Construction of a State Evolution for Kawasaki Dynamics in Continuum

Christoph Berns · Yuri Kondratiev · Oleksandr Kutoviy

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Abstract We consider conservative, non-equilibrium stochastic jump dynamics of interacting particles in continuum. These dynamics have a (grand canonical) Gibbs measure as invariant measure. The problem of existence of these dynamics is studied. The corresponding time evolution of correlation functions is constructed.

Keywords Interacting particle system · Continuous system · Kawasaki dynamics · Correlation functions · Non-equilibrium evolution

1 Introduction

The field of interacting particle systems began as a branch of probability theory in the late 1960s. The original motivation came from statistical mechanics but later it became evident that the theory of interacting particle systems can be

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Christoph Berns
IGK *"Stochastics and Real World Models"*
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
E-mail: cberns@math.uni-bielefeld.de

Yuri Kondratiev · Oleksandr Kutoviy
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
E-mail: kondrat@math.uni-bielefeld.de

Oleksandr Kutoviy
E-mail: kutoviy@math.uni-bielefeld.de

very useful in several other fields, e.g. in biology and economics (see [4] and the references therein). Interacting particle systems can be divided into lattice and continuous systems. Lattice systems are already a classical object. For a detailed discussion of lattice systems we refer e.g. to [14], [15]. Continuous systems are those in which particles can appear at any point $x \in \mathbb{R}^d$. A configuration of such a system is a subset $\gamma \subset \mathbb{R}^d$ which is locally finite and the elements of γ describe the location of the particles. The family of all such sets γ forms the configuration space $\Gamma(\mathbb{R}^d) \equiv \Gamma$.

In this article, we study the dynamics of an infinite system of point particles in the Euclidean space $\mathbb{R}^d$ which randomly hop over the space $\mathbb{R}^d$ and interact with each other. Since every particle $x \in \gamma$ can jump to some point $y \in \mathbb{R}^d$, the heuristic generator of such a process has the following form

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\; c(x,y,\gamma)(F(\gamma\backslash\{x\} \cup \{y\}) - F(\gamma)), \quad \gamma \in \Gamma. \tag{1}$$

The coefficient $c(x, y, \gamma)$ describes the rate at which the particle x of the configuration γ jumps to y. To simplify notation we continue to write x for the set $\{x\}$. In this paper, we will restrict our discussion to the dynamics with

$$c(x, y, \gamma) = a(x - y)e^{-E^{\phi}(y,\gamma)}. \tag{2}$$

Here $a \in L^1(\mathbb{R}^d)$ is a positive function and $\phi : \mathbb{R}^d \to \mathbb{R}_+$ is a (translation invariant) positive pair potential which satisfies the integrability condition:

$$C_\phi := \int_{\mathbb{R}^d} |e^{-\phi(x)} - 1| dx < \infty. \tag{3}$$

For particles located at $x \in \mathbb{R}^d$ resp. $y \in \mathbb{R}^d$, $\phi(x-y)$ describes the interaction energy between x and y. Hence, ϕ should be an even function. Due to the additive character of energy, the interaction energy between a particle located at $x \in \mathbb{R}^d$ and a configuration $\gamma \in \Gamma$ is defined by

$$E^{\phi}(x, \gamma) := \sum_{y\in\gamma} \phi(x - y). \tag{4}$$

This ansatz implies that a transition from $x \in \gamma$ to $y \in \mathbb{R}^d$ is more likely if the relative energy $E^{\phi}(y, \gamma)$ at the arrival point y is low.

The main reason for us to consider the rates of the form (2) is that any (grand canonical) Gibbs measure with potential ϕ (provided it exists) is symmetrizing (and hence invariant) for the dynamics generated by (1) with such rates, see [7]. This means that the formal generator (1) with rates (2) is symmetric in $L^2(\mu)$, where μ is a Gibbs measure with respect to (w.r.t. for short) the pair potential ϕ. We call the model of jumping particles defined by (1) and (2) the Kawasaki dynamics in continuum. In [7], the problem of existence of such dynamics was left open. The existence problem was solved in [11] (by using Dirichlet-form techniques) but only for so-called equilibrium dynamics.

In particular one obtains an existence result for almost all starting configurations w.r.t. the given stationary measure. The latter means that we can start the dynamics with any initial state which is absolutely continuous w.r.t. the symmetrizing measure (see [11]). In applications, however, one often needs to analyze the time development for different classes of initial states of the system. In these cases absolute continuity of initial states w.r.t. the stationary measure is very restrictive assumption.

In the present work we propose a construction of the non-equilibrium Kawasaki dynamics in continuum. It is worth noting that the methods used in [11] are not applicable for the construction of the corresponding evolution of states discussed in this article. Our strategy is roughly the following: we derive an evolution equation which describes the time evolution for correlation functions (Section 3). This evolution equation is an analog to the BBGKY-hierarchy for Hamiltonian dynamics, see e.g. [2], [3]. As in the case of (infinite) Hamiltonian dynamics, the computation of the n-th correlation function requires the knowledge of correlation functions of order above n. A certain dual evolution equation describes the time evolution of the objects dual to correlation functions, the so-called quasi-observables. Section 4 is devoted to the study of evolution of quasi-observables (Theorem 2). The corresponding evolution equation has the feature that the computation of the n-th component of a quasi-observable requires the knowledge of the components of order less than n. This makes a recursive computation of the evolution of the components of quasi-observables possible. The duality between quasi-observables and correlation functions allows us to transfer this evolution to correlation functions. Our main results are stated and proved in Section 5 (Theorem 3).

2 General Facts and Notions

2.1 The configuration spaces

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in the d-dimensional Euclidean space $\mathbb{R}^d$ and $\mathcal{B}_b(\mathbb{R}^d)$ the subfamily of all bounded Borel sets. The n-particle space is defined by

$$\Gamma_0^{(n)} = \Gamma_{0,\mathbb{R}^d}^{(n)} := \{\eta \subset \mathbb{R}^d \,|\, |\eta| = n\}, \quad n \in \mathbb{N}_0 = \{0,1,2,\dots\},$$

where $|\cdot|$ means the cardinality of a finite set. For $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ one defines the set $\Gamma_{0,\Lambda}^{(n)} \equiv \Gamma_\Lambda^{(n)}$ analogously. For short we write $\eta_\Lambda := \eta \cap \Lambda$. We can identify the set $\Gamma_0^{(n)}$ with the symmetrization of

$$\widetilde{\left(\mathbb{R}^d\right)^n} := \{(x_1,\dots,x_n) \in \left(\mathbb{R}^d\right)^n \big| x_i \neq x_j \text{ if } i \neq j\},$$

i.e. $\Gamma_0^{(n)} \cong \widetilde{\left(\mathbb{R}^d\right)^n}/S_n$, where S_n denotes the permutation group over $\{1,\dots,n\}$. Due to this identification we can introduce a topology $\mathcal{T}(\Gamma_0^{(n)})$ and the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0^{(n)})$ on $\Gamma_0^{(n)}$. The space of finite particle configurations is defined by

$$\Gamma_0 := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_0^{(n)}.$$

This set is equipped with the topology $\mathcal{T}(\Gamma_0)$ of disjoint unions. The space $\Gamma_{0,\Lambda} = \Gamma_\Lambda$, $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is defined analogously.

