

Unavoidable sets and harmonic measures living on small sets

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Abstract

Given a connected open set $U \neq \emptyset$ in \mathbb{R}^d , $d \geq 2$, a relatively closed set A in U is called *unavoidable in U* , if Brownian motion, starting in $x \in U \setminus A$ and killed when leaving U , hits A almost surely or, equivalently, if the harmonic measure for x with respect to $U \setminus A$ has mass 1 on A . First a new criterion for unavoidable sets is proven which facilitates the construction of smaller and smaller unavoidable sets in U . Starting with an arbitrary champagne subdomain of U (which is obtained omitting a locally finite union of pairwise disjoint closed balls $\overline{B}(z, r_z)$, $z \in Z$, satisfying $\sup_{z \in Z} r_z / \text{dist}(z, U^c) < 1$), a combination of the criterion and the existence of small nonpolar compact sets of Cantor type yields a set A on which harmonic measures for $U \setminus A$ are living and which has Hausdorff dimension $d - 2$ and, if $d = 2$, logarithmic Hausdorff dimension 1.

This can be done as well for Riesz potentials (isotropic α -stable processes) on Euclidean space and for censored stable processes on $C^{1,1}$ open subsets. Finally, in the very general setting of a balayage space (X, \mathcal{W}) on which the function 1 is harmonic (which covers not only large classes of second order partial differential equations, but also non-local situations as, for example, given by Riesz potentials, isotropic unimodal Lévy processes or censored stable processes) a construction of champagne subsets $X \setminus A$ of X with *small* unavoidable sets A is given which generalizes (and partially improves) recent constructions in the classical case.

Keywords: Harmonic measure; Brownian motion; capacity; champagne subdomain; champagne subregion; unavoidable set; Hausdorff measure; Hausdorff dimension; Riesz potential; α -stable process; Lévy process; harmonic space; balayage space; Hunt process; equilibrium measure

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1 Introduction

This paper is devoted to the construction of *small* unavoidable sets in various potential-theoretic settings (classical potential theory, Riesz potentials (isotropic α -stable processes), censored α -stable processes, harmonic spaces, balayage spaces). In particular, we shall give optimal answers to the question of how small a set may be on which harmonic measures is living.

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For the moment, let U be a non-empty connected open set in \mathbb{R}^d , $d \geq 2$. If $d = 2$ we assume that the complement of U is nonpolar (otherwise our considerations become trivial). A relatively closed subset A of U is called *unavoidable in U* if Brownian motion, starting in $U \setminus A$ and killed when leaving U , hits A almost surely or, equivalently, if $\mu_y^{U \setminus A}(A) = 1$, for every $y \in U \setminus A$, where $\mu_y^{U \setminus A}$ denotes the harmonic measure at y with respect to $U \setminus A$.

A *champagne subdomain* of U is obtained by omitting a union A of pairwise disjoint closed balls $\overline{B}(x, r_z)$, $z \in Z$, the *bubbles*, where Z is an infinite, locally finite set in U , $\sup_{z \in Z} r_z / \text{dist}(z, \partial U) < 1$ and, if U is unbounded, the radii r_z tend to 0 as $z \rightarrow \infty$. It will sometimes be convenient to write Z_A instead of Z .

For $r > 0$, let

$$\text{cap}(r) := \begin{cases} r^{d-2}, & \text{if } d \geq 3, \\ (\log^+ \frac{1}{r})^{-1}, & \text{if } d = 2. \end{cases}$$

Recently, the following has been shown (see [21, Theorem 1.1]; cf. [14, 38] for the case $U = B(0, 1)$ and $h(t) = (\text{cap}(t))^\varepsilon$).

THEOREM 1.1. *Let $h: (0, 1) \rightarrow (0, \infty)$ be such that $\liminf_{t \rightarrow 0} h(t) = 0$. Then, for every $\delta > 0$, there is a champagne subdomain $U \setminus A$ of U such that A is unavoidable in U and $\sum_{z \in Z_A} \text{cap}(r_z) h(r_z) < \delta$.*

We note that, for every champagne subdomain $U \setminus A$ of U with unavoidable A the series $\sum_{z \in Z_A} \text{cap}(r_z)$ diverges. This follows by a slight modification of arguments in [14, 38] (for the possibility of omitting finitely many bubbles, see (1b) in Lemma 2.2). Therefore the condition $\liminf_{t \rightarrow 0} h(t) = 0$ is also necessary for the conclusion in Theorem 1.1.

The proof of Theorem 1.1 given in [21] is based on a criterion for unavoidable sets which, in probabilistic terms, relies on the continuity of the paths for Brownian motion (see [21, Proposition 2.1]). We shall use a criterion which, using entry times $T_E(\omega) := \inf\{t \geq 0: X_t(\omega) \in E\}$ for Borel measurable sets E , states the following.

PROPOSITION 1.2. *Let A and B be relatively closed subsets of U and $\kappa > 0$ such that A is unavoidable in U and $P^x[T_B < T_{U^c}] \geq \kappa$, for every $x \in A$. Then B is unavoidable in U .*

Such a criterion holds as well for very general transient Hunt processes on locally compact spaces X with countable base (cf. Proposition 2.3 and its proof). Iterated application may lead to very small unavoidable sets.

Starting, for example, in our classical case with an arbitrary champagne subdomain of U with unavoidable union of bubbles (obtained by Theorem 1.1 or more simply by directly using Proposition 1.2 twice), an application of the new criterion quickly leads to the following result on the smallness of sets, where harmonic measures may live (cf. Corollary 3.3).

THEOREM 1.3. *There exists a relatively closed set A in U having the following properties:*

- *The open set $U \setminus A$ is connected.*
- *For every $x \in U \setminus A$, $\mu_x^{U \setminus A}(A) = 1$.*

- *The set A has Hausdorff dimension $d - 2$ and, if $d = 2$, logarithmic Hausdorff dimension 1.*

Let us note that smaller Hausdorff dimensions are not possible, since any set having strictly positive harmonic measure has at least Hausdorff dimension $d - 2$ and, if $d = 2$, logarithmic Hausdorff dimension 1 (see (3.2) and the subsequent lines).

A general equivalence involving arbitrary measure functions ϕ is presented in Theorem 3.2. Moreover, there are analogous results for Riesz potentials (isotropic α -stable processes) on Euclidean space (see Section 4).

On the other hand, P. W. Jones and T. H. Wolff [26] proved that harmonic measures for planar domains are always living on sets of Hausdorff dimension at most 1. Later T. H. Wolff [40] refined this by showing that there always exists a set which has full harmonic measure and σ -finite 1-dimensional measure. For simply connected domains, N. G. Makarov [30] showed that any set of Hausdorff dimension strictly less than 1 has zero harmonic measure. In fact, he found an optimal measure function such that harmonic measures are absolutely continuous with respect to the corresponding Hausdorff measure. Further results for planar domains may be found in [9, 36, 35, 10, 27, 30, 11, 37, 31, 32, 5, 39, 25, 4, 33]. For higher dimensions $d \geq 3$, J. Bourgain [8] proved that there exists an absolute constant $\gamma(d) < d$ such that harmonic measures for open sets in \mathbb{R}^d always have full mass on a set of dimension at most $\gamma(d)$. As shown by T. H. Wolff [41], $\gamma(3) > 2$.

In Section 5 we shall prove that champagne subsets with small unavoidable unions of bubbles exist in very general settings, where we have a strictly positive Green function G and a capacity function which is related to the behavior of G close to the diagonal. This even simplifies the construction given for Theorem 1.1 (in the case, where $\lim_{t \rightarrow 0} h(t) = 0$) and has applications to large classes of elliptic second order PDE's as well as to Riesz potentials, isotropic unimodal Lévy processes and censored stable processes (see Examples 5.2).

For the convenience of the reader, we add an Appendix. In a first part we discuss balayage spaces and explain their relationship with Hunt processes, sub-Markov semigroups, sub-Markov resolvents. In a second part we give a self-contained proof for the construction of small nonpolar compact sets of Cantor type (Theorem 8.10).

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2 Unavoidable sets

To work in reasonable generality let us consider a balayage space (X, \mathcal{W}) such that the function 1 is harmonic, and let $\mathfrak{X} = (\Omega, \mathfrak{M}, \mathfrak{M}_t, X_t, \theta_t, P^x)$ be an associated Hunt process (see [6] or the Appendix). This covers large classes of second order partial differential equations as well as non-local situations as, for example, given by Riesz potentials or censored stable processes.

We shall proceed in such a way that the reader who is mainly interested in classical potential theory may look at most of the following assuming that X is a connected open set U in \mathbb{R}^d , $d \geq 2$ ($\mathbb{R}^d \setminus U$ nonpolar if $d = 2$), $\mathcal{W} = \mathcal{S}^+(U) \cup \{\infty\}$, where $\mathcal{S}^+(U)$ is the set of all positive superharmonic functions on U , and \mathfrak{X} is Brownian motion on U .

Let $\mathcal{C}(X)$ be the set of all continuous real functions on X and let $\mathcal{K}(X)$ denote the set of all functions in $\mathcal{C}(X)$ having compact support. Moreover, let $\mathcal{B}(X)$ be the set of all Borel measurable numerical functions on X . Given any set \mathcal{F} of functions, let \mathcal{F}^+ (\mathcal{F}_b , respectively) denote the set of all positive (bounded, respectively) $f \in \mathcal{F}$.

We first recall some basic notions and facts on balayage we shall need. For numerical functions f on X , let

$$R_f := \inf\{u \in \mathcal{W} : u \geq f\}.$$

In particular, for every $A \subset X$ and $u \in \mathcal{W}$, let

$$R_u^A := R_{1_A u} = \inf\{v \in \mathcal{W} : v \geq u \text{ on } A\}.$$

Let $\mathcal{P}(X)$ denote the set of all potentials in $\mathcal{C}(X)$, that is,

$$(2.1) \quad \mathcal{P}(X) := \{p \in \mathcal{W} \cap \mathcal{C}(X) : \inf\{R_p^{V^c} : V \text{ relatively compact}\} = 0\}.$$

Then \mathcal{W} is the set of all limits of increasing sequences in $\mathcal{P}(X)$. A potential $p \in \mathcal{P}(X)$ is called *strict* if any two measures μ, ν on X satisfying $\int p d\mu = \int p d\nu < \infty$ and $\int q d\mu \leq \int q d\nu$ for all $q \in \mathcal{P}(X)$, coincide.

For every $x \in X$ and $A \subset X$, there exists a unique measure ε_x^A on X such that

$$\int u d\varepsilon_x^A = R_u^A(x), \quad \text{for every } u \in \mathcal{W}.$$

Of course, $\varepsilon_x^A = \varepsilon_x$, if $x \in A$. We note that in [6] the measure ε_x^A is denoted by $\overset{\circ}{\varepsilon}_x^A$, whereas there ε_x^A denotes the *swept* measure defined by $\int u d\varepsilon_x^A = \hat{R}_u^A(x) := \liminf_{y \rightarrow x} R_u^A(y)$, $u \in \mathcal{W}$ (it coincides with $\overset{\circ}{\varepsilon}_x^A$, if $x \notin A$).

In terms of the associated Hunt process, the measures ε_x^A , $x \in X$ and A Borel measurable, are the distributions of the process starting in x at the time T_A of the first entry into A (which is defined by $T_A(\omega) := \inf\{t \geq 0 : X_t(\omega) \in A\}$).

If we have to emphasize the “universe” X to avoid ambiguities, we shall add a superscript X and, for example, write ${}^X R_1^A$ and ${}^X \varepsilon_x^A$ instead of R_1^A and ε_x^A .

Given a finite measure ν on X , let $\|\nu\|$ denote its total mass.

DEFINITION 2.1. *Let A be a subset of X . It is called unavoidable (in X), if $\|\varepsilon_x^A\| = 1$, for every $x \in X$, or, equivalently, if*

$$(2.2) \quad R_1^A = 1.$$

In probabilistic terms (if A is Borel measurable): The set A is unavoidable, if

$$(2.3) \quad P^x[T_A < \infty] = 1, \quad \text{for every } x \in X.$$

The set A is called avoidable, if R_1^A is not identically 1, that is (provided A is Borel measurable), if there exists a point $x \in X$ such that $P^x[T_A < \infty] < 1$.

This is consistent with the definition we used in the Introduction, where we consider classical potential theory on \mathbb{R}^d , a connected open set $U \neq \emptyset$, and a relatively closed subset A of U . Indeed, from a probabilistic point of view, the consistency is

obvious, since $\|U\varepsilon_x^A\|$ is the probability that Brownian motion killed upon leaving U enters A during its lifetime. For an analytic proof, we recall that, for $x \in U \setminus A$,

$$\begin{aligned} {}^U R_1^A(x) &= \inf\{u(x) : u \in \mathcal{S}^+(U), u \geq 1 \text{ on } A\}, \\ \mu_x^{U \setminus A}(A) &= \inf\{v(x) : v \in \mathcal{S}^+(U \setminus A), \liminf_{y \rightarrow z} v(y) \geq 1 \text{ for } z \in A \cap \partial(U \setminus A)\} \end{aligned}$$

(cf. [3, Chapter 6]). Hence, trivially, ${}^U R_1^A(x) \geq \mu_x^{U \setminus A}(A)$. If $v \in \mathcal{S}^+(U \setminus A)$ such that $\liminf_{y \rightarrow z} v(y) \geq 1$ for every $z \in A \cap \partial(U \setminus A)$, then the function, which is equal to 1 on A and equal to $\min\{1, v\}$ on $U \setminus A$, is superharmonic on U . This yields the reverse inequality. Hence ${}^U R_1^A(x) = 1$ if and only if $\mu_x^{U \setminus A}(A) = 1$.

Returning to the setting of the balayage space (X, \mathcal{W}) , where the function 1 is harmonic, we observe some elementary facts.

LEMMA 2.2. 1. *For every unavoidable set A the following holds:*

- (a) *Every set A' in X containing A is unavoidable.*
- (b) *For every relatively compact set F in X , the set $A \setminus F$ is unavoidable.*
- (c) *If A is the union of relatively compact sets F_n , $n \in \mathbb{N}$, then, for all $x \in X$, the series $\sum_{n \in \mathbb{N}} \|\varepsilon_x^{F_n}\|$ diverges.*

2. *For every relatively compact open set V in X , the set $X \setminus V$ is unavoidable.*

Proof. 1. (a) Trivial consequence of $R_1^A \leq R_1^{A'}$.

