Uniform propagation of chaos for a class of inhomogeneous diffusions

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Abstract

We are interested in inhomogeneous diffusions in which the own law intervenes in the drift. This kind of diffusions corresponds to the hydrodynamical limit of some particle system. One also talks about propagation of chaos. It is well-known, for McKean-Vlasov diffusions, that such a propagation of chaos holds on finite interval. However, it has been proved that the lack of convexity of the external force implies that there is no uniform propagation of chaos if the diffusion coefficient is small enough. We here aim to establish a uniform propagation of chaos even if the external force is not convex, with a diffusion coefficient sufficiently large. The idea consists in combining the propagation of chaos on a finite interval with a functional inequality.

Key words and phrases: Nonlinear diffusions ; Propagation of chaos ; Feynman-Kac ; McKean-Vlasov models ; Functional inequality

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Introduction

We are interested in some inhomogeneous processes in $\mathbb{R}^d$ defined by an equation in which the own law of the process intervenes in the drift. In this work, we consider a diffusion of the form

$$\left\{ \begin{array}{l} X_t = X_0 + \sigma B_t - \int_0^t b\left(\mu_s, X_s\right) ds\,, \\ \mu_s = \mathcal{L}\left(X_s\right)\,, \end{array} \right. \tag{I}$$

where b is a function from $\mathcal{P}\left(\mathbb{R}^d\right) \times \mathbb{R}^d$ to $\mathbb{R}^d$ and $\{B_t\,;\, t \geq 0\}$ is a d-dimensional Wiener process. The assumptions are detailed subsequently. Let us just say that

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$b(\mu, x)$ does only depend on x and on some finite number of moments of μ. The infinitesimal generator of Diffusion (I) thus is

$$A\mu := \frac{\sigma^2}{2}\Delta\mu - \nabla.\left\{b\left(\mu\,,\,.\right)\mu\right\}\,.$$

The associated semi-group is denoted by $(P_t)_{t\geq 0}$. In other words, one has $\mu_t = \mu_0 P_t$. We notice that X_t, μ_t, P_t and A depend on σ. We do not write it for simplifying the reading.
An example of such diffusion is the McKean-Vlasov one:

$$X_t = X_0 + \sigma B_t - \int_0^t \nabla V\left(X_s\right) ds - \int_0^t \Big(\nabla F * \mathcal{L}\left(X_s\right)\Big)(X_s) ds\,, \qquad \text{(II)}$$

where V and F respectively are called the confining and the interacting potentials. The notation $*$ is used for denoting the convolution.

This equation is nonlinear in the sense of McKean, see [McK67, McK66].

It is well-known, see [McK67], that the law $\mathcal{L}\left(X_t\right)$ is absolutely continuous with respect to the Lebesgue measure for all $t > 0$, provided some regularity hypotheses on V and F. Moreover, its density, which is denoted by u_t, satisfies the so-called granular media equation,

$$\frac{\partial}{\partial t}u_t = \nabla.\left\{\frac{\sigma^2}{2}\nabla u_t + \left(\nabla V + \nabla F * u_t\right)u_t\right\}\,.$$

The setting of this work is restricted to the McKean-Vlasov case. However, we could apply to more general hypotheses. Let us notice that we do not assume any convex properties on the confining potential nor on the interacting one. Under easily checked assumptions, Diffusion (II) corresponds to the hydrodynamical limit of the following particle system

$$\left\{\begin{array}{l} X_t^1 = X_0^1 + \sigma B_t^1 - \int_0^t \left[\nabla V\left(X_s^1\right) + \sum_{j=1}^N \frac{1}{N}\nabla F\left(X_s^1 - X_s^j\right)\right] ds\,, \\ \vdots \\ X_t^i = X_0^i + \sigma B_t^i - \int_0^t \left[\nabla V\left(X_s^i\right) + \sum_{j=1}^N \frac{1}{N}\nabla F\left(X_s^i - X_s^j\right)\right] ds\,, \\ \vdots \\ X_t^N = X_0^N + \sigma B_t^N - \int_0^t \left[\nabla V\left(X_s^N\right) + \sum_{j=1}^N \frac{1}{N}\nabla F\left(X_s^N - X_s^j\right)\right] ds\,, \end{array}\right. \qquad \text{(III)}$$

$\left(W_t^i\right)_{t\in\mathbb{R}_+}$ being N independent d-dimensional Wiener processes. We also assume that $\left\{X_0^i\,;\, i\in\mathbb{N}^*\right\}$ is a family of independent random variables, identically distributed with common law $\mathcal{L}\left(X_0\right)$. The particles thus are excheangeable. We here focus on the first diffusion. By $\mu_t^{1,N}$, we denote the law at time t of the diffusion X^1. The associated semi-group is denoted by $\left(P_t^N\right)_{t\geq 0}$. We thus have $\mu_t^{1,N} = \mu_0 P_t^N$. Let us stress that this equality is true only if $\left\{X_0^i\,;\, i\in\mathbb{N}^*\right\}$ is a family of independent random variables, identically distributed with common

law μ_0.
One says, in this work, that simple propagation of chaos holds on interval $[0;T]$ with $T>0$ if we have the limit

$$\lim_{N\to+\infty}\sup_{0\leq t\leq T}\mathbb{W}_2\left(\mu_0P_t^N\,;\,\mu_0P_t\right)=\lim_{N\to+\infty}\sup_{0\leq t\leq T}\mathbb{W}_2\left(\mu_t^{1,N}\,;\,\mu_t\right)=0\,,$$

$\mathbb{W}_2$ standing for the Wasserstein distance. This means that X^1 is a good approximation of Diffusion (I) as N goes to infinity.

Such limit (with another distance) has been investigated for the special case of the McKean-Vlasov diffusion, see [BRTV98, BAZ99, Mal01, Mal03, Szn91]. Indeed, it is a consequence of the classical result of propagation of chaos which is proved by a coupling,

$$\sup_{t\leq T}\mathbb{E}\left\{\left|\left|X_t-X_t^1\right|\right|^2\right\}\leq\lambda\frac{e^{KT}}{N}\,,$$

with $\lambda,K>0$. Uniform propagation of chaos, that is

$$\lim_{N\to+\infty}\sup_{t\geq0}\mathbb{W}_2\left(\mu_0P_t^N\,;\,\mu_0P_t\right)=\lim_{N\to+\infty}\sup_{t\geq0}\mathbb{W}_2\left(\mu_t^{1,N}\,;\,\mu_t\right)=0\,,$$

has been proved if both confining potential V and interacting potential F are convex, see [CGM08]. The particularity here is that the coupling between the two diffusions is not necessary made with the same Brownian Motions.

