ASYMPTOTIC STATISTICS OF CYCLES IN SURROGATE-SPATIAL PERMUTATIONS

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ABSTRACT. We propose an extension of the Ewens measure on permutations by choosing the cycle weights to be asymptotically proportional to the degree of the symmetric group. This model is primarily motivated by a natural approximation to the so-called spatial random permutations recently studied by V. Betz and D. Ueltschi (hence the name "surrogate-spatial"), but it is of substantial interest in its own right. We show that under the suitable (thermodynamic) limit both measures have the similar critical behaviour of the cycle statistics characterized by the emergence of infinitely long cycles. Moreover, using a greater analytic tractability of the surrogate-spatial model, we obtain a number of new results about the asymptotic distribution of the cycle lengths (both small and large) in the full range of subcritical, critical and supercritical domains. In particular, in the supercritical regime there is a parametric "phase transition" from the Poisson–Dirichlet limiting distribution of ordered cycles to the occurrence of a single giant cycle. Our techniques are based on the asymptotic analysis of the corresponding generating functions using Pólya's Enumeration Theorem and complex variable methods.

Keywords and phrases: Spatial random permutations; Surrogate-spatial measure; Generating functions; Pólya's Enumeration Theorem; Long cycles; Poisson–Dirichlet distribution

MSC 2010: Primary 60K35; Secondary 05A15, 82B26

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Date: September 30, 2013.

1. Introduction

1.1. **Surrogate-spatial permutations.** Let $\mathfrak{S}_N$ be the symmetric group of all permutations on elements $1,\dots,N$. For any permutation $\sigma\in\mathfrak{S}_N$, denote by $C_j=C_j(\sigma)$ the *cycle counts*, that is, the number of cycles of length $j=1,\dots,N$ in the cycle decomposition of σ; clearly

$$C_j\ge 0\quad (j\ge 1),\qquad \sum_{j=1}^{N} j\,C_j=N. \tag{1.1}$$

A probability measure on $\mathfrak{S}_N$ with (multiplicative) cycle weights may now be introduced by the expression

$$\mathbb{P}_N(\sigma):=\frac{1}{N!\,H_N}\prod_{j=1}^{N} w_j^{C_j},\qquad \sigma\in\mathfrak{S}_N, \tag{1.2}$$

where H_N is the normalization constant,

$$H_N=\frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N}\prod_{j=1}^{N} w_j^{C_j}. \tag{1.3}$$

The cycle weights w_j $(j=1,\dots,N)$ in (1.2), (1.3) may in principle also depend on the degree N. In the simplest possible case one just puts $w_j\equiv 1$, resulting in the classical *uniform distribution* on permutations dating back to Cauchy (see, e.g., [1, §1.1]). Generalization with $w_j\equiv\theta>0$ is known as the *Ewens sampling formula* $\mathrm{ESF}(\theta)$, which first emerged in the study of population dynamics in mathematical biology [12]. A class of models with variable coefficients w_j's (but independent of N) was recently studied in papers [7], [11], [25], [27] (see also an extensive background bibliography therein); more general models of assemblies allowing for mild dependence of the weights on N were considered earlier by Manstavičius (see [24, p. 68]).

The *surrogate-spatial measure* $\tilde{\mathbb{P}}_N$ proposed in the present paper is a further natural generalization specified by choosing the cycle weights in the form

$$w_j(N):=\theta_j+N\kappa_j,\qquad j\in\mathbb{N}, \tag{1.4}$$

where (θ_j) and (κ_j) are given sequences with $\theta_j \geq 0$, $\kappa_j \geq 0$ $(j \in \mathbb{N})$. Thus, the general model (1.2), (1.3) specializes to

$$\tilde{\mathbb{P}}_N(\sigma) := \frac{1}{N!\,H_N} \prod_{j=1}^{N} (\theta_j + N\kappa_j)^{C_j}, \qquad \sigma \in \mathfrak{S}_N, \tag{1.5}$$

$$H_N = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \prod_{j=1}^{N} (\theta_j + N\kappa_j)^{C_j}. \tag{1.6}$$

1.2. **Spatial permutations.** Our original interest to the model (1.5), (1.6) (which also explains the proposed name "surrogate-spatial") was generated by the so-called *spatial random permutations* recently studied by Betz and Ueltschi [5], [6], defined by the family of probability measures

$$\mathbb{P}_{N,L}(\sigma) := \frac{1}{N!\,H_{N,L}} \prod_{j=1}^{N} \left(\mathrm{e}^{-\alpha_j} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \mathrm{e}^{-j\,\varepsilon(\boldsymbol{k}/L)} \right)^{C_j}, \qquad \sigma \in \mathfrak{S}_N, \tag{1.7}$$

where (α_j) is a real sequence, $L > 0$ is an additional "spatial" parameter, $\varepsilon\colon \mathbb{R}^d \to [0, \infty)$ is a certain function, and $H_{N,L}$ is the corresponding normalization factor.

The hidden spatial structure of the measure (1.7) is revealed by the fact that $\mathbb{P}_{N,L}$ emerges as the $\mathfrak{S}_N$-marginal of a suitable probability measure on a bigger space $\varLambda^N \times \mathfrak{S}_N$, where $\varLambda := [-\frac{1}{2}L, \frac{1}{2}L]^d \subset \mathbb{R}^d$; namely (cf. [6, Eq. (3.6), p. 1179])

$$\mathbb{P}_{N,L}(\sigma) = \frac{1}{N!\,H_{N,L}} \int_{\varLambda^N} \mathrm{e}^{-\mathcal{H}_N(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N;\,\sigma)}\,\mathrm{d}\boldsymbol{x}_1 \dots \mathrm{d}\boldsymbol{x}_N, \qquad \sigma \in \mathfrak{S}_N, \tag{1.8}$$

$$\mathcal{H}_N(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N; \sigma) := \sum_{j=1}^{N} V(\boldsymbol{x}_j - \boldsymbol{x}_{\sigma(j)}) + \sum_{j=1}^{N} \alpha_j C_j\,, \tag{1.9}$$

where the interaction potential $V\colon \mathbb{R}^d \to (-\infty, +\infty]$ is such that the function $\mathrm{e}^{-V(\boldsymbol{x})}$ is continuous, has *positive Fourier transform* (which implies that $V(-\boldsymbol{x}) = V(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^d$), and

$$\int_{\mathbb{R}^d} \mathrm{e}^{-V(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} = 1,$$

so that the function $f(\boldsymbol{x}) := \mathrm{e}^{-V(\boldsymbol{x})}$ can be interpreted as a probability density on $\mathbb{R}^d$.

The basic physical example is the Gaussian case with a quadratic potential (see [6, p. 1175])

$$V(\boldsymbol{x}) = \frac{1}{4\beta}\,\|\boldsymbol{x}\|^2 + c_1(\beta) d, \qquad \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d, \tag{1.10}$$

where $\|\boldsymbol{x}\| := \left(\sum_{i=1}^d x_i^2\right)^{1/2}$ is the usual (Euclidean) norm in $\mathbb{R}^d$, $\beta > 0$ is the inverse temperature, and $c_1(\beta)$ is a suitable normalization constant. According to formula (1.9), particles in a random spatial configuration $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ interact with one another via the spatial potential V only along cycles of an auxiliary permutation $\sigma \in \mathfrak{S}_N$ (see Fig. 1), whereby the existence of a cycle of length $j \in \mathbb{N}$ is either promoted or penalized depending on whether $\alpha_j < 0$ or $\alpha_j > 0$, respectively.

The link between formulas (1.7) and (1.8) is provided by the function $\varepsilon(\boldsymbol{s})$ defined by the Fourier transform

$$\mathrm{e}^{-\varepsilon(\boldsymbol{s})} = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}(\boldsymbol{x},\boldsymbol{s})}\,\mathrm{e}^{-V(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{s} \in \mathbb{R}^d, \tag{1.11}$$

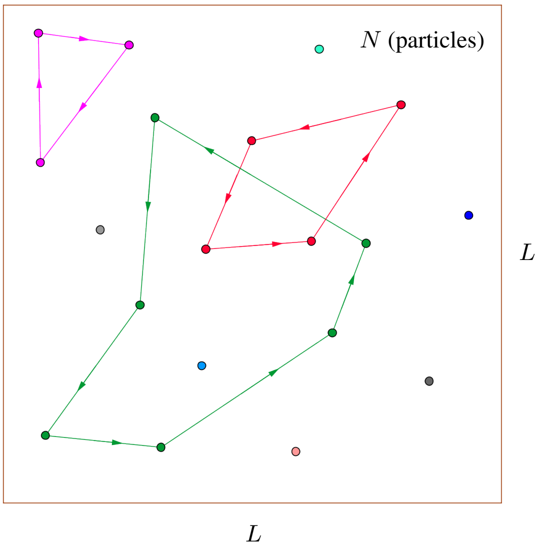

FIGURE 1. Illustration of spatial permutations.

where $(\boldsymbol{x}, \boldsymbol{s})$ denotes the inner product in $\mathbb{R}^d$. For example, the Gaussian potential (1.10) leads to a quadratic function $\varepsilon(\boldsymbol{s}) = c\|\boldsymbol{s}\|^2$ (with $c = 4\pi^2\beta$). From the assumptions on $V(\cdot)$, it readily follows that $\varepsilon(\boldsymbol{0}) = 0$, $\varepsilon(-\boldsymbol{s}) = \varepsilon(\boldsymbol{s})$ ($\boldsymbol{s} \in \mathbb{R}^d$), $\varepsilon(\boldsymbol{s}) > 0$ ($\boldsymbol{s} \neq \boldsymbol{0}$) and, by the Riemann–Lebesgue lemma, $\lim_{\boldsymbol{s}\to\infty} \varepsilon(\boldsymbol{s}) = \infty$. Since the Fourier transform (1.11) is positive, a simple lemma (see [9, Theorem 9, p. 20], [17, Lemma 7.2.1, p. 162] yields that

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon(\boldsymbol{s})}\,\mathrm{d}\boldsymbol{s} < \infty, \tag{1.12}$$

hence the Fourier inversion formula implies the dual relation

$$\mathrm{e}^{-V(\boldsymbol{x})} = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{x},\boldsymbol{s})}\,\mathrm{e}^{-\varepsilon(\boldsymbol{s})}\,\mathrm{d}\boldsymbol{s}, \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

Finally, let us assume that the function $\varepsilon(\boldsymbol{s})$ is regular enough as $\boldsymbol{s} \to \infty$, so that the integrability condition (1.12) implies the convergence (for any $L > 0$) of the series

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-\varepsilon(\boldsymbol{k}/L)} < \infty. \tag{1.13}$$

For the latter, it is sufficient that, for $\boldsymbol{s}$ large enough, $\varepsilon(\boldsymbol{s}) = \varepsilon(s_1, \dots, s_d)$ is non-decreasing in each of the variables s_i, or that there is a lower bound $\varepsilon(\boldsymbol{s}) \geq C_1 + \gamma \log \|\boldsymbol{s}\|$, with some $\gamma > d$. Importantly, condition (1.13) ensures that the spatial measure (1.7) is well defined.

The model (1.8), (1.9) is motivated by the Feynman–Kac representation of the dilute Bose gas (at least in the Gaussian case), and it has been proposed in connection with the study of the *Bose–Einstein condensation* (for more details and the background, see [3], [4], [5] and further references therein). An important question in this context, which is also interesting from the combinatorial point of view, is the possible emergence of an infinite cycle under the *thermodynamic limit*, that is, by letting $N, L \to \infty$ while keeping the density $\rho := NL^{-d}$

fixed. The (expected) fraction of points contained in infinitely long cycles can be defined as

$$\nu := \lim_{K\to\infty} \liminf_{N\to\infty} \frac{1}{N}\, \mathbb{E}_{N,L}\Biggl(\sum_{j>K} j\, C_j\Biggr). \tag{1.14}$$

Betz and Ueltschi have shown in [6] that in the model (1.7), under certain assumptions on the coefficients α_j's, the quantity ν is identified explicitly as

$$\nu = \max\left\{0,\, 1-\frac{\rho_{\rm c}}{\rho}\right\}, \tag{1.15}$$

where $\rho_{\rm c}$ is the *critical density* given by

$$\rho_{\rm c} := \sum_{j=1}^{\infty} \mathrm{e}^{-\alpha_j} \int_{\mathbb{R}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\, \mathrm{d}\boldsymbol{s} \le +\infty. \tag{1.16}$$

That is to say, infinite cycles emerge (in the thermodynamic limit) when the density ρ is greater than the critical density $\rho_{\rm c}$ (see further details in [6]).

However, the computations in [6] for the original spatial measure $\mathbb{P}_{N,L}$ are quite complicated, and it may not be entirely clear as to why the asymptotic behaviour of cycles is drastically different for $\rho < \rho_{\rm c}$ and $\rho > \rho_{\rm c}$ (even though intuition does suggest such a phase transition; see a heuristic explanation in [6, p. 1175]).

1.3. **Surrogate-spatial model as an approximation of the spatial model.** A simple observation that has motivated the present work is that the sum in (1.7) can be viewed, for each fixed j, as a Riemann sum (with mesh size $1/L$) for the corresponding integral appearing in (1.16). Using Euler–Maclaurin's (multidimensional) summation formula (see, e.g., [8, §A.4]) and recalling that $\rho = NL^{-d}$, this suggests the following plausible approximation of the cycle weights in formula (1.7),

$$\mathrm{e}^{-\alpha_j} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)} = N\kappa_j + \theta_j + o(1), \qquad N, L\to\infty, \tag{1.17}$$

with the coefficients

$$\kappa_j = \rho^{-1}\mathrm{e}^{-\alpha_j} \int_{\mathbb{R}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\, \mathrm{d}\boldsymbol{s}, \qquad j\in\mathbb{N}. \tag{1.18}$$

(Note that, due to the condition (1.12), the integral in (1.18) is finite for all $j\in\mathbb{N}$.) Thus, neglecting the o-terms in (1.17), we arrive at the surrogate-spatial model (1.4).

The ansatz (1.17) demands a few comments. In the Gaussian case with $\varepsilon(\boldsymbol{s}) = c\|\boldsymbol{s}\|^2$ it can be checked (e.g., with the help of Euler–Maclaurin's summation formula) that the expansion (1.17) holds true for any fixed $j\in\mathbb{N}$ with $\kappa_j \propto \mathrm{e}^{-\alpha_j} j^{-d/2}$ *and* $\theta_j\equiv 0$ (see more details in Section 6.1 below). In general, however, it may not be obvious that the leading correction to the principal term in (1.17) is necessarily *constant in* N.

More importantly, the expansion (1.17) with the integral coefficients (1.18) may fail to be adequate if the index $j\le N$ grows fast enough with N. For instance, again assuming the Gaussian case, for the sum in (1.17) with $j=N$ we have in dimension $d\ge 3$

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-Nc\|\boldsymbol{k}/L\|^2} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \exp\left\{-c\rho^{2/d}N^{1-2/d}\|\boldsymbol{k}\|^2\right\} \to 1, \qquad N\to\infty,$$

whereas the integral counterpart (see (1.18)) is asymptotic to $O(N^{1-2/d}) = o(1)$. This suggests that for j close to N the relation (1.17) holds with the universal (potential-free) coefficients $\kappa_j = 0,\ \theta_j = \mathrm{e}^{-\alpha_j}$. That is to say, the (Gaussian) spatial model $\mathbb{P}_{N,L}$ dynamically interpolates (up to asymptotically small correction terms) between the surrogate-spatial model $\tilde{\mathbb{P}}_N$ with $\kappa_j > 0$, $\theta_j = 0$ (for small j) and the one with $\kappa_j = 0$, $\theta_j > 0$ (for large j).

We postpone a detailed comparison of the models (1.4) and (1.7) until Section 6; for now let us just point out that both models, after a suitable calibration, share the same critical density ρ_{c} and the expected fraction of points in infinite cycles (see (1.16)); there are also qualitative similarities in the transition from the Poisson–Dirichlet distribution of large cycles to a single giant cycle (see Section 5).

1.4. **Synopsis and layout.** As will be demonstrated below, the class of surrogate-spatial measures (1.5), (1.6) produces a rich picture of the asymptotic statistics of permutation cycles as $N \to \infty$, being at the same time reasonably tractable analytically; in particular, the different asymptotic behaviour of cycles below and above the critical point can be easily understood. Furthermore, the surrogate-spatial model enables one to study in detail the asymptotics of the cycle counts C_j and of the total number of cycles $T_N = \sum_{j=1}^{N} C_j$, including the critical case (e.g., a surprising result there is that T_N may not follow a central limit theorem; see Theorem 4.4). We will also show in Section 5 that the correction terms θ_j's in (1.17) may have a significant influence on the statistics of long cycles. Thus, despite the lack of direct physical relevance of the surrogate-spatial model, it can be used as an efficient exploratory tool that may prove useful in the analysis of more complicated spatial models.

The rest of the paper is organized as follows. Section 2 contains the necessary preliminaries concerning certain generating functions, including a basic identity deriving from Pólya's Enumeration Theorem. In Section 3, with the help of complex analysis we prove some basic theorems enabling us to compute the asymptotics of (the coefficients of) the corresponding generating functions. In Section 4, we apply these techniques to study the cycle counts, the total number of cycles and also the asymptotics of lexicographically ordered cycles. In Section 5, we study the asymptotic statistics of long cycles. Finally, we compare our surrogate-spatial model with the original spatial model in Section 6.

2. Preliminaries

2.1. **Generating functions.** We use the standard notation $\mathbb{Z}$ and $\mathbb{N}$ for the sets of integer and natural numbers, respectively, and also denote $\mathbb{N}_0 := \{j \in \mathbb{Z} : j \ge 0\} = \{0\} \cup \mathbb{N}$.

For a sequence of complex numbers $(a_j)_{j\ge 0}$, its (ordinary) generating function is defined as the (formal) power series

$$g(z) := \sum_{j=0}^{\infty} a_j z^j, \qquad z \in \mathbb{C}. \tag{2.1}$$

As usual [14, §I.1, p. 19], we define the *extraction symbol* $[z^j]\, g(z) := a_j$, that is, as the coefficient of z^j in the power series expansion (2.1) of $g(z)$.

The following simple lemma known as *Pringsheim's Theorem* (see, e.g., [14, Theorem IV.6, p. 240]) is important in asymptotic enumeration where generating functions with non-negative coefficients are usually involved.

Lemma 2.1. *Assume that $a_j \ge 0$ for all $j \ge 0$, and let the series expansion* (2.1) *have a finite radius of convergence R. Then the point $z = R$ is a singularity of the function $g(z)$.*

Two special generating functions constructed with the coefficients (θ_j) and (κ_j), respectively, play a crucial role in this paper,

$$g_\theta(z) := \sum_{j=1}^{\infty} \frac{\theta_j}{j}\, z^j, \qquad g_\kappa(z) := \sum_{j=1}^{\infty} \frac{\kappa_j}{j}\, z^j. \tag{2.2}$$

As we will see, the asymptotic behaviour of the measure $\tilde{\mathbb{P}}_N$ is determined by the analytic properties of the functions $g_\theta(z)$ and $g_\kappa(z)$.

Recall that the cycle counts $C_j = C_j(\sigma)$ are defined as the number of cycles of length $j \in \mathbb{N}$ in the cycle decomposition of permutation $\sigma \in \mathfrak{S}_N$ (see the Introduction). The next well-known identity is a special case of the general *Pólya's Enumeration Theorem* [28, §16, p. 17]; we give its proof for the reader's convenience (cf., e.g., [23, p. 5]).

Lemma 2.2. *Let $(a_j)_{j\in\mathbb{N}}$ be a sequence of (real or complex) numbers. Then there is the following (formal) power series expansion*

$$\exp\left(\sum_{j=1}^{\infty} \frac{a_j z^j}{j}\right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\sigma\in\mathfrak{S}_n} \prod_{j=1}^{n} a_j^{C_j}, \tag{2.3}$$

where $C_j = C_j(\sigma)$ are the cycle counts. If either of the series in (2.3) *is absolutely convergent then so is the other one.*

Proof. Dividing all permutations $\sigma \in \mathfrak{S}_n$ into classes with the same cycle type $(c_j) := (c_1, \dots, c_n)$, that is, such that $C_j(\sigma) = c_j$ $(j = 1, \dots, n)$, we have

$$\sum_{\sigma\in\mathfrak{S}_n} \prod_{j=1}^{n} a_j^{C_j} = \sum_{(c_j)} N_{(c_j)} \prod_{j=1}^{n} a_j^{c_j}, \tag{2.4}$$

where $\sum_{(c_j)}$ means summation over non-negative integer arrays (c_j) satisfying the condition $\sum_{j=1}^{n} j\, c_j = n$ (see (1.1)) and $N_{(c_j)}$ denotes the number of permutations with cycle type (c_j).

Furthermore, allocating the elements $1, \dots, n$ to form a given cycle type (c_j) and taking into account that (i) each cycle is invariant under cyclic rotations (thus reducing the initial number $n!$ of possible allocations by a factor of $\prod_{j=1}^{n} j^{c_j}$), and (ii) cycles of the same length can be permuted among themselves (which leads to a further reduction by a factor of $\prod_{j=1}^{n} c_j!$), it is easy to obtain Cauchy's formula (see, e.g., [1, §1.1])

$$N_{(c_j)} = \frac{n!}{\prod_{j=1}^{n} j^{c_j} c_j!}. \tag{2.5}$$

Hence, using (2.4) and (2.5) and noting that $z^n = \prod_{j=1}^{n} z^{jc_j}$, the right-hand side of (2.3) is rewritten as

$$1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{1}{c_j!} \left(\frac{a_j z^j}{j}\right)^{c_j} = \prod_{j=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{a_j z^j}{j}\right)^{k} = \prod_{j=1}^{\infty} \exp\left(\frac{a_j z^j}{j}\right),$$

which coincides with the left-hand side of (2.3).

The second claim of the lemma follows by the dominated convergence theorem. □

Lemma 2.2 can be used to obtain a convenient expression for the normalization constant H_N of the measure $\tilde{\mathbb{P}}_N$ defined in (1.5). More generally, set $h_0(N) := 1$ and

$$h_n(N) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^{n} (\theta_j + N\kappa_j)^{C_j}, \qquad n \in \mathbb{N}. \tag{2.6}$$

In particular (see (1.6)) we have $H_N = h_N(N)$. By Lemma 2.2 (with $a_j = \theta_j + N\kappa_j$), it is immediately seen that the generating function of the sequence $(h_n(N))_{n\in\mathbb{N}_0}$ is given by

$$\sum_{n=0}^{\infty} h_n(N)\, z^n = \exp\left\{ \sum_{j=1}^{\infty} \frac{\theta_j + N\kappa_j}{j}\, z^j \right\} \equiv \mathrm{e}^{G_N(z)}, \tag{2.7}$$

where we define the function

$$G_N(z) := g_\theta(z) + N g_\kappa(z), \tag{2.8}$$

with $g_\theta(z)$ and $g_\kappa(z)$ as in (2.2). Hence, the coefficients $h_n(N)$ can be represented as

$$h_n(N) = [z^n]\, \mathrm{e}^{G_N(z)}, \qquad n \in \mathbb{N}_0. \tag{2.9}$$

In particular, for $n = N$ formula (2.9) specializes to

$$H_N \equiv h_N(N) = [z^N]\, \mathrm{e}^{G_N(z)}. \tag{2.10}$$

2.2. **Some simple properties of the cycle distribution.** In what follows, we use the Pochhammer symbol $(x)_n$ for the falling factorials,

$$(x)_n := x(x-1)\cdots(x-n+1) \quad (n \in \mathbb{N}), \qquad (x)_0 := 1. \tag{2.11}$$

Lemma 2.3. *For each $m \in \mathbb{N}$ and any integers $n_1, \dots, n_m \geq 0$, we have*

$$\tilde{\mathbb{E}}_N\left[\prod_{j=1}^{m} (C_j)_{n_j}\right] = \frac{h_{N-K_m}(N)}{H_N} \prod_{j=1}^{m} \left(\frac{\theta_j + N\kappa_j}{j}\right)^{n_j}, \tag{2.12}$$

where $K_m := \sum_{j=1}^{m} j n_j$ and $N \geq K_m$.

