THE BOSON STAR EQUATION WITH INITIAL DATA OF LOW REGULARITY

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Abstract. The Cauchy problem for the L^2-critical boson star equation with initial data of low regularity in spatial dimension $d = 3$ is studied. Local well-posedness in H^s for $s > 1/4$ is proved. Moreover, for radial initial data, local well-posedness is established in H^s for $s > 0$. Both results are shown to be almost optimal by providing complementary ill-posedness results.

1. Introduction and Main Results

We consider the initial value problem

$$\begin{aligned} i\partial_t u &= \sqrt{-\Delta + m^2}\, u - (|x|^{-1} * |u|^2)u \quad \text{in } (-T,T) \times \mathbb{R}^3, \\ u(0,\cdot) &= \phi \in H^s(\mathbb{R}^3). \end{aligned} \tag{1}$$

Here $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{\xi^2 + m^2}$ in Fourier space, where the constant $m \geq 0$ is a physical mass parameter, and the symbol $*$ denotes convolution in $\mathbb{R}^3$. The nonlinear dispersive evolution problem (1) arises as an effective equation describing the dynamics and gravitational collapse of relativistic boson stars; see [16, 7, 14, 8] and references therein. Given this physical background, we shall also refer to equation (1) as *the boson star equation* in the following.

We recall that the boson star equation exhibits the following conserved quantities of energy and L^2-mass, which are given by

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^3} \overline{u}\sqrt{-\Delta + m^2}\, u\, dx - \frac{1}{4}\int_{\mathbb{R}^3} (|x|^{-1} * |u|^2)|u|^2\, dx,$$

$$M(u(t)) = \int_{\mathbb{R}^3} |u|^2\, dx.$$

From these conservation laws, we see that the Sobolev space $H^{1/2}(\mathbb{R}^3)$ serve as the energy space for problem (1). Furthermore, in the case of

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vanishing mass parameter $m = 0$ in (1), we have the scaling symmetry

$$u(t,x) \mapsto u_\lambda(t,x) = \lambda^{3/2} u(\lambda t, \lambda x)$$

for fixed $\lambda > 0$. Clearly, the symmetry leaves the L^2-mass $M(u(t)) = M(u_\lambda(t))$ invariant. In this sense, the boson star equation exhibits the delicate feature of *L^2-criticality*.

Our aim is to determine minimal regularity requirements on the initial data in the Sobolev scale $H^s(\mathbb{R}^3)$ and $H^s_{rad}(\mathbb{R}^3)$ (subspace of radial functions), respectively, such that the initial value problem for the boson star equation is locally well-posed. Let us state our first main result.

Theorem 1.1. *Let $m \geq 0$.*

(i) *Let $s > 1/4$. The initial value problem* (1) *is locally well-posed for initial data in $H^s(\mathbb{R}^3)$.*
(ii) *Let $s > 0$. The initial value problem* (1) *is locally well-posed for initial data in $H^s_{rad}(\mathbb{R}^3)$.*

In both cases, the notion of local well-posedness includes existence of solutions up to some time $T > 0$, uniqueness of solutions in a certain subspace, persistence of initial regularity, and analytic dependence on the initial data.

A first well-posedness result for the boson star equation was obtained by the second author of this paper (see [12]) in H^s for $s \geq 1/2$ by energy methods. Moreover, it was shown in [12] that global well-posedness holds in $H^{1/2}$ for initial data sufficiently small in L^2. In contrast to this, see [9] for the existence of finite-time blowup solutions with smooth initial data that are large in L^2. Moreover, we refer the reader to [4, 5, 6] for further well-posedness results for the boson star equation with initial data in H^s with s slightly below $1/2$.

Our second result shows that Theorem 1.1 is essentially sharp in the following sense.

Theorem 1.2. *Let $m \geq 0$.*

(i) *Let $s < \frac{1}{4}$ and $T > 0$. If the flow map $\phi \mapsto u$ exists (in a small neighbourhood of the origin) as a map from $H^s(\mathbb{R}^3)$ to $C([0,T], H^s(\mathbb{R}^3))$, it fails to be C^3 at the origin.*
(ii) *Let $s < 0$ and $T > 0$. If the flow map $\phi \mapsto u$ exists (in a small neighbourhood of the origin) as a map from $H^s_{rad}(\mathbb{R}^3)$ to $C([0,T], H^s_{rad}(\mathbb{R}^3))$, it fails to be C^3 at the origin.*

Ill-posedness results similar to Theorem 1.2 have been proved in [17, Theorem 2] for the Benjamin-Ono equation and in [3, pp. 155–158] for the Korteweg-de Vries equation. Compared to the L^2-critical NLS, it is interesting to note that the boson star equation (1) exhibits ill-posedness for non-radial data (in the sense given above) in a regularity class above the critical space L^2. Not so surprisingly, such a phenomenom of ill-posedness above scaling is much more akin to nonlinear wave equations (see [15]).

We remark that both Theorem 1.1 and Theorem 1.2 remain true in the defocusing case, i.e. we may replace $-|x|^{-1}$ by $+|x|^{-1}$ in (1).

For radial data in $L^2(\mathbb{R}^3)$ we prove the failure of uniform continuity on balls, similar to the results in [1] for nonlinear Schrödinger and generalized Benjamin-Ono equations:

Theorem 1.3. *Let $m \geq 0$ and $T > 0$. If the flow map $\phi \mapsto u$ exists as a map from $L^2_{rad}(\mathbb{R}^3)$ to $C([0,T], L^2_{rad}(\mathbb{R}^3))$, it fails to be locally uniformly continuous.*

We refer the reader to Propositions 3.3 and 3.7 for more precise statements. The proof of Theorem 1.3 utilizes the fact that (1) has solitary wave solutions $u(t,x) = e^{it\mu}Q_\mu(x)$, where $Q_\mu \in H^{1/2}(\mathbb{R}^3)$ solves the nonlinear ellipitic equation

$$\sqrt{-\Delta + m^2}\, Q_\mu + \mu Q_\mu - (|x|^{-1} * |Q_\mu|^2)Q_\mu = 0,$$

with $\mu > 0$ given. In the case $m = 0$, the exact scaling properties simplify the analysis considerably and we can adapt an argument in [1] in order to prove Theorem 1.3. In contrast to this, the case $m > 0$ breaking exact scaling deserves an additional discussion of the elliptic problem for Q_μ.

When studying (1), it might be convenient to add the linear term mu by considering the function $e^{-itm}u$ instead of u, which turns the equation (1) into

$$i\partial_t u = (\sqrt{-\Delta + m^2} - m)u - (|x|^{-1} * |u|^2)u. \tag{2}$$

Furthermore, by rescaling u, it suffices to consider either $m = 0$ or $m = 1$ in the proofs.

The paper is organized as follows: In Section 2, we prove Theorem 1.1 concerning local well-posedness. In Section 3, we prove Theorems 1.2 and Theorem 1.3 on ill-posedness.

2. Well-posedness

Let us fix some notation. We denote the Fourier transform of a tempered distribution f both by $\mathcal{F}f$ and $\widehat{f}$, and to indicate a partial Fourier transform with respect to time and space variables we also write $\mathcal{F}_t f$ and $\mathcal{F}_x f$, respectively.