The configuration space is defined by

$$\Gamma := \{\gamma \subset \mathbb{R}^d \big| \, |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\}.$$

The space Γ is equipped with the vague topology, i.e. the smallest topology for which all mappings

$$\Gamma \ni \gamma \mapsto \langle \gamma, f\rangle := \sum_{x\in\gamma} f(x) \in \mathbb{R}$$

are continuous for any function f on $\mathbb{R}^d$ with compact support; note that the summation in $\sum_{x\in\eta} f(x)$ is taken over finitely many points of γ that belong to the support of f. In [10], it was shown that Γ with the vague topology is metrizable and becomes a Polish space (i.e. a complete separable metric space). Corresponding to this topology, the Borel σ-algebra $\mathcal{B}(\Gamma)$ is the smallest σ-algebra for which all mappings $N_\Lambda : \Gamma \to \mathbb{N}_0$, $N_\Lambda(\gamma) = |\gamma \cap \Lambda|$ are measurable, i.e.

$$\mathcal{B}(\Gamma) = \sigma(N_\Lambda | \Lambda \in \mathcal{B}(\mathbb{R}^d).$$

For every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ one can define a projection

$$p_\Lambda : \Gamma \to \Gamma_\Lambda, \; p_\Lambda(\gamma) := \gamma \cap \Lambda$$

and with respect to this projection, Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda\in\mathcal{B}_b(\mathbb{R}^d)}$, see [1] and the references therein.

2.2 Measures and functions

On $\Gamma_0^{(n)}$ we introduce a measure $\lambda^{(n)}$ by

$$\lambda^{(n)} := \frac{1}{n!}\sigma^{(n)}, \qquad \lambda^{(0)} := \delta_\emptyset$$

where $\sigma^{(n)}$ is the restriction of the Lebesgue product measure $(dx)^n$ (on $(\mathbb{R}^d)^n$) to $(\Gamma_0^{(n)}, \mathcal{B}(\Gamma_0^{(n)}))$. The combinatorial $\frac{1}{n!}$ factor takes into account the indistinguishability of the n particles. We extend the measures $\lambda^{(n)}$ to a measure λ on Γ_0 by setting

$$\lambda|_{\Gamma_0^{(n)}} = \lambda^{(n)},$$

i.e. $\lambda = \delta_\emptyset + \sum_{n\in\mathbb{N}} \frac{1}{n!}\sigma^{(n)}$. The measure λ is called the Lebesgue-Poisson measure. For any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ the restriction of λ to Γ_Λ will be denoted by λ_Λ. It holds $\lambda_\Lambda(\Gamma_\Lambda) = e^{m(\Lambda)}$, where $m(\Lambda)$ denotes the Lebesgue measure of

$\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. We define a probability measure π_Λ on Γ_Λ by $\pi_\Lambda := e^{-m(\Lambda)}\lambda_\Lambda$. It can be shown [1] that the family $\{\pi_\Lambda\}_{\Lambda\in\mathcal{B}_b(\mathbb{R}^d)}$ is consistent, and hence there exists a unique probability measure, π, on $\mathcal{B}(\Gamma)$ such that

$$\pi_\Lambda = \pi \circ p_\Lambda^{-1}, \quad \Lambda \in \mathcal{B}_b(\mathbb{R}^d).$$

The measure π is called the Poisson measure.

By $L^0_{\mathrm{ls}}(\Gamma_0)$ we denote the set of all measurable functions on Γ_0 with local support, i.e. $G \in L^0_{\mathrm{ls}}(\Gamma_0)$ if and only if $G|_{\Gamma_0\setminus\Gamma_\Lambda} \equiv 0$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exist $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. We denote the set of all bounded and measurable functions with bounded support by $B_{\mathrm{bs}}(\Gamma_0)$, i.e. $G \in B_{\mathrm{bs}}(\Gamma_0)$ if G is bounded and $G|_{\Gamma_0\setminus M} \equiv 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. We also consider the set $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ of all cylinder functions on Γ. Each $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ is characterized by the following property: $F(\gamma) = F|_{\Gamma_\Lambda}(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. Further, by $\mathcal{F}_{\mathrm{cyl}}\mathcal{P}(\Gamma)$ we denote the subspace of all cylinder functions which are polynomially bounded, i.e. $F \in \mathcal{F}_{\mathrm{cyl}}\mathcal{P}(\Gamma)$, if and only if $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ and there exists a polynomial P on $\mathbb{R}$ such that $|F(\gamma_\Lambda)| \le P(|\gamma_\Lambda|)$.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we define a Lebesgue-Poisson exponent corresponding to the one particle function f by

$$e_\lambda(f,\eta) := \prod_{x\in\eta} f(x), \quad \eta \in \Gamma_0.$$

There is the following mapping from $B_{\mathrm{bs}}(\Gamma_0)$ to $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ which plays a key role in our further considerations:

$$KG(\eta) := \sum_{\eta\Subset\gamma} G(\eta), \quad \gamma \in \Gamma. \tag{5}$$

This mapping can be interpreted as a combinatorial version of the Fourier transform and is called K-transform, see [9], [12], [13] for details. The summation in (5) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol $\eta \Subset \gamma$. The K-transform is linear, positivity preserving and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{6}$$

Here and in the sequel inclusions like $\xi \subset \eta$ hold for $\xi = \emptyset$ as well as for $\xi = \eta$. Expression (6) for the inverse K-transform is obtained by an application of the Möbius inversion formula, see e.g. [20]. Further, the K-transform maps $B_{\mathrm{bs}}(\Gamma_0)$ into $\mathcal{F}_{\mathrm{cyl}}\mathcal{P}(\Gamma)$.

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ that have finite local moments of any orders, i.e. $\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < \infty$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $n \in \mathbb{N}$. A measure ρ on Γ_0 is called locally finite iff $\rho(M) < \infty$ for all bounded sets $M \in \mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$. One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ which is dual to the K-transform, i.e. for every $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ holds

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)(K^*\mu)(d\eta). \tag{7}$$

The measure $\rho_\mu := K^*\mu$ is called the correlation measure of μ. If ρ_μ has a density w.r.t. the Lebesgue-Poisson measure λ, i.e. $d\rho_\mu = k_\mu d\lambda$, the functions

$$k_\mu^{(n)} : \left(\mathbb{R}^d\right)^n \to \mathbb{R}_+, \quad n \in \mathbb{N},$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}) & \text{if } (x_1, \ldots, x_n) \in \widetilde{\left(\mathbb{R}^d\right)^n} \\ 0 & \text{otherwise.} \end{cases}$$

are the well-known correlation functions of statistical physics, see e.g. [17], [18].