Proof. Using the definitions of $h_n(N)$ and $G_N(z)$ (see (2.6) and (2.8), respectively), differentiate the identity (2.7) n_j times with respect to θ_j for all $j = 1, \dots, m$ to obtain

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^{m} (C_j)_{n_j} \prod_{i=1}^{n} (\theta_i + N\kappa_i)^{C_i} = \mathrm{e}^{G_N(z)} \prod_{j=1}^{m} (\theta_j + N\kappa_j)^{n_j} \left(\frac{z^j}{j}\right)^{n_j}. \tag{2.13}$$

Extracting the coefficient $[z^N](\cdot)$ on both sides of (2.13) and using (1.5) we get

$$\begin{aligned}
\tilde{\mathbb{E}}_N\left[\prod_{j=1}^{m} (C_j)_{n_j}\right] &= \frac{1}{N!\, H_N} \sum_{\sigma \in \mathfrak{S}_N} \prod_{j=1}^{m} (C_j)_{n_j} \prod_{i=1}^{N} (\theta_i + N\kappa_i)^{C_i} \\
&= \frac{1}{H_N}\, [z^N] \left(\mathrm{e}^{G_N(z)} \prod_{j=1}^{m} \left(\frac{\theta_j + N\kappa_j}{j}\right)^{n_j} z^{j n_j} \right) \\
&= \prod_{j=1}^{m} \left(\frac{\theta_j + N\kappa_j}{j}\right)^{n_j} \frac{[z^{N-K_m}] \exp\{G_N(z)\}}{H_N},
\end{aligned}$$

and formula (2.12) follows on using (2.9). □

Next, let T_N be the total number of cycles,

$$T_N := \sum_{j=1}^{N} C_j. \tag{2.14}$$

Lemma 2.4. *For each $v > 0$, there is the identity*

$$\tilde{\mathbb{E}}_N\bigl[v^{T_N}\bigr] = \frac{1}{H_N}\,[z^N]\exp\{vG_N(z)\}, \tag{2.15}$$

where the expansion of the function $z \mapsto \exp\{vG_N(z)\}$ on the right-hand side of (2.15) *is understood as a formal power series.*

Proof. By definition of the measure $\tilde{\mathbb{P}}_N$ (see (1.5)) we have

$$\begin{aligned}\tilde{\mathbb{E}}_N\bigl[v^{T_N}\bigr] &= \frac{1}{N!\,H_N}\sum_{\sigma\in\mathfrak{S}_N} v^{C_1+\dots+C_N}\prod_{j=1}^{N}(\theta_j + N\kappa_j)^{C_j}\\ &= \frac{1}{N!\,H_N}\sum_{\sigma\in\mathfrak{S}_N}\prod_{j=1}^{N}(v\theta_j + Nv\kappa_j)^{C_j}.\end{aligned}$$

The last sum is analogous to the expression (2.6), only with θ_j and κ_j replaced by $v\theta_j$ and $v\kappa_j$, respectively. Hence, formula (2.9) may be used with $vG_N(z)$ in place of $G_N(z)$, thus readily yielding (2.15). ☐

One convenient way to list the cycles (and their lengths) is via the lexicographic ordering, that is, by tagging them with a suitable increasing subsequence of elements starting from 1.

Definition 2.1. For permutation $\sigma \in \mathfrak{S}_n$ decomposed as a product of cycles, let $L_1 = L_1(\sigma)$ be the length of the cycle containing element 1, $L_2 = L_2(\sigma)$ the length of the cycle containing the smallest element not in the previous cycle, etc. The sequence (L_j) is said to be *lexicographically ordered*.

It is easy to compute the joint (finite-dimensional) distribution of the lengths L_j's.

Lemma 2.5. *For each $m \in \mathbb{N}$ and any $\ell_1, \dots, \ell_m \in \mathbb{N}$ (with $\ell_0 := 0$), we have*

$$\tilde{\mathbb{P}}_N\{L_1 = \ell_1, \dots, L_m = \ell_m\} = \prod_{j=1}^{m}\frac{\theta_{\ell_j} + N\kappa_{\ell_j}}{N - \ell_1 - \dots - \ell_{j-1}} \cdot \frac{h_{N-\ell_1-\dots-\ell_m}(N)}{H_N}. \tag{2.16}$$

In particular, for $m = 1$

$$\tilde{\mathbb{P}}_N\{L_1 = \ell\} = \frac{\theta_\ell + N\kappa_\ell}{N}\cdot\frac{h_{N-\ell}(N)}{H_N}. \tag{2.17}$$

Proof. Note that there are $(N-1)\cdots(N-\ell+1) = (N-1)_{\ell-1}$ cycles of length ℓ containing the element 1, and the choice of such a cycle does not influence the cycle lengths of the remaining cycles. Using the definition (2.6) of $h_n(N)$ we get

$$\begin{aligned}\tilde{\mathbb{P}}_N\{L_1 = \ell\} &= \frac{(N-1)_{\ell-1}(\theta_\ell + N\kappa_\ell)\cdot(N-\ell)!\,h_{N-\ell}(N)}{N!\,H_N}\\ &= \frac{\theta_\ell + N\kappa_\ell}{N}\cdot\frac{h_{N-\ell}(N)}{H_N},\end{aligned}$$

which proves the lemma for $m = 1$ (see (2.17)). Similarly, for $m = 2$

$$\begin{aligned}\tilde{\mathbb{P}}_N\{L_1 = \ell_1, L_2 = \ell_2\} &= (N-1)_{\ell_1-1}(\theta_{\ell_1} + N\kappa_{\ell_1}) \cdot (N-\ell_1-1)_{\ell_2-1}(\theta_{\ell_2} + N\kappa_{\ell_2}) \\ &\qquad\times (N-\ell_1-\ell_2)!\, \frac{h_{N-\ell_1-\ell_2}(N)}{N!\, H_N} \\ &= \frac{(\theta_{\ell_1} + N\kappa_{\ell_1})(\theta_{\ell_2} + N\kappa_{\ell_2})}{N(N-\ell_1)} \cdot \frac{h_{N-\ell_1-\ell_2}(N)}{H_N}\end{aligned} \tag{2.18}$$

(cf. (2.16)). The general case $m \in \mathbb{N}$ is handled in the same manner. □

3. Asymptotic theorems for the generating function

In this section, we develop complex-analytic tools for computing the asymptotics of the coefficient $h_N(N) = H_N$ in the power series expansion of $\exp\{G_N(z)\}$ (see (2.9)). More generally, it is useful to consider expansions of the function $\exp\{vG_N(z)\}$, with some parameter $v > 0$. From Lemma 2.3 it is clear that the case $v = 1$ is of primary importance, but Lemma 2.4 suggests that information for $v \approx 1$ will also be needed for the sake of limit theorems for cycles (see Sections 4 and 5 below).

3.1. **Preliminaries and motivation.** Let us introduce notation for the "modified" derivatives of a function $z \mapsto g(z)$,

$$g^{\{n\}}(z) := z^n\, \frac{\mathrm{d}^n g(z)}{\mathrm{d}z^n}, \qquad n \in \mathbb{N}_0. \tag{3.1}$$

For the generating functions $g_\theta(z)$, $g_\kappa(z)$ (see (2.2)) it is easy to see that, for each $n \in \mathbb{N}$,

$$g_\theta^{\{n\}}(z) = \sum_{j=1}^{\infty} (j-1)_{n-1}\, \theta_j\, z^j, \qquad g_\kappa^{\{n\}}(z) = \sum_{j=1}^{\infty} (j-1)_{n-1}\, \kappa_j\, z^j, \tag{3.2}$$

where $(\cdot)_{n-1}$ is the Pochhammer symbol defined in (2.11).

Let $R > 0$ (possibly $R = +\infty$) be the radius of convergence of $g_\kappa(z)$, and hence of each of its (modified) derivatives $g_\kappa^{\{n\}}(z)$. If $R < \infty$ then, according to Pringsheim's Theorem (see Lemma 2.1), $z = R$ is a point of singularity of $g_\kappa(z)$ (and each $g_\kappa^{\{n\}}(z)$, see (3.2)). We write $g_\kappa^{\{n\}}(R) := \lim_{r\uparrow R} g_\kappa^{\{n\}}(r)$, with $g_\kappa^{\{n\}}(R) := +\infty$ if this limit is divergent. For $r \in (0, R]$, let us also denote

$$b_1(r) := g_\kappa^{\{1\}}(r) > 0, \qquad b_2(r) := g_\kappa^{\{1\}}(r) + g_\kappa^{\{2\}}(r) > 0. \tag{3.3}$$

The following simple lemma will be useful.

Lemma 3.1. *For any* $r \in (0, R]$,

$$b_1(r) \le \sqrt{g_\kappa(r)\, b_2(r)}\,, \tag{3.4}$$

where the inequality is in fact strict unless $\kappa_j = 0$ *for all* $j \ge 2$.

Proof. Using the expression (3.2) for the modified derivatives of g_κ and applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned} g_\kappa^{\{1\}}(r) = \sum_{j=1}^{\infty} \kappa_j r^j &= \sum_{j=1}^{\infty} \left(\frac{\kappa_j r^j}{j}\right)^{1/2} \left(j\kappa_j r^j\right)^{1/2} \\ &\le \left(\sum_{j=1}^{\infty} \frac{\kappa_j r^j}{j}\right)^{1/2} \left(\sum_{j=1}^{\infty} j\kappa_j r^j\right)^{1/2} \\ &= \sqrt{g_\kappa(r)} \cdot \sqrt{g_\kappa^{\{1\}}(r) + g_\kappa^{\{2\}}(r)}\,, \end{aligned}$$

and the inequality (3.4) follows in view of the notation (3.3). The equality is only possible when $\kappa_j/j = j\kappa_j$ for all $j \ge 1$, which implies that $\kappa_j \equiv 0$ for $j \ge 2$. ☐

To avoid trivial complications (or simplifications), let us impose

Assumption 3.1. The sequence (κ_j) is assumed to be *non-arithmetic*, that is, there is no integer $j_0 > 1$ such that the inequality $\kappa_j \ne 0$ would imply that j is a multiple of j_0. We also suppose that (κ_j) is non-degenerate, in that $\kappa_j > 0$ for some $j \ge 2$.

Remark 3.1. The assumption of non-arithmeticity simplifies the analysis by ensuring there is a unique maximum of (the real part of) the generating function $g_\kappa(z)$ (see Lemma 3.2 below). However, a more general case (with multiple maxima) can be treated as well, without too much difficulty. Note also that in the original spatial model (see (1.17), (1.18)) all coefficients κ_j's are positive, hence such a sequence (κ_j) is automatically non-arithmetic.

In what follows, $\Re(w)$ denotes the real part of a complex number $w \in \mathbb{C}$, and $\arg(w) \in (-\pi, \pi]$ is the principal value of its argument.

Lemma 3.2. (a) *Under Assumption* 3.1*, for each* $r \in (0, R)$ *there is a unique maximum of the function* $t \mapsto \Re(g_\kappa(r\mathrm{e}^{\mathrm{i}t}))$ *over* $t \in [-\pi, \pi]$*, attained at* $t = 0$ *and equal to* $g_\kappa(r)$*. If* $g_\kappa(R) < \infty$ *then this claim is also true with* $r = R$*.*

(b) *The same statements hold for the function* $t \mapsto \Re(g_\kappa^{\{1\}}(r\mathrm{e}^{\mathrm{i}t}))$*,* $t \in [-\pi, \pi]$.

Proof. (a) Since the coefficients κ_j are non-negative, $t = 0$ is always a point of global maximum of the function $t \mapsto \Re(g_\kappa(r\mathrm{e}^{\mathrm{i}t}))$. It is easy to see that the uniqueness of this maximum over $t \in [-\pi, \pi]$ is equivalent to the property that $g_\kappa(\cdot)$ cannot be written as $g_\kappa(z) = f(z^{j_0})$ with a holomorphic function $f(\cdot)$ and some integer $j_0 > 1$, and the latter is true because the sequence (κ_j) is non-arithmetic due to Assumption 3.1.

(b) The same considerations are valid for the function $g_\kappa^{\{1\}}(z) = \sum_{j=1}^{\infty} \kappa_j z^j$ which is again a power series with non-negative non-arithmetic coefficients. ☐

Remark 3.2. The uniqueness part of Lemma 3.2(b) may fail for $g_\kappa^{\{n\}}(z)$ with $n \ge 2$, because the non-arithmeticity property may cease to hold, like in the following example: $\kappa_1 > 0$, $\kappa_{2j} > 0$, $\kappa_{2j+1} = 0$ $(j \in \mathbb{N})$.

Remark 3.3. Lemma 3.2 is akin to the "Daffodil Lemma" in [14, §IV.6.1, Lemma IV.1, p. 266], but the latter deals with the maximum of the absolute value rather than the real part.

Setting $z = r\mathrm{e}^{\mathrm{i}t}$ with $r = |z| < R$, $t = \arg(z) \in (-\pi, \pi]$, let us consider the Taylor expansion of $g_\kappa(z)$ near $z_0 = r$ with respect to t,

$$g_\kappa(r\mathrm{e}^{\mathrm{i}t}) = g_\kappa(r) + \mathrm{i}t\,b_1(r) - \tfrac{1}{2}t^2 b_2(r) + o(t^2), \qquad t \to 0, \tag{3.5}$$

where $b_1(r)$, $b_2(r)$ are defined in (3.3). Note that $g_\kappa^{\{1\}}(0) = 0$ and the function $r \mapsto g_\kappa^{\{1\}}(r)$ is real analytic and strictly increasing for $r \geq 0$ (of course, provided that $g_\kappa(z)$ is not identically zero). Thus, the inverse of $g_\kappa^{\{1\}}(r)$ exists for $0 < r \leq R$. For $v \geq 1/g_\kappa^{\{1\}}(R)$, let r_v be the (unique) solution of the equation

$$g_\kappa^{\{1\}}(r) = v^{-1}, \qquad 0 < r \leq R. \tag{3.6}$$

In particular, for $v = 1/g_\kappa^{\{1\}}(R)$ we have $r_v = R$.

Definition 3.2. The cases $g_\kappa^{\{1\}}(R) > 1$ and $g_\kappa^{\{1\}}(R) < 1$ are termed "subcritical" and "supercritical", respectively.

This terminology will be justified in Section 4 below, in particular by Theorem 4.2, where we will demonstrate that the limiting fraction of points in infinite cycles is positive if and only if $g_\kappa^{\{1\}}(R) < 1$. In Section 6, it will also be shown that the dichotomy in Definition 3.2 corresponds to the cases $\tilde\rho < \tilde\rho_{\mathrm{c}}$ and $\tilde\rho > \tilde\rho_{\mathrm{c}}$, respectively, with $\tilde\rho$ standing for the system "density" (see Section 6.2 below).

The analytical reason for such a distinction becomes clear from the following observation. Proceeding from the expression (2.10) for H_N, with the function $G_N(z)$ defined in (2.8), Cauchy's integral formula with the contour $\gamma := \{z = r\mathrm{e}^{\mathrm{i}t},\ t \in [-\pi, \pi]\}$ $(r < R)$ gives

$$\begin{aligned}[z^N] \exp\{G_N(z)\} &= \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\exp\{G_N(z)\}}{z^{N+1}}\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\exp\{g_\theta(z)\}}{z} \exp\bigl\{N\bigl(g_\kappa(z) - \log z\bigr)\bigr\}\,\mathrm{d}z. \end{aligned} \tag{3.7}$$

Hence, the classical saddle point method (see, e.g., [10, Ch. 5]) suggests that the asymptotics of the integral (3.7) are determined by the maximum of the function $z \mapsto \Re(g_\kappa(z) - \log z)$. In turn, by Lemma 3.2(a) this is reduced to finding the maximum of the function $r \mapsto g_\kappa(r) - \log r$ for real $r > 0$, leading to the equation

$$g_\kappa'(r) - \frac{1}{r} = 0 \quad \Leftrightarrow \quad g_\kappa^{\{1\}}(r) = 1,$$

which, in view of Lemma 3.2(b), is solvable if and only if $g_\kappa^{\{1\}}(R) \geq 1$.

3.2. **The subcritical case.**

Theorem 3.3. *Assume that the generating functions $g_\theta(z)$ and $g_\kappa(z)$ both have radius of convergence $R > 0$, and suppose that $1 < g_\kappa^{\{1\}}(R) \leq \infty$. Let $f(z)$ be a function holomorphic in the open disk $|z| < R$. Then, uniformly in $v \in [v_1, v_2]$ with arbitrary constants $v_2 > 1 > v_1 > 1/g_\kappa^{\{1\}}(R)$, we have*

$$[z^N]\bigl[f(z)\exp\{vG_N(z)\}\bigr] \sim \frac{f(r_v)\exp\{vG_N(r_v)\}}{r_v^N\sqrt{2\pi N v b_2(r_v)}}, \qquad N \to \infty, \tag{3.8}$$

where $b_2(r)$ and r_v are defined in (3.3) *and* (3.6)*, respectively. In particular,*

$$H_N \sim \frac{\exp\{g_\theta(r_1) + N g_\kappa(r_1)\}}{r_1^N\sqrt{2\pi N b_2(r_1)}}, \qquad N \to \infty. \tag{3.9}$$

Proof. First of all, according to (2.10) formula (3.9) readily follows from (3.8) by setting $f(z)\equiv 1$ and $v=1$, whereby $r_v|_{v=1}=r_1$. To handle the general case, apply Cauchy's integral formula with the contour $\gamma:=\{z=r_v\mathrm{e}^{\mathrm{i}t},\ t\in[-\pi,\pi]\}$ to obtain

$$\begin{aligned}[z^N]\bigl[f(z)\exp\{vG_N(z)\}\bigr]&=\frac{1}{2\pi\mathrm{i}}\oint_\gamma\frac{f(z)\exp\{vG_N(z)\}}{z^{N+1}}\,\mathrm{d}z\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(r_v\mathrm{e}^{\mathrm{i}t})\exp\bigl\{vg_\theta(r_v\mathrm{e}^{\mathrm{i}t})+\mathrm{i}tn\bigr\}}{r_v^N}\,\exp\bigl\{N\bigl(vg_\kappa(r_v\mathrm{e}^{\mathrm{i}t})-\mathrm{i}t\bigr)\bigr\}\,\mathrm{d}t\\&=:\frac{1}{2\pi}\bigl(\mathcal{I}_N^1+\mathcal{I}_N^2+\mathcal{I}_N^3\bigr),\end{aligned}\tag{3.10}$$

where $\mathcal{I}_N^1$, $\mathcal{I}_N^2$ and $\mathcal{I}_N^3$ are the corresponding integrals arising upon splitting the interval $[-\pi,\pi]$ into three parts, with $|t|\in[0,t_N]\cup[t_N,\delta]\cup[\delta,\pi]$. Choosing $t_N=N^{-\beta}$ with $\frac13<\beta<\frac12$ and $\delta>0$ small enough, we estimate each of the integrals in (3.10) as follows.

(i) By Taylor's expansion (3.5) at $r=r_v$, with $b_1(r_v)=v^{-1}$ (see (3.3), (3.6)), we have

$$\mathcal{I}_N^1=\frac{f(r_v)\exp\{vG_N(r_v)\}}{r_v^N}\int_{-t_N}^{t_N}\exp\bigl\{-\tfrac12Nt^2vb_2(r_v)+O(Nt^3)\bigr\}\,\mathrm{d}t.\tag{3.11}$$

On the change of variables $s=\sqrt{N}\,t$, the integral in (3.11) becomes

$$\begin{aligned}&\frac{1}{\sqrt{N}}\int_{-\sqrt{N}t_N}^{\sqrt{N}t_N}\exp\bigl\{-\tfrac12vb_2(r_v)s^2+O(N^{-1/2}s^3)\bigr\}\,\mathrm{d}s\\&\qquad=\frac{\exp\{O(Nt_N^3)\}}{\sqrt{N}}\int_{-\sqrt{N}t_N}^{\sqrt{N}t_N}\exp\bigl\{-\tfrac12vb_2(r_v)s^2\bigr\}\,\mathrm{d}s\sim\sqrt{\frac{2\pi}{Nvb_2(r_v)}}\,,\end{aligned}$$

as long as $Nt_N^3\to0$. Hence, returning to (3.11) we get

$$\mathcal{I}_N^1\sim\frac{f(r_v)\exp\{vG_N(r_v)\}}{r_v^N}\sqrt{\frac{2\pi}{Nvb_2(r_v)}}\,,\qquad N\to\infty.\tag{3.12}$$

(ii) Similarly, using the expansion (3.5) for $|t|\le\delta$ we obtain

$$\begin{aligned}|\mathcal{I}_N^2|&=O(1)\,\frac{\exp\{vG_N(r_v)\}}{\sqrt{N}\,r_v^N}\int_{\sqrt{N}t_N}^{\infty}\exp\bigl\{-\tfrac12vb_2(r_v)s^2\bigr\}\,\mathrm{d}s\\&=o(1)\,\frac{\exp\{vG_N(r_v)\}}{\sqrt{N}\,r_v^N}\,,\end{aligned}\tag{3.13}$$

as long as $\sqrt{N}\,t_N\to\infty$.

(iii) For $|t|\ge\delta$, by a simple absolute value estimate we have

$$|\mathcal{I}_N^3|=\frac{O(1)}{r_v^N}\int_\delta^\pi\exp\bigl\{-Nv\Re(g_\kappa(r_v\mathrm{e}^{\mathrm{i}t}))\bigr\}\,\mathrm{d}s=O(1)\,\frac{\exp\{-Nv\mu_\kappa(\delta)\}}{r_v^N}\,,\tag{3.14}$$

where, according to Lemma 3.2(a),

$$\mu_\kappa(\delta):=\max_{|t|\in[\delta,\pi]}\Re(g_\kappa(r_v\mathrm{e}^{\mathrm{i}t}))<g_\kappa(r_v).$$

Hence, the bound (3.13) is exponentially small as compared to (3.12), and so the contribution from $\mathcal{I}_N^3$ is negligible.

Substituting the estimates (3.12), (3.13) and (3.14) into (3.10) yields the asymptotic formula (3.8) for a fixed $v > 0$. Finally, it is easy to see that all $O(\cdot)$ and $o(\cdot)$ terms used above are uniform in $v \in [v_1, v_2]$, as claimed in the theorem. This completes the proof. □

3.3. **The supercritical case.**

3.3.1. *The domain $\varDelta_0$.* Recall (see Definition 3.2) that the supercritical case occurs when $g_\kappa^{\{1\}}(R) < 1$, whereby the equation $g_\kappa^{\{1\}}(r) = 1$ is no longer solvable. To overcome this difficulty, we have to allow the contour of integration in Cauchy's integral formula akin to (3.7) to go outside the disk of convergence $|z| < R$. To make this idea more precise, we give the following definition (see Fig. 2).

Definition 3.3. For $\eta > 0$ and $\varphi \in (0, \frac{\pi}{2})$, define an open domain $\varDelta_0 = \varDelta_0(R, \eta, \varphi)$ in the complex plane by

$$\varDelta_0 := \bigl\{z \in \mathbb{C}: \, |z| < |R(1 + \eta \mathrm{e}^{\mathrm{i}\varphi})|, \ z \neq R, \ |\arg(z - R)| > \varphi\bigr\}.$$

We also introduce the notation for the radius of the outer circle, $R' := R\,|1 + \eta\mathrm{e}^{\mathrm{i}\varphi}|$, and the angle $\alpha := \arg(1 + \eta\mathrm{e}^{\mathrm{i}\varphi}) \in (0, \pi/2)$, so that $R(1 + \eta\mathrm{e}^{\mathrm{i}\varphi}) = R'\mathrm{e}^{\mathrm{i}\alpha}$.

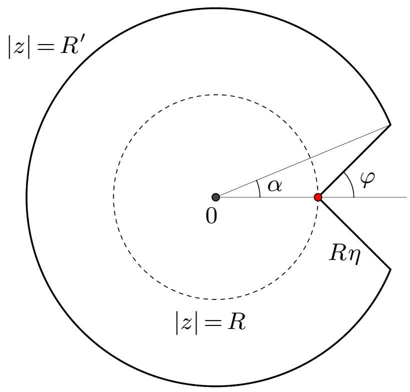

FIGURE 2. Domain $\varDelta_0 = \varDelta_0(R, \eta, \varphi)$.

Lemma 3.4. *Suppose that Assumption* 3.1 *holds, and let $g_\kappa(z)$ be holomorphic in a domain $\varDelta_0 = \varDelta_0(R, \eta, \varphi)$ as defined above, with $g_\kappa^{\{1\}}(R) < \infty$. Then there exist constants $\delta > 0$, $\varepsilon > 0$ such that, for all $z \in \varDelta_0$ with $R \le |z| \le R(1 + \delta)$, $|\arg(z)| \ge \delta$,*

$$\Re(g_\kappa(z)) \le g_\kappa(R) + \bigl(g_\kappa^{\{1\}}(R) - \varepsilon\bigr) \log \frac{|z|}{R}. \tag{3.15}$$

Proof. By Lemma 3.2(b), if $|z| = R$ and $z \neq 0$ then

$$\Re(g_\kappa^{\{1\}}(z)) < g_\kappa^{\{1\}}(R). \tag{3.16}$$

Along with $g_\kappa(z)$, the function $z \mapsto \Re(g_\kappa^{\{1\}}(z))$ is also analytic in $\varDelta_0$, and in particular it is continuous in a vicinity of the punctured circle $\{|z| = R, \ z \neq R\} \subset \varDelta_0$. Hence, by a compactness argument the inequality (3.16) also holds on a closed domain

$$\varDelta_\delta := \varDelta_0 \cap \{z : R \le |z| \le R(1 + \delta), \ |\arg(z)| \ge \delta\},$$

with $\delta > 0$ small enough. Moreover, by the continuity of $\Re(g_\kappa^{\{1\}}(z))$ the strict inequality (3.16) implies that

$$\varepsilon := g_\kappa^{\{1\}}(R) - \sup_{z\in\Delta_\delta} \Re(g_\kappa^{\{1\}}(z)) > 0.$$

Therefore, for any $z = r\,\mathrm{e}^{\mathrm{i}t} \in \Delta_\delta$ we have

$$\int_R^r \Re\bigl(g_\kappa^{\{1\}}(u\mathrm{e}^{\mathrm{i}t})\bigr)\,\frac{\mathrm{d}u}{u} \le \int_R^r \bigl(g_\kappa^{\{1\}}(R) - \varepsilon\bigr)\,\frac{\mathrm{d}u}{u} = \bigl(g_\kappa^{\{1\}}(R) - \varepsilon\bigr)\log\frac{|z|}{R}. \tag{3.17}$$

On the other hand,

$$\int_R^r \Re\bigl(g_\kappa^{\{1\}}(u\mathrm{e}^{\mathrm{i}t})\bigr)\,\frac{\mathrm{d}u}{u} = \Re\left(\int_R^r \frac{\partial}{\partial u}\, g_\kappa(u\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}u\right) = \Re\bigl(g_\kappa(z) - g_\kappa(R\mathrm{e}^{\mathrm{i}t})\bigr). \tag{3.18}$$

Hence, combining (3.17) and (3.18) we obtain

$$\Re(g_\kappa(z)) \le \Re(g_\kappa(R\,\mathrm{e}^{\mathrm{i}t})) + \bigl(g_\kappa^{\{1\}}(R) - \varepsilon\bigr)\log\frac{|z|}{R},$$

and the inequality (3.15) readily follows in view of Lemma 3.2(a). □

Motivated by the model choice $\theta_j = \mathrm{const} \ge 0$ (see Section 3.5.2 below), in the supercritical asymptotic theorems that follow we allow the generating function $g_\theta(z)$ to have a logarithmic singularity at $z = R$, of the form $g_\theta(z) \sim -\theta^* \log(1 - z/R)$ as $z \to R$ $(z \in \Delta_0)$. It turns out that there is a significant distinction between the cases $\theta^* > 0$ and $\theta^* = 0$.

