SEMICIRCLE LAW FOR A CLASS OF RANDOM MATRICES WITH DEPENDENT ENTRIES

F. GÖTZE, A. NAUMOV, AND A. TIKHOMIROV

Abstract. In this paper we study ensembles of random symmetric matrices $\mathbf{X}_n = \{X_{ij}\}_{i,j=1}^n$ with a random field type dependence, such that $\mathbb{E}\, X_{ij} = 0$, $\mathbb{E}\, X_{ij}^2 = \sigma_{ij}^2$, where σ_{ij} can be different numbers. Assuming that the average of the normalized sums of variances in each row converges to one and Lindeberg condition holds true we prove that the empirical spectral distribution of eigenvalues converges to Wigner's semicircle law.

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1. Introduction

Let $X_{jk}, 1 \le j \le k < \infty$, be triangular array of random variables with $\mathbb{E}\, X_{jk} = 0$ and $\mathbb{E}\, X_{jk}^2 = \sigma_{jk}^2$, and let $X_{jk} = X_{kj}$ for $1 \le j < k < \infty$. We consider the random matrix

$$\mathbf{X}_n = \{X_{jk}\}_{j,k=1}^n.$$

Denote by $\lambda_1 \le ... \le \lambda_n$ eigenvalues of matrix $n^{-1/2}\mathbf{X}_n$ and define its spectral distribution function by

$$\mathcal{F}^{\mathbf{X}_n}(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}(\lambda_i \le x),$$

where $\mathbb{1}(B)$ denotes the indicator of an event B. We set $F^{\mathbf{X}_n}(x) := \mathbb{E}\,\mathcal{F}^{\mathbf{X}_n}(x)$. Let $g(x)$ and $G(x)$ denote the density and the distribution function of the standard semicircle law

$$g(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}(|x| \le 2), \quad G(x) = \int_{-\infty}^x g(u)du.$$

For matrices with independent identically distributed (i.i.d.) elements, which have moments of all orders, Wigner proved in [**?**] that F_n converges to $G(x)$, later on called "Wigner's semicircle law". The result has been extended in various aspects, i.e. by Arnold in [**?**]. In the non-i.i.d. case Pastur, [**?**], showed that Lindeberg's condition is sufficient for the convergence. In [**?**] Götze and Tikhomirov proved the semicircle law for matrices satisfying martingale-type conditions for the entries.

In the majority of previous papers it has been assumed that σ_{ij}^2 are equal for all $1 \le i < j \le n$. Recently Erdős, Yau and Yin and al. study ensembles of symmetric random matrices with independent elements which satisfy $n^{-1}\sum_{j=1}^n \sigma_{ij}^2 = 1$ for all $1 \le i \le n$. See for example the survey of results in [**?**].

In this paper we study a class of random matrices with dependent entries and show that limiting distribution for $F^{\mathbf{X}_n}(x)$ is given by Wigner's semicircle law. We do not assume that the variances are equal.

Introduce the σ-algebras

$$\mathfrak{F}^{(i,j)} := \sigma\{X_{kl} : 1 \le k \le l \le n, (k,l) \ne (i,j)\}, 1 \le i \le j \le n.$$

For any $\tau > 0$ we introduce Lindeberg's ratio for random matrices as

$$L_n(\tau) := \frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}\,|X_{ij}|^2\,\mathbb{1}(|X_{ij}| \ge \tau\sqrt{n}).$$

We assume that the following conditions hold

$$\mathbb{E}(X_{ij}|\mathfrak{F}^{(i,j)}) = 0; \tag{1.1}$$

$$\frac{1}{n^2}\sum_{i,j=1}^{n} \mathbb{E}\,|\,\mathbb{E}(X_{ij}^2|\mathfrak{F}^{(i,j)}) - \sigma_{ij}^2| \to 0 \text{ as } n \to \infty; \tag{1.2}$$

$$\text{for any fixed } \tau > 0 \quad L_n(\tau) \to 0 \text{ as } n \to \infty. \tag{1.3}$$

Furthermore, we will use condition (1.3) not only for the matrix $\mathbf{X}_n$, but for other matrices as well, replacing X_{ij} in the definition of Lindeberg's ratio by corresponding elements.

For all $1 \le i \le n$ let $B_i^2 := \frac{1}{n}\sum_{j=1}^{n} \sigma_{ij}^2$. We need to impose additional conditions on the variances σ_{ij}^2 given by

$$\frac{1}{n}\sum_{i=1}^{n} |B_i^2 - 1| \to 0 \text{ as } n \to \infty; \tag{1.4}$$

$$\max_{1\le i\le n} B_i \le C, \tag{1.5}$$

where C is some absolute constant.

Remark. It is easy to see that the conditions (1.4) and (1.5) follow from the following condition

$$\max_{1\le i\le n} \left|B_i^2 - 1\right| \to 0 \text{ as } n \to \infty. \tag{1.6}$$

The main result of the paper is the following theorem

Theorem 1.1. *Let $\mathbf{X}_n$ satisfy conditions* (1.1)–(1.5). *Then*

$$\sup_x |F^{\mathbf{X}_n}(x) - G(x)| \to 0 \text{ as } n \to \infty.$$

Let us fix i, j. It is easy to see that for all $(k, l) \ne (i, j)$

$$\mathbb{E}\, X_{ij}X_{kl} = \mathbb{E}\,\mathbb{E}(X_{ij}X_{kl}|\mathfrak{F}^{(i,j)})) = \mathbb{E}\, X_{kl}\,\mathbb{E}(X_{ij}|\mathfrak{F}^{(i,j)}) = 0.$$

Hence the elements of the matrix $\mathbf{X}_n$ are uncorrelated. If we additionally assume that the elements of the matrix $\mathbf{X}_n$ are independent random variables then conditions (1.1) and (1.2) are automatically satisfied. The following theorem follows immediately from Theorem 1.1 in the case when the matrix $\mathbf{X}_n$ has independent entries.

Theorem 1.2. *Assume that the elements X_{ij} of the matrix $\mathbf{X}_n$ are independent for all $1 \le i \le j \le n$ and $\mathbb{E}\, X_{ij} = 0$, $\mathbb{E}\, X_{ij}^2 = \sigma_{ij}^2$. Assume that $\mathbf{X}_n$ satisfies conditions* (1.3)–(1.5). *Then*

$$\sup_x |F^{\mathbf{X}_n}(x) - G(x)| \to 0 \text{ as } n \to \infty.$$

The following example illustrates that without condition (1.4) convergence to Wigner's semicircle law doesn't hold.

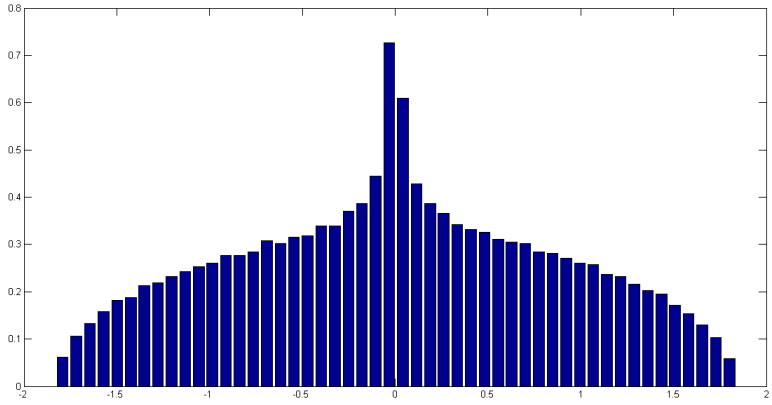

FIGURE 1. Spectrum of matrix $\mathbf{X}_n$.

