POLYTOPES AND GROUPS

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Abstract. In this note we continue to study the question of which groups lead to affinely equivalent polytopes, started in [BHNP09]. There we introduced the notation of effectively equivalent permutation groups. Here we present an example showing that the latter do unfortunately not correspond to affinely equivalent polytopes thereby answering Question 2.12 of loc. cit. Furthermore we geometrically characterize the effectively equivalent permutation groups.

1. Introduction

The permutation polytopes are an interesting class of polytopes: we can associate to any finite group G of real $(n \times n)$-permutation matrices the *permutation polytope* $P(G)$ given by the convex hull of these matrices in the vector space $\mathbb{R}^{n\times n}$. They provide various important examples, such as the Birkhoff polytope B_n, which is defined as the convex hull of all $(n \times n)$-permutation matrices [BG77, BiSa96, BLO11, DLY09]. This polytope appears in various contexts in mathematics from optimization to statistics to enumerative combinatorics (see, e.g. [Tin86, Onn93, Pak00, BaSt03, Ath05, EFRS06, BHNP09, BHNP11, BHNP12]).

A systematic study of general permutation polytopes has been started in [BHNP09]. There, the authors studied which groups lead to affinely equivalent polytopes. Recall that two polytopes P and Q are affinely equivalent if there is an affine isomorphism between the affine hulls $\mathrm{aff}(P)$ and $\mathrm{aff}(Q)$ that maps P onto Q.

Isomorphic groups do not need to have affinely equivalent permutation polytopes: For instance $\langle (12), (34)\rangle$ and $\langle (12)(34), (13)(24)\rangle$ are isomorphic groups, but the associated permutation polytopes are a quadrangle and a tetrahedron, respectively, and therefore not affinely equivalent.

The notion of isomorphism of permutation groups is too restrictive to describe the affine equivalent permutation polytopes. In [BHNP09], Section 2.1, it has been observed that there are two permutation groups which are not isomorphic as permutation groups but whose permutation polytopes are affinely equivalent: The permutation polytopes of the following permutation groups are all tetrahedrons and therefore all affinely equivalent $\langle (1234)\rangle \leq \mathrm{Sym}(4)$, $\langle (1234)(5)\rangle \leq \mathrm{Sym}(5)$, $\langle (1234)(56)\rangle \leq \mathrm{Sym}(6)$,

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$\langle(1234)(56)(78)\rangle \leq \mathrm{Sym}(8)$, $\langle(1234)(5678)\rangle \leq \mathrm{Sym}(8)$, but the underlying groups are not isomorphic as permutation groups.

The notion of isomorphism of permutation groups has been generalized to the notion of effectively equivalent permutation groups in [BHNP09]; for the definition see the next section. All the permutation groups listed in the last paragraph are effectively equivalent permutation groups. The hope was that two permutation groups are effectively equivalent if and only if the groups are isomorphic and the corresponding permutation polytopes are affinely equivalent. In this note we present an example of two permutation groups which are isomorphic as abstract groups and whose permutation polytopes are affinely equivalent, but which are not effectively equivalent; thus the raised hope has to be abandoned. We further use representation theory of groups to characterize the effectively equivalent permutation groups via their associated permutation polytopes.

The organization of the note is as follows: Next we introduce the notation and recall the relevant previous results. In the third section we present the example and in the fourth section the effectively equivalent groups are characterized. In the last section we apply this characterization to our example.

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2. Notation and previous results

The convex and the affine hull of a set S in a real vector space will be denoted by $\mathrm{conv}(S)$ and by $\mathrm{aff}(S)$, respectively.

2.1. **Representation polytopes.** Let V be a real n-dimensional vector space. Then $\mathrm{GL}(V)$ denotes the set of automorphisms of V. By choosing a basis we can identify $\mathrm{GL}(V)$ with the set $\mathrm{GL}_n(\mathbb{R})$ of invertible $n \times n$-matrices. In the same way, we identify $\mathrm{End}(V)$ with the vector space $\mathrm{Mat}_n(\mathbb{R})$ of $n \times n$-matrices.

Let G be a group. A homomorphism $\rho\colon G \to \mathrm{GL}(V)$ is called a *representation*. If the latter is injective then it is a *faithful representation*. In this case

$$P(\rho) := \mathrm{conv}(\rho(g) \ : \ g \in G) \ \subseteq \mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n \times n}$$

is called the associated *representation polytope*.