We denote dyadic numbers $\lambda = 2^l : l \in \mathbb{Z}$ by greek letters. Further, for $\lambda > 1$ we define dyadic annuli

$$\Delta_\lambda := \{\xi \in \mathbb{R}^3 : \lambda/2 < |\xi| \leq 2\lambda\}, \text{ and } \Delta_1 := \{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}.$$

We fix an even function $\beta_1 \in C_0^\infty((-2,2))$ s.t. $\beta_1(s) = 1$ if $|s| \leq 1$, and define $\beta_\lambda(s) = \beta(s\lambda^{-1}) - \beta(2s\lambda^{-1})$ for $\lambda > 1$. Next, we define the (smooth) Fourier localization operators in the standard way:

$$P_\lambda f := f_\lambda := \mathcal{F}^{-1}(\beta_\lambda(|\cdot|)\mathcal{F}f),$$

and $P_{\leq\lambda} = \sum_{\mu=1}^{\lambda} P_\mu$. For measurable sets $S \subset \mathbb{R}^n$, let χ_S denote the sharp cutoff function, i.e. define $\chi_S(x) = 1$ if $x \in S$ and zero otherwise. We set

$$P_S f = \mathcal{F}^{-1}(\chi_S \mathcal{F} f).$$

For fixed $m \geq 0$ let $U(t)$ be the linear propagator defined by

$$\mathcal{F}_x(U(t)\phi)(\xi) = e^{it\varphi_m(\xi)}\mathcal{F}_x\phi(\xi), \qquad \varphi_m(\xi) = m - \sqrt{m^2 + |\xi|^2}.$$

Definition 2.1. Let $s, b \in \mathbb{R}$. We define the space $X^{s,b}$ of tempered distributions $u \in \mathcal{S}(\mathbb{R}^{3+1})$ such that

$$\|u\|_{s,b} := \left(\int_{\mathbb{R}^{3+1}} \langle\xi\rangle^{2s}\langle\tau + |\xi|\rangle^{2b}|\mathcal{F}u(\xi,\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}} < \infty. \tag{3}$$

Further, $X^{s,b}_{rad}$ denotes the closed subspace of spatially radial distributions.

Concerning well-posedness, it suffices to consider the case $m = 0$ as long as we are considering short time scales only. Indeed, $\langle\tau + |\xi|\rangle \sim \langle\tau + \sqrt{m + |\xi|^2} - m\rangle$ and the corresponding $\|\cdot\|_{s,b}$-norms are equivalent.

We first recall Strichartz type estimates for the wave equation. They have been studied systematically in [18], see also references therein.

Lemma 2.2. *Let $b > \frac{1}{2}$.*

(i) *For any ball B_μ with radius $\mu \geq 0$ and arbitrary center, and for all $u \in X^{\frac{1}{4},b}$ it holds*

$$\|P_{B_\mu} u\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \mu^{\frac{1}{4}}\|u\|_{\frac{1}{4},b}. \tag{4}$$

(ii) *For all $u \in X^{\frac{1}{2},b}$ it holds*

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|u\|_{\frac{1}{2},b}. \tag{5}$$

(iii) *For all $\mu \geq 0$ and for all $u_1, u_2 \in X^{\frac{1}{4},b}$ it holds*

$$\|P_\mu(\widetilde{u}_1\widetilde{u}_2)\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \mu^{\frac{1}{2}}\|u_1\|_{\frac{1}{4},b}\|u_2\|_{\frac{1}{4},b}, \tag{6}$$

where $\widetilde{u}_j$ denotes either u_j or $\overline{u}_j$.

Proof. Part i) Due to [18, Theorem 4.1] and the extension lemma [10, Lemma 2.3] we have

$$\|P_{B_\mu} P_\lambda u\|_{L^4(\mathbb{R}\times\mathbb{R}^3)} \lesssim \mu^{\frac{1}{4}} \lambda^{\frac{1}{4}} \|P_\lambda u\|_{0,b}.$$

By Littlewood-Paley theory it follows that

$$\begin{aligned}\|P_{B_\mu} u\|_{L^4} &\lesssim \Big(\sum_{\lambda\geq 1} \|P_\lambda P_{B_\mu} u\|^2_{L^4(\mathbb{R}\times\mathbb{R}^3)}\Big)^{\frac{1}{2}}\\ &\lesssim \mu^{\frac{1}{4}} \Big(\sum_{\lambda\geq 1} \lambda^{\frac{1}{2}} \|P_\lambda u\|^2_{0,b}\Big)^{\frac{1}{2}} \lesssim \mu^{\frac{1}{4}} \|u\|_{\frac{1}{4},b}.\end{aligned}$$

Part ii) The Littlewood-Paley estimate and Part i) with $\mu \sim \lambda$ imply

$$\|u\|_{L^4} \lesssim \Big(\sum_{\lambda\geq 1} \|P_\lambda u\|^2_{L^4(\mathbb{R}\times\mathbb{R}^3)}\Big)^{\frac{1}{2}} \lesssim \Big(\sum_{\lambda\geq 1} \|P_\lambda u\|^2_{0,b}\Big)^{\frac{1}{2}} \lesssim \|u\|_{0,b}.$$

Part iii) We use almost orthogonality: Let us consider the collection of cubes $C_z = \mu z + [0,\mu)^3$, $z \in \mathbb{Z}^3$, which induce a disjoint covering of $\mathbb{R}^3$. We have

$$\|P_\mu(\widetilde{u}_1 \widetilde{u}_2)\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \sum_{z,z'\in\mathbb{Z}^3} \|P_\mu(P_{C_z}\widetilde{u}_1 P_{C_{z'}}\widetilde{u}_2)\|_{L^2(\mathbb{R}\times\mathbb{R}^3)}$$

For each $z \in \mathbb{Z}^3$, only those $z' \in \mathbb{Z}^3$ with $|z-z'| \lesssim 1$ yield a nontrivial contribution to the sum. Hence, by Cauchy-Schwarz and Part i) we obtain

$$\begin{aligned}&\|P_\mu(\widetilde{u}_1 \widetilde{u}_2)\|_{L^2(\mathbb{R}\times\mathbb{R}^3)}\\ \lesssim\ & \mu^{\frac{1}{2}} \Big(\sum_{z\in\mathbb{Z}^3} \|P_{B_{\sqrt{3}\mu}(\mu z)} u_1\|^2_{\frac{1}{4},b}\Big)^{\frac{1}{2}} \Big(\sum_{z'\in\mathbb{Z}^3} \|P_{B_{\sqrt{3}\mu}(\mu z')} u_2\|^2_{\frac{1}{4},b}\Big)^{\frac{1}{2}},\end{aligned}$$

where $B_{\sqrt{3}\mu}(\mu z)$ denotes the ball with radius $\sqrt{3}\mu$ and center μz, and the claim follows from $\sum_{z\in\mathbb{Z}^3} \chi^2_{B_{\sqrt{3}\mu}(\mu z)} \lesssim 1$. □

Remark 1. By (complex) interpolation of $\|u\|_{L^4_t L^4_x} \lesssim \|u\|_{\frac{1}{2},b}$ and $\|u\|_{L^2_t L^2_x} = \|u\|_{0,0}$ resp. $\|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{0,b}$ we obtain

$$\|u\|_{L_t^{\frac{8}{3}} L_x^{\frac{8}{3}}} \lesssim \|u\|_{\frac{1}{4},\frac{b}{2}} \tag{7}$$

$$\|u\|_{L_t^8 L_x^{\frac{8}{3}}} \lesssim \|u\|_{\frac{1}{4},b} \tag{8}$$

if $b > \frac{1}{2}$. The Sobolev embedding implies

$$\|u\|_{L^4_t L^4_x} \lesssim \|D^{\frac{3}{4}} u\|_{L^4_t L^2_x} \lesssim \|u\|_{\frac{3}{4},\frac{1}{4}}.$$

By interpolation with $\|u\|_{L^4_tL^4_x} \lesssim \|u\|_{\frac{1}{2},b}$ (for $b > \frac{1}{2}$) we obtain the following: For any $\delta > 0$ there exists $b < \frac{1}{2}$ such that

$$\|u\|_{L^4_tL^4_x} \lesssim \|u\|_{\frac{1}{2}+\delta,b}. \tag{9}$$

Furthermore, there is one obvious consequence from (6) which we will use later: For $s > \frac{1}{4}$, $b > \frac{1}{2}$,

$$\|P_\mu(\widetilde{u}_1\widetilde{u}_2)\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \mu^{\frac{3}{4}-s}\|u_1\|_{s,b}\|u_2\|_{s,b}. \tag{10}$$

In the case of radial data there is the following improvement, which is an immediate consequence of [18, Theorem 2.6].

