THE YOUNG BOUQUET AND ITS BOUNDARY

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Abstract. The classification results for the extreme characters of two basic "big" groups, the infinite symmetric group $S(\infty)$ and the infinite-dimensional unitary group $U(\infty)$, are remarkably similar. It does not seem to be possible to explain this phenomenon using a suitable extension of the Schur-Weyl duality to infinite dimension. We suggest an explanation of a different nature that does not have analogs in the classical representation theory.

We start from the combinatorial/probabilistic approach to characters of "big" groups initiated by Vershik and Kerov. In this approach, the space of extreme characters is viewed as a boundary of a certain infinite graph. In the cases of $S(\infty)$ and $U(\infty)$, those are the Young graph and the Gelfand–Tsetlin graph, respectively. We introduce a new related object that we call the Young bouquet. It is a poset with continuous grading whose boundary we define and compute. We show that this boundary is a cone over the boundary of the Young graph, and at the same time it is also a degeneration of the boundary of the Gelfand–Tsetlin graph.

The Young bouquet has an application to constructing infinite-dimensional Markov processes with determinantal correlation functions.

Contents

1. Introduction

We start with a brief historic survey whose goal is to explain the motivation behind our work. A description of our results starts in Section 1.5.

1.1. **Characters of $S(n)$ and $U(N)$.** The symmetric group $S(n)$ of permutations of an n-element set is a simple yet fundamental example of a noncommutative finite group. Similarly, the unitary group $U(N)$ of complex unitary matrices of size N is a basic example of a noncommutative compact group.

As is well known, the representation theory began with a sequence of papers by Frobenius that culminated in a masterful computation of the irreducible characters of $S(n)$ (see e. g. Curtis [Cur99] and references therein). An analogous result for $U(N)$ was obtained by Weyl (see [Wey39] and references therein to Weyl's earlier journal publications of the twenties).

In modern textbooks one can find different approaches to those results, but if one compares the original arguments of Frobenius and Weyl then their similarity is apparent. In essence, Weyl builds his approach following Frobenuis' path.

Furthermore, the famous Schur-Weyl duality establishes a direct link between the characters from the two families. With this duality and relatively simple additional arguments, one can derive Weyl's character formula from the formula of Frobenius and *vice versa*. One reason for that is that the characters of $S(n)$ and $U(N)$ have a common combinatorial base — the Schur symmetric functions.

Of course if one views the unitary groups $U(N)$ as a special case of the reductive Lie groups and constructs a general theory of finite-dimensional representations of those following the infinitesimal approach (replacing groups by their Lie algebras) and Cartan's theory of highest weight, then the analogy with representations of symmetric groups becomes more vague.

However, one can look at a different aspect of the theory — explicit matrix realization of representations. There are two classical results here, Young's orthogonal form for the irreducible representations of $S(n)$ and Gelfand-Tsetlin's formulas for the irreducible representations of $U(N)$. Both results are based on the existence of a basis in an irreducible representation that is connected to a chain of subgroups

$$S(1) \subset S(2) \subset \cdots \subset S(n) \quad \text{and} \quad U(1) \subset U(2) \subset \cdots \subset U(N), \tag{1.1.1}$$

respectively, and the analogy between the realizations in the Young basis and in the Gelfand-Tsetlin basis is very clear (some authors even use the term "Gelfand-Tsetlin basis" for Young's basis as well).

Thus, one observes relations between symmetric and unitary groups both on the level of characters and on the level of matrix realizations of the irreducible representations. This is surprising as the groups themselves are structurally quite different.

1.2. **Characters of $S(\infty)$ and $U(\infty)$.** One can go even further. Let us extend the group chains (1.1.1) to infinity and consider the corresponding inductive limits — the infinite symmetric group $S(\infty) := \varinjlim S(n)$ and the infinite-dimensional unitary group $U(\infty) := \varinjlim U(N)$. These two groups are neither finite nor compact, and $U(\infty)$ is not even locally compact. Nevertheless, one can modify the definition of an irreducible character in such a way that it would make perfect sense for such "big" groups. We have in mind the so-called extreme (or indecomposable) characters that correspond to finite factor representations in the sense of von Neumann. (For the finite and compact groups the extreme characters differ from the conventional irreducible ones only by normalization.)

The extreme characters of $S(\infty)$ were first considered by Thoma [Tho64], and 12 years later Voiculescu [Vo76] wrote a paper on the extreme characters of $U(\infty)$. It was discovered later (Vershik and Kerov [VK81], [VK82]; Boyer [Boy83]) that the classification of the extreme characters of both groups was implicitly contained in earlier works of Schoenberg and his followers on totally positive matrices (Aissen, Edrei, Schoenberg, and Whitney [AESW51]; Aissen, Schoenberg, and Whitney [ASW52]; Edrei [Ed52], [Ed53]).

It turns out that on the level of inductive limits the analogy between the symmetric and unitary groups becomes even more apparent. The character formulas of Thoma and Voiculescu are remarkably similar, and in the language of total positivity the character classification admits a uniform description: In both cases there exists a bijective correspondence between the extreme characters and infinite totally positive Toeplitz matrices; in the first case (for $S(\infty)$) one needs to consider only triangular matrices while in the second case (for $U(\infty)$) no restriction is necessary. In both cases the characters depend on infinitely many continuous parameters, and the set of parameters for $U(\infty)$ is roughly double of that for $S(\infty)$.

1.3. **Harmonic analysis on $S(\infty)$ and $U(\infty)$.** The term "harmonic analysis" (in noncommutative setting) usually refers to the set of questions related to the decomposition of the regular representation and its relatives on irreducibles. However, for inductive limits like $S(\infty)$ or $U(\infty)$, questions of that sort seemingly do not make sense. For example, the group $U(\infty)$ does not have a Haar measure so its regular representation simply does not exist. Nevertheless, there is a way of circumvent this obstacle and construct a whole family of representations each of which could play the role of the regular one.

The original idea is due to Pickrell [Pic87], Neretin presented its generalization in [Ner02], and further developments (detailed analysis of the representations) can be followed along Borodin and Olshanski [BO00a], [BO05a], [BO05b]; Gorin [Gor10]; Kerov, Olshanski, and Vershik [KOV93], [KOV04]; Olshanski [Ols03b], [Ols03c]; Osinenko [Osi11]. Some of these articles deal with the unitary group while the other ones deal with the symmetric group, and once again one easily sees the parallelism between the two cases. It shows in constructing extensions of the groups that allow to define analogs of the Haar measure, in defining analogs of the regular representation, and in the structure of decomposition of those.

1.4. **The Young graph and the Gelfand-Tsetlin graph.** The set of extreme characters of a given group G may be viewed as a variant of the dual object to G; for that reason we use the notation $\widehat{G}$. Vershik and Kerov ([VK81], [VK90]) were first to observe that the dual object $\widehat{S(\infty)}$ to the infinite symmetric group can be defined in purely combinatorial/probabilistic terms. More exactly, $\widehat{S(\infty)}$ serves as a "boundary" for an infinite graph called the Young graph. Similarly, $\widehat{U(\infty)}$ is the "boundary" of a different graph called the Gelfand–Tsetlin graph. (The term "boundary" carries roughly the same meaning as in the theory of Markov processes; an exact definition is given in Section 2.2.)

This interpretation leads to a fruitful connection between noncommutative harmonic analysis and probability theory: As shown in [BO09] and [BO10], the spectral measures on the dual objects $\widehat{S(\infty)}$ and $\widehat{U(\infty)}$ that arise from decomposing regular representations, serve as stationary distributions for certain Markov processes.

The Young graph, denoted as $\mathbb{Y}$, encodes branching of the irreducible characters of the group chain

$$S(1) \subset S(2) \subset \cdots \subset S(n) \subset S(n+1) \subset \cdots$$

Namely, the set of vertices of $\mathbb{Y}$ is the disjoint union of the dual objects

$$\widehat{S(1)} \sqcup \widehat{S(2)} \sqcup \cdots \sqcup \widehat{S(n)} \sqcup \widehat{S(n+1)} \sqcup \cdots .$$

Since the irreducible characters of $S(n)$ are parametrized by the Young diagrams with n boxes, the set of vertices can be identified with the set of all Young diagrams. Further, two vertices are joined by an edge if the corresponding diagrams are different by exactly one box. This definition reflects Young's branching rule: The restriction of the irreducible character of $S(n+1)$ indexed by a Young diagram ν to $S(n)$ is the sum of exactly those characters whose diagrams are obtained from ν by deleting a single box.

Similarly, the Gelfand-Tsetlin graph, denoted as $\mathbb{GT}$, encodes branching of the irreducible characters for the chain

$$U(1) \subset U(2) \subset \cdots \subset U(N) \subset U(N+1) \subset \cdots.$$

The set of vertices in $\mathbb{GT}$ is the disjoint union

$$\widehat{U(1)} \sqcup \widehat{U(2)} \sqcup \cdots \sqcup \widehat{U(N)} \sqcup U\widehat{(N+1)} \sqcup \cdots.$$

The irreducible characters of $U(N)$ are parametrized by the integer-valued vectors of length N with nonincreasing coordinates,

$$\mu = (\mu_1 \geq \cdots \geq \mu_N) \in \mathbb{Z}^N.$$

Such vectors are called signatures. According to the branching rule for irreducible characters of the unitary groups, two signatures of length N and $N+1$ are connected by an edge if their coordinates interlace:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq \mu_N \geq \lambda_{N+1}.$$

Both graphs $\mathbb{Y}$ and $\mathbb{GT}$ are graded in such a way that the edges can only join vertices of adjacent levels. In $\mathbb{Y}$, the vertices of the level $n = 1, 2, \ldots$ are those Young diagrams that have exactly n boxes, while in $\mathbb{GT}$ the vertices of level $N = 1, 2, \ldots$ are the signatures of length exactly N.

Observe that any signature $\lambda = (\lambda_1, \ldots, \lambda_N)$ can be viewed as a pair of Young diagrams (λ^+, λ^-), where the nonzero lengths of rows in λ^+ are the positive coordinates in λ, and the nonzero lengths of rows in λ^- are the absolute values of the negative coordinates in λ. This observation contains a hint at the above mentioned fact that $\widehat{U(\infty)}$ (= the boundary of $\mathbb{GT}$) has doubly many parameters comparing to $\widehat{S(\infty)}$ (= the boundary of $\mathbb{Y}$).

We now proceed to the content of the present article.

1.5. **What is the Young bouquet.** Although our last comment points to a certain similarity between $\mathbb{Y}$ and $\mathbb{GT}$, the grading of the two is totally different: n is the number of boxes of a diagram (equivalently, the sum of lengths of its rows), while N is the length of a signature (or the number of its coordinates). Even if all the coordinates of a signature λ are nonnegative, i. e. in the correspondence $\lambda = (\lambda^+, \lambda^-)$ the second diagram λ^- is empty and λ is seemingly reduced to λ^+, the quantities n and N have very different meanings.

The main idea of this paper is that in order to see a clear connection between the graphs $\mathbb{Y}$ and $\mathbb{GT}$, one needs to introduce an intermediate object. This new object, that we call the Young bouquet and denote as $\mathbb{YB}$, is not a graph. However, $\mathbb{YB}$ is a graded poset, similarly to $\mathbb{Y}$ and $\mathbb{GT}$. One new feature is that the grading in $\mathbb{YB}$ is not discrete but continuous; the grading level is marked by a positive real number. By definition, the elements of $\mathbb{YB}$ of a given level $r > 0$ are pairs (ν, r), where ν is an arbitrary Young diagram. The partial order in $\mathbb{YB}$ is defined as follows: $(\nu, r) < (\widetilde{\nu}, \widetilde{r})$ if $r < \widetilde{r}$ and diagram ν is contained in diagram $\widetilde{\nu}$ (or coincides with it).

We explain how the boundary of the Young bouquet should be understood, and show (Theorem 3.4.7) that it is a cone over the boundary of the Young graph. This establishes a connection between $\mathbb{Y}$ and $\mathbb{YB}$. We also note that the partial order in $\mathbb{YB}$ is obviously consistent with the inclusion partial order on $\mathbb{Y}$.

On the other hand, we show that $\mathbb{YB}$ can be obtained from $\mathbb{GT}$ by a degeneration procedure that can also be viewed as a kind of scaling limit transition. More exactly, one has to start with $\mathbb{GT}$'s subgraph $\mathbb{GT}^+$ consisting of signatures with nonnegative coordinates, and in the degeneration $\mathbb{GT}^+ \to \mathbb{YB}$ one renormalizes the levels, which turns the discrete grading into a continuous one.

Because of these two relationships, with $\mathbb{Y}$ and with $\mathbb{GT}$, we say that $\mathbb{YB}$ is a suitable intermediate object between $\mathbb{Y}$ and $\mathbb{GT}$.

The notion of Young bouquet is perfectly consistent with the concept of "grand canonical ensembles" of random Young diagrams: The well-known model of poissonized Plancherel measures [BDJ99] and a more general model of mixed z-measures [BO00a] become more natural when placed within the context of the Young bouquet.

1.6. **Degeneration $\mathbb{GT}^+ \to \mathbb{YB}$.** While the connection between $\mathbb{Y}$ and $\mathbb{YB}$ is fairly obvious, the degeneration $\mathbb{GT}^+ \to \mathbb{YB}$ deserves to be explained in more detail.

(a) An exact statement of what we mean by the degeneration of the graph $\mathbb{GT}^+$ to the poset $\mathbb{YB}$ is contained in Theorem 4.4.1. The statement involves a degeneration of a certain transition function that is canonically associated to $\mathbb{GT}$, to the transition function canonically associated to $\mathbb{YB}$. (Let us also mention here that our "boundary" is always the entrance boundary for a certain transition function. The graph and poset structure are mostly needed to define that transition function.)

(b) In Theorem 4.5.1 we explain in what sense the boundary of $\mathbb{YB}$ (recall that it is a cone over the boundary of $\mathbb{Y}$) can be obtained as a degeneration of the boundary of $\mathbb{GT}^+$.

(c) Theorem 4.7.1 shows that the degeneration $\mathbb{GT}^+ \to \mathbb{YB}$ is accompanied by degeneration of certain probability measures that originate in harmonic analysis on $S(\infty)$ and $U(\infty)$. This aspect of the degeneration $\mathbb{GT}^+ \to \mathbb{YB}$ can be compared to a descent in the hierarchy of the hypergeometric orthogonal polynomials.

(d) Finally, in Section 5 we discuss the spaces of monotone paths in the posets $\mathbb{Y}$, $\mathbb{YB}$, $\mathbb{GT}$, and Gibbs measures on those spaces. We show that the degeneration

$\mathbb{GT}^+ \to \mathbb{YB}$ can be described in this context as well. The finite monotone paths in $\mathbb{Y}$ and $\mathbb{GT}^+$ have well known combinatorial interpretations; these are the standard and semistandard Young tableaux, respectively. One can interpret the finite monotone paths in $\mathbb{YB}$ in a similar fashion: Those are Young diagrams filled with positive real numbers with the same monotonicity conditions along rows and columns as in the definition of the standard Young tableaux.

1.7. **An application.** In [BO10] we constructed a family of Markov processes on the dual object $\widehat{U(\infty)}$ using its identification with the boundary of $\mathbb{GT}$. On the other hand, [Ols10] contained an announcement of the existence of a similar model of Markov dynamics, where the state space is the cone over $\widehat{S(\infty)}$; in another interpretation, this is a dynamical model of determinantal processes with infinitely many particles. The construction of the Young bouquet allows one to give a simpler proof of that result of [Ols10] using the approach of [BO10]; this is a subject of the follow-up paper [BO11b].

1.8. **Acknowledgments.** A. B. was partially supported by NSF-grant DMS-1056390. G. O. was partially supported by a grant from Simons Foundation (Simons–IUM Fellowship), the RFBR-CNRS grant 10-01-93114, and the project SFB 701 of Bielefeld University.

2. Graded graphs and projective systems

2.1. **The category $\mathcal{B}$.** About the notions used in this subsection see [Mack57] and [Mey66]. A *measurable space* (also called *Borel space*) is a set with a distinguished sigma-algebra of subsets. Denote by $\mathcal{B}$ the category whose objects are *standard* measurable spaces (= standard Borel spaces) and morphisms are Markov kernels. A morphism between two objects will be denoted by a dash arrow, $X \dashrightarrow Y$, to emphasize that it is not an ordinary map. Recall that a (stochastic) *Markov kernel* $\Lambda : X \dashrightarrow Y$ between two measurable spaces X and Y is a function $\Lambda(a, A)$, where a ranges over X and A ranges over measurable subsets of Y, such that $\Lambda(a, \cdot\,)$ is a probability measure on Y for any fixed a and $\Lambda(\,\cdot\,, A)$ is a measurable function on X for any fixed A.

Below we use the short term *link* as a synonym of “Markov kernel”. The composition of two links will be read from left to right: Given $\Lambda : X \dashrightarrow Y$ and $\Lambda' : Y \dashrightarrow Z$, their composition $\Lambda\Lambda' : X \dashrightarrow Z$ is defined as

$$(\Lambda\Lambda')(x, dz) = \int_Y \Lambda(x, dy)\Lambda'(y, dz),$$

where $\Lambda(x, dy)$ and $\Lambda'(y, dz)$ symbolize the measures $\Lambda(x, \cdot\,)$ and $\Lambda'(y, \cdot\,)$, respectively.

A *projective system* in $\mathcal{B}$ is a family $\{V_i, \Lambda_i^j\}$ consisting of objects V_i indexed by elements of a linearly ordered set I (not necessarily discrete), together with links

$\Lambda_i^j : V_j \dashrightarrow V_i$ defined for any couple $i < j$ of indices, such that for any triple $i < j < k$ of indices, one has $\Lambda_j^k \Lambda_i^j = \Lambda_i^k$.

