \{-3, -2, -1, 0, 1\}$ is an admissible map, then Z_η is a wonderful B^--variety with $\eta_{Z_\eta} = \eta$.*

Proof. Suppose that Z is a wonderful B^--variety. The properties of an Enriques' B^--system imply that $\eta = \eta_Z$ satisfies all the axioms of an admissible map except for, possibly, the second part of (AM2): if $\eta(\alpha, \alpha) = 0$, then $\eta(\beta, \alpha) = 0$ for every $\beta \in \Pi$. Nevertheless, we introduce Π_η and $\mathfrak{X}_\eta$ as if η were admissible. Note that $\Pi_\eta = \{\alpha \in \Pi \mid \rho_Z(\alpha) \neq 0\}$. Regard the cone

$$\widetilde{\mathcal{C}}_0 = \{x \in Q_Z \mid \langle x, \alpha\rangle \leqslant 0 \text{ for all } \alpha \in \Pi_\eta\} \subset Q_Z.$$

By property (2) of Definition 3.2 there is a unique B^--fixed point z_0 in Z. Let $\mathcal{C}_0 \subset Q_Z$ be the maximal cone in $\mathcal{F}_Z$ corresponding to z_0. By property (3) of Definition 3.2, every T-stable prime divisor in Z containing z_0 is B-stable. Applying Lemma 3.17 we obtain $\langle \rho, \alpha\rangle \leqslant 0$ for all $\rho \in \mathcal{C}_0^1$ and $\alpha \in \Pi_\eta$, which implies $\mathcal{C}_0 \subset \widetilde{\mathcal{C}}_0$. On the other hand, let $\mathcal{C} \neq \mathcal{C}_0$ be an arbitrary maximal cone of the fan $\mathcal{F}_Z$ and let $z \in Z$ be the corresponding T-stable point. Since Z has only one B-stable point, z is B-unstable. By Lemma 3.18 the latter implies that there is a root $\alpha \in \Pi_\eta$ with $\langle c, \alpha\rangle \geqslant 0$ for all $c \in \mathcal{C}$. Therefore $\mathcal{C} \cap \widetilde{\mathcal{C}}_0^\circ = \varnothing$. Since the fan $\mathcal{F}_Z$ is complete, we obtain $\widetilde{\mathcal{C}}_0^\circ \subset \mathcal{C}_0$, hence $\widetilde{\mathcal{C}}_0 = \mathcal{C}_0$. As the cone $\mathcal{C}_0$ is simplicial, we have $\operatorname{rk} \mathfrak{X}_Z = |\Pi_\eta|$ and $\mathcal{C}_0^1 = \{\check{\alpha} \mid \alpha \in \Pi_\eta\}$. The regularity of $\mathcal{C}_0$ implies that the lattice $\mathfrak{X}_Z$ is generated by the set Π_η, whence $\mathfrak{X}_Z = \mathfrak{X}_\eta$. Consequently, every $\alpha \in \Pi$ with $\eta(\alpha, \alpha) = 0$ does not lie in $\mathfrak{X}_Z$, therefore $\eta(\beta, \alpha) = 0$ for every $\beta \in \Pi$ by (3.8). Thus the remaining part of condition (AM2) for η is proved.

Now let us prove that $Z = Z_\eta$. We have already shown that $\mathfrak{X}_Z = \mathfrak{X}_\eta$. Next, for every $\alpha \in \Pi_\eta$ we have

$$\rho_Z(\alpha) = \sum_{\gamma \in \Pi_\eta} \langle \rho_Z(\alpha), \gamma\rangle \check{\gamma} = \sum_{\gamma \in \Pi_\eta} \eta(\alpha, \gamma)\check{\gamma} = \rho_\eta(\alpha),$$

hence $\rho_Z = \rho_\eta$ and $\mathcal{F}_Z^1 \supset \mathcal{F}_\eta^1$. On the other hand, if $\rho \in \mathcal{F}_Z^1 \backslash \mathcal{C}_0^1$, then the corresponding T-stable prime divisor in Z does not contain z_0, and so it is not B^--stable and is moved by the subgroup $U_{-\alpha}$ for some $\alpha \in \Pi_Z$. In view of Lemma 3.13 the latter means that $\rho = \rho_Z(\alpha) = \rho_\eta(\alpha) \in \mathcal{F}_\eta^1$, hence $\mathcal{F}_Z^1 = \mathcal{F}_\eta^1$. At last, let us show that $\mathcal{F}_Z = \mathcal{F}_\eta$. As in the paragraph following Lemma 3.22, we introduce the number $s = s(\eta)$ and put $\Pi_i = \Pi_i(\eta)$, $\rho_i = \rho_i(\eta)$ for every $i = 1, \ldots, s$. Now let $\mathcal{C}$ be a maximal cone of the fan $\mathcal{F}_Z$ and assume that $\mathcal{C} \neq \mathcal{C}_{\Pi'}$ for every $\Pi' \in \widetilde{\Pi}_\eta$. Taking into account Remark 3.23, we conclude that there is $i \in \{1, \ldots, s\}$ such that $\mathcal{C}^1 \supset \{-\check{\alpha} \mid \alpha \in \Pi_i\} \cup \{\rho_i\}$. Since $\langle \rho_i, \alpha\rangle = 1$ for all $\alpha \in \Pi_i$ and $\langle \rho_i, \alpha\rangle \leqslant 0$ for all $\alpha \notin \Pi_i$, the element $\rho_i - \sum\limits_{\alpha \in \Pi_i} \check{\alpha}$ is contained in $\mathcal{C}_0 \cap \mathcal{C}$, the latter being a common face of $\mathcal{C}_0$ and $\mathcal{C}$. It follows that $\rho_i \in \mathcal{C}_0^1$, which is not the case. Hence $\mathcal{F}_Z = \mathcal{F}_\eta$.

Finally, let η be an admissible map and let Z_η be the corresponding smooth complete spherical B^--variety. Let $z_0 \in Z_\eta$ be the point corresponding to the cone

$$\mathcal{C}_\varnothing = \{c \in Q_\eta \mid \langle c, \alpha\rangle \leqslant 0 \text{ for all } \alpha \in \Pi_\eta\}. \tag{3.9}$$

Since $\mathcal{C}_\varnothing^1 \cap \rho_\eta(\Pi_\eta) = \varnothing$, the point z_0 is B^--fixed (see Lemma 3.18), and every T-stable irreducible closed subvariety of Z containing z_0 is B^--stable (see Lemma 3.17). Let $z \in Z_\eta$ be any T-fixed point different from z_0 and let $\mathcal{C} \in \mathcal{F}_\eta$ be the corresponding maximal cone. As $\mathcal{C} \neq \mathcal{C}_\varnothing$, there exists a root $\alpha \in \Pi_\eta$ such that $\langle \rho, \alpha\rangle \geqslant 0$ for all $\rho \in \mathcal{C}^1$. The latter implies that $-\check{\alpha} \notin \mathcal{C}^1$, hence $\rho_\eta(\alpha) \in \mathcal{C}^1$ (see Remark 3.23) and $\langle \rho, \alpha\rangle = 0$ for all $\rho \in \mathcal{C}^1 \backslash \{\rho_\eta(\alpha)\}$. By Lemma 3.18 the point z is B^--unstable. Thus Z_η is a wonderful B-variety and $\eta_{Z_\eta} = \eta$. ☐

3.5. **Relations to Luna's general classification.** For every admissible map η, we choose $H_\eta \subset B^-$ to be the stabilizer of a point of the open B^--orbit in the wonderful B^--variety Z_η. Theorem 3.4 and Proposition 3.26 imply the following result.

Theorem 3.27. *The map $\eta \mapsto H_\eta$ induces a bijection between admissible maps and conjugacy classes in B^- of strongly solvable wonderful subgroups of G contained in B^-.*

Let η be an admissible map and let Z_η be the corresponding wonderful B^--variety. Proposition 3.3(b) yields that $X_\eta = G *_{B^-} Z_\eta$ is a wonderful G-variety whose open G-orbit is isomorphic to G/H_η. The main goal of this subsection is to compute the spherical system of X_η.

Let $\phi\colon X_\eta \to G/B^-$ be the natural G-equivariant morphism. We identify Z_η with $\phi^{-1}(o)$. Let z_0 denote the unique B^--fixed point in Z_η. We also recall the notation $\Pi_\eta = \{\alpha \in \Pi \mid \eta(\alpha,\alpha) = 1\}$ and the map ρ_η given by (3.6). Let $\mathscr{S}_\eta = (\Pi^p_\eta, \Sigma_\eta, \mathcal{D}^a_\eta)$ be the spherical system of X_η and denote by $\varkappa_\eta$ the corresponding map $\mathcal{D}^a_\eta \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Sigma_\eta, \mathbb{Q})$.

Proposition 3.28. *The spherical system $\mathscr{S}_\eta$ is determined as follows:*

(a) $\Pi^p_\eta = \varnothing$.

(b) $\Sigma_\eta = \Pi_\eta$.

(c) *The set $\mathcal{D}^a_\eta$ is in bijection with the set $\Pi_\eta \cup \rho_\eta(\Pi_\eta)$. For every $\alpha \in \Pi_\eta$, let D^-_α (resp. D^+_α) be the color corresponding to α (resp. $\rho_\eta(\alpha)$). Then $\mathcal{D}(\alpha) = \{D^-_\alpha, D^+_\alpha\}$ for all $\alpha \in \Pi_\eta$, and one has $\langle \varkappa_\eta(D^+_\alpha), \beta\rangle = \eta(\alpha,\beta)$, $\langle \varkappa_\eta(D^-_\alpha), \beta\rangle = \langle \alpha^\vee, \beta\rangle - \eta(\alpha,\beta)$ for all $\beta \in \Pi_\eta$.*

Proof. (a) The open B-orbit in X_η maps onto the open B-orbit in G/B^-, whose stabilizer is well known to be B. Therefore $\Pi^p_\eta = \varnothing$.