A consequence of the uniform propagation of chaos for the nonlinear diffusion is the uniqueness of the invariant probability μ^σ and the weak convergence toward this measure. However, without convex properties, we can prove the non-uniqueness of the invariant probabilities under simple assumptions, provided that the diffusion coefficient σ is sufficiently small, see [HT10, Tug13, Tug11]. But, as pointed out in [Tug13], if σ is large enough, we have a unique invariant probability. The question thus is: does uniform propagation of chaos holds if σ is sufficiently large? Also, can we reciprocally use the convergence toward the unique invariant probability to obtain this uniform propagation of chaos?

We positively answer to the two questions.

To the best of our knowledge, the first uniform propagations of chaos estimates for mean field particle models have been developed in [DMM00] and in [DMG01] in the context of Feynman-Kac interacting jump models. Further results in this direction can be found in [DMR11] as well as in the research monographs [DM13, DM04], including exponential concentration inequalities w.r.t. the time horizon, and contraction inequalities w.r.t. several classes of relative entropy criteria.

The analysis of interacting jump particle models clearly differs from the more traditional coupling analysis of the McKean-Vlasov diffusion models developed in the present article. The common feature is to enter the stability properties of the limiting nonlinear semigroup into the estimation of the propagation of chaos

properties of the finite particle systems, to deduce L_p-mean error estimates of order $1/\sqrt{N}^{\delta}$, for any $0 < \delta < 1$ (cf. for instance theorem 2.11 in [DMM00], at the level of the empirical processes). In our context using these techniques, we obtain a variance and a W_2-estimate of order $1/N^{\delta}$, for any $0 \leq \delta < 1$. We underline that in the context of Feynman-Kac particle models, the order $1/N$ can be obtained under stronger mixing conditions, using a backward semigroup techniques. Thus, we conjecture that this decay rate is also met in our context.

The existence problem of a solution to (II) is not investigated here. We thus assume that there exists a unique strong solution $(X_t)_{t\geq 0}$. Let us just mention that it has been solved under simple assumptions. The method consists in applying a fixed point theorem, see [BRTV98, HIP08].

In a first section, we introduce the framework of the article and the first results. Then, we establish some functional inequalities in Section 2. Finally, in last section, we prove the main result that is the uniform propagation of chaos when the coefficient diffusion is sufficiently large.

Before finishing the introduction, we give the hypotheses of the paper and the main result.

Assumption (A): *We say that the confining potential V, the interacting potential F and the initial law μ_0 satisfy the set of assumptions (A) if*

(A-1) *V is a smooth function on $\mathbb{R}^d$.*

(A-2) $\liminf_{||x||\to+\infty} \nabla^2 V(x) = +\infty$.

(A-3) *there exists a strictly convex smooth function V_0 and $\vartheta > 0$ such that $\nabla V(x) = \nabla V_0(x) - \vartheta x$.*

(A-4) *the gradient ∇V is slowly increasing: there exist $m \in \mathbb{N}^*$ and $C > 0$ such that $||\nabla V(x)|| \leq C\left(1 + ||x||^{2m-1}\right)$, for all $x \in \mathbb{R}^d$.*

(A-5) *there exist a strictly convex function Θ such that $\Theta(y) > \Theta(0) = 0$ for all $y \in \mathbb{R}^d$ and $p \in \mathbb{N}$ such that the following limit holds for any $y \in \mathbb{R}^d$:* $\lim_{r\to+\infty} \frac{V(ry)}{r^{2p}} = \Theta(y)$.

(A-6) *there exist an even polynomial and strictly convex function G on $\mathbb{R}$ and a positive constant $\alpha > 0$ such that $F(x) = F_0(x) - \frac{\alpha}{2}||x||^2$ with $F_0(x) := G(||x||)$. And,* $\deg(G) =: 2n \geq 2$.

(A-7) *the $8q^2$-th moment of the measure μ_0 is finite with $q := \min\{m, n\}$.*

(A-8) *the measure μ_0 admits a $\mathcal{C}^\infty$-continuous density u_0 with respect to the Lebesgue measure. And, the entropy $-\int_{\mathbb{R}^d} u_0(x)\log(u_0(x))dx$ is finite.*

The main result (Theorem 2.4 and Theorem 3.1) of the paper can be summarized as follows

Theorem: *We assume the set of Hypotheses (A) and also $p > n$.*

Thus, there exists σ_c such that Diffusion (II) *admits a unique invariant probability μ^σ if $\sigma \geq \sigma_c$. Moreover, there exists a positive constant $C(\sigma)$ such that we have the exponential decay*

$$\mathbb{W}_2\left(\mu_t\,;\,\mu^\sigma\right) \leq \exp\left[-C(\sigma)t\right]\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right).$$

for any measure μ_0 with finite entropy. Furthermore, we have the limit

$$\lim_{N\to+\infty} \sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) = 0\,.$$

From now on, τ^* denotes the Legendre transform of any function τ from $\mathbb{R}^d$ to $\mathbb{R}$.

1 Preliminaries

Let us give the framework (definitions and basic propositions) of the current work. We begin by the Wasserstein distance.

Definition 1.1. *For any probability measures μ and ν, the Wasserstein distance between μ and ν is*

$$\mathbb{W}_2\left(\mu\,;\,\nu\right) := \sqrt{\inf\ \mathbb{E}\left\{||X-Y||^2\right\}},$$

where the infimum is taken over the random variables X and Y with law μ and ν respectively. One can also write

$$\mathbb{W}_2\left(\mu\,;\,\nu\right) = \sqrt{\inf \iint_{\mathbb{R}^d\times\mathbb{R}^d} ||x-y||^2\,\pi(dx,dy)}\,,$$

where the infimum runs over the probability measure on $\mathbb{R}^d\times\mathbb{R}^d$ with marginals equal to μ and ν.

