The Auslander conjecture for dimension less than 7

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December 28, 2012

Abstract. In 1964 L. Auslander conjectured that every crystallographic subgroup Γ of an affine group $\mathrm{Aff}(\mathbb{R}^n)$ is virtually solvable, i.e. contains a solvable subgroup of finite index. D. Fried and W. Goldman proved Auslander's conjecture for $n = 3$ using cohomological arguments. We prove the Auslander conjecture for $n < 7$. The proof is based mainly on dynamical arguments. In some cases we use the cohomological argument which we can avoid but it will significantly lengthen the proof.

1 Introduction

Let us consider the group $G_n = \mathrm{Aff}(\mathbb{R}^n)$ of affine transformations of $\mathbb{R}^n$. The group G_n is the semidirect product $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with the group of its translations. Let $l : G_n \to GL_n(\mathbb{R})$ be the natural homomorphism. Recall that $l(g)$ is called the *linear part* of the affine transformation g. Let $X \subseteq G_n$, then the set $l(X) = \{l(x), x \in X\}$ is called the *linear part* of X.

A subgroup Γ of G_n is said to act *properly discontinuously* on $\mathbb{R}^n$ if for every compact subset K of $\mathbb{R}^n$ the set $\{g \in \Gamma : gK \cap K \neq \emptyset\}$ is finite. If a group Γ consisting of isometries is a discrete subgroup of G_n, then Γ acts properly discontinuously on $\mathbb{R}^n$. But this is not true for an arbitrary infinite discrete subgroup of G_n. Indeed, consider the stabilizer of the origin $St_{G_n}(0)$. Obviously $St_{G_n}(0) \cong GL_n(\mathbb{R})$. Thus no discrete infinite subgroup of $GL_n(\mathbb{R})$ acts properly discontinuously. A subgroup Γ of G_n is called *crystallographic* if Γ acts properly discontinuously on $\mathbb{R}^n$ and the orbit space $\Gamma \setminus \mathbb{R}^n$ is compact. The study of affine crystallographic groups has a long history which goes back to Hilbert's 18th problem. More precisely Hilbert (essentially) asked if there is only finite number, up to conjugacy in Aff($\mathbb{R}^n$), of crystallographic groups Γ acting isometrically on $\mathbb{R}^n$. In a series of papers Bieberbach showed that this was so. The key result is the following famous theorem of Bieberbach. A crystallographic group Γ acting isometrically on the $n-$dimensional Euclidean space $\mathbb{R}^n$ with compact quotient $\Gamma\backslash\mathbb{R}^n$ contains a subgroup of finite index consisting of translations. In particular, such a group Γ is virtually abelian, i.e. Γ contains an abelian subgroup of finite index. Moreover, it was proved later by L. Auslader that a group Γ acting properly discontinuously and isometrically is virtually abelian [Au]. A natural way to generalize the classical problem is to broaden the class of allowed motions and consider crystallographic groups $\Gamma \subseteq$ Aff($\mathbb{R}^n$). This raises the question of the group-theoretic conditions satisfied by affine crystallographic groups. Auslander proposed the following conjecture in [Au].

The Auslander Conjecture . *Every crystallographic subgroup Γ of G_n is virtually solvable, i.e. contains a solvable subgroup of finite index.*

The proof in [Au] of this conjecture is unfortunately incorrect, but the conjecture is still an open and central problem (see Milnor [Mi2]).

It is easy to see that there exists a nilpotent, non virtually abelian affine crystallographic group. It is well known [Mo] that every discrete virtually solvable linear group and in particular every virtually solvable discrete subgroup of Aff($\mathbb{R}^n$) is virtually polycyclic. J. Milnor showed that every virtually polycyclic group can act properly discontinuously and affinely on some vector space [Mi1].

There is an additional geometric interest in properly discontinuous groups since they can be represented as fundamental groups of manifolds with certain geometric structures, namely complete flat affine manifolds. If M is a complete flat affine manifold, its universal covering manifold is isomorphic to $\mathbb{R}^n$. It follows that its fundamental group $\Gamma = \pi_1(M)$ is in a natural way a properly discontinuous torsion-free subgroup of G_n. Conversely, if Γ is a properly discontinuous torsion-free subgroup of G_n, then $\Gamma \setminus \mathbb{R}^n$ is a complete flat affine manifold M with $\pi_1(M) = \Gamma$. Therefore every virtually polycyclic group is a fundamental group of a complete affinely flat manifold by Milnor's theorem mentioned above. J. Milnor proposed the following questions in [Mi1]:

Question 1 *Let Γ be a torsion free virtually polycyclic group of rank k. Does there exists a k–dimensional compact complete affinely flat manifold M with $\pi_1(M) \cong \Gamma$?*

Now it is known that not every finitely generated nilpotent group is an affine crystallographic group [B]. This gives a negative answer to Question 1.

Question 2 *Does there exists a complete affinely flat manifold M such that $\pi_1(M)$ contains a free group ?*

In comments to the second question Milnor wrote: "I do not know if such a manifold exists even in dimension 3" and proposed "to construct a Lorentz-flat example by starting with a discrete subgroup $\mathbb{Z} * \mathbb{Z} \leq SO(2,1)$ then adding translation components to obtain a group of isometries of Lorentz 3-space; but it seems difficult to decide whether the resulting group action is properly discontinuous" [Mi2, p. 184].

G. Margulis gave a positive answer to Question 2 in dimension 3 in [M]. He constructed a free non-abelian subgroup Γ of isometries of Lorentz 3-space acting properly discontinuously on $\mathbb{R}^3$. Clearly $l(\Gamma) \subseteq SO(2,1)$. Then we proved in [AMS3, Theorem B] that for a non degenerate form B of signature (p, q) where $p = q+1$ and q is odd, there exists a free group $\Gamma \leq \text{Aff}\ (\mathbb{R}^{2q+1})$ acting properly discontinuously such that the linear part $l(\Gamma)$ of Γ is Zariski dense in $SO(B)$. Therefore in any dimension n there exists a complete affinely flat manifold M such that $\pi_1(M)$ contains a free non- abelian group.

Let B be a non-degenerate quadratic form. Set $G_B = \{x \in G_n : l(x) \in O(B)\}$. Clearly $G_B = O(B) \ltimes \mathbb{R}^n$. Let Γ be an affine crystallographic group, and suppose $\Gamma \subset G_B$ for a non degenerate quadratic form B of signature (p, q). Remark, that if $q = 0$ we have the case of isometric affine actions. D. Fried and W. Goldman in [FG] proved, that if Γ is a crystallographic subgroup of G_B where B is a non degenerate quadratic form of signature $(2, 1)$ then $l(\Gamma)$ is not Zariski dense in $O(2, 1)$. They use this theorem to deduce the Auslander conjecture for dim ≤ 3. W.Goldman and Y. Kamishima proved in [GK] that a crystallographic subgroup of G_B is virtually solvable for $q = 1$. In [AMS4] we proved that a crystallographic group $\Gamma \subseteq G_B$ is virtually solvable if B is a quadratic from of signature $(p, 2)$. F. Grunewald and G. Margulis proved in [GM] that if the linear part is a subgroup of a simple Lie group of real rank 1, then Γ is virtually solvable. This result was generalized in [To1]. Namely it was proved that if $l(\Gamma) \subseteq G$ and the semisimple part of G is a simple group of real rank 1, then Γ is virtually solvable. Finally in [S2] and [To2] it was proved, that if $l(\Gamma) \subseteq G$ and every non-abelian simple subgroup of G has real rank ≤ 1 than Γ is virtually solvable. Let us remark, that all papers [FG], [GK], [GM], [S2] and [T 1,2] basically use the same idea which was first introduced in [FG]. We call this idea "the cohomological argument" because it is based on using the virtual cohomological dimension of Γ. In contrast, [AMS 4] and [M] are based on a completely different approach namely on dynamical ideas (see also [AMS 1,2,3]).

In [To3] the author attempts to prove the Auslander conjecture for dimensions 4 and 5. Unfortunately, the proof their is incomplete. Thus the only dimensions for which there is a complete proof of the Auslander conjecture for G_n are $n \leq 3$ (see [FG, Theorem, section 2.13])

Let us mention the following result due to M. Gromov [Gr]. Let M be a connected compact Riemannian manifold. Denote by $d = d(M)$ the diameter of M, and by $c^+ = c^+(M)$ and $c^- = c^-(M)$, respectively, the upper and lower bounds of the sectional curvature of M. We set $c = c(M) = \max(c^+, c^-)$. We say that M is almost ε–flat, $\varepsilon \geq 0$ if $cd^2 < \varepsilon$. Then for sufficiently small ε the fundamental group of an ε–flat manifold is a virtually nilpotent group, i.e. contains a nilpotent subgroup of finite index. This result again shows that M being close to Euclidean has strong implications for the algebraic structure of the fundamental group $\pi_1(M)$.

In [DG1], [DG2], [CDGM] and [Me] there were studied properly discontinuous subgroups Γ of the affine group $\mathrm{Aff}(\mathbb{R}^3)$ whose linear part $l(\Gamma)$ leaves a quadratic form of signature $(2,1)$ invariant.

The aim of this paper is to prove the following theorem which was announced in [AMS2]

Main Theorem *Let Γ be a crystallographic subgroup of $\mathit{Aff}\mathbb{R}^n$ and $n < 7$, then Γ is virtually solvable.*

The proof of this theorem is based mainly on dynamical arguments. In some cases we use the cohomological argument to shorten the proofs.

Let us give a short description of the paper. In section 2 we introduce the terminology we will use throughout the paper and recall some basic results about the dynamics of the action of hyperbolic elements. We show that every element of the connected component of the Zariski closure of an affine group acting properly discontinuously has one as an eigenvalue. This simple but useful fact will be used in section 3. The goal of section 3 is to obtain a list of all possible semisimple groups S which might be a semisimple part of the Zariski closure of an affine group Γ that acts properly discontinuously for $n \leq 6$

and does not have $SO(2,1)$ as a quotient group. Using this list we prove the Auslander conjecture in dimension 4 and 5 in section 4. Actually we show a bit more. Namely, if the semisimple part of the Zariski closure of Γ is one from the list, then Γ does not act properly discontinuously. In section 5 we show that the semisimple part S of the Zariski closure of $l(\Gamma)$ cannot be $SO(3,2)$ or $SO(3) \times SL_3(\mathbb{R})$. The proof is based on the cohomolochigal argument we have mentioned above. Namely, we will compare the virtual cohomological dimension of Γ and the dimension of the symmetric space S/K, where K is a maximal compact subgroup of S. We will prove that none of these cases is possible.

The most difficult part is to show that the semisimple part of the Zariski closure of $l(\Gamma)$ is not $SO(2,1) \times SL_3(\mathbb{R})$.. This is done in section 6. We show that it is possible to change the sign of a hyperbolic element (see Main Lemma 6.7) in this case. Thus by Lemma 6.5 we conclude that the semisimple part of the Zariski closure of $l(\Gamma)$ cannot be $SO(2,1) \times SL_3(\mathbb{R})$. Hence none of the possible non-trivial semisimple groups can be the semisimple part of the Zariski closure of Γ. Therefore the semisimple part of the Zariski closure of Γ is trivial. Hence Γ is virtually solvable.

In the final section 7 we discuss Auslander's conjecture in dimension 7 and state two open problems. We believe that answers (positive or negative) to these questions are essential for further progress on Auslander's conjecture.

2 Dynamical properties of the action of hyperbolic elements.

2.1. Notation and terminology. In this section we introduce the terminology we will use throughout the paper. We also prove and recall some basic results about the dynamics of the action of hyperbolic elements [A], [AMS 1, 4].

2.2. Let V be a finite dimensional vector space over a local field k with absolute value $|\cdot|$, and let $P=\mathbb{P}(V)$ be the projective space corresponding to V. Let $g\in GL(V)$ and let $\chi_g(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i)\in k[\lambda]$ be the characteristic polynomial of the linear transformation g. Set $\Omega(g)=\{\lambda_i : |\lambda_i|=\max_{1\leq j\leq n}|\lambda_j|\}$. Put $\chi_1(\lambda)=\prod_{\lambda_i\in\Omega(g)}(\lambda-\lambda_i)$ and $\chi_2(\lambda)=\prod_{\lambda_i\notin\Omega(g)}(\lambda-\lambda_i)$. Then χ_1 and χ_2 belong to $k[\lambda]$ since the absolute value of an element is invariant under Galois automorphisms. Therefore $\chi_1(g)\in GL(V)$ and $\chi_2(g)\in GL(V)$. Let us denote $\ker(\chi_1(g))$ (resp. $\ker(\chi_2(g))$) by $V(g)$ (resp. $W(g)$) . Set $\Omega^+(g)=\{\lambda_i, 1\leq i\leq n : |\lambda_i|>1\}$ and $\Omega^-(g)=\{\lambda_i, 1\leq i\leq n : |\lambda_i|<1\}$. Let $\lambda_-(g)=\max\{|\lambda| : \lambda\in\Omega^-(g)\}$. Let $\lambda_+(g)=\min\{|\lambda| : \lambda\in\Omega^-(g)\}$. Put $\lambda(g)=\max\{\lambda_+^{-1}(g),\lambda_-(g)\}$. It is clear that $\lambda(g)=\lambda(g^{-1})$.