(b) Evidently, the closed G-orbit in X_η is just Gz_0. Let us find the canonical B-chart $(X_\eta)_B$ (see § 2.2). By definition, $(X_\eta)_B$ consists of B-orbits in X_η whose closure contains Gz_0. Since Gz_0 maps (isomorphically) onto G/B^-, every such B-orbit maps necessarily onto $Bo \simeq B/T$, which is the unique open B-orbit in G/B^-. Therefore $(X_\eta)_B \subset \phi^{-1}(Bo)$. As we have already seen in the proof of Proposition 3.3, there are B-equivariant isomorphisms $\phi^{-1}(Bo) \simeq B *_T Z_\eta \simeq U \times Z_\eta$, where the B-action on the latter variety is given by formula (3.1). We note that every T-semi-invariant rational function on Z_η naturally extends to a B-semi-invariant rational function on $U \times Z_\eta$ of the same weight, and every B-semi-invariant rational function on $U \times Z_\eta$ is obtained in this way.

Since the set Bz_0 is open in Gz_0, a B-orbit $\mathcal{O} \subset B *_T Z_\eta$ is contained in $(X_\eta)_B$ if and only if the T-orbit $\mathcal{O} \cap Z_\eta \subset Z_\eta$ contains z_0 in its closure. It follows that $(X_\eta)_B \simeq B *_T Z_0 \simeq U \times Z_0$, where $Z_0 \subset Z_\eta$ is the B-stable affine open subset corresponding to the cone $\mathcal{C}_\varnothing \in \mathcal{F}_\eta$ given by (3.9). The explicit description of the cone $\mathcal{C}_\varnothing$ yields that the weight semigroup of T-semi-invariant functions in $\mathbb{C}[Z_0]$, as well as the weight semigroup of B-semi-invariant functions in $\mathbb{C}[(X_\eta)_B]$, is generated by the set $-\Pi_\eta$. Applying Proposition 2.8, we obtain $\Sigma_\eta = \Pi_\eta$.

(c) We first note that the open G-orbit $X^0_\eta \subset X_\eta$ is isomorphic to the homogeneous bundle $G *_{B^-} Z^0_\eta$, where Z^0_η is the open B^--orbit in Z_η. We also note that the fan $\mathcal{F}^0_\eta$ corresponding to Z^0_η as a toric T-variety consists of the cones $\mathcal{C} \in \mathcal{F}_\eta$ such that $\mathcal{C} \cap \mathcal{C}_\varnothing = \{0\}$.

Let $\mathcal{D}' \subset \mathcal{D}$ be the distinguished subset of colors corresponding to ϕ (see Definition 2.24) and let $D \in \mathcal{D}'$. Since $\phi(D)$ is dense in G/B, it follows that $D \cap \phi^{-1}(Bo)$ is a prime B-stable divisor in $\phi^{-1}(Bo) \simeq B *_T Z^0_\eta \simeq U \times Z^0_\eta$, where the B-action on the latter variety is given by (3.1). It is easy to deduce that the set $\mathcal{D}'$ is in bijection with the set of T-stable divisors in Z_η^0 or, equivalently, with the set $(\mathcal{F}_\eta^0)^1$ or, equivalently, with the set $\rho_\eta(\Pi_\eta)$. It follows that the set $\mathcal{D}'$ is formed by the colors D_α^+ ($\alpha \in \Pi_\eta$) indicated in the hypothesis. Next, since $\langle \varkappa_\eta(D), \alpha\rangle = 1$ for all $\alpha \in \Sigma \cap \Pi$ and $D \in \mathcal{D}(\alpha)$, it is easily verified that

$$\mathcal{D}(\alpha) \cap \{D_\beta^+ \mid \beta \in \Pi_\eta\} = \{D_\alpha^+\}$$

for every $\alpha \in \Pi_\eta$. In view of part (b) and Proposition 1.1, for every $\alpha \in \Pi_\eta$ the set $\mathcal{D}(\alpha)\backslash\{D_\alpha^+\}$ contains the divisor D_α^- indicated in the hypothesis. It remains to notice that the axioms of an admissible map imply $\varkappa_\eta(D_\alpha^-) \ne \varkappa_\eta(D_\beta^\pm)$ for any $\alpha, \beta \in \Pi_\eta$ with $\alpha \ne \beta$. □

Remark 3.29. In fact, the fan $\mathcal{F}_\eta^0$ contains a unique maximal cone, which is generated by the set $(\mathcal{F}_\eta^0)^1 = \{\rho_\eta(\alpha) \mid \alpha \in \Pi_\eta\}$.

Corollary 3.30. *Every strongly solvable spherical system has the form* $(\varnothing, \Pi', \mathcal{D}^a)$, *where* $\Pi' \subset \Pi$.

Corollary 3.31. *Every strongly solvable wonderful subgroup of G is spherically closed.*

Proof. Let $H \subset G$ be a strongly solvable wonderful subgroup and let $\overline{H}$ be its spherical closure. Since $\Sigma_{G/H} \subset \Pi$, by Proposition 2.23 the spherical systems of G/H and $G/\overline{H}$ coincide. The proof is completed by Theorem 2.21(a). □

Remark 3.32. The converse to Corollary 3.30 is not true. For example, regard the group $G = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, take the subgroup $H_0 \simeq \mathrm{SL}_2$ diagonally embedded in G, and set $H = N_G(H_0) = Z(G)H_0$. Clearly, $N_G(H) = H$, hence H is a wonderful subgroup of G by Theorem 2.16. Let ϖ_i be the fundamental weight of the ith factor of G, $i = 1, 2, 3$. One has $\mathfrak{X}(H) \simeq (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, and the semigroup $\widehat{\Lambda}_{G/H}^+$ is freely generated by the elements $(\varpi_1 + \varpi_2, a)$, $(\varpi_2 + \varpi_3, b)$, $(\varpi_3 + \varpi_1, c)$, where a, b, c are some pairwise different elements of order 2 in $\mathfrak{X}(H)$. By Proposition 1.9 one has $\Pi_{G/H}^p = \varnothing$ and $\Sigma_{G/H} = \Pi$.

Remark 3.33. In his preprint [Lun1] Luna also proved that, for two admissible maps η, η', the corresponding strongly solvable wonderful subgroups $H_\eta, H_{\eta'}$ are conjugate in G if and only if $\Pi_\eta = \Pi_{\eta'}$ and there is a bijection $i\colon \mathcal{D}_\eta^a \to \mathcal{D}_{\eta'}^a$ such that $\varkappa_\eta = \varkappa_{\eta'} \circ i$. In other words, $H_\eta, H_{\eta'}$ are conjugate in G if and only if $\mathscr{S}_\eta = \mathscr{S}_{\eta'}$, which is a particular case of Theorem 2.21.

Let $\mathscr{S} = (\Pi^p, \Sigma, \mathcal{D}^a)$ be a strongly solvable spherical system. We note that $\Pi^p = \varnothing$ and $\Sigma \subset \Pi$ by Corollary 3.30. Let $H \subset B^-$ be a strongly solvable wonderful subgroup corresponding to $\mathscr{S}$ and let X be the wonderful embedding of G/H. We fix a subset $\mathcal{D}' \subset \mathcal{D}^a$ satisfying the conditions of Proposition 2.28. This subset gives rise to a G-equivariant morphism $\phi\colon X \to G/B^-$, which provides a wonderful B^--variety $Z = \phi^{-1}(o)$ (see Proposition 3.3). Our last goal in this subsection is to find the admissible map η corresponding to Z.

Let $\mathcal{D}$ be the set of colors of X. For every $\alpha \in \Sigma$, the set

$$\mathcal{D}(\alpha) \cap \mathcal{D}' = \{D \subset D' \mid \langle \varkappa(D), \alpha\rangle = 1\}$$

contains exactly one element, we denote it by D_α^+. The following proposition is a direct consequence of Proposition 3.28(c).

Proposition 3.34. *The admissible map η is determined as follows:*

$$\eta(\alpha,\beta) = \begin{cases} \langle \varkappa(\mathcal{D}^+_\alpha), \beta\rangle & \text{if } \alpha,\beta\in\Sigma;\\ 0 & \text{otherwise} \end{cases}$$

4. Explicit classification of strongly solvable spherical subgroups

Throughout this section the group G is assumed to be semisimple.

4.1. **Description of the approach.** Let $H \subset B^-$ be a strongly solvable subgroup of G and let $N \subset U^-$ be the unipotent radical of H. We say that H is *standardly embedded in* B^- (with respect to T) if $S = H \cap T$ is a Levi subgroup of H, so that $H = S \leftthreetimes N$. General results on Levi decompositions (see, for instance, [OniV, 6.4]) yield that every strongly solvable subgroup $H \subset B^-$ is conjugate in B^- to a subgroup standardly embedded in B^-.

In what follows, we assume that H is standardly embedded in B^- and keep the decomposition $H = S \leftthreetimes N$. Let $\tau\colon \mathfrak{X}(T) \to \mathfrak{X}(S)$ be the character restriction map from T to S. Regard the natural action of S on $\mathfrak{u}^-$. This action yields a decomposition $\mathfrak{u}^- = \bigoplus\limits_{\varphi\in\tau(\Delta^+)} \mathfrak{u}_{-\varphi}$, where $\mathfrak{u}^-_\varphi$ is the weight subspace of weight $-\varphi$ with respect to S. For every $\varphi \in \tau(\Delta^+)$, we put $\mathfrak{n}_\varphi = \mathfrak{u}^-_\varphi \cap \mathfrak{n}$ and denote by c_φ the codimension of $\mathfrak{n}^-_\varphi$ in $\mathfrak{u}^-_\varphi$.

Proposition 4.1 ([Avd2, Theorem 1])**.** *In the above notation, the following conditions are equivalent:*

(1) *H is spherical in G;*

(2) *$c_\varphi \leqslant 1$ for every $\varphi \in \Phi$, and all weights φ with $c_\varphi = 1$ are linearly independent in $\mathfrak{X}(S)$.*

Until the end of this subsection we assume that H is spherical.