As shown in [9], for $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and $G \in L^1(\Gamma_0, \rho_\mu)$, the series

$$KG(\eta) := \sum_{\eta \Subset \gamma} G(\eta)$$

is μ-a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and (7) holds. Thus, we can extend the K-transform to a mapping

$$K_\mu : L^1(\Gamma_0, d\rho_\mu) \to L^1(\Gamma, d\mu). \tag{8}$$

The following lemma will play a crucial role in many computations:

Lemma 1 (Minlos lemma) *Let $G : \Gamma_0 \mapsto \mathbb{R}$, $H : \Gamma_0 \times \cdots \times \Gamma_0 \mapsto \mathbb{R}$ be positive and measurable, then for $n \in \mathbb{N}$, $n \geq 2$:*

$$\begin{aligned} \int_{\Gamma_0} \cdots \int_{\Gamma_0} G(\eta_1 \cup \ldots \cup \eta_n) H(\eta_1, \ldots, \eta_n) \lambda(d\eta_1) \cdots \lambda(d\eta_n) \qquad (9) \\ = \int_{\Gamma_0} G(\eta) \sum_{(\eta_1, \ldots, \eta_n) \in \mathcal{P}_n^\emptyset(\eta)} H(\eta_1, \ldots, \eta_n) \lambda(d\eta), \end{aligned}$$

where $\mathcal{P}_n^\emptyset(\eta)$ denotes the family of all ordered partitions of η into n parts, which may be empty.

A proof of the Minlos lemma can be found e.g. in [16].

3 Hierarchical Equations for Kawasaki Dynamics

In this section we derive the hierarchical equations for Kawasaki dynamics which are the analog of the BBGKY-hierarchy for Hamilton dynamics. Let $\phi : \mathbb{R}^d \to \mathbb{R}_+$ be a pair potential (i.e. ϕ is a symmetric, measurable function) and $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. In [6] it was shown, that under some mild assumptions (e.g. μ is a Gibbs measure corresponding to ϕ for which the correlation functions fulfill the Ruelle bound) the generator

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\; a(x-y) e^{-E^{\phi}(y,\gamma)} \big(F(\gamma\backslash x \cup y) - F(\gamma)\big) \tag{10}$$

is μ-a.e. well defined for all cylinder functions $F \in K(B_{\mathrm{bs}}(\Gamma_0))$, moreover LF is an element in $L^1(\Gamma, d\mu)$ but we will not need this statement in the sequel.

As already mentioned, the K-transform can be regarded as a combinatorial Fourier transform. It is well known that a differential operator on $\mathbb{R}^d$ in Fourier representation is simply given by multiplication by a polynomial. More generally, a pseudo-differential operator in Fourier representation is given by multiplication by a symbol, see e.g.[8], [19]. Within our framework, we can proceed in a similar fashion. In the sequel, we define an operator $\widehat{L} := K^{-1}LK$ which we will also call the symbol corresponding to L. The advantage will be that the symbol acts on quasi-observables, i.e. on functions depending only on finitely many coordinates. The following informal consideration links the symbol with the infinitesimal generator for the evolution of correlation functions: the evolution of the initial state μ_0 of the system is informally given by

$$\frac{d}{dt} \int_{\Gamma} F(\gamma)\, d\mu_t(\gamma) = \int_{\Gamma} LF(\gamma)\, d\mu_t(\gamma), \quad t > 0, \quad F \in K\big(B_{\mathrm{bs}}(\Gamma_0)\big), \tag{11}$$

$$\mu_t\big|_{t=0} = \mu_0.$$

Suppose that the correlation function k_t of μ_t exists for each moment of time $t \geq 0$ and that (7) can be applied to both sides of (11). Then, equation (11) can be rewritten in the following form

$$\frac{d}{dt} \langle\!\langle K^{-1}F, k_t \rangle\!\rangle = \langle\!\langle K^{-1}LF, k_t \rangle\!\rangle, \quad t > 0, \quad F \in K\big(B_{\mathrm{bs}}(\Gamma_0)\big),$$

$$k_t\big|_{t=0} = k_0, \tag{12}$$

where the duality between functions on Γ_0 is given by

$$\langle\!\langle G,\, k \rangle\!\rangle := \int_{\Gamma_0} G \cdot k \, d\lambda \tag{13}$$

Next, if we substitute $F = KG$, $G \in B_{\mathrm{bs}}(\Gamma_0)$ in (12), we derive

$$\frac{d}{dt} \langle\!\langle G, k_t \rangle\!\rangle = \langle\!\langle K^{-1}LKG, k_t \rangle\!\rangle, \quad t > 0, \quad k_t\big|_{t=0} = k_0 \tag{14}$$

for all $G \in B_{\mathrm{bs}}(\Gamma_0)$. The above heuristic computation shows, that if we define $\widehat{L} := K^{-1}LK$, the dual operator $L^{\triangle} := \widehat{L}^*$ should describe the time evolution of correlation functions. For details we refer to [6].

In [6] it was shown, that for the Kawasaki system the operators $\widehat{L}$ and $L^{\triangle}$ are given by

$$
\begin{aligned}
\big(\widehat{L}G\big)(\eta) &= \sum_{x\in\eta}\sum_{\xi\subset\eta\setminus x}\int_{\mathbb{R}^d} dy\ a(x-y)e^{-E^{\phi}(y,\xi\cup x)}e_\lambda(e^{-\phi(\cdot-y)}-1,(\eta\setminus x)\setminus\xi) \\
&\quad\times\big(G(\xi\cup x)-G(\xi\cup y)\big) \\
&= \sum_{\xi\subset\eta}\sum_{x\in\xi}\int_{\mathbb{R}^d} dy\ a(x-y)e^{-E^{\phi}(y,\xi)} \\
&\quad\times\prod_{z\in\eta\setminus\xi}(e^{-\phi(y-z)}-1)\big(G(\xi\setminus x\cup y)-G(\xi)\big), \quad \eta\in\Gamma_0.
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
(L^{\triangle}k)(\eta) &= \sum_{y\in\eta}\int_{\mathbb{R}^d} dx\ \int_{\Gamma_0}\lambda(d\xi)\ k(\eta\setminus y\cup x\cup\xi)e_\lambda(e^{-\phi(y-\cdot)}-1,\xi) \\
&\quad-\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\ \int_{\Gamma_0}\lambda(d\xi)\ k(\eta\cup\xi)e_\lambda(e^{-\phi(y-\cdot)}-1,\xi).
\end{aligned} \tag{16}
$$

The above considerations suggest that the evolution of correlation functions should be described by the evolutionary problem

$$
\begin{cases} \frac{\partial}{\partial t}k_t = L^{\triangle}k_t \\ k_t|_{t=0} = k_0. \end{cases} \tag{17}
$$

Consequently, the corresponding hierarchical structure is given by the countable infinite system of equations

$$
\frac{\partial}{\partial t}k_t^{(n)} = \big(L^{\triangle}k_t\big)^{(n)},
$$

$$
k_t^{(n)} := k_t|_{\Gamma_0^{(n)}},\ \big(L^{\triangle}k_t\big)^{(n)} := \big(L^{\triangle}k_t\big)|_{\Gamma_0^{(n)}},\ n\in\mathbb{N}.
$$

To solve (17), we will use the following strategy. We begin by solving a pre-dual w.r.t. the duality (13) initial value problem

$$
\frac{\partial}{\partial t}G_t = \hat{L}G_t, \quad t>0, \quad G_t\big|_{t=0} = G_0. \tag{18}
$$

The evolution dual to the solution of (18) w.r.t. (13) will be a weak solution to (17).

