Path complexes and their homologies

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March 2013

Abstract

We introduce a new notion of a path complex and its homology and apply it to define path homologies of digraphs. We state some results about properties of path homologies for general path complexes and for digraphs.

1 Introduction

In this paper we introduce a new notion – a path complex that can be regarded as a generalization of the notion of a simplicial complex. In short, a path complex P on a finite set V is a collection of paths (=sequences of points) on V such that if a path v belongs to P then a truncated path that is obtained from v by removing either the first or the last point, is also in P. Given a path complex P, all the paths in P are called *allowed*, while the paths outside P are called *non-allowed*.

Any simplicial complex S determines naturally a path complex by associating with any simplex from S the sequence of its vertices (see Section 3 for details).

However, the main motivation for considering path complexes comes from directed graphs (digraphs). A digraph G is a pair (V, E) where V is a set as above and E is a binary relation on V that is, E is a subset of $V \times V$. If $(a, b) \in E$ then the pair (a, b) is called a directed edge; this fact is also denoted by $a \to b$. Any digraph naturally gives rise to a path complex where allowed paths go along arrows of the digraph.

One of our key observations is that any path complex P allows to define a chain complex with an appropriate boundary operator that leads to the notion of homology groups of P. We refer to this notion as a *path homology*.

In the case when P arises from a simplicial complex S, the path homology of P coincides with the simplicial homology of S. If P arises from a digraph G then we obtain a new notion: the path homology of a digraph. The path complexes of digraphs are the central objects of this paper. Although most of the results are presented for arbitrary path complexes, we always have in mind applications for digraphs. On the other hand, the notion of a path complex provides an alternative approach to the classical results about simplicial complexes.

There has been a number of attempts to define the notion of (co)homology for graphs. At a trivial level, any graph can be regarded as an one-dimensional simplicial complex, so that its simplicial homologies are defined. However, all homology groups of order 2 and higher are trivial, which makes this approach uninteresting.

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*Partially supported by SFB 701 of German Research Council and by Visiting Grants of Harvard University and MSC, Tsinghua University

†Supported by the Fundamental Research Funds for the Central Universities and the Research Funds of Renmin University of China(11XNI004)

‡Partially supported by the CONACyT Grant 98697 and SFB 701 of German Research Council

Another way to make a graph into a simplicial complex is to consider all its cliques (=complete subgraphs) as simplexes of the corresponding dimensions (cf. [11], [2]). Then higher dimensional homologies may be non-trivial, but in this approach the notion of graph looses its identity and becomes a particular case of the notion of a simplicial complex. Besides, some desirable functorial properties of homologies fail, for example, the Künneth formula is not true for Cartesian product of graphs.

Yet another approach to homologies of digraphs can be realized via Hochschild homology. Indeed, allowed paths on a digraph have a natural operation of product, which allows to define the notion of a *path algebra* of a digraph. The Hochschild homology of the path algebra is a natural object to consider. However, it was shown in [10] that Hochschild homologies of order ≥ 2 are trivial, which makes this approach not so attractive.

The path homologies of digraphs that we introduce in this paper have many advantages in comparison with the previously studied notions of graph homologies. Firstly, the homologies of all dimensions could be non-trivial. Also, the chain complex associated with a path complex has a richer structure that simplicial chain complexes. It contains not only cliques but also binary hypercubes and many other subgraphs. Secondly, this notion is well linked to graph-theoretical operations. For example, the Künneth formula is true for join of two digraphs as well as for Cartesian product of two digraphs. Thirdly, there is a dual cohomology theory with the coboundary operator that arises independently and naturally as an exterior derivative on the algebra of functions on the vertex set of the graph.

This approach to the cohomology of digraphs, that is based on a classification of [1] of exterior derivations on algebras, was developed in [4], [3] and more recently in [7]. In this note we do not discuss the coboundary operator and cohomologies because of lack of space. The reader is referred to [6] where a detailed account of all these notions can be found. Besides, all the results that are announced in this note, are proved in [6].

We feel that the notion of path homology of digraphs has a rich mathematical content and hope that it will become a useful tool in various areas of pure and applied mathematics. For example, this notion was employed in [9] to give a new elementary proof of a theorem of Gerstenhaber and Schack [5] that gives a representation of simplicial homology as a Hochschild homology. A link between path homologies of digraphs and cubical homologies was revealed in [8]. On the other hand, it is conceivable that the notion of path homology can be used in practical applications such as coverage verification in sensor networks (cf. [12]), and many others.

Let us briefly describe the structure of the paper and the main results. In Section 2 we introduce the notion of the boundary operator on paths on a finite set V. In Section 3 we define the notions of a path complex, a ∂-invariant path (element of a chain complex), and a path homology. In Section 4 we give some examples of digraphs and ∂-invariant paths there. We state some basic results about path homologies, for example, a Poincaré lemma for star-shaped digraphs (Theorem 4.4). In Section in 5 we state results on preservation of path homologies under some simple transformations of digraphs (Theorems 5.1 and 5.4).

In Section 6 we introduce the operation *join* of two path complexes and state a Künneth formula for it (Theorem 6.4). Particular cases of join are operation of taking a cone and suspension of a digraph that behave homologically in the same way as those in the classical algebraic topology. In Section 7 we introduce the notions of cross product of paths and Cartesian product of path complexes. The latter matches the notion of Cartesian product of digraph. We state a Künneth formula for Cartesian product (Theorem 7.4) and give some examples.

Technically most difficult and interesting results of this study are Theorems 6.4 and 7.4 (see [6] for details).

2 Paths on a finite set

Let V be an arbitrary non-empty finite set whose elements will be called vertices. For any non-negative integer p, an *elementary p-path* on a set V is any sequence $\{i_k\}_{k=0}^{p}$ of $p+1$ vertices of V (a priori the vertices in the path do not have to be distinct). For $p=-1$, an elementary p-path is an empty set $\emptyset$. The p-path $\{i_k\}_{k=0}^{p}$ will also be denoted simply by $i_0...i_p$, without delimiters between the vertices.

Fix a field $\mathbb{K}$ and consider a $\mathbb{K}$-linear space $\Lambda_p = \Lambda_p(V)$ that consists of all formal linear combinations of all elementary p-paths with the coefficients from $\mathbb{K}$. The elements of Λ_p are called *p-paths* on V. An elementary p-path $i_0...i_p$ as an element of Λ_p will be denoted by $e_{i_0...i_p}$. The empty set as an element of Λ_{-1} will be denoted by e.

By definition, the family $\left\{e_{i_0...i_p} : i_0, ..., i_p \in V\right\}$ is a basis in Λ_p. Each p-path v has a unique representation in the form

$$v = \sum_{i_0,...,i_p \in V} v^{i_0 \cdots i_p} e_{i_0...i_p}, \tag{2.1}$$

where $v^{i_0 \cdots i_p} \in \mathbb{K}$. For example, Λ_0 consists of all linear combinations of elements e_i that are the vertices of V, Λ_1 consists of all linear combinations of the elements e_{ij} that are pairs of vertices, etc. Note that , Λ_{-1} consists of all multiples of e so that $\Lambda_{-1} \cong \mathbb{K}$.