According to Proposition 4.1, we introduce the set

$$\Phi = \{\varphi \in \mathfrak{X}(S) \mid c_\varphi = 1\}.$$

For every $\varphi \in \Phi$, we put $\Psi_\varphi = \{\alpha \in \Delta^+ \mid \tau(\alpha) = \varphi \text{ and } \mathfrak{g}_{-\alpha} \not\subset \mathfrak{n}\}$ and

$$\Psi = \bigcup_{\varphi\in\Phi} \Psi_\varphi. \tag{4.1}$$

(Note that the union is disjoint.) Clearly,

$$\Psi = \{\alpha \in \Delta \mid \mathfrak{g}_{-\alpha} \not\subset \mathfrak{n}\}.$$

Definition 4.2. Roots in Ψ are said to be *active.*

Disjoint union (4.1) naturally determines an equivalence relation on the set Ψ, which will be denoted by $\sim$. We note that for every $\alpha, \beta \in \Psi$ one has $\alpha \sim \beta$ if and only if $\tau(\alpha) = \tau(\beta)$.

Active roots have the following property (see [Avd2, Lemma 4]): if α is an active root and $\alpha = \beta + \gamma$ for some roots $\beta, \gamma \in \Delta^+$, then exactly one of the roots β, γ is active. Taking this property into account, we say that an active root β is *subordinate* to an active root α if $\alpha = \beta + \gamma$ for some root $\gamma \in \Delta^+$. For every active root α, we denote by $F(\alpha)$ the set consisting of α and all active roots subordinate to α. An active root α is said to be *maximal* if it is not subordinate to any other active root. We denote by M the set of maximal active roots.

The following proposition plays an important role in the structure theory of strongly solvable spherical subgroups standardly embedded in B^-.

Proposition 4.3 ([Avd2, Proposition 1]). *Let $\varphi, \varphi' \in \Phi$. Suppose that roots $\alpha \in \Psi_\varphi$ and $\beta \in \Psi_{\varphi'}$ are different and $\gamma = \beta - \alpha \in \Delta^+$. Then $\Psi_\varphi + \gamma \subset \Psi_{\varphi'}$.*

Corollary 4.4. *For every $\varphi \in \Phi$, one has either $\Psi_\varphi \subset \mathrm{M}$ or $\Psi_\varphi \cap \mathrm{M} = \varnothing$.*

Corollary 4.5 ([Avd2, Lemma 3]). *The angles between the roots in* M *are pairwise non-acute. In particular, the roots in* M *are linearly independent.*

For every $\varphi \in \Phi$, the subspace $\mathfrak{n}_\varphi^- \subset \mathfrak{u}_{-\varphi}$ is the kernel of a linear function $\xi_\varphi \in (\mathfrak{u}_\varphi^-)^*$, which is uniquely determined up to proportionality. We note that $\xi_\varphi(e_{-\alpha}) \ne 0$ for all $\alpha \in \Psi_\varphi$ and $\xi_\varphi(e_{-\alpha}) = 0$ for all $\alpha \in \Delta^+ \backslash \Psi_\varphi$ with $\tau(\alpha) = \varphi$.

Proposition 4.6 ([Avd2, Proposition 2]). *Suppose that $\varphi, \varphi' \in \Phi$, $\varphi \ne \varphi'$, and $\Psi_\varphi + \gamma \subset \Psi_{\varphi'}$ for some $\gamma \in \Delta^+$. Then there is a constant $c \ne 0$ such that $\xi_\varphi(e_{-\alpha}) = c\xi_{\varphi'}([e_{-\alpha}, e_{-\gamma}])$ for all $\alpha \in \Psi_\varphi$. In particular, up to proportionality, ξ_φ is uniquely determined by $\xi_{\varphi'}$.*

Corollary 4.7. *The subspace $\mathfrak{n} \subset \mathfrak{u}^-$ is uniquely determined by linear functions ξ_φ, where φ runs over the set $\tau(\mathrm{M}) \subset \Phi$.*

Corollaries 4.5 and 4.7 yield the following result.

Theorem 4.8 ([Avd2, Theorem 2]). *Up to conjugation by an element of T, a strongly solvable spherical subgroup H standardly embedded in B^- is uniquely determined by the pair (S, Ψ). Moreover, H is explicitly recovered from (S, Ψ).*

Proposition 4.9 ([Avd2, Proposition 3]). *Let α be an active root. Then there is a unique simple root $\pi(\alpha) \in \operatorname{Supp}\alpha$ with the following property: if $\alpha = \beta + \gamma$ for some roots $\beta, \gamma \in \Delta^+$, then β (resp. γ) is active if and only if $\pi(\alpha) \notin \operatorname{Supp}\beta$ (resp. $\pi(\alpha) \notin \operatorname{Supp}\gamma$).*

This proposition provides a map $\pi \colon \Psi \to \Pi$.

Proposition 4.10. *The set $(\Psi, \sim)$ amounts to the set $(\mathrm{M}, \pi, \sim)$ with π and $\sim$ restricted to* M.

Proof. Clearly, $(\mathrm{M}, \pi, \sim)$ is determined by $(\Psi, \sim)$. Let us show the converse. Proposition 4.9 implies that for every active root α the set $F(\alpha)$ is uniquely determined by the simple root $\pi(\alpha)$. Since every active root is subordinate to a maximal active root (see [Avd2, Corollary 3(b)]), it follows that the whole set Ψ is uniquely determined by M and the restriction of π to M. At last, Proposition 4.3 and Corollary 4.4 show that for $\alpha, \beta \in \Psi$ one has $\alpha \sim \beta$ if and only if there is an element $\gamma \in \Delta^+ \cup \{0\}$ such that $\alpha + \gamma, \beta + \gamma \in \mathrm{M}$ and $\alpha + \gamma \sim \beta + \gamma$. □

The following lemma together with its corollary will be useful in § 4.3.

Lemma 4.11. *Let $\alpha \in \Delta^+$ and regard the set $I_\alpha = \{\alpha\} \cup \{\beta \in \Delta^+ \mid \alpha - \beta \in \Delta^+\}$. Then the sublattice $\mathbb{Z}I_\alpha \subset \mathbb{Z}\Delta$ contains $\operatorname{Supp}\alpha$.*

Proof. We use the same idea as in the proof of [Avd2, Lemma 6(b)]. Regard the expression $\alpha = \sum\limits_{\gamma \in \Pi} c_\gamma \gamma$ and put $\operatorname{ht}\alpha = \sum\limits_{\gamma\in\Pi} c_\gamma$. We prove the assertion by induction on $\operatorname{ht}\alpha$. If $\operatorname{ht}\alpha = 1$ then the assertion is true. Assume that $\operatorname{ht}\alpha = k$ and the assertion is proved for all $\alpha' \in \Delta^+$ with $\operatorname{ht}\alpha' < k$. By a well-known lemma from linear algebra (see [Avd2, Lemma 1]) there is a root $\alpha_0 \in \Pi$ such that $(\alpha, \alpha_0) > 0$. Then $\alpha - \alpha_0 \in \Delta^+$, whence $\alpha_0 \in I_\alpha$. Let r_0 be the reflection corresponding to α_0. Note that for every $\gamma \in \Delta$ one has $r_0(\gamma) = \gamma - \langle \gamma \,|\, \alpha_0 \rangle \alpha_0 \in \gamma + \mathbb{Z}\alpha_0$. Regard the root $\beta = r_0(\alpha) \in \Delta^+$. Clearly, $\operatorname{ht}\beta < \operatorname{ht}\alpha$ and $\operatorname{Supp}\alpha \supset \operatorname{Supp}\beta \supset \operatorname{Supp}\alpha\backslash\{\alpha_0\}$. By the induction hypothesis, $\mathbb{Z}I_\beta \supset \operatorname{Supp}\beta$. Since for every $\beta' \in I_\beta$ one has either $r_0(\beta') \in I_\alpha$ or $r_0(\beta') \in \{-\alpha_0, \alpha + \alpha_0\}$, we conclude that $I_\alpha \supset \{\alpha_0\} \cup (r_0(I_\beta)\backslash\{-\alpha_0, \alpha+\alpha_0\})$. Hence

$$\mathbb{Z}I_\alpha \supset \mathbb{Z}(r_0(I_\beta) \cup \{\alpha_0\}) \supset \mathbb{Z}(r_0(\operatorname{Supp}\beta) \cup \{\alpha_0\}) \supset \operatorname{Supp}\beta \cup \{\alpha_0\} = \operatorname{Supp}\alpha,$$

where the relation $r_0(\operatorname{Supp}\beta) \subset \operatorname{Supp}\beta + \mathbb{Z}\alpha_0$ is taken into account. □

To state the corollary, we need to introduce the set $\Pi_0 = \bigcup\limits_{\beta \in \mathrm{M}} \operatorname{Supp}\beta \subset \Pi$. Since for every $\alpha \in \Psi$ the map $\pi \colon F(\alpha) \to \operatorname{Supp}\alpha$ is bijective (see [Avd2, Corollary 6]), we have $\Pi_0 = \pi(\Psi)$.

Corollary 4.12. (a) *For every $\alpha \in \Psi$, the sublattice $\mathbb{Z}F(\alpha) \subset \mathfrak{X}(T)$ contains* $\operatorname{Supp}\alpha$.
(b) $\mathbb{Z}\Psi = \mathbb{Z}\Pi_0$.
(c) *The sublattice $\mathbb{Z}\Phi \subset \mathfrak{X}(S)$ contains $\tau(\Pi_0)$.*

The following proposition lists all possibilities for a pair $(\alpha, \pi(\alpha))$ with $\alpha \in \Psi$.

Proposition 4.13 ([Avd2, Theorem 3])**.** *For every active root α, the pair $(\alpha, \pi(\alpha))$ is contained in Table* 2.