2.3. Let g be a semisimple element in $GL(\mathbb{R}^n)$. Then the space $V=\mathbb{R}^n$ can be decomposed into the direct sum of three g–invariant subspaces $A^+(g)$, $A^-(g)$ and $A^0(g)$ determined by the condition that all eigenvalues of the restriction $g\mid A^+(g)$ (resp. $g\mid A^-(g)$, $g\mid A^0(g)$) have absolute value more than 1 (resp. less than 1, equal to 1). Put $D^+(g)=A^+(g)\oplus A^0(g)$ and $D^-(g)=A^-(g)\oplus A^0(g)$. Obviously $D^+(g)\cap D^-(g)=A^0(g)$. Let G be a subgroup of $GL(V)$. If for a semisimple element $g\in G$ we have $\dim(A^0(g))=\min\{\dim A^0(t)| \, t\in G, t \text{ is a semisimple element }\}$, then $g\in G$ is called *regular* in $G\leq GL(V)$.

2.4. Let $\|\,\cdot\,\|$ and d denote the norm and metric on $\mathbb{R}^n$ corresponding to the standard inner product on $\mathbb{R}^n$. Let $P=\mathbb{P}(\mathbb{R}^n)$ be the projective space corresponding to $\mathbb{R}^n$. Let $\|g\|_+$ be the norm of the restriction $g|_{A^-(g)}$. Denote by $\|g\|_-=\|g^{-1}\|_+$ and put $s(g)=\max\{\|g\|_+,\|g\|_-\}$. A regular element g is called *hyperbolic* if $s(g)<1$. It is clear that for a regular element g there exists a number N such that for $n>N$ the element g^n is hyperbolic. Let $\pi:\mathbb{R}^n\setminus\{0\}\to P$ be the natural projection. For a subset X of $\mathbb{R}^n$ we denote $\pi(X)=\pi(X\setminus\{0\})$.

The metric $\|\,\cdot\,\|$ on $\mathbb{R}^n$ induces a metric $\widehat{d}$ on the projective space $P=\mathbb{P}(\mathbb{R}^n)$. Thus for any point $p \in P$ and a subset $A \subseteq P$, we can define

$$\widehat{d}(p, A) = \inf_{a \in A} \widehat{d}(p, a).$$

Let A_1 and A_2 be two subsets of P. We define

$$\widehat{d}(A_1, A_2) = \inf_{a_1 \in A_1} \inf_{a_2 \in A_2} \widehat{d}(a_1, a_2)$$

and

$$\widehat{\rho}(A_1, A_2) = \inf\{R; A_2 \subseteq B(A_1, R)\, A_1 \subseteq B(A_2, R)\}$$

where $B(A, R) = \bigcup_{a \in A} B(a, R)$.
For two subspaces $W_1 \subseteq \mathbb{R}^n$ and $W_2 \subseteq \mathbb{R}^n$ we put $\widehat{d}(W_1, W_2) = \widehat{d}(\pi(W_1 \setminus \{0\}), \pi(W_2 \setminus \{0\}))$ and $\widehat{\rho}(W_1, W_2) = \widehat{\rho}(\pi(W_1 \setminus \{0\}), \pi(W_2 \setminus \{0\}))$. A hyperbolic element g is called *ε-hyperbolic* if

$$\widehat{d}(A^+(g), D^-(g)) \geq \varepsilon$$

and

$$\widehat{d}(A^-(g), D^+(g)) \geq \varepsilon.$$

Two different hyperbolic elements g_1 and g_2 are called *transversal* if $A^{\pm}(g_1) \bigcap D^{\mp}(g_2) = \{0\}$ and $A^{\pm}(g_2) \bigcap D^{\mp}(g_1) = \{0\}$. Let B be a non degenerate quadratic form and let $g \in SO(B)$ be a regular element. Since $A^+(g)$ (resp. $A^-(g)$) is the unique maximal isotropic subspace of $D^+(g)$ (resp. $D^-(g)$) it is easy to see that two hyperbolic elements g_1 and g_2 of $SO(B)$ are transversal if and only if $A^+(g_1) \bigcap A^-(g_2) = \{0\}$ and $A^+(g_2) \bigcap A^-(g_1) = \{0\}$.

Clearly g and g^{-1} are not transversal for any regular element g. Nevertheless it is quite important to be able to find an element t of a given linear group G such that g and $tg^{-1}t^{-1}$ are transversal. It is possible for example for $G = SO(B)$. In general $\dim A^+(g) \neq \dim A^-(g)$. Thus it will be impossible.

Definition 2.5 *We will say that a regular element* $g \in G \subseteq GL(V)$ *such that* $\dim A^+(g) \geq \dim A^-(g)$ *can be transformed into a transversal pair inside* G *if there exists an element* $t \in G$ *and a* g*–invariant subspace* $W \subset A^+(g)$ *such that* $V = tWt^{-1} \oplus D^+(g)$.

The next proposition shows that this property depends only on the Zariski closure G of a group G_0, and thus G_0 can be safely ignored in most of what we do.

Proposition 2.6 *Let* G_0 *be a Zariski dense subgroup of* G, $G \subset SL(V)$. *Let* $\gamma \in G_0$ *be a regular element of* G. *If* $\gamma \in G$ *can be transformed into a transversal pair inside* G *, then* $\gamma \in G_0$ *can be transformed inside* G_0 *into a transversal pair.*

Proof. By the definition above, there exist an element $t \in G$ and a g–invariant subspace $W \subset A^+(g)$ such that $V = tW \oplus D^+(g)$. Then $T \cap G_0 \neq \emptyset$ for $T = \{t \in G : tW \cap D^+(g) = \{0\}\}$ since T is not empty and Zariski open. This proves the proposition.

Let us make a simple but useful remark. Let $g \in SO(B)$. For a regular element $g \in SO(B)$, the space $A^+(g)$ (resp. $A^-(g)$) is the unique maximal isotropic subspace of $D^+(g)$ (resp. $D^-(g)$). Therefore two hyperbolic elements g_1 and g_2 are transversal if and only if $A^+(g_1) \cap A^-(g_2) = \{0\}$ and $A^-(g_1) \cap A^+(g_2) = \{0\}$. Clearly g_1 and g_2 are transversal if and only if g_1^{-1} and g_2^{-1} are transversal.

Two transversal hyperbolic elements g_1 and g_2 are called ε- *transversal*,

$$\min_{1 \leq i \neq j \leq 2} \{\widehat{d}(A^+(g_i), D^-(g_j)), \widehat{d}(A^-(g_i), D^+(g_j))\} \geq \varepsilon.$$

Let g_1 and g_2 be two transversal hyperbolic elements of $SO(B)$. By the above remark, we conclude:

(1) for every ε there exists $\delta = \delta(\varepsilon)$ such that g_1 and g_2 are ε–transversal if and only if $\widehat{d}(A^+(g_1), A^-(g_2)) > \delta$ and $\widehat{d}(A^-(g_1), A^+(g_2)) > \delta$.

Clearly

(2) g_1 and g_2 are ε–transversal if and only if g_1^{-1} and g_2^{-1} are ε– transversal.

An affine transformation is called *hyperbolic*, (respectively $\varepsilon -$ *hyperbolic*) if $l(g)$ is hyperbolic (respectively ε- hyperbolic). Recall the following useful Lemma

Lemma 2.7. [AMS 3] *There exists $s(\varepsilon) < 1$ and $c(\varepsilon)$ such that for any two ε–hyperbolic ε–transversal elements $g, h \in GL(V)$ with $s(g) < s(\varepsilon)$ and $s(h) < s(\varepsilon)$, we have*

(1) the element gh is $\varepsilon/2$–hyperbolic and is $\varepsilon/2$–transversal to both g and h;

(2) $\widehat{\rho}(A^+(gh), A^+(g)) \leq c(\varepsilon)s(g))$;

(3) $\widehat{\rho}(A^-(gh), A^-(h)) \leq c(\varepsilon)s(h))$;

(4) $s(gh) \leq c(\varepsilon)s(g)s(h)$.

Proposition 2.8 *Let Γ be an affine group acting properly discontinuously. Let g be an element of the connected component of the Zariski closure of Γ. Then $l(g)$ has 1 as an eigenvalue.*

Proof It is easy to see that for $x \in G_n$, if $l(x)$ does not have 1 as an eigenvalue then

x has a fixed point. Thus every element of an affine torsion free group acting properly discontinuously has one as an eigenvalue. Note that this is an algebraic property. It is well known that if every finitely generated subgroup of linear group Γ is finite, then Γ is finite. Hence Γ contains a finitely generated subgroup Γ_0 such that the connected components of the Zariski closure of Γ and Γ_0 coincide. By Selberg's lemma we conclude that there exists a torsion free subgroup $\Gamma_1 \leq \Gamma_0$ of finite index . It is clear that the connected component of the Zariski closure of Γ_1 is the same as the connected component of the Zariski closure of Γ_0. Since Γ_1 acts properly discontinuously the linear part $l(x)$ has one as an eigenvalue for every $x \in \Gamma_1$. Thus the same is true for every element of the Zariski closure of Γ_1. This proves the statement.

This simple proposition will help us to list all possible simple and semisiple connected Lie groups which can be a semisimple part of the Zariski closure of a subgroup of $G_n, n \leq 6$ acting properly discontinuously (see section4).

Let Γ be an affine group acting properly discontinuously. Consider a regular element $g \in \Gamma \subseteq G_n$. Then g has 1 as an eigenvalue by Proposition 2.8. Hence, there exists a g–invariant line L_g. The restriction of g to L_g is the translation by a non- zero vector t_g. Let us note that all such lines are parallel and the vector t_g does not depend on the choice of L_g. We will assume that we fixed once and for all some point q_0 in the affine space $\mathbb{R}^n$ as the origin and the g–invariant line L_g that is closest to the origin. Let us define the following affine subspaces: $E_g^+ = D^+(g) + L_g$, $E_g^- = D^-(g) + L_g$, $E_g^+ \cap E_g^- = C_g$. Let $p \in L_g$ be a point. Then $t_g = \overrightarrow{p\,gp}$. Clearly $t_g = -t_{g^{-1}}$.

Proposition 2.9 *Let $G \subset GL(V)$ be the Zariski closure of the linear part of an affine group Γ. Let S be a maximal semisimple subgroup and U be the unipotent radical of G. Assume that $G = SU$ and the space V is a direct sum $V = V_1 \oplus V_2$ of S–invariant subspaces such that*

(1) $sv = v$ for all $s \in S$and $v \in V_1$,

(2) V_1 is G–invariant

(3) Let $s \in S$ be a regular element. Then the restriction $s \mid_{V_2}$ does not have an eigenvalue of modulus one.

and

(4) the restriction $U \mid_{V_1}$ and the induced representation $U \longrightarrow GL(V/V_1)$ are trivial.

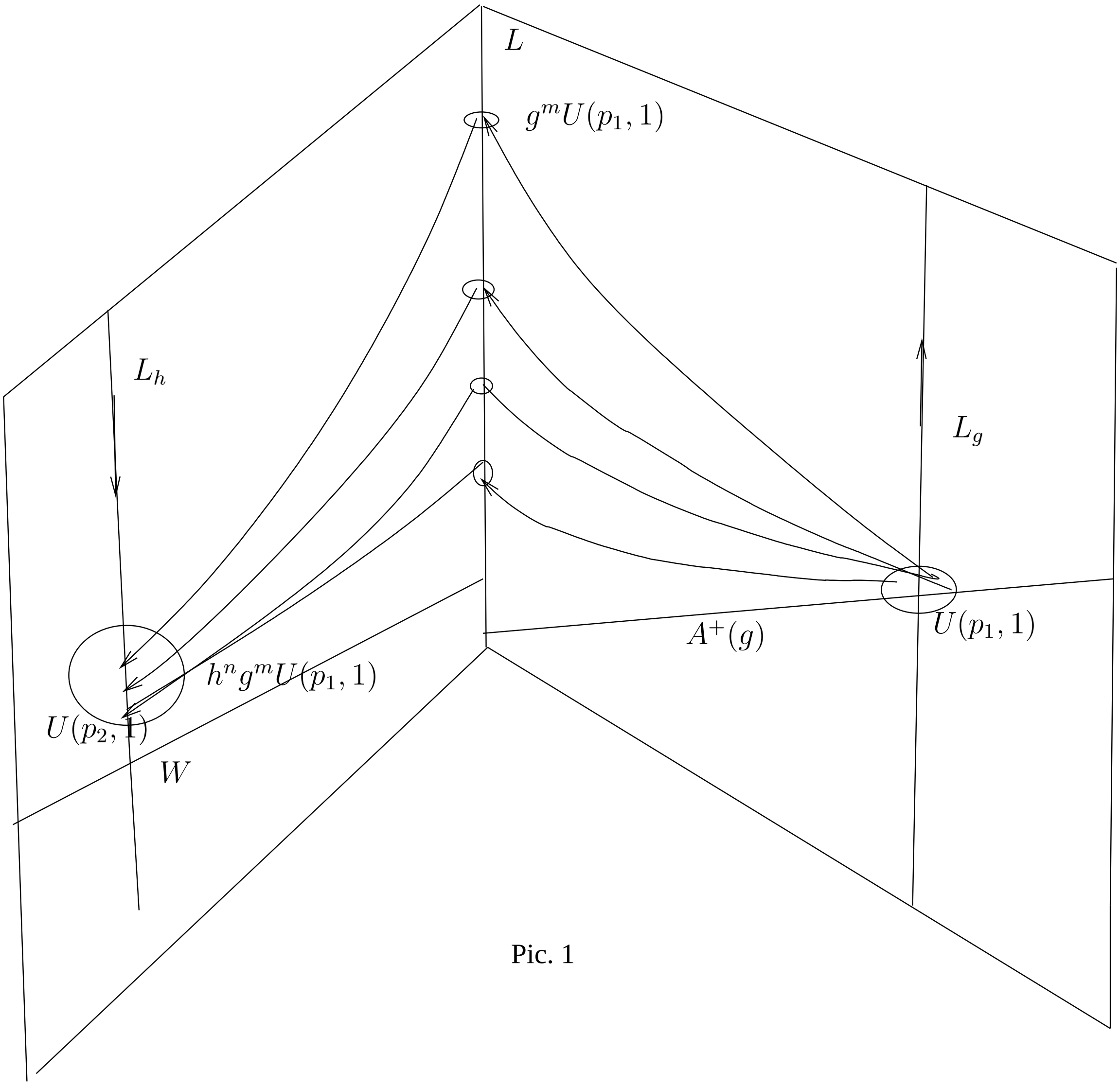

Figure 1: Transversal pairs

Suppose that there exists a regular element $g \in G$ which can be transformed into a transversal pair inside G. Then the group Γ does not act properly discontinuously.