The set of vertices of the representation polytope is precisely the set $\rho(G)$, as G acts regularly, and therefore transitively, on $\rho(G)$ by left multiplication (see [BHNP09], p. 3).

2.2. **Permutation polytopes.** An injective homomorphism $\pi : G \to \mathrm{Sym}(n)$ is called *permutation representation*. The pair (G, π) is called *permutation group*. In this case, we obtain a representation polytope as follows.

The symmetric group $\mathrm{Sym}(n)$ acts on the set $\{1,\dots,n\}$. Let V be an n-dimensional $\mathbb{R}$-vector space with basis $\{e_1,\dots,e_n\}$ and let $\mathrm{Sym}(n)$ act on this vector space by permuting the indices of the vectors in the basis. Then V is the so called *permutation module for* $\mathrm{Sym}(n)$. This module induces a representation $\Pi : \mathrm{Sym}(n) \to \mathrm{GL}(V) = \mathrm{GL}(\mathbb{R}^n)$. The homomorphism, $\Pi : g \mapsto M_g$ for all $g \in \mathrm{Sym}(n)$, identifies the symmetric group $\mathrm{Sym}(n)$ with the set of $n \times n$ *permutation matrices*, i.e. the set of matrices whose entries are 0 or 1 such that in every column and every row there is a unique 1. The representation polytope

$$P(G) := P(\pi) := P(G,\pi) := P(\Pi) = \mathrm{conv}(\Pi(G))$$

is called the *permutation polytope* associated to (G,π).

The special case $G = \mathrm{Sym}(n)$ yields the well-known nth *Birkhoff polytope* $B_n := P(\Pi(\mathrm{Sym}(n)))$ (see e.g. [BiSa96]).

Notice that every permutation polytope is in particular a representation polytope.

2.3. **Notions of equivalence of polytopes.** For a standard reference on polytopes we refer to [Zie95]. If the vertices of a polytope $R \subseteq \mathbb{R}^m$ are a subset of a full dimensional lattice Λ in $\mathbb{R}^m$, then we call P a lattice polytope. In this sense every permutation polytope is a lattice polytope, as the vertices all lie in $\mathrm{Mat}_n(\mathbb{Z})$.

As moreover, every vertex of a permutation polytope is a matrix whose entries are only 0 and 1, it is also a 0/1-*polytope*, i.e. a polytope whose vertices are in the set $\{0,1\}^d$ for some $d \in \mathbb{N}$.

There are several notations of equivalence of (lattice) polytopes (see also [Zie00]):

Definition 2.1. Two polytopes $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$ are affinely equivalent if there is an affine isomorphism of the affine hulls $\phi\colon \mathrm{aff}(P) \to \mathrm{aff}(Q)$ that maps P onto Q. For lattice equivalence we additionally require that ϕ is an isomorphism of the affine lattices $(\mathrm{aff}\, P) \cap \Lambda \to (\mathrm{aff}\, Q) \cap \Lambda'$. Combinatorial equivalence is an equivalence of the face lattices as posets.

The hierarchy of equivalence relations on lattice polytopes is as follows.

$$\begin{array}{ccccc} \text{lattice} \\ \text{equivalent} & \Rightarrow & \begin{array}{c}\text{affinely}\\ \text{equivalent}\end{array} & \Rightarrow & \begin{array}{c}\text{combinatorially}\\ \text{equivalent}\end{array} \end{array}$$

The converse implications do not hold (for examples see [Zie00, Prop. 7]).

2.4. **Notions of equivalence of groups.** When working with permutation polytopes, one would like to identify permutation groups that clearly define affinely equivalent permutation polytopes. Therefore, in [BHNP09] the notion of stable equivalence has been introduced. Here, $\mathbb{R}G$ denotes the group algebra of G with real coefficients.