(b) Let $\varphi \in \mathcal{K}(X)$ such that $1_F \leq \varphi$. Then $p := R_\varphi \in \mathcal{P}(X)$ (see [6, II.V.2]). Let $u \in \mathcal{W}$ such that $u \geq 1$ on $A \setminus F$. Then $u + p \in \mathcal{W}$ and $u + p \geq 1$ on A . So $u + p \geq R_1^A = 1$, that is, $u - 1 \geq -p$. Since the function 1 is harmonic, the function $u - 1$ is hyperharmonic and hence, by the minimum principle, $u - 1 \geq 0$ (see [6, III.6.6]). So $u \geq 1$ proving that $R_1^{A \setminus F} = 1$.

(c) Let $m \in \mathbb{N}$ and let B denote the union of all F_n , $n \geq m$. By (b), the set B is unavoidable. Hence $1 = R_1^B \leq \sum_{n \geq m} R_1^{F_n}$.

2. A consequence of (b) taking $A = X$, $F = V$. □

Iterated application of Proposition 2.3,2 will help us to construct small unavoidable sets.

PROPOSITION 2.3. *For all subsets A, B of X the following holds:*

- 1. *A is unavoidable or $\inf_{x \in X} R_1^A(x) = 0$.*
- 2. *If A is unavoidable, $\kappa > 0$, and $R_1^B \geq \kappa$ on A , then B is unavoidable.*
- 3. *Suppose that (X, \mathcal{W}) has the Liouville property, that is, every bounded harmonic function on X is constant. Then*
 - (a) *A is avoidable if and only if \hat{R}_1^A is a potential,*
 - (b) *$A \cup B$ is avoidable if and only if A and B are avoidable.*

Proof. 1. Of course, $\gamma := \inf_{x \in X} R_1^A(x) \in [0, 1]$. If $u \in \mathcal{W}$ such that $u \geq 1$ on A , then $u \geq R_1^A \geq \gamma$, hence $u - \gamma \in \mathcal{W}$ and $u - \gamma \geq 1 - \gamma$ on A . So $R_1^A - \gamma \geq R_{1-\gamma}^A$. If $u \in \mathcal{W}$ such that $u \geq 1 - \gamma$ on A , then $u + \gamma \in \mathcal{W}$ and $u + \gamma \geq 1$ on A . Therefore $R_{1-\gamma}^A + \gamma \geq R_1^A$. Thus

$$(2.4) \quad R_1^A = R_{1-\gamma}^A + \gamma.$$

Since $R_{1-\gamma}^A = (1 - \gamma)R_1^A$, (2.4) shows that $\gamma R_1^A = \gamma$. So $\gamma = 0$ or $R_1^A = 1$.

2. Suppose that $R_1^B \geq \kappa > 0$ on A , and let $u \in \mathcal{W}$ such that $u \geq 1$ on B . Then $u \geq R_1^B \geq \kappa$ on A , and therefore $u \geq R_\kappa^A$. Hence $R_1^B \geq R_\kappa^A = \kappa R_1^A$. If $R_1^A = 1$, we obtain that $R_1^B \geq \kappa$, and hence B is unavoidable by (1).

3. (a) Let h be the greatest harmonic minorant of \hat{R}_1^A . Then $0 \leq h \leq 1$ and $p := \hat{R}_1^A - h$ is a potential. By the Liouville property, h is constant. If A is avoidable, we hence see, by (1), that $h = 0$ showing that \hat{R}_1^A is the potential p .

(b) If A and B are avoidable, then the inequality $\hat{R}_1^{A \cup B} \leq \hat{R}_1^A + \hat{R}_1^B$ shows that $\hat{R}_1^{A \cup B}$ is a potential, and hence $A \cup B$ is avoidable, by (a). The converse is trivial, by (1a) in Lemma 2.2. \square

For an application in classical potential theory (and for more general harmonic spaces), the following simple consequence will be useful in combination with further applications of Proposition 2.3,2.

PROPOSITION 2.4. *Suppose that V_n , $n \in \mathbb{N}$, are relatively compact open sets covering X such that, for every $x \in V_n$, the harmonic measure $\mu_x^{V_n} = \varepsilon_x^{X \setminus V_n}$ is supported by the boundary ∂V_n . Then the union of all boundaries ∂V_n , $n \in \mathbb{N}$, is unavoidable.*

Proof. Given $x \in X$, there exists $n \in \mathbb{N}$ such that $x \in V_n$, and then, by [6, VI.2.4 and VI.9.4],

$$\varepsilon_x^{\partial V_n} = (\varepsilon_x^{X \setminus V_n})^{\partial V_n} = \varepsilon_x^{X \setminus V_n}.$$

Since the measures $\varepsilon_x^{X \setminus V_n}$ have total mass 1, by Lemma 2.2, an application of Proposition 2.3,2 (with $A = X$ and $\kappa = 1$) finishes the proof. \square

For Riesz potentials (isotropic α -stable processes) on Euclidean space we obtain the following (the reader who is primarily interested in classical potential theory may pass directly to Section 3).

PROPOSITION 2.5. *Let $X = \mathbb{R}^d$, $d \geq 1$, $0 < \alpha < \min\{d, 2\}$, and let \mathcal{W} be the set of all increasing limits of Riesz potentials $G\mu: x \mapsto \int |x - y|^{\alpha-d} d\mu(y)$ (μ finite measure on \mathbb{R}^d with compact support). Moreover, let $0 < R_1 < R_2 < \dots$ be such that $\lim_{n \rightarrow \infty} R_n = \infty$. Then the following holds:*

- (1) *If $d \geq 2$ and $\alpha > 1$, then the union of all $\partial B(0, R_n)$, $n \in \mathbb{N}$, is unavoidable.*
- (2) *For every $\delta > 0$, the union of all shells $B_n := \{x \in \mathbb{R}^d: R_n \leq |x| \leq (1 + \delta)R_n\}$, $n \in \mathbb{N}$, is unavoidable.*

Proof. (1) The boundary $S := \partial B(0, 1)$, $n \in \mathbb{N}$, is not (α -)thin at any of its points (cf. [6, VI.5.4.4]), and hence $R_1^S \in \mathcal{W}$ (in fact, $R_1^S \in \mathcal{P}(X)$). In particular,

$$\kappa := \inf\{R_1^S(x) : x \in B(0, 1)\} > 0.$$

By scaling invariance, $\inf\{R_1^{\partial B(0, R_n)}(x) : x \in B(0, R_n)\} = \kappa$, for every $n \in \mathbb{N}$. Since every $x \in \mathbb{R}^d$ is contained in some $B(0, R_n)$ (and \mathbb{R}^d is unavoidable), we see, by Proposition 2.3,2, that the union of all $\partial B(0, R_n)$, $n \in \mathbb{N}$, is unavoidable.

(2) By scaling invariance, for every $n \in \mathbb{N}$,

$$\inf\{R_1^{B_n}(x) : x \in B(0, R_n)\} = \inf\{R_1^{\{1 \leq |\cdot| \leq 1+\delta\}}(x) : x \in B(0, 1)\} > 0.$$

As in the proof of (1) we now obtain that the union of all B_n , $n \in \mathbb{N}$, is unavoidable. \square

REMARK 2.6. If $\alpha \leq 1$ ($\alpha < 1$ for $d = 1$), then, for every $R > 0$, the boundary $\partial B(0, R)$ is (α -)polar and hence $\varepsilon_x^{\partial B(0, R)} = 0$, for every $x \in \mathbb{R}^d \setminus \partial B(0, R)$ (cf. [6, VI.5.4.4]). So statement of (1) in Proposition 2.5 does not hold.

For a general balayage space, we can still say the following.

PROPOSITION 2.7. *Let (U_m) be an exhaustion of X , that is, let (U_m) be a sequence of relatively compact open sets in X such that $\bar{U}_m \subset U_{m+1}$, $m \in \mathbb{N}$, and $X = \bigcup_{m \in \mathbb{N}} U_m$. Let (k_n) be a sequence of natural numbers.*

Then there exist $m_n \in \mathbb{N}$ such that $m_n + k_n \leq m_{n+1}$, $n \in \mathbb{N}$, and the union B of the compact "shells" $B_n := \bar{U}_{m_n+1} \setminus U_{m_n+k_n}$, $n \in \mathbb{N}$, is unavoidable.

Proof. We start with $m_1 := 1$. Let $n \in \mathbb{N}$ and suppose that m_n has been chosen. We define

$$A_n := \bar{U}_{m_n} \quad \text{and} \quad V_n := U_{m_n+k_n}.$$

The functions $h_m : x \mapsto \varepsilon_x^{X \setminus V_n}(\bar{U}_m \setminus V_n)$, $m \geq m_n + k_n$, which are continuous on V_n (see [6, VI.2.10]), are increasing to 1. So there exists $m_{n+1} \geq m_n + k_n$ such that $h_{m_{n+1}} > 1/2$ on the compact A_n in V_n , and hence

$$\|\varepsilon_x^{B_n}\| \geq \varepsilon_x^{X \setminus V_n}(B_n) = h_{m_{n+1}}(x) > 1/2, \quad \text{for every } x \in A_n.$$

The claim of the proposition follows from Proposition 2.3,2, since the union of the sets A_n , $n \in \mathbb{N}$, is the whole space X , which, of course, is unavoidable. \square

3 Harmonic measures living on small sets

As in our Introduction, let $U \neq \emptyset$ be a connected open set in \mathbb{R}^d , $d \geq 2$, such that $\mathbb{R}^d \setminus U$ is nonpolar if $d = 2$, and let us consider classical potential theory on U (Brownian motion killed upon leaving U).

PROPOSITION 3.1. *Let $U \setminus A$ be a champagne subdomain of U such that A is unavoidable in U . Then the following holds.*

- (1) *For every nonpolar compact F in $B(0, 1)$, the union B of all sets $z + r_z F$, $z \in Z_A$, is relatively closed and unavoidable in U .*

(2) If $\beta \in (0, 1)$ and B denotes the union of all $\overline{B}(z, \beta r_z)$, $z \in Z_A$, then $U \setminus B$ is a champagne subdomain of U such that B is unavoidable.

Proof. Of course, (2) is a trivial consequence of (1). So let F be a nonpolar compact in $B(0, 1)$ and let B be the union of all compact sets $F_z := z + r_z F$, $z \in Z_A$. Obviously, B is relatively closed in U . Since $\sup_{z \in Z_A} r_z / \text{dist}(z, U^c) < 1$, there exists $\varepsilon > 0$ such that, for every $z \in Z_A$, the closure of $B_z := B(z, (1 + \varepsilon)r_z)$ is contained in U . By [3, Lemma 5.3.3 and Theorem 3.1.5],

$$\kappa := \inf\{B^{(0,1+\varepsilon)}\hat{R}_1^F(x) : x \in \overline{B}(0, 1)\} > 0.$$

Let u be a positive superharmonic function on U such that $u \geq 1$ on B . Then, for every $z \in Z_A$, $u \geq 1$ on $F_z := z + r_z F$ and hence, by scaling and translation invariance, $u \geq \kappa$ on $\overline{B}(z, r_z)$. Thus $u \geq \kappa$ on A . An application of Proposition 2.3,2 finishes the proof of (1). \square

For the existence of champagne subdomains of U with unavoidable bubbles which is needed for an application of Proposition 3.1, we could use Theorem 1.1. However, let us note that using Proposition 2.4, it is very easy to construct champagne subdomains $U \setminus B$, where B is unavoidable. Indeed, there exists $\kappa > 0$ such that

$$B^{(0,2)}R_1^{B(0,1/4)} \geq \kappa \quad \text{on } \overline{B}(0, 1).$$

Let (V_n) be an exhaustion of U and

$$\varepsilon_n := \frac{1}{2} \min\{\text{dist}(\partial V_n, \partial V_{n-1} \cup \partial V_{n+1}), 1/n\}, \quad n \in \mathbb{N}$$

(take $V_0 := \emptyset$). For every $n \in \mathbb{N}$, we may choose a finite set Z_n in ∂V_n such that the balls $B(z, \varepsilon_n)$, $z \in Z_n$, cover ∂V_n and the balls $\overline{B}(z, \varepsilon_n/4)$ are pairwise disjoint. Let

$$A := \bigcup_{z \in Z_n, n \in \mathbb{N}} \overline{B}(z, \varepsilon_n) \quad \text{and} \quad B := \bigcup_{z \in Z_n, n \in \mathbb{N}} \overline{B}(z, \varepsilon_n/4).$$

Then A and B are relatively closed in U and $U \setminus B$ is a champagne subdomain of U . By Proposition 2.4 and Lemma 2.2, A is unavoidable. Arguing similarly as in the proof of Proposition 3.1, we obtain that $u \geq \kappa$ on A , for every positive superharmonic function u on U such that $u \geq 1$ on B . Hence, by Proposition 2.3,2, B is unavoidable in U .

Part (1) of Proposition 3.1 indicates that harmonic measures may live on very small sets. To decide how small such sets may really be let us recall a few basic facts on measure functions and Hausdorff measures.

Any function $\phi: (0, \infty) \rightarrow (0, \infty]$ which is increasing and satisfies $\lim_{t \rightarrow 0} \phi(t) = 0$ is called *measure function*. Given such a function ϕ and a subset E of \mathbb{R}^d , we define (cf. [3, Definition 5.9.1])

$$M_\phi^{(\rho)}(E) := \inf \left\{ \sum_{n \in \mathbb{N}} \phi(r_n) : E \subset \bigcup_{n \in \mathbb{N}} B(x_n, r_n) \text{ and } r_n < \rho \text{ for each } n \right\},$$

for $\rho \in (0, \infty)$, and

$$m_\phi(E) := \lim_{\rho \rightarrow 0} M_\phi^{(\rho)}(E).$$

If ϕ, ψ are measure functions, then of course

$$(3.1) \quad m_\phi(E) \leq m_\psi(E), \quad \text{whenever } \phi \leq \psi \text{ on some interval } (0, \varepsilon).$$

The *Hausdorff dimension* of a bounded set E in \mathbb{R}^d is the infimum of all $\gamma > 0$ such that $m_{t^\gamma}(E) < \infty$ (it is at most d). Its *logarithmic Hausdorff dimension* is the infimum of all $\gamma > 0$ such that $m_{h^\gamma}(E) < \infty$, where $h(t) := (\log^+ \frac{1}{t})^{-1}$ and, as usual, $\inf \emptyset := \infty$. If the logarithmic Hausdorff dimension of E is finite, then E has Hausdorff dimension 0.

Obviously, cap is a measure function. If V is an open set in \mathbb{R}^d , $x \in V$, and E is a Borel measurable set in V^c such that $\mu_x^V(E) > 0$, then E is nonpolar and hence (cf. [3, Theorem 6.5.5 and Theorem 5.9.4])

$$(3.2) \quad m_{\text{cap}}(E) = \infty.$$

In particular, the Hausdorff dimension of E is at least $d - 2$ and, if $d = 2$, its logarithmic Hausdorff dimension is at least 1.