The Wasserstein distance can be characterized in the following way, by Brenier's theorem.

Proposition 1.2. *Let μ and ν be two probability measures. If μ is absolutely continuous with respect to the Lebesgue measure, there exists a convex function τ from $\mathbb{R}^d$ to $\mathbb{R}$ such that the following equality occurs for every bounded test function g:*

$$\int_{\mathbb{R}^d} g(x)\nu(dx) = \int_{\mathbb{R}^d} g\left(\nabla\tau(x)\right)\mu(dx)\,.$$

We write

$$\nu = \nabla\tau\#\mu\,.$$

We thus have the following characterization:

$$\mathbb{W}_2\left(\mu\,;\,\nu\right) = \sqrt{\int_{\mathbb{R}^d} ||x-\nabla\tau(x)||^2\,\mu(dx)}\,.$$

The idea-key of the paper is a WJ-inequality. Let us present the expression that we denote by J:

$$J_{V,F}\left(\nu \,|\, \mu\right) := J_{V,0}\left(\nu \,|\, \mu\right) \tag{1.1}$$
$$+ \frac{1}{2}\iint_{\mathbb{R}^{2d}} \left\langle \nabla F\left(Z(x,y)\right) - \nabla F(x-y)\,;\, Z(x,y) - (x-y)\right\rangle \mu(dx)\mu(dy)\,,$$

with $Z(x,y) := \nabla\tau(x) - \nabla\tau(y)$ and

$$J_{V,0}\left(\nu \,|\, \mu\right) := \frac{\sigma^2}{2}\int_{\mathbb{R}^d} \Big(\Delta\tau(x) + \Delta\tau^*\left(\nabla\tau(x)\right) - 2d\Big)\mu(dx) \tag{1.2}$$
$$+ \int_{\mathbb{R}^d} \left\langle \nabla V\left(\nabla\tau(x)\right) - \nabla V(x)\,;\, \nabla\tau(x) - x\right\rangle \mu(dx)\,.$$

Here, we have $\nu = \nabla\tau\#\mu$. We now present the transportation inequality on which the article is based.

Definition 1.3. *Let μ be a probability measure on $\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure and $C > 0$. We say that μ satisfies a $WJ_{V,F}(C)$-inequality if the inequality*

$$C\mathbb{W}_2\left(\nu\,;\,\mu\right)^2 \leq J_{V,F}\left(\nu \,|\, \mu\right) \tag{1.3}$$

holds for any probability measure ν on $\mathbb{R}^d$.

In the same way, we talk about $WJ_{V,0}$-inequality. Since we intend to obtain WJ-inequality for invariant probability of Diffusion (II), we only consider probability measures which are uniformly continuous with respect to the Lebesgue measure.

We now give a classical result which explains why a $WJ_{V,F}$-inequality has consequences on the long-time behavior of McKean-Vlasov diffusions (II). It is similar to Proposition 1.1 in [BGG12b]. See also [AGS08].

Proposition 1.4. *Let μ_0 and ν_0 be two probability measures on $\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure. Set $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ two McKean-Vlasov diffusions* (II) *starting with law μ_0 and ν_0. By μ_t (respectively ν_t), we denote the law of X_t (respectively Y_t).*
Thus, we have the inequality

$$\frac{1}{2}\frac{d}{dt}\mathbb{W}_2\left(\mu_t\,;\,\nu_t\right)^2 \leq -J_{V,F}\left(\nu_t \,|\, \mu_t\right)\,. \tag{1.4}$$

Consequently, if μ^σ is an invariant probability of Diffusion (II) and if μ^σ satisfies a $WJ_{V,F}(C)$-inequality, by combining Ineq. (1.3) and Ineq. (1.4), we obtain

$$\mathbb{W}_2\left(\mu_t\,;\,\mu^\sigma\right) \leq e^{-Ct}\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right)$$

for any μ_0 absolutely continuous with respect to the Lebesgue measure. This also establishes the uniqueness of the invariant probability.
Let us now remind the reader Proposition 3.5 in [BGG12a].

Proposition 1.5. *Let W be a function from $\mathbb{R}^d$ to $\mathbb{R}$ such that there exist two constants $R \geq 0$ and $K > 0$ such that $\nabla^2 W(x) \geq K$ for any $||x|| \geq R$. Let V be a continuous function from $\mathbb{R}^d$ to $\mathbb{R}$.*
Thus, there exists $C > 0$ such that for any convex function τ from $\mathbb{R}^d$ to $\mathbb{R}$, we have

$$C \int_{\mathbb{R}^d} ||\nabla\tau(x) - x||^2 \, e^{-V(x)} dx \leq \int_{\mathbb{R}^d} \left[\Delta\tau(x) + \Delta\tau^* \left(\nabla\tau(x)\right) - 2d\right] e^{-V(x)} dx$$
$$+ \int_{\mathbb{R}^d} \left\langle \nabla W \left(\nabla\tau(x)\right) - \nabla W(x) \, ; \, \nabla\tau(x) - x \right\rangle e^{-V(x)} dx$$

The other keystone of the uniform propagation of chaos is the simple propagation of chaos. We here consider it with the Wasserstein distance. Let us define it precisely.

Definition 1.6. *One says that simple propagation of chaos holds if the inequality*

$$\lim_{N \to +\infty} \sup_{t \in [0;T]} \mathbb{W}_2 \left(\mu_t \, ; \, \mu_t^{1,N}\right) = 0$$

is true for any $T > 0$, where μ_t (resp. $\mu_t^{1,N}$) is the law of Diffusion (II) *(resp. the law of the first particle in the system* (III)*).*
Uniform propagation of chaos holds if we have the limit

$$\lim_{N \to +\infty} \sup_{t \in \mathbb{R}_+} \mathbb{W}_2 \left(\mu_t \, ; \, \mu_t^{1,N}\right) = 0 \, . \tag{1.5}$$

We know that the uniform coupling limit

$$\lim_{N \to +\infty} \sup_{t \in \mathbb{R}_+} \mathbb{E} \left\{ \left|\left| X_t - X_t^{1,N} \right|\right|^2 \right\} = 0$$

implies the uniqueness of the invariant probability of Diffusion (II) and the convergence toward this stationary measure. But, what we call here uniform propagation of chaos is weaker. However, let us prove that the limit (1.5) implies the uniqueness of the invariant probability and the convergence in Wasserstein distance toward this distance.