3.3.2. *Case $\theta^* > 0$.* We first handle the case with a non-degenerate log-singularity of $g_\theta(z)$.

Theorem 3.5. *Let the generating functions $g_\theta(z)$ and $g_\kappa(z)$ both have radius of convergence $R > 0$ and be holomorphic in some domain Δ_0 as in Definition 3.3. Assume that $g_\kappa^{\{1\}}(R) < 1$ and the following asymptotic formulas hold as $z \to R$ $(z \in \Delta_0)$, with some $\theta^* \ge 0$, $\delta > 0$,*

$$g_\theta(z) = -\theta^* \log(1 - z/R) + O\bigl((1 - z/R)^\delta\bigr), \tag{3.19}$$

$$g_\kappa(z) = g_\kappa(R) - g_\kappa^{\{1\}}(R)(1 - z/R) + O\bigl((1 - z/R)^{1+\delta}\bigr). \tag{3.20}$$

Finally, let $f\colon \Delta_0 \to \mathbb{C}$ be a holomorphic function such that for some $\beta \ge 0$

$$f(z) = (1 - z/R)^{-\beta}\bigl\{1 + O\bigl((1 - z/R)^\delta\bigr)\bigr\}, \qquad z \to R \ \ (z \in \Delta_0). \tag{3.21}$$

Then, provided that $v\theta^* + \beta > 0$, we have, as $N \to \infty$,

$$[z^N]\bigl[f(z)\exp\{vG_N(z)\}\bigr] \sim \frac{\exp\{Nv g_\kappa(R)\}\cdot\bigl\{N(1 - v g_\kappa^{\{1\}}(R))\bigr\}^{v\theta^*+\beta-1}}{R^N\,\Gamma(v\theta^* + \beta)}, \tag{3.22}$$

uniformly in $v \in [v_1, v_2]$ for any $0 < v_1 < 1 < v_2 < 1/g_\kappa^{\{1\}}(R)$. In particular, for $\theta^* > 0$

$$H_N \sim \frac{\exp\{N g_\kappa(R)\}\cdot\bigl\{N(1 - g_\kappa^{\{1\}}(R))\bigr\}^{\theta^*-1}}{R^N\,\Gamma(\theta^*)}, \qquad N \to \infty. \tag{3.23}$$

Proof. In view of the identity (2.10), formula (3.23) is obtained from (3.22) by setting $f(z) \equiv 1$ (so that $\beta = 0$) and $v = 1$. Let us also observe that, according to (3.19) and (3.21),

$$v g_\theta(z) + \log f(z) = -(v\theta^* + \beta)\log(1 - z/R) + O\bigl((1 - z/R)^\delta\bigr).$$

Thus, accounting for the pre-exponential factor $f(z)$ just leads to the change $v\theta^* \mapsto v\theta^* + \beta$. With this in mind, it suffices to consider the basic case $\beta = 0$ (but now with $\theta^* > 0$).

Without loss of generality, we may and will assume (by slightly reducing the original domain Δ_0 if necessary) that both $g_\theta(z)$ and $g_\kappa(z)$ are continuous on the boundary of Δ_0

except at $z = R$. By virtue of Lemma 3.4 (and again reducing Δ_0 as appropriate), we may also assume that the inequality (3.15) is fulfilled for all $z \in \Delta_0$ such that $R \le |z| \le R'$ and $|\arg(z)| \ge \delta_0$, with some $\delta_0 > 0$, where $R' = R\,|1 + \eta\,\mathrm{e}^{\mathrm{i}\varphi}|$ (see Definition 3.3).

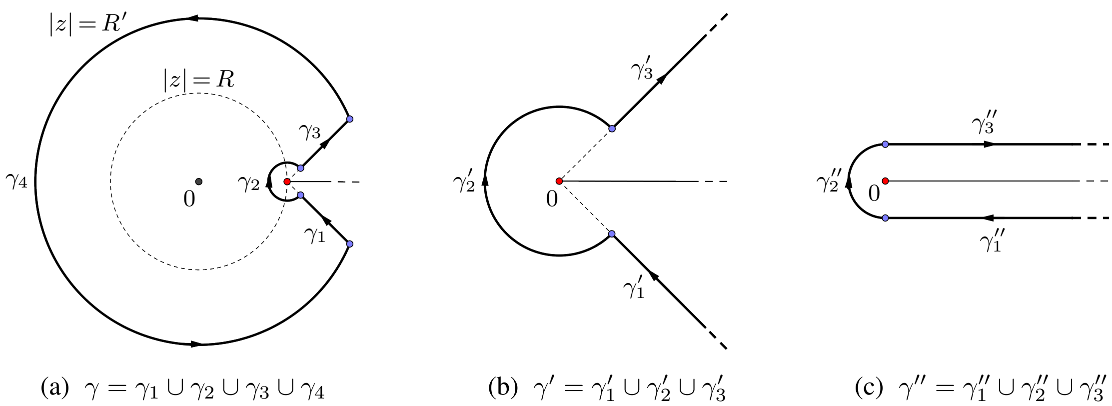

FIGURE 3. Contours used in the proof of Theorem 3.5.

Consider a continuous closed contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ (see Fig. 3(a)), where

$$\begin{aligned}
\gamma_1 &:= \bigl\{z = R(1 + (\eta - x)\,\mathrm{e}^{-\mathrm{i}\varphi}),\ \ x \in [0, \eta - N^{-1}]\bigr\},\\
\gamma_2 &:= \bigl\{z = R(1 + N^{-1}\mathrm{e}^{-\mathrm{i}t}),\ \ t \in [\varphi, 2\pi - \varphi]\bigr\},\\
\gamma_3 &:= \bigl\{z = R(1 + x\mathrm{e}^{\mathrm{i}\varphi}),\ \ x \in [N^{-1}, \eta]\bigr\},\\
\gamma_4 &:= \bigl\{z = R'\mathrm{e}^{\mathrm{i}t},\ \ t \in [\alpha, 2\pi - \alpha]\bigr\}.
\end{aligned} \tag{3.24}$$

According to the chosen parameterization in (3.24), the contour γ is traversed anti-clockwise. Then Cauchy's integral formula yields

$$\begin{aligned}
[z^N]\exp\{vG_N(z)\} &= \frac{1}{2\pi\mathrm{i}}\left(\int_{\gamma_1} + \dots + \int_{\gamma_4}\right)\frac{\exp\{vG_N(z)\}}{z^{N+1}}\,\mathrm{d}z\\
&=: \frac{1}{2\pi\mathrm{i}}\left(\mathcal{I}_N^1 + \dots + \mathcal{I}_N^4\right).
\end{aligned} \tag{3.25}$$

Now, we estimate each of the integrals in (3.25). Denote for short

$$d_1 := 1 - v g_\kappa^{\{1\}}(R) > 0. \tag{3.26}$$

(i) Let us first show that the integral $\mathcal{I}_N^4$ over the circular arc γ_4 (see (3.24)) is negligible as $N \to \infty$. Indeed, by an absolute value inequality we have

$$\begin{aligned}
|\mathcal{I}_N^4| &\le \int_{\gamma_4} \frac{\exp\{v\,\Re(G_N(z))\}}{|z|^{N+1}}\,\mathrm{d}|z|\\
&\le \frac{2\pi}{(R')^N}\,\exp\bigl\{v\mu_\theta(R') + Nv\mu_\kappa(R')\bigr\},
\end{aligned} \tag{3.27}$$

where $\mu_\theta(R') := \max_{z\in\gamma_4} \Re(g_\theta(z)) < \infty$ and, by Lemma 3.4,

$$\mu_\kappa(R') := \max_{z\in\gamma_4} \Re(g_\kappa(z)) \le g_\kappa(R) + g_\kappa^{\{1\}}(R)\cdot\log\frac{R'}{R}\,. \tag{3.28}$$

Hence, the right-hand side of the bound (3.27) is further estimated by

$$\frac{O(1)}{(R')^N}\,\exp\left\{Nv\left(g_\kappa(R)+g_\kappa^{\{1\}}(R)\cdot\log\frac{R'}{R}\right)\right\}\\ =\frac{O(1)\,\exp\{Nv\,g_\kappa(R)\}}{R^N}\,\exp\left\{-Nd_1\log\frac{R'}{R}\right\} \tag{3.29}$$

(see the notation (3.26)), which is exponentially small as compared to the right-hand side of (3.22), thanks to the inequalities $R'/R>1$ and $d_1>0$. Thus,

$$\mathcal{I}_N^4=o(1)\,\frac{\exp\{Nv\,g_\kappa(R)\}}{R^N N^{1-v\theta^*}},\qquad N\to\infty. \tag{3.30}$$

(ii) For $z\in\gamma_2$ (see (3.24)), we have

$$z=R(1+wN^{-1}),\qquad w\in\gamma_2':=\bigl\{\mathrm{e}^{-\mathrm{i}t},\ t\in[\varphi,\,2\pi-\varphi]\bigr\}. \tag{3.31}$$

Then formulas (3.19) and (3.20) yield the uniform asymptotics, as $N\to\infty$,

$$g_\theta(z)=-\theta^*\log\left(-wN^{-1}\right)+O\bigl((w/N)^{\delta}\bigr), \tag{3.32}$$

$$g_\kappa(z)=g_\kappa(R)+g_\kappa^{\{1\}}(R)\,wN^{-1}+O\bigl((w/N)^{1+\delta}\bigr). \tag{3.33}$$

We also have

$$\frac{1}{z^{N+1}}=\frac{1}{R^{N+1}}\,\exp\bigl\{-(N+1)\log\left(1+wN^{-1}\right)\bigr\}\\ =\frac{1}{R^{N+1}}\,\exp\bigl\{-w+O(wN^{-1})\bigr\},\qquad N\to\infty. \tag{3.34}$$

Collecting (3.32), (3.33) and (3.34), we obtain from (3.25) via the change of variables (3.31)

$$\mathcal{I}_N^2\sim\frac{\exp\{Nv\,g_\kappa(R)\}}{R^N N^{1-v\theta^*}}\int_{\gamma_2'}(-w)^{-v\theta^*}\,\mathrm{e}^{-d_1 w}\,\mathrm{d}w. \tag{3.35}$$

(iii) By the symmetry between the contours γ_1 and γ_3, it is easy to see that the corresponding integrals $-\mathcal{I}_N^1$ and $\mathcal{I}_N^3$ are complex conjugate to one another. Hence, it suffices to consider, say, $\mathcal{I}_N^3$. Similarly to (3.31), we reparameterize the contour γ_3 (see (3.24)) as

$$z=R(1+wN^{-1}),\qquad w\in\gamma_{3,N}':=\bigl\{x\mathrm{e}^{\mathrm{i}\varphi},\ x\in[1,\,N\eta]\bigr\}. \tag{3.36}$$

Let us split the contour $\gamma_{3,N}'$ into three parts corresponding to $x\in[1,x_N]\cup[x_N,N\eta_0]\cup[N\eta_0,N\eta]$, with $x_N\to\infty$, $x_N=o(N)$ as $N\to\infty$, and denote the respective parts of the integral $\mathcal{I}_N^3$ by $\mathcal{I}_N^{3'}$, $\mathcal{I}_N^{3''}$ and $\mathcal{I}_N^{3'''}$. Because the substitutions (3.31) and (3.36) are formally identical to one another, it is clear that the estimates (3.32), (3.33) and (3.34) hold true for (3.36) and, moreover, are uniform in w such that $|w|\le N\eta_0$, with $\eta_0>0$ small enough. Hence, the asymptotics of the integral $\mathcal{I}_N^{3'}$ is given by a formula analogous to (3.35),

$$\mathcal{I}_N^{3'}\sim\frac{\exp\{Nv\,g_\kappa(R)\}}{R^N N^{1-v\theta^*}}\int_{\gamma_3'}(-w)^{-v\theta^*}\,\mathrm{e}^{-d_1 w}\,\mathrm{d}w, \tag{3.37}$$

where (cf. (3.36))

$$\gamma_3':=\lim_{N\to\infty}\gamma_{3,N}'=\bigl\{w=x\mathrm{e}^{\mathrm{i}\varphi},\ x\in[1,\infty)\bigr\}. \tag{3.38}$$

Similarly, the integral $\mathcal{I}_N^{3''}$ is asymptotically estimated as

$$\begin{aligned}\mathcal{I}_N^{3''} &= \frac{O(1)\exp\{Nvg_\kappa(R)\}}{R^N N^{1-v\theta^*}}\int_{x_N}^{N\eta_0}\frac{\exp\{-d_1x\cos\varphi\}}{x^{v\theta^*}}\,\mathrm{d}x\\ &= \frac{O(1)\exp\{Nvg_\kappa(R)\}}{R^N N^{1-v\theta^*}}\cdot\frac{\exp\{-d_1x_N\cos\varphi\}}{(x_N)^{v\theta^*}}\\ &= o\big(\mathcal{I}_N^{3'}\big),\qquad N\to\infty,\end{aligned}\tag{3.39}$$

provided that $x_N\to\infty$ and $d_1>0$ (see (3.26)).

By the continuity of $g_\theta(\cdot)$, for z in (3.36) with $|w|\in[N\eta_0,N\eta]$ there is a uniform bound

$$\Re(g_\theta(z)) = O(1).\tag{3.40}$$

On the other hand, the estimate (3.15) of Lemma 3.4 takes the form

$$\Re(g_\kappa(z)) \le g_\kappa(R) + g_\kappa^{\{1\}}(R)\cdot\log|1+wN^{-1}|,\tag{3.41}$$

whereby

$$\log\Big|1+\frac{w}{N}\Big| = \frac{1}{2}\log\left(1+\frac{2|w|\cos\varphi}{N}+\frac{|w|^2}{N^2}\right) = \frac{|w|\cos\varphi}{N}+O(|w|^2N^{-2}).\tag{3.42}$$

Hence, using (3.34) and (3.40)–(3.42) we can adapt the estimation in (3.39) to obtain

$$\begin{aligned}\mathcal{I}_N^{3'''} &= \frac{O(1)\exp\{Nvg_\kappa(R)\}}{R^N N}\int_{N\eta_0}^{N\eta}\exp\{-d_1x\cos\varphi\}\,\mathrm{d}x\\ &= \frac{O(1)\exp\{Nvg_\kappa(R)\}}{R^N N}\exp\{-d_1N\eta_0\cos\varphi\}\\ &= o\big(\mathcal{I}_N^{3'}\big),\qquad N\to\infty,\end{aligned}\tag{3.43}$$

because $\cos\varphi>0$ and $d_1>0$ (see (3.26)).

As a result, combining the asymptotic formulas (3.37), (3.39) and (3.43) we get

$$\mathcal{I}_N^3 \sim \frac{\exp\{Nvg_\kappa(R)\}}{R^N N^{1-v\theta^*}}\int_{\gamma_3'}(-w)^{-v\theta^*}\mathrm{e}^{-d_1w}\,\mathrm{d}w.\tag{3.44}$$

Finally, collecting the contributions from $\mathcal{I}_N^1,\dots,\mathcal{I}_N^4$ (see (3.30), (3.35) and (3.44)) and returning to (3.25) yields, upon the change of the integration variable $d_1w\mapsto w$,

$$\oint_\gamma\frac{\exp\{vG_N(z)\}}{z^{N+1}}\,\mathrm{d}z \sim \frac{\exp\{Nvg_\kappa(R)\}}{R^N N^{1-v\theta^*}}\,(d_1)^{v\theta^*-1}\int_{\widetilde{\gamma}'}(-w)^{-v\theta^*}\mathrm{e}^{-w}\,\mathrm{d}w,\tag{3.45}$$

with $\widetilde{\gamma}' := d_1\gamma'$, where $\gamma'=\gamma_1'\cup\gamma_2'\cup\gamma_3'$ is defined via (3.31) and (3.38) (see Fig. 3(b)).

The integral on the right-hand side of (3.45) can be explicitly computed. Indeed, by virtue of a simple estimate

$$\big|(-w)^{-v\theta^*}\mathrm{e}^{-w}\big| \le |w|^{-v\theta^*}\exp\{-|w|\cos\arg(w)\},$$

one can apply a standard contour transformation argument to replace the contour $\widetilde{\gamma}'$ in (3.45) by the "loop" contour γ'' starting from $+\infty-\mathrm{i}c$, winding clockwise about the origin and proceeding towards $+\infty+\mathrm{i}c$ (see Fig. 3(c)), which leads to the equality

$$\int_{\gamma'}(-w)^{-v\theta^*}\mathrm{e}^{-w}\,\mathrm{d}w = \int_{\gamma''}(-w)^{-v\theta^*}\mathrm{e}^{-w}\,\mathrm{d}w = \frac{2\pi\mathrm{i}}{\Gamma(v\theta^*)},\tag{3.46}$$

according to the well-known Hankel's loop representation of the reciprocal gamma function (see, e.g., [14, §B.3, Theorem B.1, p. 745]). This completes the proof of Theorem 3.5. □

3.3.3. *Case* $\theta^* = 0$. Note that the deceptively simple case $\theta_j \equiv 0$ (leading to $\theta^* = 0$ in (3.19)) is not covered by Theorem 3.5, unless $\beta > 0$. The reason is that, in the lack of the logarithmic singularity of $g_\theta(z)$ at $z = R$, the main term in the asymptotic formula (3.22) vanishes (as suggested by the formal equality $1/\Gamma(0) = 0$). Thus, in the case $\theta^* = 0$ the singularity of $g_\kappa(z)$ at $z = R$ should become more prominent in the asymptotics. The full analysis of the competing contributions from the singularities of $g_\theta(z)$ and $g_\kappa(z)$ can be complicated (however, see Remark 3.4 below), so for simplicity let us focus on the role of $g_\kappa(z)$ assuming that $g_\theta(z)$ is regular (e.g., $g_\theta(z) \equiv 0$).

Theorem 3.6. *Let the generating function $g_\kappa(z)$ have radius of convergence $R > 0$ and be holomorphic in a domain Δ_0 as in Definition 3.3. Assume that $g_\kappa^{\{1\}}(R) < 1$ and, furthermore, there is a non-integer $s > 1$ such that $g_\kappa^{\{n\}}(R) < \infty$ for all $n < s$ while $g_\kappa^{\{n\}}(R) = \infty$ for $n > s$, and the following asymptotic expansion holds as $z \to R$ ($z \in \Delta_0$),*

$$g_\kappa(z) = \sum_{0\le n<s} \frac{(-1)^n g_\kappa^{\{n\}}(R)}{n!}\,(1 - z/R)^n + a_s\,(1 - z/R)^s + O\bigl((1 - z/R)^{s+\delta}\bigr), \quad (3.47)$$

with some $a_s > 0$, $\delta > 0$. As for the generating function $g_\theta(z)$, it is assumed to be holomorphic in the domain Δ_0 and, moreover, regular at point $z = R$. Finally, let $f\colon \Delta_0 \to \mathbb{C}$ be a holomorphic function such that, with some $\beta > 0$ and $c_\beta > 0$, as $z \to R$ ($z \in \Delta_0$),

$$f(z) = 1 + \sum_{1\le n<\beta} \frac{(-1)^n f^{\{n\}}(R)}{n!}\,(1 - z/R)^n + c_\beta\,(1 - z/R)^\beta + O\bigl((1 - z/R)^{\beta+\delta}\bigr). \quad (3.48)$$

Then, depending on the relationship between s and β, the following asymptotics hold as $N \to \infty$, uniformly in $v \in [v_1, v_2]$ for any $0 < v_1 < 1 < v_2 < 1/g_\kappa^{\{1\}}(R)$.

(i) *If $\beta > s - 1$ then*

$$[z^N]\bigl[f(z)\,\mathrm{e}^{vG_N(z)}\bigr] \sim \frac{\mathrm{e}^{vG_N(R)}\,v a_s \bigl\{N(1 - v g_\kappa^{\{1\}}(R))\bigr\}^{-s}}{R^N\,\Gamma(-s)\bigl(1 - v g_\kappa^{\{1\}}(R)\bigr)}. \quad (3.49)$$

(ii) *If β is non-integer and $\beta < s - 1$ then*

$$[z^N]\bigl[f(z)\,\mathrm{e}^{vG_N(z)}\bigr] \sim \frac{\mathrm{e}^{vG_N(R)}\,c_\beta \bigl\{N(1 - v g_\kappa^{\{1\}}(R))\bigr\}^{-\beta-1}}{R^N\,\Gamma(-\beta)}. \quad (3.50)$$

(iii) *If $\beta = s - 1$ then*

$$[z^N]\bigl[f(z)\,\mathrm{e}^{vG_N(z)}\bigr] \sim \frac{\mathrm{e}^{vG_N(R)}\bigl\{N(1 - v g_\kappa^{\{1\}}(R))\bigr\}^{-s}}{R^N} \times \left(\frac{c_\beta}{\Gamma(1-s)} + \frac{v a_s}{\Gamma(-s)\bigl(1 - v g_\kappa^{\{1\}}(R)\bigr)}\right). \quad (3.51)$$

In particular,

$$H_N \sim \frac{\mathrm{e}^{G_N(R)}\,a_s \bigl\{N(1 - g_\kappa^{\{1\}}(R))\bigr\}^{-s}}{R^N\,\Gamma(-s)\bigl(1 - g_\kappa^{\{1\}}(R)\bigr)}, \qquad N \to \infty. \quad (3.52)$$

Proof. The proof proceeds along the lines of the proof of Theorem 3.5, with suitable modifications indicated below. In what follows, we may and will assume that $0 < \delta < 1$.

Under the change of variables $z = R(1 + wN^{-1})$, with $w \in \gamma_2'$ (see (3.31)), by virtue of the expansions (3.47), (3.48) and the regularity of $g_\theta(z)$ we obtain the uniform asymptotics (cf. (3.32), (3.33))

$$g_\theta(z) = g_\theta(R) + \sum_{n=1}^{q} \frac{g_\theta^{\{n\}}(R)}{n!\,N^n}\, w^n + O(N^{-q-1}), \tag{3.53}$$

$$g_\kappa(z) = g_\kappa(R) + \sum_{n=1}^{q} \frac{g_\kappa^{\{n\}}(R)}{n!\,N^n}\, w^n + \frac{a_s(-w)^s}{N^s} + O(N^{-s-\delta}), \tag{3.54}$$

$$f(z) = 1 + \sum_{1\le n<\beta} \frac{f^{\{n\}}(R)}{n!\,N^n}\, w^n + \frac{c_\beta\,(-w)^\beta}{N^\beta} + O(N^{-\beta-\delta}),$$

with $q = \lfloor s \rfloor \ge 1$. We can also write (cf. (3.34))

$$\frac{1}{z^{N+1}} = \frac{1}{R^{N+1}}\, \exp\left\{\sum_{n=1}^{q+1} \frac{(-w)^n}{nN^{n-1}} + O(N^{-q-1})\right\}.$$

Substituting these expansions into the integral $\mathcal{I}_N^2$ over the contour γ_2 (see (3.25)), we obtain similarly to (3.35)

$$\begin{aligned}\mathcal{I}_N^2 \sim \frac{\mathrm{e}^{vG_N(R)}}{R^N N} \int_{\gamma_2'} \bigl[1 + \mathcal{P}_N(w) + c_\beta N^{-\beta}(-w)^\beta + O(N^{-\beta-\delta})\bigr] \\ \times \exp\bigl\{\mathcal{Q}_N(w) + v a_s N^{1-s}(-w)^s + O(N^{-s-\delta})\bigr\} \\ \times \exp\{-d_1 w\}\,\mathrm{d}w,\end{aligned} \tag{3.55}$$

where $d_1 = 1 - v g_\kappa^{\{1\}}(R) > 0$ (see (3.26)) and $\mathcal{P}_N(w)$, $\mathcal{Q}_N(w)$ are polynomials in w,

$$\mathcal{P}_N(w) = \sum_{1\le n<\beta} \frac{f^{\{n\}}(R)}{n!\,N^n}\, w^n,$$

$$\mathcal{Q}_N(w) = v\sum_{n=1}^{q} \frac{g_\theta^{\{n\}}(R)}{n!\,N^n}\, w^n + v\sum_{n=2}^{q} \frac{g_\kappa^{\{n\}}(R)}{n!\,N^{n-1}}\, w^n + v\sum_{n=2}^{q} \frac{(-w)^n}{nN^{n-1}}.$$

Noting that $\mathcal{Q}_N(w) = O(N^{-1})$ uniformly in $w \in \gamma_2'$, we can Taylor expand the exponential under the integral in (3.55) keeping the terms up to the order N^{1-s}. Thus, we obtain

$$\mathcal{I}_N^2 \sim \frac{\mathrm{e}^{vG_N(R)}}{R^N N} \int_{\gamma_2'} \bigl[\mathcal{R}_N(w) + v a_s N^{1-s}\,(-w)^s + c_\beta N^{-\beta}(-w)^\beta\bigr]\, \mathrm{e}^{-d_1 w}\,\mathrm{d}w,$$

where $\mathcal{R}_N(w)$ is the resulting polynomial in w.