THEOREM 3.2. *Let ϕ be a measure function. Then the following statements are equivalent:*

- (i) $\liminf_{t \rightarrow 0} \phi(t) / \text{cap}(t) = 0$.
- (ii) *There exists a relatively closed set A in U such that $m_\phi(A) = 0$, the open set $U \setminus A$ is connected, and $\mu_x^{U \setminus A}(A) = 1$, for every $x \in U \setminus A$.*

Proof. 1. Let us suppose first that (ii) holds. Assuming that, for some $\varepsilon, \delta > 0$, $\phi / \text{cap} \geq \delta$ on $(0, \varepsilon)$, we would obtain, by (3.2), that $0 = m_\phi(A) \geq \delta m_{\text{cap}}(A) = \infty$, a contradiction. Thus $\liminf_{t \rightarrow 0} \phi(t) / \text{cap}(t) = 0$, that is, (i) holds.

2. To prove that (i) implies (ii) we let $h := \phi / \text{cap}$ and suppose $\liminf_{t \rightarrow 0} h(t) = 0$. By Theorem 8.10, there exists a nonpolar compact $F \subset B(0, 1)$ of Cantor type such that $m_\phi(F) = 0$ and $B(0, 1) \setminus F$ is connected.

Let B be any union of pairwise disjoint closed balls $B(z, r_z)$, $z \in Z$, in U such that $U \setminus B$ is a champagne subdomain of U and B is unavoidable. Then the union A of all compact sets $z + r_z F$, $z \in Z$, is relatively closed in U , $U \setminus A$ is connected, and A is unavoidable in U , by Proposition 3.1 (roles of A and B interchanged). \square

Applying the implication (i) \Rightarrow (ii) to the measure function

$$\phi(t) := t^{d-2} (\log^+ \frac{1}{t})^{-1} (\log^+ \log^+ \frac{1}{t})^{-2}$$

(with $\log^+ 0 := 0$) we obtain a set A such that $m_{\text{cap}^{1+\varepsilon}}(A) = 0$, $\varepsilon > 0$, since $\text{cap}^{1+\varepsilon}(t) \leq \phi(t)$ for small t . Thus, by (3.2), Theorem 3.2 has the following immediate consequence.

COROLLARY 3.3. *There exists a relatively closed set A in U having the following properties:*

- *The open set $U \setminus A$ is connected.*
- *For every $x \in U \setminus A$, $\mu_x^{U \setminus A}(A) = 1$.*
- *The set A has Hausdorff dimension $d - 2$ and, if $d = 2$, logarithmic Hausdorff dimension 1.*

4 Application to Riesz potentials

Let us next consider Riesz potentials (isotropic α -stable processes) on Euclidean space, that is, let $X = \mathbb{R}^d$, $d \geq 1$, $0 < \alpha < \min\{d, 2\}$, and let \mathcal{W} be the set of all increasing limits of Riesz potentials $G\mu: x \mapsto \int |x-y|^{\alpha-d} d\mu(y)$ (μ finite measure on \mathbb{R}^d with compact support). The following result extends Theorem 1.1 and Theorem 1.3 (in the case $U = \mathbb{R}^d$) to Riesz potentials.

THEOREM 4.1. (1) *Let ϕ be a measure function. Then the following statements are equivalent:*

- (i) $\liminf_{t \rightarrow 0} \phi(t)t^{\alpha-d} = 0$.
 - (ii) *There exists a closed set A in \mathbb{R}^d such that $\mathbb{R}^d \setminus A$ is connected, $m_\phi(A) = 0$, and $\|\varepsilon_x^A\| = 1$, for every $x \in \mathbb{R}^d$.*
- (2) *There exists a closed set A in \mathbb{R}^d such that $\mathbb{R}^d \setminus A$ is connected, the Hausdorff dimension of A is $d - \alpha$, and $\|\varepsilon_x^A\| = 1$, for every $x \in \mathbb{R}^d$.*

Proof. If $E \subset \mathbb{R}^d$ is Borel measurable and not (α -)polar, then $m_{t^{d-\alpha}}(E) = \infty$ (see [28, Theorem III.3.14]), and therefore its Hausdorff dimension is at least $d - \alpha$. Hence the implication (ii) \Rightarrow (i) in (1) follows as in the proof of Theorem 3.2.

To prove the implication (i) \Rightarrow (ii), let B be any locally finite union of pairwise disjoint closed balls $\overline{B}(z, r_z)$ which is unavoidable. Such a set B is easily obtained. Indeed, by [6, V.4.6],

$$\kappa := \inf\{R_1^{B(0,1)}(x) : x \in B(0, 4)\} > 0.$$

Let $0 < R_1 < R_2 < \dots$ be such that $\lim_{n \rightarrow \infty} R_n = \infty$. If $\alpha \leq 1$, we assume that, for some $\delta > 0$, $(1 + \delta)R_n < R_{n+1}$, $n \in \mathbb{N}$, and define

$$B_n := \{x \in \mathbb{R}^d : R_n \leq |x| \leq (1 + \delta)R_n\}, \quad n \in \mathbb{N}.$$

If $\alpha > 1$, we omit δ , that is, we define $B_n := \partial B(0, R_n)$. By Proposition 2.5, the union of all B_n is unavoidable (in \mathbb{R}^d). Next we fix

$$0 < \beta_n \leq \frac{1}{4} \min\{\text{dist}(B_n, B_{n-1} \cup B_{n+1}), 1/n\}$$

(with $B_0 := \emptyset$) and choose a finite set Z_n in B_n such that the balls $\overline{B}(z, \beta_n)$, $z \in Z_n$, are pairwise disjoint and the balls $B(z, 4\beta_n)$ cover B_n . If $x \in B_n$, then there exists $z \in Z_n$ such that $x \in B(z, 4\beta_n)$, and hence $R_1^{B(z, \beta_n)}(x) \geq \kappa$, by translation and scaling invariance. By Proposition 2.3,2, the union B of all $B(z, \beta_n)$, $z \in Z_n$, $n \in \mathbb{N}$, is unavoidable.

Now let ϕ be any measure function such that $\liminf_{t \rightarrow 0} \phi(t)t^{\alpha-d} = 0$. There exists a compact $F \subset B(0, 1)$ of Cantor type (such that $B(0, 1) \setminus F$ is connected), which is not (α -)polar, but satisfies $m_\phi(F) = 0$ (see Theorem 8.10). Then, by [6, VI.5.1 and V.4.6],

$$\kappa' := \inf\{R_1^F(x) : x \in B(0, 1)\} > 0.$$

Let $F_z := z + r_z F$, $z \in Z$. By translation and scaling invariance, for every $z \in Z$,

$$R_1^{F_z} \geq \kappa' \quad \text{on } B(z, r_z).$$

The union A of all F_z , $z \in Z$, is closed in \mathbb{R}^d and $\mathbb{R}^d \setminus A$ is connected. Clearly, $m_\phi(A) = 0$ and A is unavoidable, by Proposition 2.3,2.

Taking $\phi(t) := t^{d-\alpha}(\log^+ \frac{1}{t})^{-1}$ we obtain that the Hausdorff dimension of A is at most $d - \alpha$, and hence equal to $d - \alpha$. \square

5 Champagne subsets of balayage spaces

In this section we shall prove results on champagne subsets of balayage spaces with small unavoidable unions of bubbles. These results will have immediate applications to various classes of harmonic spaces and to non-local theories as, for example, Riesz potentials on \mathbb{R}^d and censored stable processes on open sets in \mathbb{R}^d .

Let (X, \mathcal{W}) be a balayage space such that points are polar and the function 1 is harmonic. Let ρ be a metric on X which is compatible with the topology of X . For every $x \in X$ and $r > 0$, we define the open ball of center x and radius r by

$$B(x, r) := \{y \in X : \rho(x, y) < r\}.$$

We suppose that, for every compact K in X , there exist $0 < a \leq 1$ and $\varepsilon > 0$ such that

$$(5.1) \quad R_1^{B(x,r) \setminus \bar{B}(x,r/2)}(x) \geq a, \quad \text{for all } x \in K \text{ and } 0 < r < \varepsilon.$$

Further, we assume that we have a lower semicontinuous numerical function $G > 0$ on $X \times X$, finite and continuous off the diagonal, and, for some $\rho_0 > 0$, a strictly increasing continuous function cap on $(0, \rho_0]$ with $\lim_{r \rightarrow 0} \text{cap}(r) = 0$ such that the following holds:

- (i) For every $y \in X$, $G(\cdot, y)$ is a potential with superharmonic support $\{y\}$.
- (ii) For every $p \in \mathcal{P}(X)$, there exists a measure μ on X such that

$$(5.2) \quad p = G\mu := \int G(\cdot, y) d\mu(y).$$

- (iii) There exists a constant $c \geq 1$, such that, for every compact K in X , there exists $0 < \varepsilon \leq \rho_0$ satisfying

$$(5.3) \quad c^{-1} \leq G(x, y) \cdot \text{cap}(\rho(x, y)) \leq c, \quad \text{for all } x \in K \text{ and } y \in B(x, \varepsilon).$$

- (iv) Doubling property: There exists a constant $C > 1$ such that, for all $0 < r \leq \rho_0$,

$$(5.4) \quad \text{cap}(r) \leq C \text{cap}(r/2).$$

REMARKS 5.1. 1. If harmonic measures for relatively compact open sets V are supported by ∂V , then, by the minimum principle, (5.1) holds with $a = 1$.

2. By (i), the measure μ in (ii) is supported by the superharmonic support of p , that is, by the smallest closed set such that p is harmonic on its complement.

3. Suppose that (i) holds and that there exists a measure μ_0 on X such that, for some sub-Markov resolvent $\mathbb{V} = (V_\lambda)_{\lambda > 0}$ on X with proper potential kernel V_0 , we

have $V_0 f = G(f\mu_0)$, $f \in \mathcal{B}^+(X)$, and the set of all \mathbb{V} -excessive functions is \mathcal{W} (see Section 8.1, in particular, Theorem 8.7). Then, by [29], for every $p \in \mathcal{P}(X)$, there exists a *unique* measure μ such that (5.2) holds.

4. Of course, it is sufficient to have (5.3) for a sequence (K_n) of compact sets covering X such that each K_n is contained in the interior of K_{n+1} .

5. If there exist $c \geq 1$ and $\gamma > 0$ such that, for every compact K in X , there exists $\varepsilon > 0$ satisfying

$$c^{-1}\rho(x,y)^{-\gamma} \leq G(x,y) \leq c\rho(x,y)^{-\gamma}, \quad \text{for all } x \in K \text{ and } y \in B(x,\varepsilon),$$

then (iii) and (iv) hold with $\text{cap}(r) := r^\gamma$ and $C = 2^\gamma$.

6. Suppose that there exists $c_0 > 0$ such that, for all $x, y, z \in X$,

$$G(x,y) \leq c_0 G(y,x) \quad \text{and} \quad \min\{G(x,y), G(y,z)\} \leq c_0 G(x,z).$$

Then there *exists* a metric ρ on X and constants $c, \gamma > 0$ (see [24, Proposition 14.5], [19, pp. 1209–1212], [16]) such that ρ is compatible with the topology of X and

$$c^{-1}\rho(x,y)^{-\gamma} \leq G(x,y) \leq c\rho(x,y)^{-\gamma}, \quad \text{for all } x, y \in X.$$

EXAMPLES 5.2. Taking Euclidean distance ρ in (1) – (4), and (6):

1. $X \neq \emptyset$ connected open set in \mathbb{R}^d , classical potential theory:

a) $d \geq 3$: $c = 1 + \eta$, $\eta > 0$ arbitrary ($c = 1$ if $X = \mathbb{R}^d$), $0 < \rho_0 < \infty$ arbitrary, $\text{cap}(r) = r^{d-2}$, and $C = 2^{d-2}$.

b) $d = 2$, $\mathbb{R}^2 \setminus X$ nonpolar: $c = C = 1 + \eta$, $\eta > 0$ arbitrary, $\rho_0 = 2^{-1/\eta}$, $\text{cap}(r) = (\log(1/r))^{-1}$.

2. $X = \mathbb{R}^d$, $d \geq 1$, $0 < \alpha < \min\{2, d\}$, Riesz potentials (isotropic α -stable processes): $c = 1$, $\text{cap}(r) = r^{d-\alpha}$, $C = 2^{d-\alpha}$.

3. $X \neq \emptyset$ bounded $C^{1,1}$ open set in \mathbb{R}^d , $d \geq 2$, $\alpha \in (1, 2)$, censored α -stable process on X : $\text{cap}(r) = r^{d-\alpha}$, $C = 2^{d-\alpha}$ (see Section 7).

4. Of course, harmonic spaces given by (locally) uniformly elliptic partial differential operators of second order on open sets in \mathbb{R}^d are covered as well (having local comparison of the corresponding Green functions with the classical one).

5. Examples, where the underlying topological space is still some \mathbb{R}^m , but the metric is no longer the Euclidean metric, are given by sublaplacians on stratified Lie algebras (see [18, Theorem 1.1], where, by [24, Proposition 14.5], the quasi-metric d_N is equivalent to a power of a metric). Special cases for such sublaplacians are the Laplace-Kohn operators on Heisenberg groups \mathbb{H}_n , $n \in \mathbb{N}$ (see [6, VIII.5.7]).

6. Finally, we note that, more generally than in our second standard example, our assumptions are satisfied by isotropic unimodal Lévy processes on \mathbb{R}^d , $d \geq 3$, having a lower scaling property for the characteristic function ψ (see Section 6).

Aiming at the result in Theorem 5.5 we claim the following.

THEOREM 5.3. *Let $0 < \kappa < (cC)^{-4}$, $\eta \in (0, 1)$, and $h: (0, 1) \rightarrow (0, 1)$ satisfying $\lim_{t \rightarrow 0} h(t) = 0$. Further, let $K \neq \emptyset$ be a compact in X and let K' be a compact neighborhood of K .*

Then there exist a finite set Z in K' (even in K , if K is not thin at any of its points) and radii $0 < r_z < \min\{\eta, \rho_0\}$, $z \in Z$, such that the following holds.

- The closed balls $\overline{B}(z, r_z)$, $z \in Z$, are pairwise disjoint subsets of K' .
- The union E of all $\overline{B}(z, r_z)$, $z \in Z$, satisfies $\|\varepsilon_x^E\| \geq \kappa$, for every $x \in K$.¹
- The sum $\sum_{z \in Z} \text{cap}(r_z) h(r_z)$ is strictly smaller than η .