Proof By Proposition 2.6, there exists a regular element $g \in \Gamma$ which can be transformed into a transversal pair inside Γ. Thus there exist $t \in \Gamma$ and a subspace $\tilde{W} \subseteq D^+(g)$ such that $l(t)\tilde{W} \oplus D^+(g) = V\}$. Put $h = tg^{-1}t^{-1}$. Clearly, $W = l(t)\tilde{W} \subset A^-(h)$ and $h^m \neq g^n$ for all $n, m \in \mathbb{Z}, n, m \neq 0$. Set $E_g = A^+(g) + L_g$ and $F_h = L_h + W$. Obviously, F_h is an h–invariant affine subspace. It is easy to see that $L = E_g \cap F_h$ is a one dimensional affine space. From (1) follows that $t_g \in V_1$. Since $t_h = l(t)t_{g^{-1}}$ from (4) follows that $t_h = -t_g$. Hence two lines L_g and L_h are parallel. The line L is parallel to L_g and L_h (see Figure 1). Let φ_g (resp. φ_h) be a natural projection $\varphi_g : E_g \longrightarrow L$ along $A^+(g)$ (resp. $\varphi_h : F_h \longrightarrow L$ along W). It is a simple exercise in linear algebra to show that if $p_1, p_2 \in L_g$, $q_1, q_2 \in L_h$ such that $\varphi_g(p_i) = \varphi_h(q_i), i = 1, 2$, then $\overrightarrow{p_1p_2} = \overrightarrow{q_1q_2}$. Therefore if $p_2 = p_1 + t_g$ and $q_2 = q_1 + t_h = q_1 - t_g$ then the two vectors $\overrightarrow{\varphi_g(p_1)\varphi_g(p_2)}$ and $\overrightarrow{\varphi_h(q_1)\varphi_h(q_2)}$ have opposite directions. Consider two closed balls $U_1(p_1, 1)$ and $U_2(q_1, 1)$. Then (see [A], Lemma 1.3 and Pic. 1 above) there exist infinite sets $N_1 \subset \mathbb{N}$ and $N_2 \subset \mathbb{N}$ such that $g^n h^m U(q_1, 1) \cap U(p_1, 1) \neq \varnothing$. Thus the group Γ does not act properly discontinuously.

3 Possible linear parts

Let Γ be an affine group acting properly discontinuously. Let G be the Zariski closure of Γ and let S be a semisimple part of the connected component of G. The goal of this sectionis to give a complete list of all possible semisimple subgroups of $GL(V)$, $V = \mathbb{R}^n$ which might be a semisimple part of G. The possible semisimple subgroups of $GL(V)$, which occur in our list fulfil the following assumptions: $dimV \leq 6$, there is a simple subgroup of S of real rank ≥ 2, every regular element $s \in S$ has one as an eigenvalue. Indeed,S is a subgroup of the connected component of G. Thus by Proposition 2.8 every regular element of S has one as an eigenvalue.

It is easy to see that if $\dim S \leq \dim V \leq 6$ then $rank_{\mathbb{R}}(S) \leq 1$. Hence it is impossible.. Thus we will assume that $\dim V \leq 6 \leq \dim S$. Let us now recall a list [PV, pp 260-261] of all possible complex representations ρ of a ***simple*** Lie group S with $\dim \rho \leq 6 \leq \dim S$. In the first column the symbols SL_n, Sp_n, SO_n denote the corresponding simple Lie (algebraic) group in their simplest representation. The symbol $S^m H$ (resp. $\wedge^m H$ denotes the m^{th} symmetric (resp. exterior) power of a linear group, and $S_0^m H$ (resp. $\wedge_0^m H$, is the highest (Cartan) irreducible component of this group.

Table 1

S	$\dim \rho$	n
$SL_n, n \geq 3$	n	$n = 3, 4, 5$
$SO_n, n \neq 4, n \geq 3$	n	$n = 3, 5, 6$
Sp_{2n}	$2n$	$2, 3$
$AdSL_n$	$n^2 - 1$	$n = 2$
$S^2 Sl_n$	$n(n+1)/2$	$n = 2, 3$
$\wedge^2 SL_n, n \geq 4$	$n(n-1)/2$	$n = 4$
$\wedge^2 SO_n, n \geq 3, n \neq 4$	$n(n-1)/2$	$n = 3$
$\wedge_0^2 Sp_{2n}, n \geq 2$	$(n-1)(2n+1)$	$n = 2$

Now we will provide a list of all possible real simple groups S which might be a simple part of G. Let $\overline{V} = V \otimes_{\mathbb{R}} \mathbb{C}$ be a complex space and let $\overline{S}$ be a complex Lie group, such that S is a real form of $\overline{S}$. If the group $\overline{S}$ is simple and irreducible then $\overline{S}$ is a group listed in Table 1. Thus using [OV] we have the following list of all real simple groups S which satisfies our assumptions:

Table 2

S	$\dim \rho$
$SL_n(\mathbb{R}), n \geq 3$	$n < 6$
$SO(3,2)$	5
$Sp_4(\mathbb{R})$	4

It is easy to see that there is no simple reducible real group which satisfies our requirements.

Example Consider the group $SO_3(\mathbb{C})$. Let $\sigma : \mathbb{C} \longrightarrow M_2(\mathbb{R})$ be the natural embedding of the field $\mathbb{C}$. Put $S = SO_3(\sigma(\mathbb{C}))$. Clearly S is a simple Lie group but the group $\overline{S} = SO_3(\mathbb{C}) \times SO_3(\mathbb{C})$ is not. Moreover, S is an irreducible subgroup of $\mathbb{R}^5$ but $\overline{S}$ is a reducible subgroup of $\mathbb{C}^6$. Obviously, every regular element $s \in S$ and respectively $s \in \overline{S}$ has one as an eigenvalue. Note that $rank_{\mathbb{R}}(S) = 1$.

Assume that $\overline{S}$ is a semisimple, not simple group. Then $\overline{S}$ is the direct product of simple groups $\overline{S} = \prod_{1 \leq i \leq k} \overline{S}_i, k \geq 2$. Let $W_0 = \{v \in \overline{V} : sv = v, \forall s \in \overline{S}\}$. There exists the unique $\overline{S}$–invariant subspace $\overline{W}$ of the space $\overline{V}$ such that $\overline{V}$ is the direct sum of W_0 and $\overline{W}$. If the restriction $\overline{S}|_{\overline{W}}$ is an irreducible representation of $\overline{S}$, then it is the tensor product of S_i– irreducible representations for all $i = 1, \dots, k$. From our assumptions and the inequality $\dim \overline{V} \leq 6$ immediately follows that this is impossible. Therefore, $\overline{W}$ is the direct sum of $\overline{S}$–invariant non-trivial irreducible subspaces $\overline{W} = \Sigma_1^k W_k$, and for every $i = 1, \dots, k$ the restriction $\overline{S}|_{W_i} = \overline{S}_i$ is an irreducible subgroup of $GL(W_i)$. As we know, every regular element of S has one as an eigenvalue. Thus if the subspace W_0 is trivial, we conclude that there exists $i_0, 1 \leq i_o \leq k$ such that every regular element $s \in \overline{S}_{i_0}$ has one as an eigenvalue. Since for every $i = 1, \dots, k$ the group $\overline{S}_i$ is an irreducible subgroup of $GL(W_i)$, we can and will again use Table 1 and Table 2. This leads us to a complete list of all possible cases. Indeed, from the inequality $\dim \overline{V} \leq 6$ follows that $k \leq 3$. If $k = 3$, then $S_i = SL_2(\mathbb{R})$ for each $i = 1, 2, 3$. Since $rank_{\mathbb{R}} SL_2(\mathbb{R}) = 1$ we conclude $k \leq 2$. Assume that $\dim W_1 \leq \dim W_2$. If $\dim W_1 = 2$ then $3 \leq \dim W_2 \leq 4$, and if $\dim W_1 = 3$ then $\dim W_2 = 3$. Therefore, if $\dim W_1 = 2$ we have: $S = SL_2(\mathbb{R}) \oplus SL_3(\mathbb{R})$. For $\dim W_1 = 3$, we have $S = SL_3(\mathbb{R}) \oplus SO(2,1)$ and $SL_3(\mathbb{R}) \oplus SO(3)$.

Let V_0 be the maximal subspace in $V = \mathbb{R}^n$ such that S acts trivially on V_0. Let V_1 be the unique S–invariant subspace such that $\mathbb{R}^n = V_0 \oplus V_1$. Let $\pi_S : G \longrightarrow S$ be the natural homomorphism. We will use these notations throughout the rest of the paper.

Case 1. Assume that for every regular element $s \in S$ the restriction $s|_{V_1}$ does not have 1 as an eigenvalue. In this case, as we noted above, the subspace W_0 is non-trivial. Since $W_0 = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ the subspace V_0 is non-trivial . Hence $\dim V_1 \leq 5$. Consider the inclusion $i : S \longrightarrow GL(V_1)$ as a representation of a semisimple Lie group. Let us summarize the above arguments and give a list of all possible cases for S :

(1) $S = SL_l(\mathbb{R}), V_1 = \mathbb{R}^l, 3 \leq l \leq 5, l < n, 4 \leq n \leq 6$

(2) $Sp_4(\mathbb{R}), V_1 = \mathbb{R}^4, n = 6$

(3) $S = SL_2(\mathbb{R}) \times SL_3(\mathbb{R}), V_1 = \mathbb{R}^5, n = 6$

Case 2. Assume that for a regular element $s \in S$ the restriction $s|_{V_1}$ has 1 as eigenvalue. It easily follows from the list above that all possible cases are:

(1) $S = SO(3,2), \dim V_1 = 5, n = 5, 6$

(2) $SO(3) \times SL_3(\mathbb{R}), \dim V_1 = n = 6$

(3) $SO(2,1) \times SL_3(\mathbb{R}), \dim V_1 = n = 6.$

Remark 1. It is clear that the group S is a simple group for $n \leq 5$, and for $n = 4$ we have $S = SL_3(\mathbb{R})$.

4 The Auslander conjecture in dimensions 4 and 5

In this section we will prove the Auslander conjecture in dimension 4 and 5. Let us first explain the plan of the proof of Main Theorem. Let Γ be a crystallographic group and G be the Zariski closure of Γ. Then we have the Levi decomposition $G = SR$ where R is the solvable radical and S is a semisimple part of G. Let $S = \prod_{1 \leq i \leq k} S_k$ be the decomposition of the semisimple part into an almost direct product of simple groups. It is well known that if $rank_{\mathbb{R}}(S_i) \leq 1$ for all $1 \leq i \leq k$ then Γ is not crystallographic [S2], [To2]. Therefore from now on unless otherwise noted we will assume that

$$\max_{1 \leq i \leq k} rank_{\mathbb{R}}(S_i) \geq 2 \qquad (A)$$

Therefore using Proposition 2.8 we conclude that if S is non-trivial then S is one of the group listed in Case 1 and Case 2. We will prove the Auslander conjecture case by case. The idea is to show that the semisimple part S of the Zariski closure of Γ is trivial. Hence Γ is virtually solvable. Thus we will show that S can be none of the semisimple group, listed in Case 1 and 2. First we will show that if $dimV \leq 5$ then Γ does not act properly discontinuously. The proof for the affine space of $dim = 6$ splits into several steps. We will show that if the semisimple part is as in case 1 the group Γ does not act properly discontinuously. Then using cochomological arguments we show that in case 2 (1), (2) the group Γ is not crystallographic. Finally, using the dynamical approach we invented in [AMS 4], we will show that in case 2 (3) the group Γ is not crystallographic. Thus S is trivial. in this case.