A similar estimation holds for the integral $\mathcal{I}_N^3$ (cf. (3.44)). As a result, we get (cf. (3.45))

$$\oint_\gamma \frac{\mathrm{e}^{vG_N(z)}}{z^{N+1}}\,\mathrm{d}z \sim \frac{\mathrm{e}^{vG_N(R)}}{R^N N} \int_{\gamma'} \bigl[\mathcal{R}_N(w) + v a_s N^{1-s}\,(-w)^s + c_\beta N^{-\beta}(-w)^\beta\bigr]\, \mathrm{e}^{-d_1 w}\,\mathrm{d}w. \tag{3.56}$$

Since $\mathcal{R}_N(w)$ is an entire function of $w \in \mathbb{C}$, its contribution to the integral (3.56) vanishes,

$$\int_{\gamma'} \mathcal{R}_N(w)\,\mathrm{e}^{-d_1 w}\,\mathrm{d}w = 0. \tag{3.57}$$

Furthermore, changing the integration variable $d_1 w \mapsto w$ and transforming the contour γ' to the loop contour γ'' (see before equation (3.46)), we obtain

$$\int_{\gamma'} (-w)^s \, \mathrm{e}^{-d_1 w} \, \mathrm{d}w = (d_1)^{-1-s} \int_{\gamma''} (-w)^s \, \mathrm{e}^{-w} \, \mathrm{d}w = (d_1)^{-1-s} \, \frac{2\pi \mathrm{i}}{\Gamma(-s)}, \tag{3.58}$$

according to Hankel's identity akin to (3.46) (see [14, §B.3, Theorem B.1, p. 745]). As for the term $(-w)^\beta$ in (3.56), if $\beta \in \mathbb{N}$ then its contribution also vanishes (cf. (3.57)), but if β is non-integer then, similarly to (3.58), we have

$$\int_{\gamma'} (-w)^\beta \, \mathrm{e}^{-d_1 w} \, \mathrm{d}w = (d_1)^{-1-\beta} \, \frac{2\pi \mathrm{i}}{\Gamma(-\beta)}. \tag{3.59}$$

It remains to note that the contributions (3.58) and (3.59) enter the asymptotic formula (3.56) with the weights N^{1-s} and $N^{-\beta}$, respectively, so the relationship between the exponents $s-1$ and β determines which of these two terms is the principal one in the limit as $N \to \infty$. Accordingly, retaining one of the power terms in (3.56) (or both, if $\beta = s-1$) and dividing everything by $2\pi\mathrm{i}$ (cf. (3.25)) , we arrive at formulas (3.49), (3.50) and (3.51), respectively.

Finally, formula (3.52) follows from (3.49) by setting $f(z) \equiv 1$, $v = 1$; note that in this case the parameter β in (3.48) can be formally chosen to be any positive (integer) number, e.g. bigger than $s-1$, so formula (3.49) applies. □

Remark 3.4. Analogous considerations as in the proof of Theorem 3.6 can be used to handle the case with a *power-logarithmic* term $b_s(1-z/R)^s \log(1-z/R)$ $(b_s \neq 0)$ added to the expansion (3.47) of $g_\kappa(z)$, where the index $s > 1$ is now allowed to be *integer*; this is motivated by the choice $\kappa_j := \kappa^* j^{-q}$ with $q \in \mathbb{N}$, leading to the polylogarithm $g_\kappa(z) = \kappa^* \operatorname{Li}_{q+1}(z)$ with the asymptotics (3.113) (see Section 3.5.2 below). Furthermore, the condition of regularity of $g_\theta(z)$ at $z = R$ imposed in Theorem 3.6 is also not essential, and may be extended to include a power singularity of the form $\tilde{a}_{s_1}(1-z/R)^{s_1}$ (with $\tilde{a}_{s_1} \neq 0$ and some non-integer $s_1 > 0$) and, possibly, a power-logarithmic singularity, similarly to what was said above about $g_\kappa(z)$ (with any $s_1 > 0$). We leave the details to the interested reader.

3.4. **The critical case.**

3.4.1. *First theorems.* Here $g_\kappa^{\{1\}}(R) = 1$, and the equation $g_\kappa^{\{1\}}(r) = v^{-1}$ is still solvable for all $v \geq 1$, with the unique root $r_v \leq R$ (see (3.6)). Thus, the same argumentation may be applied as in the proof of Theorem 3.3, but for this to cover the case $v = 1$ one has to assume that $g_\theta(R) < \infty$, because the circle $z = r_v \, \mathrm{e}^{\mathrm{i}t}$ to be used in Cauchy's integral formula passes through the singularity $z = R$ if $v = 1$ (with $r_1 = R$). Recall that the function $G_N(z)$ is defined in (2.7), and $b_2(r)$ is given by (3.3).

Theorem 3.7. *Assume that both $g_\theta(z)$ and $g_\kappa(z)$ have radius of convergence $R > 0$, with $g_\theta(R) < \infty$, $g_\kappa(R) < \infty$ and, moreover, $g_\kappa^{\{1\}}(R) = 1$, $0 < g_\kappa^{\{2\}}(R) < \infty$. Let $f(z)$ be a function holomorphic in the open disk $|z| < R$ and continuous up to the boundary $|z| = R$. Then, uniformly in $v \in [1, v_0]$ with an arbitrary constant $v_0 \geq 1$, we have*

$$[z^N]\bigl[f(z) \exp\{vG_N(z)\}\bigr] \sim \frac{f(r_v) \exp\{vG_N(r_v)\}}{r_v^N \sqrt{2\pi N v b_2(r_v)}}, \qquad N \to \infty. \tag{3.60}$$

In particular,

$$H_N \sim \frac{\exp\{G_N(R)\}}{R^N \sqrt{2\pi N b_2(R)}}, \qquad N \to \infty. \tag{3.61}$$

Proof. Note that Taylor's expansion (3.5) extends to $r = R$, with $b_1(R) = g_\kappa^{\{1\}}(R) = 1$, $b_2(R) = 1 + g_\kappa^{\{2\}}(R) < \infty$ (see (3.3)). Then, on account of the continuity of $g_\theta(z)$ and $f(z)$ on each circle $|z| = r_v \le R$, it is easy to check that the proof of Theorem 3.3 is valid with any $r_v \le R$, thus leading to (3.60) (cf. (3.8)). As before, formula (3.61) follows from (3.60) by setting $f(z) \equiv 1$ and $v = 1$, making $r_v|_{v=1} = R$. □

If $g_\theta(z)$ has a logarithmic singularity at $z = R$, we can still use the same argumentation as in Theorem 3.5 as long as $v \ne 1$.

Theorem 3.8. *Assume that $g_\theta(z)$ and $g_\kappa(z)$ both have radius of convergence $R > 0$ and are holomorphic in some domain Δ_0 as in Definition 3.3. Let $g_\kappa^{\{1\}}(R) = 1,\ 0 < g_\kappa^{\{2\}}(R) < \infty$, and suppose that for some $\theta^* \ge 0,\ \delta > 0$, as $z \to R$ $(z \in \Delta_0)$,*

$$g_\theta(z) = -\theta^* \log(1 - z/R) + O\bigl((1 - z/R)^\delta\bigr), \tag{3.62}$$

$$g_\kappa(z) = g_\kappa(R) - (1 - z/R) + \frac{g_\kappa^{\{2\}}(R)}{2}(1 - z/R)^2 + O\bigl((1 - z/R)^{2+\delta}\bigr). \tag{3.63}$$

Finally, let $f\colon \Delta_0 \to \mathbb{C}$ be a holomorphic function such that for some $\beta \ge 0$

$$f(z) = (1 - z/R)^{-\beta}\bigl\{1 + O\bigl((1 - z/R)^\delta\bigr)\bigr\}, \qquad z \to R \ \ (z \in \Delta_0).$$

Then, as $N \to \infty$,

(a) *uniformly in $v \in [v_1, v_2]$ with arbitrary $v_2 \ge v_1 > 1$,*

$$[z^N]\bigl[f(z)\exp\{vG_N(z)\}\bigr] \sim \frac{f(r_v)\exp\{vG_N(r_v)\}}{r_v^N\sqrt{2\pi N v b_2(r_v)}}; \tag{3.64}$$

(b) *uniformly in $v \in [v_1, v_2]$ with arbitrary $0 < v_1 \le v_2 < 1$, provided that $v\theta^* + \beta > 0$,*

$$[z^N]\bigl[f(z)\exp\{vG_N(z)\}\bigr] \sim \frac{\exp\{Nvg_\kappa(R)\}\cdot\{N(1-v)\}^{v\theta^*+\beta-1}}{R^N\,\Gamma(v\theta^*+\beta)}. \tag{3.65}$$

Proof. (a) Like in Theorem 3.7, we can use the proof of Theorem 3.3 as long as $r_v < R$, which is guaranteed by the condition $v > 1$.

(b) Here the proof of Theorem 3.5 applies as long as $v < 1/g_\kappa^{\{1\}}(R) = 1$ (cf. (3.22)). □

Note that, unlike Theorem 3.7, both parts (a) and (b) of Theorem 3.8 do not cover the value $v = 1$ and so do not provide the asymptotics of H_N. Moreover, the asymptotic expressions on the right-hand side of (3.64) and (3.65) vanish as $v \to 1$, which suggests that no uniform statements are possible on intervals (even one-sided) that contain the point $v = 1$. Nevertheless, the case $v = 1$ can be handled by a more careful adaptation of the proof of Theorem 3.5. First, we need to compute some complex integrals emerging in the asymptotics.

3.4.2. *Two auxiliary integrals.* Let $\gamma' = \gamma'_1 \cup \gamma'_2 \cup \gamma'_3$ be a continuous contour composed of the parts defined as follows (see Fig. 3(b)),

$$\begin{aligned}
\gamma'_1 &:= \bigl\{w = -y\mathrm{e}^{-\mathrm{i}\varphi},\ y \in (-\infty, -\epsilon]\bigr\},\\
\gamma'_2 &:= \bigl\{w = \epsilon\,\mathrm{e}^{-\mathrm{i}t},\ t \in [\varphi, 2\pi - \varphi]\bigr\},\\
\gamma'_3 &:= \bigl\{w = y\mathrm{e}^{\mathrm{i}\varphi},\ y \in [\epsilon, +\infty)\bigr\},
\end{aligned} \tag{3.66}$$

with $\varphi \in (\pi/4, \pi/2)$ and $\epsilon > 0$. For any real parameter ξ, let us consider the contour integral

$$J_\xi(\sigma) := \int_{\gamma'} (-w)^{-\sigma}\exp\{-\xi w + w^2\}\,\mathrm{d}w, \qquad \sigma \in \mathbb{C}. \tag{3.67}$$

The determination of $(-w)^{-\sigma}$ in (3.67) is defined by the principal branch when w is negative real, and then extended uniquely by continuity along the contour. An absolute value estimate gives for $w \in \gamma_1' \cup \gamma_3'$

$$\bigl|(-w)^{-\sigma} \exp\{-\xi w + w^2\}\bigr| \le |w|^{-\Re(\sigma)}\, \mathrm{e}^{|\sigma|\pi} \exp\{-\xi|w|\cos\varphi + |w|^2 \cos 2\varphi\}, \qquad (3.68)$$

where $\cos 2\varphi < 0$ according to the chosen range of φ. Hence, the integral (3.67) is absolutely convergent for all complex s and, moreover, the function $\sigma \mapsto J_\xi(\sigma)$ is holomorphic in the entire complex plane $\mathbb{C}$. An explicit analytic continuation from the domain $\Re(\sigma) < 0$ is furnished through the following functional equation (which can be easily obtained from (3.67) by integration by parts),

$$2J_\xi(\sigma) = (\sigma+1)\, J_\xi(\sigma+2) - \xi J_\xi(\sigma+1), \qquad \sigma \in \mathbb{C}.$$

From the estimate (3.68) it also follows that the integral (3.67) does not depend on the choice of the angle φ and the arc radius $\epsilon > 0$; in the computation below, we will choose $\varphi = \pi/2$ and take the limit $\epsilon \to 0$.

It is straightforward to calculate a few values of $J_\xi(\sigma)$, such as

$$J_0(-1) = 0, \qquad J_0(1) = \mathrm{i}\pi, \qquad J_\xi(0) = \mathrm{i}\sqrt{\pi}\,\mathrm{e}^{-\xi^2/4}.$$

A general formula for the integral (3.67) is established in the next lemma.

Lemma 3.9. *For any $\xi \in \mathbb{R}$ and all $\sigma \in \mathbb{C}$, there is the identity*

$$J_\xi(\sigma) = \frac{\mathrm{i}\pi\, \exp\{-\xi^2/4\}}{\Gamma\bigl((\sigma+1)/2\bigr)\,\Gamma(\sigma/2)} \sum_{n=0}^{\infty} \Gamma\!\left(\frac{\sigma+n}{2}\right) \frac{\xi^n}{n!}. \qquad (3.69)$$

In particular, for $\xi = 0$ we have

$$J_0(\sigma) = \frac{\mathrm{i}\pi}{\Gamma\bigl((\sigma+1)/2\bigr)}, \qquad \sigma \in \mathbb{C}. \qquad (3.70)$$

Proof. It suffices to prove (3.69) for real $\sigma < 0$ (an extension to arbitrary $\sigma \in \mathbb{C}$ will follow by analytic continuation). As already mentioned, we can choose $\varphi = \pi/2$ in (3.66), so that the parts of the original contour $\gamma' = \gamma_1' \cup \gamma_2' \cup \gamma_3'$ take the form

$$\begin{aligned} \gamma_1' &= \bigl\{w = \mathrm{i}y,\ y \in [-\infty, -\epsilon]\bigr\}, \\ \gamma_2' &= \bigl\{w = \epsilon\,\mathrm{e}^{-\mathrm{i}t},\ t \in [\pi/2, 3\pi/2]\bigr\}, \\ \gamma_3' &= \bigl\{w = \mathrm{i}y,\ y \in [\epsilon, +\infty]\bigr\}. \end{aligned} \qquad (3.71)$$

Here the parameter $\epsilon > 0$ is arbitrary, and the idea is to send it to zero.

First of all, since we have assumed that $-\sigma > 0$, the integral over γ_2' vanishes as $\epsilon \to 0$. Next, using the parameterization (3.71) of the contour γ_1' we compute

$$\begin{aligned}\int_{\gamma_1'} (-w)^{-\sigma} \exp\{-\xi w + w^2\}\,\mathrm{d}w &= \mathrm{i}\int_{-\infty}^{-\epsilon} (-\mathrm{i}y)^{-\sigma} \exp\{-\mathrm{i}\xi y - y^2\}\,\mathrm{d}y \\ &= \mathrm{i}\int_{\epsilon}^{\infty} (\mathrm{i}y)^{-\sigma} \exp\{\mathrm{i}\xi y - y^2\}\,\mathrm{d}y \\ &= \mathrm{i}\big(\mathrm{e}^{\mathrm{i}\pi/2}\big)^{-\sigma} \int_{\epsilon}^{\infty} y^{-\sigma} \exp\{\mathrm{i}\xi y - y^2\}\,\mathrm{d}y \\ &\to \mathrm{i}\int_0^{\infty} y^{-\sigma} \exp\left\{\mathrm{i}\left(\xi y - \frac{\pi\sigma}{2}\right) - y^2\right\}\mathrm{d}y, \qquad \epsilon \to 0.\end{aligned} \tag{3.72}$$

A similar computation for γ_3' gives

$$\begin{aligned}\int_{\gamma_3'} (-w)^{-\sigma} \exp\{-\xi w + w^2\}\,\mathrm{d}w &= \mathrm{i}\int_{\epsilon}^{\infty} (-\mathrm{i}y)^{-\sigma} \exp\{-\mathrm{i}\xi y - y^2\}\,\mathrm{d}y \\ &= \mathrm{i}\big(\mathrm{e}^{-\mathrm{i}\pi/2}\big)^{-\sigma} \int_{\epsilon}^{\infty} y^{-\sigma} \exp\{-\mathrm{i}\xi y - y^2\}\,\mathrm{d}y \\ &\to \mathrm{i}\int_0^{\infty} y^{-\sigma} \exp\left\{-\mathrm{i}\left(\xi y - \frac{\pi\sigma}{2}\right) - y^2\right\}\mathrm{d}y, \qquad \epsilon \to 0.\end{aligned} \tag{3.73}$$

Combining (3.72) and (3.73), by using Euler's formula $\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi} = 2\cos\phi$ and some elementary trigonometric identities we obtain

$$\begin{aligned}\frac{1}{2}(-\mathrm{i})J_\xi(\sigma) &= \int_0^{\infty} y^{-\sigma} \cos\left(\xi y - \frac{\pi\sigma}{2}\right) \mathrm{e}^{-y^2}\,\mathrm{d}y \\ &= A_1(\xi,\sigma)\,\sin\frac{\pi(1-\sigma)}{2} + A_2(\xi,\sigma)\,\sin\frac{\pi\sigma}{2},\end{aligned} \tag{3.74}$$

where

$$A_1(\xi,\sigma) := \int_0^{\infty} y^{-\sigma} \cos(\xi y)\,\mathrm{e}^{-y^2}\,\mathrm{d}y, \qquad A_2(\xi,\sigma) := \int_0^{\infty} y^{-\sigma} \sin(\xi y)\,\mathrm{e}^{-y^2}\,\mathrm{d}y. \tag{3.75}$$

For $-\sigma > 0$ the integrals (3.75) are given by (see [15, #3.952(7, 8), p. 503])

$$A_1(\xi,\sigma) = \frac{1}{2}\Gamma\left(\frac{1-\sigma}{2}\right) \mathrm{e}^{-\xi^2/4}\,{}_1F_1\left(\frac{\sigma}{2}, \frac{1}{2}, \frac{\xi^2}{4}\right), \tag{3.76}$$

$$A_2(\xi,\sigma) = \frac{1}{2}\Gamma\left(1 - \frac{\sigma}{2}\right)\, \xi\,\mathrm{e}^{-\xi^2/4}\,{}_1F_1\left(\frac{\sigma+1}{2}, \frac{3}{2}, \frac{\xi^2}{4}\right), \tag{3.77}$$

where ${}_1F_1(a, b, z)$ is the *confluent hypergeometric function* (see [15, #9.210(1), p. 1023])

$${}_1F_1(a, b, z) := 1 + \sum_{n=1}^{\infty} \frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)}\,\frac{z^n}{n!}, \qquad z \in \mathbb{C}, \tag{3.78}$$

with $b \notin -\mathbb{N}_0$. It is easy to see (e.g., using the ratio test) that the series (3.78) converges for all $z \in \mathbb{C}$, and hence ${}_1F_1(a, b, z)$ is an entire function of z. In particular, ${}_1F_1(0, b, z) \equiv 1$.

Substituting the expressions (3.76), (3.77) into (3.74) and using (twice) the well-known complement formula for the gamma function (see, e.g., [14, §B.3, pp. 745–746])

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}\,,\qquad z\in\mathbb{C},\tag{3.79}$$

we obtain

$$(-\mathrm{i})\,J_\xi(\sigma)=\pi\,\mathrm{e}^{-\xi^2/4}\left\{\frac{{}_1F_1\bigl(\frac{\sigma}{2},\frac{1}{2},\frac{\xi^2}{4}\bigr)}{\Gamma\bigl(\frac{\sigma}{2}+\frac{1}{2}\bigr)}+\xi\,\frac{{}_1F_1\bigl(\frac{\sigma}{2}+\frac{1}{2},\frac{3}{2},\frac{\xi^2}{4}\bigr)}{\Gamma\bigl(\frac{\sigma}{2}\bigr)}\right\}.\tag{3.80}$$

Furthermore, observe from (3.78) that

$$\begin{aligned}{}_1F_1\left(\frac{\sigma}{2},\frac{1}{2},\frac{\xi^2}{4}\right)&=\frac{1}{\Gamma\bigl(\frac{\sigma}{2}\bigr)}\sum_{k=0}^{\infty}\frac{\Gamma\bigl(\frac{\sigma}{2}+k\bigr)\,\xi^{2k}}{\frac{1}{2}\cdot\frac{3}{2}\cdots\bigl(\frac{1}{2}+k-1\bigr)\,2^{2k}k!}\\&=\frac{1}{\Gamma\bigl(\frac{\sigma}{2}\bigr)}\sum_{k=0}^{\infty}\frac{\Gamma\bigl(\frac{\sigma}{2}+k\bigr)\,\xi^{2k}}{(2k)!}\,,\end{aligned}\tag{3.81}$$

and similarly

$$\xi\cdot{}_1F_1\left(\frac{\sigma+1}{2},\frac{3}{2},\frac{\xi^2}{4}\right)=\frac{1}{\Gamma\bigl(\frac{\sigma+1}{2}\bigr)}\sum_{k=0}^{\infty}\frac{\Gamma\bigl(\frac{\sigma}{2}+\frac{1}{2}+k\bigr)\,\xi^{2k+1}}{(2k+1)!}\,.\tag{3.82}$$

Now, returning to (3.80) and combining the sums in (3.81) and (3.82) we arrive at (3.69).

Finally, formula (3.70) readily follows from (3.69) by setting $\xi=0$. □

There is a simple probabilistic representation of the power series part of the expression (3.69), which will be helpful in applications (see the proof of Theorem 4.4(c-iii) below).

Lemma 3.10. *Let X be a random variable with gamma distribution $\mathrm{Gamma}\,(\sigma/2)$, that is, with probability density $f(x)=(1/\Gamma(\sigma/2))\,x^{\sigma/2-1}\,\mathrm{e}^{-x}$, $x>0$. Then the moment generating function of $\sqrt{X}$ is given by*

$$\mathbb{E}\bigl[\mathrm{e}^{\xi\sqrt{X}}\bigr]=\frac{1}{\Gamma(\sigma/2)}\sum_{n=0}^{\infty}\Gamma\left(\frac{\sigma+n}{2}\right)\frac{\xi^n}{n!}\,,\qquad \xi\in\mathbb{R}.\tag{3.83}$$

Proof. The left-hand side of (3.83) can be expanded in a series

$$\mathbb{E}\bigl[\mathrm{e}^{\xi\sqrt{X}}\bigr]=\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\,\mathbb{E}\bigl[X^{n/2}\bigr].\tag{3.84}$$

The moments of X in (3.84) are easily computed,

$$\mathbb{E}\bigl[X^{n/2}\bigr]=\frac{1}{\Gamma(\sigma/2)}\int_0^\infty x^{n/2+\sigma/2-1}\,\mathrm{e}^{-x}\,\mathrm{d}x=\frac{\Gamma\bigl((\sigma+n)/2\bigr)}{\Gamma(\sigma/2)}\,,$$

and returning to (3.84) we obtain (3.83). □

A similar argumentation as in Lemma 3.9 may be applied to the integral

$$\tilde{J}_0(\sigma;s):=\int_{\gamma'}(-w)^{-\sigma}\exp\{(-w)^s\}\,\mathrm{d}w,\qquad \sigma\in\mathbb{C},\ \ 1<s\le 2,\tag{3.85}$$

with the contour $\gamma'=\gamma_1'\cup\gamma_2'\cup\gamma_3'$ as defined in (3.66). Note that for $s=2$ the integral (3.85) is reduced to (3.70): $\tilde{J}_0(\sigma;2)=J_0(\sigma)$. The general case is handled in the next lemma.