A *limit object* of a projective system is understood in the categorical sense: This is an object $X = \varprojlim V_i$ together with links $\Lambda_i^\infty : X \dashrightarrow V_i$ defined for all $i \in I$, such that:

- $\Lambda_j^\infty \Lambda_i^j = \Lambda_i^\infty$ for all $i < j$;
- if an object Y and links $\widetilde{\Lambda}_i^\infty : Y \dashrightarrow V_i$ satisfy the similar condition, then there exists a unique link $\Lambda_X^Y : Y \dashrightarrow X$ such that $\widetilde{\Lambda}_i = \Lambda_X^Y \Lambda_i^\infty$.

General results concerning existence and uniqueness of limit objects in $\mathcal{B}$ can be found in Winkler [Wi85, Chapter 4]. See also Dynkin [Dy71], [Dy78], Kerov and Orevkova [KeOr90]. When the index set I is a subset of $\mathbb{R}$ and all spaces V_i are copies of one and the same space X, our definition of projective system turns into the classical notion of *transition function* on X (within inversion of order on I).

For a measurable space X we denote by $\mathcal{M}(X)$ the set of probability measures on X. It is itself a measurable space: the corresponding sigma-algebra is generated by the sets of the form $\{\mu \in \mathcal{M}(X) : \mu(A) \in B\}$, where $A \subseteq X$ is a measurable and $B \subseteq \mathbb{R}$ is Borel. Equivalently, the measurable structure of $\mathcal{M}(X)$ is determined by the requirement that for any bounded measurable function on X, its coupling with M should be a measurable function in M. If X is standard, then $\mathcal{M}(X)$ is standard, too.

Observe that any link $\Lambda : X \dashrightarrow Y$ gives rise to a measurable map $\mathcal{M}(X) \to \mathcal{M}(Y)$, which we write as $M \mapsto M\Lambda$. Consequently, any projective system $\{V_i, \Lambda_i^j\}$ in $\mathcal{B}$ gives rise to the conventional projective limit of sets

$$\mathcal{M}_\infty := \varprojlim_I \mathcal{M}(V_i).$$

An element of $\mathcal{M}_\infty$ is called a *coherent family of measures*: By the very definition, it is a family of probability measures $\{M_i \in \mathcal{M}(V_i) : i \in I\}$ such that for any couple $i < j$ one has $M_j \Lambda_i^j = M_i$. (In the case of a transition function, Dynkin [Dy78] terms elements of $\mathcal{M}_\infty$ *entrance laws*.)

If a limit object X exists, then there is a canonical map

$$\mathcal{M}(X) \to \mathcal{M}_\infty.$$

From now on we will gradually narrow the setting of the formalism and will finally focus on the study of some concrete examples.

2.2. **Projective chains.** Consider a particular case of a projective system, where all spaces are discrete (finite or countably infinite) and the indices range over the set $\{1, 2, \dots\}$ of natural numbers. Such a system is uniquely determined by the links Λ_N^{N+1}, $N = 1, 2, \dots$:

$$V_1 \dashleftarrow V_2 \dashleftarrow \cdots \dashleftarrow V_N \dashleftarrow V_{N+1} \dashleftarrow \cdots. \tag{2.2.1}$$

Note that a link between two discrete spaces is simply a stochastic matrix, so that $\Lambda_N^{N+1} : V_{N+1} \dashrightarrow V_N$ is a stochastic matrix whose rows are parametrized by points of V_{N+1} and columns are parametrized by points of V_N:

$$\Lambda_N^{N+1} = [\Lambda_N^{N+1}(x,y)], \quad x \in V_{N+1},\, y \in V_N,$$
$$\Lambda_N^{N+1}(x,y) \ge 0 \text{ for every } x, y, \qquad \sum_{y \in V_N} \Lambda_N^{N+1}(x,y) = 1 \text{ for every } x.$$

For arbitrary $N' > N$, the corresponding link $\Lambda_N^{N'} : V_{N'} \dashrightarrow V_N$ is a stochastic matrix of format $V_{N'} \times V_N$, which factorizes into a product of stochastic matrices corresponding to couples of adjacent indices:

$$\Lambda_N^{N'} = \Lambda_{N'-1}^{N'} \dots \Lambda_N^{N+1}.$$

We call such a projective system a *projective chain*. It gives rise to a chain of ordinary maps

$$\mathcal{M}(V_1) \leftarrow \mathcal{M}(V_2) \leftarrow \dots \leftarrow \mathcal{M}(V_N) \leftarrow \mathcal{M}(V_{N+1}) \leftarrow \cdots \tag{2.2.2}$$

Note that $\mathcal{M}(V_N)$ is a simplex whose vertices can be identified with the points of V_N, and the arrows are affine maps of simplices. In this situation a coherent family (that is, an element of $\mathcal{M}_\infty$) is a sequence $\{M_N \in \mathcal{M}(V_N) : N = 1, 2, \dots\}$ such that

$$M_{N+1}\Lambda_N^{N+1} = M_N, \qquad N = 1, 2, \dots .$$

Here we can interpret measures as row vectors, so that the left-hand side is the product of a row vector by a matrix. In more detail, the equation can be written as

$$\sum_{x \in V_{N+1}} M_{N+1}(x)\Lambda_N^{N+1}(x,y) = M_N(y), \qquad \forall y \in V_N.$$

Note that the set $\mathcal{M}_\infty$ may be empty, as the following simple example shows: Take $V_N = \{N, N+1, N+2, \dots\}$ and define Λ_N^{N+1} as the natural embedding $V_{N+1} \subset V_N$. In what follows we tacitly assume that $\mathcal{M}_\infty$ is nonempty. This holds automatically if all V_N are finite sets.

We may view $\mathcal{M}_\infty$ as a subset of the real vector space

$$L := \mathbb{R}^{V_1 \sqcup V_2 \sqcup V_3 \sqcup \dots}.$$

Here the set $V_1 \sqcup V_2 \sqcup V_3 \sqcup \dots$ is the disjoint union of V_N's. Since this set is countable, the space L equipped with the product topology is locally convex and metrizable. Clearly, $\mathcal{M}_\infty$ is a convex Borel subset of L, hence a standard Borel space.

Let V_∞ be the set of extreme points of $\mathcal{M}_\infty$. We call V_∞ the *boundary* of the chain $\{V_N, \Lambda_N^{N+1}\}$.

Theorem 2.2.1. *If $\mathcal{M}_\infty$ is nonempty, then the boundary $V_\infty \subset \mathcal{M}_\infty$ is a nonempty measurable subset (actually, a subset of type G_δ) of $\mathcal{M}_\infty$, and there is a natural bijection $\mathcal{M}_\infty \leftrightarrow \mathcal{M}(V_\infty)$, which is an isomorphism of measurable spaces.*

A proof based on Choquet's theorem is given in [Ols03c, §9], a much more general result is contained in [Wi85, Chapter 4].

By the very definition of the boundary V_∞, it comes with canonical links

$$\Lambda^\infty_N : V_\infty \dashrightarrow V_N, \qquad N = 1, 2, \dots .$$

Namely, given a point $\omega \in V_\infty \subset \mathcal{M}_\infty$, let $\{M_N\}$ stand for the corresponding sequence of measures; then, by definition,

$$\Lambda^\infty_N(\omega, x) = M_N(x), \qquad x \in V_N, \quad N = 1, 2, \dots .$$

Here, to simplify the notation, we write $\Lambda^\infty_N(\omega, x)$ instead of $\Lambda^\infty_N(\omega, \{x\})$.

From the definition of Λ^∞_N it follows that

$$\Lambda^\infty_{N+1}\Lambda^{N+1}_N = \Lambda^\infty_N, \qquad N = 1, 2, \dots .$$

Now it is easy to see that the boundary V_∞ coincides with the categorical projective limit of the initial chain (2.2.1).

Remark 2.2.2. In the context of Theorem 2.2.1, assume we are given a standard measurable space X and links $\Lambda^X_N : X \dashrightarrow V_N$, $N = 1, 2, \dots$, such that:

- $\Lambda^X_{N+1}\Lambda^{N+1}_N = \Lambda^X_N$ for all N;
- the induced map $\mathcal{M}(X) \to \mathcal{M}_\infty = \varprojlim \mathcal{M}(V_N)$ is a bijection.

Then X coincides with the boundary V_∞. Indeed, the maps $\mathcal{M}(X) \to \mathcal{M}_N$ are measurable, whence the map $\mathcal{M}(X) \to \mathcal{M}_\infty$ is measurable, too. Since $\mathcal{M}(X)$ is standard (because X is standard), the latter map is an isomorphism of measurable spaces (see [Mack57, Theorem 3.2]) and the claim becomes obvious.

Remark 2.2.3. Theorem 2.2.1 immediately extends to the case of a projective system $\{V_i, \Lambda^j_i\}$, where all V_i's are discrete spaces (finite or countable) and the directed index set I is countably generated, that is, contains a sequence $i(1) < i(2) < \dots$ such that any $i \in I$ is majorated by indices $i(N)$ with N large enough. Indeed, it suffices to observe that the space $\varprojlim \mathcal{M}(V_{i(N)})$ does not depend on the choice of $\{i(N)\}$. Such a situation is examined in Section 3, where the index set I is the halfline $\mathbb{R}_{>0}$.

2.3. **Graded and branching graphs.**

Definition 2.3.1. By a *graded graph* we mean a graph Γ with countably many vertices partitioned into *levels* enumerated by numbers $1, 2, \dots$, and such that (below $|v|$ denotes the level of a vertex v):

- if two vertices v, v' are joined by an edge then $|v| - |v'| = \pm 1$;
- multiple edges between v and v' are allowed;
- each vertex v is joined with a least one vertex of level $|v| + 1$;
- if $|v| \geq 2$, then the set of vertices of level $|v| - 1$ joined with v is finite and nonempty.

This is a natural extension of the well-known notion of a *Bratteli diagram* [Br72]: the difference between the two notions is that a Bratteli diagram has finitely many vertices at each level, whereas our definition allows countable levels.

Sometimes it is convenient to slightly modify the above definition by adding to Γ a single vertex of level 0 joined by edges with all vertices of level 1.

Example 2.3.2. The simplest nontrivial example of a graded graph is the *Pascal graph* $\mathbb{P}$, also called the *Pascal triangle*. The vertices of $\mathbb{P}$ are points (n_1, n_2) of the lattice $\mathbb{Z}^2$ with nonnegative coordinates, the edges join points with one of the coordinates shifted by ± 1, and the level is defined as the sum $|(n_1, n_2)| = n_1 + n_2$. A number of other examples are can be found in Kerov's book [Ke03] and also in Gnedin [Gn97], Gnedin and Olshanski [GO06], Kingman [Ki78].

Definition 2.3.3 (Branching graphs)**.** Given a chain of finite or compact groups embedded to each other,

$$G(1) \subset G(2) \subset \cdots \subset G(N-1) \subset G(N) \subset \dots, \tag{2.3.1}$$

one constructs a graded graph $\Gamma = \Gamma(\{G(N)\})$, called the *branching graph* of the group chain (2.3.1), as follows. The vertices of level N are the labels of the equivalence classes of irreducible representations of $G(N)$. Choose a representation π_v for each vertex v. Two vertices u and v of levels N and $N-1$, respectively, are joined by m edges if π_u enters the the decomposition of $\pi_v \downarrow G(N-1)$ with multiplicity m, with the understanding that that there are no edges if $m = 0$.

Of particular importance for us are two branching graphs: the Young graph and the Gelfand-Tsetlin graph; they are obtained from the chains of symmetric groups and compact unitary groups, respectively. These graphs are discussed below, see Sections 3.3 and 4.1.

Definition 2.3.4. Given a graded graph Γ, the *dimension* of a vertex v, denoted by $\dim v$, is defined as the number of all (monotone) paths in Γ of length $|v|-1$ starting at some vertex of level 1 and ending at v (for more detail about paths, see Section 5.1 below). Further, for an arbitrary vertex u with $|u| < |v|$, the *relative dimension* $\dim(u, v)$ is the number of (monotone) paths of length $|v| - |u|$ joining u to v. In particular, if $|u| = |v| - 1$, then $\dim(u, v)$ is the number of edges between u and v.

For instance, in the case of the Pascal graph $\Gamma = \mathbb{P}$, if $v = (n_1, n_2)$ and $u = (m_1, m_2)$, $u \neq v$, then the dimensions are binomial coefficients:

$$\dim v = \frac{(n_1+n_2)!}{n_1! n_2!}, \qquad \dim(u,v) = \begin{cases} \dfrac{(n_1+n_2-m_1-m_2)!}{(n_1-m_1)!(n_2-m_2)!}, & n_1 \geq m_1 \text{ and } n_2 \geq m_2 \\ 0, & \text{otherwise.} \end{cases}$$

Note that if Γ is a branching graph, then $\dim v$ is the dimension of the corresponding representation π_v and $\dim(u,v)$ is the multiplicity of π_u in the decomposition of representation π_v restricted to the subgroup $G(|u|)\subset G(|v|)$.

Obviously, one has

$$\dim v = \sum_{u\colon |u|=|v|-1} \dim u \, \dim(u,v).$$

This leads to

Definition 2.3.5 (Projective chains associated to graded graphs)**.** Any graded graph Γ gives rise to a chain $\{V_N, \Lambda_N^{N+1}\}$, where V_N consists of the vertices of level N and

$$\Lambda_N^{N+1}(v,u) = \frac{\dim u \cdot \dim(u,v)}{\dim v}, \qquad v\in V_{N+1}, \quad u \in V_N.$$

The boundary V_∞ of this chain is also referred to as the *boundary of the graph* Γ and denoted as $\partial\Gamma$.

More generally, for $N<N'$ we set

$$\Lambda_N^{N'} := \Lambda_{N'-1}^{N'} \dots \Lambda_N^{N+1}.$$

Then

$$\Lambda_N^{N'}(v,u) = \frac{\dim u \cdot \dim(u,v)}{\dim v}, \qquad u\in V_N, \quad v\in V_{N'}. \tag{2.3.2}$$

If Γ is a branching graph coming from a group chain (2.3.1), then the boundary $\partial\Gamma$ has a representation-theoretic meaning. Namely, it is easy to show that the points of $\partial\Gamma$ parameterize the indecomposable normalized characters of the inductive limit group $G(\infty) := \varinjlim G(N)$ (these are the normalized traces of finite factor representations of $G(\infty)$). See Thoma [Tho64], Vershik and Kerov [VK90], Voiculescu [Vo76].

Example 2.3.6 (The boundary of the Pascal graph $\mathbb{P}$)**.** The boundary $\partial\mathbb{P}$ can be identified with the closed unit interval $[0,1]\subset\mathbb{R}$ (this fact is equivalent to de Finetti's theorem, see Section 5.2 below). For $\omega\in[0,1]$ and a vertex $v=(n_1,n_2)$ of level $N=n_1+n_2$ one has

$$\Lambda_N^\infty(\omega,v) = \frac{(n_1+n_2)!}{n_1!n_2!}\omega^{n_1}(1-\omega)^{n_2}.$$

Thus $\Lambda_N^\infty(\omega,\,\cdot\,)$ is the binomial distribution on $\{0,\dots,N\}$ with parameter ω. Note also that

$$\Lambda_{N-1}^N(v,v') = \begin{cases} \dfrac{n_1}{n_1+n_2}, & v'=(n_1-1,n_2) \\[2ex] \dfrac{n_2}{n_1+n_2}, & v'=(n_1,n_2-1). \end{cases}$$

3. The Young bouquet

3.1. **The binomial projective system $\mathbb{B}$.** Here we discuss a simple example of a projective system with continuous index set. This system will serve us as a building block in a more complex construction.

Definition 3.1.1. The *binomial projective system* $\mathbb{B}$ has the index set $I = \mathbb{R}_{>0}$ (strictly positive real numbers). All the spaces V_r are discrete and are copies of the set $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ of nonnegative integers. The links are defined by formula

$$ {}^{\mathbb{B}}\Lambda_r^{r'}(n,m) = \left(1 - \frac{r}{r'}\right)^{n-m} \left(\frac{r}{r'}\right)^m \frac{n!}{(n-m)!\,m!}, \qquad n, m \in \mathbb{Z}_+. \tag{3.1.1} $$

Note that the right-hand side vanishes unless $m \le n$. For n fixed the quantities ${}^{\mathbb{B}}\Lambda_r^{r'}(n,m)$ form the binomial distribution on $\{0, 1, \dots, n\}$ with parameter r/r', which explains the name of the system.

Clearly, ${}^{\mathbb{B}}\Lambda_r^{r'}$ is a stochastic matrix. Thus, to see that the definition is correct we have only check the compatibility condition

$$ {}^{\mathbb{B}}\Lambda_{r'}^{r''}\,{}^{\mathbb{B}}\Lambda_r^{r'} = {}^{\mathbb{B}}\Lambda_r^{r''}, \qquad r'' > r' > r. $$

Or, in more detail,

$$ \sum_n {}^{\mathbb{B}}\Lambda_{r'}^{r''}(l,n)\,{}^{\mathbb{B}}\Lambda_r^{r'}(n,m) = {}^{\mathbb{B}}\Lambda_r^{r''}(l,m). $$

But this is an easy exercise.