4 Evolution Equation for Quasi-Observables

4.1 The setting

We consider the evolution equation (18) in a proper space. To each function G on Γ_0 we associate a sequence $(G^{(n)})_{n\in\mathbb{N}}$ of symmetric functions $G^{(n)}$ on $(\mathbb{R}^d)^n$ by defining

$$G^{(n)} := G|_{\Gamma_0^{(n)}}.$$

We refer to the sequence $(G^{(n)})_{n\in\mathbb{N}}$ as components or coordinates of the function G. Next, we rewrite equation (18) in components:

$$\begin{cases} \frac{d}{dt}G_t^{(n)} = (\hat{L}G_t)^{(n)} \\ G_t^{(n)}|_{t=0} = G_0^{(n)}, \quad n\in\mathbb{N}. \end{cases} \tag{19}$$

Using (15), we obtain for $(\widehat{L}G)^{(n)}$:

$$(\widehat{L}G)^{(n)}(\eta) = (L_0^{(n)}G^{(n)})(\eta) + \big(W^{(n)}(G^0,\dots,G^{(n-1)})\big)(\eta), \quad \eta\in\Gamma_0^{(n)} \tag{20}$$

with

$$(\widehat{L}_0^{(n)}G^{(n)})(\eta) = \sum_{x\in\eta}\int_{\mathbb{R}^d} dy\; a(x-y)e^{-E^\phi(y,\eta)}(G^{(n)}(\eta\backslash x\cup y) - G^{(n)}(\eta)) \tag{21}$$

and

$$W^{(n)}(G^0,\dots,G^{(n-1)}) = \sum_{k<n} W^{(n,k)}G^{(k)},$$

where

$$\begin{aligned}\big(W^{(n,k)}G^{(k)}\big)(\eta) = &\sum_{\substack{\xi\subset\eta\\|\xi|=k}}\sum_{x\in\xi}\int_{\mathbb{R}^d} dy\; a(x-y)\prod_{z\in\xi}e^{-\phi(y-z)} \\ &\times \prod_{z\in\eta\backslash\xi}(e^{-\phi(y-z)}-1)\big(G^{(k)}(\xi\backslash x\cup y) - G^{(k)}(\xi)\big),\ \eta\in\Gamma_0^{(n)}.\end{aligned}$$

Hence, equation (18) consists of a diagonal part and lower diagonal parts. Due to the special structure of (19), the following strategy for the construction of a solution of (18) is reasonable:

Fix $n\in\mathbb{N}$ and assume that $L_0^{(n)}$ generates a semigroup (in some proper Banach space). If $G_t^{(0)},\dots G_t^{(n-1)}$ are already known, the solution of (19) is given by

$$G_t^{(n)} = e^{tL_0^{(n)}}G^{(0)} + \int_0^t e^{(t-s)L_0^{(n)}}W^{(n)}(G_s^0,\dots G_s^{(n-1)})ds, \quad t>0, \tag{22}$$

where the above integral is interpreted in Bochner's sense. Hence, given G_0, we can compute the components of the solution G_t of (19) successively. In the following we realize this approach.

To solve equation (18) we need some preparations and we have to introduce the spaces where the solution will be localized at each moment of time t. As already mentioned above, we have to ensure that the operators $L_0^{(n)}$, $n \in \mathbb{N}$ induce semigroups. In the lemmas below, we verify that this is the case.

Lemma 2 *Let $n \in \mathbb{N}$. The operator*

$$(L_0^{(n)} G^{(n)})(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \; a(x-y) e^{-E^{\phi}(y,\eta)} (G^{(n)}(\eta \backslash x \cup y) - G^{(n)}(\eta)),$$
$$G^{(n)} \in \mathcal{F}_b(\Gamma_0^{(n)})$$

is the generator of a Markov (jump) process (with the state space $\Gamma_0^{(n)}$). Here $\mathcal{F}_b(\Gamma_0^{(n)})$ stands for the set of all bounded and measurable (w.r.t. $\mathcal{B}(\Gamma_0^{(n)})$) functions $G^{(n)} : \Gamma_0^{(n)} \to \mathbb{R}$.

Proof: We can rewrite the operator $L_0^{(n)}$ in the following way:

$$(L_0^{(n)} G^{(n)})(\eta) = \int_{\Gamma_0^{(n)}} (G^{(n)}(\tilde{\eta}) - G^{(n)}(\eta)) q_n(\eta, d\tilde{\eta}),$$

where q_n is given by

$$q_n(\eta, A) = \sum_{x\in\eta} \int_{\mathbb{R}^d} dy \; c(x,y,\eta) \delta_{\eta\backslash x \cup y}(A), \quad \eta \in \Gamma_0^{(n)}, \quad A \in \mathcal{B}(\Gamma_0^{(n)}). \tag{23}$$

One easily shows that

$$q_n : \Gamma_0^{(n)} \times \mathcal{B}(\Gamma_0^{(n)}) \to \mathbb{R}_+$$

is a kernel on $(\Gamma_0^{(n)}, \mathcal{B}(\Gamma_0^{(n)}))$. Further, "the total rate of jumping away from η" given by

$$\lambda_n(\eta) := q_n(\eta, \Gamma_0^{(n)} \backslash \eta), \quad \eta \in \Gamma_0^{(n)},$$

defines an element of $\mathcal{F}_b(\Gamma_0^{(n)})$. Hence

$$L_0^{(n)} : \mathcal{F}_b(\Gamma_0^{(n)}) \to \mathcal{F}_b(\Gamma_0^{(n)})$$

is a bounded operator (w.r.t. the sup-norm) and an application of the pure Markov jump process theory (see e.g. [5]) shows that $L_0^{(n)}$ is indeed the generator of a Markov process. □

The above lemma ensures that the generator $L_0^{(n)}$ induces a contraction semigroup $(p_t)_{t\geq 0}$ on $\mathcal{F}_b(\Gamma_0^{(n)})$. But in the sequel we will need that $L_0^{(n)}$ generates also a contraction semigroup in a proper L^1-space. Therefore we search for an invariant measure. Motivated by the Gibbsian approach to statistical mechanics, this measure should have the form given in the lemma below.

Lemma 3 *Define a measure* $\lambda^{(n),\phi}$ *on* $\left(\Gamma_0^{(n)}, \mathcal{B}(\Gamma_0^{(n)})\right)$ *by*

$$\lambda^{(n),\phi}(d\eta) = e^{-E^{\phi}(\eta)} \sigma^{(n)}(d\eta)$$

where $E^{\phi}(\eta)$ *denotes the energy of the configuration* $\eta \in \Gamma_0^{(n)}$ *given by*

$$E^{\phi}(\eta) := \sum_{\{x,y\} \subset \eta} \phi(x-y).$$

Then $\left(L_0^{(n)}\right)^* \lambda^{(n),\phi} = 0$ *in the distribution sense, i.e.* $\int_{\Gamma_0^{(n)}} L_0^{(n)} G^{(n)} d\lambda^{(n),\phi} = 0$ *for all functions* $G^{(n)} \in \mathcal{D}(L_0^{(n)}) = \mathcal{F}_b(\Gamma_0^{(n)})$.