Lemma 3.11. *For any* $1 < s \le 2$*, the following identity holds*

$$\tilde{J}_0(\sigma; s) = \frac{2\pi \mathrm{i}}{s\,\Gamma\Big(\dfrac{s-1+\sigma}{s}\Big)}, \qquad \sigma \in \mathbb{C}. \tag{3.86}$$

Proof. Like in the proof of Lemma 3.9, it suffices to consider the case $-\sigma > 0$. Then, similarly to (3.72) and (3.73), we obtain (cf. (3.74))

$$\tfrac{1}{2}(-\mathrm{i})\,\tilde{J}_0(\sigma; s) = \int_0^\infty y^{-\sigma} \cos\Big(\frac{\pi\sigma}{2} - y^s \sin\frac{\pi s}{2}\Big) \exp\Big\{y^s \cos\frac{\pi s}{2}\Big\}\, \mathrm{d}y. \tag{3.87}$$

Note that, according to the condition $1 < s \le 2$, we have $\cos(\pi s/2) < 0$. By the change of variable $y^s = x$ and with the help of some standard trigonometric identities, the integral on the right-hand side of (3.87) is rewritten as

$$\frac{1}{s}\Big(B_1(\sigma; s)\cos\frac{\pi\sigma}{2} + B_2(\sigma; s)\sin\frac{\pi\sigma}{2}\Big), \tag{3.88}$$

where

$$B_1(\sigma; s) := \int_0^\infty x^{-1+(1-\sigma)/s} \cos\Big(x \sin\frac{\pi s}{2}\Big) \exp\Big\{x \cos\frac{\pi s}{2}\Big\}\, \mathrm{d}y,$$
$$B_2(\sigma; s) := \int_0^\infty x^{-1+(1-\sigma)/s} \sin\Big(x \sin\frac{\pi s}{2}\Big) \exp\Big\{x \cos\frac{\pi s}{2}\Big\}\, \mathrm{d}y.$$

These integrals can be explicitly computed as follows (see [15, #3.944 (5, 6), p. 498])

$$B_1(\sigma; s) = \Gamma\Big(\frac{1-\sigma}{s}\Big)\cos\psi, \qquad B_2(\sigma; s) = \Gamma\Big(\frac{1-\sigma}{s}\Big)\sin\psi,$$

where

$$\psi := \frac{\pi(1-\sigma)(2-s)}{2s}.$$

Substituting this into (3.87) and (3.88) we get

$$\begin{aligned}
\tfrac{1}{2}(-\mathrm{i})\,\tilde{J}_0(\sigma; s) &= \frac{1}{s}\Gamma\Big(\frac{1-\sigma}{s}\Big)\Big(\cos\psi\,\cos\frac{\pi\sigma}{2} + \sin\psi\,\sin\frac{\pi\sigma}{2}\Big)\\
&= \frac{1}{s}\Gamma\Big(\frac{1-\sigma}{s}\Big)\cos\Big(\psi - \frac{\pi\sigma}{2}\Big)\\
&= \frac{1}{s}\Gamma\Big(\frac{1-\sigma}{s}\Big)\sin\frac{\pi(1-\sigma)}{s}\\
&= \frac{\pi}{s\,\Gamma\Big(\dfrac{s-1+\sigma}{s}\Big)},
\end{aligned}$$

again using the complement formula (3.79). Hence, the result (3.86) follows. □

3.4.3. *Asymptotic theorems with* $v \approx 1$. We are now in a position to obtain "dynamic" asymptotic results for the critical case in the neighbourhood of $v = 1$. For simplicity, we omit the pre-exponential factor $f(z)$ (which will not be needed). First, let us consider a "regular" case where the second derivative of $g_\kappa(z)$ at $z = R$ is finite.

Theorem 3.12. *Let $g_\theta(z)$ and $g_\kappa(z)$ satisfy the conditions of Theorem* 3.8 *with $\theta^* \ge 0$ (see* (3.62)*), for a suitable domain of holomorphicity Δ_0. Assume also that $g_\kappa^{\{1\}}(R) = 1$ and $0 < g_\kappa^{\{2\}}(R) < \infty$. Finally, let $f\colon \Delta_0 \to \mathbb{C}$ be a holomorphic function such that*

$$f(z) = 1 + O\big((1 - z/R)^\delta\big), \qquad z \to R \quad (z \in \Delta_0).$$

Then for any $u \ge 0$, as $N \to \infty$,

$$[z^N]\big[f(z)\exp\{\mathrm{e}^{-u/\sqrt{N}} G_N(z)\}\big] \sim \frac{\exp\big\{Ng_\kappa(R) - \sqrt{N}\,ug_\kappa(R) + \tfrac12 u^2 g_\kappa(R)\big\}}{R^N} \times \big(\tfrac12 N b_2(R)\big)^{(\theta^*-1)/2}\,\frac{J_{\tilde u}(\theta^*)}{2\pi\mathrm{i}}, \tag{3.89}$$

where $\tilde u := u\,\sqrt{2/b_2(R)}$ and $J_{\tilde u}(\theta^*)$ is given by formula (3.69)*. In particular,*

$$H_N \sim \frac{\exp\{Ng_\kappa(R)\}\cdot\big(\tfrac12 N b_2(R)\big)^{(\theta^*-1)/2}}{2R^N\,\Gamma\big(\frac{1+\theta^*}{2}\big)}, \qquad N \to \infty. \tag{3.90}$$

Proof. Setting $v = \mathrm{e}^{-u\sqrt{N}}$, we adapt the proof of Theorem 3.5 by making the following modifications:

(i) the angle φ in the specification (3.24) of the contour γ is chosen in the range $\varphi \in (\pi/4, \pi/2)$, so that $\cos 2\varphi < 0$;
(ii) the rescaling coefficient N^{-1} throughout (3.24) is replaced by $N^{-1/2}$;
(iii) under the change of variables $z = R(1 + wN^{-1/2})$ (cf. (3.31)), Taylor's formulas (3.33) and (3.34) are extended to the corresponding second-order expansions (in particular, using (3.47)).

Note that due to the criticality assumption, the quantity $d_1 = 1 - vg_\kappa^{\{1\}}(R)$ (see (3.26)) is reduced to zero for $v = 1$, so that the estimate (3.29) of the asymptotic contribution of the integral $\mathcal{I}_N^4$ becomes void. This can be fixed by taking advantage of an enhanced version of the inequality (3.28) provided by Lemma 3.4 (see (3.15)), leading to an improved bound

$$|\mathcal{I}_N^4| = \frac{O(1)\,\exp\{Ng_\kappa(R)\}}{R^N}\,\exp\left\{-N\varepsilon\log\frac{R'}{R}\right\}, \qquad N \to \infty.$$

Next, under the substitution $z = R(1 + wN^{-1/2})$, using (3.63) and the expansion

$$\mathrm{e}^{-u/\sqrt{N}} = 1 - \frac{u}{\sqrt{N}} + \frac{u^2}{2N} + O(N^{-3/2}), \qquad N \to \infty,$$

we have, as $N \to \infty$,

$$\mathrm{e}^{-u/\sqrt{N}} Ng_\kappa(z) = Ng_\kappa(R) - \sqrt{N}\,ug_\kappa(R) + \tfrac12 u^2 g_\kappa(R) + \sqrt{N}\,w - uw + \tfrac12\,g_\kappa^{\{2\}}(R)\,w^2 + O\big(w^{2+\delta}N^{-\delta/2}\big). \tag{3.91}$$

Combined with the expansion (cf. (3.34))

$$\frac{1}{z^{N+1}} = \frac{1}{R^{N+1}}\,\exp\big\{-\sqrt{N}\,w + \tfrac12 w^2 + O(w^3 N^{-3/2})\big\}, \qquad N \to \infty, \tag{3.92}$$

this leads to the following asymptotics of the integral $\mathcal{I}_N^2$ as $N\to\infty$ (cf. (3.35))

$$\mathcal{I}_N^2 \sim \frac{\exp\bigl\{Ng_\kappa(R)-\sqrt{N}\,ug_\kappa(R)+\frac{1}{2}u^2 g_\kappa(R)\bigr\}}{R^N} \times \bigl(\tfrac{1}{2}Nb_2(R)\bigr)^{(\theta^*-1)/2}\int_{\widetilde{\gamma}_2'}(-w)^{-\theta^*}\exp\{-\tilde{u}w+w^2\}\,\mathrm{d}w, \qquad (3.93)$$

where $\widetilde{\gamma}_2' := \gamma_2'\sqrt{b_2(R)/2}$ and $\tilde{u} := u\sqrt{2/b_2(R)}$.

The integral $\mathcal{I}_N^3$ is asymptotically evaluated in a similar fashion (cf. (3.44)), leading to the same formula as (3.93) but with the contour of integration γ_3' defined in (3.38). Note that the error terms (3.39) and (3.43), formally invalidated by $d_1=0$, can be adapted by using Lemma 3.4 as described above.

As a result, collecting the principal asymptotic terms from formula (3.93) and its other counterparts (cf. (3.37)) produces the integral $J_{\tilde{u}}(\theta^*)$, according to the notation (3.67). Hence, we arrive at the expression (3.89), as claimed. Finally, (3.90) immediately follows from (3.89) by setting $u=0$ and on using formula (3.70). □

Remark 3.5. For $\theta^*=0$, formula (3.90) gives the same asymptotics of H_N as formula (3.61) of Theorem 3.7 (note that $\Gamma(1/2)=\sqrt{\pi}$). Moreover, formula (3.89) with $\theta^*=0$ coincides with the dynamic version obtained from Theorem 3.7 by setting $v=\mathrm{e}^{-u/\sqrt{N}}$; this can be shown using calculations carried out in the proof of Theorem 4.4(a) below.

Let us now study the case with an infinite second derivative of $g_\kappa(z)$ at $z=R$.

Theorem 3.13. *Let $g_\theta(z)$ and $g_\kappa(z)$ satisfy the assumptions of Theorem* 3.8 *with $\theta^*\ge 0$ (see* (3.62)*), including the condition $g_\kappa^{\{1\}}(R)=1$, but with the expansion* (3.63) *replaced by*

$$g_\kappa(z)=g_\kappa(R)-(1-z/R)+a_s(1-z/R)^s+O\bigl((1-z/R)^{s+\delta}\bigr),$$

with $s\in(1,2)$, $a_s>0$ and $\delta>0$. Let $f\colon \Delta_0\to\mathbb{C}$ be a holomorphic function satisfying

$$f(z)=1+O\bigl((1-z/R)^{\delta}\bigr),\qquad z\to R\quad(z\in\Delta_0).$$

Then for any $u\ge 0$, as $N\to\infty$,

$$[z^N]\bigl[f(z)\exp\{\mathrm{e}^{-u/\sqrt{N}}G_N(z)\}\bigr]\sim\frac{\exp\bigl\{Ng_\kappa(R)-\sqrt{N}\,ug_\kappa(R)+\frac{1}{2}u^2g_\kappa(R)\bigr\}}{R^N}\times\frac{(Na_s)^{(\theta^*-1)/s}}{s\,\Gamma\bigl(\frac{s-1+\theta^*}{s}\bigr)}. \qquad (3.94)$$

In particular,

$$H_N\sim\frac{\exp\{Ng_\kappa(R)\}\cdot(Na_s)^{(\theta^*-1)/s}}{sR^N\,\Gamma\bigl(\frac{s-1+\theta^*}{s}\bigr)},\qquad N\to\infty. \qquad (3.95)$$

Proof. In what follows, we can assume that $0<\delta\le 2-s$. Again setting $v=\mathrm{e}^{-u\sqrt{N}}$, we adapt the proof of Theorem 3.12 by using the change of variables $z=R(1+wN^{-1/s})$. Hence, the expansions (3.91) and (3.92) are replaced, respectively, by

$$\mathrm{e}^{-u/\sqrt{N}}Ng_\kappa(z)=Ng_\kappa(R)-\sqrt{N}\,ug_\kappa(R)+\tfrac{1}{2}u^2g_\kappa(R) + N^{1-1/s}\,w+a_s(-w)^s+O\bigl(w^{s+\delta}N^{-\delta/s}\bigr)$$

and

$$\frac{1}{z^{N+1}}=\frac{1}{R^{N+1}}\,\exp\bigl\{-N^{1-1/s}\,w+O(w^2N^{1-2/s})\bigr\}.$$

Similarly as in the proof of Theorem 3.13, this leads to the asymptotics (cf. (3.93))

$$\mathcal{I}_N^2 \sim \frac{\exp\bigl\{Ng_\kappa(R)-\sqrt{N}\,u g_\kappa(R)+\frac{1}{2}u^2 g_\kappa(R)\bigr\}}{R^N} \times (Na_s)^{(\theta^*-1)/s}\int_{\widetilde{\gamma}'_2}(-w)^{-\theta^*}\exp\{(-w)^s\}\,\mathrm{d}w,$$

with the rescaled contour $\widetilde{\gamma}'_2 := a_s^{1/s}\gamma'_2$. The integral $\mathcal{I}_N^3$ is estimated similarly (cf. (3.44)). As a result, recalling the notation (3.85) we obtain

$$[z^N]\bigl[\exp\{\mathrm{e}^{-u/\sqrt{N}}G_N(z)\}\bigr] \sim \frac{\exp\bigl\{Ng_\kappa(R)-\sqrt{N}\,u g_\kappa(R)+\frac{1}{2}u^2 g_\kappa(R)\bigr\}}{R^N} \times (Na_s)^{(\theta^*-1)/s}\,\frac{\tilde{J}_0(\theta^*;s)}{2\pi\mathrm{i}}\,,$$

and (3.94) follows on using formula (3.86).

Finally, formula (3.95) follows by setting $u=0$ in (3.94). □

3.5. **Examples.** Let us give a few simple examples to illustrate the conditions of the asymptotic theorems in Sections 3.2–3.4.

3.5.1. *Constant coefficients.* For all $j\in\mathbb{N}$, let $\theta_j=\theta^*\ge 0$, $\kappa_j=\kappa^*=\mathrm{const}>0$. Then the corresponding generating functions specialize to

$$g_\theta(z) = -\theta^*\log(1-z), \qquad g_\kappa(z) = -\kappa^*\log(1-z). \tag{3.96}$$

Recalling expression (2.10) and using the binomial series expansion, we have explicitly

$$\begin{aligned} H_N &= [z^N]\exp\{G_N(z)\} = [z^N]\,(1-z)^{-(\theta^*+N\kappa^*)} \\ &= [z^N]\sum_{j=0}^{\infty}\binom{\theta^*+N\kappa^*+j-1}{j}z^j = \binom{\theta^*+N(\kappa^*+1)-1}{N}, \end{aligned}$$

and with Stirling's formula for the gamma function this yields the asymptotics

$$H_N \sim \frac{1}{\sqrt{2\pi N}}\left(\frac{\kappa^*+1}{\kappa^*}\right)^{\theta^*+N\kappa^*-1/2}(\kappa^*+1)^N, \qquad N\to\infty. \tag{3.97}$$

To apply the general machinery developed in Section 3.2, from (3.96) we find

$$g_\kappa^{\{1\}}(z) = \frac{\kappa^* z}{1-z}\,, \qquad g_\kappa^{\{2\}}(z) = \frac{\kappa^* z^2}{(1-z)^2}\,, \tag{3.98}$$

so that the expressions (3.3) specialize to

$$b_1(r) = \frac{\kappa^* r}{1-r}\,, \qquad b_2(r) = \frac{\kappa^* r}{1-r}+\frac{\kappa^* r^2}{(1-r)^2} = \frac{\kappa^* r}{(1-r)^2}\,. \tag{3.99}$$

Here $R=1$ and $g_\kappa(1)=+\infty$, $g_\kappa^{\{1\}}(1)=+\infty$, so that, according to our terminological convention in Section 3.1 (see Definition 3.2), we are always in the subcritical regime. In view of (3.98), the solution r_v of the equation (3.6) is explicitly given by

$$r_v = \frac{1}{v\kappa^*+1}\,, \qquad v>0. \tag{3.100}$$

Setting $v=1$, from (3.96), (3.99) and (3.100) we find

$$g_\theta(r_1) = \theta^*\log\frac{\kappa^*+1}{\kappa^*}\,, \qquad g_\kappa(r_1) = \kappa^*\log\frac{\kappa^*+1}{\kappa^*}\,, \qquad b_2(r_1) = \frac{\kappa^*+1}{\kappa^*}\,,$$

and Theorem 3.3 (with $v = 1$ and $n = 0$) readily yields the same result as (3.97).

3.5.2. *Polylogarithm.* To illustrate the supercritical and critical regimes, we need examples with $g_\kappa^{\{1\}}(R) < \infty$. To this end, let us set $\kappa_j := \kappa^* j^{-s}$ ($j \in \mathbb{N}$) with $\kappa^* > 0$, $s \in \mathbb{R}$, so that the corresponding generating function is proportional to the polylogarithm (see, e.g., [22])

$$g_\kappa(z) = \kappa^* \operatorname{Li}_{s+1}(z) = \kappa^* \sum_{j=1}^{\infty} \frac{z^j}{j^{s+1}}\,. \tag{3.101}$$

For $s = 0$ the model (3.101) is reduced to the case $\kappa_j \equiv \kappa^*$ considered in Section 3.5.1. Clearly, $\operatorname{Li}_s(z)$ has the radius of convergence $R = 1$ for any $s \in \mathbb{R}$, and $\operatorname{Li}_s(1) \equiv \zeta(s) < \infty$ for $s > 1$, where $\zeta(\cdot)$ is the Riemann zeta function. Differentiating (3.101) we get

$$g_\kappa^{\{1\}}(z) = \kappa^* z \operatorname{Li}'_{s+1}(z) = \kappa^* \operatorname{Li}_s(z), \tag{3.102}$$

and so the supercriticality condition reads

$$g_\kappa^{\{1\}}(1) = \kappa^* \operatorname{Li}_s(1) = \kappa^* \zeta(s) < 1 \quad \Leftrightarrow \quad s > 1, \quad \kappa^* < 1/\zeta(s). \tag{3.103}$$

Similarly,

$$g_\kappa^{\{2\}}(z) = \kappa^* \bigl(\operatorname{Li}_{s-1}(z) - \operatorname{Li}_s(z)\bigr) \tag{3.104}$$

and so

$$g_\kappa^{\{2\}}(1) = \kappa^* \bigl(\zeta(s-1) - \zeta(s)\bigr) < \infty \quad \Leftrightarrow \quad s > 2.$$

Hence, substituting (3.102) and (3.104) into (3.3) we obtain

$$b_1(r) = \kappa^* \operatorname{Li}_s(r), \qquad b_2(r) = \kappa^* \operatorname{Li}_{s-1}(r), \qquad 0 \le r \le 1.$$

Furthermore, from (3.102) we find that the root of the equation (3.6) is given by

$$r_v = \operatorname{Li}_s^{-1}\bigl((\kappa^* v)^{-1}\bigr), \qquad v \ge \frac{1}{\kappa^* \zeta(s)}\,, \tag{3.105}$$

where Li_s^{-1} is the inverse of Li_s. Differentiating the identity (3.105) we also obtain

$$\frac{r'_v}{r_v} = \frac{-\kappa^* v^{-2}}{\operatorname{Li}_{s-1}(r_v)}\,.$$

In particular,

$$r_1 = \operatorname{Li}_s^{-1}(1/\kappa^*), \qquad \frac{r'_1}{r_1} = \frac{-\kappa^*}{\operatorname{Li}_{s-1}(r_1)}\,.$$

The supercritical case $\kappa^* < 1/\zeta(s)$ is more interesting. It is known (see, e.g., [14, §IV.9, p. 237, and §VI.8, p. 408], that the polylogarithm $\operatorname{Li}_s(\cdot)$ can be analytically continued to the complex plane $\mathbb{C}$ slit along the ray $[1, +\infty)$. The asymptotic behaviour of $\operatorname{Li}_s(z)$ near $z_0 = 1$ is specified as follows (see [14, Theorem VI.7, p. 408, and §VI.20, p. 411]).

Lemma 3.14. *With the notation*

$$\varpi := -\log z \sim \sum_{n=1}^{\infty} \frac{(1-z)^n}{n}\,, \qquad z \to 1, \tag{3.106}$$

the polylogarithm $\operatorname{Li}_s(z)$ *satisfies the following asymptotic expansion as* $z \to 1$:

(a) *if* $s \in \mathbb{R} \setminus \mathbb{N}$ *then*

$$\operatorname{Li}_s(z) \sim \Gamma(1-s)\, \varpi^{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \zeta(s-n)}{n!}\, \varpi^n; \tag{3.107}$$

(b) *if* $s = q \in \mathbb{N}$ *then*

$$\mathrm{Li}_q(z) \sim \frac{(-1)^q}{(q-1)!}\,\varpi^{q-1}\,(\log\varpi - \mathrm{H}_{q-1}) + \sum_{n\ge 0,\, n\ne q-1} \frac{(-1)^n \zeta(q-n)}{n!}\,\varpi^n, \tag{3.108}$$

where $\mathrm{H}_{q-1} := \sum_{n=1}^{q-1} n^{-1}$ (*with* $\mathrm{H}_0 := 0$).

Remark 3.6. The restriction $n \ne q-1$ in the sum (3.108) respects the fact that the zeta function $\zeta(z)$ has a pole at $z = 1$.

Remark 3.7. In the simplest case $q = 1$ the expansion (3.108) specializes to

$$\mathrm{Li}_1(z) \sim -\log\varpi + \sum_{n=1}^{\infty} \frac{(-1)^n\,\zeta(1-n)}{n!}\,\varpi^n, \qquad \varpi \to 0,$$

which should be contrasted with the explicit expression $\mathrm{Li}_1(z) = -\log(1-\mathrm{e}^{-\varpi})$ $(z = \mathrm{e}^{-\varpi})$. Of course, there is no contradiction; the corresponding identity

$$\log\frac{\varpi}{1-\mathrm{e}^{-\varpi}} = \sum_{n=1}^{\infty} \frac{(-1)^n\,\zeta(1-n)}{n!}\,\varpi^n, \tag{3.109}$$

with the left-hand side defined at $\varpi = 0$ by continuity as $\log(0/0) := \log 1 = 0$, can be verified using that $\zeta(0) = -1/2$, $\zeta(1-n) = 0$ for odd $n \ge 3$ and $\zeta(1-n) = -B_n/n$ for even $n \ge 2$ (see, e.g., [14, §B 11, p. 747]), where B_n's are the *Bernoulli numbers* (see, e.g., [14, p. 268]) defined by the generating function $\varpi/(\mathrm{e}^{\varpi}-1) = \sum_{n=0}^{\infty} B_n\,\varpi^n/n!$ (note that $B_0 = 1$, $B_1 = -1/2$ and $B_n = 0$ for odd $n \ge 3$). Indeed, (3.109) is then rewritten as

$$\log\frac{\varpi}{1-\mathrm{e}^{-\varpi}} = \frac{\varpi}{2} - \sum_{n=1}^{\infty} \frac{B_n}{n!\,n}\,\varpi^n,$$

and the proof is completed by differentiation of both sides with respect to ϖ.

Explicit asymptotic expansion of $\mathrm{Li}_s(z)$ in terms of $1-z$, to any required order, can be obtained by substituting the series (3.106) into (3.107) or (3.108) as appropriate. In particular, for the generating function $g_\kappa(z)$ of the form (3.101) with $q < s < q+1$ $(q \in \mathbb{N}_0)$, from formulas (3.106) and (3.107) we get the Taylor-type expansion

$$g_\kappa(z) = \sum_{n=0}^{q} \frac{(-1)^n g_\kappa^{\{n\}}(1)}{n!}\,(1-z)^n + \kappa^*\Gamma(-s)(1-z)^s + O\bigl((1-z)^{q+1}\bigr), \tag{3.110}$$

where the coefficients

$$g_\kappa^{\{n\}}(1) = \kappa^*\,\mathrm{Li}_{s+1}^{\{n\}}(1) = \kappa^* \sum_{j=1}^{\infty} \frac{(j)_n}{j^{s+1}}, \qquad n = 0, \dots, q,$$

are expressible as linear combinations of the zeta functions $\zeta(s+1-k)$ $(k = 1, \dots, n)$,

$$\mathrm{Li}_{s+1}^{\{0\}}(1) = \zeta(s+1), \quad \mathrm{Li}_{s+1}^{\{1\}}(1) = \zeta(s), \quad \mathrm{Li}_{s+1}^{\{2\}}(1) = \zeta(s-1) - \zeta(s),$$
$$\mathrm{Li}_{s+1}^{\{3\}}(1) = \zeta(s-2) - 3\zeta(s-1) + 2\zeta(s), \quad \text{etc.}$$

Furthermore, using similar arguments it is easy to see that the expansion (3.110) can be differentiated any number of times, yielding the asymptotics

$$g_\kappa^{\{n\}}(z) = g_\kappa^{\{n\}}(1)\,\bigl\{1 + O\bigl((1-z)^{s-q}\bigr)\bigr\} \qquad (n < s), \tag{3.111}$$

$$g_\kappa^{\{n\}}(z) = \kappa^*\Gamma(n-s)\,(1-z)^{s-n}\,\bigl\{1 + O\bigl((1-z)^{q+1-s}\bigr)\bigr\} \qquad (n > s). \tag{3.112}$$

By virtue of formula (3.110), and by choosing suitable (non-integer) values of the parameter $s > 0$ in (3.101), one can easily construct examples matching the assumptions of each of the asymptotic theorems in Sections 3.3 and 3.4. Moreover, the asymptotic formulas (3.111) and (3.112) make the polylogarithm example (3.101) suitable for the setting of Section 5 below (see the assumptions (5.1), (5.2)).

The case with integer $s = q \in \mathbb{N}_0$ leads to a logarithmic term in the singular part of the asymptotic expansion at $z = 1$. Indeed, if $q = 0$ then formula (3.101) is reduced to

$$g_\kappa(z) = \kappa^* \operatorname{Li}_1(z) = \kappa^* \sum_{j=1}^{\infty} \frac{z^j}{j} = -\kappa^* \log(1-z),$$

while $q \in \mathbb{N}$ corresponds to $g_\kappa(z) = \kappa^* \operatorname{Li}_{q+1}(z)$, so that using (3.106) and (3.108) one obtains with an arbitrary $\delta \in (0,1)$ (cf. (3.110))

$$g_\kappa(z) = \sum_{n=0}^{q-1} \frac{(-1)^n g_\kappa^{\{n\}}(1)}{n!}\,(1-z)^n - \kappa^* \frac{(-1)^q}{q!}\,(1-z)^q \log(1-z) \\ + \kappa^* A_q (1-z)^q + O\bigl((1-z)^{q+\delta}\bigr), \tag{3.113}$$

where A_q is the coefficient of the term $(1-z)^q$ arising from the expansion (3.108) (with q replaced by $q+1$) upon the substitution (3.106).