For any $p \geq 0$, define the *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ is a linear operator that acts on elementary paths by

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0...\widehat{i_q}...i_p}, \tag{2.2}$$

where the hat $\widehat{i_q}$ means omission of the index i_q. For example, we have

$$\partial e_i = e, \quad \partial e_{ij} = e_j - e_i, \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}. \tag{2.3}$$

It follows that, for any $v \in \Lambda_p$,

$$(\partial v)^{j_0 \cdots j_{p-1}} = \sum_{k \in V} \sum_{q=0}^{p} (-1)^q \, v^{j_0 \cdots j_{q-1} k \, j_q \cdots j_{p-1}}. \tag{2.4}$$

For example, for any $u \in \Lambda_0$ and $v \in \Lambda_1$ we have

$$\partial u = \sum_{k \in V} u^k \quad \text{and} \quad (\partial v)^i = \sum_{k \in V} \left(v^{ki} - v^{ik}\right).$$

Set also $\Lambda_{-2} = \{0\}$ and define $\partial : \Lambda_{-1} \to \Lambda_{-2}$ to be zero.

Lemma 2.1 *We have $\partial^2 = 0$.*

For all $p, q \geq -1$ and for any two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ define their *join* $uv \in \Lambda_{p+q+1}$ as follows:

$$(uv)^{i_0 \ldots i_p j_0 \ldots j_q} = u^{i_0 \ldots i_p} v^{j_0 \ldots j_q}. \tag{2.5}$$

Clearly, join of paths is a bilinear operation that satisfies the associative law (but is not commutative). It follows from (2.5) that

$$e_{i_0...i_p} e_{j_0...j_q} = e_{i_0...i_p j_0...j_q}. \tag{2.6}$$

If $p = -2$ and $q \geq -1$ then set $uv = 0 \in \Lambda_{q-1}$. A similar rule applies if $q = -2$ and $p \geq -1$.

Lemma 2.2 (Product rule) *For all $p, q \geq -1$ and $u \in \Lambda_p$, $v \in \Lambda_q$ we have*

$$\partial (uv) = (\partial u)v + (-1)^{p+1} u\partial v. \tag{2.7}$$

We say that an elementary path $i_0...i_p$ is *non-regular* if $i_{k-1} = i_k$ for some $k = 1, ..., p$, and *regular* otherwise. For example, a 1-path ii is non-regular, while a 2-path iji is regular provided $i \neq j$. For any $p \geq -1$, consider the following subspace of Λ_p spanned by the regular elementary paths:

$$\mathcal{R}_p = \mathcal{R}_p (V) := \text{span} \left\{ e_{i_0...i_p} : i_0...i_p \text{ is regular} \right\}.$$

Note that $\mathcal{R}_p = \Lambda_p$ for $p \leq 0$ but $\mathcal{R}_p$ is strictly smaller than Λ_p for $p \geq 1$. The elements of $\mathcal{R}_p$ are called *regular* p-paths.

We would like to consider the operator ∂ on the spaces $\mathcal{R}_p$. However, ∂ is not invariant on spaces of regular paths. For example, $e_{iji} \in \mathcal{R}_2$ for $i \neq j$ while its boundary $\partial e_{iji} = e_{ji} - e_{ii} + e_{ij}$ is not in $\mathcal{R}_1$ as it has a non-regular component e_{ii}. The same applies to the notion of join of paths: the join of two regular path does not have to be regular, for example, $e_i e_i = e_{ii}$.

However, it is easy to define a *regular* boundary operator ∂ and a *regular* join that are invariant on the spaces $\mathcal{R}_p$: if after applying ∂ or join the outcome contains non-regular terms then all these terms should be discarded. A careful definition requires taking quotient over a space of non-regular paths, but we omit the obvious details. For example, we have for the non-regular operator ∂

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij},$$

whereas for the regular operator ∂

$$\partial e_{iji} = e_{ji} + e_{ij}$$

since e_{ii} is non-regular and, hence, is replaced by 0. For non-regular join we have $e_{ij} e_{ji} = e_{ijji}$ whereas for the regular join $e_{ij} e_{ji} = 0$ since e_{ijji} is non-regular.

One can show that the regular versions of ∂ and join also satisfy $\partial^2 = 0$ and the product rule (2.7), for all $u \in \mathcal{R}_p$ and $v \in \mathcal{R}_q$.

3 Path complexes

Definition 3.1 A *path complex* over a set V is a non-empty collection P of elementary paths on V with the following property: for any $n \geq 0$,

$$\text{if } i_0...i_n \in P \text{ then also the truncated paths } i_0...i_{n-1} \text{ and } i_1...i_n \text{ belong to } P. \tag{3.1}$$

The set of all n-paths from P is denoted by P_n. When a path complex P is fixed, all the paths from P are called *allowed*, whereas all the elementary paths that are not in P are called *non-allowed.*

The set P_{-1} consists of a single empty path e. The elements of P_0 (that is, allowed 0-paths) are called the *vertices* of P. Clearly, P_0 is a subset of V. By the property (3.1), if $i_0...i_n \in P$ then all i_k are vertices. Hence, we can (and will) remove from the set V all non-vertices so that $V = P_0$. The elements of P_1 (that is, allowed 1-paths) are called *edges* of P. By (3.1), if $i_0...i_n \in P$ then all 1-paths $i_{k-1} i_k$ are edges.

Example 3.2 By definition, an abstract finite simplicial complex S is a collection of subsets of a finite vertex set V that satisfies the following property:

$$\text{if } \sigma \in S \text{ then any subset of } \sigma \text{ is also in } S.$$

Let us enumerate the elements of V by distinct reals and identify any subset s of V with the elementary path that consists of the elements of s put in the (strictly) increasing order. Hence, we can regard S as a collection of elementary paths on V. Then the defining property of a simplex can be restated the following:

$$\text{if an elementary path belongs to } S \text{ then its any subsequence also belongs to } S. \tag{3.2}$$

Consequently, the family S satisfies the property (3.1) so that S is a path complex. The allowed n-paths in S are exactly the n-simplexes.

For example, a simplicial complex on Fig. 1(left) has the following path complex:

0-paths: $0, 1, ..., 8$

1-paths: $01, 02, 03, 04, 05, 06, 07, 08, 12, 34, 35, 45, 67, 68, 78$

2-paths: $012, 678, 034, 035, 045, 678$

3-paths: 0345.

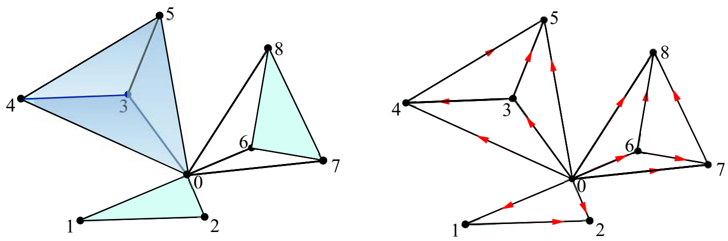


Figure 1: A simplicial complex (left) and a digraph (right)

Example 3.3 Let $G = (V, E)$ be a finite digraph, where V is a finite set of vertices and E is the set of directed edges, that is, $E \subset V \times V$. The fact that $(i, j) \in E$ will also be denoted by $i \to j$.

An elementary n-path $i_0...i_n$ on V is called allowed if $i_{k-1} \to i_k$ for any $k = 1, ..., n$. Denote by $P_n = P_n(G)$ the set of all allowed n-paths. In particular, we have $P_0 = V$ and $P_1 = E$. Clearly, the collection $\{P_n\}$ of all allowed paths satisfies the condition (3.1) so that $\{P_n\}$ is a path complex. This path complex is naturally associated with the digraph G and will be denoted by $P(G)$.

For example, a digraph on Fig. 1(right) has the following path complex:

0-paths: $0, 1, ..., 8$

1-paths: $01, 02, 03, 04, 05, 06, 07, 08, 12, 34, 35, 45, 67, 68, 78$

2-paths: $012, 678, 034, 035, 045, 067, 068, 678$

3-paths: $0345, 0678$.