Lemma 2.3. *Let $b > \frac{1}{2}$. For all $\lambda \geq \mu \geq 0$, and for all $u_1, u_2 \in X^{0,b}_{rad}$*

$$\|P_\mu(P_\lambda\widetilde{u}_1 P_\lambda\widetilde{u}_2)\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \mu\|u_1\|_{0,b}\|u_2\|_{0,b}, \tag{11}$$

where $\widetilde{u}_j$ denotes either u_j or $\overline{u}_j$.

Proof. It suffices to consider $\lambda \gg \mu$. Decompose $u_j = u_{j,low} + u_{j,high}$ into low and high modulation, i.e.

$$\operatorname{supp}\mathcal{F}u_{j,low} \subseteq \{(\tau,\xi) : |\tau + |\xi|| \leq \mu\}$$
$$\operatorname{supp}\mathcal{F}u_{j,high} \subseteq \{(\tau,\xi) : |\tau + |\xi|| > \mu\}$$

Because of

$$\operatorname{supp}\mathcal{F}(P_\mu(P_\lambda u_{1,low}P_\lambda u_{2,low})) \subseteq \{(\tau,\xi) : |\tau + |\xi|| \lesssim \mu\}$$

the estimate of [18, Theorem 2.6] with $r \sim \mu$ readily implies

$$\|P_\mu(P_\lambda\widetilde{u}_{1,low}P_\lambda\widetilde{u}_{2,low})\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \mu\|u_1\|_{0,b}\|u_2\|_{0,b}.$$

On the other hand,

$$\begin{aligned}\|P_\mu(P_\lambda\widetilde{u}_{1,low}P_\lambda\widetilde{u}_{2,high})\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} &\lesssim \mu^{\frac{3}{2}}\|P_\lambda\widetilde{u}_{1,low}P_\lambda\widetilde{u}_{2,high}\|_{L^2_tL^1_x}\\ &\lesssim \mu^{\frac{3}{2}}\|P_\lambda\widetilde{u}_{1,low}\|_{L^\infty_tL^2_x}\|P_\lambda\widetilde{u}_{2,high}\|_{L^2_tL^2_x}\\ &\lesssim \mu^{\frac{3}{2}-b}\|u_1\|_{0,b}\|u_2\|_{0,b}.\end{aligned}$$

The same argument applies in the remaining cases, since at least one factor has high modulation. □

Remark 2. By the Sobolev embeddding, we obtain

$$\|P_\mu(P_\lambda\widetilde{u}_1P_\lambda\widetilde{u}_2)\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \mu^{\frac{3}{2}}\|P_\mu(P_\lambda\widetilde{u}_1P_\lambda\widetilde{u}_2)\|_{L^2_tL^1_x} \lesssim \mu^{\frac{3}{2}}\|u_1\|_{0,\frac{1}{4}}\|u_2\|_{0,\frac{1}{4}}.$$

Interpolation with (11) implies that for any $\delta > 0$ there exists $b < \frac{1}{2}$ such that

$$\|P_\mu(P_\lambda\widetilde{u}_1P_\lambda\widetilde{u}_2)\|_{L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \mu^{1+\delta}\|u_1\|_{0,b}\|u_2\|_{0,b} \tag{12}$$

for radial u_1, u_2.

Next, we are prepared to prove

Proposition 2.4. *Let* $s > \frac{1}{4}$. *There exists* $-\frac{1}{2} < b' < 0 < \frac{1}{2} < b \leq b'+1$ *and* $\delta > 0$, *such that*

$$\| |x|^{-1} * (u_1 \overline{u}_2)\, u_3 \|_{s,b'} \lesssim T^{\delta} \|u_1\|_{s,b} \|u_2\|_{s,b} \|u_3\|_{s,b} \tag{13}$$

for all $u_j \in X^{s,b}$ *with* $\operatorname{supp}(u_j) \subset \{(t,x) : |t| \leq T\}$, $j = 1,2,3$.

Proof. Without loss we may assume $\frac{1}{4} < s < \frac{3}{8}$. We notice that in $\mathbb{R}^3$ convolution with $|x|^{-1}$ is (up to a multiplicative constant) the Fourier-multiplier $|D|^{-2}$ with symbol $|\xi|^{-2}$, which is locally integrable. By duality, it suffices to prove

$$\begin{aligned} I :=& \Big| \iint |D|^{-2}(u_1 \overline{u}_2) u_3 \langle D \rangle^s \overline{u_4} dt dx \Big| \\ \lesssim & T^{\delta} \|u_1\|_{s,b} \|u_2\|_{s,b} \|u_3\|_{s,b} \|u_4\|_{0,-b'}. \end{aligned} \tag{14}$$

It suffices to consider u_j with non-negative Fourier-transform. Then, we may split the left hand side into two terms:

$$\begin{aligned} I \lesssim & \Big| \iint \langle D \rangle^s |D|^{-2}(u_1 \overline{u}_2) u_3 \overline{u_4} dt dx \Big| + \Big| \iint |D|^{-2}(u_1 \overline{u}_2) \langle D \rangle^s u_3 \overline{u_4} dt dx \Big| \\ =: & I_1 + I_2. \end{aligned}$$

Let us first discuss the contribution of $P_{\leq 2}(u_1 \overline{u}_2)$: For any $s_1, s_2 \in \mathbb{R}$, the Hardy-Littlewood-Sobolev and Bernstein inequalities imply

$$\begin{aligned} & \Big| \iint \langle D \rangle^{s_1} |D|^{-2} P_{\leq 2}(u_1 \overline{u}_2) \langle D \rangle^{s_2} u_3 \overline{u_4} dt dx \Big| \\ \lesssim & \| |D|^{-2} P_{\leq 2}(u_1 \overline{u_2}) \|_{L^2_t L^6_x} \| P_{\leq 2}(\langle D \rangle^{s_2} u_3 \overline{u_4}) \|_{L^2_t L^{\frac{6}{5}}_x} \\ \lesssim & \| P_{\leq 2}(u_1 \overline{u_2}) \|_{L^2_t L^{\frac{6}{5}}_x} \| \langle D \rangle^{s_2} u_3 \overline{u_4} \|_{L^2_t L^1_x} \\ \lesssim & \|u_1\|_{L^4_t L^2_x} \|u_2\|_{L^4_t L^2_x} \| \langle D \rangle^{s_2} u_3 \|_{L^4_t L^2_x} \|u_4\|_{L^4_t L^2_x} \\ \lesssim & \|u_1\|_{0,\frac{1}{4}} \|u_2\|_{0,\frac{1}{4}} \|u_3\|_{s_2,\frac{1}{4}} \|u_4\|_{0,\frac{1}{4}}. \end{aligned}$$

Thus, we may henceforth assume that $P_1(u_1 \overline{u}_2) = P_1(u_3 \overline{u}_4) = 0$. First, we consider the contribution of I_1:

$$I_1 \leq \| \langle D \rangle^s |D|^{-\frac{7}{8}} (u_1 \overline{u}_2) \|_{L^2} \| |D|^{-\frac{9}{8}} (u_3 \overline{u}_4) \|_{L^2} =: I_{11} \cdot I_{12}$$

On the one hand, using (6) we obtain

$$\begin{aligned} I_{11} \lesssim & \sum_{\mu \geq 2} \mu^{s-\frac{7}{8}} \| P_\mu (u_1 \overline{u}_2) \|_{L^2} \lesssim \sum_{\mu \geq 2} \mu^{s-\frac{3}{8}} \|u_1\|_{\frac{1}{4},b} \|u_2\|_{\frac{1}{4},b} \\ \lesssim & \|u_1\|_{\frac{1}{4},b} \|u_2\|_{\frac{1}{4},b} \end{aligned}$$

On the other hand, the Hardy-Littlewood-Sobolev inequality and (8) yield

$$I_{12} \lesssim \|u_3\overline{u}_4\|_{L^2_t L^{\frac{8}{7}}_x} \lesssim \|u_3\|_{L^8_t L^{\frac{8}{3}}_x} \|u_4\|_{L^{\frac{8}{3}}_t L^2_x} \lesssim \|u_3\|_{\frac{1}{4},b}\|u_4\|_{0,\frac{1}{8}}.$$