TABLE 2. ACTIVE ROOTS

No.	Type of $\operatorname{Supp}\alpha$	α	$\pi(\alpha)$
1	any of rank n	$\alpha_1 + \alpha_2 + \ldots + \alpha_n$	$\alpha_1, \alpha_2, \ldots, \alpha_n$
2	B_n	$\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} + 2\alpha_n$	$\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$
3	C_n	$2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$	α_n
4	F_4	$2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	α_3, α_4
5	G_2	$2\alpha_1 + \alpha_2$	α_2
6	G_2	$3\alpha_1 + \alpha_2$	α_2

The notation in Table 2 is as follows. We denote by α_i the ith simple root in $\operatorname{Supp}\alpha$. In the column "$\pi(\alpha)$" we list all possibilities for $\pi(\alpha)$ for a given active root α.

4.2. **Classification.** In this subsection we present the classification of strongly solvable spherical subgroups standardly embedded in B^-, up to conjugation by elements of T, and then explain when two such subgroups are conjugate in G.

Let $\alpha \in \Delta^+$. A simple root $\delta \in \operatorname{Supp}\alpha$ is said to be *terminal with respect to* $\operatorname{Supp}\alpha$ if in the Dynkin diagram of $\operatorname{Supp}\alpha$ the node corresponding to δ is joined by an edge (possibly multiple) with exactly one node.

We now list several conditions on a pair α, β of active roots. These conditions will be used below in Definition 4.14.

(D0) $\operatorname{Supp}\alpha \cap \operatorname{Supp}\beta = \varnothing$;

(D1) $\operatorname{Supp}\alpha \cap \operatorname{Supp}\beta = \{\delta\}$, where $\pi(\alpha) \neq \delta$, $\pi(\beta) \neq \delta$, and δ is terminal with respect to both $\operatorname{Supp}\alpha$ and $\operatorname{Supp}\beta$;

(E1) $\operatorname{Supp}\alpha \cap \operatorname{Supp}\beta = \{\delta\}$, where $\delta = \pi(\alpha) = \pi(\beta)$, $\alpha - \delta \in \Delta^+$, $\beta - \delta \in \Delta^+$, and δ is terminal with respect to both $\operatorname{Supp}\alpha$ and $\operatorname{Supp}\beta$;

(D2) the Dynkin diagram of $\operatorname{Supp}\alpha \cup \operatorname{Supp}\beta$ has the form shown on Figure 1 (for some $p, q, r \geqslant 1$), $\alpha = \alpha_1 + \ldots + \alpha_p + \gamma_0 + \gamma_1 + \ldots + \gamma_r$, $\beta = \beta_1 + \ldots + \beta_q + \gamma_0 + \gamma_1 + \ldots + \gamma_r$, $\pi(\alpha) \notin \operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$, and $\pi(\beta) \notin \operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$;

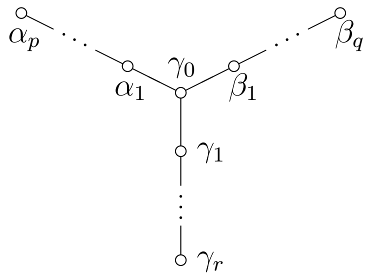

FIGURE 1

(E2) the Dynkin diagram of $\operatorname{Supp}\alpha \cup \operatorname{Supp}\beta$ has the form shown on Figure 1 (for some $p, q, r \geqslant 1$), $\alpha = \alpha_1 + \ldots + \alpha_p + \gamma_0 + \gamma_1 + \ldots + \gamma_r$, $\beta = \beta_1 + \ldots + \beta_q + \gamma_0 + \gamma_1 + \ldots + \gamma_r$, and $\pi(\alpha) = \pi(\beta) \in \operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$.

Definition 4.14. Suppose that $S \subset T$ is a subgroup, $\mathrm{M} \subset \Delta^+$ is a subset, $\pi \colon \mathrm{M} \to \Pi$ is a map, and $\sim$ is an equivalence relation on M. Let $\tau \colon \mathfrak{X}(T) \to \mathfrak{X}(S)$ be the character restriction map and put $\Pi_0 = \bigcup\limits_{\beta \in \mathrm{M}} \operatorname{Supp}\beta$.

The triple $(\mathrm{M}, \pi, \sim)$ is said to be an *ARS-set*[3] if it satisfies the following conditions:

(A) if $\alpha \in \mathrm{M}$, then $\pi(\alpha) \in \operatorname{Supp}\alpha$ and the pair $(\alpha, \pi(\alpha))$ is contained in Table 2;

(D) if $\alpha, \beta \in \mathrm{M}$ and $\alpha \nsim \beta$, then one of possibilities (D0), (D1), (D2) is realized;

(E) if $\alpha, \beta \in \mathrm{M}$ and $\alpha \sim \beta$, then one of possibilities (D0), (D1), (E1), (D2), (E2) is realized;

(C) if $\alpha \in \mathrm{M}$, then $\operatorname{Supp}\alpha \not\subset \bigcup\limits_{\delta \in \mathrm{M}\setminus\{\alpha\}} \operatorname{Supp}\delta$.

The quadruple $(S, \mathrm{M}, \pi, \sim)$ is said to be an *extended ARS-set* if $(\mathrm{M}, \pi, \sim)$ is an ARS-set and condition (T) below is satisfied:

(T) $\operatorname{Ker}\tau \cap \mathbb{Z}\Pi_0 = \mathbb{Z}\{\alpha - \beta \mid \alpha, \beta \in \mathrm{M}, \alpha \sim \beta\}$.

Remark 4.15. Thanks to Corollary 4.12(b), the original form of condition (T) from [Avd2] is equivalent to that presented in the definition.

Now let $H \subset G$ be a strongly solvable spherical subgroup standardly embedded in B^-. We retain all the notation introduced in § 4.1. We also set $\Upsilon(H) = (S, \mathrm{M}, \pi, \sim)$ and $\Upsilon_0(H) = (\mathrm{M}, \pi, \sim)$, where π and $\sim$ are restricted to M.

The following theorem provides a classification of all strongly solvable spherical subgroups in G standardly embedded in B^-, see [Avd2, Theorems 4 and 5].

Theorem 4.16. *The map $H \mapsto \Upsilon(H)$ is a bijection between strongly solvable spherical subgroups standardly embedded in B^-, up to conjugation by elements of T, and extended ARS-sets.*

To complete the presentation of the explicit classification, in the remaining part of this subsection we explain when two strongly solvable spherical subgroups $H, H' \subset G$ standardly embedded in B^- are conjugate in G. Let Ψ be the set of active roots of H.

Definition 4.17. A root $\delta \in \Psi$ is said to be *regular* if the projection of the subspace $\mathfrak{n} \subset \mathfrak{u}^-$ to the root subspace $\mathfrak{g}_\delta$ along the sum of the other root subspaces is zero.

[3]"ARS" is an abbreviation for "active root system".

Let Ψ^{reg} denote the set of regular active roots.

Definition 4.18. Suppose that $\delta \in \Psi^{\mathrm{reg}} \cap \Pi$. An *elementary transformation with center* δ (or simply an *elementary transformation*) is a transformation of the form $H \mapsto \rho_\delta H \rho_\delta^{-1}$, where $\rho_\delta \in N_G(T)$ is any element whose image in the Weyl group $N_G(T)/T$ coincides with the simple reflection associated with δ.

Evidently, in this definition the group $\rho_\delta H \rho_\delta^{-1}$ is also standardly embedded in B^-.

Theorem 4.19 ([Avd2, Theorem 6])**.** *Let $H, H' \subset G$ be strongly solvable spherical subgroups standardly embedded in B^-. The subgroups H, H' are conjugate in G if and only if there is a chain of elementary transformations taking H to H'.*

Remark 4.20. For an elementary transformation $H \mapsto H'$, the extended ARS-set of H' is explicitly expressed in terms of that of H, see [Avd2, Proposition 14].

Remark 4.21. In the following subsection we shall compute the homogeneous spherical datum corresponding to a strongly solvable spherical subgroup of G standardly embedded in B^-, see Theorem 4.27. By Theorem 2.21 this will provide an alternative approach for determining when two strongly solvable spherical subgroups $H, H' \subset G$ standardly embedded in B^- are conjugate in G.

4.3. **Computation of the invariants.** Let $H \subset G$ be a strongly solvable subgroup standardly embedded in B^-. We retain all the notation introduced in the previous subsection.

Let $L \subset \mathfrak{X}(T)$ be the sublattice generated by all elements of the form $\alpha - \beta$, where $\alpha, \beta \in \Psi$ and $\alpha \sim \beta$. Clearly, $L \subset \operatorname{Ker} \tau$.

For every $\varphi \in \Phi$, we regard the set $\pi(\Psi_\varphi)$ and put $\lambda_\varphi = \sum\limits_{\alpha \in \pi(\Psi_\varphi)} \varpi_\alpha$.

Theorem 4.22. *Suppose that G is simply connected. Then the semigroup $\widehat{\Lambda}^+_{G/H}$ is freely generated by all the elements $\Omega_\alpha = (\varpi_\alpha, -\tau(\varpi_\alpha))$, $\alpha \in \Pi$, and all the elements $\Omega_\varphi = (\lambda_\varphi, -\tau(\lambda_\varphi) + \varphi)$, $\varphi \in \Phi$.*

Proof. In the case of connected H this theorem was proved in [AvdG, Theorem 4]. (In fact, in [AvdG] the group H was assumed to be standardly embedded in B, so one has to translate those results into our settings.) But the condition of H being connected is inessential here. Indeed, it is obvious that for every $\alpha \in \Pi$ the subspace $V(\varpi_\alpha^*)^{(H^0)}_{-\tau(\varpi_\alpha)} \subset V(\varpi_\alpha^*)$ is spanned by a lowest-weight vector $w_{\varpi_\alpha^*}$, which is even B^--semi-invariant. Further, it was shown in [AvdG] that for every $\varphi \in \Phi$ the subspace $V(\lambda_\varphi^*)^{(H^0)}_{-\tau(\lambda_\varphi)+\varphi} \subset V(\lambda_\varphi^*)$ is spanned by a vector of the form $(\sum\limits_{\beta \in \Psi_\varphi} b_\beta e_\beta) \cdot v_{\lambda_\varphi^*}$, where all the coefficients b_β are non-zero. For all of these vectors to be S-semi-invariant, the only condition that matters is $L \subset \operatorname{Ker} \tau$. □

For every $\alpha \in \Pi$ (resp. $\varphi \in \Phi$), let D_α (resp. D_φ) be the color of G/H corresponding to the element Ω_α (resp. Ω_φ) of $\widehat{\Lambda}^+$.