It is easy to see that a path complex arises from a digraph if and only if it satisfies the following additional condition: if in a path $i_0...i_n$ all pairs $i_{k-1}i_k$ are allowed then the whole path $i_0...i_n$ is allowed.

We say that a path complex P is *perfect*, if any subsequence of any allowed elementary path of P is also an allowed path. We say that a path complex P is *monotone*, if there is an injective real-valued function on the vertex set of P that is strictly monotone increasing along any path from P. It is easy to show that a path complex P arises from a simplicial complex if and only if P is perfect and monotone.

Given an arbitrary path complex $P = \{P_n\}_{n=0}^{\infty}$ over a finite set V, consider for any integer $n \geq -1$ the $\mathbb{K}$-linear space $\mathcal{A}_n$ that is spanned by all the elementary n-paths from P, that is

$$\mathcal{A}_n = \mathcal{A}_n(P) = \left\{ \sum_{i_0,...,i_n \in V} v^{i_0...i_n} e_{i_0...i_n} : i_0...i_n \in P_n,\ v^{i_0...i_n} \in \mathbb{K} \right\}.$$

The elements of $\mathcal{A}_n$ are called *allowed* n-paths. By construction, $\mathcal{A}_n$ is a subspace of Λ_n. For example, $\mathcal{A}_p = \Lambda_p$ for $p \leq 0$, while $\mathcal{A}_1$ is spanned by all edges of P and can be smaller than Λ_1.

We would like to restrict the boundary operator ∂ on the spaces Λ_n to the spaces $\mathcal{A}_n$. For some path complexes it can happen that $\partial \mathcal{A}_n \subset \mathcal{A}_{n-1}$, so that the restriction is straightforward. If it is not the case then an additional construction is needed as will be explained below. The inclusion $\partial \mathcal{A}_n \subset \mathcal{A}_{n-1}$ takes place, for example, for perfect path complexes. In this case we obtain a chain complex

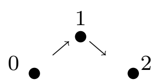
$$0 \leftarrow \mathbb{K} \leftarrow \mathcal{A}_0 \leftarrow ... \leftarrow \mathcal{A}_{n-1} \leftarrow \mathcal{A}_n \leftarrow ... \tag{3.3}$$

whose homology groups are denoted by $\widetilde{H}_n(P), n \geq -1$, and are referred to as the *reduced path homologies* of P. Consider also the truncated complex

$$0 \leftarrow \mathcal{A}_0 \leftarrow ... \leftarrow \mathcal{A}_{n-1} \leftarrow \mathcal{A}_n \leftarrow ... \tag{3.4}$$

whose homology groups are denoted by $H_n(P), n \geq 0$, and are referred to as the *path homologies* of P. For example, this construction works if the path complex P arises from a simplicial complex S. Then the path homology groups of P coincide with the corresponding simplicial homology groups of S.

Now consider a general case when $\partial \mathcal{A}_n$ does not have to be a subspace of $\mathcal{A}_{n-1}$. For example, this is the case for a digraph



where the 2-path e_{012} is allowed, while $\partial e_{012} = e_{12} - e_{02} + e_{01}$ is non-allowed because e_{02} is non-allowed.

For a general path complex P and for any $n \geq -1$, consider the following subspaces of $\mathcal{A}_n$:

$$\Omega_n = \Omega_n(P) = \{v \in \mathcal{A}_n : \partial v \in \mathcal{A}_{n-1}\}. \tag{3.5}$$

Note that $\Omega_n = \mathcal{A}_n$ for $n \leq 1$ while for $n \geq 2$ the space Ω_n can be actually smaller that $\mathcal{A}_n$. We claim that always $\partial \Omega_n \subset \Omega_{n-1}$. Indeed, if $v \in \Omega_n$ then $\partial v \in \mathcal{A}_{n-1}$ and $\partial(\partial v) = 0 \in \mathcal{A}_{n-2}$ whence it follows that $\partial v \in \Omega_{n-1}$, which was to be proved.

The elements of Ω_n are called *∂-invariant* n-paths. Thus, we obtain the chain complex of ∂-invariant paths:

$$0 \leftarrow \mathbb{K} \leftarrow \Omega_0 \leftarrow ... \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow ... \tag{3.6}$$

where all arrows are given by ∂. Consider also its *truncated* version

$$0 \leftarrow \Omega_0 \leftarrow ... \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow ... \tag{3.7}$$

Homology groups of (3.7) are referred to as the *path homology groups* of the path complex P and are denoted by $H_n(P), n \geq 0$. The homology groups of (3.6) are called the *reduced path homology groups* of P and are denoted by $\widetilde{H}_n(P), n \geq -1$.

A path complex P is called *regular* if it contains no 1-path of the form ii. Equivalently, P is regular if all the paths $i_0...i_n \in P$ are regular. For example, the path complex of a simplicial complex is always regular. The path complex of a digraph is regular if and only if the digraph is loopless, that is, if the 1-paths ii are not edges.

For a regular path complex the above construction of the spaces Ω_n allows the following variation. As the space $\mathcal{A}_n$ of allowed n-path is in this case a subspace of the space $\mathcal{R}_n$ of regular n-paths, we can replace in (3.5) a non-regular boundary operator ∂ on Λ_n by a regular boundary operator on $\mathcal{R}_n$ as described in Section 2. The resulting space Ω_n is referred to as a *regular* space of ∂-invariant paths. Hence, if the path complex P is regular then we can consider also regular versions of the chain complexes (3.6) and (3.7) and the regular versions of homology groups.

If the path complex P is perfect then we obtain $\Omega_n(P) = \mathcal{A}_n(P)$ for all n (in this case there is no difference between regular and non-regular versions). Hence, in this case the chain complex (3.6) is identical to (3.3), and (3.7) is identical to (3.4).

If $P(G)$ is the path complex of a digraph G then we use the notation $\Omega_n(G) := \Omega_n(P(G))$. The corresponding homology groups are denoted by $H_n(G), \widetilde{H}_n(G)$ and are referred to as the *path homologies of the digraph G.*

The Euler characteristic of the path complex is defined by

$$\chi(P) = \sum_{p=0}^{n} (-1)^p \dim H_p(P) \tag{3.8}$$

provided n is so big that $\dim H_p(P) = 0$ for all $p > n$. For a regular path complex P there is a regular and non-regular versions of $\chi(P)$ that do not have to match.

Let us state some simple properties of the space $\Omega_n(P)$ and $H_n(P)$.

Proposition 3.4 *(a) If $\dim \Omega_n = 0$ then $\dim \Omega_p = 0$ for all $p > n$.*

(b) If the spaces $\Omega_\bullet$ are regular then $\dim \Omega_n \leq 1$ implies that $\dim \Omega_p = 0$ for all $p > n$.

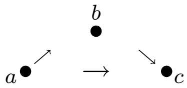
Proposition 3.5 *For any path complex P we have $\dim H_0(P) = C$, where C is the number of connected components*[1] *of P. In particular, if P is connected then $\dim H_0(P) = 1$ and, hence, $\dim \widetilde{H}_0(P) = 0$.*

4 ∂-invariant paths on digraphs

In this section, we fix a digraph $G = (V, E)$ without loops, so that its path complex $P(G)$ is regular. We deal here with the regular spaces $\Omega_n(G) = \Omega_n(P(G))$ and regular homology groups $H_n(G) = H_n(P(G))$ and $\widetilde{H}_n(G) = \widetilde{H}_n(P(G))$.

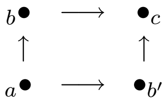
[1] A *connected component* of P is any minimal subset U of V that if $i \in U$ then U contains any vertex $j \in V$ such that ij or ji is an allowed 1-path.