Proposition 1.7. *We assume the uniform propagation of chaos, that is to say*

$$\lim_{N \to +\infty} \sup_{t \in \mathbb{R}_+} \mathbb{W}_2 \left(\mu_t \, ; \, \mu_t^{1,N}\right) = 0 \, .$$

Thus, McKean-Vlasov diffusion (II) *admits a unique invariant probability μ^σ. Moreover, we have the limit*

$$\lim_{t \to +\infty} \mathbb{W}_2 \left(\mu_t \, ; \, \mu^\sigma\right) = 0$$

for any probability measure μ_0.

Proof. We know that Diffusion (II) admits an invariant probability μ^σ, see Proposition 2.1 in [Tug11] or Proposition A.1 in [BGG12b].
Hence, we have the inequality

$$\begin{aligned}\mathbb{W}_2\left(\mu_t\,;\,\mu^\sigma\right) =&\mathbb{W}_2\left(\mu_0P_t\,;\,\mu^\sigma P_t\right)\\ \leq&\mathbb{W}_2\left(\mu_0P_t\,;\,\mu_0P_t^N\right)+\mathbb{W}_2\left(\mu_0P_t^N\,;\,\mu^\sigma P_t^N\right)\\ &+\mathbb{W}_2\left(\mu^\sigma P_t^N\,;\,\mu^\sigma P_t\right)\end{aligned}$$

Let ϵ be a positive constant. Limit (1.5) implies the existence of N such that

$$\sup_{t\geq 0}\left\{\mathbb{W}_2\left(\mu_0P_t\,;\,\mu_0P_t^N\right)+\mathbb{W}_2\left(\mu^\sigma P_t^N\,;\,\mu^\sigma P_t\right)\right\}<\frac{\epsilon}{2}\,.$$

The potentials V and F being convex at infinity, the results in [BBCG08] imply the existence of a positive constant C_N such that Diffusion (III) satisfies a Poincaré inequality with constant C_N. Consequently, for any $\epsilon>0$, there exists $T_N>0$ such that

$$\mathbb{W}_2\left(\mu_0P_t^N\,;\,\mu^\sigma P_t^N\right)<\frac{\epsilon}{2}$$

if $t\geq T_N$. This achieves the proof. □

2 Functional inequality

Bolley, Gentil and Guillin suggested a method to obtain a WJ inequality in the non-convex case. But, we proceed in a slightly different way.
In the following, μ^σ denotes an invariant probability of Diffusion (II). We know that such a diffusion exists, see Proposition 2.1 in [Tug11]. Moreover, the measure satisfies the following implicit equation

$$\mu^\sigma(dx):=\frac{\exp\left\{-\frac{2}{\sigma^2}W^\sigma(x)\right\}}{\int_{\mathbb{R}^d}\exp\left\{-\frac{2}{\sigma^2}W^\sigma(y)\right\}dy}\,dx$$

with $W^\sigma(x):=V(x)+F*\mu^\sigma(x)$. Let us give now a WJ-inequality on the measure μ^σ.

Proposition 2.1. *Under Assumption (A), the measure μ^σ satisfies a $WJ_{V_0,0}$-inequality with the constant*

$$\begin{aligned}&\mathcal{C}^\sigma:=\max_{R>0}\,\mathcal{C}^\sigma(R)>0\\ &\textit{with}\quad\mathcal{C}^\sigma(R):=\min\left\{\frac{K(R)}{3}\,;\,\frac{\sigma^2}{72R^2}e^{-\frac{2}{\sigma^2}S(R)}\,;\,\frac{K(R)}{3}\frac{3^d-2^d}{2^d}e^{\frac{2}{\sigma^2}(I(R)-S(R))}\right\},\\ &K(R):=\inf_{|x|\geq R}\nabla^2V_0(x)\,,\;I(R):=\inf_{|x|\leq 2R}W^\sigma(x)\\ &\textit{and}\quad S(R):=\sup_{|x|\leq 3R}W^\sigma(x)\,.\end{aligned}$$

The proof is left to the reader and consists in a simple adaptation of Section 5 in [BGG12a]. This proposition implies a first result about the long-time convergence of McKean-Vlasov diffusion.

Remark 2.2. *Under the hypotheses of [CGM08], there is a unique stationary measure μ_∞ and for any μ_0 with finite entropy, there exists $\lambda > 0$ such that $\mathbb{W}_2^2(\mu_t\,;\,\mu_\infty) \leq e^{-\lambda t}\mathbb{W}_2^2(\mu_0\,;\,\mu_\infty)$.*

In [CGM08], the authors prove a uniform propagation of chaos to obtain a long-time convergence.
From Proposition 2.1, we also deduce the following corollary which is central in the section.

Corollary 2.3. *Under Assumption (A), μ^σ satisfies the following inequality*

$$(\mathcal{C}^\sigma - \alpha - \vartheta)\,\mathbb{W}_2^2(\mu_t\,;\,\mu^\sigma) \leq J_{V,F}(\mu_t \mid \mu^\sigma)\,, \tag{2.1}$$

for any measure μ_0 absolutely continuous with respect to the Lebesgue measure and with finite entropy.
Particularly, if $\mathcal{C}^\sigma - \alpha - \vartheta > 0$, Diffusion (II) *admits a unique stationary measure μ^σ and for any measure μ_0 which satisfies Assumption (A), we have*

$$\mathbb{W}_2(\mu_t\,;\,\mu^\sigma) \leq \exp\left[-(\mathcal{C}^\sigma - \alpha - \vartheta)\,t\right]\mathbb{W}_2(\mu_0\,;\,\mu^\sigma)\,. \tag{2.2}$$

Proof. By Proposition 2.1, we have

$$\mathcal{C}^\sigma\mathbb{W}_2^2(\mu_t\,;\,\mu^\sigma) \leq J_{V_0,0}(\mu_t \mid \mu^\sigma)\,. \tag{2.3}$$