Definition 2.2. For a representation $\rho\colon G \to \mathrm{GL}(V)$ define the affine kernel $\ker^\circ \rho$ as

$$\ker^\circ \rho := \left\{ \sum_{g\in G} \lambda_g g \in \mathbb{R}G \mid \sum_{g\in G} \lambda_g \rho(g) = 0 \text{ and } \sum_{g\in G} \lambda_g = 0 \right\}$$

Say that a real representation $\rho'\colon G \to \mathrm{GL}(V')$ is an *affine quotient* of ρ if $\ker^\circ \rho \subseteq \ker^\circ \rho'$. Then real representations ρ_1 and ρ_2 of G are *stably equivalent*, if there are affine quotients ρ_1' of ρ_1 and ρ_2' of ρ_2 such that $\rho_1 \oplus \rho_1' \cong \rho_2 \oplus \rho_2'$ as G-representations. Two faithful real representations $\rho_i : G_i \to \mathrm{GL}(V_i)$ (for $i = 1, 2$) of finite groups are *effectively equivalent* if there exists an isomorphism $\varphi : G_1 \to G_2$ such that ρ_1 and $\rho_2 \circ \varphi$ are stably equivalent G_1-representations. Moreover, we say $G_1 \leq \mathrm{Sym}(n_1)$ and $G_2 \leq \mathrm{Sym}(n_2)$ are *effectively equivalent permutation groups* if $G_1 \hookrightarrow \mathrm{Sym}(n_1)$ and $G_2 \hookrightarrow \mathrm{Sym}(n_2)$ are effectively equivalent permutation representations.

Example 2.3. *The following representations of the group $\mathbb{Z}_4$ are stably equivalent:*

$$\begin{aligned} \langle (1234) \rangle &\leq S_4\,, & \langle (1234)(5) \rangle &\leq S_5\,, \\ \langle (1234)(56) \rangle &\leq S_6\,, & \langle (1234)(56)(78) \rangle &\leq S_8\,, \\ \langle (1234)(5678) \rangle &\leq S_8\,. \end{aligned}$$

For $K = \mathbb{R}$ or $\mathbb{C}$ we denote by $\mathrm{Irr}_K(G)$ the set of pairwise non-isomorphic irreducible K-representations, i.e. homomorphisms $G \to \mathrm{GL}(W)$ where W is a K-vector space which does not contain a proper G-invariant subspace. For instance, there is the *trivial representation*, $1_G\colon G \to \mathrm{GL}(K)$, $g \mapsto 1$. Every representation $\rho : G \to \mathrm{GL}(V)$ over K splits into irreducible representations. We denote these *irreducible factors* of ρ by $\mathrm{Irr}_K(\rho) \subseteq \mathrm{Irr}_K(G)$.

Theorem 2.4 (Baumeister et al. [BHNP09, 2.4])**.** *Two real representations are stably equivalent if and only if they contain the same non-trivial irreducible factors.*

Moreover, we showed that if ρ and $\bar{\rho}$ are two stably equivalent real representations of a finite group G, then $P(\rho)$ and $P(\bar{\rho})$ are affinely equivalent, see [BHNP09, 2.3]. If π_1 and π_2 are effectively equivalent permutation representations, then π_1 and $\pi_2 \circ \varphi$ are stably equivalent for some isomorphism $\varphi : G_1 \to G_2$. As $P(\pi_2) = P(\pi_2 \circ \varphi)$ the following holds as well:

Proposition 2.5. *The permutation polytopes related to two effectively equivalent permutation representations are affinely equivalent.*

The following example shows that effectively equivalent permutation representations do not necessarily have lattice equivalent permutation polytopes (see also [BHNP09, 2.9]):

Example 2.6. Let $G := \langle (12), (34) \rangle \leq \mathrm{Sym}(4)$ and let π_1 be the embedding $(12) \mapsto (12)(34)$, $(34) \mapsto (13)(24)$ of G into $\mathrm{Sym}(4)$. Then π_1 is the regular representation of G, that is π_1 is permutation equivalent to the action of G on G by right multiplication. We define another permutation representation $\pi_2 : G \to \mathrm{Sym}(6)$ by $(12) \mapsto (12)(34)$ and $(34) \mapsto (12)(56)$. Then $P(\Pi_i)$ is a tetrahedron (for $i = 1, 2$) and π_1 is stably equivalent to π_2. However, the vertices of $P(\Pi_2)$ do not form a lattice basis of the lattice $\mathrm{aff}(P(\Pi_2)) \cap \mathrm{Mat}_{|G|}(\mathbb{Z})$, as

$$\frac{1}{2}[\Pi_2((12)) + \Pi_2((34)) + \Pi_2((12)(34)) - \Pi_2(id)]$$

is in $\mathrm{aff}(P(\Pi_2)) \cap \mathrm{Mat}_{|G|}(\mathbb{Z})$, but it is not an integer combination of the matrices in $\Pi_2(G)$ (and therefore the volume of $P(\Pi_2)$ is $2/6$). According to [BHNP09, 2.7] the vertices of $P(\Pi_1)$ form a lattice basis of the lattice $\mathrm{aff}(P(\Pi_1)) \cap \mathrm{Mat}_{|G|}(\mathbb{Z})$ (and therefore $P(\Pi_1)$ has volume $1/6$).