Example. Let $\mathbf{X}_n$ denote a block matrix

$$\mathbf{X}_n = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{pmatrix},$$

where $\mathbf{A}$ is $m \times m$ symmetric random matrix with Gaussian elements with zero mean and unit variance, $\mathbf{B}$ is $m \times (n-m)$ random matrix with i.i.d. Gaussian elements with zero mean and unit variance. Furthermore, let $\mathbf{D}$ be a $(n-m) \times (n-m)$ diagonal matrix with Gaussian random variables on the diagonal with zero mean and unit variance. If we set $m := n/2$ then it is not difficult to check that condition (1.4) doesn't hold. We simulated the spectrum of the matrix $\mathbf{X}_n$ and illustrated a limiting distribution on Figure 1.

Remark. We conjecture that Theorem 1.1 (Theorem 1.2 respectively) holds without assumption (1.5).

Define the Levy distance between the distribution functions F_1 and F_2 by

$$L(F_1, F_2) = \inf\{\varepsilon > 0 : F_1(x-\varepsilon) - \varepsilon \le F_2(x) \le F_1(x+\varepsilon) + \varepsilon\}.$$

The following theorem formulates the Lindeberg's universality scheme for random matrices.

Theorem 1.3. *Let $\mathbf{X}_n, \mathbf{Y}_n$ denote independent symmetric random matrices with $\mathbb{E}\, X_{ij} = \mathbb{E}\, Y_{ij} = 0$ and $\mathbb{E}\, X_{ij}^2 = \mathbb{E}\, Y_{ij}^2 = \sigma_{ij}^2$. Suppose that the matrix $\mathbf{X}_n$ satisfies conditions* (1.1)–(1.4)*, and the matrix $\mathbf{Y}_n$ has independent Gaussian elements. Additionally assume that for the matrix $\mathbf{Y}_n$ conditions* (1.3)–(1.4) *hold. Then*

$$L(F^{\mathbf{X}_n}(x), F^{\mathbf{Y}_n}(x)) \to 0 \text{ as } n \to \infty.$$

In view of Theorem 1.3 to prove Theorem 1.1 it remains to show convergence to semicircle law in the Gaussian case.

Theorem 1.4. *Assume that the entries Y_{ij} of the matrix $\mathbf{Y}_n$ are independent for all $1 \le i \le j \le n$ and have Gaussian distribution with $\mathbb{E}\, Y_{ij} = 0$, $\mathbb{E}\, Y_{ij}^2 = \sigma_{ij}^2$. Assume that conditions (1.3)–(1.5) are satisfied. Then*

$$\sup_x |F^{\mathbf{Y}_n}(x) - G(x)| \to 0 \text{ as } n \to \infty.$$

For related ensembles of random covariance matrices it is well known that spectral distribution function of eigenvalues converges to the Marchenko–Pastur law. In this case Götze and Tikhomirov in [**?**] received similar results to [**?**]. Recently Adamczak, [**?**], proved the Marchenko–Pastur law for matrices with martingale structure. He assumed that the matrix elements have moments of all orders and imposed conditions similar to (1.4). Another class of random matrices with dependent entries was considered in [**?**] by O'Rourke. In a forthcoming paper we prove analogs of these theorems for random covariance matrices.

The paper organized as follows. In Section 2 we give a proof of Theorem 1.3 using the method of Stieltjes transforms. In Section 3 we prove Theorem 1.4 by the classical moment method.

Throughout this paper we assume that all random variables are defined on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathrm{Tr}(\mathbf{A})$ denote the trace of a matrix $\mathbf{A}$. For a vector $x = (x_1, ..., x_n)$ let $||x||_2 := (\sum_{i=1}^n x_i^2)^{1/2}$. We denote the operator norm of the matrix $\mathbf{A}$ by $||\mathbf{A}|| := \sup_{||x||_2=1} ||\mathbf{A}x||_2$. We will write $a \le_m b$ if there is an absolute constant C depending on m only such that $a \le Cb$.

From now on we shall omit the index n in the notation for random matrices.

2. Proof of Theorem 1.3

We denote the Stieltjes transforms of $F^{\mathbf{X}}$ and $F^{\mathbf{Y}}$ by $S^{\mathbf{X}}(z)$ and $S^{\mathbf{Y}}(z)$ respectively. Due to the relations between distribution functions and Stieltjes transforms, the statement of Theorem 1.3 will follow from

$$|S^{\mathbf{X}}(z) - S^{\mathbf{Y}}(z)| \to 0 \text{ as } n \to \infty. \tag{2.1}$$

Set

$$\mathbf{R}^{\mathbf{X}}(z) := \left(\frac{1}{\sqrt{n}}\mathbf{X} - z\mathbf{I}\right)^{-1} \text{ and } \mathbf{R}^{\mathbf{Y}}(z) := \left(\frac{1}{\sqrt{n}}\mathbf{Y} - z\mathbf{I}\right)^{-1}. \tag{2.2}$$

By definition

$$S^{\mathbf{X}}(z) = \frac{1}{n}\,\mathrm{Tr}\,\mathbb{E}\,\mathbf{R}^{\mathbf{X}}(z) \text{ and } S^{\mathbf{Y}}(z) = \frac{1}{n}\,\mathrm{Tr}\,\mathbb{E}\,\mathbf{R}^{\mathbf{Y}}(z).$$

We divide the proof of (2.1) into the two subsections 2.1 and 2.2.

Note that we can substitute τ in (1.3) by a decreasing sequence τ_n tending to zero such that

$$L_n(\tau_n) \to 0 \text{ as } n \to \infty. \tag{2.3}$$

and $\lim_{n\to\infty} \tau_n \sqrt{n} = \infty$.

2.1. **Truncation of random variables.** In this section we truncate the elements of the matrices $\mathbf{X}$ and $\mathbf{Y}$. Let us omit the indices $\mathbf{X}$ and $\mathbf{Y}$ in the notations of the resolvent and the Stieltjes transforms.

Consider some symmetric $n \times n$ matrix $\mathbf{D}$. Put $\tilde{\mathbf{X}} = \mathbf{X} + \mathbf{D}$. Let

$$\tilde{\mathbf{R}} = \left(\frac{1}{\sqrt{n}} \tilde{\mathbf{X}} - z\mathbf{I} \right)^{-1}.$$

Lemma 2.1.

$$|\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \tilde{\mathbf{R}}| \le \frac{1}{v^2} (\operatorname{Tr} \mathbf{D}^2)^{\frac{1}{2}}.$$

Proof. By the resolvent equation

$$\mathbf{R} = \tilde{\mathbf{R}} - \frac{1}{\sqrt{n}} \mathbf{R}\mathbf{D}\tilde{\mathbf{R}}. \tag{2.4}$$

For resolvent matrices we have, for $z = u + iv, v > 0$,

$$\max\{||\mathbf{R}||, ||\tilde{\mathbf{R}}||\} \le \frac{1}{v}. \tag{2.5}$$

Using (2.4) and (2.5) it is easy to show that

$$|\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \tilde{\mathbf{R}}| = \frac{1}{\sqrt{n}} |\operatorname{Tr} \mathbf{R}\mathbf{D}\tilde{\mathbf{R}}| \le \frac{1}{v^2} (\operatorname{Tr} \mathbf{D}^2)^{\frac{1}{2}}.$$