For $-b' > \frac{1}{8}$, $b > \frac{1}{2}$, this implies

$$I_1 \lesssim I_{11} + I_{12} \lesssim T^\delta \|u_1\|_{\frac{1}{4},b}\|u_2\|_{\frac{1}{4},b}\|u_3\|_{\frac{1}{4},b}\|u_4\|_{0,-b'}. \tag{15}$$

Finally, we turn to I_2: By Cauchy-Schwarz, (10) and Bernstein we obtain

$$\begin{aligned} I_2 \lesssim & \sum_{\mu\geq 2} \mu^{-2}\|P_\mu(u_1\overline{u}_2)\|_{L^2}\|P_\mu(\langle D\rangle^s u_3\overline{u}_4)\|_{L^2} \\ \lesssim & \|u_1\|_{s,b}\|u_2\|_{s,b} \sum_{\mu\geq 2} \mu^{\frac{1}{4}-s}\|P_\mu(\langle D\rangle^s u_3\overline{u}_4)\|_{L^2_t L^1_x} \\ \lesssim & \|u_1\|_{s,b}\|u_2\|_{s,b} \sum_{\mu\geq 2} \mu^{\frac{1}{4}-s}\|\langle D\rangle^s u_3\|_{L^4_t L^2_x}\|u_4\|_{L^4_t L^2_x} \\ \lesssim & T^\delta \|u_1\|_{s,b}\|u_2\|_{s,b}\|u_3\|_{s,b}\|u_4\|_{0,-b'} \end{aligned}$$

if $b' < -\frac{1}{4}$ and $b > \frac{1}{2}$, $s > \frac{1}{4}$. □

Next, we consider the radial case.

Proposition 2.5. *Let $s > 0$. There exists $-\frac{1}{2} < b' < 0 < \frac{1}{2} < b \leq b'+1$ and $\delta > 0$, such that*

$$\||x|^{-1} * (u_1\overline{u}_2)\, u_3\|_{s,b'} \lesssim T^\delta \|u_1\|_{s,b}\|u_2\|_{s,b}\|u_3\|_{s,b} \tag{16}$$

for all $u_j \in X^{s,b}_{rad}$ with $\operatorname{supp}(u_j) \subset \{(t,x) : |t| \leq T\}$.

Proof. As above, by duality, it suffices to prove

$$\begin{aligned} I := & \Big| \iint |D|^{-2}(u_1\overline{u}_2)u_3\langle D\rangle^s \overline{u_4} dtdx \Big| \\ \lesssim & T^\delta \|u_1\|_{s,b}\|u_2\|_{s,b}\|u_3\|_{s,b}\|u_4\|_{0,-b'} \end{aligned} \tag{17}$$

for $0 < s < \frac{1}{2}$. It suffices to consider u_j with non-negative Fourier-transform. Then, we may split the left hand side into two terms:

$$\begin{aligned} I \lesssim & \Big| \iint \langle D\rangle^s |D|^{-2}(u_1\overline{u}_2)u_3\overline{u_4} dtdx \Big| + \Big| \iint |D|^{-2}(u_1\overline{u}_2)\langle D\rangle^s u_3\overline{u_4} dtdx \Big| \\ =: & I_1 + I_2. \end{aligned}$$

The argument in the proof of Proposition 2.4 shows that we may henceforth assume that $P_1(u_1\overline{u}_2) = P_1(u_3\overline{u}_4) = 0$. First, we consider the contribution of I_1:

$$I_1 \leq \|\langle D\rangle^s |D|^{-1}(u_1\overline{u}_2)\|_{L^2}\||D|^{-1}(u_3\overline{u}_4)\|_{L^2} \lesssim I_{11} \cdot I_{12}$$

We have (dyadic summation)

$$I_{11} \lesssim \sum_{\lambda_1 \ll \lambda_2} \lambda_2^{s-1} \|u_{1,\lambda_1} \overline{u}_{2,\lambda_2}\|_{L^2} + \sum_{\lambda_1 \gg \lambda_2} \lambda_1^{s-1} \|u_{1,\lambda_1} \overline{u}_{2,\lambda_2}\|_{L^2}$$
$$+ \sum_{\mu \lesssim \lambda} \mu^{s-1} \|P_\mu(u_{1,\lambda} \overline{u}_{2,\lambda})\|_{L^2}$$

We start estimating the first term by using Hölder's inequality and (5):

$$\begin{aligned}\sum_{\lambda_1 \ll \lambda_2} \lambda_2^{s-1} \|u_{1,\lambda_1} \overline{u}_{2,\lambda_2}\|_{L^2} &\lesssim \sum_{\lambda_1 \ll \lambda_2} \lambda_2^{s-\frac{1}{2}} \lambda_1^{\frac{1}{2}} \|u_{1,\lambda_1}\|_{0,b} \|u_{2,\lambda_2}\|_{0,b} \\ &\lesssim \sum_{k=0}^{\infty} \sum_{0 \le l \le k} 2^{-l(\frac{1}{2}-s)} \|u_{1,2^{k-l}}\|_{s,b} \|u_{2,2^k}\|_{0,b} \\ &\lesssim \|u_1\|_{s,b} \|u_2\|_{0,b}.\end{aligned}$$

For the second term, we obtain the same result. The third term is estimated by using (11):

$$\begin{aligned}\sum_{\mu \lesssim \lambda} \mu^{s-1} \|P_\mu(u_{1,\lambda} \overline{u}_{2,\lambda})\|_{L^2} &\lesssim \sum_{\mu \lesssim \lambda} \mu^s \|u_{1,\lambda}\|_{0,b} \|u_{2,\lambda}\|_{0,b} \\ &\lesssim \|u_1\|_{s,b} \|u_2\|_{0,b}.\end{aligned}$$

Next, we decompose $I_{12} = J_1 + J_2 + J_3$. Using (9) with $\delta = s$ we see that

$$\begin{aligned}J_1 := \sum_{\lambda_3 \ll \lambda_4} \lambda_4^{-1} \|u_{3,\lambda_3} \overline{u}_{4,\lambda_4}\|_{L^2} &\lesssim \sum_{\lambda_3 \ll \lambda_4} \lambda_3^{\frac{1}{2}} \lambda_4^{-\frac{1}{2}+s} \|u_{3,\lambda_3}\|_{0,b} \|u_{4,\lambda_4}\|_{0,-b'} \\ &\lesssim \|u_3\|_{s,b} \|u_4\|_{0,-b'}\end{aligned}$$

for some $b' > -\frac{1}{2}$. Similarly,

$$J_2 := \sum_{\lambda_3 \gg \lambda_4} \lambda_3^{-\frac{1}{2}} \lambda_4^{\frac{1}{2}+s} \|u_{3,\lambda_3}\|_{0,b} \|u_{4,\lambda_4}\|_{0,-b'} \lesssim \|u_3\|_{s,b} \|u_4\|_{0,-b'}.$$

Using (12) we obtain

$$\begin{aligned}J_3 := \sum_{\mu \lesssim \lambda} \mu^{-1} \|P_\mu(u_{3,\lambda} \overline{u}_{4,\lambda})\|_{L^2} &\lesssim \sum_{\mu \lesssim \lambda} \mu^s \|u_{3,\lambda}\|_{0,b} \|u_{4,\lambda}\|_{0,-b'} \\ &\lesssim \|u_3\|_{s,b} \|u_4\|_{0,-b'}\end{aligned}$$

for some $b' > -\frac{1}{2}$. All in all, we have proved

$$I_1 \lesssim \|u_1\|_{s,b} \|u_2\|_{0,b} \|u_3\|_{s,b} \|u_4\|_{0,-b'}.$$

Concerning I_2, we obtain

$$I_2 \le \|\langle D \rangle^s |D|^{-1} (u_1 \overline{u}_2)\|_{L^2} \|\langle D \rangle^{-s} |D|^{-1} (\langle D \rangle^s u_3 \overline{u}_4)\|_{L^2} \lesssim I_{21} \cdot I_{22}$$