This example also shows that the volumes of two permutation polytopes associated to effectively equivalent permutation representations may be different.

3. The example

In this section we present the example of a group with two non effectively equivalent permutation representations such that the related permutation polytopes are affinely equivalent. But first we show the following "almost example". It consists of two permutation groups which are not stably equivalent, but whose permutation polytopes are even equal. It is not really an example to our question as the permutation groups are effectively equivalent.

3.1. **An "almost example".** Let $G = \mathrm{Alt}(6)$. Then G contains two different subgroups H_1 and H_2 which are both isomorphic to $\mathrm{Alt}(5)$, but not conjugate in G. We may choose H_1 as the stabilizer of 1 in the action of G on the set $[6] := \{1, \ldots, 6\}$. Then H_2 is transitive on $[6]$. The group G acts on both coset spaces G/H_1 and G/H_2, which yields two permutation representations π_1 and π_2. These representations are not stably equivalent, as they contain different irreducible constituents, see for instance [CNPW85], p. 4. On the other hand, as $|G : H_i| = 6$ for $i = 1, 2$, both representations π_1 and π_2 induce embeddings of G into $\mathrm{Sym}(6)$. Since in $\mathrm{Sym}(6)$ there is only one subgroup isomorphic to $\mathrm{Alt}(6)$, it follows that $\pi_1(G) = \pi_2(G)$. This implies that $\Pi_1(G) = \Pi_2(G)$ are both precisely the set of even permutation matrices. Therefore, the polytopes $P(\pi_1)$ and $P(\pi_2)$ are equal. Notice as there is an automorphism of G mapping H_1 onto H_2, the two representations π_1 and π_2 are effectively equivalent.

3.2. **The example.** Let $A = (\mathbb{Z}_2)^2$ be the direct product of two cyclic groups of order two, $B = \mathbb{Z}_4, C = \mathbb{Z}_3$ cyclic groups of order 4 and 3, and let $G := A \times B \times C$. In the following we define two different permutation

representations of G:

The permutation representation π_1. Let O_1 be the disjoint union of the right coset spaces $O_{11} := G/A$ and $O_{12} := G/(B \times C)$ and let G act by left multiplication on O_1. Then O_{11} and O_{12} are the G-orbits. The kernels of the action of G on O_{11} and O_{12} are A and $B \times C$, respectively. By Lemma 2.7 and Theorem 3.5 of [BHNP09] $P(\pi_1)$ is the combinatorial product of an 11-simplex with a 3-simplex.

Notice, if $G = H \times K$ for some subgroups H and K of G, then we can extend every irreducible complex representation φ_H of H to an irreducible complex representation φ of G by sending every element of K to the identity. Therefore, we can embed $\mathrm{Irr}_{\mathbb{C}}(H)$ into $\mathrm{Irr}_{\mathbb{C}}(G)$. In this sense $\mathrm{Irr}_{\mathbb{C}}(\Pi_1)$ is the union of $\mathrm{Irr}_{\mathbb{C}}(B \times C)$ and $\mathrm{Irr}_{\mathbb{C}}(A)$. As for an abelian group $\mathrm{Irr}_{\mathbb{C}}(G) \cong G$, see Paragraph 6, 6.4 in [Hup67], it follows that $\mathrm{Irr}_{\mathbb{C}}(\Pi_1)$ is the union of two subgroups isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_3$ and $(\mathbb{Z}_2)^2$, respectively.

The permutation representation π_2. Let O_2 be the disjoint union of the right coset spaces $O_{21} := G/B$ and $O_{22} := G/(A \times C)$ and let G act by left multiplication on O_2. Then O_{21} and O_{22} are the G-orbits. The kernels of the action of G on O_{21} and O_{22} are B and $A \times C$, respectively. By Lemma 2.7 and Theorem 3.5 of [BHNP09] $P(\pi_2)$ is again the combinatorial product of an 11-simplex with a 3-simplex.