Essentially, the idea for our proof is the following. Let μ denote the equilibrium measure for K . For $\beta > 0$, we consider a partition of K into finitely many Borel measurable sets K_z , $z \in Z$, such that $K \cap B(z, \beta/3) \subset K_z \subset K \cap B(z, \beta)$ and choose $0 < r_z < \beta/3$ such that $\text{cap}(r_z)$ is approximately $\mu(K_z)$ (possible if β is small). Then the closed balls $\overline{B}(z, r_z)$ are pairwise disjoint and the sum $\sum_{z \in Z} \text{cap}(r_z) h(r_z)$ is bounded by a multiple of $\mu(K) \max_{z \in Z} h(r_z)$, which is smaller than η provided β is sufficiently small. For every $z \in Z$, we define $\nu_z := \mu(K_z) \|\mu_z\|^{-1} \mu_z$, where μ_z is the equilibrium measure of $B(z, r_z)$. Let ν denote the sum of all ν_z , $z \in Z$. If we can show that $c_1 G\nu \leq G\mu \leq C_1 G\nu$ (which will require some effort), we may conclude that

$$R_1^E \geq c_1 G\nu \geq c_1 C_1^{-1} G\mu,$$

where $G\mu = 1$ on K , if K is not thin at any of its points.

Since K may not have this property and since we do not know if the measures μ_z have enough mass near the boundary of $B(z, r_z)$ (no problem, if harmonic measures for open sets V are supported by ∂V), we have to proceed in a more subtle way.

In a way, our approach resembles to what has been done in [2] to obtain a result on quasi-additivity of capacities. In [2], however, given equilibrium measures on well separated small sets are spread out on larger balls to obtain a one-sided estimate between the corresponding potentials, whereas we cut a measure, given on a large set, into pieces, which are concentrated on small balls and lead to a two-sided estimate.

Proof of Theorem 5.3. We fix $0 < \varepsilon < \text{dist}(K, X \setminus K')/3$ and $0 < a \leq 1$ such that (5.1) and (5.3) hold with K' instead of K . If K is not thin at any of its points, let $\varphi := 1_K$. If not, we choose $\varphi \in \mathcal{C}(X)$ such that $1_K \leq \varphi \leq 1$ and the support of φ is contained in the ε -neighborhood of K . Then R_φ is a continuous potential which is equal to 1 on K and harmonic outside the support of φ . By (5.2), there exists a measure μ on X such that $G\mu = R_\varphi$. The support L of μ is contained in the ε -neighborhood of K ; it is even a subset of K , if K is not thin at any of its points.

There exists $\gamma > 0$ such that

$$(5.5) \quad \kappa \leq \frac{1 - \gamma}{c^2 C^2 (\gamma c C + c^2 C^2)}.$$

Let $\tau := \inf_{x \in K'} G\mu(x)$. Then $0 < \tau \leq 1$ and there exists $0 < R < \varepsilon/3$ such that

$$(5.6) \quad \sup_{0 < t < R} h(t) \leq \frac{a\gamma\tau\eta}{\mu(L)} \quad \text{and} \quad \sup_{x \in L} G(1_{B(x,R)}\mu) \leq \tau' := \frac{a\gamma\tau}{cC^3},$$

where the second inequality follows from the fact that $G\mu \in \mathcal{C}_b(X)$, and hence μ does not charge points (which are polar). (Indeed, for every $x \in L$, there exists $0 < s_x < \varepsilon/3$ such that $G(1_{B(x,2s_x)}\mu) \leq \tau'$. There exist $x_1, \dots, x_m \in L$ such that the balls $B(x_1, s_{x_1}), \dots, B(x_m, s_{x_m})$ cover L . Let $R := \min\{s_{x_1}, \dots, s_{x_m}\}$. Given $x \in L$,

¹Let us note that, for $d = 2$ in Example 5.2.1, κ may be as close to 1 as we want.

there exists $1 \leq j \leq m$ such that $x \in B(x_j, s_{x_j})$, hence $B(x, R) \subset B(x_j, 2s_{x_j})$, and therefore $G(1_{B(x,R)}\mu) \leq \tau'$.

Since, by assumption, $G > 0$ and G is continuous off the diagonal, there exists $\delta > 0$ such that, for all $x \in K'$,

$$(5.7) \quad G(x, y) \leq CG(x, y'), \quad \text{if } y, y' \in K' \setminus B(x, R) \text{ and } \rho(y, y') < \delta.$$

Finally, let

$$\beta := (1/2) \min\{\delta, R, \eta, \rho_0\}.$$

There exists a finite set Z in L such that the balls $B(z, \beta/3)$, $z \in Z$, are pairwise disjoint and the balls $B(z, \beta)$ cover L . There exists a partition of L into Borel measurable sets L_z , $z \in Z$, such that

$$(5.8) \quad L \cap B(z, \beta/3) \subset L_z \subset L \cap B(z, \beta).$$

Indeed, let $Z = \{z_1, \dots, z_M\}$ and, for every $1 \leq j \leq M$, let $P'_j := L \cap B(z_j, \beta/3)$, $P''_j := L \cap B(z_j, \beta)$, and let L' be the union of the pairwise disjoint sets P'_1, \dots, P'_M . We recursively define L_{z_1}, \dots, L_{z_M} by $L_{z_1} := P'_1 \cup (P''_1 \setminus L')$ and

$$L_{z_j} := (P'_j \cup (P''_j \setminus L')) \setminus (L_{z_1} \cup \dots \cup L_{z_{j-1}}), \quad 1 < j \leq M.$$

For the moment, let us fix $z \in Z$. Since $1 \leq cG(z, \cdot) \text{cap}(\beta)$ on $B(z, \beta)$ by (5.3), we see, by (5.6) and (5.4), that

$$(a\gamma\tau)^{-1}\mu(B(z, \beta)) \leq (a\gamma\tau)^{-1}c \text{cap}(\beta)G(1_{B(z,\beta)}\mu)(z) \leq C^{-3} \text{cap}(\beta) \leq \text{cap}(\beta/8).$$

So $(a\gamma\tau)^{-1}\mu(L_z) < \text{cap}(\beta/8)$, by (5.8), and hence there exists (a unique)

$$(5.9) \quad 0 < r_z < \beta/8 \quad \text{such that} \quad \text{cap}(r_z) = (a\gamma\tau)^{-1}\mu(L_z).$$

In particular, by (5.6),

$$\sum_{z \in Z} \text{cap}(r_z) h(r_z) < \frac{\eta}{\mu(L)} \sum_{z \in Z} \mu(L_z) = \eta.$$

By (5.1), for every $z \in Z$, there exists $\varphi_z \in \mathcal{K}^+(X)$ such that $\varphi_z \leq 1_{B(z, r_z) \setminus \overline{B}(z, r_z/2)}$ and $R_{\varphi_z}(z) > a/2$. Since $R_{\varphi_z} \in \mathcal{P}(X)$ and R_{φ_z} is harmonic outside the compact $A_z := \overline{B}(z, r_z) \setminus B(z, r_z/2)$, there exists a measure μ_z on A_z such that

$$(5.10) \quad G\mu_z = R_{\varphi_z}.$$

For every $y \in A_z$, $G(z, y) \text{cap}(r_z) \leq CG(z, y) \text{cap}(r_z/2) \leq cC$, and hence, by (5.9),

$$\frac{\mu(L_z)}{2\gamma\tau} = \frac{a \text{cap}(r_z)}{2} < G\mu_z(z) \text{cap}(r_z) = \int G(z, y) \text{cap}(r_z) d\mu_z(y) \leq cC \|\mu_z\|.$$

Let

$$(5.11) \quad \nu_z := \mu(L_z) \|\mu_z\|^{-1} \mu_z.$$

Then $G\nu_z \leq 2\gamma\tau cCG\mu_z \leq 2\gamma\tau cC \leq 2\gamma cCG\mu$ on K' . By the minimum principle,

$$(5.12) \quad G\nu_z \leq 2\gamma cCG\mu.$$

Defining $\nu := \sum_{z \in Z} \nu_z$ we claim that

$$(5.13) \quad (2\gamma cC + c^2C^2)^{-1}G\nu \leq G\mu \leq (1 - \gamma)^{-1}c^2C^2G\nu.$$

Having (5.13), the proof of the proposition is easily finished. Indeed, the measure ν is supported by the union E of all closed balls $\overline{B}(z, r_z)$, $z \in Z$, and $G\nu$ is continuous. Since $G\mu \leq 1$, the first inequality and the minimum principle hence yield that

$$(5.14) \quad (2\gamma cC + c^2C^2)^{-1}G\nu \leq R_1^E.$$

Finally, (5.5), the second inequality of (5.13), and (5.14) imply that

$$\kappa 1_K \leq \kappa G\mu \leq \frac{1 - \gamma}{c^2C^2(2\gamma cC + c^2C^2)} G\mu \leq R_1^E.$$

So it remains to prove that (5.13) holds. To that end, let us fix $z \in Z$ and define

$$V := B(z, R), \quad \mu' := 1_{L \setminus V} \mu.$$

By (5.6) and the minimum principle, $G(1_V \mu) \leq \gamma G\mu$, and hence

$$(5.15) \quad G\mu' \geq (1 - \gamma)G\mu.$$

Let $x \in B(z, \beta)$, $z' \in Z \setminus \{z\}$, and $y, y' \in L_{z'}$. Then $\rho(y, y') < 2\beta \leq \delta$. If $L_{z'} \cap B(x, R) = \emptyset$, we hence conclude, by (5.7), that

$$G(x, y) \leq CG(x, y').$$

Let us suppose now that $L_{z'} \cap B(x, R) \neq \emptyset$. Then $\max\{\rho(x, y), \rho(x, y')\} < R + 2\beta$, hence $y, y' \in B(x, \varepsilon)$. If $y \in L_{z'} \setminus V$, then $\rho(x, y) \geq \rho(z, y) - \rho(x, z) \geq R - \beta \geq \beta$, and hence $\rho(x, y') < \rho(x, y) + \rho(y, y') < 3\rho(x, y)$. If $x \in \overline{B}(z, r_z)$ and $y \in \overline{B}(z', r_{z'})$, then $\rho(x, y') \leq 4\rho(x, y)$, since $\rho(x, z') \geq (\frac{2}{3} - \frac{1}{8})\beta = \frac{13}{24}\beta$, $\rho(x, y') \leq \rho(x, z') + \beta \leq \frac{37}{13}\rho(x, z')$, and $\rho(x, y) \geq \rho(x, z') - \frac{1}{8}\beta \geq \frac{10}{13}\rho(x, z')$. By the monotonicity of cap and (5.4), in both cases

$$\text{cap}(\rho(x, y')) \leq \text{cap}(4\rho(x, y)) \leq C^2 \text{cap}(\rho(x, y)),$$

and therefore, by (5.3) (with K' in place of K),

$$G(x, y) \leq c(\text{cap}(\rho(x, y)))^{-1} \leq cC^2(\text{cap}(\rho(x, y')))^{-1} \leq c^2C^2G(x, y').$$

By integration, we conclude that

$$(5.16) \quad G(1_{L_{z'} \setminus V} \mu) \leq c^2C^2 \frac{\mu(L_{z'} \setminus V)}{\|\mu_{z'}\|} G\mu_{z'} \leq c^2C^2G\nu_{z'} \quad \text{on } B(z, \beta),$$

and

$$(5.17) \quad G\nu_{z'} \leq c^2C^2G(1_{L_{z'}} \mu) \quad \text{on } \overline{B}(z, r_z).$$

Summing (5.16) over all $z' \in Z \setminus \{z\}$ we obtain that $G\mu' \leq c^2C^2G\nu$ on $B(z, \beta)$ and hence, by (5.15), $G\mu \leq (1 - \gamma)^{-1}c^2C^2G\nu$ on $B(z, \beta)$. Since the balls $B(z, \beta)$, $z \in Z$, cover the support L of μ , an application of the the minimum principle yields the second inequality of (5.13). Summing (5.17) over all $z' \in Z \setminus \{z\}$ and using (5.12), we see that $G\nu \leq 2\gamma cCG\mu + c^2C^2G\mu' \leq (2\gamma cC + c^2C^2)G\mu$ on $\overline{B}(z, r_z)$. By the minimum principle, the second inequality of (5.13) follows. \square

In the classical case (see Example 5.2.1), Theorem 5.3 implies an improved version of [21, Theorem 1.1] (recalled in this paper also as Theorem 1.1) in the (most natural) case, where $\lim_{t \rightarrow 0} h(t) = 0$.

COROLLARY 5.4. *Let U be a nonempty connected open set in \mathbb{R}^d , $d \geq 2$, such that $\mathbb{R}^d \setminus U$ is nonpolar if $d = 2$. Let (V_n) be an exhaustion of U by relatively compact open sets V_n , $n \in \mathbb{N}$, such that ∂V_n is not thin at any of its points. Finally, suppose that $h: (0, 1) \rightarrow (0, 1)$ satisfies $\lim_{t \rightarrow 0} h(t) = 0$, and let $\psi \in \mathcal{C}(U)$ such that $0 < \psi \leq \text{dist}(\cdot, U^c)$.*

Then, for every $\delta > 0$, there exist finite sets Z_n in ∂V_n and $0 < r_z < \psi(z)$, $z \in Z_n$, $n \in \mathbb{N}$, such that for the union Z of all Z_n and the union B of all $\overline{B}(z, r_z)$, $z \in Z$, the following holds:

- $U \setminus B$ is a champagne subdomain of U and B is unavoidable in U .
- $\sum_{z \in Z} \text{cap}(r_z)h(r_z) < \delta$.

Proof. By Proposition 2.4, the union A of all boundaries ∂V_n , $n \in \mathbb{N}$, is unavoidable in U . Let $\delta \in (0, 1)$ and

$$\eta_n := (1/2) \min\{2^{-n}\delta, \text{dist}(\partial V_n, \partial V_{n-1} \cup \partial V_{n+1}), \inf \psi(\partial V_n)\}, \quad n \in \mathbb{N}$$

(with $V_0 := \emptyset$).

By Theorem 5.3, we may choose $\kappa > 0$ (by Example 5.2.1, any $0 < \kappa < 2^{-4(d-2)}$ will do), finite sets Z_n in ∂V_n , and $0 < r_z < \eta_n$, $z \in Z_n$, $n \in \mathbb{N}$, such that

$$\sum_{z \in Z_n} \text{cap}(r_z)h(r_z) < \eta_n,$$

the closed balls $\overline{B}(z, r_z)$, $z \in Z_n$, are pairwise disjoint and their union E_n satisfies

$$\|U_{\varepsilon_x^{E_n}}\| \geq \kappa, \quad x \in \partial V_n.$$

Let

$$Z := \bigcup_{n \in \mathbb{N}} Z_n \quad \text{and} \quad B := \bigcup_{z \in Z} \overline{B}(z, r_z) = \bigcup_{n \in \mathbb{N}} E_n.$$

Clearly, $U \setminus B$ is a champagne subdomain of U and

$$\sum_{z \in Z} \text{cap}(r_z)h(r_z) < \sum_{n \in \mathbb{N}} 2^{-n}\delta = \delta.$$

If $x \in A$, then $x \in \partial V_n$ for some $n \in \mathbb{N}$, and hence, $\|U_{\varepsilon_x^B}\| \geq \|U_{\varepsilon_x^{E_n}}\| \geq \kappa$. So B is unavoidable, by Proposition 2.3,2, that is, $\|U_{\varepsilon_x^B}\| = 1$, for every $x \in U$. \square

Let us return to the general situation we were considering before this application of Theorem 5.3.