The estimate for I_{11} above also applies to I_{21}. As above, we decompose $I_{22} = K_1 + K_2 + K_3$. Trivial modifications of the arguments above yield

$$I_{22} \lesssim \|u_3\|_{s,b}\|u_4\|_{0,-b'},$$

and altogether we obtain

$$I_2 \lesssim \|u_1\|_{s,b}\|u_2\|_{0,b}\|u_3\|_{s,b}\|u_4\|_{0,-b'}.$$

By replacing $\|u_4\|_{0,-b'}$ by $\|u_4\|_{0,-b''}$ for some $-\frac{1}{2} < b'' < b'$ shows that one can squeeze out a factor T^δ. □

Finally, we explain how Propositions 2.4 and 2.5 imply Theorem 1.1: The general idea – due to Bourgain [2] – is well-known by now, see e.g. [10] and references therein for more details. The basic idea is to solve the integral equation

$$u(t) = \psi_T U(t)\phi + i\psi_T \int_0^t U(t-\tau)\big(|x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi\big)d\tau \qquad (18)$$

with the smooth cutoff $\psi_T(t) = \beta_1(t/T)$ and given initial data ϕ by the contraction mapping principle. For $b > \frac{1}{2}$ [10, (2.19)] implies

$$\|\psi_T U(t)\phi\|_{s,b} \lesssim T^{\frac{1}{2}-b}\|\phi\|_{H^s}$$

Also, [10, Lemma 2.1] implies that

$$\|\psi_T \int_0^t U(t-\tau)f(\tau)d\tau\|_{s,b} \lesssim T^{1-b+b'}\|f\|_{s,b'}.$$

if $-\frac{1}{2} < b' < 0 < \frac{1}{2} < b \leq b'+1$. Now, due to Propositions 2.4 and 2.5, it is an easy exercise to verify that the right hand side in (18) defines a contraction in an appropriate closed ball in $X^{s,b} \subset C([0,T];H^s(\mathbb{R}^3))$ and $X^{s,b}_{rad} \subset C([0,T];H^s_{rad}(\mathbb{R}^3))$, respectively. Hence, it has a fixed point. Uniqueness in $X^{s,b}$ resp. $X^{s,b}_{rad}$ and smooth (real analytic) dependence on the initial data are immediate consequences.

3. Ill-posedness

3.1. **Proof of Theorem 1.2.** As explained in [17, Section 2.2] (for the quadratic Benjamin-Ono equation), it suffices to prove the following:

Proposition 3.1. *For fixed* $0 < t \leq 1$ *and* $s < \frac{1}{4}$ *the inequality*

$$\Big\|\int_0^t U(t-\tau)\big(|x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi\big)d\tau\Big\|_{H^s(\mathbb{R}^3)} \lesssim \|\phi\|^3_{H^s(\mathbb{R}^3)} \qquad (19)$$

fails to hold for all $\phi \in H^s(\mathbb{R}^3)$ *(in any neighborhood of the origin).*

Proof. Let $1 \leq \mu \ll \lambda$. We will choose $\mu = \mu(\lambda) = \delta\lambda^{\frac{1}{2}}$ for fixed $0 < \delta \ll 1$. Define the cube

$$W_\lambda^\pm = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \colon |\xi_1 \mp \lambda| \leq \mu,\ |\xi_2|, |\xi_3| \leq \mu\},$$

which is centered at $\pm\lambda e_1$ with sidelength 2μ. Let ϕ be the inverse Fourier transform of the characteristic function $\chi_{W_\lambda^+}$ of W_λ^+. Obviously, $\|\phi\|_{H^s(\mathbb{R}^3)} \approx \mu^{\frac{3}{2}}\lambda^s$.

Next, we consider

$$F_t(\xi) := \mathcal{F}_x\Big(\int_0^t U(t-\tau)\big(|x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi\big)d\tau\Big)(\xi).$$

Our aim is to prove that for $0 < t \ll 1$ and all $\xi \in \frac{1}{4}W_\lambda^+$

$$|F_t(\xi)| \gtrsim |t|\mu^4. \tag{20}$$

Assuming that (20) holds, the claim follows since the validity of (19) implies

$$|t|\mu^{\frac{11}{2}}\lambda^s \lesssim \|\langle\xi\rangle^s F_t(\xi)\|_{L^2_\xi(\mathbb{R}^3)} \lesssim \mu^{\frac{9}{2}}\lambda^{3s}$$

which for $\mu = \delta\lambda^{\frac{1}{2}}$ is equivalent to

$$|t|\delta \lesssim \lambda^{2s-\frac{1}{2}},$$

which can hold for fixed $t, \delta > 0$ and $\lambda \to \infty$ only if $s \geq \frac{1}{4}$. Hence, it suffices to establish (20): Similarly to [17, p.985] we compute

$$\begin{aligned} &F_t(\xi)\\ =&ce^{it\varphi_m(\xi)}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(e^{itr_m(\xi_1,\xi_2,\xi)}-1)}{ir_m(\xi_1,\xi_2,\xi)}\frac{\chi_{W_\lambda^+}(\xi_1)\chi_{W_\lambda^-}(\xi_2)}{|\xi_1+\xi_2|^2}\chi_{W_\lambda^+}(\xi-\xi_1-\xi_2)d\xi_1 d\xi_2, \end{aligned}$$

where

$$r_m(\xi_1,\xi_2,\xi) = \varphi_m(\xi_1) - \varphi_m(\xi_2) + \varphi_m(\xi-\xi_1-\xi_2) - \varphi_m(\xi).$$

We notice that in the domain of integration we have

$$\big||\xi_1| - |\xi_2| + |\xi-\xi_1-\xi_2| - |\xi|\big| \lesssim \frac{\mu^2}{\lambda},$$

hence, for each fixed $m \geq 0$,

$$|r_m(\xi_1,\xi_2,\xi)| \lesssim \frac{1}{\lambda} + \frac{\mu^2}{\lambda} \ll 1 \text{ (by choosing } \delta > 0 \text{ small enough).}$$

Therefore, if $\xi \in \frac{1}{4}W_\lambda^+$ we have

$$|F_t(\xi)| \gtrsim |t|\int_{\frac{1}{4}W_\lambda^+}\int_{\frac{1}{4}W_\lambda^-}|\xi_1+\xi_2|^{-2}d\xi_1 d\xi_2 \gtrsim |t|\mu^4,$$

and the proof is complete. Notice that (by multiplying the above function ϕ by a small fixed parameter) we can provide such a counterexample in any neighbourhood of the origin. □

Proposition 3.2. *For fixed $T > 0$ and $s < 0$ the inequality*

$$\sup_{t\in[0,T]} \Big\| \int_0^t U(t-\tau)\big(|x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi\big)d\tau \Big\|_{H^s(\mathbb{R}^3)} \lesssim \|\phi\|^3_{H^s(\mathbb{R}^3)} \quad (21)$$

fails to hold for all radial $\phi \in H^s(\mathbb{R}^3)$ (in any neighborhood of the origin).

Proof. Let $T > 0$ be fixed. For $\lambda \gg 1$ we define the annulus

$$A_\lambda = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \colon \lambda \le |\xi| \le 2\lambda\},$$

and let ϕ be the inverse Fourier transform of the characteristic function χ_{A_λ} of A_λ. Obviously, ϕ is radial and $\|\phi\|_{H^s(\mathbb{R}^3)} \approx \lambda^{s+\frac{3}{2}}$. As above we consider

$$F_t(\xi) := \mathcal{F}_x\Big(\int_0^t U(t-\tau)\big(|x|^{-1} * |U(\tau)\phi|^2 U(\tau)\phi\big)d\tau\Big)(\xi).$$