Remark 4.23. Under the natural morphism $G/H \to G/B^-$, every color D_α ($\alpha \in \Pi$) is the preimage of a color of G/B^-, and every color D_φ ($\varphi \in \Phi$) maps dominantly to G/B^-.

We denote by $\widehat{S}$ the subgroup of T defined by the vanishing of all elements in $\mathscr{L}$. Clearly, $\widehat{S}$ is the largest subgroup of T normalizing N. We set $\widehat{H} = \widehat{S} \leftthreetimes N$.

Proposition 4.24. *The spherical closure $\overline{H}$ of H coincides with $\widehat{H}$.*

Proof. To prove the assertion, we apply Proposition 2.13. For every $\alpha \in \Pi$, the stabilizer in G of the line $\langle v_{\varpi_\alpha^*} \rangle = V(\varpi_\alpha^*)^{(H)}_{-\tau(\varpi_\alpha)}$ is the parabolic subgroup $P^-_{\Pi \setminus \{\alpha\}}$. The intersection of such parabolic subgroups over all $\alpha \in \Pi$ is B^-, whence $\overline{H} \subset B$. On the other hand, by [Avd3, Theorem 3] one has $N_G(H) \cap B^- = N_G(H^0) \cap B^- = \widehat{H}$. (Here we also use that $N_G(H) = N_G(H^0)$, see [BriP, Corollary 5.2].) As was already mentioned in the proof of Theorem 4.22, for every $\varphi \in \Phi$ the line $V(\lambda_\varphi^*)^{(H)}_{-\tau(\lambda_\varphi)+\varphi}$ is $\widehat{H}$-semi-invariant. $\square$

Corollary 4.25. (a) *A strongly solvable wonderful subgroup of G contained in B^- is uniquely determined by its unipotent radical.*

(b) *There is a bijection between ARS-sets and conjugacy classes in B^- of strongly solvable wonderful subgroups of G contained in B^-.*

Proof. Recall that by Corollary 3.31 every strongly solvable wonderful subgroup is spherically closed.

(a) (compare with the proof of [Avd2, Lemma 32]) Let H be a strongly solvable wonderful subgroup of G standardly embedded in B^- and let N be the unipotent radical of H. Then by Proposition 4.24 a Levi subgroup of H is recovered as a Levi subgroup of $N_{B^-}(N)$.

(b) Let H, H' be two strongly solvable wonderful subgroups standardly embedded in B^- and let $\Upsilon_0(H), \Upsilon_0(H')$ be the corresponding ARS-sets. Suppose that $H' = bHb^{-1}$ for some $b \in B^-$. Then by [Avd2, Proposition 13] one has $b \in N_G(T) \cap B^- = T$, whence $\Upsilon_0(H) = \Upsilon_0(H')$. $\square$

According to Corollary 4.12(c), for every $\alpha \in \Pi_0$ and every $\varphi \in \Phi$ we introduce integers $J(\varphi, \alpha)$ in such a way that

$$\tau(\alpha) = \sum_{\varphi \in \Phi} J(\varphi, \alpha)\varphi. \tag{4.2}$$

Lemma 4.26. *The union $\pi(\Psi) = \bigcup\limits_{\varphi \in \Phi} \pi(\Psi_\varphi)$ is disjoint.*

Proof. This is a direct consequence of [Avd2, Lemma 10]. $\square$

In view of Lemma 4.26, every root $\alpha \in \Pi_0$ determines a unique weight $\varphi[\alpha] \in \Phi$ such that $\alpha \in \pi(\Phi_{\varphi[\alpha]})$.

Theorem 4.27. *The principal combinatorial invariants of G/H are determined as follows:*

(a) $\Lambda_{G/H} = \mathbb{Z}\Pi_0 + \operatorname{Ker} \tau$;

(b) $\Pi^p_{G/H} = \varnothing$;

(c) $\Sigma_{G/H} = \Pi_0$;

(d) *The set $\mathcal{D}^a_{G/H}$ consists of all divisors D_α, $\alpha \in \Pi_0$, and all divisors D_φ, $\varphi \in \Phi$; moreover, $\langle \varkappa(D_\alpha), \mu \rangle = \langle \alpha^\vee, \mu \rangle - J(\varphi[\alpha], \mu)$ for every $\alpha \in \Pi_0$ and $\mu \in \mathbb{Z}\Pi_0 + \operatorname{Ker}\tau$; and $\langle \varkappa(D_\varphi), \mu \rangle = J(\varphi, \mu)$ for every $\varphi \in \Phi$ and $\mu \in \mathbb{Z}\Pi_0 + \operatorname{Ker}\tau$.*

Proof. Without loss of generality we may assume that G is simply connected.

(a) Let $\mu \in \mathfrak{X}(T)$. By Proposition 1.6, $\mu \in \Lambda_{G/H}$ if and only if there is an expression

$$(\mu, 0) = \sum_{\alpha \in \Pi} c_\alpha \Omega_\alpha + \sum_{\varphi \in \Phi} c_\varphi \Omega_\varphi, \tag{4.3}$$

where c_α and c_φ are integers for all $\alpha \in \Pi$ and $\varphi \in \Phi$.

First suppose that $\mu \in \Lambda_{G/H}$. Then (4.3) implies the following equality in the group $\mathfrak{X}(S)$:

$$0 = \sum_{\alpha \in \Pi} c_\alpha \tau(\varpi_\alpha) + \sum_{\varphi \in \Phi} c_\varphi \tau(\lambda_\varphi) - \sum_{\varphi \in \Phi} c_\varphi \varphi.$$

For every $\varphi \in \Phi$, fix any root $\alpha_\varphi \in \Psi$ with $\tau(\alpha_\varphi) = \varphi$. Then

$$\sum_{\alpha \in \Pi} c_\alpha \varpi_\alpha + \sum_{\varphi \in \Phi} c_\varphi \lambda_\varphi - \sum_{\varphi \in \Phi} c_\varphi \alpha_\varphi \in \operatorname{Ker} \tau.$$

Hence in view of (4.3) one has

$$\mu = \sum_{\alpha \in \Pi} c_\alpha \varpi_\alpha + \sum_{\varphi \in \Phi} c_\varphi \lambda_\varphi \in \sum_{\varphi \in \Phi} c_\varphi \alpha_\varphi + \operatorname{Ker} \tau \subset \mathbb{Z}\Psi + \operatorname{Ker} \tau = \mathbb{Z}\Pi_0 + \operatorname{Ker} \tau,$$

where the latter equality holds by Corollary 4.12(b).

Conversely, suppose that $\mu \in \mathbb{Z}\Pi_0 + \operatorname{Ker} \tau = \mathbb{Z}\Psi + \operatorname{Ker} \tau$. Then by Proposition 4.1 there is a unique expression

$$\tau(\mu) = \sum_{\varphi \in \Phi} J(\varphi, \mu) \varphi, \tag{4.4}$$

where $J(\varphi, \mu) \in \mathbb{Z}$ for all $\varphi \in \Phi$. Hence

$$(0, \tau(\mu)) = \sum_{\varphi \in \Phi} J(\varphi, \mu)(0, \varphi) = \sum_{\varphi \in \Phi} J(\varphi, \mu)\Big(\sum_{\alpha \in \pi(\Psi_\varphi)} \Omega_\varphi - \Omega_\alpha\Big) = \sum_{\varphi \in \Phi} J(\varphi, \mu)\Omega_\varphi - \sum_{\varphi \in \Phi} \sum_{\alpha \in \pi(\Psi_\varphi)} J(\varphi, \mu)\Omega_\alpha.$$

By Lemma 4.26, the double sum in the latter expression is just a sum over the set $\pi(\Psi) = \Pi_0$, whence

$$(0, \tau(\mu)) = \sum_{\varphi \in \Phi} J(\varphi, \mu)\Omega_\varphi - \sum_{\alpha \in \Pi_0} J(\varphi[\alpha], \mu)\Omega_\alpha.$$

Taking into account the expression $(\mu, -\tau(\mu)) = \sum\limits_{\alpha \in \Pi} \langle \alpha^\vee, \mu \rangle \Omega_\alpha$, we finally obtain

$$(\mu, 0) = \sum_{\alpha \in \Pi} \langle \alpha^\vee, \mu \rangle \Omega_\alpha - \sum_{\alpha \in \Pi_0} J(\varphi[\alpha], \mu)\Omega_\alpha + \sum_{\varphi \in \Phi} J(\varphi, \mu)\Omega_\varphi, \tag{4.5}$$

whence $\mu \in \Lambda_{G/H}$.

(b)–(d) The knowledge of the semigroup $\widehat{\Lambda}^+_{G/H}$ in combination with Proposition 1.9 yields that $\Pi^p_{G/H} = \varnothing$, $\Pi \cap \Sigma_{G/H} = \Pi_0$, and the set $\mathcal{D}^a_{G/H}$ consists of all the divisors D_α with $\alpha \in \Pi_0$ and all the divisors D_φ, where $\varphi \in \Phi$. By Proposition 1.7, the values on $\Lambda_{G/H}$ of elements in $\varkappa(\mathcal{D}^a_{G/H})$ are read off from expression (4.5). It remains to prove that $\Sigma_{G/H} = \Pi \cap \Sigma_{G/H}$. To this end, we first note that $\Sigma_{G/H} = \Sigma_{G/\overline{H}}$ by Proposition 2.23. As

$\Lambda_{G/\overline{H}} = \mathbb{Z}\Sigma_{G/\overline{H}}$ (see Proposition 2.9), it suffices to show that $\Lambda_{G/\overline{H}}$ is generated by the set Π_0. But the latter is true by part (a) applied to $G/\overline{H}$ since $\operatorname{Ker}\tau_{\widehat{S}} = \mathscr{L} \subset \mathbb{Z}\Pi_0$. $\square$

4.4. **A connection with Luna's 1993 approach.** Theorem 3.27 together with Corollary 4.25(b) imply that there is a natural bijection between admissible maps and ARS-sets. The goal of this subsection is to find an explicit description of this bijection.