Set $\Gamma_\infty = \bigcap_{1 \leq i \leq \infty} \Gamma_i$, where $\Gamma_1 = [\Gamma, \Gamma]$, $\Gamma_i = [\Gamma, \Gamma_{i-1}]$. Let G_1 be the Zariski closure of

$l(\Gamma_\infty)$. We often will pass to Γ_∞ and G_1 because in some cases it is easier to prove the stronger statement. Namely that Γ does not act properly discontinuously .

Step 1 $dimV = 4$

We will prove that in this case Γ does not act properly discontinuously under all our assumptions. The unique possible case is $S = SL_3(\mathbb{R})$. Let R be the solvable radical of G and let U be the unipotent radical of G. The space V is a direct sum of two S–invariant subspaces V_0 and V_1 such that $V_0 = \{v \in V : sv = v \text{ for all } s \in S\}$. Obviously, there are two possibilities: (a) $(u-e)v \in V_0$, and $uv_0 = v_0$ for all $u \in U, v \in V, v_0 \in V_0$; (b) $uv_1 = v_1$ and $(u-e)v \in V_1$ for all $u \in U, v \in V, v_1 \in V_1$. Clearly $S = SL_3(\mathbb{R})$ is a semisimple part of G_1 and G_1 fulfils the requirements of Proposition 2.9 in case (a). Hence the subgroup Γ_∞ of Γ does not act properly discontinuously. Obviously the same is true for Γ.

(b) In this case, Γ_∞ is an affine group which acts properly discontinuously on V_1 and $SL_3(\mathbb{R})$ is the linear part of Γ_∞. This contradicts Proposition 2.1. Thus under the assumption (A), we have proved

Proposition " $\dim V = 4$***".*** *If Γ acts properly discontinuously, then Γ is virtually solvable.*

Thus we have

Proposition 4.1 *Let Γ be a crystallographic group, $\Gamma \subseteq Aff\mathbb{R}^4$ then Γ is virtually solvable.*

Step 2 $n = 5, V = \mathbb{R}^5$

Note that if $n = 5$ and $S = SO(3,2)$, then by Theorem B [AMS 3] the group Γ does not act properly discontinuously. Hence for $n = 5$ we can assume that $S = SL_3(\mathbb{R}), SL_4(\mathbb{R}), Sp_4(\mathbb{R})$. As in Step 1, we will prove that Γ does not act properly discontinuously. If $S = SL_4(\mathbb{R})$ or $Sp_4(\mathbb{R})$ then by the same arguments as in Step 1, we conclude that Γ does not act properly discontinuously. Let $S = SL_3(\mathbb{R})$. There exists a chain of length ≤ 3 of G–invariant subspaces of V. Recall that G_1 is the Zariski closure of Γ_∞. Then $S = SL_3(\mathbb{R})$ is the semisimple part of G_1. Let U be the unipotent radical of the linear part of G_1. Then there exists an $l(G_1)$ –invariant chain $\{0\} \subset W_0 \subset V$ such that $S \mid_{W_0} = SL_3$ and the restriction $U \mid_{W_0}$ and the induced representation $U \longrightarrow GL(V/W_0)$ are trivial. Thus we can apply Proposition 2.9 and conclude that Γ_∞ does not act properly discontinuously. Hence the same it true for Γ. Thus under the assumption (A), we proved

Proposition **"**$\dim V = 5$**".** *If Γ acts properly discontinuously, then Γ is virtually solvable.*

Thus as above we have

Proposition 4.2 *Let Γ be a crystallographic group, $\Gamma \subseteq Aff\mathbb{R}^5$ then Γ is virtually solvable.*

5 The Auslander conjecture in dimension 6. The cohomological argument.

In this section we will show that Case 1 and Case 2 (1), (2) are impossible. We will start with the following

Proposition 5.1. *Assume that S is as in the Case 1 (1), (2) Then the group Γ does not act properly discontinuously*

Proof The proof is a verbatim repetition of the proof given in Step 2.

Proposition 5.2 *Assume that $S = SL_2(\mathbb{R}) \times SL_3(\mathbb{R})$. Then the group Γ does not act* *properly discontinuously.*

Proof Recall that G_1 is the Zariski closure of the linear part of Γ_∞. Then G_1 is a product of a semisimple group S and the unipotent radical U. Let $\{0\} \subset W_0 \subset W_1 \subset \cdots \subset W_k \subset W_{k+1} = \mathbb{R}^n$ be a chain of G_1–invariant subspaces such that for every $u \in U$ and $i, 0 \leq i \leq k$, we have $(u-e)W_{i+1} \subset W_i$. It is easy to conclude that $k \leq 2$ and that there exists one case which can not be reduced to Proposition 2.9. Namely, the restriction $S \mid W_0$ and the induced representation $S \longrightarrow GL(W_2/W_1)$ are non trivial. Hence the induced representation $S \longrightarrow GL(W_1/W_0)$ is trivial. It is obvious that there exist S–invariant spaces U_1 and U_2 such that $V = W_0 \oplus U_1 \oplus U_2$, $W_0 \oplus U_1 = W_1$. Let us prove Proposition 5.2 assuming that $S \mid W_0 = SL_3(\mathbb{R})$, $S \mid U_1 = I$ and $S \mid U_2 = SL_2(\mathbb{R})$. In case $S \mid W_0 = SL_2(\mathbb{R})$, $S \mid U_1 = I$ and $S \mid U_2 = SL_3(\mathbb{R})$ one can prove that Γ does not act properly discontinuously using the same arguments.

Let $g \in \Gamma_\infty$ be a regular element. We can and will assume that $l(g) \in S$. Let $g_0 = l(g) \mid W_0$, $g_1 = l(g) \mid U_1$ and $g_2 = l(g) \mid U_2$. We can assume that $dim A^+(g_0) = 2$. Note that $dim A^+(g_2) = 1$ and $A^0(g) = U_1$. Let W be a one dimensional $l(g)$–invariant subspace of $A^+(g_0)$. Then there exists $t \in \Gamma_\infty$ such that $l(t)W \oplus A^+(g_0) = W_0$ and $l(t)A^+(g_2) \oplus A^+(g_2) \oplus W_1 = V$. Put $h = tg^{-1}t^{-1}$. Let us show that there are two balls of a radius one $U(p_1, 1)$ and $U(p_2, 1)$ where $p_1 \in L_g$ and $p_2 \in L_h$ such that for infinitely many $m, n \in \mathbb{N}$ we have $h^m g^n U(p_1, 1) \cap U(p_2, 1) \neq \emptyset$. Let $A = l(t)A^+(g_2) + l(t)W$ and $D = A + L_h$. Clearly D is an h–invariant affine space and $\dim(D \cap E^+(g)) = 1$ Let $L = D \cap E^+(g)$. Obviously the natural projection $\pi_1 : L_g \longrightarrow L$ and $\pi_2 : L_h \longrightarrow L$ are affine isomorphisms. Set $\theta : L_g \longrightarrow L_h$ where $\theta = \pi_2^{-1} \circ \pi_1$. It is easy to check, that $\theta(t_g) = -t_h$. It is easy to see that there exists N_1 such that for $m > N_1$ we have $g^m U(p_1, 1) \cap L \neq \emptyset$. It is obvious that there exists N_2 such that for $n > N_2$ we have $h^n(g^m U(p_1, 1) \cap L) \cap U(p_2, 1) \neq \emptyset$. Therefore for $n, m > \max N_1, N_2$ we have $h^m g^n U(p_1, 1) \cap U(p_2, 1) \neq \emptyset$. Since $g^m \neq h^n$ for all $n, m \in \mathbb{Z}, n, m \neq 0$ the group Γ does not act properly discontinuously.

Proposition 5.3. *Assume that S is as in Case 2 (1),(2). Then the group Γ is not a crystallographic group.*

Proof . Let us first explain the main idea of the proof. Since the subgroup $\Gamma \subseteq Aff(\mathbb{R}^n)$ is a crystallographic group, the virtual cohomological dimension $vcd(\Gamma)$ of Γ is $\dim \mathbb{R}^n = n$. Hence $vcd(\Gamma) = 6$. As a first step we will show that $vcd(\Gamma) \leq \dim(S/K)$, where S/K- the symmetric space of S. Then we compare $\dim S/K$ and $vcd(\Gamma)$ in the cases $S = SO(3) \times SL_3(\mathbb{R})$, $S = SO(3,2)$ and come to the conclusion that $\dim S/K \geq vcd(\Gamma)$. This will lead to a contradiction.

Let us first show that $vcd(\Gamma) \leq \dim(S/K)$. Recall that R is the solvable radical of G. Let U be the unipotent radical of G. It is easy to see that in Case 2 (2) , we have $R = U$. Let $\Gamma_r = R \cap \Gamma$ and let R_1 be the Zariski closure of Γ_r. Then the group R_1 is a normal subgroup in G since Γ_r is a normal subgroup in Γ. By [S2, Proposition 2], we have that Γ_r is a co-compact lattice in R_1. Set $W = R_1 q_0$ where q_0 is an origin point. We have $sW = W$ for $s \in S$, since $sq_0 = q_0$ and R_1 is a normal subgroup of G. Then we have the natural linear representation $\rho : S \to End(T_{q_0})$, where T_{q_0} is the tangent space of W at the point q_0. It is clear that possible numbers for $\dim(T_{q_0})$ are $\{0, 3, 6\}$ if $S = SO(3) \times SL_3(\mathbb{R})$ and $\{0, 1, 5, 6\}$ if $S = SO(3,2)$. Let us show that $\dim(T_{q_0}) = 0$. Assume that $\dim(T_{q_0}) = 6$. Then $R_1 q_0 = \mathbb{R}^6$. Therefore, Γ_r is a crystallographic group. On the other hand Γ_r is a subgroup of a crystallographic group Γ which acts on the same affine space. Then the index $|\Gamma/\Gamma_r|$ is finite, a contradiction. Hence $\dim(T_{q_0}) < 6$. We will treat the two cases $S = SO(3) \times SL_3(\mathbb{R})$ and $SO(3,2)$ separately.

Let $S = SO(3) \times SL_3(\mathbb{R})$ and $\dim(T_{q_0}) = 3$. Then G is a subgroup of the following group $\widetilde{G} = \{X : X \in GL_7(\mathbb{R})\}$, where

$$X = \begin{pmatrix} A & B & v_1 \\ 0 & C & v_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and $A \in SO(3), B \in SL_3(\mathbb{R}), v_1, v_2 \in \mathbb{R}^3$ or $A \in SL_3(\mathbb{R}), B \in SO(3), v_1, v_2 \in \mathbb{R}^3$. Obviously G and $\widetilde{G}$ have the same semisimple part, the solvable radical R of G is unipotent and if $X \in R$ then

$$X = \begin{pmatrix} I_3 & B & v_1 \\ 0 & I_3 & v_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since R_1 is a normal subgroup of G and $\dim W = 3$, we conclude that there are two possible cases for R_1, namely, $R_1 = \{X, X \in R : v_1 = 0\}$ or $R_1 = \{X, X \in R : v_2 = 0\}$. Obviously in both cases W is a $\Gamma-$ invariant *affine* subspace of dimension 3 and Γ acts as a crystallographic group on W. This contradiction proves that $\dim W = 0$. By Auslander's theorem [R], $\pi_S(\Gamma)$ is a discrete subgroup of S. Since the intersection $\Gamma \cap R$ is trivial, $\pi_S(\Gamma)$ and Γ are isomorphic. Hence $vcd(\Gamma) = vcd(\pi_S(\Gamma)) \leq \dim S/K$, where K is a maximal compact subgroup in S. Thus $vcd(\Gamma) \leq 5$. On the other hand, $vcd(\Gamma) = 6$ a contradiction.

Let us now show that Case 2 (1) is also impossible. This will prove the proposition. We will prove first that $\dim W = 0$. Recall that $W = R_1 q_0$. As we concluded above, there are three possible cases for $\dim W$, namely, $\dim W = 0, 1, 5$. Assume that $\dim W = 1$. Then the natural representation $\rho : S \to End(T_{q_0})$ is trivial. Clearly, $S = SO(3,2)$ is an irreducible subgroup of $GL(V_1)$. Therefore we conclude that if X is an element in the normal subgroup R_1 of G, then

$$X = \begin{pmatrix} 1 & w & a \\ 0 & I_5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $w \in \mathbb{R}^5, a \in \mathbb{R}$. Thus, W is an affine $\Gamma-$ invariant subspace in $\mathbb{R}^6$. Therefore we have a natural homomorphism $\theta : \Gamma \longrightarrow \text{Aff}(\mathbb{R}^6/W)$. By [S2, Lemma 4], $\Gamma/\Gamma_r = \theta(\Gamma)$ is a crystallographic subgroup in $\text{Aff}(\mathbb{R}^6/W)$. Obviously, the semisimple part of the Zariski closure of $\theta(\Gamma)$ is $SO(3,2)$ and $\mathbb{R}^6/W = \mathbb{R}^5$. By [AMS 3, Theorem A] this is impossible.