THEOREM 5.5. *Let $h: (0, 1) \rightarrow (0, 1)$ be such that $\lim_{t \rightarrow 0} h(t) = 0$, let $\delta > 0$ and $\psi \in \mathcal{C}(X)$, $\psi > 0$.*

Then there exist a locally finite set Z in X and $0 < r_z < \psi(z)$, $z \in Z$, such that the closed balls $\overline{B}(z, r_z)$ are pairwise disjoint, the union of all $\overline{B}(z, r_z)$ is unavoidable, and $\sum_{z \in Z} \text{cap}(r_z)h(r_z) < \delta$.

Proof. Let us choose an exhaustion (U_m) of X . By Proposition 2.7, there exist $m_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that $m_n + 4 \leq m_{n+1}$ and the union A of the compact "shells" $K_n := \overline{U}_{m_{n+1}} \setminus U_{m_n+4}$ is unavoidable. For every $n \in \mathbb{N}$, the compact

$$K'_n := \overline{U}_{m_{n+1}+1} \setminus U_{m_n+3}$$

is a neighborhood of K_n , and the sets K'_n , $n \in \mathbb{N}$, are pairwise disjoint. Assuming without loss of generality that $\delta < 1$ we define

$$\eta_n := \min\{2^{-n}\delta, \inf \psi(K'_n)\}.$$

Let $0 < \kappa < (cC)^{-4}$. By Theorem 5.3, there are finite sets Z_n in K'_n and $0 < r_z < \eta_n$, $n \in \mathbb{N}$, such that

$$\sum_{z \in Z_n} \text{cap}(r_z)h(r_z) < \eta_n,$$

the closed balls $\overline{B}(z, r_z)$, $z \in Z_n$, are pairwise disjoint, contained in K'_n , and their union E_n satisfies

$$\|\varepsilon_x^{E_n}\| \geq \kappa, \quad x \in K_n.$$

The proof is finished in a similar way as the proof of Corollary 5.4. \square

Applying Theorem 5.5 to Example 5.2.2 we obtain the following.

COROLLARY 5.6. *Let $X = \mathbb{R}^d$, $d \geq 1$, $0 < \alpha < \min\{d, 2\}$, and let \mathcal{W} be the convex cone of all increasing limits of Riesz potentials $G\mu: x \mapsto \int |x - y|^{\alpha-d} d\mu(y)$ (μ finite measure on \mathbb{R}^d with compact support). Let $h: (0, 1) \rightarrow (0, 1)$ be such that $\lim_{t \rightarrow 0} h(t) = 0$, and let $\psi \in \mathcal{C}(X)$, $\psi > 0$.*

Then, for every $\delta > 0$, there is a locally finite set Z in X and $0 < r_z < \psi(z)$, $z \in Z$, such that the closed balls $\overline{B}(z, r_z)$ are pairwise disjoint, the union of all $\overline{B}(z, r_z)$ is unavoidable, and $\sum_{z \in Z} r_z^{d-\alpha} h(r_z) < \delta$.

REMARK 5.7. Let $\alpha \in (1, 2)$ and $0 < R_1 < R_2 < \dots$ such that $\lim_{n \rightarrow \infty} R_n = \infty$. Using Proposition 2.5 it is easy to see that (as in the classical case) we may choose $Z = \bigcup_{n \in \mathbb{N}} Z_n$, where $Z_n \subset \partial B(0, R_n)$ and r_z is the same for all $z \in Z_n$.

6 Application to isotropic unimodal semigroups

To cover more general isotropic unimodal Lévy processes, we study (in a purely analytic way) the following isotropic situation.

Let $\mathbb{P} = (P_t)_{t>0}$ be a right continuous sub-Markov semigroup on \mathbb{R}^d , $d \geq 1$, and let V_0 denote the potential kernel of \mathbb{P} (see Section 8.1):

$$V_0 f(x) = \int_0^\infty P_t f(x) dt, \quad f \in \mathcal{B}^+(\mathbb{R}^d), \quad x \in \mathbb{R}^d.$$

Further, let g be a decreasing numerical function on $[0, \infty)$ such that $0 < g < \infty$ on $(0, \infty)$, $\lim_{r \rightarrow 0} g(r) = g(0) = \infty$, $\lim_{r \rightarrow \infty} g(r) = 0$, and

$$(6.1) \quad \int_0^1 g(r)r^{d-1} dr < \infty$$

(later on we shall replace (6.1) by the stronger property (6.6)). We suppose that

$$(6.2) \quad G_0 := g(|\cdot|) \in E_{\mathbb{P}} \quad \text{and} \quad V_0 f(x) = G_0 * f(x) = \int G_0(x-y)f(y) dy,$$

for all $f \in \mathcal{B}^+(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ (where $E_{\mathbb{P}}$ is the set of all \mathbb{P} -excessive functions).

REMARKS 6.1. 1. *Of course, the assumptions, including (6.6), are satisfied in the classical case and by Riesz potentials with $g(r) = r^{\alpha-d}$, $\alpha \in (0, 2)$, $\alpha < d$.*

2. *Let us note that (6.2) holds with $G_0 = \int_0^\infty p_t dt$, if there exists a Borel measurable function $(t, x) \mapsto p_t(x)$ on $(0, \infty) \times \mathbb{R}^d$ such that each p_t is radial and decreasing (that is, $p_t(x) \leq p_t(y)$ if $|y| \leq |x|$), $p_s * p_t = p_{s+t}$, for all $s, t > 0$, and $P_t f = p_t * f$, for every $f \in \mathcal{B}^+(\mathbb{R}^d)$.*

3. *In particular, our hypotheses, including (6.6), are satisfied by the transition semigroups of the isotropic unimodal Lévy processes $\mathfrak{X} = (X_t, P^x)$ studied in [15, 34] (see also [23]). It is assumed that the characteristic function ψ for such a process \mathfrak{X} (given by $e^{-t\psi(x)} = E^0[e^{i(x, X_t)}]$, $t > 0$, $x \in \mathbb{R}^d$) satisfies a weak lower scaling condition: There exist $\alpha > 0$, $0 \leq C_L \leq 1$, and $R_0 > 0$ such that*

$$(6.3) \quad \psi(\lambda x) \geq C_L \lambda^\alpha \psi(x), \quad \text{for all } \lambda \geq 1 \text{ and } x \in B(0, R_0)^c$$

(see [15, p. 2] and [34, (1.4)]). *Having shown that $g(r) \approx r^{-d}\psi(1/r)^{-1}$ (see [15, Proposition 1, Theorem 3] or [34, Lemma 2.1]), condition (6.3) implies that (6.8) holds (which in turn leads to (6.6)).*

Examples in the case $d \geq 3$ are listed in [34, p. 3]; for $d \leq 2$, see [34, Section 6].

4. *For the general possibility of constructing new examples by subordination see Theorem 8.9.*

LEMMA 6.2. *For every bounded $f \in \mathcal{B}^+(\mathbb{R}^d)$ having compact support, the function $V_0 f$ is contained in $E_{\mathbb{P}} \cap C(\mathbb{R}^d)$ and vanishes at infinity.*

Proof. Since $V_0(\mathcal{B}^+(\mathbb{R}^d)) \subset E_{\mathbb{P}}$, the statement follows immediately from (6.1), (6.2), and our transience property $\lim_{r \rightarrow \infty} g(r) = 0$. \square

The next result as well as Theorem 6.6 is of interest in its own right.

THEOREM 6.3. 1. $(\mathbb{R}^d, E_{\mathbb{P}})$ is a balayage space.

2. Every point in \mathbb{R}^d is polar.

3. Borel measurable finely open $U \neq \emptyset$ have strictly positive Lebesgue measure.

4. If \mathbb{P} is a Markov semigroup, then the constant 1 is harmonic.

Proof. 1. Consequence of Lemma 6.2 and Theorems 8.2 and 8.3.

2. Since $G_0 \in E_{\mathbb{P}}$, the origin is polar. By translation invariance, every point in \mathbb{R}^d is polar.

3. Let $\mathbb{V} = (V_\lambda)_{\lambda > 0}$ be the resolvent of \mathbb{P} and let U be a Borel measurable finely open set and $x \in U$. By Theorem 8.2, there exists $\lambda > 0$ such that $V_\lambda(x, U) > 0$. The proof is finished, since $V_0(x, \cdot) \geq V_\lambda(x, \cdot)$ and $V_0(x, \cdot)$ is absolutely continuous with respect to Lebesgue measure (having the density $y \mapsto G(x-y)$).

4. True, by [6, III.7.6]. \square

The following proposition will be useful for us (and shows that any open set satisfying an exterior cone condition is regular for the Dirichlet problem):

PROPOSITION 6.4. *Let $z, z_0 \in \mathbb{R}^d$, $z \neq z_0$ and $0 < r < |z - z_0|$. Then the open set $U_0 := \text{conv}(\{z\} \cup B(z_0, r)) \setminus \{z\}$ is not thin at z , that is, z is contained in the fine closure of U .²*

Proof. We may assume without loss of generality that $z = 0$. Let $R := |z_0|$. There exist $z_1, \dots, z_m \in \partial B(0, R)$ such that the balls $B(z_j, r)$, $0 \leq j \leq m$, cover $\partial B(0, R)$. Then $B(0, R) \setminus \{0\}$ is covered by the union of the sets $U_j := \text{conv}(\{0\} \cup B(z_j, r)) \setminus \{0\}$, $0 \leq j \leq m$. Since the origin is polar, it is contained in the fine closure of $B(0, R) \setminus \{0\}$, hence in the fine closure of one of these sets. By radial invariance, the origin is contained in the fine closure of U_0 . \square

COROLLARY 6.5. *Every open ball $B(x, r)$, $x \in \mathbb{R}^d$, $r > 0$, is finely dense in the closed ball $\overline{B}(x, r)$.*

In particular, for all $Z \subset \mathbb{R}^d$ and $r_z > 0$, $z \in Z$, the union of all $B(z, r_z)$ is unavoidable if and only if the union of all $\overline{B}(z, r_z)$ is unavoidable.

Proof. Let $x \in \mathbb{R}^d$, $r > 0$, $z \in \partial B(x, r)$. Then $\text{conv}(\{z\} \cup B(x, r/2)) \setminus \{z\} \subset B(x, r)$. Thus, by Proposition 6.4, the point z is contained in the fine closure of $B(x, r)$. \square

For all $x, y \in \mathbb{R}^d$, let

$$G(x, y) := G_0(x - y).$$

Then G is symmetric (that is, $G(x, y) = G(y, x)$, $x, y \in \mathbb{R}^d$), continuous outside the diagonal, and it is a Green function for $(\mathbb{R}^d, E_{\mathbb{P}})$:

THEOREM 6.6. 1. *For every $y \in \mathbb{R}^d$, $G_y := G(\cdot, y)$ is a potential with superharmonic support $\{y\}$.*

2. *Let μ be a measure on \mathbb{R}^d . Then $G\mu := \int G_y d\mu(y) \in E_{\mathbb{P}}$ and, provided $G\mu$ is a potential,³ the support of μ is the superharmonic support of $G\mu$.*

3. *For every potential p on \mathbb{R}^d , there exists a (unique) measure μ on \mathbb{R}^d such that $p = G\mu$.*

Proof. Having (1) we shall obtain (2) and (3) from (6.2) and [22, Lemma 2.1 and Theorem 4.1], since using Lemma 6.2 we may construct $f \in \mathcal{C}^+(\mathbb{R}^d)$, $f > 0$, such that $V_0 f \in \mathcal{C}(\mathbb{R}^d)$. We note that (3) also follows from [29].

To prove (1) we may assume without loss of generality that $y = 0$. We know that $G_0 \in E_{\mathbb{P}}$ is not only lower semicontinuous, but also radial and decreasing. Therefore, by Proposition 6.4, G_0 is continuous on $\mathbb{R}^d \setminus \{0\}$. Indeed, let $z \in \mathbb{R}^d$. Then

$$\gamma := \inf\{G_0(x) : |x| < |z|\} = \limsup_{x \rightarrow z} G_0(x).$$

By Corollary 6.5, $G_0(z) \geq \gamma$, since G_0 is finely continuous. So G_0 is continuous at z . Moreover, G_0 vanishes at infinity, since $\lim_{r \rightarrow \infty} g(r) = 0$. Hence G_0 is a potential.

²The fine topology is the coarsest topology for which all functions in \mathcal{W} are continuous, and $\text{conv}(A)$ denotes the convex hull of A .

³For the definition of potentials on balayage spaces see Section 2.

To see that G_0 is harmonic on $\mathbb{R}^d \setminus \{0\}$, let us fix a bounded open set $U \neq \emptyset$ in \mathbb{R}^d and a bounded open neighborhood W of \overline{U} such that $0 \notin \overline{W}$. We define

$$r := \frac{1}{3} \min\{\text{dist}(\overline{U}, W^c), \text{dist}(0, \overline{W})\}, \quad v := V_0 1_{B(0,r)} = \int_{B(0,r)} G_y dy,$$

and fix $x \in U$. Since $\overline{W} \cap \overline{B}(0, r) = \emptyset$, we know that

$$(6.4) \quad \varepsilon_x^{W^c}(v) = v(x)$$

(immediate consequence of [6, II.7.1] or, probabilistically, from the strong Markov property for a corresponding Hunt process). Further, for every $y \in B(0, r)$, $G_y \in E_{\mathbb{P}}$ and $y + U \subset W$, hence $\varepsilon_x^{W^c}(G_y) \leq \varepsilon_x^{(y+U)^c}(G_y) \leq G_y(x)$. So we obtain that

$$\varepsilon_x^{W^c}(v) = \int_{B(0,r)} \varepsilon_x^{W^c}(G_y) dy \leq \int_{B(0,r)} \varepsilon_x^{(y+U)^c}(G_y) dy \leq \int_{B(0,r)} G_y(x) dy = v(x).$$

Having (6.4) we see that $\varepsilon_x^{(y+U)^c}(G_y) = G_y(x)$ for almost every $y \in B(0, r)$, where, by translation invariance, $\varepsilon_x^{(y+U)^c}(G_y) = \varepsilon_{x-y}^{U^c}(G_0) = \hat{R}_{G_0}^{U^c}(x-y)$. So, for almost every $y \in B(0, r)$,

$$(6.5) \quad G_0(x-y) = \hat{R}_{G_0}^{U^c}(x-y).$$

By fine continuity and Theorem 6.3,3, (6.5) holds for *every* $y \in B(0, r)$. In particular, $G_0(x) = \hat{R}_{G_0}^{U^c}(x) = \varepsilon_x^{U^c}(G_0)$. This finishes the proof. \square

To get property (5.3) (even with $c = 1$) it suffices to define

$$\text{cap}(r) := g(r)^{-1}, \quad r > 0.$$

For every ball B let $|B|$ denote the Lebesgue measure of B and let λ_B denote normalized Lebesgue measure on B (the measure on B having density $1/|B|$ with respect to Lebesgue measure). From now on, let us replace (6.1) by the following stronger hypothesis.