Let $H \subset G$ be a strongly solvable wonderful subgroup standardly embedded in B^-. Let $(\mathrm{M}, \pi, \sim)$ (resp. η) be the ARS-set (resp. admissible map) corresponding to H. Recall that by Proposition 4.10 the set $(\mathrm{M}, \pi, \sim)$ amounts to the pair $(\Psi, \sim)$.

We first express η in terms of the pair $(\Psi, \sim)$.

Clearly, B^-/H is a toric T-variety whose weight lattice $\mathfrak{X} = \mathfrak{X}_{B^-/H}$ is generated by the set Π_0. Let $\mathcal{F} = \mathcal{F}_{B^-/H}$ be the corresponding fan in $Q_{B^-/H} = \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{X}, \mathbb{Q})$.

The natural projection $B^-/H \to T/S$ yields a T-equivariant isomorphism $B^-/H \simeq T *_S U^-/N$. Hence T-stable prime divisors in B^-/H are in natural bijection with S-stable prime divisors in U^-/N. By [Mon, Lemma 1.4], there is an S-equivariant isomorphism $U^-/N \simeq \mathfrak{u}^-/\mathfrak{n}$. As an S-module, $\mathfrak{u}^-/\mathfrak{n}$ is isomorphic to the direct sum $\bigoplus\limits_{\varphi\in\Phi} \mathbb{C}_{-\varphi}$, where $\mathbb{C}_{-\varphi}$ is the one-dimensional S-module on which S acts by the character $-\varphi$. Clearly, S-stable prime divisors in $\bigoplus\limits_{\varphi\in\Phi} \mathbb{C}_{-\varphi}$ are just the coordinate hyperplanes, hence they are in natural bijection with the set Φ. For every $\varphi \in \Phi$, let D_φ be the prime divisor in B^-/H corresponding to φ via the above-mentioned natural bijections. Let also f_φ denote the element of $\mathcal{F}^1$ corresponding to D_φ.

Lemma 4.28. *For every $\varphi \in \Phi$ and every $\alpha \in \Psi$, one has*

$$\langle f_\varphi, \alpha\rangle = \begin{cases} 1 & \text{if } \alpha \in \Psi_\varphi; \\ 0 & \text{if } \alpha \notin \Psi_\varphi. \end{cases} \tag{4.6}$$

In particular, $\langle f_\varphi, \operatorname{Ker}\tau\rangle = 0$ for every $\varphi \in \Phi$.

Proof. For every $\varphi \in \Phi$, fix an arbitrary root $\alpha_\varphi \in \Psi_\varphi$. Then the weight semigroup of T-semi-invariant regular functions on D_φ is generated by the set $\{\alpha_{\varphi'} \mid \varphi' \in \Phi\backslash\{\varphi\}\} \cup \operatorname{Ker}\tau$. Since $\mathfrak{X} = \mathbb{Z}\Psi$ by Corollary 4.12(b), the required result follows from the definition of f_φ. $\square$

Now recall that the fan $\mathcal{F}$ coincides with the fan $\mathcal{F}^0_\eta$ introduced in the proof of Proposition 3.28.

Lemma 4.29. *For every $\alpha \in \Pi_0$, one has $\rho_\eta(\alpha) = f_{\varphi[\alpha]}$.*

Proof. Fix a root $\delta \in \Phi_{\varphi[\alpha]}$ such that $\pi(\delta) = \alpha$. By Corollary 4.12(a), one has $\alpha \in \mathbb{Z}F(\delta)$. Proposition 4.9 yields $\alpha \in \delta + \mathbb{Z}\{F(\delta)\backslash\{\delta\}\}$. Since $\tau(\delta') \ne \tau(\delta)$ for all $\delta' \in F(\delta)\backslash\{\delta\}$ (see [Avd2, Lemma 5(a)]), by formula (4.6) we obtain $\langle f_{\varphi[\alpha]}, \alpha\rangle = 1$. Let $\gamma \in \Pi_0$ be such that $f_{\varphi[\alpha]} = \rho_\eta(\gamma)$. Then formula (3.6) yields $\eta(\gamma, \alpha) = 1$, whence $\rho_\eta(\gamma) = \rho_\eta(\alpha)$. $\square$

Theorem 4.30. *Let $\alpha, \beta \in \Pi$. Then*

$$\eta(\alpha, \beta) = \begin{cases} J(\varphi[\alpha], \beta) & \text{if } \alpha, \beta \in \Pi_0; \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Proof. Fix a root $\beta \in \Pi_0$ and write $\tau(\beta) = \sum\limits_{\varphi \in \Phi} J(\varphi, \beta)\varphi$ according to (4.2). In view of formula (3.6), Lemma 4.29, and Lemma 4.28 one has

$$\eta(\alpha, \beta) = \langle \rho_\eta(\alpha), \beta \rangle = \langle f_{\varphi[\alpha]}, \beta \rangle = J(\varphi[\alpha], \beta).$$

To complete the proof, it remains to notice that $\eta(\alpha, \beta) = 0$ whenever at least one of the roots α, β is not in $\mathfrak{X} = \mathbb{Z}\Pi_0$. □

Remark 4.31. Using the classification and properties of active roots, one can prove that the map η defined by formula (4.7) is indeed an admissible map, that is, it satisfies axioms (AM1)–(AM5). However it would be interesting to obtain a classification-free proof of that.

Our next goal is to establish the converse part of the relation between the two approaches. The following lemma plays a key role in that.

Lemma 4.32. *Let α be an active root and let β be a linear combination of simple roots in Π_0 with non-negative integer coefficients. Suppose that $\tau(\beta) = \tau(\alpha)$. Then β is an active root.*

Proof. For $\beta = \alpha$ there is nothing to prove, therefore we assume $\beta \ne \alpha$. By condition (T) there are maximal active roots $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ and integers $p_1, \ldots, p_k$ such that

$$\tau(\alpha_1) = \tau(\beta_1), \ldots, \tau(\alpha_k) = \tau(\beta_k)$$

and

$$\beta = \alpha + p_1(\alpha_1 - \beta_1) + \ldots + p_k(\alpha_k - \beta_k).$$

Without loss of generality we may assume that $p_i > 0$ for all i and $\alpha_i \ne \beta_j$ for all i, j. By condition (C), for every $i = 1, \ldots, k$ there is a simple root $\gamma_i \in \operatorname{Supp} \beta_i$ that is contained in none of $\operatorname{Supp} \alpha_j$ and none of $\operatorname{Supp} \beta_j$ with $\beta_j \ne \beta_i$. This implies that $\gamma_i \in \operatorname{Supp} \alpha$, whence $\alpha \in F(\beta_i)$ for all $i = 1, \ldots, k$. Further, for $\beta_i \ne \beta_j$ one has $\gamma_i \in \operatorname{Supp} \alpha \subset \operatorname{Supp} \beta_j$, which contradicts the condition $\gamma_i \notin \operatorname{Supp} \beta_j$. It follows that $\beta_1 = \ldots = \beta_k$ and $p_1 + \ldots + p_k = 1$, whence $k = 1$, $p_1 = 1$, and $\beta = \alpha + \alpha_1 - \beta_1 = \alpha_1 - \beta_1'$, where $\beta_1' = \beta_1 - \alpha$. Since $\alpha \in F(\beta_1)$, one has either $\beta_1' = 0$ or $\beta_1' \in \Delta^+ \backslash \Psi$. The first case yields $\beta = \alpha_1 \in \Psi$. In the second case we obtain $\operatorname{Supp} \beta_1' \subset \operatorname{Supp} \alpha_1$. Since $\tau(\alpha_1) = \tau(\beta_1)$, one of possibilities (E1) or (E2) holds for α_1 and β_1. A simple analysis of these possibilities yields $\beta = \alpha_1 - \beta_1' \in \Psi$. □

We now recover the pair $(\Psi, \sim)$ from the admissible map η. We set $\Pi_\eta = \{\alpha \in \Pi \mid \eta(\alpha, \alpha) = 1\}$ and recall the map $\rho_\eta \colon \Pi_\eta \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\Pi_\eta, \mathbb{Z})$ given by (3.6).

Theorem 4.33. (a) *Equivalence classes of the set Ψ are in bijection with the set $\rho_\eta(\Pi_\eta)$.*

(b) *For a fixed $\rho_0 \in \rho_\eta(\Pi_\eta)$, an element $\alpha \in \mathbb{Z}^+\Pi_\eta$ is an active root in the corresponding equivalence class if and only if it satisfies the following system of linear equations:*

$$\begin{cases} \langle \rho_0, \alpha \rangle = 1; \\ \langle \rho, \alpha \rangle = 0 \qquad \text{for all } \rho \in \rho_\eta(\Pi_\eta) \backslash \{\rho_0\}. \end{cases}$$

Proof. (a) Equivalence classes of the set Ψ are in bijection with the set Φ. In turn, the map $\varphi \mapsto f_\varphi$ is a bijection between Φ and $\mathcal{F}^1 = \rho_\eta(\Pi_\eta)$.