□

We split the matrix entries as $X = \hat{X} + \check{X}$, where $\hat{X} := X \mathbb{1}(|X| < \tau_n \sqrt{n})$ and $\check{X} := X \mathbb{1}(|X| \ge \tau_n \sqrt{n})$. Define the matrix $\hat{\mathbf{X}} = \{\hat{X}_{ij}\}_{i,j=1}^n$. Let

$$\hat{\mathbf{R}}(z) := \left(\frac{1}{\sqrt{n}} \hat{\mathbf{X}} - z\mathbf{I} \right)^{-1} \text{ and } \hat{S}(z) = \frac{1}{n} \mathbb{E} \operatorname{Tr} \hat{\mathbf{R}}(z).$$

By Lemma 2.1

$$|S(z) - \hat{S}(z)| \le \frac{1}{v^2} \left(\frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E}\, X_{ij}^2 \mathbb{1}(|X_{ij}| \ge \tau_n \sqrt{n}) \right)^{1/2} = v^{-2} L_n^{\frac{1}{2}}(\tau_n).$$

From (2.3) we conclude that

$$|S(z) - \hat{S}(z)| \to 0 \text{ as } n \to \infty.$$

Introduce the centralized random variables $\overline{X}_{ij} = \hat{X}_{ij} - \mathbb{E}(\hat{X}_{ij}|\mathfrak{F}^{(i,j)})$ and the matrix $\overline{\mathbf{X}} = \{\overline{X}_{ij}\}_{i,j=1}^n$. Let

$$\overline{\mathbf{R}}(z) := \left(\frac{1}{\sqrt{n}}\overline{\mathbf{X}} - z\mathbf{I}\right)^{-1} \text{ and } \overline{S}(z) = \frac{1}{n}\,\mathbb{E}\,\mathrm{Tr}\,\overline{\mathbf{R}}(z).$$

Again by Lemma 2.1

$$|\hat{S}(z) - \overline{S}(z)| \le \frac{1}{v^2}\left(\frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}\,X_{ij}^2\,\mathbb{1}(|X_{ij}| \ge \tau_n\sqrt{n})\right)^{1/2} = v^{-2}L_n^{\frac{1}{2}}(\tau_n).$$

In view of (2.3) the right hand side tends to zero as $n \to \infty$.

Now we show that (1.2) will hold if we replace $\mathbf{X}$ by $\overline{\mathbf{X}}$. For all $1 \le i \le j \le n$

$$\begin{aligned}&\mathbb{E}(\overline{X}_{ij}^2|\mathfrak{F}^{(i,j)}) - \sigma_{ij}^2\\ &= \mathbb{E}(\overline{X}_{ij}^2|\mathfrak{F}^{(i,j)}) - \mathbb{E}(\hat{X}_{ij}^2|\mathfrak{F}^{(i,j)}) - \mathbb{E}(\check{X}_{ij}^2|\mathfrak{F}^{(i,j)}) + \mathbb{E}(X_{ij}^2|\mathfrak{F}^{(i,j)}) - \sigma_{ij}^2.\end{aligned}$$

By the triangle inequality and (1.2), (2.3)

$$\begin{aligned}&\frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}\,|\,\mathbb{E}(\overline{X}_{ij}^2|\mathfrak{F}^{(i,j)}) - \sigma_{ij}^2|\\ &\le \frac{1}{n^2}\sum_{i,j=1}^n \mathbb{E}\,|\,\mathbb{E}(X_{ij}^2|\mathfrak{F}^{(i,j)}) - \sigma_{ij}^2| + 2L_n(\tau_n) \to 0 \text{ as } n \to \infty.\end{aligned} \tag{2.6}$$

It is also not very difficult to check that the condition (1.4) holds true for the matrix $\mathbf{X}$ replaced by $\overline{\mathbf{X}}$.

Similarly, one may truncate the elements of the matrix $\mathbf{Y}$ and consider the matrix $\overline{\mathbf{Y}}$ with the entries $Y_{ij}\,\mathbb{1}(|Y_{ij}| \le \tau_n\sqrt{n})$. Then one may check that

$$\frac{1}{n^2}\sum_{i,j=1}^n |\,\mathbb{E}\,\overline{Y}_{ij}^2 - \sigma_{ij}^2| \to 0 \text{ as } n \to \infty. \tag{2.7}$$

In what follows assume from now on that $|X_{ij}| \le \tau_n\sqrt{n}$ and $|Y_{ij}| \le \tau_n\sqrt{n}$. We shall write $\mathbf{X}, \mathbf{Y}$ instead of $\overline{\mathbf{X}}$ and $\overline{\mathbf{Y}}$ respectively.

2.2. **Universality of the spectrum of eigenvalues.** To prove (2.1) we will use a method introduced in [**?**]. Define the matrix $\mathbf{Z} := \mathbf{Z}(\varphi) := \mathbf{X}\cos\varphi + \mathbf{Y}\sin\varphi$. It is easy to see that $\mathbf{Z}(0) = \mathbf{X}$ and $\mathbf{Z}(\pi/2) = \mathbf{Y}$. Set $\mathbf{W} := \mathbf{W}(\varphi) := n^{-1/2}\mathbf{Z}$ and

$$\mathbf{R}(z,\varphi) := (\mathbf{W} - z\mathbf{I})^{-1}. \tag{2.8}$$

Introduce the Stieltjes transform

$$S(z,\varphi) := \frac{1}{n}\sum_{i=1}^n \mathbb{E}[\mathbf{R}(z,\varphi)]_{ii}.$$

Note that $S(z,0)$ and $S(z,\pi/2)$ are the Stieltjes transforms $S^{\mathbf{X}}(z)$ and $S^{\mathbf{Y}}(z)$ respectively.

Obviously we have

$$S(z,\frac{\pi}{2}) - S(z,0) = \int_0^{\frac{\pi}{2}} \frac{\partial S(z,\varphi)}{\partial \varphi} d\varphi. \tag{2.9}$$

To simplify the arguments we will omit arguments in the notations of matrices and Stieltjes transforms. We have

$$\frac{\partial \mathbf{W}}{\partial \varphi} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial Z_{ij}}{\partial \varphi} \mathbf{e}_i \mathbf{e}_j^T,$$

where we denote by $\mathbf{e}_i$ the column vector with 1 in position i and zeros in the other positions. We may rewrite the integrand in (2.9) in the following way

$$\begin{aligned} \frac{\partial S}{\partial \varphi} &= -\frac{1}{n} \mathbb{E} \operatorname{Tr} \mathbf{R} \frac{\partial \mathbf{W}}{\partial \varphi} \mathbf{R} \\ &= -\frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \operatorname{Tr} \mathbf{R} \frac{\partial Z_{ij}}{\partial \varphi} \mathbf{e}_i \mathbf{e}_j^T \mathbf{R} \\ &= \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \frac{\partial Z_{ij}}{\partial \varphi} u_{ij}, \end{aligned} \tag{2.10}$$

where $u_{ij} = -[\mathbf{R}^2]_{ji}$.