Here $\mathrm{Irr}_{\mathbb{C}}(\Pi_2)$ is the union of $\mathrm{Irr}_{\mathbb{C}}(A \times C)$ and $\mathrm{Irr}_{\mathbb{C}}(B)$ and therefore, the union of two subgroups isomorphic to $(\mathbb{Z}_2)^2 \times \mathbb{Z}_3$ and $\mathbb{Z}_4$.

It follows that $P(\pi_1)$ and $P(\pi_2)$ are affinely equivalent. In $\mathrm{Irr}_{\mathbb{C}}(\Pi_1)$ there is an irreducible representation of order 12, while every element in $\mathrm{Irr}_{\mathbb{C}}(\Pi_2)$ has order at most 6. This shows that the induced real representations Π_1 and $\Pi_2 \circ \varphi$ do not contain the same irreducible factors for every automorphism $\varphi \in \mathrm{Aut}(G)$. Thus π_1 and π_2 are not effectively equivalent.

4. Characterization: effectively equivalence

Let $G = (G, \pi)$ be a permutation group of degree n with permutation module V and representation $\Pi : G \to \mathrm{GL}(V)$. Then the associated permutation polytope is $P = P(\Pi) = \mathrm{conv}(M_g \mid g \in G)$. The affine hull of the polytope is

$$\mathrm{aff}(\Pi(G)) = \{\sum_{g\in G} \lambda_g M_g \mid \lambda_g \in \mathbb{R}, \sum_{g\in G} \lambda_g = 1\} = E_n + U_\Pi, \text{ where}$$

$$U_\Pi := \{\sum_{g\in G} \lambda_g M_g \mid \lambda_g \in \mathbb{R}, \sum_{g\in G} \lambda_g = 0\} = \{\sum_{g\in G} \lambda_g (M_g - M_e) \mid \lambda_g \in \mathbb{R}\}$$
$$= \langle M_g - M_h \mid g, h \in G\rangle_{\mathbb{R}} \leq \mathrm{End}(V).$$

The $\mathbb{R}$-vector space U_Π is a G-module through the definition:

$$hM := M_h M = \Pi(h)M, \text{where } h \in G \text{ and } M \in U_\Pi,$$

as Π is a group homomorphism from G into $\mathrm{GL}_n(\mathbb{R})$. Notice, that if in particular $M = \Pi(g)$, then $hM = \Pi(hg)$.

In order to nicely describe the structure of U_Π we introduce more notation. For $\chi \in \mathrm{Irr}_\mathbb{R}(G)$ let $V(\chi)$ be an irreducible $\mathbb{R}G$-module with character χ. Then $\mathrm{End}_{\mathbb{R}G}(V(\chi))$ is either isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and thus $d_\chi := 1/\dim_\mathbb{R} \mathrm{End}_{\mathbb{R}G}(V(\chi))$ is either 1, $\frac{1}{2}$ or $\frac{1}{4}$. Set

$$V_\Pi := \{\sum_{g\in G} \lambda_g M_g \mid \lambda_g \in \mathbb{R}\} \text{ and } \epsilon_\Pi := \sum_{g\in G} M_g.$$

Then $V_\Pi = U_\Pi \oplus \mathbb{R}\epsilon_G$.

Theorem 4.1. *Let $G = (G, \pi)$ be a permutation group of degree n with permutation module V. Then U_Π is isomorphic to*

$$\sum_{\chi\in \mathrm{Irr}_\mathbb{R}(\Pi)\setminus\{1_G\}} (d_\chi \cdot \chi(1))V(\chi)$$

as an $\mathbb{R}G$-module, where for d a natural number $dV(\chi)$ is the direct sum of d $\mathbb{R}G$-modules which are isomorphic to $V(\chi)$.

Proof. Extend Π linearly to an $\mathbb{R}$-algebra epimorphism $\Pi : \mathbb{R}G \to V_\Pi$ by setting $\Pi(\sum_{g\in G} \lambda_g g) := \sum_{g\in G} \lambda_g \Pi(g)$. By Maschke's Theorem, the group algebra $\mathbb{R}G$ is semi-simple. Thus by a theorem by Wedderburn (see for instance [Hup67] Chapter V, Hauptsatz 4.4) we get $\mathbb{R}G = \oplus_{i=1}^m A_i$ with A_i simple.