Let us call by a *triangle* a sequence of three distinct vertices $a, b, c \in V$ such that $a \to b, b \to c, a \to c$:



Note that a triangle determines a 2-path $e_{abc} \in \Omega_2$ as $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$.

Let us called by a *square* a sequence of four distinct vertices $a, b, b', c \in V$ such that $a \to b, b \to c, a \to b', b' \to c$:



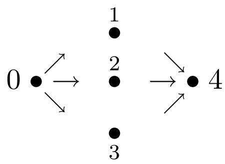
Note that a square determines a 2-path $v := e_{abc} - e_{ab'c} \in \Omega_2$ as $v \in \mathcal{A}_2$ and

$$\partial v = (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) = e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1.$$

Proposition 4.1 *Assume that a digraph $G = (V, E)$ contains no squares (as subgraphs). Then* $\dim \Omega_2 (G)$ *is equal to the number of distinct triangles in G, and* $\dim \Omega_p (G) = 0$ *for all $p > 2$. In particular, if G contains neither triangle nor square then* $\dim \Omega_p (G) = \dim H_p (G) = 0$ *for all $p \geq 2$.*

In the presence of squares one cannot relate directly $\dim \Omega_2$ to the number of squares and triangles since there may be a linear dependence between them as in the next example.

Example 4.2 In the following digraph



there are three squares $0, 1, 2, 4$, $0, 1, 3, 4$, and $0, 2, 3, 4$, which determine three ∂-invariant paths

$$e_{014} - e_{024}, \quad e_{024} - e_{034}, \quad e_{034} - e_{014}.$$

These paths are linearly dependent as their sum is equal to 0. It is easy to see that $\dim \Omega_2 = 2$. For this digraph all homologies are trivial.

Also, in the presence of squares one may have non-trivial Ω_p for arbitrary p as one can see from numerous examples in the subsequent sections.

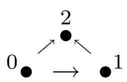
A *snake* of length p is a digraph with $p + 1$ vertices, say $0, 1, ..., p$, and with the edges $i\,(i+1)$ and $i\,(i+2)$ (see Fig. 2). In particular, any triple $i\,(i+1)\,(i+2)$ is a triangle.

A snake of length p contains a ∂-invariant p-path $v = e_{01\ldots p}$. Indeed, this path is obviously allowed, its boundary

$$\partial v = \sum_{k=0}^{p} (-1)^k e_{0\ldots\widehat{k}\ldots p}$$

is also allowed (because $(k-1)\,(k+1)$ is an edge), whence $v \in \Omega_p$.

Let us define for any $n \geq 0$ a *simplex-digraph* Sm_n as follows: its set of vertices is $\{0, 1, ..., n\}$ and the edges are $i \to j$ for all $i < j$. For example, we have

$\mathrm{Sm}_1 = {}^0\bullet \to \bullet^1$, $\quad \mathrm{Sm}_2 =$  ,

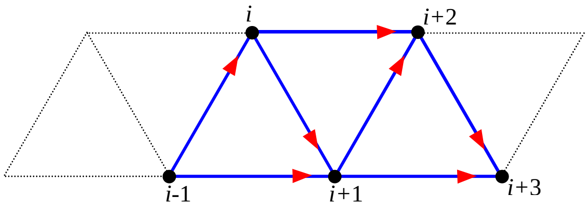


Figure 2: A snake

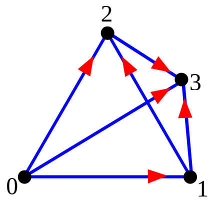


Figure 3: A 3-simplex digraph Sm_3

and Sm_3 is shown on Fig. 3.

Since a simplex contains a snake as a subgraph, the n-path $v = e_{01\ldots n}$ is ∂-invariant on Sm_n .

Definition 4.3 We say that a digraph G is *star-shaped* if there is a vertex a (called a star center) such that $a \to b$ for all $b \neq a$. Similarly, a digraph G is called inverse star-shaped if if there is a vertex a (called a star center) such that $b \to a$ for all $b \neq a$

For example, any simplex-digraph is star-shaped and inverse star-shaped.

Theorem 4.4 (A Poincaré lemma) *If G is a (inverse) star-shaped digraph, then all reduced homologies $\widetilde{H}_n(G)$ are trivial.*

For example, all reduced homologies of Sm_n are trivial.

We say that a digraph $G = (V, E)$ is a *cycle-graph* if it is connected (as an undirected graph) and every vertex had the degree 2. For a cycle-graph we have $\dim H_0(G) = 1$ and $\dim \Omega_0(G) = |V| = |E| = \dim \Omega_1(G)$.

Proposition 4.5 *Let G be a cycle-graph. Then*

$$\dim \Omega_p(G) = 0 \quad \forall p \geq 3 \quad \textit{and} \ \dim H_p(G) = 0 \ \forall p \geq 2.$$

If G is a triangle or a square then

$$\dim \Omega_2(G) = 1, \ \dim H_1(G) = 0, \quad \chi = 1$$

whereas otherwise

$$\dim \Omega_2(G) = 0, \quad \dim H_1(G) = 1, \quad \chi = 0.$$

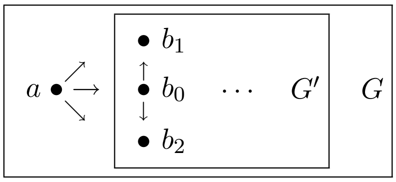
In the latter case, the spanning element of $H_1(G)$ is the 1-path σ such that

$$\sigma^{i(i+1)} = \begin{cases} 1, & \textit{if } i\,(i+1) \ \textit{is an edge} \\ -1, & \textit{if } (i+1)\,i \ \textit{is an edge,} \end{cases} \tag{4.1}$$

and all other components of σ vanish.

5 Homologies of subgraphs

Theorem 5.1 *Suppose that a digraph G has a vertex a with n outcoming edges $a \to b_0, a \to b_1, ..., a \to b_{n-1}$ and no incoming edges. Assume also that $b_0 \to b_i$ for all $i \geq 1$:*



Denote by G' the digraph that is obtained from G by removing the vertex a with all adjacent edges. Then $H_p(G) \cong H_p(G')$ for any $p \geq 0$.

The same is true if a vertex a has n incoming edges $b_0 \to a, b_1 \to a, ..., b_{n-1} \to a$ and no outcoming edges, while $b_i \to b_0$ for all $i \geq 1$.

Corollary 5.2 *Let a digraph G be a tree* (that is, the underlying undirected graph is a tree)*. Then $H_p(G) = 0$ for all $p \geq 1$.*

Example 5.3 Consider a digraph G as shown in Fig. 4.

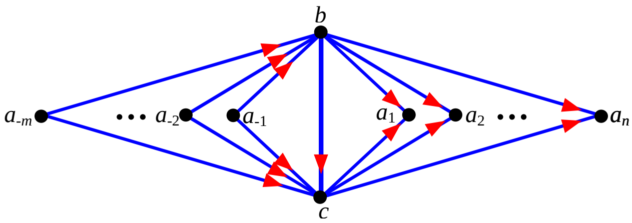
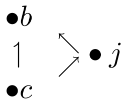


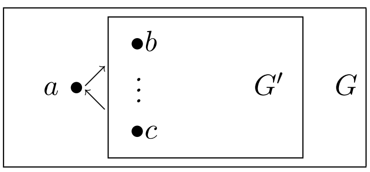
Figure 4: A digraph with many triangles and squares

Each of the vertices a_k satisfies the hypotheses of Theorem 5.1 with $n = 2$ (either with incoming or outcoming edges). Removing successively the vertices a_k, we see that all the homologies of G are the same as those of the remaining graph ${}^b\bullet \to \bullet^c$. Since it is a star-shaped graph, we obtain $\dim H_0 = 1$ and $\dim H_p = 0$ for all $p \geq 1$. In particular, $\chi = 1$.