For all $1 \le i \le j \le n$ introduce the random variables

$$\xi_{ij} := Z_{ij}, \quad \hat{\xi}_{ij} := \frac{\partial Z_{ij}}{\partial \varphi} = -\sin\varphi X_{ij} + \cos\varphi Y_{ij},$$

and the sets of random variables

$$\xi^{(ij)} := \{\xi_{kl} : 1 \le k \le l \le n, (k,l) \ne (i,j)\}.$$

Using Taylor's formula one may write

$$u_{ij}(\xi_{ij}, \xi^{(ij)}) = u_{ij}(0, \xi^{(ij)}) + \xi_{ij} \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)}) + \mathbb{E}_\theta \, \theta(1-\theta) \xi_{ij}^2 \frac{\partial^2 u_{ij}}{\partial \xi_{ij}^2}(\theta \xi_{ij}, \xi^{(ij)}),$$

where θ has a uniform distribution on $[0,1]$ and is independent of $(\xi_{ij}, \xi^{(ij)})$. Multiplying both sides of the last equation by $\hat{\xi}_{ij}$ and taking mathematical expectation on both sides we have

$$\begin{aligned} \mathbb{E} \, \hat{\xi}_{ij} u_{ij}(\xi_{ij}, \xi^{(ij)}) &= \mathbb{E} \, \hat{\xi}_{ij} u_{ij}(0, \xi^{(ij)}) + \mathbb{E} \, \hat{\xi}_{ij} \xi_{ij} \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)}) \\ &+ \mathbb{E} \, \theta(1-\theta) \hat{\xi}_{ij} \xi_{ij}^2 \frac{\partial^2 u_{ij}}{\partial \xi_{ij}^2}(\theta \xi_{ij}, \xi^{(ij)}). \end{aligned} \tag{2.11}$$

By independence of Y_{ij} and $\xi^{(ij)}$ we get

$$\mathbb{E} \, Y_{ij} u_{ij}(0, \xi^{(ij)}) = \mathbb{E} \, Y_{ij} \, \mathbb{E} \, u_{ij}(0, \xi^{(ij)}) = 0. \tag{2.12}$$

By the properties of conditional expectation and condition (1.1)

$$\mathbb{E}\, X_{ij} u_{ij}(0, \xi^{(ij)}) = \mathbb{E}\, u_{ij}(0, \xi^{(ij)})\, \mathbb{E}(X_{ij} | \mathfrak{F}^{(i,j)}) = 0. \tag{2.13}$$

By (2.11), (2.12) and (2.13) we can rewrite (2.10) in the following way

$$\frac{\partial S}{\partial \varphi} = \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}\, \hat{\xi}_{ij} \xi_{ij} \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)}) + \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}\, \theta(1-\theta) \hat{\xi}_{ij} \xi_{ij}^2 \frac{\partial^2 u_{ij}}{\partial \xi_{ij}^2}(\theta \xi_{ij}, \xi^{(ij)})$$
$$= \mathbb{A}_1 + \mathbb{A}_2.$$

It is easy to see that

$$\hat{\xi}_{ij} \xi_{ij} = -\frac{1}{2} \sin 2\varphi X_{ij}^2 + \cos^2 \varphi X_{ij} Y_{ij} - \sin^2 \varphi X_{ij} Y_{ij} + \frac{1}{2} \sin 2\varphi Y_{ij}^2.$$

The random variables Y_{ij} are independent of X_{ij} and $\xi^{(ij)}$. Using this fact we conclude that

$$\mathbb{E}\, X_{ij} Y_{ij} \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)}) = \mathbb{E}\, Y_{ij}\, \mathbb{E}\, X_{ij} \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)}) = 0, \tag{2.14}$$

$$\mathbb{E}\, Y_{ij}^2 \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)}) = \sigma_{ij}^2\, \mathbb{E}\, \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)}). \tag{2.15}$$

By the properties of conditional mathematical expectation we get

$$\mathbb{E}\, X_{ij}^2 \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)}) = \mathbb{E}\, \frac{\partial u_{ij}}{\partial \xi_{ij}}(0, \xi^{(ij)})\, \mathbb{E}(X_{ij}^2 | \mathfrak{F}^{(i,j)}). \tag{2.16}$$

A direct calculation shows that the derivative of $u_{ij} = -[\mathbf{R}^2]_{ji}$ is equal to

$$\frac{\partial u_{ij}}{\partial \xi_{ij}} = \left[\mathbf{R}^2 \frac{\partial \mathbf{Z}}{\partial \xi_{ij}} \mathbf{R} \right]_{ji} + \left[\mathbf{R} \frac{\partial \mathbf{Z}}{\partial \xi_{ij}} \mathbf{R}^2 \right]_{ji}$$
$$= \frac{1}{\sqrt{n}} [\mathbf{R}^2 \mathbf{e}_i \mathbf{e}_j^T \mathbf{R}]_{ji} + \frac{1}{\sqrt{n}} [\mathbf{R}^2 \mathbf{e}_j \mathbf{e}_i^T \mathbf{R}]_{ji} + \frac{1}{\sqrt{n}} [\mathbf{R} \mathbf{e}_i \mathbf{e}_j^T \mathbf{R}^2]_{ji} + \frac{1}{\sqrt{n}} [\mathbf{R} \mathbf{e}_j \mathbf{e}_i^T \mathbf{R}^2]_{ji}$$
$$= \frac{1}{\sqrt{n}} [\mathbf{R}^2]_{ji} [\mathbf{R}]_{ji} + \frac{1}{\sqrt{n}} [\mathbf{R}^2]_{jj} [\mathbf{R}]_{ii} + \frac{1}{\sqrt{n}} [\mathbf{R}]_{ji} [\mathbf{R}^2]_{ji} + \frac{1}{\sqrt{n}} [\mathbf{R}]_{jj} [\mathbf{R}^2]_{ii}.$$

Using the obvious bound for the spectral norm of the matrix resolvent $||\mathbf{R}|| \leq v^{-1}$ we get

$$\left| \frac{\partial u_{ij}}{\partial \xi_{ij}} \right| \leq \frac{C}{\sqrt{n} v^3}. \tag{2.17}$$

From (2.14)–(2.17) and (2.6)–(2.7) we deduce

$$|\mathbb{A}_1| \leq \frac{C}{n^2 v^3} \sum_{i,j=1}^{n} \mathbb{E}\, |\, \mathbb{E}(X_{ij}^2 | \mathfrak{F}^{(i,j)}) - \sigma_{ij}^2 | \to 0 \text{ as } n \to \infty. \tag{2.18}$$

It remains to estimate $\mathbb{A}_2$. We calculate the second derivative of u_{ij}

$$\frac{\partial^2 u_{ij}}{\partial \xi_{ij}^2} = -2 \left[\mathbf{R}^2 \frac{\partial \mathbf{W}}{\partial \xi_{ij}} \mathbf{R} \frac{\partial \mathbf{W}}{\partial \xi_{ij}} \mathbf{R} \right]_{ji} - 2 \left[\mathbf{R} \frac{\partial \mathbf{W}}{\partial \xi_{ij}} \mathbf{R}^2 \frac{\partial \mathbf{W}}{\partial \xi_{ij}} \mathbf{R} \right]_{ji}$$
$$- 2 \left[\mathbf{R} \frac{\partial \mathbf{W}}{\partial \xi_{ij}} \mathbf{R} \frac{\partial \mathbf{W}}{\partial \xi_{ij}} \mathbf{R}^2 \right]_{ji} = \mathbb{T}_1 + \mathbb{T}_2 + \mathbb{T}_3.$$