However, by definition, the quantity $J_{V,F}(\mu_t \mid \mu^\sigma)$ is equal to

$$\begin{aligned} J_{V,F}(\mu_t \mid \mu^\sigma) =& J_{V_0,F_0}(\mu_t \mid \mu^\sigma) - \vartheta\int_{\mathbb{R}^d} ||\nabla\phi_t(x) - x||^2\,\mu^\sigma(dx) \\ &- \frac{\alpha}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} ||(\nabla\phi_t(x) - \nabla\phi_t(y)) - (x-y)||^2\,\mu^\sigma(dx)\mu^\sigma(dy) \\ \geq& J_{V_0,0}(\mu_t \mid \mu^\sigma) - (\alpha+\vartheta)\int_{\mathbb{R}^d} ||\nabla\phi_t(x) - x||^2\,\mu^\sigma(dx) \end{aligned}$$

which with (2.3) gives (2.1). Here, the convex function ϕ_t is defined by $\mu_t = \nabla\phi_t\#\mu^\sigma$.
The uniqueness of the stationary measure if $\mathcal{C}^\sigma - \alpha - \vartheta > 0$ and the exponential decay in (2.2) are given by Proposition 1.4. □

Let us note that the inequality

$$\inf_{\mathbb{R}^d}\nabla^2 V > \sup_{\mathbb{R}^d} -\nabla^2 F$$

implies the uniqueness of the stationary measure μ^σ and the exponential convergence toward μ^σ for any $\sigma > 0$. Such a result has already been proven in

[CMV03].
Reciprocally, the inequality

$$\inf_{\mathbb{R}^d} \nabla^2 F > \sup_{\mathbb{R}^d} -\nabla^2 V$$

is not sufficient to ensure the uniqueness and the convergence result. We need also the center of mass to be fixed, see [Tug10, CMV03].

Theorem 2.4. *We assume the set of Hypotheses (A) and also $p > n$.*
Thus, there exists σ_c such that $\mathcal{C}^\sigma > \alpha + \vartheta$ for any $\sigma \geq \sigma_c$. Consequently, if σ is large enough, Diffusion (II) *admits a unique invariant probability μ^σ. Moreover, it satisfies the exponential decay* (2.2) *for any measure μ_0 with finite entropy.*

Proof. In order to prove it, we first admit the following limit

$$\lim_{\sigma \to +\infty} \frac{1}{\sigma^2} \int_{\mathbb{R}^d} ||x||^{2n} \, \mu^\sigma(dx) = 0 \tag{2.4}$$

for a family $\{\mu^\sigma \, ; \, \sigma \geq 1\}$, μ^σ being any invariant probability of Diffusion (II). In a first step, we prove that Limit (2.4) implies the statement of Theorem 2.4. In a second step, we prove (2.4).

Step 1. We admit the limit (2.4). We remind the reader the following equality

$$W^\sigma(x) = V(x) + F * \mu^\sigma(x) \, .$$

Moreover, the hypotheses on F imply

$$|F * \mu^\sigma(x)| \leq C \left(1 + ||x||^{2n}\right) \left(1 + \int_{\mathbb{R}^d} ||y||^{2n} \, \mu^\sigma(dy)\right)$$

so that we have the limit

$$\lim_{\sigma \to +\infty} \frac{1}{\sigma^2} \sup_{||x|| \leq 3R} ||W^\sigma(x)|| = 0 \, .$$

Thus, for any $R > 0$, the quantities $\exp\left[-\frac{2}{\sigma^2} S(R)\right]$ and $\exp\left[\frac{2}{\sigma^2} \left(I(R) - S(R)\right)\right]$ go to 1 as σ goes to infinity. We remind the reader that $I(R)$ and $S(R)$ are defined in Proposition 2.1. We obtain the following limit for any $R > 0$:

$$\lim_{\sigma \to \infty} \mathcal{C}^\sigma(R) = \frac{K(R)}{3} \min\left\{1 \, ; \, \frac{3^d - 2^d}{2^d}\right\} ,$$

where $K(R) = \inf_{||x|| \geq R} \nabla^2 V_0(x)$. By assumption, the quantity $K(R)$ goes to infinity as R goes to infinity. We take R_0 such that

$$\frac{K(R_0)}{3} \min\left\{1 \, ; \, \frac{3^d - 2^d}{2^d}\right\} > 2(\alpha + \vartheta) \, .$$

Then, we take σ_c large enough such that $\mathcal{C}^\sigma(R_0) > \frac{1}{2}\lim_{\xi\to\infty}\mathcal{C}^\xi(R_0)$ for any $\sigma \geq \sigma_c$. Thus, we have the inequality

$$\mathcal{C}^\sigma \geq \mathcal{C}^\sigma(R_0) > (\alpha + \vartheta)$$

for any $\sigma \geq \sigma_c$. Consequently, if Limit (2.4) is satisfied, the statement of the theorem is proved.

Step 2. We now achieve the proof by establishing Limit (2.4). *It is in this step that we use the hypothesis $p > n$.* We proceed a *reducto ad absurdum*. Let us assume the existence of a positive constant C and an increasing sequence $(\sigma_k)_{k\in\mathbb{N}}$ which goes to infinity such that for any $k \in \mathbb{N}$, Diffusion (II) admits an invariant probability μ^{σ_k} satisfying

$$\eta_{2n}(k) := \int_{\mathbb{R}^d} ||x||^{2n}\, \mu^{\sigma_k}(dx) \geq C\sigma_k^2\,.$$

In particular, we deduce that the sequence $(\eta_{2n}(k))_{k\in\mathbb{N}}$ goes to infinity as k goes to infinity. Since μ^{σ_k} is a stationary measure, we have

$$\eta_{2n}(k) = \frac{\int_{\mathbb{R}^d} ||x||^{2n} \exp\left\{-\frac{2}{\sigma_k^2}\left[V(x) + F * \mu^{\sigma_k}(x)\right]\right\} dx}{\int_{\mathbb{R}^d} \exp\left\{-\frac{2}{\sigma_k^2}\left[V(x) + F * \mu^{\sigma_k}(x)\right]\right\} dx}\,.$$