A pair cb of distinct vertices on a graph is called a *semi-edge* if $c \not\to b$ but there is a vertex j such that $c \to j$ and $j \to b$ as on the diagram:



Theorem 5.4 *Let the field $\mathbb{K}$ has characteristic* 0*. Suppose that a graph (V, E) has a vertex a such that there is only one outcoming edge $a \to b$ from a and only one incoming edge $c \to a$, where $b \neq c$. Denote by G' the digraph that is obtained from G by removing the vertex a and the adjacent edges $a \to b, c \to a$:*



Then the following is true.

(a) *For any* $p \geq 2$*,*

$$\dim H_p(G) = \dim H_p(G'). \tag{5.1}$$

(b) *If* cb *is an edge or a semi-edge in* G' *then* (5.1) *is satisfied also for* $p = 0, 1$*, that is, for all* $p \geq 0$*.*

(c) *If* cb *is neither edge nor semi-edge in* G'*, but* b, c *belong to the same connected component of* G' *then*

$$\dim H_1(G) = \dim H_1\left(G'\right) + 1$$

and $\dim H_0(G) = \dim H_0(G')$.

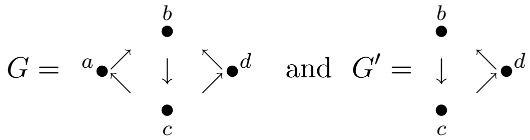
(d) *If* b, c *belong to different connected components of* G' *then*

$$\dim H_1(G) = \dim H_1(G')$$

and $\dim H_0(G) = \dim H_0(G') - 1$.

Consequently, in the case (b), $\chi(G) = \chi(G')$, *whereas in the cases* (c) *and* (d), $\chi(G) = \chi(G') - 1$.

Example 5.5 Consider the graphs



Since cb is semi-edge in G' we have case (b) so that all homologies of G and G' are the same. Removing further vertex d we obtain a digraph $^{b}\bullet \to \bullet^{c}$ that will be denoted by G''. It is a star-shaped graph with all $\dim H_p(G'') = 0$ for $p \geq 1$. Since cb is neither edge nor semi-edge in G'', but the graph is connected, we conclude by case (c) that

$$H_p\left(G'\right) = H_p\left(G''\right) \text{ for } p \geq 2,$$

and

$$\dim H_1\left(G'\right) = \dim H_1\left(G''\right) + 1 = 1.$$

It follows that $\dim H_p(G) = 0$ for $p \geq 2$ and $\dim H_1(G) = 1$.

Example 5.6 Consider a digraph as on Fig. 5 (an *anti-snake*).

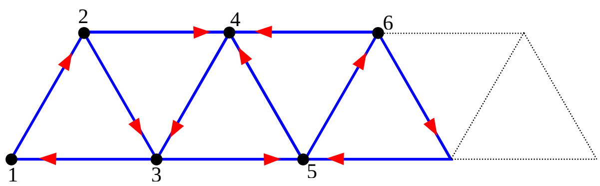


Figure 5: An anti-snake

We start building this graph with $1 \to 2$. Since 21 is neither edge nor semi-edge, adding a path $2 \to 3 \to 1$ increases $\dim H_1$ by 1 and preserves other homologies. Since 23 is an edge, adding a path $2 \to 4 \to 3$ preserves all homologies. Since 34 is neither edge nor semi-edge, adding a path $3 \to 5 \to 4$ increases $\dim H_1$ by 1 and preserves other homologies. Similarly, adding a path $5 \to 6 \to 4$ preserves all homologies.

One can repeat this pattern arbitrarily many times. By doing so we construct a digraph with a prescribed positive value of $\dim H_1$ while keeping $\dim H_p = 0$ for all $p \geq 2$. Consequently, the Euler characteristic χ can take arbitrary negative values.

Example 5.7 Consider a digraph on Fig. 1(right). By Theorem 5.1, we can remove the vertices 5 and 8 (and their adjacent edges) without change of homologies. Then by the same theorem we can remove 4 and 7. By Theorem 5.4 we can remove the vertex 1. The resulting graph with the vertices $0, 2, 3, 6$ is star-shaped, so that by Theorem 4.4 the homology groups H_p are trivial for all $p \geq 1$, while $\dim H_0 = 1$.

6 Join of path complexes

In this and next sections we use slightly different way of denoting the path spaces associated with a given path complex as we will have to consider path complexes on more than one set. Given a finite set X, denote by $P(X)$ a path complex on X. The space $\mathcal{A}_p(P(X))$ of all allowed p-paths will be denoted shortly by $\mathcal{A}_p(X)$. Similarly, the space $\Omega_p(P(X))$ of all ∂-invariant p-paths will be denoted by $\Omega_p(X)$. Similar notation will apply to all other relevant notions including path homologies $H_p(X)$, etc.

Definition 6.1 Given two disjoint finite sets X, Y and their path complexes $P(X), P(Y)$, set $Z = X \sqcup Y$ and define a path complex $P(Z)$ as follows: $P(Z)$ consists of all paths of the form uv where $u \in P(X)$ and $v \in P(Y)$. The path complex $P(Z)$ is called a *join* of $P(X), P(Y)$ and is denoted by $P(Z) = P(X) * P(Z)$.

The operation $*$ on the path complexes is obviously non-commutative but associative. An example of the path $uv \in P(Z)$ is shown on Fig. 6(left). Note that each of u, v can be empty so that all allowed paths on X and Y will also be allowed on Z.

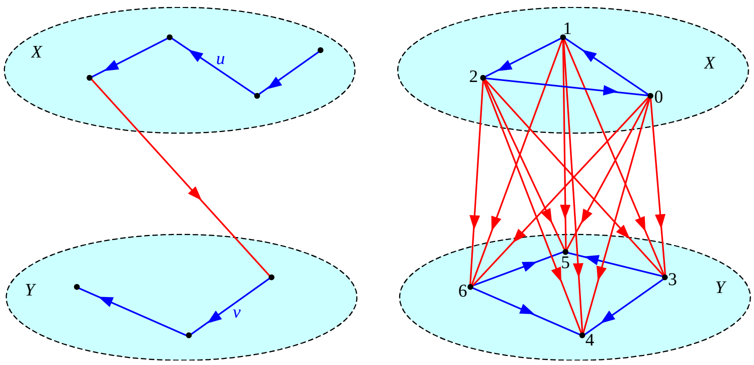


Figure 6: Join of two paths (left) and join of two digraphs (right)

Example 6.2 Let X, Y be two digraphs with disjoint set of vertices. Consider the digraph Z whose the set of vertices is $X \sqcup Y$, while the set of edges of Z consists of all the edges of X and Y, as well as of all the edges $x \to y$ for all $x \in X$ and $y \in Y$. The digraph Z is called a *join* of X and Y and is denoted by $X * Y$. An example of a join of two digraphs is shown on Fig. 6(right).

Let $P(Z)$ be the path complex arising from the digraph structure of Z. Then it is obvious from the definition that $P(Z)$ is the join of $P(X)$ and $P(Y)$ so that $P(X*Y) = P(X)*P(Y)$. Hence, the operation of joining of digraphs is compatible with the operation of joining path complexes.