Assume that $\dim W = 5$. Again consider the orbit space $\mathbb{R}^6/W$. Let T_W be a tangent space of $\mathbb{R}^6/W$ at the point W. We show in [S2, Proof of Theorem A] that one is an eigenvalue of $\pi(g)$ for every element $g \in G$, where $\pi : G \longrightarrow T_W$ is the natural representation $\pi : G \longrightarrow T_W$. Since $\dim \mathbb{R}^6/W = 1$ the representation π is trivial. Note that from this in particular follows, that R is a unipotent group. By direct calculation we conclude that there are two possible cases for the normal subgroup R_1 in G namely, $R_1 = \{X, X \in GL_7(\mathbb{R})\}$ such that

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_5 & v \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where $v \in \mathbb{R}^5$, or

$$X = \begin{pmatrix} 1 & v^t J & a \\ 0 & I_5 & v \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where J is the involution such that $A^t J A = J$ for every $A \in SO(3,2)$, $v \in \mathbb{R}^5$ and $a = v^t J v$.

If elements in R_1 are as in (1), then evidently W is an affine subspace of $\mathbb{R}^6$. Thus by the same argument we used in case 2 (2) we conclude that W is an affine Γ–invariant subspace and Γ_r is a crystallographic subgroup of $\mathrm{Aff}\,(W)$. On the other hand, Γ_r is a subgroup of a crystallographic group Γ which acts on the same affine space W. Then the index Γ/Γ_r is finite, a contradiction.

Suppose that elements in R_1 are as in (2). Consider the orbit space $\mathbb{R}^6/W$. By [S2, Lemma 4], $\widehat{\Gamma} = \Gamma/\Gamma_r$ is a crystallographic group which acts on $\mathbb{R}^6/W$. Clearly, $\Gamma W \subset (l(\Gamma)Z(G) \cap N_G(W))\,W$. Obviously the commutator $[\widehat{\Gamma}, \widehat{\Gamma}]$ acts trivially on the orbit space $\mathbb{R}^6/W$ which is impossible. Therefore $W = 0$. Hence $R_1 = \{e\}$ and the restriction of the homomorphism $\pi_S : G \longrightarrow S = G/R$ onto Γ is an isomorphism. By Auslander's theorem [R], the projection $\pi_S(\Gamma)$ is a discrete subgroup in S and $vcd(\pi_S(\Gamma)) = vcd(\Gamma) = 6$. On the other hand $vcd(\pi_S(\Gamma)) \leq \dim S/K$, where K is a maximal subgroup in S. Obviously, $\dim S/K = 6$. Hence $vcd(\pi_S(\Gamma)) = \dim S/K$. Therefore $\pi_S(\Gamma)$ is a co-compact lattice in S. We can apply the Margulis rigidity theorem, since $rank_{\mathbb{R}}(S) = 2$ and conclude that there exists a $g \in \Gamma$ such that $\Gamma_1 = g\Gamma g^{-1} \cap S$ is a subgroup of finite index in Γ. Since $\Gamma_1 \leqslant S$ we have $\Gamma_1 p_0 = p_0$. Thus Γ does not act properly discontinuously.

Remark 2. Using the dynamical ideas and results from [AMS 4] we can prove that if

S is as in Proposition 5.3, i.e as in Case 2, (1), (2) then the group Γ does not act *properly discontinuously.*

6 The Auslander conjecture in dimension 6. Dynamical arguments

6.1. Orientation. The dynamical approach we have used [AMS3] and will use here is based on the so called "Margulis's sign" of an affine transformation. The case $S = SO(2,1) \times SL_3(\mathbb{R})$ needs other tools, namely a new version of the Margulis sign. We will need to introduce it for the natural representation of S which goes roughly saying by ignoring the $SL_3(\mathbb{R})$–factor. We then have a lemma similar to the cases of $SO(k+1,k)$, namely lemma 6.7, which says that if a group acts properly discontinuously, then opposite signs are impossible.

Now we will recall the important definition of sign of an affine transformation. This definition was first introduced by G. Margulis [M] for $n = 3$ Then it was generalized in [AMS3] for the case in which the signature of the quadratic form is $(k+1,k)$ and finally for an arbitrary quadratic form in [AMS4]. We will follow along the lines of [AMS 4]. Let B be a quadratic form of signature (p,q), $p \geq q, p+q = n$. Let v be a vector in $\mathbb{R}^n$, $v = x_1 v_1 + \cdots + x_p v_p + y_1 w_1 + \cdots + y_q w_q$, where $v_1, v_2, \ldots, v_p,\ w_1, w_2, \ldots, w_q$ is a basis of $\mathbb{R}^n$. We can and will assume that

$$B(v,v) = x_1^2 + \cdots + x_p^2 - y_1^2 - \cdots - y_q^2.$$

Consider the set Ψ of all maximal B–isotropic subspaces. Let X be the subspace spanned by $\{v_1, v_2, \ldots, v_p\}$ and Y be the subspace spanned by $\{w_1, w_2, \ldots, w_q\}$. It is clear that $\mathbb{R}^n = X \oplus Y$. Define the cone

$$\mathbb{C}_B = \{v \in \mathbb{R}^n | B(v,v) < 0\}.$$

Clearly $Y \backslash \{0\} \subset \mathbb{C}_B$. We have the two projections

$$\pi_X : \mathbb{R}^n \longrightarrow X \text{ and } \pi_{\mathrm{Y}} : \mathbb{R}^{\mathrm{n}} \longrightarrow \mathrm{Y}$$

along Y and X, respectively. The restriction of π_Y to $W \in \Psi$ is a linear isomorphism $W \longrightarrow Y$. Hence if we fix an orientation on Y, then we have also fixed an orientation on each $W \in \Psi$. For $V \in \Psi$, let us denote the B–orthogonal subspace of V by $W^\perp = \{z \in \mathbb{R}^n \ ; \ B(z, W) = 0\}$. We have $W \subset W^\perp$ since W is B–isotropic. We also have

$$\dim W^\perp = \dim W + (p - q) = p.$$

The restriction of π_X to $W^\perp$ is a linear isomorphism $W^\perp \longrightarrow X$. Hence if we fix an orientation on X, then we have also fixed an orientation on $W^\perp$ for each $W \in \Psi$. Thus we have orientations on both W and $W^\perp$ and we have naturally induced an orientation on any subspace $\widehat{W}$, such that $W^\perp = W \oplus \widehat{W}$. If $V_1 \in \Psi$ and $V_2 \in \Psi$ are transversal, then $V_0 = V_1^\perp \cap V_2^\perp$ is a subspace that is transversal to both V_1 and V_2; therefore $V_0 \oplus V_1 = V_1^\perp$ and $V_0 \oplus V_2 = V_2^\perp$. So there are two orientations ω_1 and ω_2 on V_0, where ω_i is defined if we consider V_0 as a subspace in $V_i^\perp$. We have [see AMS3, Lemma 2.1]

Lemma 6.2. *The orientations defined above on V_0 are the same if q is even and they are opposite if q is odd, i.e. $\omega_1 = (-1)^q \omega_2$.*

Let us explain this in the special case when $p = k + 1, q = k$.

Example 6.3 . Let V_1 and V_2 be the maximal isotropic subspaces spanned by the vectors $\{w_1 + v_1, \ldots, w_k + v_k\}$ and $\{w_1 - v_1, \ldots, w_k - v_k\}$ respectively. Since for every $i = 1, \ldots, k$ we have $\pi_Y(w_i \pm v_i)) = w_i, i = 1, \ldots, k$, we conclude that $w_1 + v_1, \ldots, w_k + v_k$ (resp. $w_1 - v_1, \ldots, w_k - v_k$) is a positively oriented basis of V_1 (resp. V_2) Then $V_1^\perp \cap V_2^\perp$ is spanned by the vector v_{k+1}. Let $v^0(V_1^\perp) \in V_1^\perp \cap V_2^\perp$ and $v^0(V_2^\perp) \in V_1^\perp \cap V_2^\perp$ such that $\{w_1 + v_1, \ldots, w_k + v_k, v^0(V_1^\perp)\}$ (resp. $\{w_1 - v_1, \ldots, w_k - v_k, v^0(V_2^\perp)\}$) is a positively oriented base of $V_1^\perp$ (resp. $V_2^\perp$.) We have $v^0(V_1^\perp) = (-1)^k v^0(V_2^\perp)$ since $\pi_X(w_i + v_i) = v_i$ and $\pi_X(w_i - v_i) = -v_i$ for all $i, i = 1, \ldots, k$. In particular, $v^0(V_1^\perp) = -v^0(V_2^\perp)$ when $k = 1$. The general case follows since any pair of maximal B–isotropic transversal subspaces of $\mathbb{R}^n$ is of the form $(g(V_1), g(V_2))$ for some $g \in SO(B)$.

6.4 Sign. Let us recall now the definition of the *sign* of an affine element. Let $g \in G_n$ be a regular element with $l(g) \in SO(B)$ where B is a non-degenerate form on $\mathbb{R}^n$ of signature $(k+1, k)$. Obviously, the subspaces $A^+(g)$ and $A^-(g)$ are maximal B- isotropic subspaces, $D^+(g) = A^+(g)^\perp$ and $D^-(g) = A^-(g)^\perp$. Following the procedure above for the element g we choose and fix a vector v_+ with the following property $v_+ = v^0(D^+(g))$. Let $q \in \mathbb{R}^n$. Let us point out that we choose an orientation on the line $A^0(g)$ according to the orientation coming from $D^+(g)$ Thus the orientation we have to take on $A^0(g^{-1})$ comes from $D^+(g^{-1}) = D^-(g)$. We will denote a corresponding vector by v_-. Set

$$\alpha(g) = B(gq - q, v_+)/B(v_+, v_+)^{1/2}.$$

It is clear, that $\alpha(g)$ does not depend on the point $q \in \mathbb{R}^n$ and $\alpha(g) = \alpha(x^{-1}gx)$ for every $x \in G_n$ such that $l(x) \in SO(B)$. Consider now any regular element g and let us show that $\alpha(g^{-1}) = (-1)^k\alpha(g)$. Indeed by Example 6.3, $v_- = v^0(D^+(g^{-1})) = v^0(D^-(g)) = (-1)^k v^0(D^+(g)) = (-1)^k v_+$. Let $q \in \mathbb{R}^n$ be a point. We have $\alpha(g^{-1}) = B(g^{-1}q - q, v_-)/B(v_-, v_-)^{1/2} = (-1)^k B(g^{-1}q - q, v_+)/B(v_+, v_+)^{1/2} = (-1)^{k+1}B(q - g^{-1}q, v_+)/B(v_+, v_+)^{1/2}$. Put $p = g^{-1}q$. Hence $\alpha(g^{-1}) = (-1)^{k+1}\alpha(g)$. Note that $\alpha(g) = \alpha(g^{-1})$ if $k = 1$. Recall that $\alpha(g)$ is called the sign of g (see [AMS3]).