Remark 3.1.2. Setting $r = e^{-t}$ we may view the binomial projective system as a time-stationary transition function on $\mathbb{Z}_+$:

$$ p(s,n;t,m) = \left(1 - e^{s-t}\right)^{n-m} e^{(s-t)m} \frac{n!}{(n-m)!\,m!}, \qquad s < t, \quad n, m \in \mathbb{Z}_+. $$

By virtue of Remark 2.2.3 we may speak about the boundary $\partial\mathbb{B}$ of the binomial system. This boundary is described in the following theorem:

Theorem 3.1.3. *The boundary of the binomial projective system $\mathbb{B}$ is the space $\mathbb{R}_+ := \{x \in \mathbb{R} : x \ge 0\}$ with the links ${}^{\mathbb{B}}\Lambda_r^{\infty} : \mathbb{R}_+ \dashrightarrow \mathbb{Z}_+$ defined by the Poisson distributions*

$$ {}^{\mathbb{B}}\Lambda_r^{\infty}(x,m) = e^{-rx}\frac{(rx)^m}{m!}, \qquad x \in \mathbb{R}_+, \quad m \in \mathbb{Z}_+. $$

Before proceeding to the proof of the theorem we will prove two simple lemmas.

Lemma 3.1.4. *Let $r > 0$ and $k \in \mathbb{Z}_+$ be fixed. For any $r' > r$, the function*

$$ x \mapsto \left(1 - \frac{r}{r'}\right)^{r'x} x^k $$

belongs to the Banach space $C_0(\mathbb{R}_+)$ of continuous functions on $\mathbb{R}_+$ vanishing at infinity, with the supremum norm. In the limit as parameter r' goes to $+\infty$, this function converges in the metric of $C_0(\mathbb{R}_+)$ to the function

$$x \mapsto e^{-rx}x^k.$$

Proof. Clearly, the convergence holds uniformly on x in any bounded interval $[0, a]$. On the other hand, it is easy to estimate the tail of the pre-limit function for x near infinity: As $x \to +\infty$, the function tends to 0 uniformly on $r' \gg r$, because

$$\left(1 - \frac{r}{r'}\right)^{r'} = e^{-r}\left(1 + O(1/r')\right), \qquad r' \text{ large.}$$

This proves the lemma. □

Lemma 3.1.5. *For any $r > 0$, the map $M \mapsto M_r := M\, {}^{\mathbb{B}}\Lambda^\infty_r$ from $\mathcal{M}(\mathbb{R}_+)$ to $\mathcal{M}(\mathbb{Z}_+)$ is injective.*

Proof. Indeed, given $M \in \mathcal{M}(\mathbb{R}_+)$, its image M_r under ${}^{\mathbb{B}}\Lambda^\infty_r$ is given by

$$M_r(m) = \frac{1}{m!}\int_{\mathbb{R}_+} M(dx)e^{-rx}(rx)^m, \qquad m \in \mathbb{Z}_+.$$

The trivial estimate

$$e^{-rx}\frac{(rx)^m}{m!} \le 1, \qquad x \in \mathbb{R}_+,$$

entails $e^{-rx}x^m \le m!r^{-m}$. Since M is a probability measure, this implies that the mth moment of measure $M(dx)e^{-rx}$ does not exceed $m!r^{-m}$. It follows that the exponential generating function for the moments is analytic in the open disc of radius r, which guarantees that the corresponding moment problem is definite. Therefore, the initial measure $M(dx)e^{-rx}$ is recovered from its moments uniquely, so that M is determined by M_r uniquely. □

The following corollary will be used in [BO11b].

Corollary 3.1.6. *For any fixed $r > 0$, the linear span of the functions $e^{-rx}x^m$, $m = 0, 1, 2, \dots$, is dense in $C_0(\mathbb{R}_+)$.*

Proof. The dual space to $C_0(\mathbb{R}_+)$ is the space of finite signed measures on $\mathbb{R}_+$. Therefore, it suffices to prove that if M is a signed measure such that $e^{-rx}M$ is orthogonal to all polynomials, then $M = 0$. To do this write M as the difference of two finite positive measures M' and M''. The assumption on M means that measures $M'(dx)e^{-rx}$ and $M''(dx)e^{-rx}$ have the same moments. Then the argument in the proof of Lemma 3.1.5 shows that these measures are equal. Therefore $M' = M''$ and $M = 0$. □

Proof of Theorem 3.1.3. It is easy to check the relations

$${}^{\mathbb{B}}\Lambda^\infty_{r'}\, {}^{\mathbb{B}}\Lambda^{r'}_r = {}^{\mathbb{B}}\Lambda^\infty_r, \qquad r' > r. \tag{3.1.2}$$

They determine a Borel map

$$\mathcal{M}(\mathbb{R}_+) \to \mathcal{M}_\infty = \varprojlim \mathcal{M}(V_r), \quad M \mapsto \{M_r\}, \quad M_r := M\, {}^{\mathbb{B}}\Lambda^\infty_r.$$

By virtue of Remark 2.2.2 it suffices to prove that this map is a bijection.

By Lemma 3.1.5, it is injective; even more, $M \mapsto M_r$ is injective for any fixed $r > 0$.

We proceed to the proof that the map $M \mapsto \{M_r\}$ is surjective. Fix an element $\{M_r : r > 0\}$ of the projective limit space $\mathcal{M}_\infty$. Let us show that it comes from some probability measure $M \in \mathcal{M}(\mathbb{R}_+)$. The idea is that M arises as a scaling limit of the measures $M_{r'}$ as $r' \to +\infty$.

Write the compatibility relation $M_{r'}\, {}^{\mathbb{B}}\Lambda^{r'}_r = M_r$ in the form

$$\langle M_{r'}, {}^{\mathbb{B}}\Lambda^{r'}_r(\,\cdot\,, m)\rangle = M_r(m), \qquad \forall m \in \mathbb{Z}_+, \tag{3.1.3}$$

where ${}^{\mathbb{B}}\Lambda^{r'}_r(\,\cdot\,, m)$ is viewed as the function $l \mapsto {}^{\mathbb{B}}\Lambda^{r'}_r(l, m)$ on $\mathbb{Z}_+$. Fix r and m and let parameter r' go to $+\infty$. Embed $\mathbb{Z}_+$ into $\mathbb{R}_+$ via the map

$$\varphi_{r'} : l \mapsto x := (1/r')l$$

that depends on r'. Denote by $\widetilde{M}_{r'}$ the pushforward of $M_{r'}$ under $\varphi_{r'}$; this is a probability measure on $\mathbb{R}_+$. Next, rewrite the expression

$${}^{\mathbb{B}}\Lambda^{r'}_r(l, m) = \left(1 - \frac{r}{r'}\right)^{l-m} \left(\frac{r}{r'}\right)^m \frac{l!}{(l-m)!m!}$$

as a function of variable $x := \varphi_{r'}(l)$:

$${}^{\mathbb{B}}\Lambda^{r'}_r(l, m) = \frac{r^m}{m!} \cdot \left(1 - \frac{r}{r'}\right)^{r'x-m} x \left(x - \frac{1}{r'}\right) \dots \left(x - \frac{m-1}{r'}\right) \tag{3.1.4}$$

Here x ranges over the grid $\varphi_{r'}(\mathbb{Z}_+) = (1/r')\mathbb{Z}_+ \subset \mathbb{R}_+$, but the expression in the right-hand side of (3.1.4) makes sense for all $x \in \mathbb{R}_+$. By Lemma 3.1.4, this expression, as a function of variable $x \in \mathbb{R}_+$, belongs to $C_0(\mathbb{R}_+)$ and converges, as parameter r' goes to $+\infty$, to the function

$$x \mapsto e^{-rx}\frac{(rx)^m}{m!} = {}^{\mathbb{B}}\Lambda^\infty_r(x, m)$$

in the metric of $C_0(\mathbb{R}_+)$.

On the other hand, the set of sub-probability measures on $\mathbb{R}_+$ is compact in the vague topology (the topology of convergence on functions from $C_0(\mathbb{R}_+)$). Therefore, the family $(\widetilde{M}_{r'})$ has a nonempty set of partial vague limits as $r' \to +\infty$. Choose any such limit M. Then we may pass to a limit in (3.1.3), which gives us

$$\langle M, {}^{\mathbb{B}}\Lambda^\infty_r(\,\cdot\,, m)\rangle = M_r(m), \qquad \forall m \in \mathbb{Z}_+, \quad \forall r > 0,$$

which in turn implies that M is actually a probability measure. This concludes the proof of the theorem. □

The following example is used below in Section 3.5.

Example 3.1.7. Fix parameter $c>0$. For any $r>0$ define a probability measure ${}^{\mathbb{B}}M_r^{(c)}$ on $\mathbb{Z}_+$ by

$$ {}^{\mathbb{B}}M_r^{(c)}(m)=(1+r)^{-c}\frac{(c)_m}{m!}\left(\frac{r}{1+r}\right)^m, \qquad m\in\mathbb{Z}_+, $$

where $(c)_m:=c(c+1)\dots(c+m-1)$. This is a negative binomial distribution. A direct check shows that the family $\{{}^{\mathbb{B}}M_r^{(c)}\}_{r>0}$ is compatible with the links ${}^{\mathbb{B}}\Lambda_r^{r'}$, so that this family is an element of the projective limit space $\mathcal{M}_\infty$ associated with the system $\mathbb{B}$. The corresponding limit measure on the boundary $\partial\mathbb{B}=\mathbb{R}_+$ is the gamma distribution with parameter c; it has density $(\Gamma(c))^{-1}x^{c-1}e^{-x}$ with respect to the Lebesgue measure.

3.2. **Thoma's simplex, Thoma's cone, and symmetric functions.** The *Thoma simplex* is the subspace Ω of the infinite product space $\mathbb{R}_+^\infty\times\mathbb{R}_+^\infty$ formed by all couples (α,β), where $\alpha=(\alpha_i)$ and $\beta=(\beta_i)$ are two infinite sequences such that

$$ \alpha_1\ge\alpha_2\ge\dots\ge0, \qquad \beta_1\ge\beta_2\ge\dots\ge0 \tag{3.2.1} $$

and

$$ \sum_{i=1}^\infty\alpha_i+\sum_{i=1}^\infty\beta_i\le1. \tag{3.2.2} $$

We equip Ω with the product topology inherited from $\mathbb{R}_+^\infty\times\mathbb{R}_+^\infty$. Note that in this topology, Ω is a compact metrizable space.

The *Thoma cone* $\widetilde\Omega$ is the subspace of the infinite product space $\mathbb{R}_+^\infty\times\mathbb{R}_+^\infty\times\mathbb{R}_+$ formed by all triples $\omega=(\alpha,\beta,\delta)$, where $\alpha=(\alpha_i)$ and $\beta=(\beta_i)$ are two infinite sequences and δ is a nonnegative real number, such that the couple (α,β) satisfies (3.2.1) and the following modification of the inequality (3.2.2)

$$ \sum_{i=1}^\infty\alpha_i+\sum_{i=1}^\infty\beta_i\le\delta. $$

We set $|\omega|=\delta$.

Note that $\widetilde\Omega$ is a locally compact space in the product topology inherited from $\mathbb{R}_+^\infty\times\mathbb{R}_+^\infty\times\mathbb{R}_+$. The space $\widetilde\Omega$ is also metrizable and has countable base. Every subset of the form $\{\omega\in\widetilde\Omega : |\omega|\le\text{const}\}$ is compact. Therefore, a sequence of points ω_n goes to infinity in $\widetilde\Omega$ if and only if $|\omega_n|\to\infty$.

We will identify Ω with the subset of $\widetilde\Omega$ formed by triples $\omega=(\alpha,\beta,\delta)$ with $\delta=1$. The name "Thoma cone" given to $\widetilde\Omega$ is justified by the fact that $\widetilde\Omega$ may be viewed as the cone with the base Ω: the ray of the cone passing through a base point $(\alpha,\beta)\in\Omega$ consists of the triples $\omega=(r\alpha,r\beta,r)$, $r\ge0$.

More generally, for $\omega=(\alpha,\beta,\delta)\in\widetilde\Omega$ and $r>0$ we set $r\omega=(r\alpha,r\beta,r\delta)$.

Let Sym denote the graded algebra of symmetric functions over the base field $\mathbb{R}$ (see, e.g., [Ma95], [Sa01]). As an abstract algebra, Sym is isomorphic to the polynomial algebra $\mathbb{R}[p_1, p_2, \dots]$, where the generators p_k are the power sums in formal variables $x_1, x_2, \dots$,

$$p_k = \sum_{i=1}^{\infty} x_i^k, \qquad \deg p_k = k.$$

Here we employ the (conventional) realization of Sym as the subalgebra in $\mathbb{R}[[x_1, x_2, \dots]]$ formed by symmetric power series in countably many variables, of bounded total degree, see [Sa01].

However, this realization is not used in what follows. Instead, we embed Sym into the algebra of continuous functions on the Thoma cone by setting

$$p_k(\omega) = \begin{cases} \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k, & k = 2, 3, \dots \\ |\omega|, & k = 1, \end{cases}$$

where ω ranges over $\widetilde{\Omega}$.

In more detail, every element $F \in \mathrm{Sym}$ is uniquely written as a polynomial in $p_1, p_2, \dots$; then we define $F(\omega)$ as the same polynomial in numeric variables $p_1(\omega), p_2(\omega), \dots$. Note that the above expressions with $k \geq 2$ are the super power sums in variables (α_i) and $(-\beta_i)$, see [Ma95, §I.3, Ex. 23].

Another system of generators in Sym is provided by the *complete homogeneous symmetric functions* $h_1, h_2, \dots$ whose relation with p_k's can be conveniently written in the form

$$H(t) = \exp(P(t)),$$

where $H(t) = 1 + \sum_{k\geq 1} h_k t^k$ and $P(t) = \sum_{k \geq 1} p_k t^k / k$ are suitable generating functions.

Hence, under the embedding of Sym into $C(\widetilde{\Omega})$ described above, we have

$$1 + h_1(\omega)t + h_2(\omega)t^2 + \dots = e^{\gamma t} \prod_{i=1}^{\infty} \frac{1+\beta_i t}{1 - \alpha_i t}, \tag{3.2.3}$$

where $\omega = (\alpha, \beta, \delta)$, and $\gamma := \delta - \sum_{i \geq 1} (\alpha_i + \beta_i) \geq 0$.

A distinguished *linear* basis of Sym is formed by the *Schur functions*. We denote them by S_μ, where the index μ ranges over $\mathbb{Y}$. The Schur functions are homogeneous elements, $\deg S_\mu = |\mu|$, and they can be expressed through the complete homogeneous symmetric functions by the *Jacobi-Trudi formula*

$$S_\mu = \det \big[h_{\mu_i - i + j} \big]_{i,j=1}^{\ell},$$

where $\ell = \ell(\mu)$ is the number of nonzero parts of μ, and we assume that $h_0 = 1$, $h_{-1} = h_{-2} = \dots = 0$. Thus, the functions $S_\mu \in C(\widetilde{\Omega})$ are given by

$$S_\mu(\omega) = \det \big[h_{\mu_i - i + j}(\omega) \big]_{i,j=1}^{\ell},$$

where $h_k(\omega)$ are determined by (3.2.3).

3.3. **The Young graph $\mathbb{Y}$.** Consider the group chain (2.3.1), where the nth group is the symmetric group $S(n)$ formed by permutations of the set $\{1,\dots,n\}$. The embedding $S(n)\subset S(n+1)$ is defined by identifying $S(n)$ with the subgroup of $S(n+1)$ fixing the point $n+1$. The branching graph associated with this group chain is called the *Young graph* and denoted by $\mathbb{Y}$. The vertices of $\mathbb{Y}$ are the Young diagrams including the empty diagram $\varnothing$ at level 0. The level of a Young diagram λ equals the number of its boxes, and two diagrams are joined by a (simple) edge if they differ by a single box. This agrees with general Definition 2.3.3 by virtue of the *Young branching rule* for irreducible representations of symmetric groups (see, e.g., [Sa01, Theorem 2.8.3]).

Young diagrams are usually identified with *partitions* and written in the partition notation, $\lambda=(\lambda_1,\lambda_2,\dots)$. Here, by definition, λ_i equals the number of boxes in the ith row of λ. We set $|\lambda|=\sum\lambda_i$; this is the same as the number of boxes in the diagram λ.

The dimension function in the Young graph has a nice combinatorial meaning: $\dim\lambda$ coincides with the number of standard tableaux of shape λ. For this quantity there are several nice explicit formulas, e.g., the hook formula (see [Sa01, Theorem 3.10.2]).

Consider the projective chain defined by the Young graph:

$$\mathbb{Y}_0 \dashleftarrow \mathbb{Y}_1 \dashleftarrow \mathbb{Y}_2 \dashleftarrow \cdots$$

with the links ${}^{\mathbb{Y}}\Lambda^{m+1}_m:\mathbb{Y}_{m+1}\dashrightarrow\mathbb{Y}_m$ defined by (below $\mu\in\mathbb{Y}_m$ and $\nu\in\mathbb{Y}_{m+1}$)

$$ {}^{\mathbb{Y}}\Lambda^{m+1}_m(\nu,\mu)=\begin{cases}\dim\mu/\dim\nu, & \mu\subset\nu,\\ 0, & \text{otherwise}\end{cases}$$

(the notation $\mu\subset\nu$ means that μ is a subdiagram of ν; since $|\nu|=|\mu|+1$, this is equivalent to saying that μ is obtained from ν be removing a box).