By making the transformation $x := (\eta_{2n}(k))^{\frac{1}{2n}}\, y$, we obtain

$$1 = \frac{\int_{\mathbb{R}^d} ||y||^{2n} \exp\left\{-\frac{2}{\widehat{\sigma_k}^2}\left[\frac{V\left((\eta_{2n}(k))^{\frac{1}{2n}}\, y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}} + \frac{F*\mu^{\sigma_k}\left((\eta_{2n}(k))^{\frac{1}{2n}}\, y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}}\right]\right\} dy}{\int_{\mathbb{R}^d} \exp\left\{-\frac{2}{\widehat{\sigma_k}^2}\left[\frac{V\left((\eta_{2n}(k))^{\frac{1}{2n}}\, y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}} + \frac{F*\mu^{\sigma_k}\left((\eta_{2n}(k))^{\frac{1}{2n}}\, y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}}\right]\right\} dy} \tag{2.5}$$

with $\widehat{\sigma_k} := \frac{\sigma_k}{\sqrt{\eta_{2n}(k)}}\,(\eta_{2n}(k))^{-\frac{p-n}{2n}} \leq \frac{1}{\sqrt{C}}\,(\eta_{2n}(k))^{-\frac{p-n}{2n}} \to 0$ as k goes to infinity.

By hypotheses, we have the two following limits for any $y \in \mathbb{R}^d$:

$$\lim_{k\to+\infty} \frac{F * \mu^{\sigma_k}\left((\eta_{2n}(k))^{\frac{1}{2n}}\, y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}} = 0 \quad \text{and} \quad \lim_{k\to+\infty} \frac{V\left((\eta_{2n}(k))^{\frac{1}{2n}}\, y\right)}{(\eta_{2n}(k))^{\frac{p}{n}}} = \Theta(y)\,,$$

the function Θ being strictly convex and such that $\Theta(y) > \Theta(0) = 0$ for any $y \neq 0$. Consequently, by applying Lemma A.2 in [Tug11], the right hand term in (2.5) goes to 0 as k goes to infinity. Nevertheless, the left hand term is equal to 1. The initial assumption of Step 2 is absurd. This achieves the proof. □

Let us remark that Theorem 2.4 goes further than the results in [Tug13] concerning the uniqueness of the invariant probability for sufficiently large σ. Moreover, it provides, with Corollary 2.3 a method for simulating a lower-bound of the critical value above which there is a unique invariant probability. Nevertheless, this method needs more computation than those described in [Tug13].

3 Main results

In this paragraph, we prove the main results. We remind the reader the work of Cattiaux, Guillin and Malrieu. But also, we know that there is not a uniform propagation of chaos since there are several stationary measures. Here, we prove that there is this propagation of chaos uniform with sufficiently large σ.

In all this section, we assume the following inequality (simple propagation of chaos):

$$\sup_{0\leq t\leq T} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) \leq \frac{\exp\left[\psi(T)\right]}{N}\,,$$

for any $T>0$. Here, the function ψ is nondecreasing. Let us remark that this inequality is satisfied thanks to the classical coupling result

$$\sup_{0\leq t\leq T} \mathbb{E}\left\{\left|\left|X_t - X_t^{1,N}\right|\right|^2\right\} \leq \lambda\frac{\exp\left[CT\right]}{N}\,.$$

In this case, we can write $\psi(T)=CT+\log(\lambda)$.

Theorem 3.1. *Under the Hypotheses of Theorem 2.4, if $\sigma\geq\sigma_c$, we have the uniform propagation of chaos. In other words, for any measure μ_0 with finite entropy, we have the limit*

$$\lim_{N\to+\infty}\sup_{t\geq 0}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right)=0\,. \tag{3.1}$$

Moreover, we can compute the rate of convergence. By $C(\sigma)$, we denote the expression $\mathcal{C}^\sigma-\alpha-\vartheta$.

First case: *The quantity $\frac{C(\sigma)t}{\psi(t)}$ goes to $\lambda\in\mathbb{R}_+^*\bigcup\{+\infty\}$ as t goes to infinity. Thus, for all $\delta>0$, we have:*

$$\lim_{N\to+\infty} N^{\frac{1}{1+1/\lambda}-\delta}\sup_{t\geq 0}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right)=0\,. \tag{3.2}$$

Second case: *The quantity $\frac{C(\sigma)t}{\psi(t)}$ goes to 0 as t goes to infinity. Thus, for all $\delta>0$, we have:*

$$\lim_{N\to+\infty}\exp\left\{C(\sigma)\psi^{-1}\left[(1-\delta)\log(N)\right]\right\}\sup_{t\geq 0}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right)=0\,. \tag{3.3}$$

Proof. **Step 1.** Let t be a positive real. The idea is to consider the nonlinear diffusion (II) starting with law $\mu_t^{1,N}$. In other words, let $X_0^{t,N}$ be a random variable with law $\mu_t^{1,N}$. We look at the diffusion

$$\left\{\begin{array}{l} X_s^{t,N}=X_0^{t,N}+\sigma\left(B_{t+s}-B_t\right)-\int_0^s\left(\nabla V+\nabla F*\mu_r^{t,N}\right)\left(X_r^{t,N}\right)dr\,,\\ \mu_r^{t,N}=\mathcal{L}\left(X_r^{t,N}\right)\,.\end{array}\right. \tag{3.4}$$

Let T be a positive real. The law at time T of Diffusion (3.4) is $\mu_t^{1,N}P_T$.