3.5.3. *Perturbed polylogarithm.* A natural extension of the polylogarithm example considered in the previous section is furnished by setting $\kappa_j := \kappa^* j^{-s}\,(1+\xi(j))$ ($j \in \mathbb{N}$), where $\kappa^* > 0$, $s \ge 0$, $1+\xi(j) \ge 0$ for all $j \in \mathbb{N}$, and the perturbation function $z \mapsto \xi(z)$ is assumed to be analytic in the half-plane $\Re(z) > \frac{1}{4}$ and to satisfy there the estimate $\xi(z) = O(z^{-\epsilon})$, with some $\epsilon > 0$. It follows that the corresponding generating function

$$g_\kappa(z) = \kappa^* \sum_{j=1}^{\infty} \frac{1+\xi(j)}{j^{s+1}}\, z^j = \kappa^* \operatorname{Li}_{s+1}(z) + \kappa^* \sum_{j=1}^{\infty} \frac{\xi(j)}{j^{s+1}}\, z^j \tag{3.114}$$

is analytic in the disk $|z| < 1$, with singularity at $z = 1$. Furthermore (see (3.102)),

$$g_\kappa^{\{1\}}(z) = \kappa^* \operatorname{Li}_s(z) + \kappa^* \sum_{j=1}^{\infty} \frac{\xi(j)}{j^s}\, z^j. \tag{3.115}$$

As suggested by the principal term $\operatorname{Li}_s(z)$ in (3.115), the criticality occurs if $s > 1$ (cf. (3.103)); indeed, with a suitable constant $c > 0$ we have for sufficiently small κ^*

$$g_\kappa^{\{1\}}(1) = \kappa^* \operatorname{Li}_s(1) + \kappa^* \sum_{j=1}^{\infty} \frac{\xi(j)}{j^s} \le \kappa^* \bigl\{\zeta(s) + c\,\zeta(s+\epsilon)\bigr\} < 1. \tag{3.116}$$

Lemma 3.15. *Under the above conditions on $\xi(z)$ (with $s \ge 0$), the function $g_\kappa(z)$ can be analytically continued to the slit complex plane $\mathbb{C} \setminus [1,+\infty)$.*

Proof. Since the claim is valid for the polylogarithm $\operatorname{Li}_{s+1}(z)$ (see Section 3.5.2), from (3.114) we see that it suffices to prove the lemma for the series $\sum_{j=1}^{\infty} a(j)\, z^j$, where $a(w) := \xi(w)/w^{s+1}$. Clearly, for any $\delta \in (0,\pi)$ we have the estimate $a(w) = O\bigl(\mathrm{e}^{(\pi-\delta)|w|}\bigr)$. Hence, the Lindelöf theorem (see [14, § IV.8, p. 237] yields the integral representation

$$\sum_{j=1}^{\infty} a(j)\, z^j = -\frac{1}{2\pi \mathrm{i}} \int_{(1/2)-\mathrm{i}\infty}^{(1/2)+\mathrm{i}\infty} a(w)\,(-z)^w \frac{\pi}{\sin \pi w}\, \mathrm{d}w, \tag{3.117}$$

which provides an analytic continuation of the series to the domain $\mathbb{C} \setminus \{z : |\arg(z)| \le \delta\}$. Since $\delta > 0$ can be taken arbitrarily small, this implies the analyticity in $\mathbb{C}$ slit along the ray $[0, +\infty)$. To complete the proof, it remains to recall that $\sum_j a(j)\, z^j$ is analytic in $|z| < 1$. □

The next question is the asymptotic behaviour of the function (3.114) as $z \to 1$ with $z \in \mathbb{C}\setminus[1, +\infty)$. Application of the asymptotic formulas (3.107) and (3.108) in the particular case $\xi(z) = z^{-\epsilon}$ (leading to $g_\kappa(z) = \kappa^*\{\mathrm{Li}_{s+1}(z) + \mathrm{Li}_{s+\epsilon+1}(z)\}$) motivates and illustrates the following expansions, which now have to be finite (up to the order of $(1 - z)^s$) due to the limited information about the perturbation function $\xi(z)$.

Lemma 3.16. *Suppose that $\xi(z)$ satisfies the same conditions as in Lemma* 3.15*, and let $q \in \mathbb{N}_0$ be such that $q \le s < q + 1$. Then, as $z \to 1$ so that $z \in \mathbb{C} \setminus [1, +\infty)$,*

(a) *for $s \notin \mathbb{N}_0$ (i.e., $q < s < q + 1$),*

$$g_\kappa(z) = \sum_{n=0}^{q} \frac{(-1)^n g_\kappa^{\{n\}}(1)}{n!}\,(1-z)^n + \kappa^*\Gamma(-s)(1-z)^s + O\bigl((1-z)^{s+\epsilon_*}\bigr), \tag{3.118}$$

where $\epsilon_* = \epsilon$ if $s + \epsilon < q + 1$, $\epsilon_* = 1$ if $s + \epsilon > q + 1$, and ϵ_* is any number in $(0, \epsilon)$ if $s + \epsilon = q + 1$;

(b) *for $s = q \in \mathbb{N}_0$,*

$$g_\kappa(z) = \sum_{n=0}^{q-1} \frac{(-1)^n g_\kappa^{\{n\}}(1)}{n!}\,(1-z)^n - \kappa^*\,\frac{(-1)^q}{q!}\,(1-z)^q \log(1-z) + \kappa^* B_q\,(1-z)^q + O\bigl((1-z)^{q+\epsilon_*}\bigr), \tag{3.119}$$

where B_q is some constant, $\epsilon_* = \epsilon$ if $\epsilon < 1$ and ϵ_* is any number in $(0, \epsilon)$ if $\epsilon \ge 1$.

The expansions (3.118) *and* (3.119) *can be differentiated q times.*

Sketch of proof. This result is of marginal significance for our purposes, as it will only be used for illustration in Section 6.3. Its full proof is quite tedious but follows very closely the proof of a similar result for the polylogarithm $\mathrm{Li}_{s+1}(z)$ (see details in [14, §VI.8]). Thus, we opt to derive the expansion (3.118) only for *real* $z \uparrow 1$; an extension to complex $z \in \mathbb{C} \setminus [1, +\infty)$ is based on the Lindelöf integral representation (3.117).

Observe that it suffices to prove (3.118) for $s \in (0, 1)$; the case of an arbitrary (non-integer) $s > 0$ may then be handled via a suitable (q-fold) integration over the interval $[z, 1]$. To this end, using the substitution $z = \mathrm{e}^{-\varpi}$ (with $\varpi \downarrow 0$ as $z \uparrow 1$), from (3.114) we obtain

$$g_\kappa(z) = g_\kappa(1) - \kappa^* \sum_{j=1}^{\infty} \frac{1 - \mathrm{e}^{-j\varpi}}{j^{s+1}} - \kappa^* \sum_{j=1}^{\infty} \frac{\xi(j)(1 - \mathrm{e}^{-j\varpi})}{j^{s+1}}. \tag{3.120}$$

The first series in (3.120) may be rewritten as a Riemann integral sum

$$\varpi^s \sum_{j=1}^{\infty} \frac{1 - \mathrm{e}^{-j\varpi}}{(j\varpi)^{s+1}}\,\varpi = \varpi^s \int_0^\infty \frac{1 - \mathrm{e}^{-x}}{x^{s+1}}\,\mathrm{d}x + O(\varpi), \qquad \varpi \downarrow 0, \tag{3.121}$$

where the asymptotics on the right-hand side can be obtained using Euler–Maclaurin's summation formula. The integral in (3.121) is easily computed via integration by parts,

$$\frac{1}{(-s)} \int_0^\infty (1 - \mathrm{e}^{-x})\,\mathrm{d}(x^{-s}) = \frac{1}{s} \int_0^\infty x^{-s}\,\mathrm{e}^{-x}\,\mathrm{d}x = \frac{\Gamma(1-s)}{s} = -\Gamma(-s). \tag{3.122}$$

Next, using the estimate $\xi(j) = O(j^{-\epsilon})$ and assuming that $s + \epsilon < 1$, the second series in (3.120) is estimated, similarly to (3.121) and (3.122), by

$$O(1)\sum_{j=1}^{\infty}\frac{1-\mathrm{e}^{-j\varpi}}{j^{s+\epsilon+1}} = O(\varpi^{s+\epsilon}), \qquad \varpi \downarrow 0. \tag{3.123}$$

Finally, collecting (3.120), (3.121), (3.122) and (3.123), we obtain

$$g_\kappa(z) = g_\kappa(1) + \kappa^*\Gamma(-s)\,\varpi^s + O(\varpi^{s+\epsilon}),$$

and the formula (3.118) (with $0 < s < 1$ and real $z \uparrow 1$) immediately follows by the substitution $\varpi = -\log z = 1 - z + O\bigl((1-z)^2\bigr)$. □

4. Asymptotic statistics of cycles

Throughout this section, we assume that the generating functions $g_\theta(z)$ and $g_\kappa(z)$ satisfy the hypotheses of a suitable asymptotic theorem in Section 3 — namely, Theorem 3.3 for the subcritical case ($g_\kappa^{\{1\}}(R) > 1$), Theorems 3.5 or 3.6 for the supercritical case ($g_\kappa^{\{1\}}(R) < 1$), and Theorems 3.7, 3.12 or 3.13 for the critical case ($g_\kappa^{\{1\}}(R) = 1$).

Let us define the quantity r_* as

$$r_* := \begin{cases} r_1, & g_\kappa^{\{1\}}(R) \ge 1,\\ R, & g_\kappa^{\{1\}}(R) \le 1,\end{cases} \tag{4.1}$$

where $r_1 = r_v|_{v=1}$ is the (unique) solution of the equation (3.6) with $v = 1$. Note that

$$\begin{aligned} g_\kappa^{\{1\}}(r_*) &= 1 \quad \text{if} \quad g_\kappa^{\{1\}}(R) \ge 1,\\ g_\kappa^{\{1\}}(r_*) &< 1 \quad \text{if} \quad g_\kappa^{\{1\}}(R) < 1.\end{aligned} \tag{4.2}$$

4.1. **Cycle counts.**

Our first result treats the asymptotics of the cycle counts C_j's (i.e., the numbers of cycles of length $j \in \mathbb{N}$, respectively, in a random permutation $\sigma \in \mathfrak{S}_N$).

Theorem 4.1. *Let r_* be as defined in* (4.1).

(a) *For each $m \in \mathbb{N}$ and any integers $n_1, \dots, n_m \ge 0$, we have*

$$\lim_{N\to\infty} N^{-(n_1+\dots+n_m)}\,\tilde{\mathbb{E}}_N\biggl[\prod_{j=1}^{m}(C_j)_{n_j}\biggr] = \prod_{j=1}^{m}\biggl(\frac{\kappa_j\, r_*^j}{j}\biggr)^{n_j}. \tag{4.3}$$

In particular, the random variables (C_j/N) are asymptotically independent and, for each $j \in \mathbb{N}$, there is the convergence in probability

$$\frac{C_j}{N} \xrightarrow{\ p\ } \frac{\kappa_j\, r_*^j}{j}, \qquad N \to \infty. \tag{4.4}$$

(b) *If, for some $j \in \mathbb{N}$, $\kappa_j = 0$ but $\theta_j > 0$ then for any integer $n \ge 0$*

$$\lim_{N\to\infty}\tilde{\mathbb{E}}_N\bigl[(C_j)_n\bigr] = \biggl(\frac{\theta_j\, r_*^j}{j}\biggr)^{n}. \tag{4.5}$$

Hence, C_j converges weakly to a Poisson law with parameter $\theta_j\, r_*^j/j$. The asymptotic independence of C_j with other cycle counts (*normalized or not, as appropriate*) *is preserved.*

Remark 4.1. If both $\kappa_j = 0$ and $\theta_j = 0$ then, by the definition (1.5) of the measure $\tilde{\mathbb{P}}_N$, $C_j = 0$ almost surely (a.s.).

Proof of Theorem 4.1. If all $\kappa_j > 0$ then Lemma 2.3 implies that for any $n_1, \dots, n_m \in \mathbb{N}_0$

$$N^{-(n_1+\dots+n_m)}\, \tilde{\mathbb{E}}_N\left[\prod_{j=1}^{m} (C_j)_{n_j}\right] \sim \frac{h_{N-K_m}(N)}{H_N} \prod_{j=1}^{m} \left(\frac{\kappa_j}{j}\right)^{n_j}, \qquad N\to\infty, \tag{4.6}$$

where $K_m = \sum_{j=1}^m j n_j$. Furthermore, note that

$$h_{N-K_m}(N) = \bigl[z^{N-K_m}\bigr]\, \mathrm{e}^{G_N(z)} = \bigl[z^N\bigr]\bigl[z^{K_m}\, \mathrm{e}^{G_N(z)}\bigr]. \tag{4.7}$$

Hence, applying one of Theorems 3.3, 3.5, 3.6, 3.7, 3.12 or 3.13 as appropriate (each one with the pre-exponential function $f(z) = z^{K_m}$), we get

$$\frac{h_{N-K_m}(N)}{H_N} \to r_*^{K_m}, \qquad N\to\infty. \tag{4.8}$$

Combining (4.6) and (4.8) we obtain formula (4.3), which also entails the asymptotic independence. Finally, the convergence (4.4) follows from (4.3) by the method of moments.

Similarly, for $\kappa_j = 0,\ \theta_j > 0$ we have

$$\tilde{\mathbb{E}}_N\bigl[(C_j)_n\bigr] \to \left(\frac{\theta_j\, r_*^j}{j}\right)^n, \qquad N\to\infty,$$

where the limit is the n-th factorial moment of the corresponding Poisson distribution. ☐

Remark 4.2. Formally, the limiting result (4.4) suggests that the total proportion of points contained in finite cycles, $N^{-1}\sum_{j=1}^\infty j\, C_j$, is asymptotically given by (see (4.2))

$$\sum_{j=1}^{\infty} \kappa_j\, r_*^j = g_\kappa^{\{1\}}(r_*) \begin{cases} = 1 & \text{if } \ g_\kappa^{\{1\}}(R) \ge 1,\\ < 1 & \text{if } \ g_\kappa^{\{1\}}(R) < 1,\end{cases}$$

which indicates the emergence of an infinite cycle in the supercritical case (i.e., $g_\kappa^{\{1\}}(R) < 1$, see Definition 3.2). This observation is elaborated below (see Theorems 4.2 and 4.3).

4.2. **Fraction of points in long cycles.** By analogy with the spatial case (see (1.14)), let us define the similar quantities in the surrogate-spatial model to capture the expected fraction of points in long cycles,

$$\tilde{\nu}_K := \liminf_{N\to\infty} \frac{1}{N}\, \tilde{\mathbb{E}}_N\left(\sum_{j>K} j\, C_j\right), \qquad \tilde{\nu} := \lim_{K\to\infty} \tilde{\nu}_K. \tag{4.9}$$

Theorem 4.2. *The quantity* $\tilde{\nu}_K$ ($K \in \mathbb{N}$) *defined in* (4.9) *exists as a limit and is explicitly given by*

$$\tilde{\nu}_K = \lim_{N\to\infty} \frac{1}{N}\, \tilde{\mathbb{E}}_N\left(\sum_{j>K} j\, C_j\right) = 1 - \sum_{j=1}^{K} \kappa_j\, r_*^j\,, \tag{4.10}$$

where r_* *is defined in* (4.1). *Moreover,*

$$\tilde{\nu} = \lim_{K\to\infty} \tilde{\nu}_K = \begin{cases} 0, & g_\kappa^{\{1\}}(R) \ge 1,\\ 1 - g_\kappa^{\{1\}}(R) > 0, & g_\kappa^{\{1\}}(R) < 1.\end{cases} \tag{4.11}$$

Proof. Noting that $\sum_{j=1}^\infty j\, C_j = N$ and applying Theorem 4.1(a), we get

$$\tilde{\nu}_K = 1 - \lim_{N\to\infty} \frac{1}{N}\, \tilde{\mathbb{E}}_N\left(\sum_{j\le K} j\, C_j\right) = 1 - \sum_{j=1}^{K} \kappa_j\, r_*^j,$$

which proves (4.10). Hence, using (4.2) we obtain

$$\tilde{\nu} = \lim_{K\to\infty} \tilde{\nu}_K = 1 - \sum_{j=1}^{\infty} \kappa_j\, r_*^j = 1 - g_\kappa^{\{1\}}(r_*),$$

which is reduced to (4.11) thanks to (4.2). □

Theorem 4.2 can be complemented by a similar statement about the convergence of the (random) proportion of points in long cycles, rather than its expected value.

Theorem 4.3. *Under the sequence of probability measures $\tilde{\mathbb{P}}_N$, for any finite $K \in \mathbb{N}$ there is the convergence in probability*

$$\frac{1}{N}\sum_{j>K} j\, C_j \xrightarrow{\ p\ } \tilde{\nu}_K, \qquad N \to \infty, \tag{4.12}$$

where $\tilde{\nu}_K$ is identified in (4.10).

Proof. Similarly to the proof of Theorem 4.2, by the probability convergence part of Theorem 4.1 (a) and according to (4.10) we have

$$\frac{1}{N}\sum_{j>K} j\, C_j = 1 - \frac{1}{N}\sum_{j=1}^{K} j\, C_j \xrightarrow{\ p\ } 1 - \sum_{j=1}^{K} \kappa_j\, r_*^j = \tilde{\nu}_K,$$

and the limit (4.12) is proved. □

4.3. **Total number of cycles.** The next result is a series of weak limit theorems (in the subcritical, supercritical and critical cases, respectively) for fluctuations of the total number of cycles T_N (see (2.14)). As stipulated at the beginning of Section 4, we work under the conditions of suitable asymptotic theorems from Section 3, which will be applied without the pre-exponential factor (i.e., with $f(z) \equiv 1$). Note that in all but one cases the limiting distribution is normal, whereas in the critical case with $\theta^* > 0$ (part (c-iii)) the answer is more complicated (and more interesting).

In what follows, the notation $\xrightarrow{d}$ indicates convergence in distribution (with respect to the sequence of measures $\tilde{\mathbb{P}}_N$), and $\mathcal{N}(0,1)$ denotes the standard normal law (i.e., with mean 0 and variance 1).

Theorem 4.4. (a) *Let $1 < g_\kappa^{\{1\}}(R) \le \infty$. Then, under the conditions of Theorem* 3.3*,*

$$\frac{T_N - N g_\kappa(r_1)}{\sqrt{N\bigl[g_\kappa(r_1) - 1/b_2(r_1)\bigr]}} \xrightarrow{\ d\ } \mathcal{N}(0,1), \qquad N \to \infty, \tag{4.13}$$

where r_1 is the root of (3.6) *with $v = 1$, $b_2(r)$ is defined in* (3.3)*, and $g_\kappa(r_1) - 1/b_2(r_1) > 0$.*

(b) *If $g_\kappa^{\{1\}}(R) < 1$ then, under the conditions of either of Theorems* 3.5 *or* 3.6*,*

$$\frac{T_N - N g_\kappa(R)}{\sqrt{N g_\kappa(R)}} \xrightarrow{\ d\ } \mathcal{N}(0,1), \qquad N \to \infty. \tag{4.14}$$

(c) *Let $g_\kappa^{\{1\}}(R) = 1$.*

(c-i) *Under the conditions of Theorem* 3.7*,*

$$\frac{T_N - N g_\kappa(R)}{\sqrt{N\bigl[g_\kappa(R) - 1/b_2(R)\bigr]}} \xrightarrow{\ d\ } \mathcal{N}(0,1), \qquad N \to \infty,$$

where $g_\kappa(R) - 1/b_2(R) > 0$*.*

(c-ii) *Under the conditions of Theorem* 3.13*,*

$$\frac{T_N - N g_\kappa(R)}{\sqrt{N g_\kappa(R)}} \xrightarrow{d} \mathcal{N}(0,1), \qquad N \to \infty. \tag{4.15}$$

(c-iii) *Under the conditions of Theorem* 3.12*,*

$$\frac{T_N - N g_\kappa(R)}{\sqrt{N\bigl[g_\kappa(R) - 1/b_2(R)\bigr]}} \xrightarrow{d} Z - \sqrt{\frac{2X}{g_\kappa(R)\, b_2(R) - 1}}\,, \qquad N \to \infty, \tag{4.16}$$

where Z *is a standard normal random variable and* X *is an independent random variable with gamma distribution* $\mathrm{Gamma}\,(\theta^*/2)$*.*

Proof. (a) Using Lemma 2.4 and applying Theorem 3.3 (see (3.8)) we obtain as $N \to \infty$, uniformly in v in a neighbourhood of $v_0 = 1$,

$$\tilde{\mathbb{E}}_N\bigl[v^{T_N}\bigr] = \frac{[z^N]\exp\{vG_N(z)\}}{[z^N]\exp\{G_N(z)\}} \sim \frac{\Phi(v)\exp\{N\Psi(v)\}}{\Phi(1)\exp\{N\Psi(1)\}}\,, \tag{4.17}$$

where we set for short

$$\Phi(v) := \frac{\exp\{v g_\theta(r_v)\}}{\sqrt{v b_2(r_v)}}\,, \qquad \Psi(v) := v g_\kappa(r_v) - \log r_v. \tag{4.18}$$

Using the definition of r_v (see (3.6)), from (4.18) we find

$$\begin{aligned} \Psi'(v) &= g_\kappa(r_v) + v g'_\kappa(r_v)\, r'_v - \frac{r'_v}{r_v} \\ &= g_\kappa(r_v) + \frac{r'_v}{r_v}\bigl(v g_\kappa^{\{1\}}(r_v) - 1\bigr) = g_\kappa(r_v). \end{aligned} \tag{4.19}$$

Differentiating (4.19) once more gives

$$\Psi''(v) = g'_\kappa(r_v)\, r'_v = g_\kappa^{\{1\}}(r_v)\,\frac{r'_v}{r_v} = \frac{r'_v}{v r_v}\,. \tag{4.20}$$

On the other hand, differentiating the identity $r_v g'_\kappa(r_v) = v^{-1}$ (see equation (3.6)), we get

$$r'_v\, g'_\kappa(r_v) + r_v\, g''_\kappa(r_v)\, r'_v = -v^{-2},$$

which yields, on account of (3.1) and (3.3),

$$\frac{r'_v}{r_v} = -\frac{1}{v^2 b_2(r_v)}\,. \tag{4.21}$$

Hence, using (4.19), (4.20) and (4.21) we obtain the expansion of $\Psi(v)$ around $v_0 = 1$,

$$\Psi(v) = \Psi(1) + g_\kappa(r_1)(v-1) - \frac{1}{2b_2(r_1)}\,(v-1)^2 + o\bigl((v-1)^2\bigr). \tag{4.22}$$

Substituting $v = \mathrm{e}^{u/\sqrt{N}} = 1 + uN^{-1/2} + \frac{1}{2}u^2 N^{-1} + O(N^{-3/2})$ into (4.22) gives

$$N\Psi\bigl(\mathrm{e}^{u/\sqrt{N}}\bigr) = N\Psi(1) + \sqrt{N}\, g_\kappa(r_1)\, u + \tfrac{1}{2}\bigl(g_\kappa(r_1) - 1/b_2(r_v)\bigr)\, u^2 + O(N^{-1/2}).$$

Besides, for the function $\Phi(\cdot)$ from (4.18) we have

$$\Phi\bigl(\mathrm{e}^{u/\sqrt{N}}\bigr) \to \Phi(1), \qquad N \to \infty.$$

Therefore, returning to (4.17) we get, as $N\to\infty$,

$$\tilde{\mathbb{E}}_N\bigl[\exp\bigl\{uT_N/\sqrt{N}\,\bigr\}\bigr]\sim\exp\bigl\{N\bigl[\Psi(\mathrm{e}^{u/\sqrt{N}})-\Psi(1)\bigr]\bigr\}$$
$$\sim\exp\bigl\{\sqrt{N}\,g_\kappa(r_1)\,u+\tfrac12\bigl(g_\kappa(r_1)-1/b_2(r_1)\bigr)\,u^2\bigr\}.$$

The statement (4.13) now follows by a standard convergence theorem based on convergence of moment generating functions and the well-known fact that if a random variable Z is standard normal then its moment generating function is given by $\mathbb{E}\bigl[\exp\{uZ\}\bigr]=\exp\{u^2/2\}$.

It remains to check that the limit variance $g_\kappa(r_1)-1/b_2(r_1)$ is positive. But this follows by Lemma 3.1, yielding

$$g_\kappa(r_1)\,b_2(r_1)>\bigl(g_\kappa^{\{1\}}(r_1)\bigr)^2\equiv 1,$$

according to the definition of r_1 as the root of the equation (3.6) with $v=1$.

(b) If $\theta^*>0$ then, similarly as in (4.17), (4.18), we use Lemma 2.4 and Theorem 3.5 to obtain as $N\to\infty$, uniformly in v in a neighbourhood of $v_0=1$,

$$\tilde{\mathbb{E}}_N\bigl[v^{T_N}\bigr]\sim N^{\theta^*(v-1)}\exp\bigl\{N(v-1)\,g_\kappa(R)\bigr\}\,\frac{\Gamma(\theta^*)(1-vg_\kappa^{\{1\}}(R))^{v\theta^*-1}}{\Gamma(v\theta^*)(1-g_\kappa^{\{1\}}(R))^{\theta^*-1}}\,.$$

Again substituting $v=\mathrm{e}^{u/\sqrt{N}}$, the rest of the proof proceeds similarly as in part (a), giving

$$\tilde{\mathbb{E}}_N\bigl[\exp\bigl\{uT_N/\sqrt{N}\,\bigr\}\bigr]\sim\exp\bigl\{N\bigl(\mathrm{e}^{u/\sqrt{N}}-1\bigr)\,g_\kappa(R)\bigr\}$$
$$\sim\exp\bigl\{\sqrt{N}\,g_\kappa(R)\,u+\tfrac12 g_\kappa(R)\,u^2\bigr\},\qquad(4.23)$$

and (4.14) follows. Likewise, if $\theta^*=0$ then by Theorem 3.6 we have from (3.49) and (3.52)

$$\tilde{\mathbb{E}}_N\bigl[v^{T_N}\bigr]\sim v\exp\bigl\{(v-1)\,G_N(R)\bigr\}\left(\frac{1-vg_\kappa^{\{1\}}(R)}{1-g_\kappa^{\{1\}}(R)}\right)^{-s-1},\qquad N\to\infty,\qquad(4.24)$$

and the substitution $v=\exp\bigl\{u/\sqrt{N}\,\bigr\}$ in (4.24) yields the same asymptotics (4.23).