Example 6.3 Let X and Y be the vertex sets of finite simplicial complexes $S(X)$ and $S(Y)$. Let us construct a simplicial complex $S(Z)$ with the vertex set $Z = X \sqcup Y$ as follows. Assuming that $|X| = n$ and $|Y| = m$, embed the set X (together with all simplexes from $S(X)$) into a hyperplane $h^{n-1} \subset \mathbb{R}^{n+m-1}$ and Y – into a hyperplane $h^{m-1} \subset R^{n+m-1}$, where the hyperplanes h^{n-1}, h^{m-1} are orthogonal and non-intersecting. For any two simplexes $\sigma_1 \in S(X)$ and $\sigma_2 \in S(Y)$, define their join $\sigma_1 * \sigma_2$ as the convex hull of σ_1 and σ_2 embedded in $\mathbb{R}^{n+m-1}$ as above (see Fig. 7).

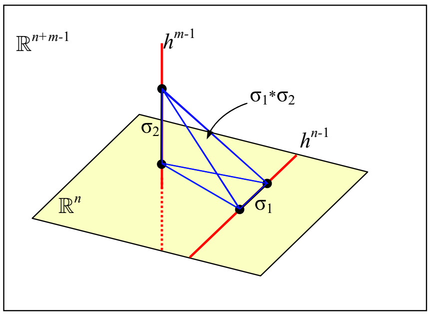


Figure 7: A join $\sigma_1 * \sigma_2$ of two one-dimensional simplexes σ_1, σ_2 (case $n = m = 2$)

Due to a general position of σ_1 and σ_2, the join $\sigma_1 * \sigma_2$ is also a simplex. Then $S(Z)$ is a collection of all simplexes $\sigma_1 * \sigma_2$ with $\sigma_1 \in S(X)$ and $\sigma_2 \in S(Y)$. We refer to $S(Z)$ as a join of simplicial complexes $S(X), S(Y)$ and denote it by $S(X) * S(Y)$.

Equivalently, one can define $S(Z)$ in an abstract way without embedding into a Euclidean space. Indeed, considering simplexes as sequences of vertices, we can say that $S(Z)$ consists of all simplexes of the form $[x_0, ..., x_p, y_0, ..., y_q]$ where $[x_0, ..., x_p] \in S(X)$ and $[y_0, ..., y_q] \in S(Y)$. It is clear that $S(Z)$ is a simplicial complex as it satisfies the defining property (3.2). It is also obvious that the path complexes $P(X), P(Y), P(Z)$ of the simplicial complexes $S(X), S(Y), S(Z)$, respectively, satisfy $P(Z) = P(X) * P(Y)$. Hence, the operation of joining of simplicial complexes is compatible with the operation of joining path complexes.

Given a regular path complex $P(X)$ on a finite set X, we consider as before its regular chain complex $\{\Omega_n(X)\}_{n \geq -1}$ and its slight modification $\{\Omega'_n\}_{n \geq 0}$ where $\Omega'_n(X) \equiv \Omega_{n-1}(X)$.

Theorem 6.4 *Let X, Y be two finite non-empty sets and $P(X)$ and $P(Y)$ be regular path complexes on X and Y respectively. Set $Z = X \sqcup Y$ and consider the join path complex $P(Z) = P(X) * P(Y)$. Then we have the following isomorphism of the chain complexes:*

$$\Omega_\bullet(Z) \cong \Omega'_\bullet(X) \otimes \Omega_\bullet(Y), \tag{6.1}$$

where the mapping $\Omega'_\bullet(X) \otimes \Omega_\bullet(Y) \to \Omega_\bullet(Z)$ is given by $u \otimes v \mapsto uv$.

It follows from (6.1) that, for any $r \geq -1$,

$$\Omega_r(Z) \cong \bigoplus_{\{p,q \geq -1: p+q=r-1\}} (\Omega_p(X) \otimes \Omega_q(Y)) \tag{6.2}$$

and, for any $r \geq 0$,

$$\widetilde{H}_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r-1\}} \left(\widetilde{H}_p(X) \otimes \widetilde{H}_q(Y)\right) \tag{6.3}$$

(a Künneth formula for join).

Example 6.5 Consider the digraph $Z = X * Y$ as on Fig. 6(right). In this case we have by Proposition 4.5 that all homologies $\widetilde{H}_p(X)$ and $\widetilde{H}_q(Y)$ are trivial except for

$$\begin{aligned} H_1(X) &= \operatorname{span}\{e_{01} + e_{12} + e_{20}\}, \\ H_1(Y) &= \operatorname{span}\{e_{35} - e_{65} + e_{64} - e_{34}\}. \end{aligned}$$

Therefore, all $\widetilde{H}_r(Z)$ are trivial except for $H_3(Z)$ that is generated by a single element

$$e_{0135} - e_{0165} + e_{0164} - e_{0134} + e_{1235} - e_{1265} + e_{1264} - e_{1234} + e_{2035} - e_{2065} + e_{2064} - e_{2034}.$$

A *cone* over a digraph X is a digraph $\operatorname{Cone} X$ that is obtained from X by adding one more vertex a and all the edges of the form $b \to a$ for all $b \in X$. The vertex a is called the cone vertex. Clearly, we have $\operatorname{Cone} X = X * Y$ where Y consists of a single vertex a.

Proposition 6.6 *For any digraph X, we have for any $r \geq 0$*

$$\Omega_r(\operatorname{Cone} X) \cong \Omega_{r-1}(X), \tag{6.4}$$

where the isomorphism is given by the mapping $u \mapsto ue_a$ from $\Omega_{r-1}(X)$ to $\Omega_r(\operatorname{Cone} X)$ where a is the cone vertex. Furthermore, all the reduced homologies of $\operatorname{Cone} X$ *are trivial.*

Example 6.7 Clearly, a simplex-digraph Sm_n can be regarded as a cone over Sm_{n-1} (cf. Section 4). Since $\Omega_0(\mathrm{Sm}_0)$ is spanned by a 0-path e_0, we obtain by induction from (6.4) that $\Omega_n(\mathrm{Sm}_n)$ is spanned by a path $e_{01\ldots n}$.

A *suspension* over a digraph X is a digraph $\operatorname{Sus} X$ that is obtained from X by adding two vertices a, b and all the edges $c \to a$ and $c \to b$ for all $c \in X$. The vertices a, b are called the suspension vertices. Clearly, we have $\operatorname{Sus} X = X * Y$ where Y is a digraph that consists of two vertices a, b and no edges.

Theorem 6.8 *For any digraph X we have, for any $r \geq 0$,*

$$\Omega_r(\operatorname{Sus} X) \cong \Omega_{r-1}(X) \otimes \operatorname{span}\{e_a, e_b\}, \tag{6.5}$$

where a, b are the suspension vertices and the isomorphism is given by the mappings $u \otimes e_a \mapsto ue_a$ and $u \otimes e_b \mapsto ue_b$. Furthermore, we have

$$\widetilde{H}_r(\operatorname{Sus} X) \cong \widetilde{H}_{r-1}(X), \tag{6.6}$$

where the isomorphism is given by the mapping $u \mapsto u(e_a - e_b)$. Consequently, we have $\chi(\operatorname{Sus} X) = 2 - \chi(X)$.

In particular, having examples of digraphs X with arbitrary negative values of χ (cf. Example 5.6), we obtain examples of digraphs $\operatorname{Sus} X$ with arbitrary positive values of χ.

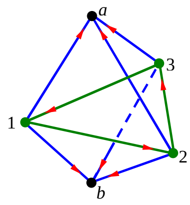


Figure 8: A graph S_2 based on a 3-vertex cycle S

Example 6.9 Let S be any cycle-graph that is neither triangle nor square. We regards S as an analog of a circle. Define S_n inductively by $S_1 = S$ and $S_{n+1} = \operatorname{Sus} S_n$. Then S_n can be regarded as n-dimensional sphere-graph. An example of S_2 is shown on Fig. 8.