Using this approach we define now the sign of a regular element g of the group Γ for the case that the semisimple part of the Zariski closure of Γ is $SO(2,1) \times SL_3(\mathbb{R})$. Recall that $V = V_1 \oplus V_2$, $S\ |_{V_1} = SO(2,1)$ and $S\ |_{V_2} = SL_3(\mathbb{R})$. We will also assume that our standard inner product (see 2.4) is chosen so that subspaces V_1 and V_2 are orthogonal. As the first step we have to choose the positive vector $v_g, v_g \in A^0(g)$. Let $g \in S$ be a regular element. Let $\widehat{g}$ be the restriction $g\ |_{V_1}$. Obviously $A^0(g) = A^0(\widehat{g})$. Set $v_g =$

$v_+/B(v_+.v_+)^{1/2}$ Let $g \in G$ be a regular element, then there exists unique $u \in U$ such that $h = ugu^{-1} \in S$. Set $v_g = uv_h$. There is a simple geometrical explanation of this definition. Let $\pi : V \longrightarrow V_1$ be the natural projection onto V_1 along V_2. We have the corresponding homomorphism $\widehat{\pi} : G \longrightarrow SO(2,1)$. It is easy to see that the restriction of π onto $A^0(g)$ gives an isomorphism onto $A^0(\widehat{\pi}(g))$ and $\pi(v_g) = v_{\widehat{\pi}(g)}$. Let $\tau_g : V \longrightarrow L_g$ be the natural projection of the affine space V onto the line L_g along the subspace $A^+(g) \oplus A^-(g)$, where g is a regular affine element . There exists a unique $\alpha \in \mathbb{R}$ such that $\tau_g(p) - p = \alpha v_g$. We set $\alpha(g) = \alpha$. Clearly $\alpha(g) = B(\pi(\tau_g(p) - p), \pi(v_g))$ where B is the form of signature $(2,1)$ on V_1 fixed by $SO(2,1)$ since $\pi(v_g) = v_{\widehat{\pi}(g)}$. Obviously $\alpha(g)$ does not depend on the chosen point $\alpha(g^{-1}) = \alpha(g)$ and $\alpha(g^n) = |n|\alpha(g)$. For more details see [AMS4, p. 5]

Let us now explain the main application of these definitions. Let g and h be two regular transversal elements. Then $A^-(h) \oplus D^+(g) = V$ and $dim(D^-(h) \cap D^+(g)) = 1$. Let $A = D^-(h) \cap D^+(g)$. Let L be the corresponding line $L = E_g^+ \cap E_h^-$. There exist affine isomorphisms $\pi_1 : L_g \longrightarrow L$ and $\pi_2 : L_h \longrightarrow L$. By the above arguments for $p \in L_g, q \in L_h$ the vectors $\pi_2(hq-q)$ and $\pi_1(gp-p)$ have opposite directions if $\alpha(g)\alpha(h) < 0$. Then as in the proof of Theorem A [AMS3], we conclude that there exist infinitely many positive numbers n, m and two balls $B(p,1)$ and $B(q,1)$ such that $h^m g^n B(p,1) \bigcap B(q,1) \neq \emptyset$. Thus the following statement is true

Lemma 6.5 . *Assume that S as in the Case 2 (3), and there are two hyperbolic transversal elements $g, h \in \Gamma$ such that $\alpha(g)\alpha(h) < 0$. Then the group Γ does not act properly discontinuously.*

6.6 To construct transversal elements of the group Γ with opposite sign is more difficult here than in the case when the semisimple part is $SO(2,1)$ (see [S2]). To make products transversal, one needs a quantitative version of hyperbolicity and transversality, see Lemma 2.7. Thus we construct the appropriate set $M \subseteq \Gamma$ of hyperbolic elements to insure that a given hyperbolic element $\gamma \in \Gamma$ will be at least $\varepsilon = \varepsilon(M)$–transversal to

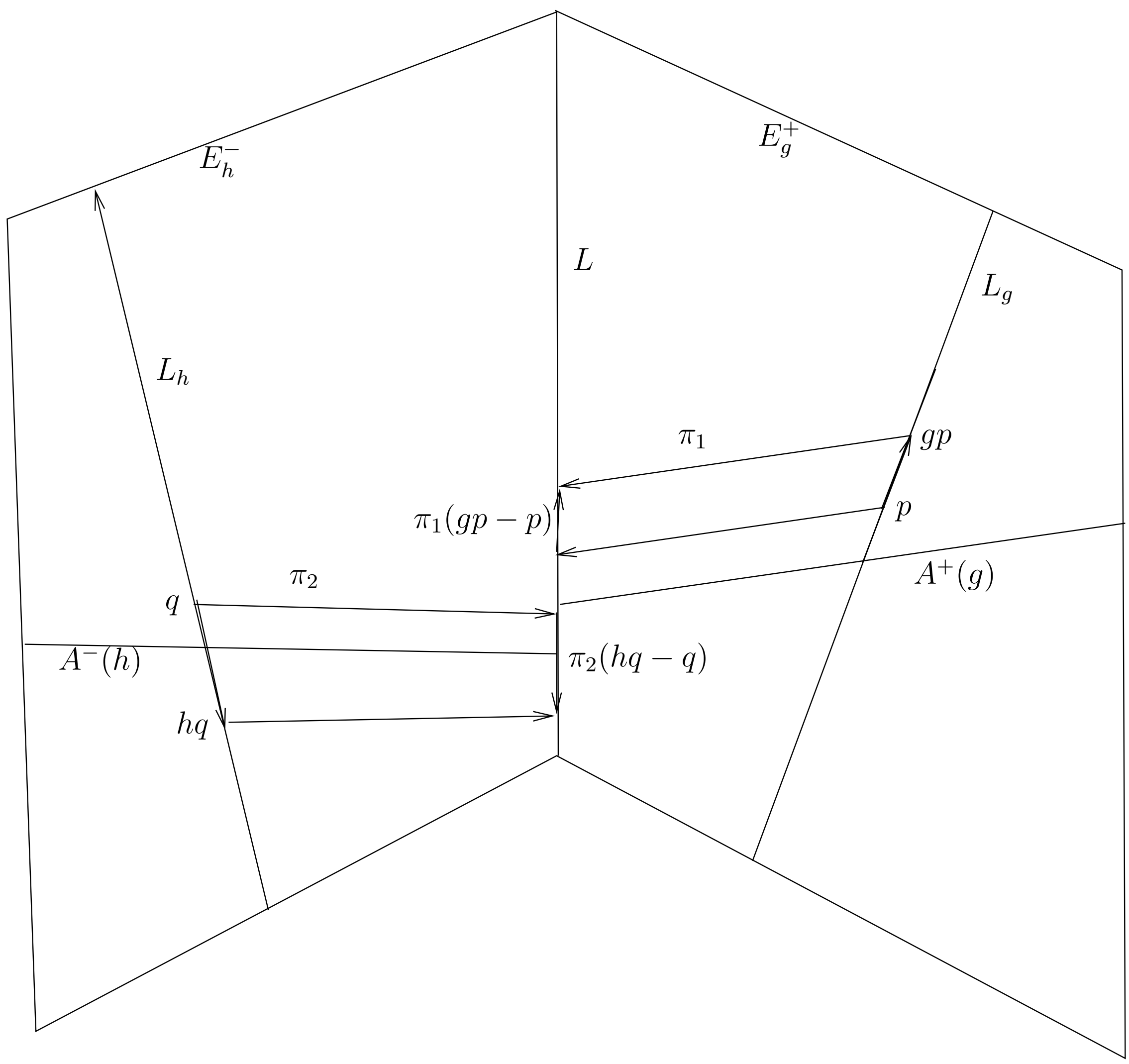

Figure 2: Positive and negative parts

some element of M. Moreover, the set M will be the union of two sets M_1 and M_2. If the number of eigenvalues of γ greater than 1 is two (resp. three) then $\gamma \in \Gamma$ will be at least ε–transversal to some element of M_1 (resp. M_2). This is close to the strategy we used in [AMS1].

Recall that v_1, v_2, w_1 is a basis of V_1 such that for any vector $v \in V_1$, $v = x_1v_1 +$

$x_2v_2 + y_1w_1$ we have $B(v,v) = x_1^2 + x_2^2 - y_1^2$. We will use notations and definitions from ***6.1***. Let U be a maximal B-isotropic subspace of V_1 and let v be a vector from U such that U is spanned by v and $\pi_Y(v) = w_1$ Let v_0 be a vector from $U^\perp \cap X$ such that $B(v_0, v_0) = 1$ and the basis $\pi_X(v), v_0$ has the same orientation as v_1, v_2. Let W be a maximal B-isotropic subspace, $W \neq U$. Then $\dim(U^\perp \cap W^\perp) = 1$. There exists a unique vector $w_0(W)$ in $U^\perp \cap W^\perp$ and $\widehat{v} \in U$ such that $w_0(W) = v_0 + \widehat{v}$. Obviously there exists a unique number $\alpha(W)$ such that $\widehat{v} = \alpha(W)v$. Set $\Phi_U^+ = \{W \in \Phi |\, \alpha(W) > 0\}$ and $\Phi_U^- = \{W \in \Phi |\, \alpha(W) < 0\}$. Since $v_0 \in X$, we have $B(v_0, w_1) = 0$. Therefore $B(w_0(W)), w_1) = \alpha(W)$, $B(v, w_1) = -\alpha(W)$. Hence for every vector $w \in \Phi_U^+$ (resp. $w \in \Phi_U^-$) we have $B(w, w_1) < 0$ (resp.$B(w, w_1) > 0$). Since $v_0 \in X$, we have $B(v_0, w_1) = 0$. Therefore $B(w_0(W)), w_1) = \alpha(W)$. We conclude :

(1). For every any $W \in \Phi_U^+$ (resp. $W \in \Phi_U^-$) we have $B(w_0(W), w_1) > 0$ (resp. $B(w_0(W), w_1) < 0$).

(2). Let W_1, W_2, W_3, W_4 be maximal B–isotropic subspaces of V_1 such that $w_1 \in (W_1 + W_2) \bigcap (W_3 + W_4)$.

Then for every maximal B–isotropic subspace U of V_1 there exists an $i_0 \in \{1, 2, 3, 4\}$ such that $W_{i_0} \in \Phi_U^-$. Let $d = \min_{1 \leq i \neq j \leq 4}\{d(W_i, W_j)\}$. There exists $\delta = \delta(d)$ such that if $\widehat{W_i}$ are B–maximal isotropic subspaces of V_1, , $1 \leq i \leq 4$ and $d(\widehat{W_i}, W_i) \leq \delta$ for $1 \leq i \leq 4$ then for every maximal B–isotropic subspace U of V_1 there exist an $i_0 \in \{1, 2, 3, 4\}$ such that $\widehat{W}_{i_0} \in \Phi_U^-$.

(3) Assume that $W_1 \in \Phi_U^+$ and $W_2 \in \Phi_U^-$. Let $\varepsilon = \min\{d(W_1, U), d(W_2, U)\}$. Then there exists a $\delta = \delta(\varepsilon)$ such that if $\widehat{U}$ is a maximal B–isotropic subspace with $d(\widehat{U}, U) < \delta$ we have $\widehat{W_1} \in \Phi_{\widehat{U}}^+$ and $\widehat{W_2} \in \Phi_{\widehat{U}}^-$ for any maximal B–isotropic subspaces $\widehat{W_1}, \widehat{W_2}$ with $d(\widehat{W_1}, W_1) < \delta$ and $d(\widehat{W_2}, W_2) < \delta$.

Lemma 6.7 *Let $\widehat{\Gamma} \subset GL(V_1)$ be a Zariski dense subgroup of $SO(2,1)$. Then there exist four transversal hyperbolic elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that we have $B(v,v) < 0$ for every*

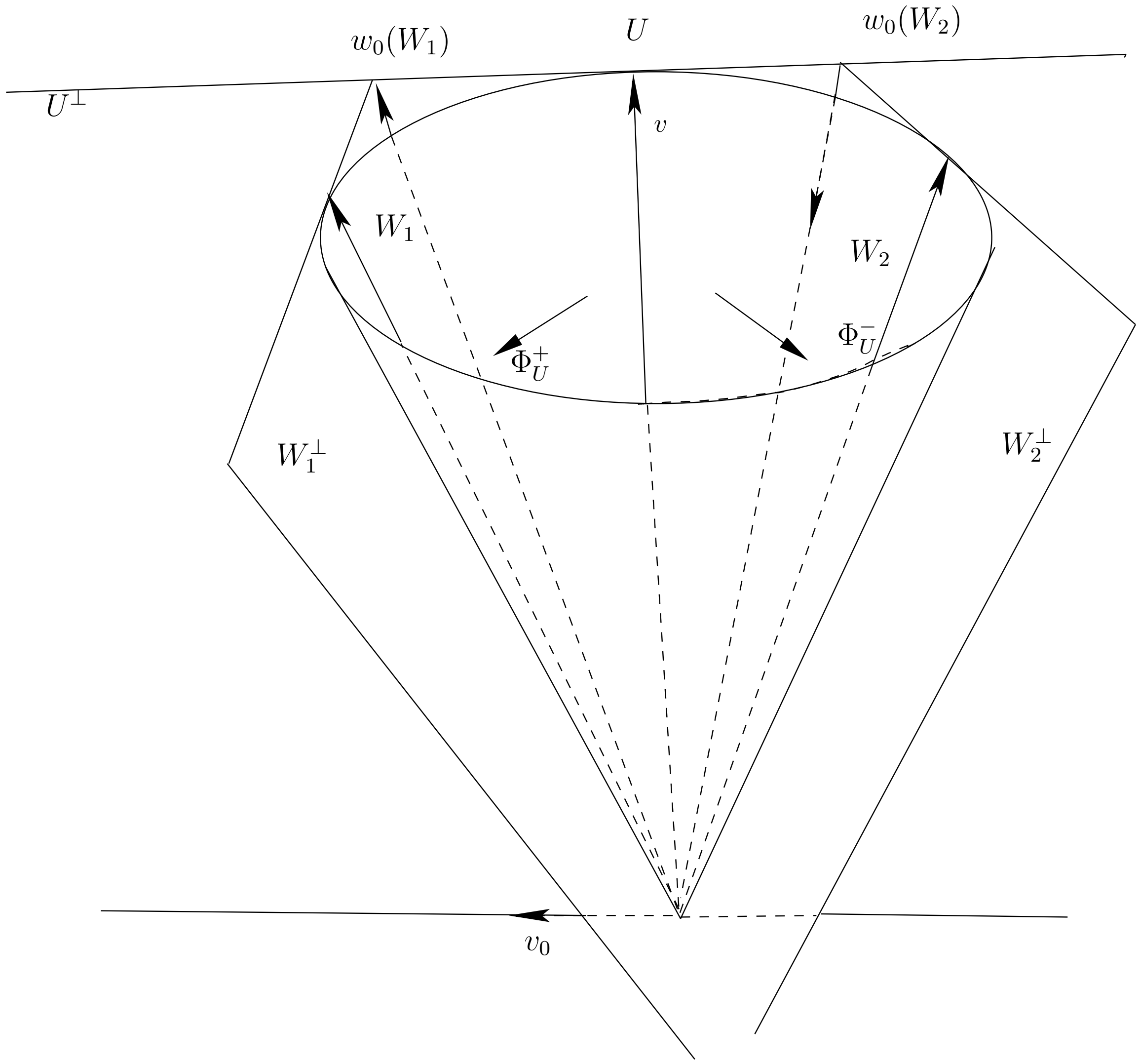

Figure 3: Positive and negative parts

non- zero vector

$$v \in (A^+(\gamma_1) + A^+(\gamma_2)) \bigcap (A^+(\gamma_3) + A^+(\gamma_4))$$