For $t = \delta\lambda^{-1}$ with $0 < \delta \ll 1$ (such that $t < T$) and $\xi \in \frac{1}{4}A_\lambda$ we will prove

$$|F_t(\xi)| \gtrsim \delta\lambda^3. \quad (22)$$

The estimate (21) in conjuction with (22) implies

$$\delta\lambda^{s+\frac{9}{2}} \lesssim \|\langle\xi\rangle^s F_t(\xi)\|_{L^2_\xi(\mathbb{R}^3)} \lesssim \lambda^{3s+\frac{9}{2}}$$

which can hold for fixed $\delta > 0$ and $\lambda \to \infty$ only if $s < 0$. It remains to prove (22): As above (cp. [17, p.985]) we compute

$$F_t(\xi) = ce^{it\varphi_m(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(e^{itr_m(\xi_1,\xi_2,\xi)}-1)}{ir_m(\xi_1,\xi_2,\xi)} \frac{\chi_{A_\lambda}(\xi_1)\chi_{A_\lambda}(\xi_2)}{|\xi_1+\xi_2|^2}\chi_{A_\lambda}(\xi-\xi_1-\xi_2)d\xi_1 d\xi_2,$$

where

$$r_m(\xi_1,\xi_2,\xi) = \varphi_m(\xi_1) - \varphi_m(\xi_2) + \varphi_m(\xi-\xi_1-\xi_2) - \varphi_m(\xi).$$

Obviously, in the domain of integration we have

$$|tr_m(\xi_1,\xi_2,\xi)| \lesssim |t\lambda| \ll 1.$$

Therefore, if $\xi \in \frac{1}{4}A_\lambda$ and $t = \delta\lambda^{-1}$ we have

$$|F_t(\xi)| \gtrsim \delta\lambda^{-1} \int_{\frac{1}{4}A_\lambda}\int_{A_\lambda} |\xi_1+\xi_2|^{-2} d\xi_1 d\xi_2 \gtrsim \delta\lambda^3,$$

and the proof is complete. As above, we can provide such a counterexample in any neighbourhood of the origin by multiplying ϕ by a small parameter. $\square$

3.2. **Proof of Theorem 1.3: Failure of Uniform Continuity.** By exploiting the fact that (1) exhibits solitary wave solutions, we show failure of (local) uniform continuity of the solution map $\phi \mapsto u(t)$ in L^2. In the case $m = 0$, the exact scaling symmetry of (1) simplifies the analysis considerably. From [12, Appendix A.2] (see also [14]) we recall the existence of radial ground state solutions $Q \in H^{\frac{1}{2}}(\mathbb{R}^3)$ of

$$\sqrt{-\Delta}\, Q + Q - (|x|^{-1} * |Q|^2)Q = 0. \tag{23}$$

We divide this subsection by treating first the massless case $m = 0$, followed by a discussion of the more complicated situation when $m > 0$ hold in (1). The following result concerns the massless case. To treat the case $m > 0$, we need more elaborate arguments worked out below.

Proposition 3.3. *Suppose that* $m = 0$ *holds in* (1) *and let* $t > 0$. *Then the map* $\phi \mapsto u(t)$ *fails to be uniformly continuous for initial data in the set*

$$M = \{\phi \in L^2(\mathbb{R}^3) \colon \phi \text{ radial },\ \|\phi\|_{L^2}^2 \geq \|Q\|_{L^2}^2\}$$

with respect to the L^2*-norm.*

Proof. We adapt the arguments in [1] to our setting here. Suppose that $m = 0$ holds in (1) and suppose that Q is a ground state solution (23). By scaling, we have that $Q_\mu(x) = \mu^{\frac{3}{2}} Q(\mu x)$ with $\mu > 0$ solves

$$\sqrt{-\Delta}\, Q_\mu + \mu Q_\mu - (|x|^{-1} * |Q_\mu|^2) Q_\mu = 0.$$

Note that $\|Q_\mu\|_{L^2} = \|Q\|_{L^2}$ reflecting the L^2-criticality of (1).

For $\mu_1, \mu_2 > 0$ we consider the solutions

$$u_{\mu_1}(t,x) = e^{it\mu_1} Q_{\mu_1}(x), \quad u_{\mu_2}(t,x) = e^{it\mu_2} Q_{\mu_2}(x).$$

and we set

$$I_{\mu_1,\mu_2}(t) = \|u_{\mu_1}(t) - u_{\mu_2}(t)\|_{L^2}^2.$$

Note that

$$\begin{aligned} I_{\mu_1,\mu_2}(t) &= \|Q_{\mu_1}\|_{L^2}^2 + \|Q_{\mu_2}\|_{L^2}^2 - 2\mathrm{Re}\, \langle e^{it\mu_1} Q_{\mu_1}, e^{it\mu_2} Q_{\mu_2} \rangle \\ &= 2\|Q\|_{L^2}^2 - 2\mathrm{Re} \left\{ e^{it(\mu_1 - \mu_2)} \left(\frac{\mu_2}{\mu_1}\right)^{3/2} \int_{\mathbb{R}^3} Q(x) Q\left(\left(\frac{\mu_1}{\mu_2}\right) x\right) dx \right\} \\ &= 2\|Q\|_{L^2}^2 - 2\cos(t(\mu_1 - \mu_2)) \left(\frac{\mu_2}{\mu_1}\right)^{3/2} \int_{\mathbb{R}^3} Q(x) Q\left(\left(\frac{\mu_1}{\mu_2}\right) x\right) dx \end{aligned}$$

Let $t > 0$ be fixed and choose the sequences

$$\mu_1(n) = \frac{\pi}{2t}(n+1)^2, \quad \mu_2(n) = \frac{\pi}{2t} n^2, \quad \text{with } n \in \mathbb{N}.$$

This choice implies that $t(\mu_1(n) - \mu_2(n)) = \frac{\pi}{2}(2n+1)$ and thus $\cos(t(\mu_1(n) - \mu_2(n))) = 0$. Therefore,

$$I_{\mu_1(n),\mu_2(n)}(t) = 2\|Q\|_{L^2}^2 \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, since $\mu_1(n)/\mu_2(n) \to 1$ as $n \to +\infty$, we easily see that

$$\lim_{n \to +\infty} I_{\mu_1(n),\mu_2(n)}(0) = 0.$$

But this show that the solution $u(t)$ cannot depend in a uniformly continuous way on the initial datum ϕ in the L^2-topology.

This completes the proof of Proposition 3.3. □

We now turn to the case when $m > 0$ holds in (1). Without loss of generality, we assume that $m = 1$ throughout the following. To derive an illposedness result that is analogue to Proposition 3.3, we utilize solitary wave solutions $u(t,x) = e^{it\mu} Q_\mu(x)$, where $Q_\mu \in H^{1/2}(\mathbb{R}^3)$ are radial positive solutions to

$$\sqrt{-\Delta + 1}\, Q_\mu + \mu Q_\mu - (|x|^{-1} * |Q_\mu|^2) Q_\mu = 0. \tag{24}$$

Here $\mu > 0$ can be chosen arbitrarily; in contrast to [14], where μ arises as a Lagrange multiplier. However, we need some auxiliary results about a class of Q_μ that arise from a suitable variational problem. In particular, we are ultimately interested in the limit of large $\mu > 0$ and we will show that

$$R_\mu(x) = \mu^{-3/2} Q_\mu(\mu^{-1} x)$$

converges strongly in L^2 (up to subsequences) as $\mu \to \infty$ to some positive radial solution $R \in H^{1/2}$ that solves $\sqrt{-\Delta}\, R + R - (|x|^{-1} * |R|^2) R = 0$.

Note that $R_\mu \in H^{1/2}$ solves

$$\sqrt{-\Delta + \mu^{-2}}\, R_\mu + R_\mu - (|x|^{-1} * |R_\mu|^2) R_\mu = 0 \tag{25}$$

if and only if $Q_\mu \in H^{1/2}$ solves (24). In order to construct solutions R_μ (for any $\mu > 0$ given), we define the functional $F_\mu : H^{1/2}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$F_\mu(u) = \int_{\mathbb{R}^3} \overline{u} \sqrt{-\Delta + \mu^{-2}}\, u \, dx + \int_{\mathbb{R}^3} |u|^2 \, dx.$$

We consider the following minimization problem

$$F_\mu^* = \inf\{ F_\mu(u) : u \in H^{1/2}(\mathbb{R}^3),\ V(u) = 1 \}, \tag{26}$$

where

$$V(u) = \int_{\mathbb{R}^3} (|x|^{-1} * |u|^2)|u|^2 \, dx.$$

By interpolation inequalities, it is easy to see that $F_\mu^* > 0$ holds. In fact, the infimum is attained.