Proof: By means of the Minlos lemma (Lemma 1) we derive:

$$\begin{aligned}
&\int_{\Gamma_0^{(n)}} L_0^{(n)} G^{(n)} d\lambda^{(n),\phi} \\
&= \int_{\Gamma_0^{(n)}} \lambda^{(n)}(d\eta) e^{-E^{\phi}(\eta \setminus x \cup y)} \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \; a(x-y) e^{-E^{\phi}(x, \eta \setminus x \cup y)} G^{(n)}(\eta) \\
&\quad - \int_{\Gamma_0^{(n)}} \lambda^{(n)}(d\eta) e^{E^{\phi}(\eta)} \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \; a(x-y) e^{-E^{\phi}(y,\eta)} G^{(n)}(\eta) \\
&= \int_{\Gamma_0^{(n)}} \lambda^{(n)}(d\eta) e^{-E^{\phi}(\eta)} \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \; a(x-y) e^{-E^{\phi}(y,\eta)} G^{(n)}(\eta) \\
&\quad - \int_{\Gamma_0^{(n)}} \lambda^{(n)}(d\eta) e^{-E^{\phi}(\eta)} \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \; a(x-y) e^{-E^{\phi}(y,\eta)} G^{(n)}(\eta) = 0,
\end{aligned}$$

where we have used the identity

$$e^{-E^{\phi}(x,\eta \setminus x \cup y)} e^{-E^{\phi}(\eta \setminus x \cup y)} = e^{-E^{\phi}(y,\eta)} e^{-E^{\phi}(\eta)}. \quad \square$$

The above lemma implies that the measure $\lambda^{(n),\phi}$ is infinitesimally invariant. Since the domain $\mathcal{D}(L_0^{(n)})$ is the whole space $\mathcal{F}_b(\Gamma_0^{(n)})$, the generator $L_0^{(n)}$ induces a C_0-semigroup in $\mathcal{F}_b(\Gamma_0^{(n)})$. Further, the space $\mathcal{F}_b(\Gamma_0^{(n)})$ separates probability measures, hence it follows from the above lemma that the measure $\lambda^{(n),\phi}$ is also global invariant. Therefore the generator $L_0^{(n)}$ induces a contraction semigroup in $L^p(d\lambda^{(n),\phi})$, $p \geq 1$.

Next, we solve a general evolution problem in which the evolution operator has a diagonal and lower diagonal parts (as in (20)). We assume certain bounds and a special additive structure of the lower diagonal part. Afterwards we show that the evolution problem for the Kawasaki dynamics fits in this scheme.

4.2 Abstract statements

Given a constant $C > 0$ and a pair potential ϕ we define the space

$$\mathcal{L}_{C,\phi} := L^1(\Gamma_0, d\lambda_C^\phi),$$

where the measure λ_C^ϕ has the density $C^{|\cdot|}e^{-E^\phi(\cdot)}$ w.r.t. the Lebesgue-Poisson measure λ. By X_n^ϕ we denote the space

$$X_n^\phi := L^1(\Gamma_0^{(n)}, d\lambda^{(n),\phi}).$$

We denote the norm of the space X_n^ϕ by $\|\cdot\|_{X_n^\phi}$. Moreover, we define for $\alpha > 0$ and $C > 0$ a Banach space $\mathcal{I}_{\alpha,C}^\phi$ consisting of all functions $G = (G^{(n)})_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} X_n^\phi$ for which the norm

$$\|G\|_{\mathcal{I}_{\alpha,C}^\phi} := \sup_{n\in\mathbb{N}} \frac{\|G^{(n)}\|_{X_n^\phi} C^n}{\alpha^n n!} \tag{24}$$

is finite. In the next theorem, we study an abstract evolutionary problem on quasi-observables, having in mind the application to the Kawasaki dynamics w.r.t. a pair potential ϕ.

Theorem 1 *Consider the evolution problem* (18) *and assume that the components of* $\widehat{L}G$ *have the following form:*

$$(\widehat{L}G)^{(n)} = L_0^{(n)}G^{(n)} + \sum_{k<n} W^{(n,k)}G^{(k)},$$

where $L_0^{(n)}$ *generates a contraction semigroup in* X_n^ϕ *(for some positive pair potential* ϕ*) and the following bound holds*

$$\|W^{(n,k)}G^{(k)}\|_{X_n^\phi} \le Ak\binom{n}{k}B^{n-k}\|G^{(k)}\|_{X_k^\phi}, \quad n \in \mathbb{N}, \quad k < n \tag{25}$$

for some $A, B > 0$*. Let* $\alpha, C > 0$ *and define* C_t*,* $t \in \mathbb{R}_+$ *by*

$$C_t := \frac{C}{1 + Ae^{\frac{CB}{\alpha}}t}. \tag{26}$$

Then the following holds: if G_0 *is an element of* $\mathcal{I}_{\alpha,C}^\phi$ *then* G_t *is an element of* $\mathcal{I}_{\alpha,C_t}^\phi$*,* $t > 0$ *and the bound*

$$\|G_t\|_{\mathcal{I}_{\alpha,C_t}^\phi} \le \|G_0\|_{\mathcal{I}_{\alpha,C}^\phi} \tag{27}$$

holds for all $t > 0$*.*

Proof: Consider the component-wise solution

$$G_t^{(n)} = e^{tL_0^{(n)}} G^{(0)} + \int_0^t e^{(t-s)L_0^{(n)}} W^{(n)}(G_s^{(0)}, \dots G_s^{(n-1)}) ds, \quad t > 0$$
$$G_t^{(0)} = G_0^{(0)}$$

of (18). We proof by induction that for all $n \in \mathbb{N}$ it holds:

$$\|G_t^{(n)}\|_{X_n^\phi} \le \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} n! \left(\frac{\alpha}{C_t}\right)^n, \quad t \ge 0. \tag{28}$$

Clearly, from (28) follows (27). Since $G_0 \in \mathcal{I}^\phi_{\alpha,C}$, it follows

$$\|G_0^{(n)}\|_{X_n} \le \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} \left(\frac{\alpha}{C}\right)^n n!. \tag{29}$$

Suppose that

$$\|G_t^{(k)}\|_{X_k^\phi} \le \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} k! \left(\frac{\alpha}{C_t}\right)^k \tag{30}$$

for $t > 0$ and $k < n$ holds.