Assumption. *There exist $C_G \geq 1$ and $0 < r_0 \leq \infty$ such that, for every $0 < r < r_0$,*

$$(6.6) \quad d \int_0^r s^{d-1} g(s) ds \leq C_G r^d g(r)$$

or, equivalently,

$$(6.7) \quad G \lambda_{B(0,r)}(0) = \frac{1}{|B(0,r)|} \int_{B(0,r)} G_y(0) dy \leq C_G g(r).$$

Let us note that (6.6) holds with constant $C_G = (d/\alpha)C$, if g has the following *decay property*: There exists $C > 0$ such that

$$(6.8) \quad g(\gamma r) \leq C \gamma^{\alpha-d} g(r), \quad \text{for all } 0 < \gamma < 1 \text{ and } 0 < r < r_0.$$

Indeed, if (6.8) holds and $0 < r < r_0$, then

$$\int_0^r s^{d-1} g(s) ds = r^d \int_0^1 \gamma^{d-1} g(\gamma r) d\gamma \leq Cr^d g(r) \int_0^1 \gamma^{d-1} \gamma^{\alpha-d} d\gamma = \alpha^{-1} Cr^d g(r)$$

(of course, the argument shows that we still get (6.6), if $\gamma^{\alpha-d}$ in (6.8) is replaced by any $f(\gamma) \geq 0$ with $\int_0^1 \gamma^{d-1} f(\gamma) d\gamma < \infty$).

Moreover, we observe that, by (6.7),

$$(6.9) \quad G\lambda_{B(0,r)}(x) \leq C_G g(r) \quad \text{for all } x \in \mathbb{R}^d \text{ and } 0 < r < r_0.$$

Indeed, defining $B := B(0, r)$ and $A := B \cap B(x, r)$ we obtain, by symmetry, that $\int_A G_y(x) d\lambda_B(y) = \int_A G_y(0) d\lambda_B(y)$. Further, $G_y(x) \leq g(r) \leq G_y(0)$, if $y \in B \setminus A$. Therefore $\int G_y(x) d\lambda_B(y) \leq \int G_y(0) d\lambda_B(y)$.

And, last but not least, since $g(r/2) \leq g$ on $(0, r/2)$ and $d \int_0^{r/2} s^{d-1} ds = (r/2)^d$, we get the following *doubling property*: There exists $1 \leq C_D \leq 2^d C_G$ such that

$$(6.10) \quad g(r/2) \leq C_D g(r), \quad \text{for every } 0 < r < r_0.$$

So cap satisfies (5.4) with $C = C_D$.

Another consequence of (6.6) is the following result (cf. [34, Lemmas 2.5, 2.7]; if $r_0 = \infty$ and $|x| \geq r > 0$, then $g(|x| + r)/g(r) \geq C_D^{-1} g(|x|)/g(r)$).

PROPOSITION 6.7. *For every $B := B(0, r)$, $0 < r < r_0/4$, the following holds.*

1. *For every $x \in \mathbb{R}^d$,*

$$\frac{g(|x|)}{g(r)} \geq R_1^B(x) \geq C_G^{-1} \frac{g(|x| + r)}{g(r)}.$$

2. *Let μ be the equilibrium measure for B , that is, $R_1^B = G\mu$. Then*

$$C_G^{-1} \text{cap}(r) \leq \|\mu\| \leq C_D \text{cap}(r).$$

3. *Property (5.1) holds with $a = C_D^{-2} C_G^{-1}$ and $\varepsilon = r_0$.*

Proof. The first inequality in (1) holds, since $G_0 \in E_{\mathbb{P}}$ and $g(r)^{-1} G_0 \geq 1$ on B . Moreover, we know that $G\lambda_B \in E_{\mathbb{P}} \cap \mathcal{C}(\mathbb{R}^d)$ and $G\lambda_B$ vanishes at infinity. By (6.9) and the minimum principle (or [6, II.7.1]),

$$(6.11) \quad R_1^B \geq (C_G g(r))^{-1} G\lambda_B.$$

For every $x \in \mathbb{R}^d$, we have $g(|x - y|) \geq g(|x| + r)$, $y \in B$, and hence

$$(6.12) \quad G\lambda_B(x) \geq g(|x| + r).$$

Clearly, (6.11) and (6.12) imply the second inequality in (1).

Using the doubling property (6.10), we conclude from (6.12) that

$$(6.13) \quad G\lambda_B(x) \geq g(|x| + r) \geq g(2r) \geq C_D^{-1} g(r), \quad \text{for every } x \in B.$$

Moreover, $G\mu = R_1^B = 1$ on B , hence, by Fubini's theorem and the symmetry of G ,

$$1 = \int G\mu d\lambda_B = \int \left(\int G(y, x) d\mu(x) \right) d\lambda_B(y) = \int G\lambda_B d\mu.$$

Therefore (2) follows from (6.9) and (6.12).

Finally, by (1), we have $R_1^B \geq C_G^{-1}g(4r)/g(r) \geq C_D^{-2}C_G^{-1}$ on $\partial B(0, 3r)$. If $x \in \mathbb{R}^d$ and $x' := x + (0, \dots, 0, 3r)$, then $|x' - x| = 3r$, $B(x', r) \subset B(x, 4r) \setminus \overline{B}(x, 2r)$, and hence, using translation invariance,

$$R_1^{B(x, 4r) \setminus \overline{B}(x, 2r)}(x) \geq R_1^{B(x', r)}(x) \geq C_D^{-2}C_G^{-1}.$$

□

We now obtain the following.

THEOREM 6.8. *Let $\mathbb{P} = (P_t)_{t>0}$ be a right continuous sub-Markov semigroup on \mathbb{R}^d , $d \geq 1$, such that the function $G_0 : x \mapsto g(|x|)$ is \mathbb{P} -excessive, where $g > 0$ is decreasing, $0 < g < \infty$ on $(0, \infty)$, $\lim_{r \rightarrow 0} g(r) = g(0) = \infty$, $\lim_{r \rightarrow \infty} g(r) = 0$, and (6.6) holds. Further, suppose that the potential kernel V_0 of \mathbb{P} is given by convolution with G_0 .*

1. *Let $h : (0, 1) \rightarrow (0, 1)$, $\lim_{t \rightarrow 0} h(t) = 0$, let $\delta > 0$ and $\varphi \in \mathcal{C}(\mathbb{R}^d)$, $\varphi > 0$. Then there exist a locally finite set Z in \mathbb{R}^d and $0 < r_z < \varphi(z)$, $z \in Z$, such that the closed balls $\overline{B}(z, r_z)$ are pairwise disjoint, the union of all $\overline{B}(z, r_z)$ is unavoidable, and $\sum_{z \in Z} \text{cap}(r_z)h(r_z) < \delta$.*
2. *Let $Z \subset \mathbb{R}^d$ and $r_z > 0$, such that the union of all $\overline{B}(z, r_z)$, $z \in Z \subset \mathbb{R}^d$, is unavoidable. Then*

$$(6.14) \quad \sum_{z \in Z} g(|z|) \text{cap}(r_z) = \sum_{z \in Z} \frac{g(|z|)}{g(r_z)} = \infty.$$

In particular, $\sum_{z \in Z} \text{cap} r_z = \infty$.

Proof. 1. Consequence of Theorem 5.5.

2. By Lemma 2.2,(c), $\sum_{z \in Z} R_1^{\overline{B}(z, r_z)}(0) = \infty$. By Proposition 6.7,1 and translation invariance, $R_1^{\overline{B}(z, r_z)}(0) \leq g(|z|)/g(r_z)$, for every $z \in Z$. So (6.14) holds.

The proof will be finished using Lemma 2.2,(b). However, not having assumed that the set Z is locally finite, we have to work a little. Clearly, $\sum_{z \in Z} \text{cap}(r_z) = \infty$, unless the set of all $z \in Z$ such that $\text{cap}(r_z) \geq \text{cap}(1)$ is finite. By Lemma 2.2,(b), it hence suffices to consider the case, where $\text{cap}(r_z) < \text{cap}(1)$ and hence $r_z < 1$, for every $z \in Z$. But then, of course, the balls $\overline{B}(z, r_z)$ with $|z| < 1$ are contained in $B(0, 2)$. Hence, applying Lemma 2.2(b) once more, we may assume without loss of generality that $|z| \geq 1$ for every $z \in Z$. Having (6.14) we now immediately see that $\sum_{z \in Z} \text{cap} r_z = \infty$, since $g(|z|) \leq g(1)$, whenever $|z| \geq 1$. □

7 Application to censored stable processes

Throughout this section let U be a (non-empty) bounded $C^{1,1}$ open set in \mathbb{R}^d , $d \geq 2$, and $\alpha \in (1, 2)$. Let \mathfrak{X} be the censored α -stable process on U (see [7, 12, 13]) and let $E_{\mathfrak{X}}$ denote the set of all excessive functions for \mathfrak{X} .

We claim that $(U, E_{\mathfrak{X}})$ is a balayage space satisfying the assumptions made in Section 5 and that the following analogue of Theorems 4.1 and 5.6 holds.

- THEOREM 7.1.** (1) *Let $\psi \in \mathcal{C}(U)$, $0 < \psi \leq \text{dist}(\cdot, U^c)$, and $h: (0, 1) \rightarrow (0, 1)$ with $\lim_{t \rightarrow 0} h(t) = 0$. Then, for every $\delta > 0$, there is a locally finite set Z in U and $0 < r_z < \psi(z)$, $z \in Z$, such that the closed balls $\overline{B}(z, r_z)$ are pairwise disjoint, the union of all $\overline{B}(z, r_z)$ is unavoidable in U , and $\sum_{z \in Z} r_z^{d-\alpha} h(r_z) < \delta$.*
- (2) *Let ϕ be a measure function with $\liminf_{t \rightarrow 0} \phi(t)t^{\alpha-d} = 0$. Then there exists a relatively closed set A in U such that A is unavoidable, $m_\phi(A) = 0$ and, for every connected component D of U , the set $D \setminus A$ is connected.*
- (3) *There exists a relatively closed set A in U such that A is unavoidable, the Hausdorff dimension of A is $d - \alpha$, and, for every connected component D of U , the set $D \setminus A$ is connected.*

For $x, y \in U$, let

$$G_0(x, y) := |x - y|^{\alpha-d} \quad \text{and} \quad \delta_U(x) := \text{dist}(x, U^c).$$

Let V_0 be the potential kernel of \mathfrak{X} . By [12, p. 599 and Theorem 1.3], there exists a unique (symmetric) function $G: U \times U \rightarrow (0, \infty]$ such that G is continuous off the diagonal, $G = \infty$ on the diagonal, and

$$\int_U G(\cdot, y) f(y) dy = V_0 f, \quad f \in \mathcal{B}^+(U).$$

Moreover, there exists $c > 1$ such that, defining $\Psi(x, y) := \delta_U(x)\delta_U(y)|x - y|^{-2(\alpha-1)}$,

$$(7.1) \quad c^{-1} \min\{1, \Psi\} G_0 \leq G \leq c \min\{1, \Psi\} G_0 \quad \text{on } U \times U.$$

In particular, if $x \in U$, $\varepsilon \in (0, 1)$ with $\varepsilon^{\alpha-1} < \delta_U(x)/4$, and $y \in \overline{B}(x, 2\varepsilon)$, then

$$(7.2) \quad c^{-1} G_0(x, y) \leq G(x, y) \leq c G_0(x, y)$$

(we have $\delta_U(y) \geq \delta_U(x) - \varepsilon > \delta_U(x)/2$ and hence $\Psi(x, y) > (1/2)\delta_U(x)^2 \varepsilon^{-2(\alpha-1)} > 1$).

LEMMA 7.2. *For every measure μ on U the following holds:*

- (i) $G\mu \leq cG_0\mu$.
- (ii) *If $z \in U$, $\varepsilon \in (0, 1)$ with $\varepsilon^{\alpha-1} \leq \delta_U(z)/5$, and μ is supported by $\overline{B}(z, \varepsilon)$, then $G_0\mu \leq cG\mu$ on $\overline{B}(z, \varepsilon)$.*
- (iii) *If $G_0\mu \in \mathcal{C}(U)$, then $G\mu \in \mathcal{C}(U)$.*

Proof. (i) and (ii) are immediate consequences of (7.1) and (7.2). To prove (iii), we introduce $\tilde{G}: U \times U \rightarrow (0, \infty]$ such that $\tilde{G} \geq G_0$ and $G + \tilde{G} = (c + 1)G_0$. Then the functions G, \tilde{G} are lower semicontinuous. So, for every measure μ on U , the functions $G\mu, \tilde{G}\mu$ are lower semicontinuous, by Fatou's lemma, and their sum is $(c + 1)G_0\mu$. Thus $G\mu \in \mathcal{C}(U)$, if $G_0\mu \in \mathcal{C}(U)$. \square

PROPOSITION 7.3. *The potential kernel V_0 is a strong Feller kernel, that is, $V_0(\mathcal{B}_b(U)) \subset \mathcal{C}_b(U)$.*

Proof. Let $f \in \mathcal{B}_b^+(U)$. It is well known (and easily verified) that the function $\int_U G_0(\cdot, y)f(y) dy$ is continuous and bounded. So, by Lemma 7.2, $V_0f \in \mathcal{C}_b(U)$. \square

COROLLARY 7.4. *$(U, E_{\mathfrak{X}})$ is a balayage space.*

Proof. Since $V_0(\mathcal{B}^+(U)) \subset E_{\mathfrak{X}}$ (see [6, II.3.8.2] or [17, Proposition 2.2.11]) and, for every $v \in E_{\mathfrak{X}}$, there exist $f_n \in \mathcal{B}_b^+(U)$, $n \in \mathbb{N}$, such that $V_0f_n \uparrow v$ (see [6, II.3.11] or [17, Theorem 2.2.12]), we see that $E_{\mathfrak{X}}$ satisfies property (ii) of (B_4) .