Since $\chi(S) = 0$ by Proposition 4.5, it follows that $\chi(S_n) = 0$ if n is odd and $\chi(S_n) = 2$ if n is even. Theorem 6.8 also implies that $\dim H_n(S_n) = \dim H_1(S) = 1$, which gives an example of a non-trivial H_n with an arbitrary n.

Let v be an 1-path on S that spans $H_1(S)$ (see Section 4). If S_{n+1} is a suspension of S_n on the vertices a_n, b_n then we obtain by induction that the spanning element of $H_n(S_n)$ is

$$v\left(e_{a_1} - e_{b_1}\right)\left(e_{a_2} - e_{b_2}\right) \ldots \left(e_{a_{n-1}} - e_{b_{n-1}}\right).$$

For example, if S is a cycle-graph on Fig. 8 with $V = \{1, 2, 3\}$ and $E = \{12, 23, 31\}$, then $v = e_{12} + e_{23} + e_{31}$, whence the spanning element of $H_2(S_2)$ is

$$v\left(e_a - e_b\right) = \left(e_{12a} + e_{23a} + e_{31a}\right) - \left(e_{12b} + e_{23b} + e_{31b}\right).$$

Example 6.10 Another example of a 2-dimensional sphere-graph G is shown on Fig. 9.

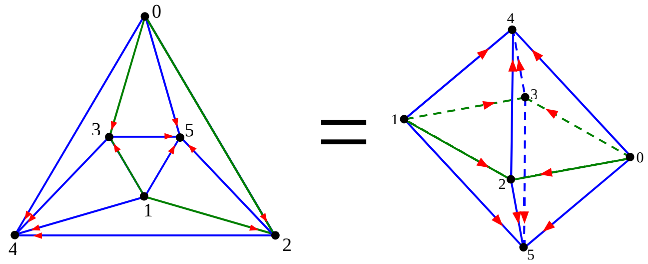


Figure 9: An octahedron digraph

Indeed, we have $G = \operatorname{Sus} G'$ where G' is the subgraph with vertices $\{0, 1, 2, 3\}$ that is a cycle-graph. Applying Proposition 4.5 to compute homology groups of G' and then Theorem 6.8, we obtain

$$\dim H_0(G) = 1, \ \dim H_1(G) = 0, \ \dim H_2(G) = 1, \ \dim H_p(G) = 0 \ \text{ for } p \geq 3. \tag{6.7}$$

Consequently, $\chi(G) = 2$. Furthermore, the spanning element of $H_2(G)$ is

$$e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}$$

that is some sense represents the surface of the octahedron.

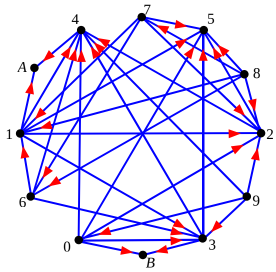


Figure 10: A digraph with 12 vertices and 32 edges.

Example 6.11 Consider a digraph on Fig. 10.

Removing successively the vertices $A, B, 8, 9, 6, 7$ by Theorem 5.1, we obtain a digraph G' with the vertex set $\{0, 1, 2, 3, 4, 5\}$ that has the same homologies as G. The digraph G' is the same as the one on Fig. 9. Hence, we obtain by (6.7) that $\dim H_2(G) = 1$ while $H_p(G) = \{0\}$ for $p = 1$ and $p > 2$. The following closed 2-path spans the non-trivial homology class in $H_2(G)$:

$$e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}.$$

In other words, this 2-path determines a 2-dimensional hole in G given by the octahedron. Note that on Fig. 10 this octahedron is hardy visible, but it can be worked out purely algebraically using the above tools.

7 Cartesian product of path complexes

In this section all path complexes are regular and their chain complexes are always regular and truncated.

Given two finite sets X, Y, consider their Cartesian product $Z = X \times Y$. Let $z = z_0 z_1 ... z_r$ be a regular elementary r-path on Z, where $z_k = (x_k, y_k)$ with $x_k \in X$ and $y_k \in Y$. We say that the path z is *step-like* if, for any $k = 1, ..., r$, either $x_{k-1} = x_k$ or $y_{k-1} = y_k$. In fact, exactly one of these conditions holds as z is regular.

Any step-like path z on Z determines regular elementary paths x on X and y on Y by projection. More precisely, x is obtained from z by taking the sequence of all X-components of the vertices of z and then by collapsing in it any subsequence of repeated vertices to one vertex. The same rule applies to y. By construction, the projections x and y are *regular* elementary paths on X and Y, respectively. If the projections of z are $x = x_0 ... x_p$ and $y = y_0 ... y_q$ then $p + q = r$ (cf. Fig. 11(left)).

Every vertex (x_i, y_j) of a step-like path z can be represented as a point (i, j) of $\mathbb{Z}^2$ so that the whole path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting the points $(0, 0)$ and (p, q). Define the *elevation* $L(z)$ of the path z as the number of cells in $\mathbb{Z}^2_+$ below the staircase $S(z)$ (the shaded area on Fig. 11(right)).

Given paths $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ with some $p, q \geq 0$, define a path $u \times v$ on Z by the following rule: for any step-like elementary $(p+q)$-path z on Z, the component $(u \times v)^z$ is defined by

$$(u \times v)^z = (-1)^{L(z)} u^x v^y, \tag{7.1}$$

where x and y are the projections of z onto X and Y, respectively, and u^x and v^y are the corresponding components of u and v. For non-step-like paths z set $(u \times v)^z = 0$. The path $u \times v$ is called the *cross product* of u and v. It follows that $u \times v \in \mathcal{R}_{p+q}(Z)$.

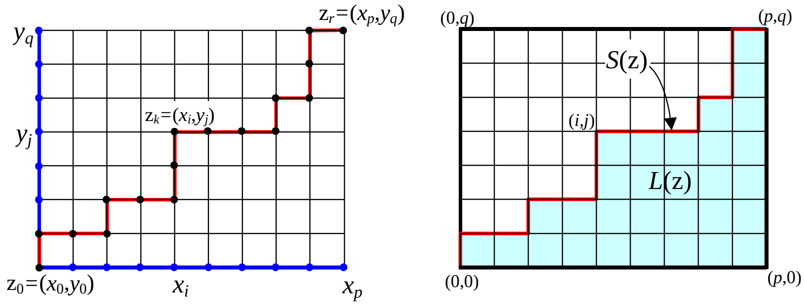


Figure 11: Left: a step-like path z and its projections x and y. Right: a staircase $S(z)$ and its elevation $L(z)$ (here $L(z) = 30$).

For given elementary regular p-path x on X and q-path y on Y, denote by $\Pi_{x,y}$ the set of all step-like paths z on Z whose projections on X and Y are x and y respectively. It follows from (7.1) that

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z. \tag{7.2}$$

It is not difficult to see that the cross product is associative.

Example 7.1 Let us denote the vertices of X by letters a, b, c etc and the vertices of Y by integers $0, 1, 2$, etc so that the vertices of Z can be denoted as the fields on the chessboard, for example, $a0, b1$ etc. Then we have

$$e_a \times e_{01} = e_{a0a1}, \quad e_{ab} \times e_0 = e_{a0b0}$$

$$e_{ab} \times e_{01} = e_{a0b0b1} - e_{a0a1b1}$$

$$e_{abc} \times e_{01} = e_{a0b0c0c1} - e_{a0b0b1c1} + e_{a0a1b1c1}$$

$$\begin{aligned} e_{abc} \times e_{012} &= e_{a0b0c0c1c2} - e_{a0b0b1c1c2} + e_{a0b0b1b2c2} \\ &\quad + e_{a0a1b1c1c2} - e_{a0a1b1b2c2} + e_{a0a1a2b2c2} \end{aligned}$$

etc (cf. Fig. 12).