Since A_i is simple, either $\Pi|A_i$ is injective or $\Pi(A_i) = 0$. Thus V_Π is semi-simple as well and we can assume that there is an integer $k \leq m$ such that $V_\Pi \cong \oplus_{i=1}^k A_i$. Moreover, we can assume that $A_1 = \mathbb{R}\epsilon$ with $\epsilon = \sum_{g\in G} g$. Since $\Pi(\epsilon) = \epsilon_\Pi$, we have $U_\Pi = \oplus_{i=2}^k A_i$.

Let $\chi \in Irr_\mathbb{R}(G)$ be an irreducible representation of G. Then $V(\chi)$ is a composition factor of the G-module V if and only if there is an index $1 \leq i \leq k$ such that the $\mathbb{R}$-linear extension of χ does not vanish on A_i (or equivalently, $A_i = e_\chi \mathbb{R}G$ with

$$e_\chi = \sum_{\psi\in Irr_\mathbb{C}(G|\chi)} e_\psi = \sum_{\psi\in Irr_\mathbb{C}(G|\chi)} \frac{1}{|G|}\psi(1) \sum_{g\in G} \overline{\psi(g)}g$$

the central idempotent corresponding to χ). In this case,

$$\begin{aligned}\dim_\mathbb{R} A_i &= \dim_\mathbb{R} End_{\mathbb{R}G}(V(\chi)) \cdot (\dim_{End_{\mathbb{R}G}(V(\chi))} V(\chi)))^2 \\ &= (1/d_\chi) \cdot (d_\chi \cdot \chi(1))^2 = d_\chi \cdot \chi(1)^2\end{aligned}$$

and $A_i \cong \mathrm{End}_\mathbb{R}(V(\chi)) \cong (d_\chi \cdot \chi(1))V(\chi)$ as an $\mathbb{R}G$-module, see [Hup67], Chapter V, Satz 4.5. Since ϵ is the central idempotent corresponding to the trivial representation, the claim follows. □

The following lemma shows that affine maps between permutation polytopes are always induced by linear maps of their linear hulls.

Lemma 4.2. *If V is a $\mathbb{R}$-vectorspace and $\psi : \operatorname{aff}(P(\Pi)) \to V$ is an affine map, then ψ can be uniquely lifted to a linear map $\Psi : V_\Pi \to V$.*

Proof. Since $P(\Pi)$ contains a basis of V_Π, the space V_Π is the affine hull of $\{0\}$ and $P(\Pi)$. Moreover, every element in $\operatorname{aff}(P(\Pi))$ is a matrix whose rows and columns all have sum 1. This shows that $0 \notin \operatorname{aff}(P(\Pi))$. Thus there is a unique affine map $\Psi : V_\Pi \to V$ such that $\Psi|\operatorname{aff}(P(\Pi)) = \psi$ and $\Psi(0) = 0$. Since $\Psi(0) = 0$, the map Ψ is linear. □

Theorem 4.1 implies the following characterization of the effectively equivalent permutation groups:

Theorem 4.3. *Let $G_1 = (G, \pi_1)$ and $G_2 = (G, \pi_2)$ be two permutation groups. Then the following are equivalent:*

(a) *G_1 and G_2 are effectively equivalent.*
(b) *There is a group automorphism φ of G and an isomorphism $\Phi : U_{\Pi_1} \to U_{\Pi_2}$ with $\Phi(gu) = \varphi(g)\Phi(u)$ for all $u \in U_{\Pi_1}$ and all $g \in G$.*
(c) *There is a group automorphism φ of G and an isomorphism $\Phi : V_{\Pi_1} \to V_{\Pi_2}$ with $\Phi(gu) = \varphi(g)\Phi(u)$ for all $u \in V_{\Pi_1}$ and all $g \in G$.*
(d) *There is an affine isomorphism $\Phi : \operatorname{aff}(P(\Pi_1)) \to \operatorname{aff}(P(\Pi_2))$ which maps $P(\Pi_1)$ to $P(\Pi_2)$ and which restricted to $\Pi_1(G)$ is a group homomorphism.*

Proof. Suppose that (b) holds, then by Theorem 4.1 $U_{\Pi_2 \circ \varphi}$ and U_{Π_1} must have the same irreducible constituents. Thus $\Pi_2 \circ \varphi$ and Π_1 are stably equivalent and G_1 and G_2 are effectively equivalent; so (a) holds. Statements (b) and (c) are equivalent since V_{Π_i} and U_{Π_i} only differ by the trivial $\mathbb{R}G$-module.