Next let $x, y \in U$, $x \neq y$. Since $G(x, y) < \infty = \liminf_{z \rightarrow y} G(z, y)$, there exists $0 < \varepsilon < \delta_U(y)$ such that $G(x, z) < G(y, z)$, for all $z \in B(y, \varepsilon)$. Then $v := V_01_{B(y, \varepsilon)} \in E_{\mathfrak{X}} \cap \mathcal{C}(U)$ and $v(x) < v(y)$. Moreover, $v \rightarrow 0$ at infinity, by (7.1).

Since $1 \in E_{\mathfrak{X}}$, we conclude that $E_{\mathfrak{X}}$ satisfies (B_4) . Thus $(U, E_{\mathfrak{X}})$ is a balayage space, by Corollary 8.6. \square

Proof of Theorem 7.1. Let us first verify that the balayage space $(U, E_{\mathfrak{X}})$ satisfies the assumptions made in Section 5. To that end let ρ_0 be the diameter of U and $\text{cap}(r) := r^{d-\alpha}$, $0 < r \leq \rho_0$. Clearly, cap is strictly increasing on $(0, \rho_0]$ and (5.4) holds with $C := 2^{d-\alpha}$. Further, the estimate (5.3) follows immediately from (7.2), and the representation (5.2) of potentials is given by Remark 5.1.3.

The harmonicity of $G(\cdot, y)$, $y \in U$, on $U \setminus \{y\}$ is already stated in [12, p. 599]. We may as well get it from the fact that taking $r_n := \delta_U(y)/n$, $n \in \mathbb{N}$, the functions $h_n := (\lambda^d(B(y, r_n)))^{-1}V_01_{B(y, r_n)}$ are harmonic on $U \setminus \overline{B}(y, r_n)$ and converge to $G(\cdot, y)$ locally uniformly on $U \setminus \{y\}$. By (7.1), each function $G(\cdot, y)$, $y \in U$, tends to zero at ∂U , and hence is a potential.

To obtain (5.1), let us fix a compact K in U and choose $\varepsilon \in (0, 1)$ such that $\varepsilon^{a-1} < \delta_U/5$ on K . Let $x \in K$, $0 < r < \varepsilon$, and let μ denote the equilibrium measure for $A := \overline{B}(x, (3/4)r) \setminus B(x, (2/3)r)$ with respect to Riesz potentials, that is, μ is the (unique) measure on A satisfying $G_0\mu \in \mathcal{C}(U)$ and $G_0\mu = 1$ on A . By translation and scaling invariance, the value $\beta := G_0\mu(x)$ does not depend on x and r . Of course, $G_0\mu \leq 1$, by the minimum principle (for Riesz potentials). By Lemma 7.2, $p := c^{-1}G\mu \in \mathcal{P}(U)$ and $p \leq G_0\mu$. Hence, by the minimum principle (for $(U, E_{\mathfrak{X}})$), $p \leq R_1^{B(x, r) \setminus \overline{B}(x, r/2)}$. In particular, $R_1^{B(x, r) \setminus \overline{B}(x, r/2)}(x) \geq c^{-2}\beta$.

So we may apply Theorem 5.3 and obtain (1) in Theorem 7.1. To prove (2) let ϕ be a measure function such that $\liminf_{t \rightarrow 0} \phi(t)t^{\alpha-d} = 0$. By Theorem 8.10, there exists a compact F in $B(0, 1)$ such that $B(0, 1) \setminus F$ is connected, $m_\phi(F) = 0$, and F is nonpolar with respect to Riesz potentials. So there exists a measure $\nu \neq 0$ on F such that $G_0\nu \in \mathcal{C}(\mathbb{R}^d)$ and $G_0\nu \leq 1$. Let

$$\gamma := \inf\{G_0\nu(x) : x \in \overline{B}(0, 1)\}.$$

We now choose a locally finite set Z in U and $0 < r_z < \delta_U(z)/5$, $z \in Z$, such that the closed balls $\overline{B}(z, r_z)$ are pairwise disjoint and the union of all $\overline{B}(z, r_z)$, $z \in Z$, is unavoidable in U . Let A be the union of all compact sets $F_z := z + r_z F$, $z \in Z$. Clearly, A is relatively closed in U and, for every connected component D of U , the set $D \setminus A$ is connected. Moreover, $m_\phi(A) = 0$, since $m_\phi(F) = 0$.

To prove that A is unavoidable, let $z \in Z$ and let T_z denote the transformation $x \mapsto z + r_z x$ on \mathbb{R}^d . Then the measure $\nu_z := r_z^{d-\alpha} T_z(\nu)$ is supported by $T_z(F) = F_z$, $G_0 \nu_z \in \mathcal{C}(\mathbb{R}^d)$, $G_0 \nu_z \leq 1$ on \mathbb{R}^d , and $G_0 \nu_z \geq \gamma$ on $\overline{B}(z, r_z)$. By Lemma 7.2, $G \nu_z \in \mathcal{C}(U)$, $c^{-1} G \nu_z \leq G_0 \nu_z \leq 1$ on U , and $G \nu_z \geq c^{-1} G_0 \nu_z \geq c^{-1} \gamma > 0$ on $\overline{B}(z, r_z)$. Since ν_z is supported by F_z , we see, by the minimum principle, that $R_1^{F_z} \geq c^{-1} G \nu_z$. In particular,

$$R_1^{F_z} \geq c^{-2} \gamma \quad \text{on } \overline{B}(z, r_z).$$

Thus A is unavoidable, by Proposition 2.3,2.

As before, (3) is a consequence of (2) considering $\phi(t) := t^{d-\alpha} (\log^+ \frac{1}{t})^{-1}$. \square

REMARK 7.5. As in Theorem 4.1, the condition $\liminf_{t \rightarrow 0} \phi(t) t^{\alpha-d} = 0$ in (2) of Theorem 7.1 is necessary for the statement.

8 Appendix

8.1 Balayage spaces

Throughout this section let X be a locally compact space with countable base. As before, let $\mathcal{C}(X)$ be the set of all continuous real functions on X , let $\mathcal{K}(X)$ denote the set of all functions in $\mathcal{C}(X)$ having compact support, and let $\mathcal{B}(X)$ be the set of all Borel measurable numerical functions on X .

In probabilistic terms, the theory of balayage spaces is the theory of Hunt processes with proper potential kernel on X such that every excessive function is the supremum of its continuous excessive minorants and there are two strictly positive continuous excessive functions u, v such that u/v vanishes at infinity (Corollary 8.6).

We shall introduce balayage spaces by properties of their positive hyperharmonic functions (see [17, Definition 1.1.3]) and give characterizations in terms of excessive functions for sub-Markov resolvents (see [6, II.3.11, II.4.7, II.7.8] and [17, Theorem 2.2.12, Theorem 2.3.4, Corollary 2.3.7]) and for sub-Markov semigroups (see [6, II.4.9 and II.8.6] and [17, Corollary 2.3.8]). For a characterization by properties of an associated family of harmonic kernels (given by $H_U(x, \cdot) := \varepsilon_x^{U^c}$) see [6, III.2.8 and III.6.11] or [17, Theorems 5.1.2 and 5.3.11]. Numerous examples are given in [6]; see also [19, Examples 10.1].

Let \mathcal{W} be a convex cone of positive lower semicontinuous numerical functions on X . The coarsest topology on X which is at least as fine as the initial topology and for which all functions of \mathcal{W} are continuous is the (\mathcal{W}) -fine topology. For every function $v: X \rightarrow [0, \infty]$, the largest finely lower semicontinuous minorant of v is denoted by \hat{v}^f .

DEFINITION 8.1. (X, \mathcal{W}) is a balayage space, if the following holds:

(B₁) \mathcal{W} is σ -stable, that is, $\sup v_n \in \mathcal{W}$ for every increasing sequence (v_n) in \mathcal{W} .

(B₂) $\widehat{\inf \mathcal{V}}^f \in \mathcal{W}$, for every subset \mathcal{V} of \mathcal{W} .

(B₃) If $u, v', v'' \in \mathcal{W}$ such that $u \leq v' + v''$, there exist $u', u'' \in \mathcal{W}$ such that $u = u' + u''$, $u' \leq v'$, and $u'' \leq v''$.

(B₄) (i) \mathcal{W} is linearly separating.⁴

(ii) For every $w \in \mathcal{W}$, $w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) : v \leq w\}$.

(iii) There are strictly positive $u, v \in \mathcal{W} \cap \mathcal{C}(X)$ such that $u/v \rightarrow 0$ at infinity.

A sub-Markov resolvent $\mathbb{V} = (V_\lambda)_{\lambda>0}$ on X is a family of kernels V_λ on X such that, for every $\lambda > 0$, the kernel λV_λ is sub-Markov (that is, $\lambda V_\lambda 1 \leq 1$) and $V_\lambda = V_\mu + (\mu - \lambda)V_\lambda V_\mu$, for all $\lambda, \mu \in (0, \infty)$. The kernel $V_0 := \sup_{\lambda>0} V_\lambda$ is called the *potential kernel* of \mathbb{V} . The resolvent \mathbb{V} is *right continuous*, if $\lim_{\lambda \rightarrow \infty} \lambda V_\lambda \varphi = \varphi$, for every $\varphi \in \mathcal{K}(X)$. It is *strong Feller*, if $V_\lambda(\mathcal{B}_b(X)) \subset \mathcal{C}_b(X)$, for every $\lambda > 0$.

A function $u \in \mathcal{B}^+(X)$ is \mathbb{V} -*excessive*, if $\sup_{\lambda>0} \lambda V_\lambda u = u$. Let $E_{\mathbb{V}}$ denote the set of all \mathbb{V} -excessive functions. The convex cone $E_{\mathbb{V}}$ contains $V_0(\mathcal{B}^+(X))$ and satisfies (B₁).

THEOREM 8.2. *For every sub-Markov resolvent \mathbb{V} on X , the following statements are equivalent.*

1. $(X, E_{\mathbb{V}})$ is a balayage space.

2. The resolvent \mathbb{V} is right continuous, and $E_{\mathbb{V}}$ satisfies (B₄).

Moreover, if $(X, E_{\mathbb{V}})$ is a balayage space, then $\lim_{\lambda \rightarrow \infty} \lambda V_\lambda(x, U) = 1$ for all $x \in X$ and Borel measurable fine neighborhoods U of x .

THEOREM 8.3. *Suppose that \mathbb{V} is a sub-Markov resolvent such that its potential kernel V_0 is proper, that is, there exists $g \in \mathcal{B}^+(X)$, $g > 0$, such that $V_0 g < \infty$. Then $E_{\mathbb{V}}$ is the set of all limits of increasing sequences in $V(\mathcal{B}^+(X))$, and $E_{\mathbb{V}}$ is linearly separating.*

COROLLARY 8.4. *Let $\mathbb{V} = (V_\lambda)_{\lambda>0}$ be a right continuous strong Feller sub-Markov resolvent on X such that V_0 is proper and there are strictly positive functions $u, v \in E_{\mathbb{V}} \cap \mathcal{C}(X)$ such that $u/v \rightarrow 0$ at infinity. Then $(X, E_{\mathbb{V}})$ is a balayage space.*

A family $\mathbb{P} = (P_t)_{t>0}$ of kernels on X is a *sub-Markov semigroup* if $P_t 1 \leq 1$ and $P_{s+t} = P_s P_t$, for all $s, t > 0$. It is *right continuous* if $\lim_{t \rightarrow 0} P_t \varphi = \varphi$, for all $\varphi \in \mathcal{K}(X)$. A function $u \in \mathcal{B}^+(X)$ is \mathbb{P} -*excessive*, if $\sup_{t>0} P_t u = u$. Let $E_{\mathbb{P}}$ denote the set of all \mathbb{P} -excessive functions. If \mathbb{P} is right continuous (measurability of $(t, x) \mapsto P_t f(x)$, $f \in \mathcal{K}(X)$, would be sufficient), the connection to an associated sub-Markov resolvent $\mathbb{V} = (V_\lambda)_{\lambda>0}$ is given by $V_\lambda = \int_0^\infty e^{-\lambda t} P_t dt$, $\lambda > 0$, we have $E_{\mathbb{P}} = E_{\mathbb{V}}$ (see [6, II.3.13] or [17, Corollary 2.2.14]), and the kernel $V_0 = \sup_{\lambda>0} V_\lambda = \int_0^\infty P_t dt$ is also called the *potential kernel* of \mathbb{P} .

COROLLARY 8.5. *For every sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ on X the following holds.*

⁴That is, for all $x, y \in X$, $x \neq y$, and $\lambda \in [0, \infty)$, there exists $v \in \mathcal{W}$ such that $v(x) \neq \lambda v(y)$.

1. $(X, E_{\mathbb{P}})$ is a balayage space if and only if \mathbb{P} is right continuous and $E_{\mathbb{P}}$ satisfies (B_4) .
2. If \mathbb{P} is right continuous, the potential kernel of \mathbb{P} is proper, the resolvent \mathbb{V} of \mathbb{P} (or even \mathbb{P} itself) is strong Feller, and there are strictly positive functions $u, v \in E_{\mathbb{P}} \cap \mathcal{C}(X)$ such that $u/v \rightarrow 0$ at infinity, then $(X, E_{\mathbb{P}})$ is a balayage space.

Finally, given a Hunt process $\mathfrak{X} = (\Omega, \mathfrak{M}, \mathfrak{M}_t, X_t, \theta_t, P^x)$ on X , the transition kernels P_t , $t > 0$, defined by $P_t f(x) := E^x(f \circ X_t)$, $f \in \mathcal{B}^+(X)$, form a right continuous sub-Markov semigroup \mathbb{P} on X . By definition, $E_{\mathfrak{X}} := E_{\mathbb{P}}$, and $V_0 = \int_0^\infty P_t dt$ is the potential kernel of \mathfrak{X} .

COROLLARY 8.6. *Let $\mathfrak{X} = (\Omega, \mathfrak{M}, \mathfrak{M}_t, X_t, \theta_t, P^x)$ be a Hunt process on X . Then $(X, E_{\mathfrak{X}})$ is a balayage space if and only if $E_{\mathfrak{X}}$ satisfies (B_4) .*

Conversely, the following holds (see [6]).