Let's expand the term $\mathbb{T}_1$

$$\mathbb{T}_1 = -2\left[\mathbf{R}^2\frac{\partial \mathbf{W}}{\partial \xi_{ij}}\mathbf{R}\frac{\partial \mathbf{W}}{\partial \xi_{ij}}\mathbf{R}\right]_{ji} = \mathbb{T}_{11} + \mathbb{T}_{12} + \mathbb{T}_{13} + \mathbb{T}_{14}, \tag{2.19}$$

where we denote

$$\mathbb{T}_{11} = -\frac{2}{n}[\mathbf{R}^2]_{ji}[\mathbf{R}]_{ji}[\mathbf{R}]_{ji}, \quad \mathbb{T}_{12} = -\frac{2}{n}[\mathbf{R}^2]_{ji}[\mathbf{R}]_{jj}[\mathbf{R}]_{ii},$$
$$\mathbb{T}_{13} = -\frac{2}{n}[\mathbf{R}^2]_{jj}[\mathbf{R}]_{ii}[\mathbf{R}]_{ji}, \quad \mathbb{T}_{14} = -\frac{2}{n}[\mathbf{R}^2]_{jj}[\mathbf{R}]_{ij}[\mathbf{R}]_{ii}.$$

Using again the bound $||\mathbf{R}|| \leq v^{-1}$ we can show that

$$\max(|\mathbb{T}_{11}|, |\mathbb{T}_{12}|, |\mathbb{T}_{13}|, |\mathbb{T}_{14}|) \leq \frac{C}{nv^4}.$$

From the expansion (2.19) and the bounds of $\mathbb{T}_{1i}, i = 1, 2, 3, 4$ we conclude that

$$|\mathbb{T}_1| \leq \frac{C}{nv^4}.$$

Repeating the above arguments one can show that

$$\max(|\mathbb{T}_2|, |\mathbb{T}_3|) \leq \frac{C}{nv^4}.$$

Finally we have

$$\left|\frac{\partial^2 u_{ij}}{\partial \xi_{ij}^2}(\theta\xi_{ij}, \xi^{(ij)})\right| \leq \frac{C}{nv^4}.$$

Using the assumption $|\xi_{ij}| \leq \tau_n\sqrt{n}$ and the condition (1.4) we deduce the bound

$$|\mathbb{A}_2| \leq \frac{C\tau_n}{v^4}. \tag{2.20}$$

We may turn τ_n to zero and conclude the statement of Theorem 1.3 from (2.9), (2.10), (2.18) and (2.20).

3. Proof of Theorem 1.4

We prove the theorem using the moment method. It is easy to see that the moments of $F^{\mathbf{Y}}(x)$ can be rewritten as normalized traces of powers of $\mathbf{Y}$:

$$\int_{\mathbb{R}} x^k dF^{\mathbf{Y}}(x) = \mathbb{E}\,\frac{1}{n}\operatorname{Tr}\left(\frac{1}{\sqrt{n}}\mathbf{Y}\right)^k, \ k \geq 1.$$

It is sufficient to prove that

$$\mathbb{E}\,\frac{1}{n}\operatorname{Tr}\left(\frac{1}{\sqrt{n}}\mathbf{Y}\right)^k = \int_{\mathbb{R}} x^k dG(x) + o_k(1).$$

for $k \geq 1$, where $o_k(1)$ tends to zero as $n \to \infty$ for any fixed k.

It is well known that the moments of semicircle law are given by the Catalan numbers

$$\beta_k = \int_{\mathbb{R}} x^k dG(x) = \begin{cases} \frac{1}{m+1}\binom{2m}{m}, & k = 2m \\ 0, & k = 2m+1. \end{cases}$$

Furthermore we shall use the notations and the definitions from [**?**]. A graph is a triple (E, V, F), where E is the set of edges, V is the set of vertices, and F is a function, $F : E \to V \times V$. Let $\boldsymbol{i} = (i_1, ..., i_k)$ be a vector taking values in $\{1, ..., n\}^k$. For a vector $\boldsymbol{i}$ we define a Γ-graph as follows. Draw a horizontal line and plot the numbers $i_1, ..., i_k$ on it. Consider the distinct numbers as vertices, and draw k edges e_j from i_j to $i_{j+1}, j = 1, ..., k$, using $i_{k+1} = i_1$ by convention. Denote the number of distinct i_j's by t. Such a graph is called a $\Gamma(k, t)$-graph.

Two $\Gamma(k, t)$-graphs are said to be isomorphic if they can be converted each other by a permutation of $(1, ..., n)$. By this definition, all Γ-graphs are classified into isomorphism classes. We shall call the $\Gamma(k, t)$-graph canonical if it has the following properties:
1) Its vertex set is $\{1, ..., t\}$;
2) Its edge set is $\{e_1, ..., e_k\}$;
3) There is a function g from $\{1, ..., k\}$ onto $\{1, ..., t\}$ satisfying $g(1) = 1$ and $g(i) \leq \max\{g(1), ..., g(i-1)\} + 1$ for $1 < i \leq k$;
4) $F(e_i) = (g(i), g(i+1))$, for $i = 1, ..., k$, with the convention $g(k+1) = g(1) = 1$.

It is easy to see that each isomorphism class contains one and only one canonical Γ-graph that is associated with a function g, and a general graph in this class can be defined by $F(e_j) = (i_{g(j)}, i_{g(j+1)})$. It is easy to see that each isomorphism class contains $n(n-1)...(n-t+1)$ $\Gamma(k, t)$-graphs.

We shall classify all canonical graphs into three categories. Category 1 consists of all canonical $\Gamma(k, t)$-graphs with the property that each edge is coincident with exactly one other edge of opposite direction and the graph of noncoincident edges forms a tree. It is easy to see if k is odd then there are no graphs in category 1. If k is even, i.e. $k = 2m$, say, we denote a $\Gamma(k, t)$-graph by $\Gamma_1(2m)$. Category 2 consists of all canonical graphs that have at least one edge with odd multiplicity. We shall denote the graph from this category by $\Gamma_2(k, t)$. Finally, category 3 consists of all other canonical graphs, which we denote by $\Gamma_3(k, t)$.

It is known, see [**?**, Lemma 2.4], that the number of $\Gamma_1(2m)$-graphs is equal to $\frac{1}{m+1}\binom{2m}{m}$.

We expand the traces of powers of $\mathbf{Y}$ in a sum

$$\operatorname{Tr}\left(\frac{1}{\sqrt{n}}\mathbf{Y}\right)^k = \frac{1}{n^{k/2}} \sum_{i_1, i_2, ..., i_k} Y_{i_1 i_2} Y_{i_2 i_3} ... Y_{i_k i_1}, \tag{3.1}$$

where the summation is taken over all sequences $\boldsymbol{i} = (i_1, ..., i_k) \in \{1, ..., n\}^k$.

For each vector $\boldsymbol{i}$ we construct a graph $G(\boldsymbol{i})$ as above. We denote by $Y(\boldsymbol{i}) = Y(G(\boldsymbol{i}))$.

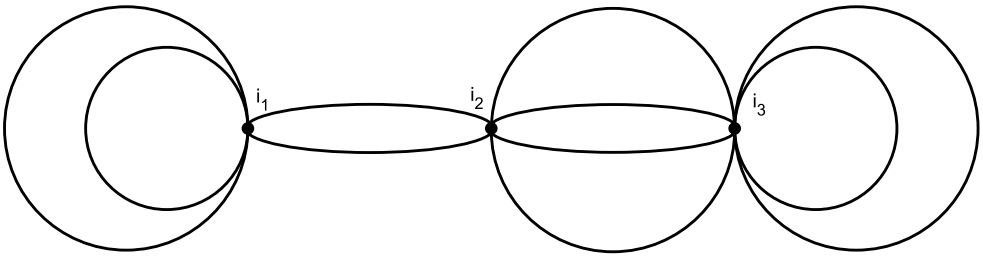

FIGURE 2. Graph $\Gamma_3(10,3)$.