Lemma 3.4. *For any $\mu > 0$, there exists a radial positive minimizer $T_\mu \in H^{1/2}(\mathbb{R}^3)$ for problem (26). Moreover, the function T_μ satisfies*

$$\sqrt{-\Delta + \mu^{-2}}\, T_\mu + T_\mu - \theta(|x|^{-1} * |T_\mu|^2) T_\mu = 0,$$

where $\theta = \theta(T_\mu) > 0$ is some Lagrange multiplier. In particular, the function $R_\mu = \theta^{1/2} T_\mu \in H^{1/2}(\mathbb{R}^3)$ is a positive radial solution of (25).

Proof. This follows from standard variational arguments. We provide a brief sketch of the main steps, where we use rearrangement inequalities to gain compactness of minimizing sequences.

For $\sigma \geq 1$ given, we embed (26) into the family of variational problems

$$I_\mu(\sigma) = \inf\{F_\mu(u) : u \in H^{1/2},\ V(u) = \sigma\}.$$

Clearly, we have that $I_\mu(\sigma) \geq 0$. Moreover, by scaling, we readily check that $I_\mu(\sigma) = \sigma^{1/2} I_\mu(1)$ and hence $I_\mu(\sigma) \geq I_\mu(1)$ for $\sigma \geq 1$. Now, by rearrangement inequalities (see, e. g., [13]) we have that $F_\mu(u^*) \leq F_\mu(u)$ and $V(u^*) \geq V(u)$, where u^* denotes the symmetric-decreasing rearrangement of $u \in H^{1/2}$. Therefore, any minimizing sequence (u_n) for problem (26) can be replaced by (u_n^*) without loss of generality. Hence we assume $u_n^* = u_n$ from now on. Since $F_\mu(u_n) \to F_\mu^*$, we have that $\|u_n\|_{H^{1/2}} \lesssim 1$. Thus, by passing to a subsequence, we can assume that $u_n \rightharpoonup u_*$ weakly in $H^{1/2}$ and $u_n \to u_*$ strongly in L^p_{loc} for $p \in [1,3)$. Furthermore, by using that $\|u_n\|_{L^2} \lesssim 1$ and the fact that $u_n = u_n(|x|)$ are radial and monotone decreasing in $|x|$, we deduce the pointwise bound $|u_n(x)| \lesssim |x|^{-3/2}$. Using this decay bound together with $u_n \to u_*$ strongly in $L^{12/5}_{\mathrm{loc}}$, we deduce that $u_n \to u_*$ strongly in $L^{12/5}$. Thus, by the Hardy-Littlewood-Sobolev inequality, this implies that $V(u_n) \to V(u_*) = 1$. In particular, we have that $u_* \not\equiv 0$ holds. Next, by the lower semi-continuity of F_μ with respect to weak convergence in $H^{1/2}$, we deduce that $\lim_{n\to\infty} F_\mu(u_n) \geq F_\mu(u_*)$. Hence $T_\mu := u_* \geq 0$ is a radial and nonnegative minimizer for problem (26) and satisfies the corresponding Euler-Lagrange equation with some mutliplier $\theta \in \mathbb{R}$. The positivity of $\theta > 0$ follows from integrating the Euler-Lagrange equation against T_μ. In fact, we easily see that $\theta \leq 0$ is not possible for $T_\mu \not\equiv 0$. Finally, we note that $T_\mu(x) > 0$ is in fact positive by adapting an argument in [12]. □

Next we derive uniform bounds for the minimizers T_μ given above when taking in the limit $\mu \to \infty$. Hence, without loss generality, we consider the case $\mu \geq 1$, say. We have the following uniform L^2 bounds.

Lemma 3.5. *Suppose that $\mu \geq 1$ and let $R_\mu = \theta^{1/2} T_\mu \in H^{1/2}(\mathbb{R}^3)$ be given as in Lemma 3.4 above. Then we have the uniform bounds*

$$1 \lesssim \|R_\mu\|_{L^2} \leq \|R_\mu\|_{H^{1/2}} \lesssim 1.$$

Proof. We derive approriate bounds for $T_\mu \in H^{1/2}$ and $\theta = \theta(T_\mu) > 0$ as follows.

First, we claim that

$$\|T_\mu\|_{H^{1/2}} \lesssim 1. \tag{27}$$

To see this, we simply note that T_μ minimizes problem (26). Therefore, by taking some fixed function $w \in H^{1/2}$ such that $V(w) = 1$, we deduce that

$$\begin{aligned} F_\mu(T_\mu) \leq F_\mu(w) &\leq \int_{\mathbb{R}^3} \overline{w} \sqrt{-\Delta} w \, dx + (1 + \mu^{-1}) \int_{\mathbb{R}^3} |w|^2 \, dx \\ &\lesssim 1 + \mu^{-1} \lesssim 1, \end{aligned}$$

using the operator inequality $\sqrt{-\Delta + \mu^{-2}} \leq \sqrt{-\Delta} + \mu^{-1}$, which directly follows in Fourier space. From this we deduce that the bound (27) holds true.

Next, we derive an upper bound for $\theta > 0$ as follows. By integrating the Euler-Lagrange equation for T_μ against T_μ, we deduce that

$$\theta = F_\mu(T_\mu) \lesssim 1, \tag{28}$$

using that $V(T_\mu) = 1$. Combining (27) and (28), we find that $R_\mu = \theta^{1/2} T_\mu$ satisfies

$$\|R_\mu\|_{L^2} \leq \|R_\mu\|_{H^{1/2}} \lesssim \theta^{1/2} \|T_\mu\|_{H^{1/2}} \lesssim 1.$$

It remains to show the uniform lower

$$\|R_\mu\|_{L^2} \gtrsim 1. \tag{29}$$

Indeed, this can be seen by exploiting a Hardy type inequality as follows. Let H denote the self-adjoint operator

$$H = \sqrt{-\Delta + \mu^{-2}} - (|x|^{-1} * |R_\mu|^2).$$

Note that -1 is an eigenvalue of H, since $H R_\mu = -R_\mu$ holds. Furthermore, by radiality of R_μ and Newton's theorem (see, e. g., [13]), we have the pointwise bound

$$\int_{\mathbb{R}^3} \frac{|R_\mu(y)|^2}{|x - y|} \, dy \leq \frac{\|R_\mu\|_{L^2}^2}{|x|}.$$

Suppose now that $\|R_\mu\|_{L^2}^2 \leq (2/\pi)$ was true. In view of the simple fact that $\sqrt{-\Delta + \mu^{-2}} \geq \sqrt{-\Delta}$, we obtain the operator inequality

$$H \geq \sqrt{-\Delta} - \frac{\|R_\mu\|_{L^2}^2}{|x|} \geq 0,$$

by Hardy's inequality $|x|^{-1} \leq \frac{\pi}{2}\sqrt{-\Delta}$, see [11]. But the nonnegativity of H contradicts the fact that -1 is an eigenvalue of H. Therefore we deduce that (29) holds, which completes the proof of Lemma 3.5. □

Next, we derive the following strong convergence result for the family R_μ.

Lemma 3.6. *Suppose that $\mu_n \to \infty$ as $n \to \infty$ and let $T_{\mu_n} \in H^{1/2}(\mathbb{R}^3)$ be a sequence of minimizers as given by Lemma 3.4. Furthermore, we denote $R_{\mu_n} = \theta_n^{1/2} T_{\mu_n} \in H^{1/2}(\mathbb{R}^3)$ with $\theta_n = \theta(T_{\mu_n})$ be as above. Then, after passing to a subsequence if necessary, we have that*

$$R_{\mu_n} \to R \text{ strongly in } L^2(\mathbb{R}^3) \text{ as } n \to \infty,$$

where $R \in H^{1/2}(\mathbb{R}^3)$ is a positive radial solution of

$$\sqrt{-\Delta}\, R + R - (|x|^{-1} * |R|^2)R = 0,$$

and it holds that $\|R\|_{L^2}^2 \geq \|Q\|_{L^2}^2$ (i. e. the critical L^2-mass for the boson star equation, see (23)).