Using (30) and the assumptions, we get for $t > 0$:

$$\begin{aligned}
&\| \int_0^t e^{(t-s)L_0^{(n)}} W^{(n,k)} G_s^{(k)} ds \|_{X_n^\phi} \\
&\le \int_0^t \|W^{(n,k)} G_s^{(k)}\|_{X_n^\phi}\, ds \\
&\le Ak \binom{n}{k} B^{n-k} \int_0^t \|G_s^{(k)}\|_{X_k^\phi}\, ds \\
&\le Ak \binom{n}{k} B^{n-k} \int_0^t \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} k! \left(\frac{\alpha}{C}\right)^k (1 + Ae^{\frac{BC}{\alpha}} s)^k\, ds \\
&= Ak \binom{n}{k} k! B^{n-k} \left(\frac{\alpha}{C}\right)^k \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} \left(\frac{(1 + Ae^{\frac{CB}{\alpha}} t)^{k+1} - 1}{(k+1)Ae^{\frac{CB}{\alpha}}}\right) \\
&\le\ An! \left(\frac{\alpha}{C}\right)^n \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} \left(\frac{CB}{\alpha}\right)^{n-k} \frac{1}{(n-k)!} \left(\frac{(1 + Ae^{\frac{CB}{\alpha}} t)^{k+1} - 1}{Ae^{\frac{CB}{\alpha}}}\right)
\end{aligned} \tag{31}$$

As a consequence

$$\begin{aligned}
&\| \sum_{k<n} \int_0^t e^{(t-s)L_0^{(n)}} W^{(n,k)} G_s^{(k)}\, ds \|_{X_n^\phi} \\
&\le An! \left(\frac{\alpha}{C}\right)^n \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} \left(\frac{(1 + Ae^{\frac{CB}{\alpha}} t)^n - 1}{Ae^{\frac{CB}{\alpha}}}\right) \sum_{k<n} \frac{B^{n-k}}{(n-k)!} \frac{C^{n-k}}{\alpha^{n-k}} \\
&\le An! \left(\frac{\alpha}{C}\right)^n \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} \left(\frac{(1 + Ae^{\frac{CB}{\alpha}} t)^n - 1}{Ae^{\frac{CB}{\alpha}}}\right) e^{\frac{CB}{\alpha}} \\
&= n! \left(\frac{\alpha}{C}\right)^n \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}} \big((1 + Ae^{\frac{CB}{\alpha}} t)^n - 1\big).
\end{aligned} \tag{32}$$

By means of (32) and the initial bound (29), we can estimate the norm of the component $G_t^{(n)}$:

$$\begin{aligned}
\|G_t^{(n)}\|_{X_n^\phi} &\le \|G_0^{(n)}\|_{X_n^\phi} + n!\|G_0\|_{\mathcal{I}_{\alpha,C}^\phi}\left(\frac{\alpha}{C}\right)^n \left((1+Ae^{\frac{CB}{\alpha}}t)^n - 1\right) \\
&\le \|G_0\|_{\mathcal{I}_{\alpha,C}^\phi}\left(\frac{\alpha}{C}\right)^n n! \\
&\quad + n!\|G_0\|_{\mathcal{I}_{\alpha,C}^\phi}\left(\frac{\alpha}{C}\right)^n \left((1+Ae^{\frac{CB}{\alpha}}t)^n - 1\right) \\
&= \|G_0\|_{\mathcal{I}_{\alpha,C}^\phi} n!\left(\frac{\alpha}{C_t}\right)^n .
\end{aligned}$$

This shows (28). $\square$

4.3 The Kawasaki system

In this subsection we will show that the Kawasaki dynamics w.r.t. a positive pair potential ϕ fulfills all assumptions of Theorem 1. To this end, we write

$$W^{(n,k)} = W_1^{(n,k)} + W_2^{(n,k)}$$

with

$$\begin{aligned}
(W_1^{(n,k)}G^{(k)})(\eta) = \sum_{\substack{\xi\subset\eta \\ |\xi|=k}} \sum_{x\in\xi} \int_{\mathbb{R}^d} dy\ a(x-y) \prod_{z\in\xi} e^{-\phi(y-z)} \\
\times \prod_{z\in\eta\setminus\xi} (e^{-\phi(y-z)} - 1) G^{(k)}(\xi\setminus x \cup y)
\end{aligned}$$

and

$$\begin{aligned}
(W_2^{(n,k)}G^{(k)})(\eta) = -\sum_{\substack{\xi\subset\eta \\ |\xi|=k}} \sum_{x\in\xi} \int_{\mathbb{R}^d} dy\ a(x-y) \prod_{z\in\xi} e^{-\phi(y-z)} \\
\times \prod_{z\in\eta\setminus\xi} (e^{-\phi(y-z)} - 1) G^{(k)}(\xi).
\end{aligned}$$

In the following we use the abbreviations $dx^{(n)}$ for $dx_1 \dots dx_n$, $n \in \mathbb{N}$ and $dx^{(n-k)}$ for $dx_{k+1} \dots dx_n$, $k < n$. Without loss of generality we can assume that the Euclidean mean $\langle a \rangle$ of a is equal to 1. Further, for $\eta = \{x_1, \dots, x_n\}$ and $y \in \mathbb{R}^d$ we use the notation $E(\eta) = E_n(x_1, \dots, x_n)$ resp. $E(y,\eta) = E(y|x_1, \dots x_n)$.

Now we are in a position to estimate the second term of $W^{(n,k)}$:

$$
\begin{aligned}
\|W_2^{(n,k)}G^{(k)}\|_{X_n^\phi} &\le \binom{n}{k} k \int_{\left(\mathbb{R}^d\right)^n} dx^{(n)} \int_{\mathbb{R}^d} dy\ a(x_1-y) \prod_{i=1}^{k} e^{-\phi(x_i-y)} \\
&\quad \times \prod_{i=k+1}^{n} |e^{-\phi(x_i-y)}-1| e^{-E_n(x_1,\ldots,x_n)} |G^{(k)}(x_1,\ldots,x_k)| \\
&\le \binom{n}{k} k \int_{\left(\mathbb{R}^d\right)^{n-k}} dx^{(n-k)} \prod_{i=k+1}^{n} |e^{-\phi(x_i-y)}-1| \\
&\quad \times \int_{\left(\mathbb{R}^d\right)^k} dx^{(k)} |G^{(k)}(x_1,\ldots,x_k)| e^{-E_k(x_1,\ldots,x_k)} \int_{\mathbb{R}^d} dy\ a(x_1-y) \\
&\le \binom{n}{k} k C_\phi^{n-k} \langle a\rangle \|G^{(k)}\|_{X_k^\phi} = \binom{n}{k} k C_\phi^{n-k} \|G^{(k)}\|_{X_k^\phi}. \qquad (33)
\end{aligned}
$$

For $W_1^{(n,k)}$ we obtain the same bound, since it holds:

$$
\begin{aligned}
&\int_{\left(\mathbb{R}^d\right)^n} dx^{(n)} e^{-E_n(x_1,\ldots,x_n)} \int_{\mathbb{R}^d} dy\ a(x_1-y) G^{(k)}(x_2,\ldots x_k,y) \\
&\quad \times \prod_{i=1}^{k} e^{-\phi(x_i-y)} \prod_{i=k+1}^{n} |e^{-\phi(x_i-y)}-1| \\
&\le \int_{\left(\mathbb{R}^d\right)^{n-k}} dx^{(n-k)} \prod_{i=k+1}^{n} |e^{-\phi(x_i-y)}-1| \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^{(k-1)d}} dx_2 \ldots dx_k \\
&\quad \times e^{-E(y|x_2\ldots,x_k)} e^{-E_{k-1}(x_2\ldots,x_k)} G^{(k)}(x_2,\ldots x_k,y) \int_{\mathbb{R}^d} dx_1\ e^{-\phi(x_1-y)} a(x_1-y) \\
&\le C_\phi^{n-k} \int_{\left(\mathbb{R}^d\right)^k} dz_1 \ldots dz_k\ e^{-E_k(z_1,\ldots,z_k)} G^{(k)}(z_1,\ldots,z_k) \\
&= C_\phi^{n-k} \|G^{(k)}\|_{X_k^\phi}. \qquad (34)
\end{aligned}
$$