More generally, for any $n>m$ the link ${}^{\mathbb{Y}}\Lambda^n_m:\mathbb{Y}_n\dashrightarrow\mathbb{Y}_m$ is defined as the composition

$$ {}^{\mathbb{Y}}\Lambda^n_m={}^{\mathbb{Y}}\Lambda^n_{n-1}\dots{}^{\mathbb{Y}}\Lambda^{m+1}_m$$

and has the form

$$ {}^{\mathbb{Y}}\Lambda^n_m(\nu,\mu)=\frac{\dim\mu\cdot\dim(\mu,\nu)}{\dim\nu},\qquad \nu\in\mathbb{Y}_n,\quad \mu\in\mathbb{Y}_m, \tag{3.3.1}$$

where $\dim(\mu,\nu)$ is defined as the number of standard tableaux of skew shape ν/μ if $\mu\subset\nu$, and 0 otherwise.

For a Young diagram λ, its *modified Frobenius coordinates* $(a_1,\dots,a_d;b_1,\dots,b_d)$ are defined as follows: d is the number of diagonal boxes in λ; a_i is equal to $\frac12$ plus the number of boxes in the ith row, on the right of the ith diagonal box; likewise, b_i is equal to $\frac12$ plus the number of boxes in the ith column, below the ith diagonal box. Note that

$$\sum_{i=1}^{d}(a_i + b_i) = |\lambda|.$$

We embed the set $\mathbb{Y}$ into $\widetilde{\Omega}$ through the map

$$\lambda \mapsto \omega_\lambda := ((a_1, \dots, a_d, 0, 0, \dots),\, (b_1, \dots, b_d, 0, 0, \dots),\, |\lambda|).$$

Obviously, $|\omega_\lambda| = |\lambda|$.

Recall that for any $\mu \in \mathbb{Y}$ we denote by S_μ the corresponding Schur symmetric function.

Lemma 3.3.1. *In the algebra* Sym*, there exist elements FS_μ indexed by diagrams $\mu \in \mathbb{Y}$ and characterized by the properties that*

$$FS_\mu = S_\mu + \textit{lower degree terms}$$

and

$$l^{\downarrow m} \frac{\dim(\mu, \lambda)}{\dim \nu} = FS_\mu(\omega_\lambda), \qquad \lambda \in \mathbb{Y}, \quad l = |\lambda|, \tag{3.3.2}$$

where

$$l^{\downarrow m} = l(l-1)\dots(l-m+1).$$

Proof. See [ORV03, Section 2]. The result is actually a reformulation of [OO97, Theorem 8.1]. The elements FS_μ are called the *Frobenius-Schur functions*. □

Corollary 3.3.2. *Fix m and $\mu \in \mathbb{Y}_m$. For large l and $\lambda \in \mathbb{Y}_l$*

$$\frac{\dim(\mu, \lambda)}{\dim \nu} = S_\mu(l^{-1}\omega_\lambda) + O(l^{-1}),$$

where the bound $O(l^{-1})$ for the rest term depends on m and μ but is uniform on λ.

Proof. Observe that for any homogeneous element $F \in \mathrm{Sym}$, one has

$$F(\omega) = O\left(|\omega|^{\deg F}\right), \tag{3.3.3}$$

where the bound depends only on F. Indeed, it suffices to check this for the generators p_k and then the assertion is immediate from the very definition of $p_k(\omega)$.

By Lemma 3.3.1, the expansion of FS_μ on homogeneous components has the form

$$FS_\mu = S_\mu + \sum_{k=0}^{m-1} F_k$$

where $F_0, \dots, F_{m-1}$ are some homogeneous elements with $\deg F_k = k$; their explicit form is inessential. Hence,

$$FS_\mu(\omega_\lambda) = S_\mu(\omega_\lambda) + \sum_{k=0}^{m-1} F_k(\omega_\lambda) = l^m \left(S_\mu(l^{-1}\omega_\lambda) + \sum_{k=0}^{m-1} \frac{1}{l^{m-k}} F_k(l^{-1}\omega_\lambda) \right).$$

Therefore,

$$\frac{\dim(\mu,\lambda)}{\dim\nu} = \frac{1}{l^{\downarrow m}} FS_\mu(\omega_\lambda) = \frac{l^m}{l^{\downarrow m}} \left(S_\mu(l^{-1}\omega_\lambda) + \sum_{k=0}^{m-1} \frac{1}{l^{m-k}} F_k(l^{-1}\omega_\lambda) \right).$$

Taking into account equality $l = |\omega_\lambda|$ and applying (3.3.3) we see that the asymptotics of this expression is indeed $S_\mu(l^{-1}\omega_\lambda) + O(l^{-1})$. □

Lemma 3.3.3. *For $m = 1, 2, \dots$ there exist links ${}^{\mathbb{Y}}\Lambda^\infty_m : \Omega \dashrightarrow \mathbb{Y}_m$ defined by*

$${}^{\mathbb{Y}}\Lambda^\infty_m(\omega,\mu) = \dim\mu \cdot S_\mu(\omega), \qquad \omega \in \Omega, \quad \mu \in \mathbb{Y}_m.$$

They satisfy the compatibility relation

$${}^{\mathbb{Y}}\Lambda^\infty_{m+1}\,{}^{\mathbb{Y}}\Lambda^{m+1}_m = {}^{\mathbb{Y}}\Lambda^\infty_m, \qquad m = 1, 2, \dots . \tag{3.3.4}$$

Proof. The key observation is that any point $\omega \in \Omega$ can be approximated by an appropriate sequence of points of the form $l^{-1}\omega_\lambda$, where $l \to \infty$ and $\lambda \in \mathbb{Y}_l$ varies together with l. Fix m and $\mu \in \mathbb{Y}_m$. Since the function $S_\mu(\omega)$ is continuous on Ω, the preceding lemma implies that ${}^{\mathbb{Y}}\Lambda^\infty_m(\omega,\mu) \ge 0$ for any $\omega \in \Omega$.

The same approximation argument shows that

$$\sum_{\mu\in\mathbb{Y}_m} {}^{\mathbb{Y}}\Lambda^\infty_m(\omega,\mu) = 1,$$

because the sum is finite and the similar relation holds for ${}^{\mathbb{Y}}\Lambda^l_m$.

Likewise, the limit transition as $l \to \infty$ in

$$\sum_{\nu\in\mathbb{Y}_{m+1}} {}^{\mathbb{Y}}\Lambda^l_{m+1}(\lambda,\nu)\,{}^{\mathbb{Y}}\Lambda^{m+1}_m(\nu,\mu) = {}^{\mathbb{Y}}\Lambda^l_m(\lambda,\mu),$$

proves the required compatibility relation. □

Theorem 3.3.4. *The links ${}^{\mathbb{Y}}\Lambda^\infty_m : \Omega \dashrightarrow \mathbb{Y}_m$ introduced above make it possible to identify the boundary $\partial\mathbb{Y}$ of the Young graph $\mathbb{Y}$ with the Thoma simplex Ω.*

Proof. We use the same argument as in the proof of Theorem 3.1.3. To some extent, the situation is even simpler because Ω is a compact space.

By virtue of Lemma 3.3.3 the links ${}^{\mathbb{Y}}\Lambda^\infty_m : \Omega \dashrightarrow \mathbb{Y}_m$ define a map

$$\mathcal{M}(\Omega) \to \mathcal{M}_\infty := \varprojlim \mathcal{M}(\mathbb{Y}_m),$$

and we have to check that is bijective.

The functions $F(\omega)$ on Ω coming from elements $F \in \operatorname{Sym}$ form a real algebra that contains 1 and separates points. By Stone-Weierstrass' theorem, this algebra is dense in $C(\Omega)$. Hence, every measure M on Ω is uniquely determined by its pairings $\langle M, F\rangle$. Since the Schur functions S_μ form a basis in Sym, M is uniquely determined by its pairings with the functions ${}^{\mathbb{Y}}\Lambda^\infty_m(\,\cdot\,,\mu)$. This proves injectivity.

To prove surjectivity, fix an element $(M_m) \in \mathcal{M}_\infty$. For each l consider the embedding

$$\varphi_l : \mathbb{Y}_l \to \Omega, \qquad \varphi_l(\lambda) := l^{-1}\omega_\lambda, \quad \lambda \in \mathbb{Y}_l.$$

It takes M_l to a probability measure $\widetilde{M_l}$ on Ω.

Next, by virtue of Lemma 3.3.1, the compatibility relation $M_l {}^{\mathbb{Y}}\Lambda^l_m = M_m$ can be rewritten as

$$\dim\mu \, \langle \widetilde{M_l}, S_\mu \rangle + O(l^{-1}) = M_m(\mu).$$

Let M stand for any partial weak limit of the sequence $(\widetilde{M_l})$ as $l \to \infty$. Then the above relation implies

$$\dim\mu \, \langle M, S_\mu \rangle = M(\mu)$$

which is equivalent to $M \, {}^{\mathbb{Y}}\Lambda^\infty_m = M_m$. This concludes the proof. □

This result is closely related to Thoma's theorem on the characters of the infinite symmetric group $S(\infty)$. The above proof follows the approach of the paper Kerov, Okounkov, and Olshanski [KOO98], which in turn develops the ideas of Vershik and Kerov [VK81], [VK90]; see also Kerov' monograph [Ke03].

3.4. **The Young bouquet $\mathbb{YB}$.** The set $\mathbb{Y}$ of Young diagrams is a poset with respect to the partial order defined by inclusion of diagrams. That is, a diagram μ is smaller than a diagram ν if μ is contained in ν. Equivalently, in the partition notation, $\mu_i \le \nu_i$ for all i, where at least one inequality is strict. As a poset, $\mathbb{Y}$ is a lattice, and for this reason it is often called the *Young lattice*. There is an obvious relation between the order on $\mathbb{Y}$ and the graph structure.

We are going to define a (partially) continuous analog of the Young lattice $\mathbb{Y}$.

Definition 3.4.1. The *Young bouquet* is the poset $(\mathbb{YB}, <)$ defined as follows.

1) The set $\mathbb{YB}$ is the wedge sum of countably many rays indexed by all Young diagrams $\mu \in \mathbb{Y}$ (whence the term "bouquet", which is a synonym for wedge sum). More precisely, $\mathbb{YB}$ is obtained from the direct product space $\mathbb{Y} \times \mathbb{R}_+$ (where $\mathbb{R}_+ = [0, +\infty)$) by gluing together all the points $(\mu, 0)$ into a single point, denoted as $(\varnothing, 0)$.

2) The partial order in $\mathbb{YB}$ comes from the conventional partial order in the Young lattice $\mathbb{Y}$ and the conventional order in $\mathbb{R}_+$. That is, an element $(\mu, r) \in \mathbb{YB}$ is declared to be smaller than another element (ν, r') if $r < r'$ and $\mu \subseteq \nu$; then we write $(\mu, r) < (\nu, r')$ or $(\nu, r') > (\mu, r)$.

For an element $(\mu, r) \in \mathbb{YB}$ we write $|(\mu, r)| := r$ and call this number the *level* of (μ, r). Let $\mathbb{YB}_r$ denote the subset of elements of level r. The stratification $\mathbb{YB} = \sqcup_{r \ge 0} \mathbb{YB}_r$ is viewed as a continuous analog of grading. Unless otherwise stated, below we assume $r > 0$ and identify each level set $\mathbb{YB}_r$ with $\mathbb{Y}$.

Definition 3.4.2. With every couple $r'>r>0$ we associate a matrix ${}^{\mathbb{YB}}\Lambda^{r'}_r$ of format $\mathbb{Y}\times\mathbb{Y}$, with the entries

$$\begin{aligned} {}^{\mathbb{YB}}\Lambda^{r'}_r(\nu,\mu) &= {}^{\mathbb{B}}\Lambda^{r'}_r(n,m)\,{}^{\mathbb{Y}}\Lambda^{n}_m(\nu,\mu) && (3.4.1)\\ &= \Big(1-\frac{r}{r'}\Big)^{n-m}\Big(\frac{r}{r'}\Big)^m\frac{n!}{(n-m)!\,m!}\,\frac{\dim\mu\,\dim(\mu,\nu)}{\dim\nu}, && (3.4.2)\end{aligned}$$

where $n:=|\nu|$ and $m:=|\mu|$ and the matrices right-hand side of (3.4.1) are defined in (3.1.1) and (3.3.1).

Due to the factor $(n-m)!$ in the denominator and the factor $\dim(\mu,\nu)$ in the numerator ${}^{\mathbb{YB}}\Lambda^{r'}_r(\nu,\mu)$ vanishes unless $(\mu,r)<(\nu,r')$.

From (3.4.1) one sees that ${}^{\mathbb{YB}}\Lambda^{r'}_r$ is a stochastic matrix, because it is composed from two auxiliary stochastic matrices. In other words, given ν, the random diagram μ can be drown in two steps: First, we choose its size m according to the binomial distribution ${}^{\mathbb{B}}\Lambda^{r'}_r(n,\,\cdot\,)$ and then μ is specified inside $\mathbb{Y}_m$ according to the probabilities from the second stochastic matrix. Thus, ${}^{\mathbb{YB}}\Lambda^{r'}_r$ is a link $\mathbb{Y}\dashrightarrow\mathbb{Y}$.

The new links satisfy the compatibility relation

$${}^{\mathbb{YB}}\Lambda^{r''}_{r'}\,{}^{\mathbb{YB}}\Lambda^{r'}_r={}^{\mathbb{YB}}\Lambda^{r''}_r,\qquad r''>r'>r,$$

because the auxiliary links satisfy analogous compatibility relations. Thus, we get a projective system formed by the levels $\mathbb{YB}_r=\mathbb{Y}$, $r>0$, of the Young bouquet with the links ${}^{\mathbb{YB}}\Lambda^{r'}_r$. By definition, the *boundary of the Young bouquet* is the boundary of this projective system. We aim to show that this boundary is the Thoma cone $\widetilde{\Omega}$.

Let $(0,0,0)$ denote the origin of the Thoma cone; this is the only point $\omega\in\widetilde{\Omega}$ with $|\omega|=0$. To every $\omega\in\widetilde{\Omega}\setminus\{(0,0,0)\}$ we assign the point $\widehat{\omega}=|\omega|^{-1}\omega$ in the Thoma simplex Ω. The map $\omega\mapsto(|\omega|,\widehat{\omega})$ is a bijection between $\Omega\setminus\{(0,0,0)\}$ and the "cylinder" $\mathbb{R}_{>0}\times\Omega$.

Definition 3.4.3. Let $r>0$. For $\omega=(x,\widehat{\omega})\in\widetilde{\Omega}\setminus\{(0,0,0)\}$ and $\mu\in\mathbb{Y}_m$ we set

$$\begin{aligned}{}^{\mathbb{YB}}\Lambda^{\infty}_r(\omega,\mu)&={}^{\mathbb{B}}\Lambda^{\infty}_r(x,m)\,{}^{\mathbb{Y}}\Lambda^{\infty}_m(\widehat{\omega},\mu)\\ &=e^{-rx}\frac{(rx)^m}{m!}\dim\mu\cdot S_\mu(\widehat{\omega})\\ &=e^{-r|\omega|}\frac{r^m}{m!}\dim\mu\cdot S_\mu(\omega).\end{aligned}$$

We extend this definition to the origin $\omega=(0,0,0)$ by continuity, which gives

$${}^{\mathbb{YB}}\Lambda^{\infty}_r((0,0,0),\mu)=\begin{cases}1, & \mu=\varnothing,\\ 0, & \mu\ne\varnothing.\end{cases}$$

Lemma 3.4.4. *For every $r>0$, ${}^{\mathbb{YB}}\Lambda^{\infty}_r$ is a link $\widetilde{\Omega}\dashrightarrow\mathbb{Y}$.*

(ii) *The links ${}^{\mathbb{YB}}\Lambda^{\infty}_r$ satisfy the compatibility relation*

$${}^{\mathbb{YB}}\Lambda^{\infty}_{r'}\,{}^{\mathbb{YB}}\Lambda^{r'}_r={}^{\mathbb{YB}}\Lambda^{\infty}_r,\qquad r'>r>0.$$

Proof. (i) We have to check that ${}^{\mathbb{YB}}\Lambda^{\infty}_{r}(\omega,\,\cdot\,)$ is a probability measure on $\mathbb{Y}$ for every $\omega\in\widetilde\Omega$. Consider separately the cases $\omega\ne(0,0,0)$ and $\omega=(0,0,0)$. In the first case, the claim follows from the factorization of ${}^{\mathbb{YB}}\Lambda^{\infty}_{r}(\omega,\mu)$; this quantity is represented as the probability to select μ through a 2-step procedure directed by two probability distributions. In the second case the claim is obvious, for ${}^{\mathbb{YB}}\Lambda^{\infty}_{r}((0,0,0),\,\cdot\,)$ is the delta measure at $\mu=\varnothing$.

(ii) We have to check that

$$({}^{\mathbb{YB}}\Lambda^{\infty}_{r'}\,{}^{\mathbb{YB}}\Lambda^{r'}_{r})(\omega,\,\cdot\,)={}^{\mathbb{YB}}\Lambda^{\infty}_{r}(\omega,\,\cdot\,),\qquad r'>r>0,$$

for any $\omega\in\widetilde\Omega$. Consider again the same two case: $|\omega|=0$ and $|\omega|>0$. In the first case, both sides are delta measures at $\varnothing\in\mathbb{Y}$. In the second case we use the factorization property of the links ${}^{\mathbb{YB}}\Lambda^{\infty}_{r}$ and ${}^{\mathbb{YB}}\Lambda^{r'}_{r}$ and the compatibility relations for the auxiliary links, see (3.1.2) and (3.3.4). □

Lemma 3.4.5 (cf. Lemma 3.1.5)**.** *Fix $r>0$ and assume that M' and M'' are two finite Borel measures on $\widetilde\Omega$ such that*

$$\int_{\widetilde\Omega}M'(d\omega)e^{-r|\omega|}F(\omega)=\int_{\widetilde\Omega}M''(d\omega)e^{-r|\omega|}F(\omega)\tag{3.4.3}$$

for all $F\in\operatorname{Sym}$. Then $M'=M''$.