Step 2. The triangular inequality implies

$$\begin{aligned}\mathbb{W}_2\left(\mu_{T+t}\,;\,\mu_{T+t}^{1,N}\right) \leq & \mathbb{W}_2\left(\mu_{T+t}\,;\,\mu^{\sigma}\right) + \mathbb{W}_2\left(\mu^{\sigma}\,;\,\mu_t^{1,N}P_T\right) \\ & + \mathbb{W}_2\left(\mu_t^{1,N}P_T\,;\,\mu_{T+t}^{1,N}\right).\end{aligned}$$

Step 3. The last term is bounded by applying the simple propagation of chaos:

$$\begin{aligned}\mathbb{W}_2\left(\mu_t^{1,N}P_T\,;\,\mu_{T+t}^{1,N}\right) &= \mathbb{W}_2\left(\mu_t^{1,N}P_T\,;\,\mu_t^{1,N}P_T^N\right) \\ &\leq \frac{\exp\left[\psi(T)\right]}{N}.\end{aligned}$$

The first term can be bounded like so:

$$\mathbb{W}_2\left(\mu_{T+t}\,;\,\mu^{\sigma}\right) \leq e^{-C(\sigma)(T+t)}\mathbb{W}_2\left(\mu_0\,;\,\mu^{\sigma}\right).$$

We proceed in a similar way with the second term

$$\begin{aligned}\mathbb{W}_2\left(\mu^{\sigma}\,;\,\mu_t^{1,N}P_T\right) &\leq e^{-C(\sigma)T}\mathbb{W}_2\left(\mu^{\sigma}\,;\,\mu_t^{1,N}\right) \\ &\leq e^{-C(\sigma)T}\left(\mathbb{W}_2\left(\mu^{\sigma}\,;\,\mu_t\right) + \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right)\right) \\ &\leq e^{-C(\sigma)(T+t)}\mathbb{W}_2\left(\mu^{\sigma}\,;\,\mu_0\right) + e^{-C(\sigma)T}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right).\end{aligned}$$

Consequently, we have:

$$\begin{aligned}\mathbb{W}_2\left(\mu_{T+t}\,;\,\mu_{T+t}^{1,N}\right) \leq & e^{-C(\sigma)T}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) + 2e^{-C(\sigma)(T+t)}\mathbb{W}_2\left(\mu_0\,;\,\mu^{\sigma}\right) \\ & + \frac{\exp\left[\psi(T)\right]}{N}.\end{aligned}$$

We now take the supremum for t running between $(k-1)T$ and kT and we obtain

$$\begin{aligned}\sup_{kT\leq t\leq (k+1)T}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) \leq & e^{-C(\sigma)T}\sup_{(k-1)T\leq t\leq kT}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) \\ & + 2e^{-C(\sigma)kT}\mathbb{W}_2\left(\mu_0\,;\,\mu^{\sigma}\right) + \frac{\exp\left[\psi(T)\right]}{N}.\end{aligned} \tag{3.5}$$

We denote $\lambda_k(T) := \sup_{kT\leq t\leq (k+1)T}\mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right)$ and $\gamma := 2\mathbb{W}_2\left(\mu_0\,;\,\mu^{\sigma}\right)$. The inequality (3.5) can be written in the following way:

$$\lambda_k(T) \leq e^{-C(\sigma)T}\lambda_{k-1}(T) + \gamma e^{-C(\sigma)kT} + \frac{\exp\left[\psi(T)\right]}{N}.$$

Step 4. By elementary computations, we have:

$$\lambda_k(T) \leq \frac{1}{1-e^{-C(\sigma)T}}\frac{\exp\left[\psi(T)\right]}{N} + k\gamma e^{-C(\sigma)kT}.$$

By taking $T > \frac{1}{C(\sigma)}$, the sequence $\left(ke^{-C(\sigma)kT}\right)_{k\geq 1}$ is decreasing. Thus, we have $\sup_{k\geq 0} ke^{-C(\sigma)kT} = e^{-C(\sigma)T}$. Consequently, we deduce

$$\sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) = \sup_{k\geq 0} \lambda_k(T) \leq \frac{\exp\left[\psi(T)\right]}{1-e^{-C(\sigma)T}} \frac{1}{N} + \gamma e^{-C(\sigma)T}\,. \tag{3.6}$$

Let $\epsilon > 0$ be arbitrarily small. We take $T > \frac{1}{C}\log\left(\frac{2\gamma}{\epsilon}\right)$ so that $\gamma e^{-C(\sigma)T} < \frac{\epsilon}{2}$. Then, by taking N large enough, we have $\frac{\exp[\psi(T)]}{1-e^{-C(\sigma)T}}\frac{1}{N} < \frac{\epsilon}{2}$. This implies $\sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) < \epsilon$ if N is large enough. This proves Limit (3.1).

Step 5. We now prove the rate of convergence result. Let $\delta > 0$ be arbitrarily small.

Step 5.1. We look at the first case. Inequality (3.6) holds for any $T > 0$. We take $T_N := \frac{1}{C(\sigma)}\left(\frac{1}{1+1/\lambda}\log(N)\right)$. We immediately deduce $\frac{1}{1-e^{-C(\sigma)T_N}} \leq 2$ for N large enough. We thus have:

$$\frac{\exp\left[\psi\left(T_N\right)\right]}{N}\frac{1}{1-e^{-C(\sigma)T_N}} \leq 2\frac{\exp\left[\frac{1}{1+1/\lambda}\log(N)\frac{\psi(T_N)}{C(\sigma)T_N}\right]}{N}\,.$$

For N large enough, the quantity $\frac{\psi(T_N)}{C(\sigma)T_N}$ is less than $\frac{1}{\lambda} + \frac{\delta}{2}\left(1+\frac{1}{\lambda}\right)$ so that the quantity $\frac{\exp[\psi(T_N)]}{N}$ is less than $N^{-(\frac{1}{1+1/\lambda}-\frac{\delta}{2})}$. We deduce

$$\lim_{N\to+\infty} N^{\frac{1}{1+1/\lambda}-\delta}\frac{\exp\left[\psi\left(T_N\right)\right]}{1-e^{-\varphi(T_N)}}\frac{1}{N} = 0\,.$$

The second term, that is $\gamma e^{-C(\sigma)T_N}$, is equal to $\gamma N^{\frac{1}{1+1/\lambda}}$ so

$$\lim_{N\to+\infty} N^{\frac{1}{1+1/\lambda}-\delta}\vartheta e^{-C(\sigma)T_N} = 0\,.$$

This achieves the proof of Limit (3.2).