(b) Clearly, $\Pi_0 = \Pi_\eta$, so every active root is contained in $\mathbb{Z}^+\Pi_\eta$. By Lemma 4.28, for every $\varphi \in \Phi$, an element $\alpha \in \mathbb{Z}\Pi_\eta$ satisfies $\tau(\alpha) = \varphi$ if and only if it satisfies equalities (4.6). By Lemma 4.32, every element $\alpha \in \mathbb{Z}\Pi_\eta$ with $\tau(\alpha) = \varphi$ is in fact an active root. $\square$

Remark 4.34. Proposition 3.34 combined with Theorem 4.33 provide an explicit method for determining the strongly solvable wonderful subgroup starting from its spherical system. This method was suggested by D. Luna as a conjecture in a private note addressed to the author.

Example 4.35. Suppose that $\eta(\alpha, \beta) = 0$ for all $\alpha, \beta \in \Pi$. Then $\Pi_\eta = \varnothing$, hence $\Psi = \varnothing$ and the corresponding strongly solvable wonderful subgroup is just B^-.

Example 4.36. Supose that

$$\eta(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Pi_\eta = \Pi$, $\rho_\eta(\Pi_\eta) = \{\breve{\alpha} \mid \alpha \in \Pi\}$, and $\Psi = \Pi$ with pairwise non-equivalent roots. The corresponding strongly solvable wonderful subgroup is $T \leftthreetimes (U^-, U^-)$.

Other examples can be found in § 5.

4.5. **Computation of extended ARS-sets via homogeneous spherical data.** In this subsection we assume G to be simply connected.

Let $\mathscr{H} = (\Pi^p, \Lambda, \Sigma, \mathcal{D}^a)$ be a homogeneous spherical datum and let $H \subset G$ be a spherical subgroup corresponding to $\mathscr{H}$.

Proposition 4.37. *The subgroup H is strongly solvable if and only if $\Pi^p = \varnothing$, $\Sigma \subset \Pi$, and the triple $(\Pi^p, \Sigma, \mathcal{D}^a)$ with $\varkappa$ restricted to $\mathbb{Z}\Sigma$ is a strongly solvable spherical system.*

Proof. From Theorem 4.27 we know that the conditions $\Pi^p = \varnothing$ and $\Sigma \subset \Pi$ are necessary for H to be strongly solvable. Under these conditions, by Proposition 2.23 the spherical system of the spherical closure $\overline{H}$ of H coincides with $(\Pi^p, \Sigma, \mathcal{D}^a)$. It remains to notice that H is strongly solvable if and only if $\overline{H}$ is such. $\square$

In what follows we assume that H is strongly solvable. By Proposition 4.37, there exists a subset $\mathcal{D}'_{G/\overline{H}} \subset \mathcal{D}^a_{G/\overline{H}}$ satisfying the conditions of Proposition 2.28. This subset is the distinguished subset of colors of a uniquely determined G-equivariant morphism $G/\overline{H} \to G/B = G/B^-$. Hence we have a natural G-equivariant morphism $G/H \to G/B^-$ and may assume $H \subset B^-$. Moreover, it may be also assumed that H is standardly embedded in B^-, so that $H = S \leftthreetimes N$, where $S \subset T$ and $N \subset U^-$. Let $\tau, \Phi, \Psi, \ldots$ be as in § 4.1.

Let $\mathcal{D}'_{G/H} \subset \mathcal{D}^a_{G/H}$ be the subset corresponding to $\mathcal{D}'_{G/\overline{H}}$ under the natural bijection between $\mathcal{D}_{G/H}$ and $\mathcal{D}_{G/\overline{H}}$. Then the set $\mathcal{D}_{G/H} \backslash \mathcal{D}'_{G/H}$ contains exactly $|\Pi|$ elements. More precisely, for every $\alpha \in \Pi$ the set $\mathcal{D}^\circ = \mathcal{D}_{G/H}(\alpha) \cap (\mathcal{D}_{G/H} \backslash \mathcal{D}'_{G/H})$ contains exactly one element, we denote it by D_α. For every $\alpha \in \Pi$, the indecomposable element of $\widehat{\Lambda}^+_{G/H}$ corresponding to D_α has the form $(\varpi_\alpha, -\tau(\varpi_\alpha))$, see Theorem 4.22 and Remark 4.23.

We recall that in § 1.3 we introduced the notation $\mathbb{Z}^{|\mathcal{D}|}$ for the free Abelian group generated by the set $\mathcal{D} = \mathcal{D}_{G/H}$. By Proposition 1.14, the homomorphism

$$\nu \colon \mathbb{Z}^{|\mathcal{D}|} \to \mathfrak{X}(H) \simeq \mathfrak{X}(S)$$

given by $D \mapsto \chi_D$ is surjective. Proposition 1.15 says that $\operatorname{Ker}\nu$ is generated by elements $\sum\limits_{D\in\mathcal{D}} \langle \varkappa(D), \mu\rangle D$, where μ runs over a basis of $\Lambda_{G/H}$.

Let $\mathbb{Z}^{|\mathcal{D}^\circ|} \subset \mathbb{Z}^{|\mathcal{D}|}$ be the subgroup generated by the set $\mathcal{D}^\circ$. We identify this subgroup with $\mathfrak{X}(T)$ via the isomorphism given by $D_\alpha \mapsto -\omega_\alpha$. Modulo this identification, the subgroup S is recovered as follows.

Proposition 4.38. *One has* $\operatorname{Ker}\tau = \mathbb{Z}^{|\mathcal{D}^0|} \cap \operatorname{Ker}\nu$.

Proof. Since $\chi_{D_\alpha} = -\tau(\varpi_\alpha)$, the restriction of the map ν to $\mathbb{Z}^{|\mathcal{D}^\circ|}$ is surjective. □

For every $D \in \mathcal{D}'_{G/H}$, we set $\Pi_D = \{\alpha \in \Pi \mid \langle \varkappa(D), \alpha\rangle = 1\}$.

Theorem 4.39. (a) *One has* $\Phi = \{\nu(D - \sum\limits_{\alpha\in\Pi_D} D_\alpha) \mid D \in \mathcal{D}'_{G/H}\} \subset \mathfrak{X}(S)$.

(b) *For a given* $\varphi \in \Phi$, *one has* $\Psi_\varphi = \{\alpha \in \mathbb{Z}^+\Sigma \mid \tau(\alpha) = \varphi\}$.

Proof. The set Φ is in natural bijection with the set $\mathcal{D}'_{G/H}$. Under this bijection, a weight $\varphi \in \Phi$ corresponds to the color D_φ such that the respective indecomposable element of $\widehat{\Lambda}^+_{G/H}$ is $(\lambda_\varphi, -\tau(\lambda_\varphi)+\varphi)$. Evidently, $\varphi = \nu(D_\varphi - \sum\limits_{\alpha\in supp\,\lambda_\varphi} D_\alpha)$. To prove part (a) it remains to notice that $supp\,\lambda_\varphi = \Pi_{D_\varphi}$. Part (b) is a direct consequence of Lemma 4.32. □

5. Strongly solvable wonderful subgroups in small rank

In this section we list all strongly solvable spherical systems for all semisimple groups G of rank at most 2 and also for any simple group of type A_3. Since every strongly solvable wonderful subgroup of G is spherically closed (see Corollary 3.31), these lists depend only on the Dynkin diagram of G. We recall that by Corollary 3.30 every strongly solvable spherical system has the form $(\varnothing, \Pi', \mathcal{D}^a)$ for some subset $\Pi' \subset \Pi$.

For every strongly solvable spherical system, we indicate all distinguished subsets of colors satisfying the conditions of Proposition 2.28. For every such subset of colors, we also indicate the corresponding admissible map and the corresponding set of active roots divided into equivalence classes.

It turns out that in all cases it is enough to list all strongly solvable spherical systems $(\varnothing, \Pi', \mathcal{D}^a)$ with $\Pi' = \Pi$ (according to the general theory, these are called *cuspidal*, see [Lun4, § 3.4]), since all other strongly solvable spherical systems (together with all possible admissible maps and sets of active roots) naturally come from strongly solvable spherical systems corresponding to proper subdiagrams of the Dynkin diagram of Π. We note that cuspidal strongly solvable spherical systems are uniquely determined by the set $\mathcal{D}^a$, and in this case the set $\mathcal{D}$ of colors of the corresponding wonderful G-variety coincides with $\mathcal{D}^a$.

For every G, the only strongly solvable spherical system $(\varnothing, \Pi', \mathcal{D}^a)$ with $\Pi' = \varnothing$ coincides with $(\varnothing, \varnothing, \varnothing)$ and corresponds to the wonderful variety G/B^-. In this case, the admissible map vanishes and the set of active roots is empty.

If $\operatorname{rk} G = 1$, then the only cuspidal strongly solvable spherical system has the form $(\varnothing, \{\alpha\}, \{D^+, D^-\})$, where α is the unique simple root of G and $\langle \varkappa(D^+), \alpha\rangle = \langle \varkappa(D^-), \alpha\rangle = 1$. The subsets $\{D^+\}, \{D^-\} \subset \mathcal{D}^a$ are the only distinguished subsets of colors satisfying the conditions of Proposition 2.28, both determine the same admissible map η given by $\eta(\alpha, \alpha) = 1$ and the same set of active roots, which is just $\{\alpha\}$.

If $\operatorname{rk} G = 2$, then, depending on the type of the Dynkin diagram of Π, the information about all cuspidal strongly solvable spherical systems is presented in Table 3 (type $\mathsf{A}_1 \times \mathsf{A}_1$), Table 4 (type A_2), Table 5 (type B_2), and Table 6 (type G_2).

At last, the information about all cuspidal strongly solvable spherical systems in the case where G is of type A_3 is presented in Table 7.