Proof Since $\widehat{\Gamma}$ is Zariski dense in $SO(2,1)$ there are four transversal hyperbolic elements

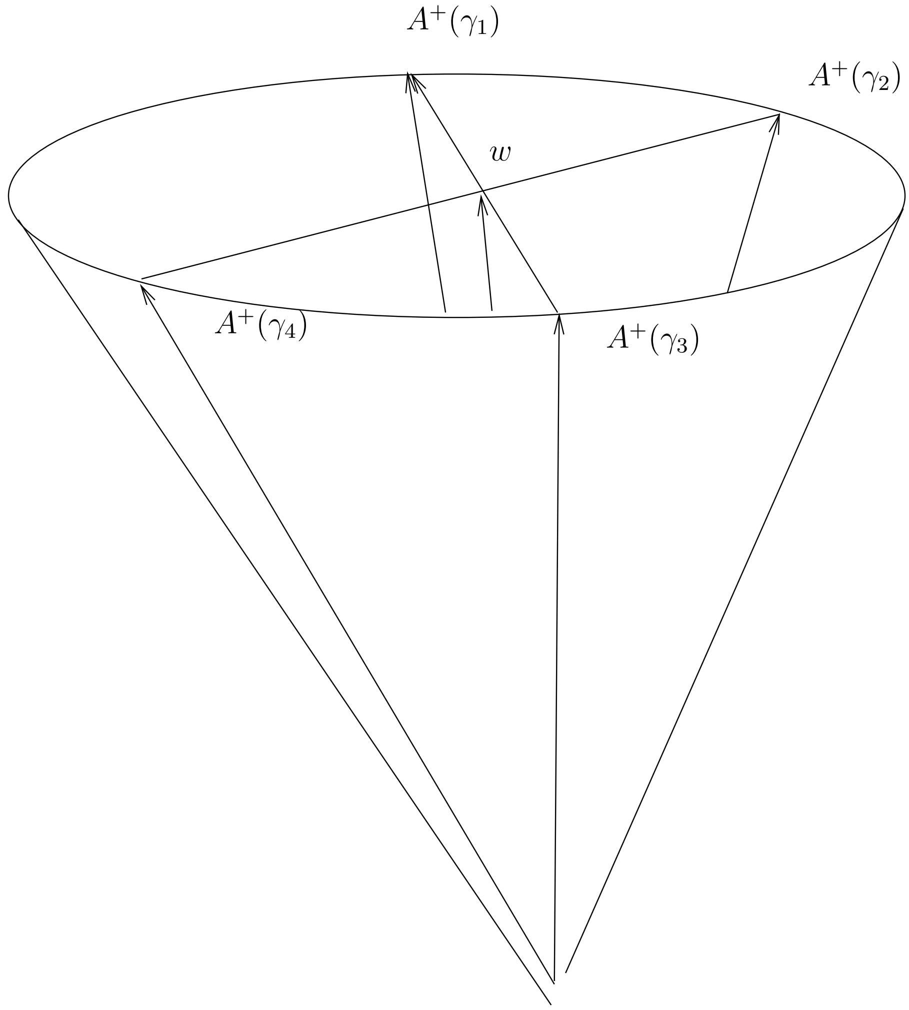

Figure 4: Configuration

[AMS1]. It is enough now to give an order of these four elements such that a vector $v \in (A^+(\gamma_1) + A^+(\gamma_2)) \bigcap (A^+(\gamma_3) + A^+(\gamma_4))$ will be inside the cone $\mathbb{C}_B$. (see Fig.4). Thus $B(v, v) < 0$ for any $v \in (A^+(\gamma_1) + A^+(\gamma_2)) \bigcap (A^+(\gamma_3) + A^+(\gamma_4))$ which proves the lemma.

Since any two vectors of V_1 of the same hyperbolic length are conjugate, we can and will assume that we choose and fix four hyperbolic elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ of Γ such that $w_1 \in (A^+(\theta_1(\gamma_1)) + A^+(\theta_1(\gamma_2))) \bigcap (A^+(\theta_1(\gamma_3)) + A^+(\theta_1(\gamma_4)))$ where $\theta_1 : \Gamma \longrightarrow SO(2,1) \subset$

$GL(V_1)$. Let $\theta_2 : \Gamma \longrightarrow SL_3(\mathbb{R}) \subset GL(V_2)$. Set $A_i = A^+(\theta_1(\gamma_i)),\ for i = 1, 2, 3, 4$.

Lemma 6.8 *For any point $A_i, i = 1, 2, 3, 4$ and positive δ there exist sets $S_i = \{g_{i1}, g_{i2}, g_{i3}\} \subset \Gamma$ and $T_i = \{h_{i1}, h_{i2}, h_{i3}\} \subset \Gamma$ and positive real numbers $\varepsilon, q < 1$, such that*

1. $\widehat{d}(A^-(\theta_1(g_{ik})), A_i) < \delta$, $\widehat{d}(A^-(\theta_1(h_{ik})), A_i) < \delta$;
2. g_{ik} *and* h_{ik} *are ε-hyperbolic,* $k = 1, 2, 3$;
3. $\max_{1\leq i\leq 4, 1\leq k\leq 3}\{s(g_{ik}), s(h_{ik})\} < q$;
4. *Let i be an index $1 \leq i \leq 4$. Then for every $k = 1, 2, 3$ we have* $\dim A^-(\theta_2(g_{ik})) = 2 \dim A^-(\theta_2(h_{ik})) = 1$;
5. *For every index i, $1 \leq i \leq 4$ we have* $\bigcap_{1\leq k\leq 3} A^-(\theta_2(g_{ik})) = \{0\}$;
6. *For every index i, $1 \leq i \leq 4$ we have* $\dim(A^-(\theta_2(h_{i1}))+A^-(\theta_2(h_{i2}))+A^-(\theta_2(h_{i3}))) = 3$.

Proof Obviously it is enough to prove the statement for one point. Let us do it for A_1. It is easy to show that there exists a hyperbolic element γ of Γ such that $\theta_1(\gamma)$ and $\theta_1(\gamma^{-1})$ are transversal to $\theta_1(\gamma_1)$ and there is no proper $\theta_2(\gamma_1)$–invariant subspace which is a subspace of a proper $\theta_2(\gamma)$–invariant subspace and there is no proper $\theta_2(\gamma)$– invariant subspace which is a subspace of a proper $\theta_2(\gamma_1)$–invariant subspace . We will also assume that $\theta_2(\gamma)$ has three eigenvalues of different norms [AMS1]. In that case all of them are real numbers. Put $\gamma_n = \gamma_1^n\gamma\gamma_1^{-n}$. We can assume that $\dim A^-(\gamma) = 2$ otherwise we can take γ^{-1} instead of γ. Let us first show that for some positive numbers n_1, n_2, n_3 we have $\cap_{1\leq i\leq 3}A^-(\theta_2(\gamma_{n_i})) = \{0\}$. Since for $n \neq m$ we have $A^-(\theta_2(\gamma_n)) \neq A^-(\theta_2(\gamma_m))$ there are positive numbers n_1 and n_2 such that $\dim A^-(\theta_2(\gamma_{n_1})) \cap A^-(\theta_2(\gamma_{n_2}) = 1$. Let v be a non -zero vector of this intersection. If $\theta_2(\gamma_1)^{-n}v \in A^-(\theta_2(\gamma))$ for infinitely many positive n

then the proper $\theta_2(\gamma)$–invariant subspace $A^-(\theta_2(\gamma))$ contains a $\theta_2(\gamma_1)$– invariant subspace. This contradicts our assumptions. Thus, by the choice of γ and γ_1 there exists an n_3 such that $\theta_2(\gamma_1)^{-n_3}v \notin A^-(\theta_2(\gamma))$ Therefore $v \notin \theta_2(\gamma_1)^{n_3}A^-(\theta_2(\gamma)) = A^-(\theta_2(\gamma_{n_3}))$. Clearly $A^-(\theta_2(\gamma_{n_1+m}))\cap A^-(\theta_2(\gamma_{n_2+m}))\cap A^-(\theta_2(\gamma_{n_3+m})) = \{0\}$ for all positive numbers m. Since the projective space PV is compact we can and will assume that $A^+(\gamma_{n_i+m}) \longrightarrow X_i^+$, $A^-(\gamma_{n_i+m}) \longrightarrow X_i^-$ for $m \longrightarrow \infty$ and $i = 1, 2, 3$. By standard arguments [MS], [AMS 1], we conclude that there exists a hyperbolic element γ_0 such that $\widehat{d}(A^+(\gamma_0), X_i^-) > 0$ and $\widehat{d}(A^-(\gamma_0), X_i^+) > 0$ for all $i = 1, 2, 3$. Let $\widehat{\varepsilon} = \min_{1\le i\le 3}\{\widehat{d}(A^+(\gamma_0), X_i^-), \widehat{d}(A^-(\gamma_0), X_i^+) > 0, \delta\}$. Thus there exists an $M \in \mathbb{N}$ such that for $m \ge M$ the elements γ_0 and γ_{n_i+m} are $\widehat{\varepsilon}/2$–transversal. Let $q_1 = \max\{s(\gamma_0\}),\ s(\gamma_{n_1+m}), s(\gamma_{n_2+m}), s(\gamma_{n_3+m})\}$. From Lemma 2.7 follows that for every positive δ and big m we have $\widehat{\rho}(A_1, A^-(\theta_1(\gamma_m))) \le \delta/4$. Fix such m and denote $g_i = \gamma_{n_i+m}$ for $i = 1, 2, 3$. For every $i, i = 1, 2, 3$ the element g_i is regular. Thus g_i^m is a hyperbolic element for big $m \in \mathbb{N}$. Hence $s(g_i^m) < 1$ for all $i, i = 1, 2, 3$. Recall that $A^+(g^m) = A^+(g), A^-(g^m) = A^-(g), A^0(g^m) = A^0(g)$ for all positive numbers m. We will not introduce new notations and assume that $s(g_i) < 1$ for all $i, i = 1, 2, 3$. Clearly $q_1 < 1$. From [MS], [AMS 1] follows that for a big positive number n we have $\widehat{\rho}(A^+(\gamma_0^n\gamma_i^n, A^+(\gamma_0)) \le q_1^{n/2}$, $\widehat{\rho}(A^-(\gamma_0^n\gamma_i^n), A^-(\gamma_i)) \le q_1^{n/2}$ and $s(\gamma_0^n\gamma_i^n) \le q_1^{n/2}$ for $i = 1, 2, 3$. Therefore there exist positive numbers N_1 and ε such that for all $i = 1, 2, 3$ the element $\gamma_0^n\gamma_i^n$ is $\varepsilon/2$-hyperbolic and $\widehat{\rho}(A_1, A^+(\theta_1(\gamma_0^n\gamma_i^n)) < \delta/2$ for $n > N_1$. Since $A^-(\gamma_0^n\gamma_i^n) \longrightarrow A^-(\gamma_i)$ for $n \longrightarrow \infty$, there exists $N \in \mathbb{N}$ such that $\dim A^-(\theta_2(\gamma_0^n\gamma_i^n)) =$ $\dim A^-(\theta_2(\gamma_i^n)) = 2, i = 1, 2, 3$ and $\cap_{1\le i\le 3}A^-(\theta_2(\gamma_0^n\gamma_i^n)) = \{0\}$ for $n > N_2$. Take $n > \max\{N_1, N_2\}$ and set $g_{1k} = \gamma_0^n\gamma_k^n, k = 1, 2, 3.$. Following the same way one can show that there is a set $T_1 = \{h_{11}, h_{12}, h_{13}\}$ with properties 1-4, 6. This proves Lemma 6.8.