Proof. Step 1. Let $\bar M'$ and $\bar M''$ stand for the pushforwards of M' and M'' under the projection $\widetilde\Omega\to\mathbb{R}_+$ defined as $\omega\mapsto|\omega|$. We claim that $\bar M'=\bar M''$.

Indeed, recall that $p_1(\omega)=|\omega|$. Taking $F=p_1^k$ we get

$$\int_{\mathbb{R}_+}\bar M'(dx)e^{-rx}x^k=\int_{\mathbb{R}_+}\bar M''(dx)e^{-rx}x^k,\qquad\forall k\in\mathbb{Z}_+.$$

Now the argument of Lemma 3.1.5 shows that $\bar M'=\bar M''$.

Step 2. Without loss of generality we may assume that M' and M'' have no atom at the origin of the Thoma cone. Indeed, if M' has an atom at the origin, then $\bar M'$ has an atom of the same mass at the point $0\in\mathbb{R}_+$. Since $\bar M'=\bar M''$, the measure M'' has the same atom as M', so that we may simply remove it.

Step 3. The previous step allows us to transfer the measures M' and M'' from the cone $\widetilde\Omega$ to the cylinder $\mathbb{R}_{>0}\times\Omega$ with coordinates $(x,\widehat\omega)$, where $x=|\omega|\in\mathbb{R}_{>0}$ and $\widehat\omega=|\omega|^{-1}\omega\in\Omega$. Step 1 tells us that the projections of the both measures on coordinate x are one and the same measure $\bar M:=\bar M'=\bar M''$ on $\mathbb{R}_{>0}$. Let us disintegrate $\bar M'$ and $\bar M''$ with respect to $\bar M$ (see e.g. Theorem 8.1 in [Pa67] on the existence of the conditional distributions). Then we get two families $\{Q'_x\}$ and $\{Q''_x\}$ of probability measures on Ω, indexed by points $x\in\mathbb{R}_{>0}$. These families are defined uniquely, modulo $\bar M$-null sets.

We claim that

$$\int_{\Omega}Q'_x(d\widehat\omega)G(\widehat\omega)=\int_{\Omega}Q''_x(d\widehat\omega)G(\widehat\omega)\tag{3.4.4}$$

for all $G \in \mathrm{Sym}$ and all x outside an appropriate $\bar{M}$-null set that does not depend on G.

Indeed, since Sym possesses a countable homogeneous basis (for instance, the basis of Schur functions) it suffices to prove that (3.4.4) holds for any given homogeneous function G and for all x outside a $\bar{M}$-null set possibly dependent on G.

Then substitute $F = p_1^k G$ into the initial equality (3.4.3) and denote by m the degree of G. We get the equality

$$\int_{\mathbb{R}_+} \bar{M}(dx) e^{-rx} x^{m+k} \left\{ \int_\Omega Q'_x(d\widehat{\omega}) G(\widehat{\omega}) \right\} = \int_{\mathbb{R}_+} \bar{M}(dx) e^{-rx} x^{m+k} \left\{ \int_\Omega Q''_x(d\widehat{\omega}) G(\widehat{\omega}) \right\},$$

which holds for any $k \in \mathbb{Z}_+$. This means equality of moments for two measures, each of which is is the product of $\bar{M}(dx)e^{-rx}x^m$ and a bounded function. The same argument as in step 1 shows that these two measures are the same, which proves (3.4.4).

Step 4. The functions from Sym are dense in $C(\Omega)$ because they separate points and the space Ω is compact. Together with (3.4.4) this implies that $Q'_x = Q''_x$ almost everywhere with respect to $\bar{M}$. We conclude that $M' = M''$. □

Recall that $\widetilde{\Omega}$ is a locally compact space. Let $C_0(\widetilde{\Omega})$ stand for the real Banach space of continuous functions on $\widetilde{\Omega}$ vanishing at infinity, with the supremum norm.

Corollary 3.4.6 (cf. Corollary 3.1.6)**.** *For any fixed $r > 0$, the set of functions of the form $e^{-rx}F$ with F ranging over* Sym *is dense in $C_0(\widetilde{\Omega})$.*

Proof. We argue as in Corollary 3.1.6, with appeal to 3.4.5 instead of Lemma 3.1.5. □

Theorem 3.4.7. *The Thoma cone $\widetilde{\Omega}$ together with the collection of links ${}^{\mathbb{YB}}\Lambda^\infty_r : \widetilde{\Omega} \dashrightarrow \mathbb{Y}$, $r > 0$, is the boundary of the Young bouquet.*

Proof. We follow the scheme of the proof of Theorem 3.1.3. By virtue of Lemma 3.4.4, the links ${}^{\mathbb{YB}}\Lambda^\infty_r$ define a Borel map $M \mapsto (M_r)_{r>0}$ from $\mathcal{M}(\widetilde{\Omega})$ to the projective limit space constructed from the system $\{V_r = \mathbb{Y}, {}^{\mathbb{YB}}\Lambda^{r'}_r\}$. According to Remark 2.2.2, it suffices to prove that this map is a bijection. We divide this claim into two parts, injectivity and surjectivity.

The injectivity claim follows from Lemma 3.4.5 or Corollary 3.4.6, which say that even the map $M \mapsto M_r$ with any fixed $r > 0$ is injective.

We proceed to the proof of the surjectivity claim. Write the compatibility relation $M_{r'} {}^{\mathbb{YB}}\Lambda^{r'}_r = M_r$ in the form

$$\langle M_{r'}, {}^{\mathbb{YB}}\Lambda^{r'}_r(\,\cdot\,, \mu)\rangle = M_r(\mu), \qquad \forall \mu \in \mathbb{Y},$$

where ${}^{\mathbb{YB}}\Lambda^{r'}_r(\,\cdot\,, \mu)$ is viewed as the function $\lambda \mapsto {}^{\mathbb{YB}}\Lambda^{r'}_r(\lambda, \mu)$ on $\mathbb{Y}$. Fix r and μ and let parameter r' go to $+\infty$. Embed $\mathbb{Y}$ into $\widetilde{\Omega}$ via the map

$$\varphi_{r'} : \lambda \mapsto (1/r')\omega_\lambda$$

that depends on r'. Denote by $\widetilde{M}_{r'}$ the pushforward of $M_{r'}$ under $\varphi_{r'}$; this is a probability measure on $\widetilde{\Omega}$. Next, regard ${}^{\mathbb{YB}}\Lambda^{r'}_r(\lambda,\mu)$ as a function of variable $\omega := \varphi_{r'}(\lambda)$. Of course, this function is initially defined only on the discrete subset $\varphi_{r'}(\mathbb{Y}) \subset \widetilde{\Omega}$, but we will see that it admits a natural extension to a continuous function on the whole $\widetilde{\Omega}$ depending also on parameter r'. The key fact proved below is that the latter function lies in the Banach space $C_0(\widetilde{\Omega})$ and converges, as $r' \to \infty$, to the function $\omega \mapsto {}^{\mathbb{YB}}\Lambda^{\infty}(\omega,\mu)$ in the metric of that space. Once this is established, the desired result follows. Indeed, take as M an arbitrary partial limit of $\{\widetilde{M}_{r'}, r' \to +\infty\}$ with respect to the vague topology; this is a sub-probability measure on $\widetilde{\Omega}$. Then we may pass to the limit in the above equation, which gives us

$$\langle M, {}^{\mathbb{YB}}\Lambda^{\infty}_r(\,\cdot\,,\mu)\rangle = M_r(\mu), \qquad \forall \mu \in \mathbb{Y}, \quad \forall r > 0,$$

which in turn implies that M is a probability measure. This concludes the proof modulo the claim concerning the function ${}^{\mathbb{YB}}\Lambda^{r'}_r(\,\cdot\,,\mu)$ and its convergence to ${}^{\mathbb{YB}}\Lambda^{\infty}_r(\,\cdot\,,\mu)$ in the metric of $C_0(\widetilde{\Omega})$.

Now let us prove that claim. Write again the explicit expression for ${}^{\mathbb{YB}}\Lambda^{r'}_r(\lambda,\mu)$:

$${}^{\mathbb{YB}}\Lambda^{r'}_r(\lambda,\mu) = \left(1-\frac{r}{r'}\right)^{l-m}\left(\frac{r}{r'}\right)^{m}\frac{l!}{(l-m)!\,m!}\,\frac{\dim\mu\,\dim(\mu,\lambda)}{\dim\lambda},$$

where, as usual, $l = |\lambda|$ and $m = |\mu|$. By virtue of (3.3.2),

$$\frac{l!}{(l-m)!}\,\frac{\dim\mu\,\dim(\mu,\lambda)}{\dim\lambda} = FS_\mu(\omega_\lambda) = S_\mu(\omega_\lambda) + \sum_{k=0}^{m-1} F_k(\omega_\lambda),$$

where $F_0,\dots,F_{m-1}$ are the same homogeneous elements of Sym with $\deg F_k = k$ as in the proof of Corollary 3.3.2.

Setting $\omega := (1/r')\omega_\lambda$ (and keeping in mind that ω depends both on λ and r') we may rewrite the above equality as

$$\frac{1}{(r')^m}\frac{l!}{(l-m)!}\,\frac{\dim\mu\,\dim(\mu,\lambda)}{\dim\lambda} = S_\mu(\omega) + \sum_{k=0}^{m-1}\frac{1}{(r')^{m-k}}F_k(\omega).$$

Now, returning to ${}^{\mathbb{YB}}\Lambda^{r'}_r(\lambda,\mu)$, we may write it as

$${}^{\mathbb{YB}}\Lambda^{r'}_r(\lambda,\mu) = \frac{r^m\dim\mu}{m!}\cdot\left(1-\frac{r}{r'}\right)^{l-m}\left(S_\mu(\omega) + \sum_{k=0}^{m-1}\frac{1}{(r')^{m-k}}F_k(\omega)\right).$$

Since $l = |\omega_\lambda| = r'|\omega|$, we finally get a nice formula

$${}^{\mathbb{YB}}\Lambda^{r'}_r(\lambda,\mu) = \frac{r^m\dim\mu}{m!}\cdot\left(1-\frac{r}{r'}\right)^{r'|\omega|-m}\left(S_\mu(\omega) + \sum_{k=0}^{m-1}\frac{1}{(r')^{m-k}}F_k(\omega)\right).$$

In this formula ω is assumed to be related to λ via relation $\omega = (1/r')\omega_\lambda$ but the right-hand side is well defined as a function on the whole space $\widetilde{\Omega}$. We have to prove that this function is continuous, vanishes at infinity, and in the limit as $r' \to \infty$ converges to

$$ {}^{\mathbb{YB}}\Lambda_r^\infty(\omega, \mu) = \frac{r^m \dim \mu}{m!} \cdot e^{-r|\omega|} S_\mu(\omega) $$

in the metric of the Banach space $C_0(\widetilde{\Omega})$. But this follows from Lemma 3.1.4 by virtue of the bound (3.3.3). □

3.5. **Z-Measures on $\mathbb{YB}$.** Introduce some notation. For $z \in \mathbb{C}$ and $\mu \in \mathbb{Y}$, set

$$ (z)_\mu = \prod_{(i,j)\in\mu} (z + j - i), $$

where the product is taken over the boxes (i, j) of diagram μ (here i are j stand for the row and column numbers of the box). This is a generalization of the Pochhammer symbol: In the particular case when $\mu = (m)$ is a one-row diagram, we get $(z)_\mu = (z)_m = z(z + 1) \dots (z + m - 1)$.

Definition 3.5.1. Let us say that a couple $(z, z') \in \mathbb{C}^2$ of complex parameters is *admissible* if $z \neq 0$, $z' \neq 0$, and $(z)_\mu(z')_\mu \geq 0$ for all $\mu \in \mathbb{Y}$.

Obviously, the set of admissible values is invariant under symmetries $(z, z') \to (z', z)$ and $(z, z) \to (-z, -z')$; the latter holds because $(-z)_\mu = (-1)^{|\mu|}(z)_{\mu'}$.

It is not difficult to get an explicit description of the admissible range of the parameters (z, z'), see [BO06, Proposition 1.2]. One can represent it as the union of the following three subsets or *series*:

- The *principal series* is $\{(z, z') \colon z' = \bar{z} \in \mathbb{C} \setminus \mathbb{Z}\}$.
- The *complementary series* is $\cup_{k\in\mathbb{Z}}\{(z, z') \colon k < z, z' < k + 1\}$.
- The *degenerate series* comprises the set

$$ \{(z, z') = (k, k + b - 1) \colon k = 1, 2, \dots;\ b > 0\} $$

  together with its images under the symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The reason why the values $z = 0$ and $z' = 0$ are excluded is that then $(z)_\mu(z')_\mu$ vanishes for all $\mu \neq \varnothing$, which is a trivial case. Note that $zz' > 0$ for any admissible couple (z, z').

Definition 3.5.2. The *z-measure* with admissible parameters (z, z') and additional parameter $r > 0$ is the measure on $\mathbb{Y}$ given by

$$ {}^{\mathbb{YB}}M_r^{(z,z')}(\mu) = (1 + r)^{-zz'}(z)_\mu(z')_\mu \left(\frac{r}{1 + r}\right)^{|\mu|} \left(\frac{\dim \mu}{|\mu|!}\right)^2, \qquad \mu \in \mathbb{Y}. $$

Proposition 3.5.3. *The z-measures are probability measures, and they are compatible with the links* ${}^{\mathbb{YB}}\Lambda_r^{r'} : \mathbb{Y} \dashrightarrow \mathbb{Y}$:

$$ {}^{\mathbb{YB}}M_{r'}^{(z,z')}\, {}^{\mathbb{YB}}\Lambda_r^{r'} = {}^{\mathbb{YB}}M_r^{(z,z')}, \qquad r' > r > 0. $$

Proof. Set $c = zz'$ and $m = |\mu|$. The measure ${}^{\mathbb{YB}}M_r^{(z,z')}$ can be written in the form

$$ {}^{\mathbb{YB}}M_r^{(z,z')}(\mu) = {}^{\mathbb{B}}M_r^{(c)}(m)\, {}^{\mathbb{Y}}M_m^{(z,z')}(\mu), $$

where the first factor in the right-hand side has been defined in Example 3.1.7 and the second factor is defined by

$$ {}^{\mathbb{Y}}M_m^{(z,z')}(\mu) = \frac{(z)_\mu (z')_\mu}{(c)_m} \frac{(\dim \mu)^2}{m!}. $$

It is known that for each $m \in \mathbb{Z}_+$, ${}^{\mathbb{Y}}M_m^{(z,z')}$ is a probability measure on $\mathbb{Y}_m$ and the family $\{{}^{\mathbb{Y}}M_m^{(z,z')}\}_{m\in\mathbb{Z}_+}$ is compatible with the links ${}^{\mathbb{Y}}\Lambda_m^n : \mathbb{Y}_n \dashrightarrow Y_m$, see [Ols03a], [BO00b]. Together with Example 3.1.7 this implies the proposition. □

The z-measures play a key role in harmonic analysis on the infinite symmetric group, see the survey [Ols03b] and references therein.

4. Connection with the Gelfand–Tsetlin graph

4.1. **The Gelfand–Tsetlin graph $\mathbb{GT}$.** For $N = 1, 2, \dots$ define a *signature* of length N as an N-tuple of nonincreasing integers $\mu = (\mu_1 \ge \dots \ge \mu_N) \in \mathbb{Z}^N$, and denote by $\mathbb{GT}_N$ the set of all such signatures. Elements of $\mathbb{GT}_N$ parameterize irreducible representations of the compact unitary group $U(N)$ ("signature" is another name for "highest weight" in the special case of the group $U(N)$, see, e.g., [Wey39], [Zhe70].) We will also use for elements $\mu \in \mathbb{GT}_N$ a more detailed notation $[\mu, N]$.

Write $[\mu, N] \prec [\nu, N+1]$ if $\nu_j \ge \mu_j \ge \nu_{j+1}$ for all meaningful values of indices. These inequalities are well-known to be equivalent to the condition that the restriction of the ν-representation of $U(N+1)$ to $U(N)$ contains a μ-component (then the multiplicity of this component equals 1).

Definition 4.1.1. Set $\mathbb{GT} = \bigsqcup_{N\ge 1} \mathbb{GT}_N$, and equip $\mathbb{GT}$ with edges that join any two signatures μ and ν such that $\mu \prec \nu$ or $\nu \prec \mu$. This turns $\mathbb{GT}$ into a graph that is called the *Gelfand-Tsetlin graph*. It will be denoted by the same symbol $\mathbb{GT}$.

By the very definition, $\mathbb{GT}$ is a branching graph with countable levels. It arises from the chain $U(1) \subset U(2) \subset \cdots$ of compact unitary groups just as the Young graph arises from the chain of symmetric groups $S(1) \subset S(2) \subset \cdots$. As in the Young graph $\mathbb{Y}$, all edges in $\mathbb{GT}$ are simple; this is because the restriction of an irreducible representation of $U(N+1)$ to the subgroup $U(N)$ is always multiplicity free.

The dimension function in $\mathbb{GT}$ will be denoted by the symbol Dim. We have

$$ \operatorname{Dim}[\mu, N] = \prod_{1\le i<j\le N} \frac{\mu_i - \mu_j - i + j}{j - i}. $$

This is classical Weyl's formula for the dimension of irreducible representations of the unitary groups.