The notation used in Tables 3–7 is as follows. If G is simple, then α_i denotes the ith simple root of G. If G is of type $\mathsf{A}_1 \times \mathsf{A}_1$, then α_1, α_2 denote two different simple roots of G. Each matrix in the column "$\mathcal{D}^a$" represents the set $\mathcal{D}^a$ of a cuspidal strongly solvable spherical system. The elements of $\mathcal{D}^a$ are in bijection with rows of the matrix. For every $D \in \mathcal{D}^a$, the ith element in the corresponding row is the value $\langle \varkappa(D), \alpha_i \rangle$. In the column "DSC" (the heading is an abbreviation for "distinguished subsets of colors") we list all subsets of $\mathcal{D}^a$ satisfying the conditions of Proposition 2.28. Each subset is given by numbers of the corresponding rows of the matrix in the column "$\mathcal{D}^a$". For every such subset $\mathcal{D}'$, in the column "Admissible map" (resp. "Active roots") we indicate the matrix of the corresponding admissible map (resp. the equivalence classes of active roots) determined by $\mathcal{D}'$.

Table 3. Cuspidal strongly solvable spherical systems in type $\mathsf{A}_1 \times \mathsf{A}_1$

No.	$\mathcal{D}^a$	DSC	Admissible map	Active roots
1	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$1,3$ or $1,4$ or $2,3$ or $2,4$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\{\alpha_1\}, \{\alpha_2\}$
2	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$	3	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\{\alpha_1, \alpha_2\}$

Table 4. Cuspidal strongly solvable spherical systems in type A_2

No.	$\mathcal{D}^a$	DSC	Admissible map	Active roots
1	$\begin{bmatrix}1&0\\0&1\\1&-1\\-1&1\end{bmatrix}$	$1,2$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\{\alpha_1\},\{\alpha_2\}$
		$1,4$	$\begin{bmatrix}1&0\\-1&1\end{bmatrix}$	$\{\alpha_1+\alpha_2\},\{\alpha_2\}$
		$2,4$	$\begin{bmatrix}1&-1\\0&1\end{bmatrix}$	$\{\alpha_1\},\{\alpha_1+\alpha_2\}$
2	$\begin{bmatrix}1&1\\1&-2\\-2&1\end{bmatrix}$	1	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\{\alpha_1,\alpha_2\}$

Table 5. Cuspidal strongly solvable spherical systems in type B_2

No.	$\mathcal{D}^a$	DSC	Admissible map	Active roots
1	$\begin{bmatrix}1&0\\0&1\\1&-1\\-2&1\end{bmatrix}$	$1,2$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\{\alpha_1\},\{\alpha_2\}$
		$1,4$	$\begin{bmatrix}1&0\\-2&1\end{bmatrix}$	$\{\alpha_1+2\alpha_2\},\{\alpha_2\}$
		$2,3$	$\begin{bmatrix}1&-1\\0&1\end{bmatrix}$	$\{\alpha_1\},\{\alpha_1+\alpha_2\}$
2	$\begin{bmatrix}1&0\\-1&1\\1&-1\\-1&1\end{bmatrix}$	$1,2$ or $1,4$	$\begin{bmatrix}1&0\\-1&1\end{bmatrix}$	$\{\alpha_1+\alpha_2\},\{\alpha_2\}$
3	$\begin{bmatrix}1&1\\1&-2\\-3&1\end{bmatrix}$	1	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\{\alpha_1,\alpha_2\}$

Table 6. Cuspidal strongly solvable spherical systems in type G_2

No.	$\mathcal{D}^a$	DSC	Admissible map	Active roots
1	$\begin{bmatrix}1&0\\0&1\\1&-1\\-3&1\end{bmatrix}$	$1,2$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\{\alpha_1\},\{\alpha_2\}$
		$1,4$	$\begin{bmatrix}1&0\\-3&1\end{bmatrix}$	$\{\alpha_1+3\alpha_2\},\{\alpha_2\}$
		$2,3$	$\begin{bmatrix}1&-1\\0&1\end{bmatrix}$	$\{\alpha_1\},\{\alpha_1+\alpha_2\}$
2	$\begin{bmatrix}1&0\\-1&1\\1&-1\\-2&1\end{bmatrix}$	$1,2$	$\begin{bmatrix}1&0\\-1&1\end{bmatrix}$	$\{\alpha_1+\alpha_2\},\{\alpha_2\}$
		$1,4$	$\begin{bmatrix}1&0\\-2&1\end{bmatrix}$	$\{\alpha_1+2\alpha_2\},\{\alpha_2\}$
3	$\begin{bmatrix}1&1\\1&-2\\-4&1\end{bmatrix}$	1	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	$\{\alpha_1,\alpha_2\}$

Table 7. Cuspidal strongly solvable spherical systems in type A_3

No.	$\mathcal{D}^a$	DSC	Admissible map	Active roots
1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	$1,2,3$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$
		$1,2,6$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$	$\{\alpha_1\}$, $\{\alpha_2+\alpha_3\}$, $\{\alpha_3\}$
		$1,3,5$	$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$	$\{\alpha_1+\alpha_2\}$, $\{\alpha_2\}$, $\{\alpha_2+\alpha_3\}$
		$2,3,4$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\{\alpha_1\}$, $\{\alpha_1+\alpha_2\}$, $\{\alpha_3\}$
		$2,4,6$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$	$\{\alpha_1\}$, $\{\alpha_1+\alpha_2+\alpha_3\}$, $\{\alpha_3\}$
2	$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$	$1,2,3$	$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$	$\{\alpha_1+\alpha_2+\alpha_3\}$, $\{\alpha_2+\alpha_3\}$, $\{\alpha_3\}$
		$1,2,6$	$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\{\alpha_1+\alpha_2\}$, $\{\alpha_2\}$, $\{\alpha_3\}$
		$1,5,6$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_2+\alpha_3\}$
		$4,5,6$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$	$\{\alpha_1\}$, $\{\alpha_1+\alpha_2\}$, $\{\alpha_1+\alpha_2+\alpha_3\}$
3	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$	$1,2$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\{\alpha_1\}$, $\{\alpha_2,\alpha_3\}$
		$2,3$	$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\{\alpha_1\}$, $\{\alpha_1+\alpha_2,\alpha_3\}$
4	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$	$1,2$	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\{\alpha_1,\alpha_3\}$, $\{\alpha_2\}$
		$1,4$	$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$	$\{\alpha_1+\alpha_2,\alpha_2+\alpha_3\}$, $\{\alpha_2\}$

No.	$\mathcal{D}^a$	DSC	Admissible map	Active roots
5	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	$1,2$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\{\alpha_1,\alpha_2\}$, $\{\alpha_3\}$
		$1,5$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$	$\{\alpha_1,\alpha_2{+}\alpha_3\}$, $\{\alpha_3\}$
6	$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$	$1,2$	$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\{\alpha_1{+}\alpha_2,\alpha_3\}$, $\{\alpha_2\}$
		$1,4$	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$	$\{\alpha_1,\alpha_2{+}\alpha_3\}$, $\{\alpha_2\}$
7	$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$	$1,2$	$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$	$\{\alpha_1,\alpha_3\}$, $\{\alpha_1{+}\alpha_2,\alpha_2{+}\alpha_3\}$
8	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$	1	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\{\alpha_1,\alpha_2,\alpha_3\}$

Appendix A. Homogeneous bundles

In this appendix we recall the construction of a homogeneous bundle. Let L be a group, K a subgroup of L, and let Z be an arbitrary K-variety. By definition, the homogeneous bundle $L*_K Z$ over L/K associated with Z is the quotient set of $L\times Z$ by the action of K given by the formula $k(l,z)=(lk^{-1},kz)$, where $k\in K$, $l\in L$, $z\in Z$. Clearly, $L*_K Z$ is equipped with an action of L induced by the natural action on $L\times Z$ by left translation of the first factor. Moreover, there is a natural L-equivariant map $L*_K Z\to L/K$, which justifies the term "homogeneous bundle over L/K".

Theorem A.1 (see [Bia, Corollary 2], [PopV, Theorem 4.19])**.** *If Z is covered by K-stable quasiprojective open subsets, then the set $L*_K Z$ has a structure of an algebraic variety such that the quotient map $L\times Z\to L*_K Z$ by the action of K is a geometric quotient*[4].

The assumptions of Theorem A.1 are fulfilled in case of connected K and normal Z (see [Sum, Lemma 8]) or in case of quasiprojective Z. This turns out to be enough for all homogeneous bundles considered in this paper to be algebraic varieties in the indicated sense.

Below we list a few properties of homogeneous bundles.

Proposition A.2 (see [Tim, 2.1])**.** *If an L-variety X admits an L-equivariant morphism $\phi\colon X\to L/K$, then $X\simeq L*_K Z$, where $Z=\phi^{-1}(eK)$.*

[4]See the definition of a geometric quotient, for instance, in [PopV, 4.2].

Proposition A.3 (see [PopV, Proposition 4.22] or [Tim, 2.1]). *The variety $L *_K Z$ is smooth (resp. normal) if and only if Z is smooth (resp. normal).*

Proposition A.4. *Let $P \subset G$ be a parabolic subgroup and let Z be a P-variety. The variety $G *_P Z$ is complete if and only if Z is complete.*

The proof of this proposition provided below was communicated to the author by D. A. Timashev.

Proof. We may assume that $P \supset B$. Let P^- be the parabolic subgroup opposite to P and let P_u^- be its unipotent radical. Regard the natural morphism $\phi\colon G *_P Z \to G/P$. Since $Z \simeq \phi^{-1}(o)$, Z is complete whenever $G *_P Z$ is complete. Conversely, assume that Z is complete. To prove that $G *_P Z$ is complete, by definition one has to show that for every algebraic variety W the projection morphism $(G *_P Z) \times W \to W$ is closed. Since G/P is complete, it suffices to show that the morphism

$$(G *_P Z) \times W \to (G/P) \times W$$

extending ϕ is closed. As G/P is covered by a finite number of shifts of the open subset $P^- o = P_u^- o \simeq P_u^-$, the problem reduces to showing that the morphism

$$\phi^{-1}(P_u^- o) \times W \to P_u^- o \times W$$

is closed. But $\phi^{-1}(P_u^- o) \simeq P_u^- \times Z$, and so the required result follows from the completeness of Z. □

For a more detailed discussion of the construction of a homogeneous bundle see [Bia], [PopV, 4.8], or [Tim, 2.1].

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