For chosen sets $S_i = \{g_{i1}, g_{i2}, g_{i3}\} \subset \Gamma$ and $T_i = \{h_{i1}, h_{i2}, h_{i3}\} \subset \Gamma, i = 1, 2, 3, 4$ we will define the following constants. For any one dimensional subspace U of V_1, we have

$\sum_{1\le i\le 4, 1\le k\le 3}\widehat{d}(U, A^-(\theta_1(g_{ik}))) > 0$. Since the projective space is compact we have

$$\inf_{U\in V_1}\sum_{1\le i\le 4,1\le k\le 3}\widehat{d}(U, A^-(\theta_1(g_{ik}))) > 0.$$

Set

$$d_1^{(S)} = \inf_{U\in V_1}\sum_{1\le i\le 4,1\le k\le 3}\widehat{d}(U, (A^-(\theta_1(g_{ik})))/100.$$

By the same arguments there exists a positive constant $d_1^{(T)}$, such that

$$d_1^{(T)} = \inf_{U\subset V_1}\sum_{1\le i\le 4,1\le k\le 3}\widehat{d}(U, A^-\theta_1((h_{ik})))/100.$$

Let U be a one dimensional subspace of V_2. From 5, Lemma 6.8 follows that $\sum_{1\le i\le 4,1\le k\le 3}\widehat{d}(U, A^-(\theta_2(g_{ik}))) > 0$. Thus

$$\inf_{U\subset V_2}\sum_{1\le i\le 4,1\le k\le 3}\widehat{d}(U, A^-(\theta_2(g_{ik}))) > 0.$$

Set

$$d_2^{(S)} = \inf_{U\subset V_2}\sum_{1\le i\le 4,1\le k\le 3}\widehat{d}(U, A^-(\theta_2(g_{ik})))/100.$$

Let U be a two dimensional subspace of V_2. From 6, Lemma 6.8 follows that $\sum_{1\le i\le 4,1\le k\le 3}\widehat{d}(U, A^-(\theta_2(h_{ik}))) > 0$. Now by the same arguments as above there exists a positive

$$d_2^{(T)} = \inf_{U\subset V_2}\sum_{1\le i\le 4,1\le k\le 3}\widehat{d}(U, A^-(\theta_2(h_{ik})))/100,$$

Main Lemma 6.9 *There are two hyperbolic elements of the group* Γ *such that* $\alpha(g)\alpha(h) < 0$.

Proof. We have to prove that there are two elements with opposite sign. Since we can and will assume that there exists a hyperbolic element of positive sign, we will prove that there exists an element with negative sign.

Let $S_i = \{g_{i1}, g_{i2}, g_{i3}\} \subset \Gamma$, $T_i = \{h_{i1}, h_{i2}, h_{i3}\} \subset \Gamma$ and positive real numbers $\varepsilon, q < 1$ be as in Lemma 6.8. Assume that we choose a positive δ in Lemma 6.8 , (1) such that $\delta \leq \delta(d)/4$ where $d = \min_{1 \leq i \neq j \leq 4}\{\widehat{d}(A_i, A_j)\}$ (see Lemma 6.8, (2)). Set

$$\varepsilon_1 = \max\{d_1^{(S)}, d_1^{(T)}, d_2^{(S)}, d_2^{(T)}\}.$$

Let K be a compact subset of V such that $\Gamma K = V$. Denote by L the ray $L = \{tw_1, t \in \mathbb{R}, t > 0\}$. We may assume that $K \bigcap L \neq \emptyset$. Then there exist a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of elements of Γ and a sequence of points $p_n \in L$ such that

(1) $\gamma_n^{-1} p_n \in K$

(2) $d(p_n, \gamma_n^{-1} p_n) \to \infty$ when $n \to \infty$.

Set $k_n = \gamma_n^{-1} p_n \in K$. It is easy to see that for $n \to \infty$ we have

(3) $\gamma_n k_n - k_n / d(\gamma_n k_n, k_n) \to w_1$.

By [AMS1] there exist an $\varepsilon_2 = \varepsilon(\Gamma)$ and a finite set $S(\Gamma) =$
$\{g_1, \ldots, g_m\} \subset \Gamma$ such that for every $\gamma \in \Gamma$ there exists $g_i, i = i(\gamma), 1 \leq i \leq m$ and $M = M(\varepsilon_2)$ such that the element γg_i^m is ε_2–hyperbolic and $s(\gamma g_i^m) < s(g_i)^{m/2}$ for $m > N$. We can choose an infinite subsequence γ_{n_k} such that the element $g_i \in S(\Gamma)$ is the same for all γ_{n_k}. Assume that this is g_1. Put $r_m = g_1^{-t} k_m$ - a point of the compact set $K_1 = g_1^{-t} K$. Then for a fixed t we have

(4) $g_1^{-t} \gamma_n^{-1} p_n \in K_1$

(5) $\gamma_n g_1^t r_n - r_n / d(\gamma_n g_1^t r_n, r_n) \to w_1$ for $n \to \infty$.

Thus we assume that there exists a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of ε_2–hyperbolic elements of Γ, a compact set K, $K \cap L \neq \emptyset$ and a sequence of points $k_n \in K$ which fulfil properties (1),(2),(3). The projective space PV is compact. Thus we can and will assume that the sequences $\{A^+(\gamma_n)\}_{n\in\mathbb{N}}$ and $\{A^-(\gamma_n)\}_{n\in\mathbb{N}}$ converge. Let $A^+(\gamma_n) \longrightarrow A^+$ when $n \longrightarrow \infty$ and $A^-(\gamma_n) \longrightarrow A^-$ when $n \longrightarrow \infty$.

There are two cases. For infinitely many $n \in \mathbb{N}$ we have

$$\dim A^-(\theta_2(\gamma_n) = 2 \qquad (1),$$

or for infinitely many $n \in \mathbb{N}$ we have

$$\dim A^-(\theta_2(\gamma_n) = 1 \qquad (2).$$

In case (1) we will consider the sets $S_i, i = 1,2,3,4$, in case (2) we will consider the sets $T_i, i = 1,2,3,4$ and will use the following procedure.

Assume that for infinitely many $n \in \mathbb{N}$ we have $\dim A^-(\theta_2(\gamma_n) = 2$. From [AMS 1] follows that there exists a hyperbolic element γ_0 such that γ_0 and g_{ik} are transversal for all $g_{ik} \in S_i, i = 1,2,3,4,\ A^+(\gamma_0) \cap A^- = \{0\}$ and $A^-(\gamma_0) \cap A^+ = \{0\}$. Thus there exists ε_3 such that for all n hyperbolic elements γ_n and γ_0, g_{ik} and γ_0, are ε_3–transversal where $g_{ik} \in S_i, i = 1,2,3,4$. From Lemma 2.7 follows that there exists a positive number $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ the element $\gamma_n\gamma_0^m$ is $\varepsilon_3/4$-hyperbolic and $\widehat{\rho}(A^-(\gamma_n\gamma_0^m)), A^-(\gamma_0)) \leq q_2^m < \varepsilon/8$ and $s(\gamma_n\gamma_0^m) \leq s(\gamma_0^m)^{m/2}$ for $m \geq M$. Thus $\widehat{d}(A^-(\gamma_n\gamma_0^m)), A^+(g_{ik})) > \varepsilon_3/2$ for all $g_{ik} \in S_i, i = 1,2,3,4$, $n \in \mathbb{N}$, $m \geq M$. There exists an M_1 such that for $m \geq M_1$ we have $s(\gamma_0^m)^{m/2}\max\{\varepsilon_1,\varepsilon_3\} \leq \min\{\varepsilon_1,\varepsilon_3\}/8$. Fix $m > M_1, m \in \mathbb{N}$ and set $\widehat{\gamma}_n = \gamma_n\gamma_0^m$. Obviously $\min_{n\in\mathbb{N},1\leq i\leq 4,1\leq k\leq 3}\widehat{d}(A^+(\theta_1(\widehat{\gamma}_n)), A^-(\theta_1(g_{ik})) > \varepsilon_1$ and $\min_{n\in\mathbb{N},1\leq i\leq 4,1\leq k\leq 3}\widehat{d}(A^+(\theta_2(\widehat{\gamma}_n)), A^-(\theta_2(g_{ik})) > \varepsilon_1$. From 6.6.(2) follows that there exists an index $i_0, 1 \leq i_0 \leq 4$, such that$A^-(\theta_1(g_{i_0k}) \in \Phi^-_{A^+(\theta_1(\widehat{\gamma}_n))}$. Without loss of generality, we will assume that $i_0 = 1$. Clearly $\min_{n\in\mathbb{N},1\leq k\leq 3}\widehat{d}(A^+(\theta_1(\widehat{\gamma}_n)), A^-(\theta_2(g_{1k})) > \varepsilon_1/10$. Then for some k we have $\widehat{d}(A^+(\theta_1(\widehat{\gamma}_n)), A^-(\theta_2(g_{1k})) > \varepsilon_1/30$. Assume that this is hold for $k = 1$. Thus $\widehat{d}(A^+(\theta_1(\widehat{\gamma}_n)), A^-(\theta_2(g_{11})) > \varepsilon_1/30$. On the other hand we know that $\widehat{d}(A^-(\widehat{\gamma}_n), A^+(g_{11})) > \varepsilon_3/2$. From Lemma 2.7 and [MS], [AMS 1] follows that there exists a positive number N_0 such that if $\overline{\gamma}_n = \widehat{\gamma}_n g_{11}^{2N_0}$, $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}/10$ we have

(6) $\overline{\gamma}_n$ are ε- hyperbolic elements for all positive numbers n.

(7) $A^+(\theta_1(\overline{\gamma}_n)) \in \Phi_{A^-(\theta_1(\overline{\gamma}_n))}$.

(8) There exist a compact set K_0 and a sequence of points $\{k_n\}_{n\in\mathbb{N}} \subset K_0$ such that $\overline{\gamma}_n(k_n) \in L$ and $d(\overline{\gamma}_n(k_n), k_n) \longrightarrow \infty$ when $n \longrightarrow \infty$.

Therefore, $\overline{\gamma}_n(k_n) - k_n / d(\overline{\gamma}_n(k_n), k_n) \longrightarrow w_1$ when $n \longrightarrow \infty$. From (7) immediately follows that $\alpha(\overline{\gamma}_n) \longrightarrow B(v_{\theta_1(\overline{\gamma}_n)}, w_1) = -1$. Therefore there exists $\overline{\gamma}_n$ such that $\alpha(\overline{\gamma}_n) < 0$. Let $g \in \Gamma$ be an element with $\alpha g > 0$. If $\dim A^-(\theta_2(g)) = \dim A^+(\theta_2(\overline{\gamma}_n))$ set $h = \overline{\gamma}_n^{-1}$. Then $\alpha(h) < 0$ and $\dim A^-(\theta_2(g)) + \dim A^+(\theta_2(h)) = 3$. Otherwise set $h = \overline{\gamma}_n$. It is easy to see that there exists $t \in \Gamma$ such that g and tht^{-1} are transversal. Since $\alpha(tht^{-1}) = \alpha(h)$ we have proved that there are two transversal elements in Γ with opposite sign.

Proposition 6.10 *Assume that S as in the Case 2 (3) . Then Γ is not a crystallographic group.*

Proof follows immediately from Lemma 6.5 and the Main Lemma 6.9.

Remark 3 .It is possible to show that there is an affine group $\Gamma \subseteq \mathbb{R}^6$ acting properly discontinuously such that the linear part of Γ is Zariski dense in $SO(2,1) \times SL_3(\mathbb{R})$.

Proof of the Main Theorem Let G be the Zariski closure of the group Γ. Assume that the semisimple part S of G is not trivial. From [S 2], [To 2] follows that the real rank of at least one simple group is ≥ 2. Let $\dim V \leq 5$. From Proposition 4.1 and 4.2 follows that this is impossible, therefore a crystallographic group Γ is virtually solvable if $\dim V \leq 5$. Let $\dim V = 6$. From Propositions 5.1, 5.2 , 5.3 and 6.10 follows that $S = \{e\}$. Therefore the group Γ is virtually solvable.

Remark 4 . Actually we can prove the following proposition. Let Γ be an affine group which acts properly discontinuously. Then Γ is virtually solvable if and only if the linear part of the Zariski closure of Γ does not have $SO(2,1)$ as a quotient group.

7 The Auslander conjecture in dimension 7.

We would like to state the following important problems

Problem 1 *Does there exist a crystallographic group* $\Gamma \subseteq Aff(\mathbb{R}^7)$ *such that* $l(\Gamma)$ *is Zariski dense in* $SO(4,3)$ *?*

We believe that this question is crucial for the further progress on the Auslander conjecture.

Let G be the simplest representation of a simple Lie group of type G_2. It is well known that G is a proper subgroup of $O(4,3)$.

Problem 2 *Does there exist a crystallographic group* $\Gamma \subseteq Aff(\mathbb{R}^7)$ *such that* $l(\Gamma)$ *is Zariski dense in* G *?*

We think that these problems are very difficult . The cohomological argument used in the proof of Proposition 5.3 does not work here since the virtual cohomological dimension of Γ is 7 and dimensions of corresponding symmetric spaces are ≥ 8 . Note that by 6.4 $\alpha(\gamma) = \alpha(\gamma^{-1})$. Thus there is no simple way to change the sign of a hyperbolic element of $SO(4,3)$.

We can show that the negative answer to Problem 2 will lead to a proof of the following conjecture

Conjecture. *Let* G *be a connected Lie group. Assume that the real rang of any simple non-commutative connected subgroup of* G *is* ≤ 2. *If a crystallographic group* Γ *is a subgroup of* G *than* Γ *is virtually solvable.*

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