More generally, for $N' > N$ we write $\operatorname{Dim}([\mu, N], [\nu, N'])$ for the relative dimension. According to the general definition (2.3.2), the links between various levels of $\mathbb{GT}$ have the form

$$ {}^{\mathbb{GT}}\Lambda^{N'}_N([\nu, N'], [\mu, N]) = \frac{\operatorname{Dim}[\mu, N] \operatorname{Dim}([\mu, N], [\nu, N'])}{\operatorname{Dim}[\nu, N']}. \tag{4.1.1} $$

4.2. **The boundary of the Gelfand-Tsetlin graph.** Consider the space $\widetilde{\Omega} \times \widetilde{\Omega}$, the direct product of two copies of the Thoma cone. Its elements are pairs (ω^+, ω^-), where $\omega^\pm = (\alpha^\pm, \beta^\pm, \delta^\pm) \in \widetilde{\Omega}$. It is convenient to introduce auxiliary parameters

$$ \gamma^\pm := \delta^\pm - \sum_{i \ge 1} (\alpha_i^\pm + \beta_i^\pm) \ge 0. $$

To any pair (ω^+, ω^-) we assign a function on the unit circle $\{u \in \mathbb{C} : |u| = 1\}$ by

$$ \Phi(u; \omega^+, \omega^-) = e^{\gamma^+(u-1) + \gamma^-(u^{-1}-1)} \prod_{i \ge 1} \frac{1 + \beta_i^+(u-1)}{1 - \alpha_i^+(u-1)} \frac{1 + \beta_i^-(u^{-1}-1)}{1 - \alpha_i^-(u^{-1}-1)}. $$

This function is analytic in an open neighborhood of the unit circle, where it can be written as a Laurent series

$$ \Phi(u; \omega^+, \omega^-) = \sum_{n=-\infty}^{\infty} \varphi_n(\omega^+, \omega^-) u^n. $$

For $\mu \in \mathbb{GT}_N$ set

$$ {}^{\mathbb{GT}}\Lambda^\infty_N(\omega^+, \omega^-; \mu) = \operatorname{Dim}[\mu, N] \cdot \det\left[\varphi_{\mu_i - i + j}(\omega^+, \omega^-)\right]_{i,j=1}^N. \tag{4.2.1} $$

Theorem 4.2.1. *The boundary $\partial\mathbb{GT}$ of the Gelfand-Tsetlin graph can be identified with the subset in $\widetilde{\Omega} \times \widetilde{\Omega}$ determined by the condition $\beta_1^+ + \beta_1^- \le 1$, with links ${}^{\mathbb{GT}}\Lambda^\infty_N : \partial\mathbb{GT} \to \mathbb{GT}_N$ given by* (4.2.1).

For the history and various proofs of this deep result (which we propose to call the *Edrei–Voiculescu theorem*), see Borodin and Olshanski [BO11a], Boyer [Boy83], Edrei [Ed53], Okounkov and Olshanski [OO98], Vershik and Kerov [VK82].

Note that $\partial\mathbb{GT}$ is a closed subset in $\widetilde{\Omega} \times \widetilde{\Omega}$, thus it is a locally compact space.

If one replaces the condition $\beta_1^+ + \beta_1^- \le 1$ by the weaker one of $\beta_1^\pm \le 1$ then (4.2.1) would still define boundary points, but each boundary point would correspond to multiple pairs (ω^+, ω^-).

4.3. **The subgraph $\mathbb{GT}^+ \subset \mathbb{GT}$.** A signature $\mu \in \mathbb{GT}_N$ is said to be *nonnegative* if all its coordinates $\mu_1, \dots, \mu_N$ are nonnegative. Of course, it suffices to require $\mu_N \ge 0$. The nonnegative signatures span a subgraph $\mathbb{GT}^+ = \bigsqcup_{N \ge 1} \mathbb{GT}^+_N$ in $\mathbb{GT}$. In what follows we will be concerned exclusively with this subgraph.

Note that a nonnegative signature may be viewed as a Young diagram. More precisely, given a Young diagram $\mu \in \mathbb{Y}$ and a positive integer N, the signature

$[\mu, N]$ is well defined if and only if $\ell(\mu)$, the number of nonzero rows in μ, does not exceed N.

Let μ and ν be two Young diagrams with $\ell(\mu) \le N$ and $\ell(\nu) \le N+1$, so that vertices $[\mu, N]$ and $[\nu, N+1]$ in $\mathbb{GT}^+$ are well defined. Then these vertices are joined by an edge, that is, $[\mu, N] \prec [\nu, N+1]$ if and only if $\mu \subseteq \nu$ and the skew diagram ν/μ is a *horizontal strip*, meaning that ν/μ has at most one box in each column.

This implies, in particular, that if $[\mu, N]$ and $[\nu, N']$ are in $\mathbb{GT}^+$ and $N' > N$, then $\operatorname{Dim}([\mu, N], [\nu, N'])$ vanishes unless $\mu \subseteq \nu$.

Given Theorem 4.2.1, it is not hard to see that the boundary $\partial\mathbb{GT}^+$ can be identified with the subset of $\partial\mathbb{GT}$ determined by $\omega^- = (0,0,0)$.

4.4. **Degeneration $\mathbb{GT}^+ \to \mathbb{YB}$.** The next theorem says that the projective system corresponding to the Young bouquet $\mathbb{YB}$ can be obtained from the projective system corresponding to the Gelfand–Tsetlin graph (or rather its part $\mathbb{GT}^+$) via a scaling limit transition turning the discrete scale of levels numbered by $1, 2, \dots$ into a continuous one parametrized by $\mathbb{R}_{>0}$.

Note that the links ${}^{\mathbb{YB}}\Lambda^{r'}_r$ depend on parameters $r' > r$ only through their ratio r'/r.

Theorem 4.4.1. *Fix arbitrary positive numbers $r' > r > 0$ and arbitrary two Young diagrams μ and ν such that $\mu \subseteq \nu$. Let two positive integers $N' > N$ go to infinity in such a way that $N'/N \to r'/r$. Then*

$$\lim {}^{\mathbb{GT}}\Lambda^{N'}_N([\nu, N'], [\mu, N]) = {}^{\mathbb{YB}}\Lambda^{r'}_r(\nu, \mu). \tag{4.4.1}$$

Proof. The idea is to express all the dimensions entering the left- and right-hand sides through Schur functions and their specializations.

In what follows the brackets $(\,\cdot\,,\,\cdot\,)$ denote the canonical inner product in Sym; with respect to this product, the Schur functions form an orthonormal basis. By (1^N) we denote the N-tuple $(1, \dots, 1)$. We set $m = |\mu|$ and $n = |\nu|$. Denote by $S_{\nu/\mu}$ the skew Schur function indexed by the skew diagram ν/μ, see Section I.5 in [Ma95].

Here are the necessary formulas:

$$\operatorname{Dim}[\mu, N] = S_\mu(1^N), \qquad \operatorname{Dim}([\mu, N], [\nu, N']) = S_{\nu/\mu}(1^{N'-N}) \tag{4.4.2}$$

$$\dim \mu = (p_1^m, S_\mu), \qquad \dim(\mu, \nu) = (p_1^{n-m}, S_{\nu/\mu}). \tag{4.4.3}$$

Both in (4.4.2) and (4.4.3) the first equality is a particular case of the second one. The first relation in (4.4.2) follows from the fact that the irreducible characters of the unitary groups are given by the Schur polynomials, and the second equation follows from the combinatorial formula for the skew Schur functions, see e.g. [Ma95, I(5.12)]. As for (4.4.3), we first note that $\dim(\mu, \nu) = (S_\mu p_1^{n-m}, S_\nu)$ by the simplest instance of the Pieri rule [Ma95, I(5.16)]. Then the equality $(S_\mu p_1^{n-m}, S_\nu) = (p_1^{n-m}, S_{\nu/\mu})$ follows from [Ma95, Chapter I, (5.1)].

Observe that $p_k(1^N)=N$ for all $k=1,2,\dots$. Therefore, if F is a monomial in $p_1,p_2,\dots$, then $F(1^N)$ equals N raised to the number of letters in F. This number is strictly less than $\deg F$ unless F is a power of p_1. It follows that if $F\in\operatorname{Sym}$ is a homogeneous element, then for large N

$$F(1^N)=[F:p_1^d]\,N^d+O(N^{d-1}),\qquad d:=\deg F,$$

where $[F:p_1^d]$ denotes the coefficient of p_1^d in the expansion of F on monomials in $p_1,p_2,\dots$. Next, since the monomials in $p_1,p_2,\dots$ form an orthogonal basis, we have

$$[F:p_1^d]=\frac{(F,p_1^d)}{(p_1^d,p_1^d)}=\frac{(F,p_1^d)}{d!}$$

and finally

$$F(1^N)=\frac{(F,p_1^d)}{d!}\,N^d+O(N^{d-1}),\qquad d:=\deg F. \tag{4.4.4}$$

Now we proceed to the proof of (4.4.1). By virtue of (4.1.1) and (4.4.2)

$$\begin{aligned}{}^{\mathbb{GT}}\Lambda^{N'}_N([\nu,N'],[\mu,N])&=\frac{\operatorname{Dim}[\mu,N]\operatorname{Dim}([\mu,N],[\nu,N'])}{\operatorname{Dim}[\nu,N']}\\&=\frac{S_\mu(1^N)S_{\nu/\mu}(1^{N'-N})}{S_\nu(1^{N'})}\end{aligned} \tag{4.4.5}$$

Applying (4.4.4) to the ordinary and skew Schur functions entering (4.4.5) we get

$$\begin{aligned}S_\mu(1^N)&=\frac{(S_\mu,p_1^m)}{m!}N^m+O(N^{m-1})\\&=\left(\frac{r}{r'}\right)^m\frac{(S_\mu,p_1^m)}{m!}(N')^m+O((N')^{m-1})\end{aligned} \tag{4.4.6}$$

$$S_\nu(1^{N'})=\frac{(S_\nu,p_1^n)}{n!}(N')^n+O((N')^{n-1}) \tag{4.4.7}$$

$$\begin{aligned}S_{\nu/\mu}(1^{N'-N})&=\frac{(S_{\nu/\mu},p_1^{n-m})}{(n-m)!}(N'-N)^{n-m}+O((N'-N)^{n-m-1})\\&=\left(1-\frac{r}{r'}\right)^{n-m}\frac{(S_{\nu/\mu},p_1^{n-m})}{(n-m)!}(N')^{n-m}+O((N')^{n-m-1}).\end{aligned} \tag{4.4.8}$$

Plugging (4.4.6), (4.4.7), and (4.4.8) into (4.4.5) we get

$$\left(1-\frac{r}{r'}\right)^{n-m}\left(\frac{r}{r'}\right)^m\frac{n!}{(n-m)!\,m!}\,\frac{(S_\mu,p_1^m)(S_{\nu/\mu},p_1^{n-m})}{(S_\nu,p_1^n)}+O(1/N').$$

Applying (4.4.3) we may rewrite this as

$$\left(1-\frac{r}{r'}\right)^{n-m}\left(\frac{r}{r'}\right)^m\frac{n!}{(n-m)!\,m!}\,\frac{\dim\mu\,\dim(\mu,\nu)}{\dim\nu}+O(1/N').$$

Comparing with (3.4.2) we see that this is exactly the right-hand side of (4.4.1), within $O(1/N')$. □

4.5. **Degeneration of the boundary.** From Theorem 4.4.1 it is natural to expect that there should exist a limit procedure that turns $\partial\mathbb{GT}^+$ into $\partial\mathbb{YB}$, and our closest goal is to exhibit this procedure.

Each point $\omega \in \partial\mathbb{YB} = \widetilde{\Omega}$ defines a coherent system of measures $\{{}^{\mathbb{YB}}M_r\}_{r>0}$ on the levels $\mathbb{YB}_r = \mathbb{Y}$ of $\mathbb{YB}$. Similarly, each point $(\omega^+, \omega^-) \in \partial\mathbb{GT}$ defines a coherent system of measures $\{{}^{\mathbb{GT}}M_N\}_{N\ge 1}$ on the levels $\mathbb{GT}_N$ of $\mathbb{GT}$. We are about to show that the former family of coherent systems can be obtained from the latter one by taking $\omega^- = \underline{0} = (0, 0, 0)$ (since we want to start from $\partial\mathbb{GT}^+$ rather than from $\partial\mathbb{GT}$) and appropriate $\omega^+ = \omega^+(\epsilon)$ depending on a small parameter $\epsilon > 0$.

As before, we identify nonnegative signatures and Young diagrams.

Theorem 4.5.1. *Fix $\omega \in \widetilde{\Omega}$. For any $\mu \in \mathbb{Y}$ and any $r > 0$, the following limiting relation holds: If $N(\epsilon) \sim r\epsilon^{-1}$ as $\epsilon \to +0$ then*

$$\lim_{\epsilon\to+0} {}^{\mathbb{GT}}\Lambda^{\infty}_{N(\epsilon)}(\epsilon\omega, \underline{0}; \mu) = {}^{\mathbb{YB}}\Lambda^{\infty}_{r}(\omega, \mu).$$

Proof. In the special case $\omega^- = \underline{0}$, the function $u \mapsto \Phi(u; \omega^+, \omega^-) = \Phi(u; \omega^+, \underline{0})$ is not just holomorphic in an neighborhood of the unit circle $|u| = 1$, but in a neighborhood of the unit disc $|u| \le 1$. Indeed, all the factors that involve $\alpha_i^-, \beta_i^-, \gamma^-$ disappear, and the Laurent series turns into a Taylor series. Thus, all the coefficients φ_n with $n < 0$ vanish.

This reduces the $N \times N$ determinant in (4.2.1) to a determinant of size $\ell = \ell(\mu)$ that does not depend on N:

$$\det\big[\phi_{\mu_i-i+j}(\omega^+, \underline{0})\big]_{i,j=1}^{N} = \det\big[\phi_{\mu_i-i+j}(\omega^+, \underline{0})\big]_{i,j=1}^{\ell} \cdot \big(\varphi_0(\omega^+, \underline{0})\big)^{N-\ell}. \qquad (4.5.1)$$

This follows from the fact that the (i, j)-entry of the matrix in the left-hand side of (4.2.1) vanishes for $i > j > \ell$.

Let us now rewrite the expression for $\Phi(u; \omega^+, \underline{0})$ assuming that $\beta_1^+ < 1$ (this will be automatically satisfied for $\omega^+ = \epsilon\omega$ with small ϵ). We have

$$\Phi(u; \omega^+, \underline{0}) = e^{\gamma^+(u-1)} \prod_{i=1}^{\infty} \frac{1 + \beta_i^+(u-1)}{1 - \alpha_i^+(u-1)} = e^{-\gamma^+} \prod_{i=1}^{\infty} \frac{1 - \beta_i^+}{1 + \alpha_i^+} \cdot e^{\gamma^+ u} \prod_{i=1}^{\infty} \frac{1 + \widetilde{\beta}_i^+ u}{1 - \widetilde{\alpha}_i^+ u},$$

where

$$\widetilde{\alpha}_i = \frac{\alpha_i^+}{1 + \alpha_i^+}, \quad \widetilde{\beta}_i^+ = \frac{\beta_i^+}{1 - \beta_i^+}, \qquad i \ge 1.$$

Let us substitute $\omega^+ = \epsilon\omega$, where $\omega = (\alpha, \beta, \delta) \in \widetilde{\Omega}$. We obtain

$$\Phi(u; \omega^+, \underline{0}) = e^{-\epsilon\delta}(1 + O(\epsilon^2)) \cdot e^{\epsilon\gamma u} \prod_{i=1}^{\infty} \frac{1 + (\epsilon + O(\epsilon^2))\beta_i u}{1 - (\epsilon + O(\epsilon^2))\alpha_i u},$$

where $\gamma = \delta - \sum_{i\ge1}(\alpha_i + \beta_i)$. All the $O(\epsilon^2)$ terms above are uniform in $i \ge 1$.

Hence,

$$\varphi_n(\epsilon\omega, \underline{0}) = h_n(\omega)\epsilon^n + O(\epsilon^{n+1}), \quad n \geq 1, \qquad \varphi_0(\epsilon\omega, \underline{0}) = e^{-\epsilon\delta}(1 + O(\epsilon^2)).$$

Taking $N = N(\epsilon) \sim r\epsilon^{-1}$ we see that the determinant (4.5.1) is asymptotically equal to

$$\det\left[h_{\mu_i-i+j}(\omega)\right]_{i,j=1}^{\ell} \epsilon^{|\mu|} e^{-r\delta} = S_\mu(\omega)\, \epsilon^{|\mu|} e^{-r|\omega|},$$

and, using the hook formula for $\operatorname{Dim}[\mu, N]$ and $\dim\mu$,

$$\operatorname{Dim}[\mu, N] = \frac{\dim\mu}{|\mu|!} \cdot (N)_\mu \sim \frac{\dim\mu}{|\mu|!} \cdot N^{|\mu|} \sim \frac{\dim\mu}{|\mu|!} r^{|\mu|} \cdot \epsilon^{-|\mu|}. \tag{4.5.2}$$

When we multiply these two expressions the factors $\epsilon^{\pm|\mu|}$ cancel out, and we obtain exactly

$$^{\mathbb{YB}}\Lambda_r^\infty(\omega, \mu) = \frac{r^{|\mu|}\dim\mu}{|\mu|!}\, e^{-r|\omega|} S_\mu(\omega).$$

□

4.6. **ZW-Measures on $\mathbb{GT}$.** Let $\mathcal{Z} \subset \mathbb{C}^2$ be the disjoint union of the following three sets:

$$\{(z, z') \in \mathbb{C}^2 \setminus \mathbb{R}^2 \mid z' = \bar{z}\}, \tag{4.6.1}$$

$$\{(z, z') \in \mathbb{R}^2 \mid \exists m \in \mathbb{Z},\, m < z, z < m + 1\}, \tag{4.6.2}$$

$$\{(z, z') \in \mathbb{R}^2 \mid \exists m \in \mathbb{Z},\, z = m,\, z' > m - 1, \quad \text{or} \quad z' = m,\, z > m - 1\} \tag{4.6.3}$$

Note that if $(z, z') \in \mathcal{Z}$, then $z + z'$ is real.