Then we may split the moments of $F^{\mathbf{Y}}(x)$ into three terms

$$\mathbb{E}\,\frac{1}{n}\,\mathrm{Tr}\left(\frac{1}{\sqrt{n}}\mathbf{Y}\right)^k = \frac{1}{n^{k/2+1}}\sum_{\boldsymbol{i}}\mathbb{E}\,Y_{i_1i_2}Y_{i_2i_3}...Y_{i_ki_1} = S_1 + S_2 + S_3,$$

where

$$S_j = \frac{1}{n^{k/2+1}}\sum_{\Gamma(k,t)\in C_j}\sum_{G(\boldsymbol{i})\in\Gamma(k,t)}\mathbb{E}[Y(G(\boldsymbol{i}))],$$

and the summation $\sum_{\Gamma(k,t)\in C_j}$ is taken over all canonical $\Gamma(k,t)$-graphs in category C_j and the summation $\sum_{G(\boldsymbol{i})\in\Gamma(k,t)}$ is taken over all isomorphic graphs for a given canonical graph.

From the independence of Y_{ij} and $\mathbb{E}\,Y_{ij}^{2s-1} = 0, s \geq 1$, it follows that $S_2 = 0$.

For the graphs from categories C_1 and C_3 we introduce further notations. Let us consider the $\Gamma(k,t)$-graph $G(\boldsymbol{i})$. Without loss of generality we assume that $i_l, l = 1,...,t$ are distinct coordinates of the vector $\boldsymbol{i}$ and define a vector $\boldsymbol{i}_t = (i_1,...,i_t)$. We also set $G(\boldsymbol{i}_t) := G(\boldsymbol{i})$. Let $\tilde{\boldsymbol{i}}_t = (i_1,...,i_{q-1},i_{q+1},...,i_t)$ and $\hat{\boldsymbol{i}}_t = (i_1,...,i_{p-1},i_{p+1},...,i_{q-1},i_{q+1},...,i_t)$ be vectors derived from $\boldsymbol{i}_t$ by deleting the elements in the position q and p,q respectively. We additionally assume that the coordinates of $\hat{\boldsymbol{i}}_t$ do not coincide with i_p. We denote the graph without the vertex i_q and all edges linked to it by $G(\tilde{\boldsymbol{i}}_t)$. If the vertex i_q is incident to a loop we denote by $G'(\boldsymbol{i}_t)$ the graph with this loop removed. By $\tilde{G}(\boldsymbol{i}_t)$ we mean the graph derived from $G(\boldsymbol{i}_t)$ by deleting the edge from i_p to i_q taking into account the multiplicity.

Now we will estimate the term S_3. For a graph from category C_3 we know that k has to be even, i.e. $k = 2m$, say. We illustrate the example of a $\Gamma_3(k,t)$-graph in Figure 2. This graph corresponds to the term $Y(G(\boldsymbol{i}_3)) = Y_{i_1i_1}^2Y_{i_1i_2}^2Y_{i_2i_3}^4Y_{i_3i_3}^2$.

We mention that $\mathbb{E}\,Y_{i_pi_q}^{2s} \leq_s \sigma_{i_pi_q}^{2s}$. Hence we may rewrite the terms which correspond to the graphs from category C_3 via variances.

Each graph G from C_3 we can decompose into two graphs $G = G_1 \cup G_2$ in the following way. We will paint all edges into two colours such that all coincident edges have the same colour. We choose the graph G_1 such that one of the following cases holds true:
i) graph G_1 consists from only one vertex and all incident to it loops;
ii) graph G_1 consists from two vertices and the edge between them with the multiplicity greater then two;
iii) Each edge of the graph G_1 coincide with exactly one edge of the opposite direction and the graph of non coincident edges forms a simple cycle.

We may assume that the sum of multiplicities of all edges from G_1 is equal $2s$. It remains to consider the remaining $2(m-s)$ edges from the graph G_2.

Denote by V_1, E_1 and V_2, E_2 the sets of coordinates and edges of the graphs G_1 and G_2 respectively. We define $\boldsymbol{i}_t^1 := (i_l, i_l \in V_1)$. Now we may fix the graph G_1 and consider the graph G_2.

a) If the set E_2 is empty, then one should go to the step c). Otherwise, we can take the vertex $i_q \in V_2$ and consider the following opportunities:

(1) There is a loop from G_2 incident to the vertex $i_q \in V_2$ with the multiplicity $2a, a \geq 1$. In this case we estimate
$$\frac{1}{n^{m+1}} \sum_{G(\boldsymbol{i}) \in \Gamma(2m,t)} \mathbb{E}[Y(G(\boldsymbol{i}))] \leq_a \frac{1}{n^{m+1}} \sum_{\boldsymbol{i}_t} \mathbb{E}[Y(G'(\boldsymbol{i}_t))] \sigma_{i_q i_q}^{2a}.$$
Applying a times the inequality $n^{-1}\sigma_{i_q i_q}^2 \leq B_{i_q}^2$ and the condition (1.5), we delete all loops incident to this vertex;
(2) There are no loops incident to the vertex $i_q \in V_2 \setminus V_1$, but i_q is connected with only one vertex $i_p \in V_2$ by an edge of the graph G_2 and the multiplicity of this edge is equal $2b, b \geq 1$. In this case we estimate
$$\frac{1}{n^{m+1}} \sum_{G(\boldsymbol{i}) \in \Gamma(2m,t)} \mathbb{E}[Y(G(\boldsymbol{i}))] \leq_b \frac{1}{n^{m+1}} \sum_{\tilde{\boldsymbol{i}}_t} \mathbb{E}[Y(G(\tilde{\boldsymbol{i}}_t))] \sum_{i_q=1}^{n} \sigma_{i_p i_q}^{2b}.$$
Here we may use $b-1$ times the inequality $n^{-1}\sigma_{i_p i_q}^2 \leq B_{i_p}^2$ and condition (1.5) and consequently delete all coinciding edges except two. Then we may again apply condition (1.5) to delete i_q.
(3) There are no loops incident to some vertex from V_2 and no vertices in $V_2 \setminus V_1$ which are connected with only one vertex from V_2. Then we can take any two vertices, let's say i_q and i_p from V_2 and estimate
$$\frac{1}{n^{m+1}} \sum_{G(\boldsymbol{i}) \in \Gamma(2m,t)} \mathbb{E}[Y(G(\boldsymbol{i}))] \leq_b \frac{1}{n^{m+1}} \sum_{\boldsymbol{i}_t} \mathbb{E}[Y(\tilde{G}(\boldsymbol{i}_t))] \sigma_{i_p i_q}^{2c},$$
where the multiplicity of the edge between i_p and i_q is equal $2c, c \geq 1$. Here we may use c times the inequality $n^{-1}\sigma_{i_p i_q}^2 \leq B_{i_p}^2$ and the condition (1.5) and consequently delete all coinciding edges between i_p and i_q.

b) go to step a);
c) It is easy to see that each time on the step a) we use the same bound (1.5). Hence we will have

$$\frac{1}{n^{m+1}}\sum_{G(\boldsymbol{i})\in\Gamma(2m,t)}\mathbb{E}[Y(G(\boldsymbol{i}))] \le_m \frac{C^{2(m-s)}n^{m-s}}{n^{m+1}}\sum_{\boldsymbol{i}_t^1}\mathbb{E}[Y(G_1(\boldsymbol{i}_t^1))] \tag{3.2}$$

It remains to estimate the right hand side of (3.2). In the case i) we may estimate

$$\begin{aligned}&\frac{1}{n^{s+1}}\sum_{i_1=1}^{n}\mathbb{E}\,Y_{i_1i_1}^{2s} \le_s \frac{1}{n^{s+1}}\sum_{i_1=1}^{n}\sigma_{i_1i_1}^{2s} \le \frac{C^{2(s-1)}}{n^2}\sum_{i_1=1}^{n}\sigma_{i_1i_1}^2\\ &\le C^{2(s-1)}\tau_n^2 + \frac{C^{2(s-1)}}{n^2}\sum_{i_1=1}^{n}\mathbb{E}\,Y_{i_1i_1}^2\,\mathbb{1}(|Y_{i_1i_1}|\ge\tau_n\sqrt{n})\\ &\le C^{2(s-1)}\tau_n^2 + C^{2(s-1)}L_n(\tau_n) = o_s(1),\end{aligned}$$

where we have used the inequality $n^{-1}\sigma_{i_1i_1}^2 \le B_{i_1}^2$ and (1.5).