Proof. For notational convenience, we write $R_n = R_{\mu_n}$ in what follows.

By the bounds in Lemma 3.5, we can assume that $R_n \rightharpoonup R$ weakly in $H^{1/2}$. Moreover, by local Rellich compactness, we can have that $R_n \to R$ strongly in L^2_{loc}. We will upgrade this to strong convergence in L^2, by deriving a uniform decay estimate for R_n as follows.

We rewrite the equation satisfied by R_n as

$$\sqrt{-\Delta + \mu_n^{-2}}\, R_n = f_n$$

with $f_n(x) = (V_n(x) - 1)R_n(x)$ and $V_n(x) = (|\cdot|^{-1} * |R_n|^2)(x)$. Since R_n are radial functions and $\|R_n\|_{L^2} \lesssim 1$, we derive from Newton's theorem [13, Theorem 9.7] the uniform pointwise bound $V_n(x) \leq \|R_n\|_{L^2}^2 |x|^{-1} \lesssim |x|^{-1}$. Moreover, by the fact that $R_n(x) > 0$ is positive, we deduce that

$$f_n^+(x) := \max\{0, f_n(x)\} \equiv 0 \quad \text{for} \quad |x| \gtrsim 1.$$

Now, let $G_\mu(x, y)$ be the convolution kernel associated to $(-\Delta + \mu^{-2})^{-1/2}$. From well-known facts (see e.g. [13, p. 183, formula (11)]) we have the explicit formula

$$\begin{aligned} G_\mu(x) &= \int_0^\infty e^{-t\sqrt{-\Delta+\mu^{-2}}}(x,y)\,dt \\ &= \frac{\mu^{-4}}{2\pi^2}\int_0^\infty \frac{t}{t^2+|x-y|^2} K_2\left(\mu^{-2}\sqrt{|x-y|^2+t^2}\right)\,dt, \end{aligned}$$

where K_2 is the modified Bessel function of the third kind. From the fact that $K_\nu(z) \lesssim |z|^{-\nu}$ for $\operatorname{Re}\nu > 0$, we easily deduce the uniform bound

$$0 < G_\mu(x,y) \lesssim |x-y|^{-4}.$$

Hence, by using the posivity of R_n and the fact that f_n^+ have compact support in a fixed large ball independent of n, we obtain from $R_\mu = (-\Delta+\mu^{-2})^{-1/2} f_n$ that

$$\begin{aligned} 0 < R_n(x) &\leq \int_{\mathbb{R}^3} G_{\mu_n}(x-y) f_n^+(y)\,dy \leq \int_{|y|\lesssim 1} G_{\mu_n}(x-y) f_n^+(y)\,dy \\ &\leq |x|^{-4}\int_{|y|\lesssim 1} f_n^+(y)\,dy \lesssim |x|^{-4} \quad \text{for} \quad |x| \gtrsim 1. \end{aligned}$$

In the last step, we used that

$$\int_{|y|\lesssim 1} f_n^+(y)\,dy \lesssim \int_{|y|\lesssim 1} \frac{1}{|y|} R_n(y)\,dy \lesssim 1,$$

thanks to Newton's theorem and $\|R_n\|_{L^2} \lesssim 1$ again.

In summary, we have shown that

$$R_n(x) \lesssim |x|^{-4} \quad \text{for } |x| \gtrsim 1 \text{ and } n \geq 1.$$

Using this decay bound (which is square integrable at infinity), we easily see that $R_n \to R$ stronlgy in L^2_{loc} implies that

$$R_n \to R \text{ stronlgy in } L^2(\mathbb{R}^3) \text{ as } n \to \infty.$$

In view of the lower bound in Lemma 3.5, we deduce that $R \not\equiv 0$ holds. By passing to the limit in the equation satisfied by R_n, we deduce that R satisfies the equation displayed in Lemma 3.6.

Finally, integrating the equation satisfied by R against

$$\Lambda R = x\cdot\nabla R + \frac{3}{2}R = \frac{\mathrm{d}}{\mathrm{d}a} a^{\frac{3}{2}} R(ax)\Big|_{a=1}$$

we obtain that

$$\int_{\mathbb{R}^3} R\sqrt{-\Delta}\,R\,dx = \frac{1}{2}\int_{\mathbb{R}^3} (|x|^{-1} * |R|^2)|R|^2\,dx.$$

Recall the interpolation estimate

$$\int_{\mathbb{R}^3} (|x|^{-1} * |f|^2)|f|^2 \, dx \le C_{\mathrm{opt}} \left(\int_{\mathbb{R}^3} \overline{f} \sqrt{-\Delta} f \right) \left(\int_{\mathbb{R}^3} |f|^2 \right)$$

for all $f \in H^{1/2}$, where the optimal constant is given by $C_{\mathrm{opt}} = 2/\|Q\|_{L^2}^2$; see, e. g., [12, Appendix] for this fact. Hence we deduce that $\|R\|_{L^2}^2 \ge \|Q\|_{L^2}^2$. This completes the proof of Lemma 3.6. □

Having the result of Lemma 3.6 at hand, we can now prove the following illposedness following in the case of positive mass parameter $m > 0$.

Proposition 3.7. *Suppose that $m > 0$ holds in (1), and let $t > 0$. Then the map $\phi \mapsto u(t)$ fails to be uniformly continuous for initial data in the set*

$$M = \{\phi \in L^2 : \phi \text{ radial } , \|\phi\|_{L^2}^2 \ge K\}$$

with respect to the L^2-norm, where $K \ge \|Q\|_{L^2}^2$ is some universal constant.

Proof. Recall that we can assume $m = 1$ without loss of generality. For $\mu_1, \mu_2 > 0$ given, we consider the solitary wave solutions

$$u_{\mu_1}(t,x) = e^{it\mu_1} Q_{\mu_1}(t,x), \quad u_{\mu_2}(t,x) = e^{it\mu_2} Q_{\mu_2}(t,x),$$

where $Q_\mu(x)$ are radial positive solutions to (24) that are given by $Q_\mu = \mu^{3/2} R_\mu(\mu x)$ with R_μ taken from Lemma 3.4. Following the proof of Proposition 3.3, we define

$$I_{\mu_1,\mu_2}(t) = \|u_{\mu_1}(t) - u_{\mu_2}(t)\|_{L^2}^2.$$

Similarly as above, we find that

$$I_{\mu_1,\mu_2}(t) = \|Q_{\mu_1}\|_{L^2}^2 + \|Q_{\mu_2}\|_{L^2}^2 - 2\cos(t(\mu_1 - \mu_2)) \left(\frac{\mu_1}{\mu_2} \right)^{3/2} \int_{\mathbb{R}^3} R_{\mu_1}(x) R_{\mu_2} \left(\left(\frac{\mu_1}{\mu_2} x \right) \right) dx$$

Now let $t > 0$ be given. Define the sequence $\mu(n) = \frac{\pi}{2t} n^2$ for $n \in \mathbb{N}$, which ensure that $\cos(t(\mu(n+1) - \mu(n)) = 0$ for all $n \in \mathbb{N}$. By Lemma 3.6 and after possibly passing to a subsequence, we have that $R_{\mu(n)} \to R$ strongly in L^2 with a radial positive function $R \not\equiv 0$. Hence, we conclude

$$\lim_{n \to \infty} I_{\mu(n+1),\mu(n)}(t) = 2\|R\|_{L^2}^2 \neq 0.$$

On the other hand, we have

$$
\begin{aligned}
&\lim_{n\to\infty} I_{\mu(n+1),\mu(n)}(0) \\
&= 2\|R\|_{L^2}^2 - 2\lim_{n\to\infty}\Big(\frac{\mu(n+1)}{\mu(n)}\Big)^{\frac{3}{2}} \int_{\mathbb{R}^3} R_{\mu(n)}(x) R_{\mu(n+1)}\Big(\frac{\mu(n+1)}{\mu(n)} x\Big)\, dx \\
&= 2\|R\|_{L^2}^2 - 2\int_{\mathbb{R}^3} |R|^2\, dx = 0.
\end{aligned}
$$

This completes the proof of Proposition 3.7. □

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