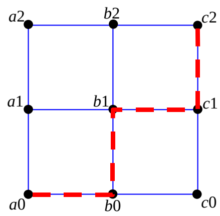


Figure 12: The staircase $a0b0b1c1c2$ has elevation 1. Hence, $e_{a0b0b1c1c2}$ enters the product $e_{abc} \times e_{012}$ with the negative sign.

Proposition 7.2 (Product rule) *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial(u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v). \tag{7.3}$$

Definition 7.3 Given two finite sets X and Y with path complexes $P(X)$ and $P(Y)$ respectively, define on the set $Z = X \times Y$ a path complex $P(Z)$ as follows: the elements of $P(Z)$ are step-like paths on Z whose projections on X and Y belong to $P(X)$ and $P(Y)$, respectively. The path complex $P(Z)$ is called the Cartesian product of the path complexes $P(X)$ and $P(Y)$ and is denoted by $P(X) \boxplus P(Y)$.

It is not difficult to see that the Cartesian product of the path complexes is associative. If x and y are elementary allowed paths on X and Y, respectively, then all the paths $z \in \Pi_{x,y}$ are allowed on Z. It clearly follows from (7.2) that the cross product of allowed paths is allowed. Furthermore, the product rule (7.3) implies that the cross product of ∂-invariant paths is ∂-invariant. The next theorem gives a complete description of ∂-invariant paths on Z.

Theorem 7.4 *Let $P(X)$ and $P(Y)$ be two regular path complexes. Then for their Cartesian product $P(Z) = P(X) \boxplus P(Y)$ the following isomorphism of chain complexes holds:*

$$\Omega_\bullet(Z) \cong \Omega_\bullet(X) \otimes \Omega_\bullet(Y) \tag{7.4}$$

where the mapping $\Omega_\bullet(X) \otimes \Omega_\bullet(Y) \to \Omega_\bullet(Z)$ is given by $u \otimes v \mapsto u \times v$.

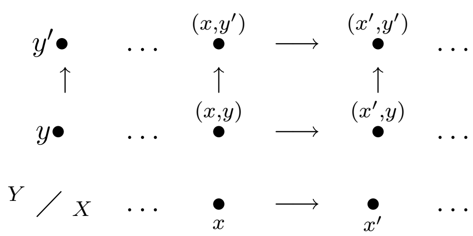
Consequently, we obtain a Künneth formula

$$H_\bullet(Z) \cong H_\bullet(X) \otimes H_\bullet(Y), \tag{7.5}$$

that is, for any $r \geq 0$,

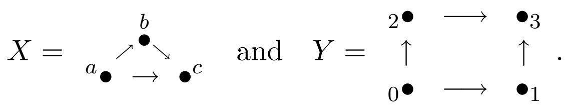
$$H_r(Z) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} \left(H_p(X) \otimes H_q(Y)\right). \tag{7.6}$$

Let X be a digraph. For simplicity of notation, we denote the set of vertices of X by the same letter X, and the set of edges by E_X. Given two digraphs X and Y, their Cartesian product is the digraph $Z = X \boxplus Y$ where the set of vertices of Z is the Cartesian product of the set of vertices of X and Y, while the set E_Z of edges is defined as follows: $(x, y) \to (x', y')$ if and only if either $x \to x'$ and $y = y'$ and $y \to y'$ and $x = x'$:



Clearly, any allowed path on Z is step-like, and its projections onto X and Y are also allowed. Hence, the path complex of the digraph Z is the Cartesian product of the path complexes of the digraphs X and Y.

Example 7.5 Let $Z = X \boxplus Y$ where



(see Fig. 13).

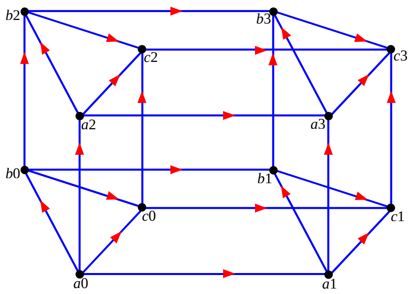


Figure 13: Cartesian product of a triangle and a square

Taking the cross product of ∂-invariant paths e_{abc} and $e_{013} - e_{023}$, we obtain the following ∂-invariant path on Z:

$$\begin{aligned}
&e_{a0b0c0c1c3} - e_{a0b0b1c1c3} + e_{a0b0b1b3c3} \\
&+e_{a0a1b1c1c3} - e_{a0a1b1b3c3} + e_{a0a1a3b3c3} \\
&-e_{a0b0c0c2c3} + e_{a0b0b2c2c3} - e_{a0b0b2b3c3} \\
&-e_{a0a2b2c2c3} + e_{a0a2b2b3c3} - e_{a0a2a3b3c3}
\end{aligned}$$

For any digraph X, the *cylinder* over X is the digraph

$$\operatorname{Cyl} X := X \boxplus {}^{0}\bullet \rightarrow \bullet^{1}.$$

Assuming that the vertices of X are enumerated by $0, 1, ..., n-1$, we can enumerate the vertices of $\operatorname{Cyl} X$ by $0, 1, ..., 2n-1$ using the following rule: $(x, 0)$ is assigned the number x, while $(x, 1)$ is assigned $x + n$.

Define the operation of *lifting* paths from X to $\operatorname{Cyl} X$ as follows: for any regular path v on X, the lifted path is denoted by $\widehat{v}$ and is defined by $\widehat{v} = v \times e_{01}$. Since e_{01} is ∂-invariant on Y, we obtain that if $v \in \Omega_p(X)$ then $\widehat{v} \in \Omega_{p+1}(\operatorname{Cyl} X)$. For example, if $v = e_{i_0 \dots i_p}$ then

$$\widehat{v} = e_{i_0 \dots i_p} \times e_{01} = \sum_{k=0}^{p} (-1)^{p-k} e_{i_0 \dots i_k (i_k+n) \dots (i_p+n)}. \tag{7.7}$$

Example 7.6 The cylinder over the graph $X = {}^{0}\bullet \rightarrow \bullet^{1}$ is a square:

$$\begin{array}{ccc}
{}^{2}\bullet & \longrightarrow & \bullet^{3} \\
\uparrow & & \uparrow \\
{}^{0}\bullet & \longrightarrow & \bullet^{1}
\end{array}$$

Lifting a ∂-invariant 1-path e_{01} on X we obtain the following ∂-invariant 2-path on the square: $e_{013} - e_{023}$. The cylinder over a square is a 3-cube that is shown in Fig. 14.

Lifting the 2-path $e_{013} - e_{023}$ by (7.7), we obtain the following ∂-invariant 3-path on the 3-cube:

$$e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}.$$

Defining further n-cube for any positive integer n by

$$\operatorname{Cube}_n = \operatorname{Cyl} \operatorname{Cube}_{n-1},$$

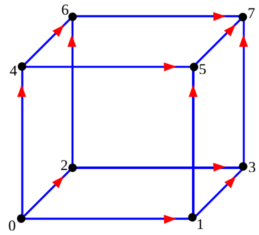


Figure 14: A 3-cube

we see that Cube_n determines a ∂-invariant n-path that is a lifting of a ∂-invariant $(n-1)$-path from Cube_{n-1} and that is an alternating sum of $n!$ elementary terms. It is easy to show that these terms correspond to partitioning of a solid n-cube into $n!$ simplexes.

By (7.6) all homology groups of Cube_n are trivial except for H_0.

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