In the case ii) we will use the bound

$$\begin{aligned}&\frac{1}{n^{s+1}}\sum_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}\mathbb{E}\,Y_{i_1i_2}^{2s} \le_s \frac{1}{n^{s+1}}\sum_{i_1,i_2=1}^{n}\sigma_{i_1i_2}^{2s} \le \frac{C^{2(s-2)}}{n^3}\sum_{i_1,i_2=1}^{n}\sigma_{i_1i_2}^4\\ &\le C^{2(s-2)}\frac{\tau_n^2}{n^2}\sum_{i_1,i_2=1}^{n}\sigma_{i_1i_2}^2 + \frac{C^{2(s-2)}}{n^3}\sum_{i_1,i_2=1}^{n}\sigma_{i_1i_2}^2\,\mathbb{E}\,Y_{i_1i_2}^2\,\mathbb{1}(|Y_{i_1i_2}|\ge\tau_n\sqrt{n})\\ &\le C^{2(s-1)}\tau_n^2 + C^{2(s-1)}L_n(\tau_n) = o_s(1),\end{aligned}$$

where we have used the inequality $n^{-1}\sigma_{i_1i_2}^2 \le B_{i_1}^2$ and (1.5).

It remains to consider the case iii) only. We need to introduce further notation for this situation. We may redenote the vertices from the set V_1 and assume that $\boldsymbol{i}_t^1 = \boldsymbol{i}_s := (i_1,...,i_s)$. By $G_1(\boldsymbol{i}_s)$ we denote the graph $G_1(\boldsymbol{i}_t^1)$. Using the previous notations of $\tilde{\boldsymbol{i}}_s, \hat{\boldsymbol{i}}_s$ and $\tilde{G}_1(\boldsymbol{i}_s)$ we set $p=1$ and $q=2$. Finally by $\hat{G}_1(\tilde{\boldsymbol{i}}_s)$ we denote the graph derived from $\tilde{G}_1(\boldsymbol{i}_s)$ by deleting the vertex i_2 and the edge between i_2 and some another vertex $i_x, 2<x\le s$. We may write

$$\frac{1}{n^{s+1}}\sum_{\boldsymbol{i}_s}\mathbb{E}\,Y(G_1(\boldsymbol{i}_s)) \le \frac{\tau_n^2}{n^s}\sum_{\boldsymbol{i}_s}\mathbb{E}\,Y(\tilde{G}_1(\boldsymbol{i}_s)) \tag{3.3}$$

$$+\frac{1}{n^{s+1}}\sum_{\boldsymbol{i}_s}\mathbb{E}\,Y(\tilde{G}_1(\boldsymbol{i}_s))\,\mathbb{E}\,Y_{i_1i_2}^2\,\mathbb{1}(|Y_{i_1i_2}|\ge\tau_n\sqrt{n}). \tag{3.4}$$

First we estimate the right hand side of (3.3). The graph of the non coincident edges of $\tilde{G}_1(\boldsymbol{i}_s)$ forms a tree. We can sequently delete all vertices and edges from $\tilde{G}_1(\boldsymbol{i}_s)$ using the assumption (1.5) on each step. We derive the bound

$$\frac{\tau_n^2}{n^s}\sum_{\boldsymbol{i}_s}\mathbb{E}\,Y(\tilde{G}_1(\boldsymbol{i}_s) \le C^{2(s-1)}\tau_n^2. \tag{3.5}$$

For the term (3.4) we may write

$$\begin{aligned}(3.6)\qquad &\frac{1}{n^{s+1}}\sum_{\boldsymbol{i}_s}\mathbb{E}\,Y(\tilde{G}_1(\boldsymbol{i}_s))\,\mathbb{E}\,Y_{i_1i_2}^2\,\mathbb{1}(|Y_{i_1i_2}|\ge\tau_n\sqrt{n})\\ &\le\frac{1}{n^{s+1}}\sum_{\boldsymbol{i}_s}\mathbb{E}\,Y(\hat{G}_1(\tilde{\boldsymbol{i}}_s))\sigma_{i_2,i_x}^2\,\mathbb{E}\,Y_{i_1i_2}^2\,\mathbb{1}(|Y_{i_1i_2}|\ge\tau_n\sqrt{n})\\ &\le\frac{C^2}{n^s}\sum_{i_1,i_2=1}^{n}\mathbb{E}\,Y_{i_1i_2}^2\,\mathbb{1}(|Y_{i_1i_2}|\ge\tau_n\sqrt{n})\sum_{\hat{\boldsymbol{i}}_s}\mathbb{E}\,Y(\hat{G}_1(\tilde{\boldsymbol{i}}_s)).\end{aligned}$$

Again using (1.5) one may show that

$$(3.7)\qquad \sum_{\hat{\boldsymbol{i}}_s}\mathbb{E}\,Y(\hat{G}_1(\tilde{\boldsymbol{i}}_s))\le C^{2(s-2)}n^{s-2}.$$

By (3.6) and (3.7) we have

$$(3.8)\qquad \frac{1}{n^{s+1}}\sum_{\boldsymbol{i}_s}\mathbb{E}\,Y(\tilde{G}_1(\boldsymbol{i}_s))\,\mathbb{E}\,Y_{i_1i_2}^2\,\mathbb{1}(|Y_{i_1i_2}|\ge\tau_n\sqrt{n})\le C^{2(s-1)}L_n(\tau_n).$$

From (3.5) and (3.8) we derive the estimate

$$\frac{1}{n^{s+1}}\sum_{\boldsymbol{i}_s}\mathbb{E}\,Y(G_1(\boldsymbol{i}_s))\le C^{2(s-1)}\tau_n^2+C^{2(s-1)}L_n(\tau_n).$$

Finally for the cases i)–iii) we will have

$$\frac{1}{n^{m+1}}\sum_{G(\boldsymbol{i})\in\Gamma(2m,t)}\mathbb{E}[Y(G(\boldsymbol{i}))]\le_m C^{2(m-1)}(\tau_n^2+L_n(\tau_n))=o_m(1).$$

As an example we recommend to check this algorithm for the graph in Figure 2.

It is easy to see that the number of different canonical graphs in C_3 is of order $O_m(1)$. Finally for the term S_3 we get

$$S_3=o_m(1).$$

It remains to consider the term S_1. For a graph from category C_1 we know that k has to be even, i.e. $k=2m$, say. In the category C_1 using the notations of $\boldsymbol{i}_t,\tilde{\boldsymbol{i}}_t$ and $\hat{\boldsymbol{i}}_t$ we take $t=m+1$.