Denote by $\mathcal{D}_{\text{adm}}$ the subset in $\mathbb{C}^4$ formed by all quadruples (z, z', w, w') of complex numbers such that:

- $(z, z') \in \mathcal{Z}$, $(w, w') \in \mathcal{Z}$;
- $z + z' + w + w' > -1$;
- if both couples (z, z') and (w, w') belong to subsets (4.6.3) with indices m and $\widetilde{m}$, respectively, then it is additionally required that $m + \widetilde{m} \geq 1$.

Definition 4.6.1. The *zw-measure* on $\mathbb{GT}_N$ with parameters $(z, z', w, w') \in \mathcal{D}_{\text{adm}}$ is given by

$$^{\mathbb{GT}}M_N^{(z,z',w,w')}(\mu) = C_N^{(z,z',w,w')} \cdot \Pi_N^{(z,z',w,w')}(\mu) \cdot (\operatorname{Dim}[\mu, N])^2$$

where μ ranges over $\mathbb{GT}_N$,

$$C_N^{(z,z',w,w')} = \prod_{i=1}^{N} \frac{\Gamma(z + w + i)\Gamma(z + w' + i)\Gamma(z' + w + i)\Gamma(z' + w' + i)\Gamma(i)}{\Gamma(z + z' + w + w' + i)}$$

is a normalization constant, and

$$\Pi_N^{(z,z',w,w')}(\mu) = \prod_{i=1}^{N} \frac{1}{\Gamma(z-\mu_i+i)\Gamma(z'-\mu_i+i)\Gamma(w+N+1+\mu_i-i)\Gamma(w'+N+1+\mu_i-i)}.$$

Note that the measure does not change under transposition $z \leftrightarrow z'$ or $w \leftrightarrow w'$. If both couples (z,z') and (w,w') belong to subset (4.6.1) or subset (4.6.2), that is, none of the four parameters is an integer, then the expression for $\Pi_N^{(z,z',w,w')}(\mu)$ is strictly positive for all $\mu \in \mathbb{GT}_N$. If some of the parameters are integers, then $\Pi_N^{(z,z',w,w')}(\mu)$ vanishes for some signatures μ. Moreover, if both (z,z') and (w,w') are in subset (4.6.3), then it may even happen that the normalizing constant has a singularity, but in such a case the singularity actually disappears after multiplication by $\Pi_N^{(z,z',w,w')}(\mu)$ due to cancellation with zeros arising from appropriate $(1/\Gamma(\,\cdot\,))$-factors in $\Pi_N^{(z,z',w,w')}(\mu)$. Thus, the whole expression for ${}^{\mathbb{GT}}M_N^{(z,z',w,w')}(\mu)$ makes sense for all $(z,z',w,w') \in \mathcal{D}_{\mathrm{adm}}$. For more detail, see [Ols03c, Section 7] and [BO05a, Section 3].

Proposition 4.6.2. *The zw-measures are probability measures, and they are compatible with the links* ${}^{\mathbb{GT}}\Lambda_N^{N+1} : \mathbb{GT}_{N+1} \dashrightarrow \mathbb{GT}_N$.

A proof can be found in [Ols03c, Section 7].

Similarly to z-measures that arise in harmonic analysis on the infinite symmetric group, the zw-measures play a key role in harmonic analysis on the infinite-dimensional unitary group, see [Ols03c], [BO05a], [BO05b].

4.7. **Degeneration of zw-measures to z-measures.** It is convenient to rewrite the expression for the zw-measures in a slightly different form:

$${}^{\mathbb{GT}}M_N^{(z,z',w,w')}(\mu) = {}^{\mathbb{GT}}M_N^{(z,z',w,w')}(0^N) \cdot \widetilde{\Pi}_N^{(z,z',w,w')}(\mu) \cdot (\mathrm{Dim}[\mu,N])^2$$

where $0^N = (0,\dots,0) \in \mathbb{GT}_N$ is the zero signature,

$${}^{\mathbb{GT}}M_N^{(z,z',w,w')}(0^N) = \prod_{i=1}^{N} \frac{\Gamma(z+w+i)\Gamma(z+w'+i)\Gamma(z'+w+i)\Gamma(z'+w'+i)\Gamma(i)}{\Gamma(z+z'+w+w'+i)\Gamma(z+i)\Gamma(z'+i)\Gamma(w+i)\Gamma(w'+i)},$$

and

$$\begin{aligned}\widetilde{\Pi}_N^{(z,z',w,w')}(\mu) = &\prod_{i=1}^{N} \frac{\Gamma(z+i)\Gamma(z'+i)}{\Gamma(z-\mu_i+i)\Gamma(z'-\mu_i+i)} \\ &\times \prod_{i=1}^{N} \frac{\Gamma(w+N+1-i)\Gamma(w'+N+1-i)}{\Gamma(w+N+1+\mu_i-i)\Gamma(w'+N+1+\mu_i-i)}\end{aligned} \tag{4.7.1}$$

Assume that $w=0$ while w' is real positive and large enough. Then the expression for ${}^{\mathbb{GT}}M_N^{(z,z',w,w')}(0^N)$ simplifies:

$$ {}^{\mathbb{GT}}M_N^{(z,z',w,w')}(0^N)=\prod_{i=1}^{N}\frac{\Gamma(z+w'+i)\Gamma(z'+w'+i)}{\Gamma(z+z'+w'+i)\Gamma(w'+i)}\,, $$

Since w' is assumed to be large, this quantity is nonsingular.

Further, observe that if $\mu_N<0$, then $\widetilde{\Pi}_N^{(z,z',w,w')}(\mu)$ vanishes because

$$ \left.\frac{\Gamma(w+N+1-i)}{\Gamma(w+N+1+\mu_i-i)}\right|_{w=0,\,i=N}=\frac{1}{\Gamma(1+\mu_N)}=0 $$

and this zero cannot be cancelled after multiplication by ${}^{\mathbb{GT}}M_N^{(z,z',w,w')}(0^N)$. This means that the zw-measure with $w=0$ and w' real and large enough is concentrated on nonnegative signatures. Thus, we may assume that the measure lives on the set $\mathbb{GT}_N^+$, which we regard as a subset of $\mathbb{Y}$.

Observe that if (z,z') is admissible in the sense explained in Section 3.5, parameter w equals 0, and parameter w' is real and large enough, then the quadruple (z,z',w,w') belongs to the set $\mathcal{D}_{\text{adm}}$ so that the corresponding zw-measure ${}^{\mathbb{GT}}M_N^{(z,z',0,w')}$ is well defined for all N.

Theorem 4.7.1. *Fix an arbitrary admissible couple (z,z') of parameters and let parameter w equal 0 while parameter w' is positive and goes to $+\infty$. Assume that N varies together with w' in such a way that $N\sim rw'$ with an arbitrary fixed $r>0$. Then the corresponding zw-measures ${}^{\mathbb{GT}}M_N^{(z,z',0,w')}$ weakly converge to the z-measure ${}^{\mathbb{YB}}M_r^{(-z,-z')}$ on $\mathbb{Y}$.*

Proof. It suffices to prove that for any fixed $\mu\in\mathbb{Y}$ and $w'=w'(N)\sim r^{-1}N$,

$$ \lim_{N\to\infty}{}^{\mathbb{GT}}M_N^{(z,z',0,w')}(\mu)={}^{\mathbb{YB}}M_r^{(-z,-z')}(\mu). $$

Step 1. Let us prove this for $\mu=\varnothing$, which amounts to

$$ \lim_{N\to\infty}\prod_{i=1}^{N}\frac{\Gamma(z+w'+i)\Gamma(z'+w'+i)}{\Gamma(z+z'+w'+i)\Gamma(w'+i)}=(1+r)^{-zz'},\qquad w'=w'(N)\sim r^{-1}N. $$

Stirling's formula implies

$$ \frac{\Gamma(z+w'+i)}{\Gamma(w'+i)}=(w'+i)^{z}\left(1+\frac{z(z-1)}{2(w'+i)}+O\left(N^{-2}\right)\right), $$
$$ \frac{\Gamma(z'+w'+i)}{\Gamma(z+z'+w'+i)}=(z'+w'+i)^{-z}\left(1-\frac{z(z-1)}{2(z'+w'+i)}+O\left(N^{-2}\right)\right). $$

(Note that, since w' is a large positive number, the arguments of the complex numbers $z'+w'+i$ are small.) Hence,

$$\frac{\Gamma(z+w'+i)\Gamma(z'+w'+i)}{\Gamma(z+z'+w'+i)\Gamma(w'+i)} = \left(\frac{w'+i}{z'+w'+i}\right)^{z}\left(1+O(N^{-2})\right).$$

Next, Taylor series type argument shows that

$$\left(\frac{w'+i}{z'+w'+i}\right)^{z} = \left(1-\frac{zz'}{w'+i}\right)\left(1+O(N^{-2})\right).$$

Thus,

$$\prod_{i=1}^{N}\frac{\Gamma(z+w'+i)\Gamma(z'+w'+i)}{\Gamma(z+z'+w'+i)\Gamma(w'+i)} = \prod_{i=1}^{N}\left(1-\frac{zz'}{w'+i}\right)\cdot\left(1+O(N^{-1})\right).$$

But

$$\prod_{i=1}^{N}\left(1-\frac{zz'}{w'+i}\right) = \prod_{i=1}^{N}\frac{-zz'+w'+i}{w'+i} = \frac{\Gamma(-zz'+w'+N+1)\Gamma(w'+1)}{\Gamma(-zz'+w'+1)\Gamma(w'+N+1)}$$

$$= \frac{\Gamma(-zz'+w'+N+1)}{\Gamma(w'+N+1)}\,\frac{\Gamma(w'+1)}{\Gamma(-zz'+w'+1)} \sim \left(\frac{r^{-1}}{r^{-1}+1}\right)^{zz'} = (1+r)^{-zz'},$$

which gives the desired result.

Note that on the last step we used the well-known asymptotic relation

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b}, \qquad x>0 \text{ large.} \tag{4.7.2}$$

Step 2. It remains to prove that

$$\lim_{N\to\infty}\frac{{}^{\mathbb{GT}}M_N^{(z,z',0,w')}(\mu)}{{}^{\mathbb{GT}}M_N^{(z,z',0,w')}(\varnothing)} = \frac{{}^{\mathbb{YB}}M_r^{(-z,-z')}(\mu)}{{}^{\mathbb{YB}}M_r^{(-z,-z')}(\varnothing)}, \qquad w'=w'(N)\sim r^{-1}N,$$

which amounts to

$$\lim_{N\to\infty}\left(\widetilde{\Pi}_N^{(z,z',0,w')}(\mu)\,(\mathrm{Dim}[\mu,N])^2\right) = (-z)_\mu(-z')_\mu\left(\frac{r}{1+r}\right)^{|\mu|}\left(\frac{\dim\mu}{|\mu|!}\right)^2$$

Recall that $\widetilde{\Pi}_N^{(z,z',0,w')}(\mu)$ is given by formula (4.7.1), which involves two products over $i=1,\dots,N$. Observe that the ith factor in each of the two products equals 1 when $\mu_i=0$. Since μ is a Young diagram, this allows us to restrict each of the products to indices $i=1,\dots,\ell$, where ℓ stands for the number of nonzero rows in μ. Since ℓ does not depend on N, we may examine the asymptotics of the factors corresponding to each i separately. Using again (4.7.2) we get

$$\widetilde{\Pi}_N^{(z,z',0,w')}(\mu) \sim \prod_{i=1}^{\ell} \frac{\Gamma(z+i)\Gamma(z'+i)}{\Gamma(z-\mu_i+i)\Gamma(z'-\mu_i+i)} \cdot \left(\frac{1}{N^2(r^{-1}+1)}\right)^{|\mu|}$$
$$= (-z)_\mu(-z')_\mu \left(\frac{r}{1+r}\right)^{|\mu|} N^{-2|\mu|}.$$

Further, (4.5.2) gives

$$(\mathrm{Dim}[\mu,N])^2 \sim \left(\frac{\dim\mu}{|\mu|!}\right)^2 \cdot N^{2|\mu|}.$$

Therefore,

$$\lim_{N\to\infty} \frac{{}^{\mathbb{GT}}M_N^{(z,z',0,w')}(\mu)}{{}^{\mathbb{GT}}M_N^{(z,z',0,w')}(\varnothing)} = (-z)_\mu(-z')_\mu \left(\frac{r}{1+r}\right)^{|\mu|} \left(\frac{\dim\mu}{|\mu|!}\right)^2$$
$$= \frac{{}^{\mathbb{YB}}M_r^{(-z,-z')}(\mu)}{{}^{\mathbb{YB}}M_r^{(-z,-z')}(\varnothing)}.$$

□

5. Gibbs measures on the path space

5.1. Gibbs measures. Let Γ be a graded graph (Definition 2.3.1). By a (monotone) *path* in Γ we mean a finite or infinite collection

$$v_1, e_{12}, v_2, e_{23}, v_3, \dots$$

where $v_1, v_2, \dots$ are vertices of Γ such that $|v_{i+1}| = |v_i| + 1$ and $e_{i,i+1}$ is an edge between v_i and v_{i+1}. Since we do not consider more general paths, the adjective "monotone" will be omitted. If the graph has no multiple edges, then every path is uniquely determined by its vertices, but when multiple edges occur it is necessary to specify which of the edges between every two consecutive vertices is selected.

Unless otherwise stated, we will assume that the paths start at the lowest level of the graph. Then the *path space* $\mathcal{T} = \mathcal{T}(\Gamma)$ is defined as the set of all infinite paths.

A *cylinder set* in $\mathcal{T}$ is the subset of infinite paths with a prescribed initial part of finite length. We equip $\mathcal{T}$ with the Borel structure generated by the cylinder sets.

Definition 5.1.1. Let P be a probability measure P on $\mathcal{T}$. Let us call P a *Gibbs measure* if any two initial finite paths with the same endpoint are equiprobable. Equivalently, the measure of any cylinder set depends only on the endpoint of the initial part that defines the set.

This kind of measures on the path space was introduced by Vershik and Kerov [VK81] under the name of *central measures*.

As above, consider the projective chain $\{V_N, \Lambda_N^{N+1}\}$ associated with the graph Γ, so that V_N is the set of vertices of level $N = 1, 2, \dots$.

Proposition 5.1.2. *There is a natural bijective correspondence between the Gibbs measures on the path space and coherent systems of measures*

$$\{M_N\}_{N\ge 1} \in \mathcal{M}_\infty = \varprojlim \mathcal{M}(V_N).$$

Proof. Indeed, given a Gibbs measure P, define for each N a probability measure $M_N \in \mathcal{M}(V_N)$ as follows: For any $v \in V_N$, $M_N(v)$ equals the probability that the infinite random path distributed according to P passes through v. The measures M_N are compatible with the links Λ_N^{N+1} by the very construction of these links. Therefore, the sequence (M_N) determines an element of $\mathcal{M}_\infty$. The inverse map, from $\mathcal{M}_\infty$ to Gibbs measures, is obtained by making use of Kolmogorov's extension theorem. □

Together with Theorem 2.2.1 this implies

Corollary 5.1.3. *There is a bijection between the Gibbs measures on the path space of Γ and the probability measures on the boundary $\partial\Gamma$.*

Note that the random paths distributed according to a Gibbs measure can be viewed as trajectories of a Markov chain with discrete time N that flows backwards from $+\infty$ to 0 and transition probabilities Λ_N^{N+1}. Then $\partial\Gamma$ plays the role of the entrance boundary, and probability measures on $\partial\Gamma$ turn into entrance laws for the Markov chain.

5.2. **Examples of path spaces for graded graphs.** (a) For the Pascal graph $\Gamma = \mathbb{P}$, the path space can be identified with the space $\{0,1\}^\infty$ of infinite binary sequences. Under this identification, the Gibbs measures are just the exchangeable measures on $\{0,1\}^\infty$, and the claim of Corollary 5.1.3 turns into the classical de Finetti theorem: exchangeable probability measures on $\{0,1\}^\infty$ are parametrized by probability measures on $[0,1]$.

(b) Consider the Young graph $\Gamma = \mathbb{Y}$. Recall that for a Young diagram $\lambda \in \mathbb{Y}$, a standard Young tableau of shape λ is a filling of the boxes of λ by numbers $1, 2, \dots, |\lambda|$ in such a way that the numbers increase along each row from left to right and along each column from top to bottom.

Let us also define an *infinite Young diagram* as an infinite subset $\widetilde{\lambda} \subseteq \mathbb{N} \times \mathbb{N}$ (where $\mathbb{N} := \{1, 2, \dots\}$) such that if $(i,j) \in \widetilde{\lambda}$, then $\widetilde{\lambda}$ contains all pairs (i', j') with $i' \le i$, $j' \le j$. An *infinite standard tableau* of shape $\widetilde{\lambda}$ is an assignment of a positive integer to any pair $(i,j) \in \widetilde{\lambda}$ in a such a way that all positive integers are used, and they increase in both i and j. If we only pay attention to where the integers $1, 2, \dots, n$ are located, we will observe a Young tableau whose shape is a Young diagram $\lambda \subset \widetilde{\lambda}$ with n boxes. Let us call this finite tableau the *n-truncation* of the original infinite one.