Step 5.2. We now look at the second case. Here, we obtain

$$\sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) \leq \frac{N^{-\frac{\delta}{2}}}{1-e^{-C(\sigma)T_N}} + \gamma e^{-C(\sigma)\psi^{-1}\left(\left(1-\frac{\delta}{2}\right)\log(N)\right)}\,,$$

by taking $T_N := \psi^{-1}\left(\left(1-\frac{\delta}{2}\right)\log(N)\right)$. This implies Limit (3.3). □

The second case, that is to say when $\lim_{t\to+\infty}\frac{\psi(t)}{t} = +\infty$, does not hold with McKean-Vlasov diffusion. However, the current work aims to be applied for more general diffusions.

This uniform propagation of chaos has a consequence on the interacting particle system. Let us denote by $\left(\mathcal{P}_t^N\right)_{t\geq 0}$ the semi-group associated to Diffusion (III). Consequently, we have

$$\mathcal{L}\left(X_t^1,\cdots,X_t^N\right) = \mathcal{L}\left(X_0^1,\cdots,X_0^N\right)\mathcal{P}_t^N\,.$$

We recall that, in the current work, $\mathcal{L}\left(X_0^1,\cdots,X_0^N\right) = \mu_0^{\otimes N}$.

Corollary 3.2. *Under the hypotheses of Theorem 3.1, for any measures μ_0 and ν_0 with finite entropy, we have the following inequality for any $N \in \mathbb{N}^*$ and $t \geq 0$:*

$$\mathbb{W}_2\left(\mu_0 P_t^N \,;\, \nu_0 P_t^N\right) \leq e^{-C(\sigma)t}\mathbb{W}_2\left(\mu_0 \,;\, \nu_0\right) + 2\eta(N)$$

with $\lim_{N\to+\infty}\eta(N) = 0$. *Moreover, we have*

$$\mathbb{W}_2\left(\mu_0^{\otimes N}\mathcal{P}_t^N \,;\, \nu_0^{\otimes N}\mathcal{P}_t^N\right) \leq e^{-C(\sigma)t}\mathbb{W}_2\left(\mu_0 \,;\, \nu_0\right) + 2\eta(N)\,.$$

It is a simple application of Theorem 3.1.

We remind the reader that in [CGM08] (Theorem 3.2), the authors obtain a uniform propagation of chaos of the form

$$\sup_{t\geq 0}\mathbb{E}\left\{\left|\left|X_t - X_t^{1,N}\right|\right|^2\right\} \leq \frac{K}{N^{-(1-\rho)}}\,,$$

with $0 < \rho < 1$. However, by using a method similar to the one of the proof of Theorem 3.1, we obtain a better inequality with the Wasserstein distance.

Corollary 3.3. *Under the hypotheses of Theorem 3.2 in [CGM08], for any $\sigma > 0$, we have the following uniform propagation of chaos result:*

$$\lim_{N\to\infty} N^{1-\delta}\sup_{t\geq 0}\mathbb{W}_2\left(\mu_t \,;\, \mu_t^{1,N}\right) = 0$$

for any $0 < \delta < 1$.

Proof. By proceeding exactly like in Lemma 5.4 in [BRTV98], there exists $K > 0$ such that the following inequality holds:

$$\sup_{0\leq t\leq T}\mathbb{E}\left\{\left|\left|X_t - X_t^{1,N}\right|\right|^2\right\} \leq \frac{KT^2}{N} \tag{3.7}$$

for any $T > 0$. Here, the unique difference is the presence of a confining potential but since this potential is convex, we can proceed similarly. Inequality (3.7) implies that $\psi(T) = \log(K) + 2\log(T)$.

Now, since $\alpha + \vartheta > 0$, any invariant probability μ^σ satisfies a $WJ_{V,F}$-inequality with a constant $C(\sigma) > 0$. Consequently, we have

$$\lim_{t\to+\infty}\frac{C(\sigma)t}{\psi(t)} = +\infty\,.$$

We apply Theorem 3.1 and we obtain the statement of Corollary 3.3 for any $\delta > 0$. □

The rate that we obtain is better. However, the coupling that we proceed is not necessary with the same Brownian Motions.

In Theorem 3.1, we remark that the stationary measure plays a particular role in the WJ-inequality. Consequently, we can obtain a better result if we start from $\mu_0 = \mu^\sigma$, the unique stationary measure.

Corollary 3.4. *Under the Hypotheses of Theorem 3.1, if $\mathcal{C}^\sigma > \alpha + \vartheta$, we have the following uniform propagation of chaos result if we start from $\mu_0 = \mu^\sigma$:*

$$\sup_{t\geq 0} \mathbb{W}_2\left(\mu^\sigma\,;\,\mu^\sigma P_t^N\right) \leq \frac{K}{N}\,,$$

K being a positive constant.

Proof. We recall Inequality (3.6).

$$\sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) \leq \frac{\exp\left[\psi(T)\right]}{1-e^{-CT}}\frac{1}{N} + \gamma e^{-CT}\,.$$

Here, $\gamma = 2\mathbb{W}_2\left(\mu_0\,;\,\mu^\sigma\right) = 0$. Thus, we immediately obtain

$$\sup_{t\geq 0} \mathbb{W}_2\left(\mu^\sigma\,;\,\mu^\sigma P_t^N\right) \leq \frac{\exp\left[\psi(T)\right]}{1-e^{-CT}}\frac{1}{N}\,,$$

for any $T > 0$. □

We stress that the contraction which holds in Assumption (A) links the measure μ_t with the invariant probability μ^σ. But we do not have any contraction between $\mu_0 P_t$ and $\nu_0 P_t$. Let us note that if we had the contraction inequality

$$\mathbb{W}_2\left(\mu_t\,;\,\nu_t\right) \leq e^{-Ct}\mathbb{W}_2\left(\mu_0\,;\,\nu_0\right)\,,$$

we would have the following uniform propagation of chaos

$$\sup_{t\geq 0} \mathbb{W}_2\left(\mu_t\,;\,\mu_t^{1,N}\right) \leq \frac{K}{N}\,,$$

K being a positive constant. However, To obtain such an inequality requires the strictly uniform convexity of V, see [SvR05]. And, in this case, we directly have the uniform propagation of chaos without using the convergence.

Remark 3.5. *We can extend all of these results to any other non-linear models like* (I) *provided that we have the simple propagation of chaos between the interacting particle system and the hydrodynamical limit.*

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