Combining (33) and (34) yields

$$
\|W^{(n,k)}G^{(k)}\|_{X_n^\phi} \le 2\binom{n}{k} k C_\phi^{(n-k)} \|G^{(k)}\|_{X_k^\phi}. \tag{35}
$$

The estimate (35) implies that all assumptions of Theorem 1 are fulfilled for a Kawasaki dynamics (with $B = C_\phi$, ϕ the interacting potential and $A = 2$) and we get the following

Theorem 2 *Let $C > 0$, $\alpha > 0$ be arbitrary and fixed. Consider the evolutionary problem* (18) *(Kawasaki dynamics w.r.t. a positive pair potential ϕ) with initial data $G_0 \in \mathcal{I}^\phi_{\alpha,C}$. Then the solution $G_t = (G_t^{(n)})_{n\in\mathbb{N}}$ of* (18) *given by* (22) *belongs to $G_t \in \mathcal{I}^\phi_{\alpha,C_t}$, $t > 0$ where C_t is given by*

$$
C_t = \frac{C}{1 + 2e^{\frac{CC_\phi}{\alpha}} t} \tag{36}
$$

and C_ϕ is given by (3)*. Further the bound*

$$\|G_t\|_{\mathcal{I}^\phi_{\alpha,C_t}} \le \|G_0\|_{\mathcal{I}^\phi_{\alpha,C}}$$

holds for all $t \ge 0$.

Remark 1 We can define a propagator $\widehat{P}_t$ by

$$\widehat{P}_t : \mathcal{I}^\phi_{\alpha,C} \to \mathcal{I}^\phi_{\alpha,C_t}, \quad \widehat{P}_t G := G_t$$

where G_t is the solution of (18) with initial data G. This propagator describes the time evolution of quasi-observables.

5 Evolution of Correlation Functions

In this section we construct the time evolution for correlation functions. Since quasi-observables and correlation functions are dual to each other (in the same manner as observables and states are dual to each other), we will construct this evolution as dual evolution to quasi-observables.

The natural space in which the evolution of correlation functions (for a Kawasaki dynamics w.r.t. a potential ϕ) takes place is the space $\mathcal{K}_{C,\phi}$, $C > 0$ and ϕ the interacting potential, defined by

$$\mathcal{K}_{C,\phi} := \{k : \Gamma_0 \to \mathbb{R} |\ k \cdot C^{-|\cdot|} e^{E^\phi(\cdot)} \in L^\infty(\Gamma_0, d\lambda)\},$$

with the norm $\|k\|_{\mathcal{K}_{C,\phi}} := \|k(\cdot)C^{-|\cdot|}e^{E^\phi(\cdot)}\|_{L^\infty(\Gamma_0,d\lambda)}$. For technical reasons we have to introduce some additional spaces. Recall that for $C > 0$ the space $\mathcal{L}_{C,\phi}$ was defined as

$$\mathcal{L}_{C,\phi} := L^1(\Gamma_0, C^{|\cdot|} d\lambda^\phi_C).$$

Then $\mathcal{K}_{C,\phi}$ is the dual space to $\mathcal{L}_{C,\phi}$ w.r.t. to the duality (13). We also remind that the space $\mathcal{I}^\phi_{\alpha,C}$ consists of all measurable functions G on Γ_0, such that $\|G\|_{\mathcal{I}^\phi_{\alpha,C}} < \infty$ holds (cf.(24)). For $\alpha \in (0,1)$, we obtain the inclusion

$$\mathcal{L}_{\frac{C}{\alpha},\phi} \subset \mathcal{I}^\phi_{\alpha,C} \subset \mathcal{L}_{C,\phi} \tag{37}$$

since it holds firstly:

$$\frac{C^n}{n!}\|G^{(n)}\|_{X^\phi_n} \le \alpha^n \|G\|_{\mathcal{I}^\phi_{\alpha,C}}, \quad n \in \mathbb{N}, G \in \mathcal{I}^\phi_{\alpha,C},$$

which implies

$$\|G\|_{\mathcal{L}_{C,\phi}} \le \frac{1}{1-\alpha}\|G\|_{\mathcal{I}^\phi_{\alpha,C}} < \infty.$$

Because of that we obtain

$$\mathcal{I}^\phi_{\alpha,C} \subset \mathcal{L}_{C,\phi}. \tag{38}$$

Secondly, it holds for $G \in \mathcal{L}_{\frac{C}{\alpha},\phi}$

$$\|G\|_{\mathcal{I}^{\phi}_{\alpha,C}} \leq \|G\|_{\mathcal{L}_{\frac{C}{\alpha},\phi}},$$

and hence

$$\mathcal{L}_{\frac{C}{\alpha},\phi} \subset \mathcal{I}^{\phi}_{\alpha,C}.$$

Altogether we obtain (37).

We also consider a functional space $\mathcal{J}^{\phi}_{\alpha,C}$ which consists of all functions $k = (k^{(n)})_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \mathbb{X}^{\phi}_{n}$ for which

$$\|k\|_{\mathcal{J}^{\phi}_{\alpha,C}} := \sum_{n=0}^{\infty} \frac{\alpha^n}{C^n} \|k^{(n)}\|_{\mathbb{X}^{\phi}_{n}} < \infty$$

holds. Here

$$\mathbb{X}^{\phi}_{n} := \{k^{(n)} : \Gamma^{(n)}_0 \to \mathbb{R} \,|\, k^{(n)} e^{E^{\phi}_n} \in L^{\infty}(\Gamma^{(n)}_0, d\sigma^{(n)})\}$$

and $\|k^{(n)}\|_{\mathbb{X}^{\phi}_{n}} := \|k^{(n)} e^{E^{\phi}_n}\|_{L^{\infty}(\Gamma^{(n)}_0,\sigma^{(n)})}$. Let $G \in \mathcal{I}^{\phi}_{\alpha,C}$ and $k \in \mathcal{J}^{\phi}_{\alpha,C}$. It follows

$$\begin{aligned}
|\langle\langle k, G\rangle\rangle| &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma^{(n)}_0} |k^{(n)}||G^{(n)}| d\sigma^{(n)} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma^{(n)}_0} |k^{(n)}| e^{E^{\phi}_n} |G^{(n)}| d\lambda^{(n),\phi} \\
&\leq \sum_{n=0}^{\infty} \frac{1}{n!} \|G^{(n)}\|_{X^{\phi}_n} \|k^{(n)}\|_{\mathbb{X}^{\phi}_n} \\
&\leq \sum_{n=0}^{\infty} \frac{C^n}{\alpha^n n!} \|G^{(n)}\|_{X^{\phi}_n} \frac{\alpha^n}{C^n} \|k^{(n)}\|_{\mathbb{X}^{\phi}_n} \\
&\leq \left(\sup_{n\in\mathbb{N}} \|G^{(n)}\|_{X^{\phi}_n} \frac{C^n}{\alpha^n n!}\right) \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{C^n} \|k^{(n)}\|_{\mathbb{X}^{\phi}_n}\right) \\
&= \|G\|_{\mathcal{I}^{\phi}_{\alpha,C}} \|k\|_{\mathcal{J}^{\phi}_{\alpha,C}}.
\end{aligned} \tag{39}$$