(c-i) From (3.60) similarly as in part (a) we obtain the asymptotic relation (4.17) that holds uniformly in a right neighbourhood of $v_0=1$. The rest of the proof is an exact copy of that in part (a), except that we must use the substitution $v=\mathrm{e}^{u/\sqrt{N}}$ with $u\ge 0$.

(c-ii) We use Lemma 2.4 and the dynamic asymptotics (3.94) to obtain

$$\tilde{\mathbb{E}}_N\bigl[\exp\bigl\{-uT_N/\sqrt{N}\,\bigr\}\bigr]\sim\exp\bigl\{-\sqrt{N}\,g_\kappa(R)\,u+\tfrac12 g_\kappa(R)\,u^2\bigr\},$$

which immediately implies (4.15).

(c-iii) Again using Lemma 2.4, from the asymptotic relation (3.89) and with the help of Lemma 3.10 we get

$$\tilde{\mathbb{E}}_N\bigl[\exp\bigl\{-uT_N/\sqrt{N}\,\bigr\}\bigr]\sim\exp\bigl\{-\sqrt{N}\,g_\kappa(R)\,u+\tfrac12\bigl(g_\kappa(r_1)-1/b_2(r_1)\bigr)\,u^2\bigr\}$$
$$\times\,\mathbb{E}\bigl[\exp\bigl\{-u\sqrt{2/b_2(R)}\,(-X^{1/2})\bigr\}\bigr],$$

where X has the distribution $\mathrm{Gamma}\,(\theta^*/2)$. Hence, the result (4.16) follows. □

Remark 4.3. To summarize the content of Theorem 4.4, the sequence of "phase transitions" manifested by the limit distribution of $T_N^*:=N^{-1/2}\bigl(T_N-Ng_\kappa(R)\bigr)$ is as follows. In the subcritical domain ($g_\kappa^{\{1\}}(R)>1$), T_N^* is asymptotically normal with asymptotic variance $g_\kappa(r_1)-1/b_2(r_1)$ (Theorem 4.4(a)). This is consistent with the critical case ($g_\kappa^{\{1\}}(R)=1$) with finite $g_\theta(R)$ and $g_\kappa^{\{2\}}(R)$, whereby $r_1=R$ (Theorem 4.4(c-i)). Quite surprisingly, if $g_\theta(z)$ acquires logarithmic singularity with exponent $\theta^* > 0$, the central limit theorem breaks down as the existing normal component of the limit is reduced by the square root of an independent gamma-distributed random variable (Theorem 4.4(c-iii)). In particular, the limit distribution of T_N^* gets negatively skewed; for instance, its expected value equals $-\Gamma((\theta^*+1)/2)/\Gamma(\theta^*/2) < 0$ whilst $\mathbb{E}[T_N^*] \equiv 0$. Thus, although we prove weak convergence using the moment generating functions, the limit in this case cannot be established by the plain method of moments. However, when the function $g_\kappa(z)$ becomes more singular at $z = R$, with $g_\kappa^{\{2\}}(R) = \infty$, the limit of T_N^* reverts to a normal distribution but with a bigger variance, $g_\kappa(R)$ (Theorem 4.4(c-ii)), which continues to hold in the supercritical regime $g_\kappa^{\{1\}}(R) < 1$ (Theorem 4.4(b)).

Despite a complicated structure of the weak limit, the total number of cycles T_N in all cases satisfies the following simple law of large numbers, as $N \to \infty$.

Corollary 4.5. *Under the sequence of probability measures $\tilde{\mathbb{P}}_N$, in each of the cases in Theorem* 4.4 *there is the convergence in probability*

$$\frac{T_N}{N} \xrightarrow{p} g_\kappa(r_*), \qquad N \to \infty, \tag{4.25}$$

where r_* is given by (4.1).

Remark 4.4. The result (4.25) for $T_N = \sum_j C_j$ is formally consistent with Theorem 4.1(a), since the limits (4.4) of the normalized cycle counts C_j/N sum up exactly to the right-hand side of (4.25) (cf. (2.2)):

$$\sum_j \frac{\kappa_j\, r_*^j}{j} = g_\kappa(r_*).$$

Remark 4.5. The uniform asymptotic formula (4.17) can also be used to derive large deviation results for T_N (cf. [25, §4]), but we do not enter into further details here.

4.4. **Lexicographic ordering of cycles.** We can now find the asymptotic (finite-dimensional) distribution of the cycle lengths L_j (see Definition 2.1). Note that no normalization is needed.

Theorem 4.6. *For each $m \in \mathbb{N}$, the random variables $L_1, \dots, L_m$ are asymptotically independent as $N \to \infty$ and, moreover, for any $\ell_1, \dots, \ell_m \in \mathbb{N}$*

$$\lim_{N\to\infty} \tilde{\mathbb{P}}_N\{L_1 = \ell_1, \dots, L_m = \ell_m\} = \prod_{j=1}^{m} \kappa_{\ell_j}\, r_*^{\ell_j}, \tag{4.26}$$

where r_* is defined in (4.1).

Proof. By Lemma 2.5 we have, as $N \to \infty$,

$$\tilde{\mathbb{P}}_N\{L_1 = \ell_1, \dots, L_m = \ell_m\} \sim \prod_{j=1}^{m} \kappa_j \cdot \frac{h_{N-\ell_1-\dots-\ell_m}(N)}{H_N}.$$

Similarly to (4.7), with $K_m := \ell_1 + \dots + \ell_m$ we have

$$h_{N-K_m}(N) = [z^N]\big[z^{K_m}\,\mathrm{e}^{G_N(z)}\big].$$

Hence, by a suitable asymptotic theorem from Section 3 (i.e., one of Theorems 3.3, 3.5, 3.6, 3.7, 3.8 or 3.12 as appropriate, each with $f(z) = z^{K_m}$), we get

$$\frac{h_{N-K_m}(N)}{H_N} \sim r_*^{K_m}, \qquad N \to \infty.$$

Substituting this into (4.7) we obtain the limit (4.26). □

Using (4.2), observe that if $g_\kappa^{\{1\}}(R) \ge 1$ then the limit (4.26) defines a proper (m-dimensional) distribution, because

$$\sum_{\ell=1}^{\infty} \kappa_\ell \, r_*^\ell = g_\kappa^{\{1\}}(r_*) = 1,$$

but if $g_\kappa^{\{1\}}(R) < 1$ then this distribution is deficient (i.e., with the total mass less than 1). Clearly, the reason for this is the possible emergence of infinite cycles as $N \to \infty$ (see Theorem 4.2). However, noting from (4.26) that, for each $j \in \mathbb{N}$,

$$\lim_{M\to\infty} \lim_{N\to\infty} \tilde{\mathbb{P}}_N\{L_j \le M\} = \lim_{M\to\infty} \sum_{\ell=1}^{M} \kappa_\ell R^\ell = g_\kappa^{\{1\}}(R),$$

and therefore

$$\lim_{M\to\infty} \lim_{N\to\infty} \tilde{\mathbb{P}}_N\{L_1 \le M, \dots, L_m \le M\} = \bigl(g_\kappa^{\{1\}}(R)\bigr)^m,$$

we obtain the conditional limit (assuming that $\ell_j \le M$ for $j = 1, \dots, m$)

$$\begin{aligned}\lim_{M\to\infty} \lim_{N\to\infty} &\tilde{\mathbb{P}}_N\{L_1 = \ell_1, \dots, L_m = \ell_m \,|\, L_1 \le M, \dots, L_m \le M\} \\ &= \lim_{M\to\infty} \lim_{N\to\infty} \frac{\tilde{\mathbb{P}}_N\{L_1 = \ell_1, \dots, L_m = \ell_m\}}{\tilde{\mathbb{P}}_N\{L_1 \le M, \dots, L_m \le M\}} = \bigl(g_\kappa^{\{1\}}(R)\bigr)^{-m} \prod_{j=1}^{m} \kappa_{\ell_j} R^{\ell_j},\end{aligned}$$

which gives a proper limiting distribution.

5. LONG CYCLES

The ultimate goal of Section 5 is to characterize the asymptotic behaviour of longest cycles in the surrogate-spatial model. We will follow the classical approach of Kingman [20] and Vershik and Shmidt [30], and study first the asymptotic extreme value statistics of cycles under lexicographic ordering and then deduce from this the asymptotics of cycles arranged in the decreasing order of their lengths.

5.1. **Analytic conditions on the generating functions.** For the most part (until Section 5.6), the generating functions $g_\theta(z)$ and $g_\kappa(z)$ will be assumed to satisfy the hypotheses of Theorem 3.5, including the asymptotic formulas (3.19) and (3.20) (with some $\theta^* > 0$). In particular, $g_\kappa^{\{1\}}(R) < 1$ so we are in the supercritical regime. Furthermore, let us assume that there is a non-integer $s > 1$ such that $g_\kappa^{\{n\}}(R) < \infty$ for all $n < s$ while $g_\kappa^{\{n\}}(R) = \infty$ for $n > s$; moreover, there exist a constant $a_s > 0$ and a sequence $\delta_n > 0$ such that, as $z \to R$, $z \in \varDelta_0$ (see Definition 3.3),

$$g_\kappa^{\{n\}}(z) = g_\kappa^{\{n\}}(R) \bigl\{1 + O\bigl((1 - z/R)^{\delta_n}\bigr)\bigr\} \qquad (n < s), \tag{5.1}$$

$$g_\kappa^{\{n\}}(z) = \frac{\Gamma(-s+n)\, a_s}{\Gamma(-s)} (1 - z/R)^{s-n} \bigl\{1 + O\bigl((1 - z/R)^{\delta_n}\bigr)\bigr\} \qquad (n > s). \tag{5.2}$$

In addition to (3.19), we also assume that

$$g_\theta^{\{n\}}(z) = \frac{\theta^*(n-1)!}{(1 - z/R)^n} \bigl\{1 + O\bigl((1 - z/R)^{\delta_n}\bigr)\bigr\}, \qquad n \in \mathbb{N}. \tag{5.3}$$

In Section 5.6 we will also consider the case $\theta^* = 0$, using the results of Theorem 3.6.

Remark 5.1. The principal term in the asymptotic expression (5.3) may be formally obtained by differentiating the condition (3.19). Similarly, the asymptotic formulas (5.1) and (5.2) are underpinned by the expansion (3.47).

5.2. **Reminder: Poisson–Dirichlet distribution.** Our first aim is to show that, under the assumptions stated in Section 5.1 (most importantly, with $\theta^* > 0$), the descending order statistics of the cycle lengths converge to the so-called *Poisson–Dirichlet distribution* PD(θ^*). The Poisson–Dirichlet distribution was introduced by Kingman [19, §5] as the weak limit of order statistics of a symmetric Dirichlet distribution (with parameter α) on N-dimensional simplex Δ_N as $N \to \infty$, $\alpha N \to \theta > 0$. Such a limit can be identified as the distribution law $\mathcal{L}$ of the normalized points $\sigma_1 > \sigma_2 > \cdots$ of a Poisson process on $(0, \infty)$ with rate $\theta\, x^{-1} \mathrm{e}^{-x}$,

$$\mathrm{PD}(\theta) = \mathcal{L}(\sigma_1/\sigma,\, \sigma_2/\sigma,\, \dots),$$

where $\sigma := \sigma_1 + \sigma_2 + \cdots < \infty$ (a.s.); note that σ has gamma distribution $\mathrm{Gamma}\,(\theta)$ and is independent of the sequence (σ_j/σ) (see, e.g., [21, §9.3] or [1, §5.7] for more details).

An explicit formula for the finite-dimensional probability density of PD(θ), first obtained by Watterson [31, §2] (see also [1, p. 113]), is quite involved but is of no particular interest to us. There is, however, an equivalent descriptive definition of the Poisson-Dirichlet distribution PD(θ) through the so-called $\mathrm{GEM}(\theta)$ distribution (named after Griffiths, Engen and McCloskey; see, e.g., [1, p. 107]), which is the joint distribution of the random variables

$$Y_1 := B_1, \qquad Y_n := B_n \prod_{j=1}^{n} (1 - B_j) \qquad (n \ge 2), \tag{5.4}$$

where (B_n) is a sequence of independent identically distributed (i.i.d.) random variables with the beta distribution $\mathrm{Beta}\,(1, \theta)$ (i.e., with the probability density $\theta(1-x)^{\theta-1}$, $0 \le x \le 1$). From the definition (5.4), it is straightforward to obtain the finite-dimensional probability density of $\mathrm{GEM}(\theta)$ (see [1, p. 107]),

$$f_\theta(x_1, \dots, x_m) = \frac{\theta^m (1 - x_1 - \cdots - x_m)^{\theta-1}}{\prod_{j=1}^{m-1} (1 - x_1 - \cdots - x_j)}, \qquad x_1, \dots, x_m \ge 0, \quad x_1 + \cdots + x_m < 1.$$

Remark 5.2. The sequence (5.4) has a geometric interpretation as a *stick-breaking process*, whereby at each step an i.i.d. $\mathrm{Beta}\,(1, \theta)$-distributed fraction is removed from the remaining "stick" (see, e.g., [18, §3] for the background and further references).

The remarkable link of the $\mathrm{GEM}(\theta)$ with the PD(θ), discovered by Tavaré [29, Theorems 4 and 6] (see also [1, §5.7]), is as follows.

Lemma 5.1. *The decreasing order statistics* $Y^{(1)} \ge Y^{(2)} \ge \cdots$ *of the* $\mathrm{GEM}(\theta)$ *sequence* (5.4) *have precisely the* PD(θ) *distribution.*

Note that since the joint distribution of Y_n's is continuous, the order statistics $Y^{(n)}$ are in fact distinct (a.s.).

5.3. **Modified stick-breaking process.** Let (U_n), (B_n) be two sequences of i.i.d. random variables each, also independent of one another, where U_n's have uniform distribution on $[0, 1]$ and B_n's have beta distribution $\mathrm{Beta}\,(1, \theta^*)$ with parameter $\theta^* > 0$. Setting $\eta_0 := 1$, let us define inductively the random variables

$$\xi_n := \mathbb{1}_{\{U_n \le u^*(\eta_{n-1})\}}, \qquad D_n := \xi_n B_n, \qquad \eta_n := \prod_{j=1}^{n} (1 - D_j) \qquad (n \in \mathbb{N}), \tag{5.5}$$

where $u^*(x) := \tilde{\nu}x/(1-\tilde{\nu}+\tilde{\nu}x)$ $(x \in [0,1])$ and $\mathbb{1}_A$ denotes the indicator of event A. Note that ξ_n's are Bernoulli random variables (with values 0 and 1), adapted to the filtration $\mathcal{F}_n := \sigma\{(U_j, B_j),\, 1 \le j \le n\}$ and with the conditional distribution

$$\mathbb{P}\{\xi_n = 1 \,|\, \mathcal{F}_{n-1}\} = u^*(\eta_{n-1}) \equiv \frac{\tilde{\nu}\,\eta_{n-1}}{1-\tilde{\nu}+\tilde{\nu}\,\eta_{n-1}} \qquad (n \in \mathbb{N}), \tag{5.6}$$

where $\mathcal{F}_0$ is a trivial σ-algebra. In particular, $\mathbb{P}\{\xi_1 = 1\} = \tilde{\nu}$, $\mathbb{P}\{\xi_1 = 0\} = 1-\tilde{\nu}$.

Finally, let us consider the random variables

$$X_n := \eta_{n-1} D_n \qquad (n \in \mathbb{N}). \tag{5.7}$$

Noting that, for all $n \in \mathbb{N}$, we have $0 < B_n < 1$ (a.s.), from (5.5) it is clear that

$$\forall n \in \mathbb{N}, \qquad 0 \le D_n < 1 \quad \text{and} \quad 0 < \eta_n \le 1 \qquad \text{(a.s.)}, \tag{5.8}$$

which also implies, due to (5.7), that

$$\forall n \in \mathbb{N}, \qquad 0 \le X_n < 1 \qquad \text{(a.s.)}.$$

Remark 5.3. The random sequence (X_n) may be interpreted as a *modified stick-breaking process with delays* (cf. Remark 5.2), whereby the original interval $[0,1]$ ("stick") is divided into two parts: (i) $[0,\tilde{\nu}]$ which is subject to a subsequent breaking, and (ii) $[\tilde{\nu},1]$ which stays intact. At each step, the breaking is only enabled if an independent point chosen at random in the remaining stick falls in the breakable part, otherwise the process is idle; the breaking, when it occurs, acts as the removal of a fraction of the current breakable part, independently sampled from $\mathrm{Beta}\,(1,\theta^*)$.

Lemma 5.2. *The random sequence $(X_n)_{n\in\mathbb{N}}$ satisfies the a.s.-identities*

$$\sum_{j=1}^{n} X_j = 1-\eta_n \quad (n \in \mathbb{N}), \qquad \sum_{j=1}^{\infty} X_j = 1. \tag{5.9}$$

Proof. Recalling (5.7), we have

$$\sum_{j=1}^{n} X_j = \sum_{j=1}^{n} (\eta_{j-1}-\eta_j) = 1-\eta_n, \tag{5.10}$$

which proves the first formula in (5.9). To establish the second one, in view of (5.10) we only need to check that $\lim_{n\to\infty} \eta_n = 0$ a.s. To this end, note that $0 \le D_j \le 1$, so that the sequence (η_n) is non-increasing and therefore converges to a (possibly random) limit $\eta \ge 0$. To show that $\eta = 0$ a.s., consider

$$\mathbb{E}\,[\eta_n \,|\, \mathcal{F}_{n-1}] = \eta_{n-1}\bigl(1-\mathbb{E}\,[D_n \,|\, \mathcal{F}_{n-1}]\bigr). \tag{5.11}$$

Recalling the definition of (D_n) (see (5.5)) and noting that ξ_n and B_n are mutually independent, with B_n also independent of $\mathcal{F}_{n-1}$, we obtain, using (5.6) and replacing η_{n-1} in the denominator with its upper bound 1,

$$\begin{aligned}\mathbb{E}\,[D_n \,|\, \mathcal{F}_{n-1}] = \mathbb{E}\,[\xi_n B_n \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}\,[\xi_n \,|\, \mathcal{F}_{n-1}]\cdot\mathbb{E}\,[B_n]\\ &= \frac{\tilde{\nu}\,\eta_{n-1}}{1-\tilde{\nu}+\tilde{\nu}\,\eta_{n-1}}\,\mathbb{E}\,[B_1] \ge \tilde{\nu}\,\eta_{n-1}\,\mathbb{E}\,[B_1].\end{aligned}$$

Returning to (5.11) and taking the expectation, we obtain

$$\mathbb{E}\,[\eta_n] \le \mathbb{E}\,[\eta_{n-1}] - \tilde{\nu}\,\mathbb{E}\,[B_1]\cdot\mathbb{E}\,[\eta_{n-1}^2].$$

Passing to the limit as $n \to \infty$ and using the monotone convergence theorem, we deduce that $\mathbb{E}[\eta^2] \le 0$, hence $\eta = 0$ a.s. This completes the proof of the lemma. □

Remark 5.4. In terms of the modified stick-breaking process (see Remark 5.2), the second equality in (5.9) means that, with probability 1, the total fraction removed from the breakable part of the stick has full measure. This is a generalization of the similar property of the standard stick-breaking process (5.4).

Lemma 5.3. *With probability* 1*, infinitely many* X_n*'s are non-zero.*

Proof. By Lemma 5.2, $\eta_n \to 0$ (a.s.) as $n \to \infty$. In view of the last formula in (5.5), this implies that $\sum_n \log(1 - D_n) = -\infty$ (a.s.), which is equivalent to $\sum_n D_n = +\infty$ (a.s.). Hence, $D_n > 0$ infinitely often (a.s.), and the claim of the lemma now readily follows from (5.7) since $\eta_n > 0$ a.s. (see (5.8)).

The fact that $D_n > 0$ infinitely often (a.s.) can be established more directly as follows. Put $\tau_0 := 0$ and define inductively the successive hitting times

$$\tau_n := \min\{j > \tau_{n-1} : \ \xi_j = 1\}, \qquad n \in \mathbb{N}, \tag{5.12}$$

with the usual convention $\min \emptyset := +\infty$. Since $B_n > 0$ (a.s.), from (5.5) we see that $D_{\tau_n} > 0$ (a.s.), while $D_j = 0$ for $\tau_{n-1} < j < \tau_n$, so it remains to verify that (τ_n) is an a.s.-infinite sequence. Indeed, from the definition of η_n's it follows that

$$\eta_j \equiv \eta_{\tau_{n-1}} \quad (\tau_{n-1} \le j < \tau_n), \qquad \eta_{\tau_n} < \eta_{\tau_{n-1}}.$$

Together with formulas (5.5) and (5.6), this implies that, conditionally on $\mathcal{F}_{\tau_{n-1}}$, the random variable $\tau_n - \tau_{n-1}$ is time to first success in a sequence of independent Bernoulli trials ξ_j with success probability $u^*(\eta_{\tau_{n-1}}) > 0$ (see (5.6)), so it has the geometric distribution

$$\mathbb{P}\{\tau_n - \tau_{n-1} = k \,|\, \mathcal{F}_{\tau_{n-1}}\} = \bigl(1 - u^*(\eta_{\tau_{n-1}})\bigr)^{k-1} u^*(\eta_{\tau_{n-1}}), \qquad k = 1, 2, \dots,$$

and in particular $\tau_n - \tau_{n-1} < \infty$ a.s. From the product structure of the filtration $(\mathcal{F}_n)$ it is also clear that the waiting times $\tau_n - \tau_{n-1}$ are mutually independent ($n \in \mathbb{N}$). Hence, it follows that all τ_n's are a.s.-finite, as required. □

Let $X^{(1)} \ge X^{(2)} \ge \cdots$ be the order statistics built from the sequence (X_n) (see (5.7)) by arranging the entries in descending order. Recall that $X_n \ge 0$; moreover, by Lemma 5.3 infinitely many entries are positive (a.s.) and, in addition, $\sum_n X_n = 1$ according to Lemma 5.2. It follows that, with probability 1, there is an infinite sequence of order statistics $X^{(n)} > 0$, each well defined up to possible ties of at most finite multiplicities (however, it will be clear from the next lemma and the continuity of the Poisson–Dirichlet distribution that positive order statistics $X^{(n)}$ are in fact a.s.-distinct).

In particular, the sequence $X^{(1)} \ge X^{(2)} \ge \cdots > 0$ is not affected by any zero entries among X_n's, which therefore can be removed from (X_n) prior to ordering. But, according to the definition of the random times τ_n (see (5.12)) and by formulas (5.5) and (5.7), successive non-zero entries among $X_1, X_2, \dots$ are precisely given by

$$X_{\tau_1} = B_{\tau_1}, \qquad X_{\tau_n} = B_{\tau_n} \prod_{j=1}^{n-1} (1 - B_{\tau_j}) \quad (n \ge 2), \tag{5.13}$$

where (B_{τ_n}) are i.i.d. random variables with beta distribution $\mathrm{Beta}(1, \theta^*)$. The latter claim can be easily verified using the total probability formula and mutual independence of the waiting times $\tau_n - \tau_{n-1}$ pointed out in the alternative proof of Lemma 5.3; for instance,

$$\begin{aligned}\mathbb{P}\{B_{\tau_1} > x_1, B_{\tau_2} > x_2\} &= \sum_{k_1=1}^{\infty} \sum_{k_2>k_1} \mathbb{P}\{\tau_1 = k_1,\, \tau_2 - \tau_1 = k_2 - k_1\}\, \mathbb{P}\{B_{k_1} > x_1, B_{k_2} > x_2\} \\ &= (1-x_1)^{\theta^*}(1-x_2)^{\theta^*} \sum_{k_1=1}^{\infty} \mathbb{P}\{\tau_1 = k_1\} \sum_{\ell=1}^{\infty} \mathbb{P}\{\tau_2 - \tau_1 = \ell\} \\ &= (1-x_1)^{\theta^*}(1-x_2)^{\theta^*}.\end{aligned}$$

Now, comparing (5.13) with (5.4) and using Lemma 5.1, we arrive at the following result.

Lemma 5.4. *The sequence of descending order statistics $(X^{(n)})$ has the Poisson–Dirichlet distribution* PD(θ^*) *with parameter $\theta^* > 0$.*

In conclusion of this subsection, let us prove some moment identities for the random variables X_j's.