We illustrate on the left part of Figure 3 an example of the tree of noncoincident edges of a $\Gamma_1(2m)$-graph for $m=5$. The term corresponding to this tree is $Y(G(\boldsymbol{i}_6))=Y_{i_1i_2}^2Y_{i_2i_3}^2Y_{i_2i_4}^2Y_{i_1i_5}^2Y_{i_5i_6}^2$.

We denote by $\sigma^2(\boldsymbol{i}_{m+1})=\sigma^2(G(\boldsymbol{i}_{m+1}))$ the product of m numbers $\sigma_{i_si_t}^2$, where $i_s,i_t,s<t$ are vertices of the graph $G(\boldsymbol{i}_{m+1})$ connected by edges of this graph. In our example, $\sigma^2(\boldsymbol{i}_{m+1})=\sigma^2(\boldsymbol{i}_6)=\sigma_{i_1i_2}^2\sigma_{i_2i_3}^2\sigma_{i_2i_4}^2\sigma_{i_1i_5}^2\sigma_{i_5i_6}^2$.

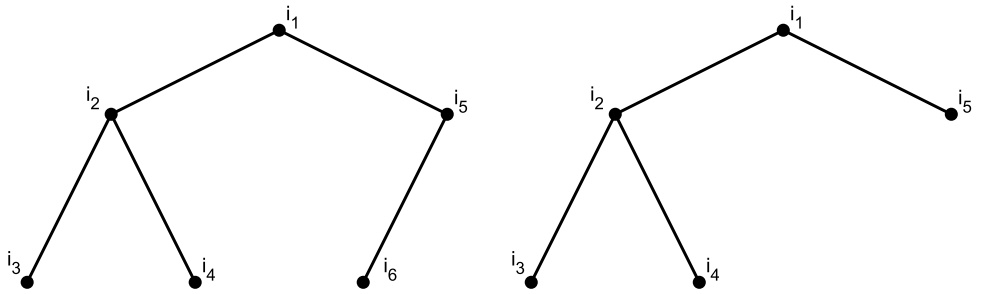

FIGURE 3. On the left, the tree of noncoincident edges of a $\Gamma_1(10)$-graph is shown. On the right, the tree of noncoincident edges of a $\Gamma_1(10)$-graph with deleted leaf i_6 is shown.

If $m = 1$ then $\sigma^2(\boldsymbol{i}_2) = \sigma^2_{i_1 i_2}$ and

$$\frac{1}{n^2} \sum_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{n} \sigma^2_{i_1 i_2} = \frac{1}{n} \sum_{i_1=1}^{n} \left[\frac{1}{n} \sum_{i_2=1}^{n} \sigma^2_{i_1 i_2} - 1 \right] + 1 + o(1), \tag{3.9}$$

where we have used $n^{-2} \sum_{i_1=1}^{n} \sigma^2_{i_1 i_1} = o(1)$. By (1.4) the first term is of order $o(1)$. The number of canonical graphs in C_1 for $m = 1$ is equal to 1. We conclude for $m = 1$ that

$$S_1 = n^{-2} \sum_{\Gamma_1(2)} \sum_{\substack{i_1, i_2 = 1 \\ i_1 \neq i_2}}^{n} \sigma^2_{i_1 i_2} = 1 + o(1),$$

Now we assume that $m > 1$. Let's consider the tree of non-coincident edges of the graph. We can find a leaf in the tree, let's say i_q, and a vertex i_p, which is connected to i_q by an edge of this tree. We have $\sigma^2(\boldsymbol{i}_{m+1}) = \sigma^2(\tilde{\boldsymbol{i}}_{m+1}) \cdot \sigma^2_{i_p i_q}$, where $\sigma^2(\tilde{\boldsymbol{i}}_{m+1}) = \sigma^2(G(\tilde{\boldsymbol{i}}_{m+1}))$.

In our example we can take the leaf i_6. On the right part of Figure 3 we have drawn the tree with deleted leaf i_6. We have $\sigma^2_{i_p i_q} = \sigma^2_{i_5 i_6}$ and $\sigma^2(\tilde{\boldsymbol{i}}_6) = \sigma^2_{i_1 i_2} \sigma^2_{i_2 i_3} \sigma^2_{i_2 i_4} \sigma^2_{i_1 i_5}$.

$$\begin{aligned}
&\frac{1}{n^{m+1}} \sum_{\boldsymbol{i}_{m+1}} \sigma^2(\boldsymbol{i}_{m+1}) = \frac{1}{n^{m+1}} \sum_{\tilde{\boldsymbol{i}}_{m+1}} \sigma^2(\tilde{\boldsymbol{i}}_{m+1}) \sum_{i_q=1}^{n} \sigma^2_{i_p i_q} + o_m(1) \\
&= \frac{1}{n^m} \sum_{\tilde{\boldsymbol{i}}_{m+1}} \sigma^2(\tilde{\boldsymbol{i}}_{m+1}) \left[\frac{1}{n} \sum_{i_q=1}^{n} \sigma^2_{i_p i_q} - 1 \right] \qquad (3.10) \\
&+ \frac{1}{n^m} \sum_{\tilde{\boldsymbol{i}}_{m+1}} \sigma^2(\tilde{\boldsymbol{i}}_{m+1}) \qquad (3.11) \\
&+ o_m(1),
\end{aligned}$$

where we have added some graphs from category C_3 and use the similar bounds as for S_3 term. Now we will show that the term (3.10) is of order $o_m(1)$. Note that

$$\frac{1}{n^m}\sum_{\tilde{\boldsymbol{i}}_{m+1}}\sigma^2(\tilde{\boldsymbol{i}}_{m+1})\left|\frac{1}{n}\sum_{i_q=1}^{n}\sigma^2_{i_pi_q}-1\right| \tag{3.12}$$
$$=\frac{1}{n}\sum_{i_p=1}^{n}\left|\frac{1}{n}\sum_{i_q=1}^{n}\sigma^2_{i_pi_q}-1\right|\frac{1}{n^{m-1}}\sum_{\hat{\boldsymbol{i}}_{m+1}}\sigma^2(\tilde{\boldsymbol{i}}_{m+1}).$$

We can sequentially delete leafs from the tree and using (1.5) write the bound

$$\frac{1}{n^{m-1}}\sum_{\hat{\boldsymbol{i}}_{m+1}}\sigma^2(\tilde{\boldsymbol{i}}_{m+1})\le C^{2(m-1)}. \tag{3.13}$$

By (3.13) and (1.4) we have shown that (3.10) is of order $o_m(1)$. For the second term (3.11) we can repeat the above procedure and stop after $m-1$ steps when we arrive at only two vertices in the tree. In the last step we can use the result (3.9). Finally we get

$$S_1=\frac{1}{n^{m+1}}\sum_{\Gamma_1(2m)}\sum_{\boldsymbol{i}_{m+1}}\sigma^2(\boldsymbol{i}_{m+1})=\frac{1}{m+1}\binom{2m}{m}+o_m(1),$$

which proves Theorem 1.4.

F. Götze, Faculty of Mathematics, Bielefeld University, Bielefeld, Germany

E-mail address: goetze@math.uni-bielefeld.de

A. Naumov, Faculty of Mathematics, Bielefeld University, Bielefeld, Germany, and Faculty of Computational Mathematics and Cybernetics, Moscow State University, Moscow, Russia

E-mail address: naumovne@gmail.com, anaumov@math.uni-bielefeld.de

A. Tikhomirov, Department of Mathematics, Komi Research Center of Ural Branch of RAS, Syktyvkar, Russia

E-mail address: tichomir@math.uni-bielefeld.de