Hence $G \mapsto \langle\langle k, G\rangle\rangle$ is a bounded linear functional on $\mathcal{I}^{\phi}_{\alpha,C}$ and therefore

$$\mathcal{J}^{\phi}_{\alpha,C} \subset (\mathcal{I}^{\phi}_{\alpha,C})^*.$$

Next we observe that for $k \in \mathcal{J}^{\phi}_{\alpha,C}$ holds:

$$\frac{\alpha^n}{C^n} \|k^{(n)}\|_{\mathbb{X}^{\phi}_n} \leq \|k\|_{\mathcal{J}^{\phi}_{\alpha,C}}, \quad n \in \mathbb{N}.$$

It follows

$$\|k\|_{\mathcal{K}_{\frac{C}{\alpha},\phi}} \leq \|k\|_{\mathcal{J}^{\phi}_{\alpha,C}}$$

which implies

$$\mathcal{J}^{\phi}_{\alpha,C} \subset \mathcal{K}^{\phi}_{\frac{C}{\alpha}}.$$

At the same time, for $k \in \mathcal{K}_{C,\phi}$

$$\|k^{(n)}\|_{\mathbb{X}^{\phi}_n} \leq C^n \|k\|_{\mathcal{K}_{C,\phi}}, \quad n \in \mathbb{N}.$$

Using this we conclude

$$\|k\|_{\mathcal{J}^{\phi}_{\alpha,C}} = \sum_{n=0}^{\infty} \frac{\alpha^n}{C^n} \|k^{(n)}\|_{\mathbb{X}^{\phi}_n} \leq \sum_{n=0}^{\infty} \frac{\alpha^n}{C^n} C^n \|k\|_{\mathcal{K}_{C,\phi}} = \frac{1}{1-\alpha} \|k\|_{\mathcal{K}_{C,\phi}}.$$

Therefore

$$\mathcal{K}_{C,\phi} \subset \mathcal{J}^{\phi}_{\alpha,C} \subset \mathcal{K}_{\frac{C}{\alpha},\phi}. \tag{40}$$

Further by means of (37) we get

$$\mathcal{K}_{C,\phi} \subset (\mathcal{I}^{\phi}_{\alpha,C})^* \subset \mathcal{K}_{\frac{C}{\alpha},\phi}. \tag{41}$$

Altogether

$$\mathcal{K}_{C,\phi} \subset \mathcal{J}^{\phi}_{\alpha,C} \subset (\mathcal{I}^{\phi}_{\alpha,C})^* \subset \mathcal{K}_{\frac{C}{\alpha},\phi}.$$

Having disposed of these preliminary steps, we can now turn to the construction of the evolution of correlation functions:

Theorem 3 *Let $C_0 > 0$ and $\alpha \in (0,1)$ be arbitrary and fixed. Define a time horizon $T > 0$ by*

$$T := \frac{1}{2} e^{-\frac{C_0 C_\phi}{\alpha}}.$$

Let $k_0 \in \mathcal{K}_{C_0,\phi}$. Then for all $t < T$ there exists $k_t \in \mathcal{K}_{\frac{C_t^*}{\alpha},\phi}$, where

$$C_t^* := \frac{C_0}{1 - 2e^{\frac{C_0 C_\phi}{\alpha}} t}, \tag{42}$$

such that for all $G \in \mathcal{I}^{\phi}_{\alpha,C_t^*}$ it holds

$$\langle\langle k_t, G\rangle\rangle = \langle\langle k_0, G_t\rangle\rangle. \tag{43}$$

Here G_t is the solution $(G_s)_{s\geq 0}$ of (18) with initial data $G_0 = G$ evaluated at $s = t$.

Proof: Let $k_0 \in \mathcal{K}_{C_0,\phi} \subset \left(\mathcal{I}^{\phi}_{\alpha,C_0}\right)^*$ and $t < T$. Take any $G_0 \in \mathcal{I}^{\phi}_{\alpha,C_t^*}$. By Theorem 2, there exists an evolution $G_0 \to G_\tau$ for any $\tau > 0$ such that $G_\tau \in \mathcal{I}^{\phi}_{\alpha,C_\tau}$, where

$$C_\tau = \frac{C_t^*}{1 + 2e^{\frac{C_\phi C_t^*}{\alpha}} \tau}.$$

From the definition of C_t^* we conclude that $C_t = C_\tau|_{\tau=t} = C_0$. Thus, $G_t \in \mathcal{I}^\phi_{\alpha,C_0}$. Moreover, since $k_0 \in \mathcal{K}_{C_0,\phi} \subset \mathcal{J}^\phi_{\alpha,C_0}$ and $G_t \in \mathcal{I}^\phi_{\alpha,C_0}$, we obtain by means of (27) and (39):

$$|\langle\langle k_0, G_t\rangle\rangle| \le \|k_0\|_{\mathcal{J}^\phi_{\alpha,C_0}} \|G_t\|_{\mathcal{I}^\phi_{\alpha,C_0}} \le \|k_0\|_{\mathcal{J}^\phi_{\alpha,C_0}} \|G_0\|_{\mathcal{I}^\phi_{\alpha,C_t^*}}.$$

Therefore, the mapping $G_0 \to \langle\langle k_0, G_t\rangle\rangle$ is a linear continuous functional on the space $\mathcal{I}^\phi_{\alpha,C_t^*}$. Consequently, there exists $k_t \in \left(\mathcal{I}^\phi_{\alpha,C_t^*}\right)^* \subset \mathcal{K}_{\frac{C_t^*}{\alpha},\phi}$ (cf. (41)) such that, for any $G \in \mathcal{I}^\phi_{\alpha,C_t^*}$

$$\langle\langle k_t, G\rangle\rangle = \langle\langle k_0, G_t\rangle\rangle. \quad \square$$

Remark 2 The evolution k_t, $t \in [0,T)$ describes the time evolution of the initial correlation function k_0. We can regard k_t as the weak solution to (17), i.e.

$$\frac{d}{dt}\langle\langle k_t, G\rangle\rangle = \langle\langle k_0, \hat{L}G_t\rangle\rangle$$

provided $\langle\langle k_0, G_t\rangle\rangle$ is differentiable on any $[0,T'] \subset [0,T)$ and the corresponding derivative can be represented in the form $\langle\langle k_0, \hat{L}G_t\rangle\rangle$.

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