Clearly, the infinite paths in the Young graph are in one-to-one correspondence with the infinite Young tableaux. The initial finite parts of such a path are described by the various trancations of the corresponding tableau. The condition of a measure on infinite Young tableaux being Gibbs consists in the requirement that the probability of observing a prescribed truncation depends only on the shape of the truncation (and not on its filling).

(c) Let us proceed to the Gelfand-Tsetlin graph $\Gamma = \mathbb{GT}$. By definition, an infinite path in $\mathbb{GT}$ is a sequence $\lambda^{(1)} \prec \lambda^{(2)} \prec \dots$ with $\lambda^{(N)} \in \mathbb{GT}_N$. If one defines $x_i^j = \lambda_i^{(j)}$ then one easily sees that such paths are in one-to-one correspondence with the infinite triangular arrays $\{x_i^j \mid 1 \le i \le j,\ j \ge 1\}$ of integers that satisfy the inequalities

$$x_i^{j+1} \ge x_i^j \ge x_{i+1}^{j+1}$$

for all meaningful indices (i, j). Such arrays are called infinite *Gelfand-Tsetlin schemes*. The initial finite parts of infinite paths in a similar way give rise to finite Gelfand-Tsetlin schemes. Infinite Gelfand-Tsetlin schemes are also in one-to-one correspondence to certain tilings of a half-plane by lozenges, see the introduction to [BF08].

If we restrict ourselves to the subgraph $\mathbb{GT}^+ \subset \mathbb{GT}$, then the signatures can be identified with Young diagrams, and infinite paths may be viewed as infinite *semi-standard* Young tableaux, where "semi-standard" refers to the condition that the filling numbers are only required to *weakly* increase along rows, and they are also not required to exhaust all positive integers. The finite paths of length N then turn into semi-standard Young tableaux whose shape has no more than N rows.

(d) Other examples of Gibbs measures on path spaces related to exchangeability can be found in Kingman [Ki78] (exchangeable partitions of $\mathbb{N}$), Gnedin [Gn97] (exchangeable ordered partitions of $\mathbb{N}$), Gnedin and Olshanski [GO06] (exchangeable orderings of $\mathbb{N}$).

5.3. **Path spaces for $\mathbb{B}$ and $\mathbb{YB}$.** Similarly to the case of graded graphs described above, one can define Gibbs measures on paths corresponding to more general projective systems. Without going into general definitions, let us describe the outcome in the cases of the binomial system $\mathbb{B}$ and the Young bouquet $\mathbb{YB}$.

Recall that the levels of the binomial system $\mathbb{B}$ are labelled by numbers $r \in \mathbb{R}_{>0}$ (strictly positive real numbers), and each level consists of points $m \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$. It is convenient to denote these points as pairs $(m, r) \in \mathbb{Z}_+ \times \mathbb{R}_{>0}$, and also add the point $(0, 0)$ at level 0.

An infinite path in $\mathbb{B}$ can be viewed as an integer-valued function $m = m(r)$, $m(0) = 0$, that is weakly increasing, left-continuous, and has only jumps of size 1: $m(r + 0) - m(r) \in \{0, 1\}$ for any $r > 0$.

The jump locations for $m(r)$ form a (possibly empty) increasing sequence $r_1 < r_2 < \dots$ tending to $+\infty$. Thus, a path in $\mathbb{B}$ may be encoded by a locally finite point configuration in the space $\mathbb{R}_+$ of nonnegative real numbers.

A probability measure on the infinite paths in $\mathbb{B}$ (equivalently, point configurations in $\mathbb{R}_+$) is *Gibbs* if for any $n \geq 0$, under the condition that a segment $[0, r] \subset \mathbb{R}_+$ contains exactly n jumps at $r_1, \dots, r_n$, the distribution of their locations is proportional to the Lebesgue measure $dr_1 \cdots dr_n$.

One shows that coherent systems on $\mathbb{B}$ are in one-to-one correspondence with the Gibbs measures as defined above.

The extreme Gibbs measure corresponding to a point $x \in \mathbb{R}_+ = \partial\mathbb{B}$, $x \neq 0$, corresponds to the Poisson process on $\mathbb{R}_+$ with constant intensity x. The extreme Gibbs measure corresponding to $x = 0$ is the delta-measure on the path $m(r) \equiv 0$.

A general Gibbs measure is thus a (possibly continuous) convex combination of the delta-measure at the zero path and a random mix of the Poisson processes with constant intensities, also known as a doubly stochastic Poisson process, or a Cox process.

Let us proceed to $\mathbb{YB}$. The construction is a combination of those for $\mathbb{Y}$ and for $\mathbb{B}$.

Recall that an element of $\mathbb{YB}$ is a pair $(\lambda, r) \in \mathbb{Y} \times \mathbb{R}_+$ with the condition that $\lambda = \varnothing$ if $r = 0$. A path in $\mathbb{YB}$ is defined as a monotonically increasing Young diagram-valued function $\lambda(r)$, $\lambda(r') \supseteq \lambda(r)$ for $r' > r$, such that $(|\lambda(r)|, r)$ is a path in $\mathbb{B}$.

Such a path can be encoded by a *generalized* standard Young tableau, whose shape is a finite or infinite Young diagram and filling numbers are positive reals (strictly increasing along rows and columns) that have no finite accumulation points.

A finite initial part of a path is then given by the following data: a real number $r > 0$, an integer $n \geq 0$, a collection of n numbers $0 < r_1 < \cdots < r_n \leq r$, and a standard Young tableau whose shape has n boxes. The Gibbs property consists in requiring that the distribution of the coordinates $r_1, \dots, r_n$ is proportional to the Lebesgue measure $dr_1 \cdots dr_n$ on the polytope in $\mathbb{R}_+^n$ cut out by the inequalities that guarantee row and column monotonicity of the coordinates.

Once again, the Gibbs measures are in one-to-one correspondence with the probability measures on $\widetilde{\Omega} = \partial\mathbb{YB}$.

Every probability measure M on the boundary $\partial\mathcal{B}$ or $\partial\mathbb{YB}$ serves as the entrance law of a Markov process on $\mathbb{Z}_+$ or $\mathbb{Y}$, respectively, with "time" r ranging from $+\infty$ to 0 (a more conventional picture is obtained by taking as time $t := -\log r$). The trajectories of this process are the paths as described above, and the Gibbs measure corresponding to M is the law of the process.

5.4. **Path degeneration $\mathbb{GT}^+ \to \mathbb{YB}$.** To conclude, let us see how the degeneration $\mathbb{GT}^+ \to \mathbb{YB}$ described in Theorem 2.2.1 works on the level of Gibbs measures on paths. Consider all finite paths in $\mathbb{GT}^+$ that have a given nonnegative signature $[\lambda, N]$ as their final point. They may be viewed as semi-standard Young tableaux of shape λ filled with (some of the) numbers $1, \dots, N$. By definition of the Gibbs property, all those tableaux must have equal probabilities for any Gibbs measure on the path space of $\mathbb{GT}^+$.

Let us further consider the asymptotics when λ stays fixed and $N = rL$ with a fixed $r > 0$ and $L \gg 1$. Then if we take the random path in $\mathbb{GT}^+$ that ends at $[\lambda, N]$ and divide the entries in the corresponding Young tableau by L, we will observe a random generalized Young tableau of shape λ with filling numbers not exceeding r, or a finite path in $\mathbb{YB}$ ending at (λ, r). Its asymptotic distribution will be proportional to the Lebesgue measure on the polytope of the filling numbers, and this is exactly what is required by the Gibbs property on $\mathbb{YB}$.

References

[AESW51] M. Aissen, A. Edrei, I. J. Schoenberg, and A. Whitney, *On the generating functions of totally positive sequences*. Proc. Nat. Acad. Sci. USA **37** (1951), 303–307.

[ASW52] M. Aissen, I. J. Schoenberg, and A. Whitney, *On the generating functions of totally positive sequences I*. J. Analyse Math. **2** (1952), 93–103.

[BDJ99] J. Baik, P. Deift, and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*. J. Amer. Math. Soc. **12** (1999), 1119–1178 [arXiv:math/9810105].

[BF08] A. Borodin and P. L. Ferrari, *Anisotropic growth of random surfaces in 2+1 dimensions*. arXiv:0804.3035.

[BO00a] A. Borodin and G. Olshanski, *Distributions on partitions, point processes, and the hypergeometric kernel*. Commun. Math. Phys. **211** (2000), no. 2, 335–358; arXiv:math/9904010.

[BO00b] A. Borodin and G. Olshanski, *Harmonic functions on multiplicative graphs and interpolation polynomials*. Electronic J. Comb. **7** (2000), Paper #R28; math/9912124.

[BO05a] A. Borodin and G. Olshanski, *Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes*. Ann. of Math. (2) **161** (2005), no. 3, 1319–1422.

[BO05b] A. Borodin and G. Olshanski, *Representation theory and random point processes*. In: A. Laptev (ed.), *European congress of mathematics* (ECM), Stockholm, Sweden, June 27–July 2, 2004. Zürich: European Mathematical Society, 2005, pp. 73–94.

[BO06] A. Borodin and G. Olshanski, *Markov processes on partitions*. Probab. Theory Rel. Fields **135** (2006), 84–152; arXiv:math-ph/0409075.

[BO09] A. Borodin and G. Olshanski, *Infinite-dimensional diffusions as limits of random walks on partitions*. Probab. Theory Rel. Fields **144** (2009), 281–318; arXiv:0706.1034.

[BO10] A. Borodin and G. Olshanski, *Markov processes on the path space of the Gelfand-Tsetlin graph and on its boundary*, arXiv:1009.2029 .

[BO11a] A. Borodin and G. Olshanski, *The boundary of the Gelfand–Tsetlin graph: A new approach*. Preprint arXiv:1109.1412.

[BO11b] A. Borodin and G. Olshanski, paper in preparation.

[Boy83] R. P. Boyer, *Infinite traces of AF-algebras and characters of* $U(\infty)$. J. Operator Theory **9** (1983), 205–236.

[Br72] O. Bratteli, *Inductive limits of finite-dimensional* C^**-algebras*. Trans. Amer. Math. Soc. bf171 (1972), 195–234.

[Cur99] C. W. Curtis, *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer*. Amer. Math. Soc., Providence, RI, 1999.

[Dy71] E. B. Dynkin, *Initial and final behavior of trajectories of Markov processes*. Uspehi Mat. Nauk **26** (1971), no. 4, 153–172 (Russian); English translation: Russian Math. Surveys **26** (1971), no. 4, 165–185.

[Dy78] E. B. Dynkin, *Sufficient statistics and extreme points*. Ann. Probab. **6** (1978), 705–730.

[Ed52] A. Edrei, *On the generating functions of totally positive sequences II*. J. Analyse Math., **2** (1952), 104–109.

[Ed53] A. Edrei, *On the generating function of a doubly infinite, totally positive sequence*. Trans. Amer. Math. Soc. **74** (1953), 367–383.

[Gn97] A. V. Gnedin, *The representation of composition structures*. Ann. Prob. **25** (1997), 1437–1450.

[GO06] A. Gnedin and G. Olshanski, *Coherent permutations with descent statistic and the boundary problem for the graph of zigzag diagrams*. Internat. Math. Research Notices **2006** (2006), Article ID 51968.

[Gor10] V. Gorin, *Disjointness of representations arising in the problem of harmonic analysis on an infinite-dimensional unitary group*. Funct. Anal. Appl. **44** (2010), no. 2, 92–105; arXiv:0805.2660.

[Ke03] S. V. Kerov, *Asymptotic representation theory of the symmetric group and its applications in analysis*. Translations of Mathematical Monographs, **219**. American Mathematical Society, Providence, RI, 2003.

[KOO98] S. Kerov, A. Okounkov, and G. Olshanski, *The boundary of Young graph with Jack edge multiplicities*. Intern. Mathematics Research Notices, 1998, no. 4, 173–199; arXiv:q-alg/9703037.

[KOV93] S. Kerov, G. Olshanski, and A. Vershik, *Harmonic analysis on the infinite symmetric group. A deformation of the regular representation*. Comptes Rendus Acad. Sci. Paris, Sér. I **316** (1993), 773–778.

[KOV04] S. Kerov, G. Olshanski, and A. Vershik, *Harmonic analysis on the infinite symmetric group*. Invent. Math. **158** (2004), 551–642; arXiv:math/0312270.

[KeOr90] S. V. Kerov, O. A. Orevkova, *Random processes with common cotransition probabilities*. Zapiski Nauchnyh Seminarov LOMI **184** (1990), 169–181 (Russian); English translation in J. Math. Sci. (New York) **68** (1994), no. 4, 516–525.

[Ki78] J. F. C. Kingman, *The representation of partition structures*. J. London Math. Soc. **18** (1978), 374–380.

[Ma95] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*. Clarendon Press Oxford, 1995.

[Mack57] G. W. Mackey, *Borel structure in groups and their duals*. Trans. Amer. Math. Soc. **85** (1957), 134–165.

[Mey66] P.-A. Meyer, *Probability and potentials*. Blaisdell, 1966.

[Ner02] Yu. A. Neretin, *Hua-type integrals over unitary groups and over projective limits of unitary groups*. Duke Math. J. **114** (2002), 239–266; arXiv:math-ph/0010014.

[OO97] A. Okounkov and G. Olshanski, *Shifted Schur functions*. Algebra i Analiz **9** (1997), no. 2, 73–146 (Russian); English version: St. Petersburg Mathematical J., 9 (1998), 239–300; arXiv:q-alg/9605042.

[OO98] A. Okounkov and G. Olshanski, *Asymptotics of Jack polynomials as the number of variables goes to infinity*. Intern. Math. Res. Notices 1998, no. 13, 641–682.

[Ols03a] G. Olshanski, *Point processes related to the infinite symmetric group*. In: The orbit method in geometry and physics: in honor of A. A. Kirillov (Ch. Duval et al., eds.), Progress in Mathematics **213**, Birkhäuser, 2003, pp. 349393; arXiv:math.RT/9804086.

[Ols03b] G. Olshanski, *An introduction to harmonic analysis on the infinite symmetric group*. In: Asymptotic Combinatorics with Applications to Mathematical Physics (A. Vershik, ed.). Springer Lecture Notes in Math. **1815**, 2003, 127–160.

[Ols03c] G. Olshanski, *The problem of harmonic analysis on the infinite-dimensional unitary group*. J. Funct. Anal. **205** (2003), no. 2, 464–524.

[Ols10] G. Olshanski, *Laguerre and Meixner symmetric functions, and infinite-dimensional diffusion processes*. Zapiski Nauchnyh Seminarov POMI **378** (2010), 81–110. Reproduced in J. Math. Sci. (New York) **174** (2011), No. 1, 41–57; arXiv:1009.2037.

[Ols11] G. Olshanski, *Laguerre and Meixner orthogonal bases in the algebra of symmetric functions*. Intern. Math. Research Notices, to appear; arXiv:1103.5848.

[ORV03] G. Olshanski, A. Regev, and A. Vershik, *Frobenius-Schur functions*. In: Studies in memory of Issai Schur (A. Joseph, A. Melnikov, R. Rentschler, eds.). Progress in Mathematics **210**, pp. 251–300. Birkhäuser, 2003; arXiv:math/0110077.

[Osi11] A. Osinenko, *Harmonic analysis on the infinite-dimensional unitary group*. To appear.

[Pa67] K. R. Parthasarathy, *Probability measures on metric spaces*. Academic Press, Inc., New York-London 1967.

[Pic87] D. Pickrell, *Measures on infinite dimensional Grassmann manifold*. J. Func. Anal. **70** (1987); 323–356.

[Sa01] B. E. Sagan, *The symmetric group. Representations, combinatorial algorithms, and symmetric functions*. Second edition, Springer, 2001.

[Tho64] E. Thoma, *Die unzerlegbaren, positive–definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe*. Math. Zeitschr., **85** (1964), 40–61.

[VK81] A. M. Vershik and S. V. Kerov, *Asymptotic theory of characters of the symmetric group*. Funct. Anal. Appl. **15** (1981), 246–255.

[VK82] A. M. Vershik and S. V. Kerov, *Characters and factor representations of the infinite unitary group*. Doklady AN SSSR **267** (1982), no. 2, 272–276 (Russian); English translation: Soviet Math. Doklady **26** (1982), 570–574.

[VK90] A. M. Vershik and S. V. Kerov, *The Grothendieck group of infinite symmetric group and symmetric functions (with the elements of the theory of K_0-functor for AF-algebas)*. In: Representations of Lie groups and related topics. Advances in Contemp. Math., vol. 7 (A. M. Vershik and D. P. Zhelobenko, editors). Gordon and Breach, N.Y., London etc. 1990, 39–117.

[Vo76] D. Voiculescu, *Représentations factorielles de type* II_1 *de* $U(\infty)$. J. Math. Pures et Appl. **55** (1976), 1–20.

[Wey39] H. Weyl, *The classical groups. Their invariants and representations*. Princeton Univ. Press, 1939; 1997 (fifth edition).

[Wi85] G. Winkler, *Choquet order and simplices. With applications in probabilistic models*. Springer Lect. Notes Math. **1145**, 1985.

[Zhe70] D. P. Zhelobenko, *Compact Lie groups and their representations*, Nauka, Moscow, 1970 (Russian); English translation: Transl. Math. Monographs 40, Amer. Math. Soc., Providence, RI, 1973.

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