Suppose that (a) holds. Then there is an automorphism φ of G such that Π_1 and $\Pi_2 \circ \varphi$ are stably equivalent. Thus if $(\lambda_g)_{g\in G}$ is a family of real numbers with $\sum_{g\in G} \lambda_g = 0$, then by Definition 2.2

$$\sum_{g\in G} \lambda_g \Pi_1(g) = 0 \text{ if and only if } \sum_{g\in G} \lambda_g \Pi_2(\varphi(g)) = 0.$$

Thus the map

$$\Phi : \operatorname{aff}(P(\Pi_1)) \to \operatorname{aff}(P(\Pi_2)), \sum_{g\in G} \lambda_g \Pi_1(g) \mapsto \sum_{g\in G} \lambda_g \Pi_2(\varphi(g)),$$

where $\sum_{g\in G} \lambda_g = 1$, is a well-defined affine map, since if $(\lambda_g)_{g\in G}$ and $(\mu_g)_{g\in G}$ are two families of real numbers with

$$\sum_{g\in G} \lambda_g = 1 = \sum_{g\in G} \mu_g$$

and

$$\sum_{g\in G} \lambda_g \Pi_1(g) = \sum_{g\in G} \mu_g \Pi_1(g),$$

then $\sum_{g\in G}(\lambda_g - \mu_g) = 0$ and $\sum_{g\in G}(\lambda_g - \mu_g)\Pi_1(g) = 0$. Therefore,

$$\sum_{g\in G}(\lambda_g - \mu_g)\Pi_2(\varphi(g)) = 0$$

and

$$\sum_{g\in G}\lambda_g\Pi_2(\varphi(g)) = \sum_{g\in G}\mu_g\Pi_2(\varphi(g)).$$

By the same argument, Φ is injective, and as the image of Φ affinely spans $\mathrm{aff}(P(\pi_2))$, the map Φ is surjective as well. As Π_1, Π_2 and φ are group homomorphisms, the restriction of Φ on $\Pi_1(G)$ is a group homomorphism onto $\Pi_2(G)$. This shows that (a) implies (d).

Suppose that (d) holds. We want to show (c). First note that Φ maps $\Pi_1(G)$ bijectively onto $\Pi_2(G)$ (since these are vertices of the corresponding polytopes and Φ is an affine isomorphism) and thus induces a group isomorphism between $\Pi_1(G)$ and $\Pi_2(G)$ which we will also call Φ. Then $\varphi := \Pi_2^{-1} \circ \Phi \circ \Pi_1$ is a group automorphism of G. If $u = \sum_{g\in G}\lambda_g\Pi_1(g)$ with $\sum_{g\in G}\lambda_g = 1$ and $x \in G$, then

$$\Phi(xu) = \Phi(x\sum_{g\in G}\lambda_g\Pi_1(g)) = \Phi(\sum_{g\in G}\lambda_g x\Pi_1(g)) = \sum_{g\in G}\lambda_g\Phi(\Pi_1(xg)) =$$

$$\sum_{g\in G}\lambda_g\Phi(\Pi_1(x))\Phi(\Pi_1(g)) = \sum_{g\in G}\lambda_g\Pi_2(\varphi(g))\Phi(\Pi_1(g)).$$

This equals by the definition of the action of G on U_{Π_2} (see the beginning of the forth section) the sum

$$\sum_{g\in G}\lambda_g\varphi(x)\Phi(\Pi_1(g)) = \varphi(x)\sum_{g\in G}\lambda_g\Phi(\Pi_1(g)) = \varphi(x)\Phi(u).$$

By 4.2 we can extend Φ to a linear isomorphism $\Psi : V_{\Pi_1} \to V_{\Pi_2}$ for which one easily sees that $\Psi(gu) = \varphi(g)\Psi(u)$ holds for all $g \in G$ and all $u \in V_{\Pi_1}$. □

The equivalence between (a) and (d) yields another possibility to describe effectively equivalence. If P and Q are two polytopes, G a group which acts as automorphism group on both P and Q and if φ an automorphism of G, then an affine isomorphism $\Phi : \mathrm{aff}(P) \to \mathrm{aff}(Q)$ with $\Phi(P) = Q$ is called an affine (G, φ)-isomorphism if $\Phi(gx) = \varphi(g)\Phi(x)$ holds for all $g \in G$ and all vertices x of P. If π is a permutation representation of a finite group G, then left multiplication defines a natural action of G on $P(\Pi)$ as we saw above. Then the equivalence between (a) and (d) of 4.3 gives us:

Corollary 4.4. *If $G_1 = (G, \pi_1)$ and $G_2 = (G, \pi_2)$ are two permutation groups, then they are effectively equivalent if and only if there is an automorphism φ of G and an affine (G, φ)-isomorphism between the corresponding permutation polytopes.*

5. Application of Theorem 4.3 to the example

In this section we apply our characterization of the effectively equivalent permutation groups given in Theorem 4.3 to give a new, direct and more geometric proof of the fact that the permutation groups (G, π_1) and (G, π_2) presented in Example 3.2 are not effectively equivalent. We continue to use the notation introduced in Example 3.2.

Suppose that (G, π_1) and (G, π_2) are effectively equivalent. Then according to Theorem 4.3 there is an affine isomorphism $\Phi : \mathrm{aff}(P(\Pi_1)) \to \mathrm{aff}(P(\Pi_2))$ which maps $P(\Pi_1)$ to $P(\Pi_2)$ and which restricted to $\Pi_1(G)$ is a group homomorphism.

Let H be a subgroup of G such that $\Pi_1(H)$ is the set of vertices of a face of the polytope $P(\Pi_1)$. Then $\Pi_1(H)$ is a subgroup of $\Pi_2(G)$ which implies that $\Phi(\Pi_1(H))$ is a subgroup of $\Pi_2(G)$. As Φ is an affine isomorphism from $\mathrm{aff}(P(\Pi_1))$ to $\mathrm{aff}(P(\Pi_2))$ as well, the set $\Phi(\Pi_1(H))$ is the set of vertices of a face of the polytope $P(\Pi_2)$.

Now we count all the faces of $P(\Pi_i)$ which have 24 vertices and whose set of vertices is a subgroup of $\Pi_i(G)$ (for $i = 1, 2$). The polytope $P(\Pi_i)$ is the product of an 11-simplex with a 3-simplex. Therefore every face of $P(\Pi_i)$ has the shape $E \times F$ where E is a face of the 11-simplex and F a face of the 3-simplex (for $i = 1, 2$).

The faces of $P(\Pi_1)$ given by a subgroup of size 24. In this case $G = H_1 \times H_2$ where $H_1 \cong \mathbb{Z}_{12}$ and $H_2 \cong \mathbb{Z}_2^2$; and $P(\Pi_1) = P(H_1) \times P(H_2)$. Further $P(H_1)$ is an 11-simplex and $P(H_2)$ a 3-simplex. If H is a subgroup of G such that $\Pi_1(H)$ is the set of vertices of a face with 24 vertices, then $H = K_1 \times K_2$ such that K_i is a subgroup of H_i (for $i = 1, 2$) and such that $|H| = |K_1| \cdot |K_2| = 24$. Then either $K_1 = H_1$ and K_2 of order 2 or K_1 is of order 6 and $K_2 = H_2$. As there is just one subgroup of order 6 in H_1 and three subgroups of order 2 in H_2, it follows that the 24-vertex faces which are coming from a subgroup are precisely three faces of the shape of a prisma over an 11-simplex and one face which is the product of a 5-simplex with a 3-simplex.

The faces of $P(\Pi_2)$ given by a subgroup of size 24. Here we have the factorization $G = M_1 \times M_2$ where $M_1 \cong \mathbb{Z}_2^2 \times 3$ and $M_2 \cong \mathbb{Z}_4$. The polytope $P(\Pi_2) = P(M_1) \times P(M_2)$ is the product of an 11-simplex and a 3-simplex. If H is a subgroup of G such that $\Pi_2(H)$ is the set of vertices of a face with 24 vertices, then as above $H = K_1 \times K_2$ such that K_i is a subgroup of M_i (for $i = 1, 2$) and such that $|H| = |K_1| \cdot |K_2| = 24$. In this case there are three subgroups of M_1 of size 6 and precisely one subgroup of K_2 of size 2. Therefore, the 24-vertex faces which are coming from a subgroup are precisely one prisma over an 11-simplex and three faces which are the product of a 5-simplex and a 3-simplex.

This contradicts the fact that Φ maps every face of $P(\Pi_1)$ which is induced

by a subgroup of G isomorphically onto a face of $P(\Pi_1)$ which is induced by a subgroup of G. Thus (G, π_1) and (G, π_2) are not effectively equivalent.

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