THEOREM 8.7. *Let (X, \mathcal{W}) be a balayage space such that $1 \in \mathcal{W}$, and let $p \in \mathcal{P}(X)$ be a bounded strict potential.⁵ Then there exists a unique right continuous strong Feller sub-Markov resolvent $\mathbb{V} = (V_\lambda)_{\lambda > 0}$ on X such that $V_0 1 = p$ and $E_{\mathbb{V}} = \mathcal{W}$. Moreover, \mathbb{V} is the resolvent of a right continuous sub-Markov semigroup $\mathbb{P} = (P_t)_{t > 0}$ on X , and \mathbb{P} is the transition semigroup of a Hunt process \mathfrak{X} on X .*

REMARK 8.8. *The assumption $1 \in \mathcal{W}$ is not very restrictive. Indeed, let (X, \mathcal{W}) be an arbitrary balayage space, let $v \in \mathcal{W} \cap \mathcal{C}(X)$, $v > 0$, and $\widetilde{\mathcal{W}} := \{u/v : u \in \mathcal{W}\}$. Then $(X, \widetilde{\mathcal{W}})$ is a balayage space such that $1 \in \widetilde{\mathcal{W}}$.*

Finally, let us mention the possibility of constructing new examples by subordination with convolution semigroups $(\mu_t)_{t > 0}$ on $(0, \infty)$, that is, families of measures μ_t on $(0, \infty)$ such that $\|\mu_t\| \leq 1$, $\mu_s * \mu_t = \mu_{s+t}$, for all $s, t \in (0, \infty)$, and $\lim_{t \rightarrow 0} \mu_t = \varepsilon_0$ (that is, $\lim_{t \rightarrow 0} \mu_t(\varphi) = \varphi(0)$, for all $\varphi \in \mathcal{K}((0, \infty))$). The following result is contained [6, V.3.6, V.3.7]).

THEOREM 8.9. *Let $(\mu_t)_{t > 0}$ be a convolution semigroup on $(0, \infty)$ such that $\kappa := \int_0^\infty \mu_t dt$ is a Radon measure which is absolutely continuous with respect to Lebesgue measure on $(0, \infty)$.*

Moreover, let \mathbb{P} be a sub-Markov semigroup on X with strong Feller resolvent such that $(X, E_{\mathbb{P}})$ is a balayage space, and define kernels P_t^μ , $t > 0$, by

$$P_t^\mu f := \int P_s f d\mu_t(s), \quad f \in \mathcal{B}^+(X).$$

Then $\mathbb{P}^\mu = (P_t^\mu)_{t > 0}$ is a sub-Markov semigroup on X with strong Feller resolvent, and $(X, E_{\mathbb{P}^\mu})$ is a balayage space.⁶

⁵See Section 2 for definitions.

⁶If $\|\mu_t\| = 1$, for every $t > 0$, and \mathbb{P} is a Markov semigroup, then, of course, \mathbb{P}^μ is a Markov semigroup as well.

8.2 Nonpolar compact sets of Cantor type

For the convenience of the reader, we first present a self-contained construction of small nonpolar compact sets of Cantor type for classical potential theory as well as for Riesz potentials on Euclidean space (the result itself is a special case of [1, Theorem 5.4.1]).

Let $d \geq 1$ and $0 < \alpha \leq 2$ with $\alpha < 1$ if $d = 1$, and let

$$\text{cap}(r) := \left(\log \frac{1}{r}\right)^{-1} \quad \text{and} \quad G(x, y) := \log \frac{1}{|x - y|},$$

if $\alpha = d = 2$. In the other cases, let

$$(8.1) \quad \text{cap}(r) := r^{d-\alpha} \quad \text{and} \quad G(x, y) := |x - y|^{\alpha-d}.$$

THEOREM 8.10. *Let $d \geq 2$ and let ϕ be a measure function such that*

$$\liminf_{t \rightarrow 0} \phi(t) / \text{cap}(t) = 0.$$

Then there is a nonpolar compact $F \subset B(0, 1)$ of Cantor type such that $m_\phi(F) = 0$ and $B(0, 1) \setminus F$ is connected.

Let us first establish a simple scaling property for potentials of Lebesgue measure on cubes (see (8.2) below). Let

$$K := [-(2\sqrt{d})^{-1}, (2\sqrt{d})^{-1}]^d$$

so that the diagonal of K has length 1. For every $a > 0$, let

$$\mathcal{Q}_a := \{x + aK : x \in \mathbb{R}^d\},$$

and, for every cube Q in \mathbb{R}^d , let μ_Q denote normalized Lebesgue measure on Q . There exists a constant $c = c(d)$ such that

$$(8.2) \quad c^{-1} \leq \text{cap}(a) \|G\mu_Q\|_\infty \leq c \quad (0 < a \leq 1/e, Q \in \mathcal{Q}_a).$$

Indeed, let us assume for the moment that $\alpha = d = 2$. Since the function $t \mapsto t \log \frac{1}{t}$ is increasing on $(0, 1/e]$, we obtain that, for all $0 < \alpha < \beta \leq 1/e$,

$$(8.3) \quad 2\pi\beta^2 \log \frac{1}{\beta} \geq 2\pi \int_\alpha^\beta t \log \frac{1}{t} dt = \int_{B(0,\beta) \setminus B(0,\alpha)} \log \frac{1}{|x|} dx \geq 2\pi(\beta - \alpha)\alpha \log \frac{1}{\alpha}.$$

Knowing that $B(0, a/(2\sqrt{2})) \setminus B(0, a/(4\sqrt{2})) \subset aK$ and $aK \subset B(x, a)$, for $x \in aK$, the estimate (8.2) follows easily from (8.3). In the other cases, for every $a > 0$, $\|G\mu_{x+aK}\|_\infty = \|G\mu_{aK}\|_\infty = a^{\alpha-d} \|G\mu_K\|_\infty$, since μ_{aK} is the image of μ_K under the scaling $x \mapsto ax$.

We prove Theorem 8.10 by a recursive construction of a decreasing sequence of finite unions F_m of cubes and probability measures μ_m on F_m , $m \in \mathbb{N}$. Of course, we shall finally define $F := \bigcap_{m \in \mathbb{N}} F_m$.

Let $K_1 := (1/e)K$ and $c_1 := \|G\mu_{K_1}\|_\infty$. We start with $F_1 = K_1$ and the measure $\mu_1 := \mu_{K_1}$. Let us suppose that $m \in \mathbb{N}$ and that after $m - 1$ construction steps we have a probability measure

$$\mu_m = \frac{1}{M} \sum_{1 \leq i \leq M} \mu_{Q_i} \quad \text{on } F_m = Q_1 \cup \cdots \cup Q_M,$$

where $M \in \mathbb{N}$, $0 < a \leq 1/e$ and $Q_1, \dots, Q_M \in \mathcal{Q}_a$ are pairwise disjoint such that, for every $1 \leq i \leq M$,

$$(8.4) \quad \frac{1}{M} \|\mu_{Q_i}\|_\infty \leq 2^{-(m-1)} c_1$$

(true for $m = 1$ with $M = 1$ and $a = 1/e$).

Our m -th construction step is the following: For $n \in \mathbb{N}$ and $0 < r < (1/2)a/n$ (to be fixed below), we “cut” each Q_i into n^d cubes Q_{i1}, \dots, Q_{in^d} in $\mathcal{Q}_{a/n}$ in the canonical way. For each $1 \leq j \leq n^d$, let Q'_{ij} be the cube in \mathcal{Q}_r having the same center as Q_{ij} (see Figure 1).

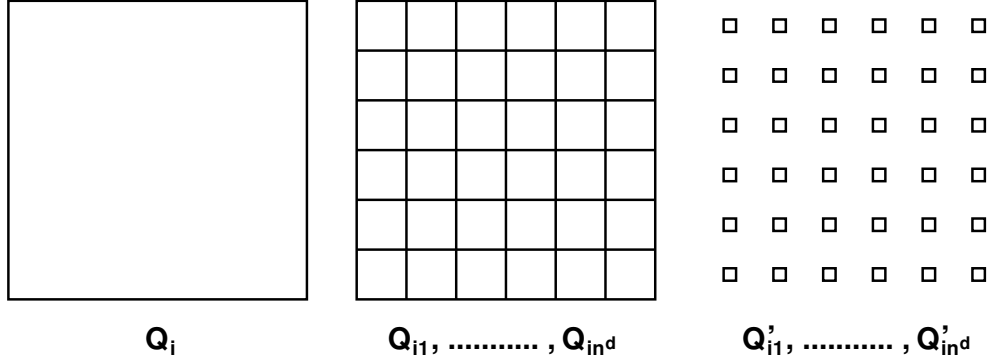


Figure 1. The construction step

Finally, let

$$\mu_{m+1} := \frac{1}{M} \sum_{1 \leq i \leq M} \nu_i, \quad \text{where } \nu_i := \frac{1}{n^d} \sum_{1 \leq k \leq n^d} \mu_{Q'_{ik}}.$$

We note that each ν_i is a probability measure on Q_i , $1 \leq i \leq M$, and that μ_{m+1} is a probability measure on

$$(8.5) \quad F_{m+1} := \bigcup_{1 \leq i \leq M, 1 \leq j \leq n^d} Q'_{ij}.$$

Let $h := \phi/\text{cap}$. We may choose $n \in \mathbb{N}$ and $r \in (0, 1/m)$ such that the following holds:

- (i) For all $i, j \in \{1, \dots, M\}$, $i \neq j$, $|G\nu_j - G\mu_{Q_j}| < 2^{-m} c_1$ on Q_i ,
- (ii) $h(r) < 1/(3mMc^2 \text{cap}(a))$,
- (iii) $2c^2 \text{cap}(a) < n^d \text{cap}(r) < 3c^2 \text{cap}(a)$ and $2r < a/n$.

Indeed, by uniform approximation, there exists $n_0 \in \mathbb{N}$ such that (i) holds if $n \geq n_0$. We may assume without loss of generality that $n^d/(n-1)^d < 3/2$ and $3c^2 \text{cap}(a) < n^d \text{cap}(a/(2n))$ for all $n \geq n_0$. Since $\lim_{t \rightarrow 0} \text{cap}(t) = 0$ and $\liminf_{t \rightarrow 0} h(t) = 0$, there exists $r \in (0, 1/m)$ such that $n_0^d \text{cap}(r) < c^2 \text{cap}(a)$ and (ii) holds. Let $n \in \mathbb{N}$ be minimal such that $n^d \text{cap}(r) > 2c^2 \text{cap}(a)$. Then $n > n_0$, $(n-1)^d \text{cap}(r) < 2c^2 \text{cap}(a)$, and hence $n^d \text{cap}(r) < (3/2)(n-1)^d \text{cap}(r) < 3c^2 \text{cap}(a)$. Moreover, $\text{cap}(r) < 3c^2 \text{cap}(a)n^{-d} < \text{cap}(a/(2n))$, and hence $r < a/(2n)$. So (iii) holds as well.

Since each Q'_{ij} is contained in an open ball of radius $r < 1/m$, we obtain, by (ii) and (iii), that

$$(8.6) \quad M_\phi^{1/m}(F_{m+1}) \leq Mn^d \phi(r) = Mn^d \text{cap}(r)h(r) < 3Mc^2 \text{cap}(a)h(r) < 1/m.$$

For the moment, let us fix $1 \leq i \leq M$, $1 \leq j \leq n^d$, and consider

$$(8.7) \quad Q := Q'_{ij} \in \mathcal{Q}_r.$$

Let $1 \leq k \leq n^d$, $k \neq j$. If $x \in Q$, $y \in Q_{ik}$ and $y' \in Q'_{ik}$ (see Figure 2 for a case, where Q and Q_{ik} are close to each other), then $|x - y'| \geq (2\sqrt{d})^{-1}a/n$ and $|y' - y| \leq a/n$, hence $|x - y| \leq |x - y'| + |y' - y| \leq (1 + 2\sqrt{d})|x - y'|$ and $|x - y'|^{-1} \leq (1 + 2\sqrt{d})|x - y|^{-1}$.

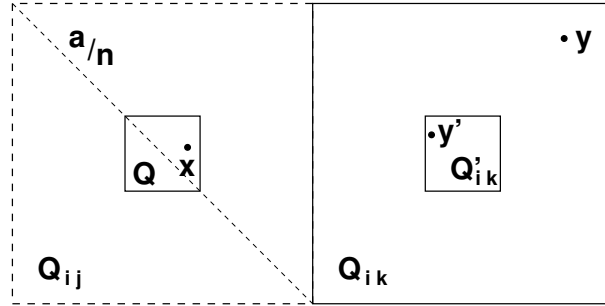


Figure 2. The cubes Q , Q_{ik} and Q'_{ik}

If $d = 2$, then $(1 + 2\sqrt{d})|x - y|^{-1} \leq e^2|x - y|^{-1} \leq |x - y|^{-3}$. Hence, defining $C := \max\{3, (1 + 2\sqrt{d})^{d-\alpha}\}$, we obtain that $G\mu_{Q'_{ik}} \leq CG\mu_{Q_{ik}}$ on Q . Thus

$$(8.8) \quad G\nu_i = n^{-d}G\mu_Q + n^{-d} \sum_{1 \leq k \leq n^d, k \neq j} G\mu_{Q'_{ik}} \leq n^{-d}G\mu_Q + CG\mu_{Q_i} \quad \text{on } Q.$$

By our induction hypothesis (8.4), we have $(1/M)\|\mu_{Q_i}\|_\infty \leq 2^{-(m-1)}c_1$, and hence, by (8.2) and (iii),

$$\left\| \frac{1}{Mn^d} G\mu_Q \right\|_\infty \leq \frac{c}{Mn^d \text{cap}(r)} < \frac{c^{-1}}{2M \text{cap}(a)} \leq \frac{1}{2} \left\| \frac{1}{M} G\mu_{Q_i} \right\|_\infty \leq 2^{-m}c_1$$

(which allows us to continue our construction). So (8.8) leads to the inequality

$$\frac{1}{M}G\nu_i \leq 2^{-m}c_1 + C2^{-(m-1)}c_1 \quad \text{on } Q.$$

By (i), we know that

$$G\mu_{m+1} = \frac{1}{M} \left(G\nu_i + \sum_{j \neq i} G\nu_j \right) \leq \frac{1}{M}G\nu_i + G\mu_m + 2^{-m}c_1 \quad \text{on } Q_i.$$

Therefore $G\mu_{m+1} \leq G\mu_m + (C + 1)2^{-(m-1)}c_1$ on Q . Recalling the definitions of Q and F_{m+1} (see (8.7) and (8.5)) and using the minimum principle, we finally see that

$$(8.9) \quad G\mu_{m+1} \leq G\mu_m + (C + 1)2^{-(m-1)}c_1.$$

Clearly, (8.9) implies that the sequence $(G\mu_m)$ is bounded. As announced above, let F denote the intersection of the decreasing sequence (F_m) . Since the sequence (μ_m) is weakly convergent to a probability measure μ on F satisfying $G\mu \leq \sup_{m \in \mathbb{N}} G\mu_m$, we obtain that F is nonpolar. By (8.6), $m_\phi(F) = 0$ finishing the proof of Theorem 8.10.

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