Lemma 5.5. *For each $n \in \mathbb{N}$, as $N \to \infty$,*

$$\mathbb{E}[X_1^n] = \tilde{\nu}\, \frac{B(n+1, \theta^*)}{B(1, \theta^*)} \equiv \frac{\tilde{\nu}\, n!\, \Gamma(\theta^*+1)}{\Gamma(\theta^*+n+1)}\,. \tag{5.14}$$

Furthermore, for all $n_1, n_2 \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\bigl[X_1^{n_1} X_2^{n_2}(1-\tilde{\nu}X_1)\bigr] &= \frac{\tilde{\nu}^2 B(n_1+1, \theta^*+n_2+1)\, B(n_2+1, \theta^*)}{(B(1,\theta^*))^2} \\ &\equiv \frac{\theta^*\, \tilde{\nu}^2\, n_1!\, n_2!\, \Gamma(\theta^*+1)}{\Gamma(\theta^*+n_1+n_2+2)}\,,\end{aligned} \tag{5.15}$$

while for $n_1 = 0$ and any $n_2 \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\bigl[X_2^{n_2}(1-\tilde{\nu}X_1)\bigr] &= \left(1 - \tilde{\nu} + \tilde{\nu}\, \frac{B(1, n_2+1+\theta^*)}{B(1,\theta^*)}\right) \frac{\tilde{\nu}\, B(n_2+1, \theta^*)}{B(1,\theta^*)} \\ &\equiv \bigl\{\theta^* + (n_2+1)(1-\tilde{\nu})\bigr\}\, \frac{\tilde{\nu}\, n_2!\, \Gamma(\theta^*+1)}{\Gamma(\theta^*+n_2+2)}\,.\end{aligned} \tag{5.16}$$

Proof. Using the definitions (5.5) and (5.7), we obtain

$$\begin{aligned}\mathbb{E}[X_1^n] &= \mathbb{E}[\xi_1 B_1^n] = \mathbb{E}[\xi_1] \cdot \mathbb{E}[B_1^n] \\ &= \frac{\tilde{\nu}}{B(1,\theta^*)} \int_0^1 x^n (1-x)^{\theta^*-1}\, \mathrm{d}x = \frac{\tilde{\nu}\, B(n+1, \theta^*)}{B(1,\theta^*)}\,,\end{aligned}$$

which proves the first part of formula (5.14). The second part follows by substituting the well-known representation $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

To prove (5.15), let us first compute the conditional expectation

$$\mathbb{E}\bigl[X_1^{n_1} X_2^{n_2}(1-\tilde{\nu}X_1)\,|\,\mathcal{F}_1\bigr] = D_1^{n_1}(1-D_1)^{n_2}(1-\tilde{\nu}D_1)\, \mathbb{E}\bigl[D_2^{n_2}\,|\,\mathcal{F}_1\bigr], \tag{5.17}$$

where, according to (5.5) and (5.6),

$$\begin{aligned}\mathbb{E}\bigl[D_2^{n_2}\,|\,\mathcal{F}_1\bigr] = \mathbb{E}\bigl[\xi_2 B_2^{n_2}\,|\,\mathcal{F}_1\bigr] &= \frac{\tilde{\nu}\, \eta_1}{1-\tilde{\nu}+\tilde{\nu}\, \eta_1} \cdot \mathbb{E}[B_2^{n_2}] \\ &= \frac{\tilde{\nu}\,(1-D_1)}{1-\tilde{\nu}D_1} \cdot \frac{B(n_2+1, \theta^*)}{B(1,\theta^*)}\,.\end{aligned}$$

Hence, on taking the expectation of (5.17), we obtain

$$\begin{aligned}\mathbb{E}\big[X_1^{n_1}X_2^{n_2}(1-\tilde{\nu}X_1)\big] &= \frac{\tilde{\nu}\,B(n_2+1,\theta^*)}{B(1,\theta^*)}\,\mathbb{E}\big[D_1^{n_1}(1-D_1)^{n_2+1}\big]\\ &= \frac{\tilde{\nu}\,B(n_2+1,\theta^*)}{B(1,\theta^*)}\,\mathbb{E}\big[\xi_1^{n_1}B_1^{n_1}(1-\xi_1B_1)^{n_2+1}\big].\end{aligned}\tag{5.18}$$

For $n_1 \geq 1$ we have $\xi_1^{n_1} = \xi_1$, leading to

$$\begin{aligned}\mathbb{E}\big[\xi_1^{n_1}B_1^{n_1}(1-\xi_1B_1)^{n_2+1}\big] &= \mathbb{P}\{\xi_1=1\}\,\mathbb{E}\big[B_1^{n_1}(1-B_1)^{n_2+1}\big]\\ &= \frac{\tilde{\nu}}{B(1,\theta^*)}\int_0^1 x^{n_1}(1-x)^{\theta^*+n_2}\,\mathrm{d}x = \frac{\tilde{\nu}\,B(n_1+1,\theta^*+n_2+1)}{B(1,\theta^*)},\end{aligned}\tag{5.19}$$

and the substitution of (5.19) into (5.18) gives the first line of formula (5.15). The second line can again be obtained by expressing the beta function through the gamma function.

For $n_1 = 0$, formula (5.18) is reduced to

$$\mathbb{E}\big[X_2^{n_2}(1-\tilde{\nu}X_1)\big] = \frac{\tilde{\nu}\,B(n_2+1,\theta^*)}{B(1,\theta^*)}\,\mathbb{E}\big[(1-\xi_1B_1)^{n_2+1}\big],\tag{5.20}$$

and similarly as before we find

$$\begin{aligned}\mathbb{E}\big[(1-\xi_1B_1)^{n_2+1}\big] &= \mathbb{P}\{\xi_1=0\} + \mathbb{P}\{\xi_1=1\}\,\mathbb{E}\big[(1-B_1)^{n_2+1}\big]\\ &= (1-\tilde{\nu}) + \tilde{\nu}\int_0^1 (1-x)^{\theta^*+n_2}\,\mathrm{d}x\\ &= 1-\tilde{\nu} + \frac{\tilde{\nu}\,B(1,\theta^*+n_2+1)}{B(1,\theta^*)}.\end{aligned}$$

Together with (5.20) this proves the first line of (5.16), while the second line then follows by usual manipulations with the beta function. □

5.4. **Cycles under lexicographic ordering.** The next theorem characterizes the asymptotic (finite-dimensional) distributions of normalized cycle lengths (L_j) under the lexicographic ordering introduced in Definition 2.1. Owing to the normalization proportional to N, only long cycles (i.e., of length comparable to N) survive in the limit as $N \to \infty$. This result should be contrasted with Theorem 4.6 that deals with non-normalized cycle lengths, thus revealing an asymptotic loss of mass in the supercritical regime due to long cycles.

Theorem 5.6. *For each* $m \in \mathbb{N}$,

$$\frac{1}{N\tilde{\nu}}\,(L_1,\dots,L_m) \xrightarrow{d} (X_1,\dots,X_m)\,,\qquad N\to\infty,\tag{5.21}$$

where $\tilde{\nu} > 0$ *is given by* (4.11) *and the random variables* X_j*'s are as defined in* (5.7)*. In particular,* $L_1/(N\tilde{\nu})$ *converges in distribution to a random variable* X_1 *with atom* $1-\tilde{\nu}$ *at zero and an absolutely continuous component on* $(0,1)$ *with density* $\tilde{\nu}\,\theta^*(1-x)^{\theta^*-1}$.

For the proof of the theorem, we first need to establish the following lemma.

Lemma 5.7. *For each* $n \in \mathbb{N}$,

$$\lim_{N\to\infty}\frac{1}{(N\tilde{\nu})^n}\,\tilde{\mathbb{E}}_N\big[L_1^n\big] = \frac{\tilde{\nu}\,n!\,\Gamma(\theta^*+1)}{\Gamma(\theta^*+n+1)}\,.\tag{5.22}$$

Furthermore, for any $n_1, n_2 \in \mathbb{N}$

$$\lim_{N\to\infty} \frac{1}{(N\tilde{\nu})^{n_1+n_2}}\, \tilde{\mathbb{E}}_N\bigl[L_1^{n_1} L_2^{n_2} (1 - L_1/N)\bigr] = \frac{\theta^* \tilde{\nu}^{\,2}\, n_1!\, n_2!\, \Gamma(\theta^*+1)}{\Gamma(\theta^*+n_1+n_2+2)}\,, \tag{5.23}$$

while for $n_1 = 0$ *and any* $n_2 \in \mathbb{N}$

$$\lim_{N\to\infty} \frac{1}{(N\tilde{\nu})^{n_2}}\, \tilde{\mathbb{E}}_N\bigl[L_2^{n_2} (1 - L_1/N)\bigr] = \frac{\tilde{\nu}\,\{\theta^* + (n_2+1)(1-\tilde{\nu})\}\, n_2!\, \Gamma(\theta^*+1)}{\Gamma(\theta^*+n_2+2)}\,. \tag{5.24}$$

Remark 5.5. The subtlety of Lemma 5.7 is hidden in the fact that the asymptotics (5.22), (5.23), (5.24) are invalidated if either of n, n_1, n_2 takes the value 0; for instance, formula (5.24) cannot be readily deduced from (5.23) by setting $n_1 = 0$. An explanation lies in the asymptotic separation of "short" and "long" cycles, leading to the emergence of an atom of mass $1 - \tilde{\nu}$ at zero in the limiting distribution of $(L_1/N)^n$ for $n > 0$.

Proof of Lemma 5.7. Let us first consider $\tilde{\mathbb{E}}_N\bigl[(L_1-1)_n\bigr]$, where $(\cdot)_n$ is the Pochhammer symbol defined in (2.11). Using the distribution of L_1 obtained in Lemma 2.5 (see (2.17)) and recalling formulas (2.9) and (3.2), we get for each $n \in \mathbb{N}$

$$\begin{aligned}
\tilde{\mathbb{E}}_N\bigl[(L_1-1)_n\bigr] &= \frac{1}{NH_N}\sum_{\ell=1}^{\infty} (\ell-1)_n\,(\theta_\ell + N\kappa_\ell)\, h_{N-\ell}(N) \\
&= \frac{1}{NH_N}\sum_{\ell=1}^{\infty} (\ell-1)_n\,(\theta_\ell + N\kappa_\ell)\, [z^{N-\ell}]\bigl[\mathrm{e}^{G_N(z)}\bigr] \\
&= \frac{1}{NH_N}\,[z^N]\left[\sum_{\ell=1}^{\infty} (\ell-1)_n\,(\theta_\ell + N\kappa_\ell)\, z^{\ell}\, \mathrm{e}^{G_N(z)}\right] \\
&= \frac{1}{NH_N}\,[z^N]\bigl[G_N^{\{n+1\}}(z)\, \mathrm{e}^{G_N(z)}\bigr].
\end{aligned} \tag{5.25}$$

In view of the formula $\tilde{\nu} = 1 - g_\kappa^{\{1\}}(R)$ (see (4.11)), Theorem 3.5 with function $f(z) = g_\theta^{\{n+1\}}(z)$ and $\beta = n+1$ (see (5.3)) gives

$$\frac{1}{NH_N}\,[z^N]\bigl[g_\theta^{\{n+1\}}(z)\, \mathrm{e}^{G_N(z)}\bigr] \sim (N\tilde{\nu})^n \frac{\theta^* n!\, \Gamma(\theta^*)}{\Gamma(\theta^*+n+1)}\, \tilde{\nu}. \tag{5.26}$$

Furthermore, using the asymptotic expansions (5.1), (5.2) and applying Theorem 3.5 with $f(z) = g_\kappa^{\{n+1\}}(z)$ and $\beta = \max\{0, n+1-s\}$, we obtain

$$\frac{1}{NH_N}\,[z^N]\bigl[N g_\kappa^{\{n+1\}}(z)\, \mathrm{e}^{G_N(z)}\}\bigr] = O(1) + O(N^{n+1-s}) = o(N^n), \tag{5.27}$$

due to the conditions $s > 1$, $n > 0$. Thus, substituting (5.26) and (5.27) into (5.25) yields

$$\tilde{\mathbb{E}}_N\bigl[(L_1-1)_n\bigr] \sim (N\tilde{\nu})^n \frac{n!\, \Gamma(\theta^*+1)}{\Gamma(\theta^*+n+1)}\, \tilde{\nu},$$

which implies (5.22).

We now turn to (5.24) and (5.23). Using the joint distribution of L_1 and L_2 (see (2.16)), it follows by a similar computation as in (5.25) that, for any integers $n_1 \ge 0$, $n_2 > 0$,

$$\tilde{\mathbb{E}}_N\bigl[(L_1-1)_{n_1}(L_2-1)_{n_2}(1-L_1/N)\bigr] = \frac{1}{N^2 H_N}\,[z^N]\bigl[G_N^{\{n_1+1\}}(z)\, G_N^{\{n_2+1\}}(z)\, \mathrm{e}^{G_N(z)}\bigr]. \tag{5.28}$$

As before, we can work out the asymptotics of (5.28) by using Theorem 3.5 with the function

$$\begin{aligned} f(z) &= G_N^{\{n_1+1\}}(z)\, G_N^{\{n_2+1\}}(z) \\ &= g_\theta^{\{n_1+1\}}(z)\, g_\theta^{\{n_2+1\}}(z) + N g_\kappa^{\{n_1+1\}}(z)\, g_\theta^{\{n_2+1\}}(z) \\ &\quad + N g_\theta^{\{n_1+1\}}(z)\, g_\kappa^{\{n_2+1\}}(z) + N^2 g_\kappa^{\{n_1+1\}}(z)\, g_\kappa^{\{n_2+1\}}(z). \end{aligned} \tag{5.29}$$

The singularity of each term in (5.29) (and the respective index β, see (3.21)) is specified from formulas (5.1), (5.2) and (5.3). First of all, using (5.3) we have

$$g_\theta^{\{n_1+1\}}(z)\, g_\theta^{\{n_2+1\}}(z) \sim \frac{(\theta^*)^2\, n_1!\, n_2!}{(1-z/R)^{n_1+n_2+2}}$$

(i.e., $\beta = n_1 + n_2 + 2$), so formulas (3.22), (3.23) of Theorem 3.5 give

$$\frac{1}{N^2 H_N}\, [z^N]\big[g_\theta^{\{n_1+1\}}(z)\, g_\theta^{\{n_2+1\}}(z)\, \mathrm{e}^{G_N(z)}\big] \sim \frac{\theta^*\, \Gamma(\theta^*+1)\, n_1!\, n_2!}{\Gamma(\theta^*+n_1+n_2+2)}\, (N\tilde{\nu})^{n_1+n_2}, \tag{5.30}$$

which coincides with the asymptotics (5.23).

For $n_1 \ge 1$, contributions from other terms in (5.29) are negligible as compared to $N^{n_1+n_2}$. Indeed, from (5.1) and (5.2) we get

$$g_\kappa^{\{n_1+1\}}(z)\, g_\kappa^{\{n_2+1\}}(z) = O\big((1-z/R)^{-\beta}\big)$$

with

$$\beta = \max\{0, n_1+1-s, n_2+1-s, n_1+n_2+2-2s\} < n_1+n_2,$$

thanks to the condition $s > 1$. Hence, by Theorem 3.5 we have

$$\frac{1}{N^2 H_N}[z^N]\big[N^2 g_\kappa^{\{n_1+1\}}(z)\, g_\kappa^{\{n_2+1\}}(z)\, \mathrm{e}^{G_N(z)}\big] = o(N^{n_1+n_2}). \tag{5.31}$$

Similarly, using (5.1), (5.2) and (5.3) we get

$$g_\theta^{\{n_1+1\}}(z)\, g_\kappa^{\{n_2+1\}}(z) = \frac{\theta^* n_1!}{(1-z/R)^{n_1+1}}\, \big\{O(1) + O\big((1-z/R)^{s-n_2-1}\big)\big\},$$

and Theorem 3.5 with

$$\beta = \max\{n_1+1, n_1+n_2+2-s\} < n_1+n_2+1$$

again gives

$$\frac{1}{N^2 H_N}[z^N]\big[N g_\theta^{\{n_1+1\}}(z)\, g_\kappa^{\{n_2+1\}}(z)\, \mathrm{e}^{G_N(z)}\big] = o(N^{n_1+n_2}). \tag{5.32}$$

By symmetry, the same estimate (5.32) holds for the term $g_\kappa^{\{n_1+1\}}(z)\, g_\theta^{\{n_2+1\}}(z)$ with $n_1 \ge 1$, $n_2 \ge 1$. Thus, substituting (5.30), (5.31) and (5.32) into (5.28), we obtain (5.23).

The case $n_1 = 0$ requires more care; here we have (see (5.1) and (5.3))

$$g_\kappa^{\{1\}}(z)\, g_\theta^{\{n_2+1\}}(z) \sim \frac{g_\kappa^{\{1\}}(R)\, \theta^* n_2!}{(1-z/R)^{n_2+1}},$$

and so Theorem 3.5 with $\beta = n_2 + 1$ yields

$$\frac{1}{N^2 H_N}[z^N]\big[N g_\kappa^{\{1\}}(z)\, g_\theta^{\{n_2+1\}}(z)\, \mathrm{e}^{G_N(z)}\big] \sim \frac{g_\kappa^{\{1\}}(R)\, \Gamma(\theta^*+1)\, n_2!}{\Gamma(\theta^*+n_2+1)}\, (N\tilde{\nu})^{n_2}, \tag{5.33}$$

which is of the same order as the right-hand side of (5.30) (with $n_1 = 0$). Hence, adding up the contributions (5.30) and (5.33) and recalling that $g_\kappa^{\{1\}}(R) = 1 - \tilde{\nu}$, we obtain (5.24). □

Remark 5.6. As should be clear from the proof, the factor $1 - L_1/N = N^{-1}(N - L_1)$ is included in (5.23) and (5.24) in order to cancel the denominator $N - \ell_1$ in formula (2.18) of the two-dimensional distribution of (L_1, L_2). As suggested by the general formula (2.16) of Lemma 2.5, an extension of Lemma 5.7 to the m-dimensional case $L_1, \dots, L_m$ requires the inclusion of the product $\prod_{j=1}^{m-1}(N - L_1 - \dots - L_j)$. The corresponding calculations are tedious but straightforward, and follow the same pattern as for $m = 2$. A suitable extension is also possible for Lemma 5.5.

Proof of Theorem 5.6. For the sake of clarity, we consider only the case $m = 2$ (i.e., involving the joint distribution of L_1, L_2); computations in the general case require an extension of Lemmas 5.5 and 5.7 (see Remark 5.6) and can be carried out along the same lines.

By the continuous mapping theorem and according to the definitions (5.5) and (5.7), the convergence (5.21) with $m = 2$ is equivalent to

$$\frac{1}{N\tilde{\nu}}\bigl(L_1, L_2(1 - L_1/N)\bigr) \xrightarrow{d} \bigl(X_1, X_2(1 - \tilde{\nu}X_1)\bigr), \qquad N \to \infty.$$

By the method of moments, it suffices to show that for any $n_1, n_2 \in \mathbb{N}_0$, as $N \to \infty$,

$$\tilde{\mathbb{E}}_N\bigl[L_1^{n_1} L_2^{n_2} (1 - L_1/N)^{n_2}\bigr] \sim (N\tilde{\nu})^{n_1+n_2}\, \mathbb{E}\bigl[X_1^{n_1} X_2^{n_2} (1 - \tilde{\nu}X_1)^{n_2}\bigr]. \tag{5.34}$$

First, for $n_1 = n_2 = 0$ the relation (5.34) is trivial, since both sides are reduced to 1. If $n_2 = 0$ and $n_1 > 0$ then (5.34) readily follows from the relations (5.14) (Lemma 5.5) and (5.22) (Lemma 5.7).

To cover the case $n_2 \ge 1$, let us prove a more general asymptotic relation

$$\tilde{\mathbb{E}}_N\bigl[L_1^{n_1} L_2^{n_2} (1 - L_1/N)^{q}\bigr] \sim (N\tilde{\nu})^{n_1+n_2}\, \mathbb{E}\bigl[X_1^{n_1} X_2^{n_2} (1 - \tilde{\nu}X_1)^{q}\bigr], \tag{5.35}$$

valid for all $n_1 \in \mathbb{N}_0$ and $n_2, q \in \mathbb{N}$. We argue by induction on q. For $q = 1$, the relation (5.35) is verified by comparing formulas (5.15) and (5.23) (for $n_1 \ge 1$) or (5.16) and (5.24) (for $n_1 = 0$). Now suppose that (5.35) is true for some $q \ge 1$. Expanding

$$(1 - L_1/N)^{q+1} = (1 - L_1/N)^q - N^{-1} L_1 (1 - L_1/N)^q$$

and using the induction hypothesis (5.35), we get

$$\begin{aligned}
&\tilde{\mathbb{E}}_N\bigl[L_1^{n_1} L_2^{n_2} (1 - L_1/N)^{q+1}\bigr] \\
&\qquad = \tilde{\mathbb{E}}_N\bigl[L_1^{n_1} L_2^{n_2} (1 - L_1/N)^{q}\bigr] - N^{-1}\, \tilde{\mathbb{E}}_N\bigl[L_1^{n_1+1} L_2^{n_2} (1 - L_1/N)^{q}\bigr] \\
&\qquad \sim (N\tilde{\nu})^{n_1+n_2} \Bigl\{\mathbb{E}\bigl[X_1^{n_1} X_2^{n_2} (1 - \tilde{\nu}X_1)^{q}\bigr] - \tilde{\nu}\, \mathbb{E}\bigl[X_1^{n_1+1} X_2^{n_2} (1 - \tilde{\nu}X_1)^{q}\bigr]\Bigr\} \\
&\qquad = (N\tilde{\nu})^{n_1+n_2}\, \mathbb{E}\bigl[X_1^{n_1} X_2^{n_2} (1 - \tilde{\nu}X_1)^{q+1}\bigr],
\end{aligned}$$

which verifies (5.35) for $q + 1$, and therefore for all $q \ge 1$. This completes the proof of Theorem 5.6. □

5.5. **Poisson–Dirichlet distribution for the cycle order statistics.** Let us now consider the cycle lengths without the lexicographic ordering, and arrange them in descending order.

Definition 5.1. For a permutation $\sigma \in \mathfrak{S}_N$, let $L^{(1)} = L^{(1)}(\sigma)$ be the length of the longest cycle in σ, $L^{(2)} = L^{(2)}(\sigma)$ the length of the second longest cycle in σ, etc.

Let us first prove a suitable "cut-off" lemma for lexicographically ordered cycles.

Lemma 5.8. *For any $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \tilde{\mathbb{P}}_N\left(\bigcup_{j>n}\{N^{-1}L_j > \varepsilon\}\right) = 0. \tag{5.36}$$

Proof of Lemma 5.8. Fix a $K \in \mathbb{N}$ and note that, for all $N \geq K/\varepsilon$,

$$\begin{aligned}\tilde{\mathbb{P}}_N\left(\bigcup_{j>n}\{N^{-1}L_j > \varepsilon\}\right) &= \tilde{\mathbb{P}}_N\left(\bigcup_{j>n}\{N^{-1}L_j\,\mathbb{1}_{\{L_j>K\}} > \varepsilon\}\right)\\ &\leq \tilde{\mathbb{P}}_N\left\{\frac{1}{N}\sum_{j>n} L_j\,\mathbb{1}_{\{L_j>K\}} > \varepsilon\right\}.\end{aligned} \tag{5.37}$$

Furthermore, noting that

$$\sum_{j>n} L_j\,\mathbb{1}_{\{L_j>K\}} = \sum_{j=1}^{\infty} L_j\,\mathbb{1}_{\{L_j>K\}} - \sum_{j=1}^{n} L_j\,\mathbb{1}_{\{L_j>K\}}, \tag{5.38}$$

by Theorems 4.3 and 5.6 we have, as $N \to \infty$,

$$\frac{1}{N}\sum_{j=1}^{\infty} L_j\,\mathbb{1}_{\{L_j>K\}} \xrightarrow{p} \tilde{\nu}_K, \qquad \frac{1}{N}\sum_{j=1}^{n} L_j\,\mathbb{1}_{\{L_j>K\}} \xrightarrow{d} \tilde{\nu}\sum_{j=1}^{n} X_j\,.$$

Returning to (5.38), this gives

$$\sum_{j>n} L_j\,\mathbb{1}_{\{L_j>K\}} \xrightarrow{d} \tilde{\nu}_K - \tilde{\nu}\sum_{j=1}^{n} X_j, \qquad N \to \infty.$$

Hence, recalling that $\lim_{K\to\infty}\tilde{\nu}_K = \tilde{\nu}$ (see (4.9)), from (5.37) we obtain

$$\begin{aligned}\limsup_{N\to\infty} \tilde{\mathbb{P}}_N\left(\bigcup_{j>n}\{N^{-1}L_j > \varepsilon\}\right) &\leq \liminf_{K\to\infty} \mathbb{P}\left\{\tilde{\nu}\sum_{j=1}^{n} X_j \leq \tilde{\nu}_K - \varepsilon\right\}\\ &= \mathbb{P}\left\{\sum_{j=1}^{n} X_j \leq 1 - \varepsilon/\tilde{\nu}\right\}.\end{aligned}$$

Finally, passing here to the limit as $n \to \infty$ and noting that, by Lemma 5.2, $\sum_{j=1}^{\infty} X_j = 1$ (a.s.), we arrive at (5.36), as claimed. □

The next theorem is our main result in this subsection. Recall that the parameter $\theta^* > 0$ is involved in the assumption (5.3).

Theorem 5.9. *In the sense of convergence of finite-dimensional distributions,*

$$\frac{1}{N\tilde{\nu}}\left(L^{(1)}, L^{(2)}, \dots\right) \xrightarrow{d} \mathrm{PD}(\theta^*), \qquad N \to \infty,$$

where $\mathrm{PD}(\theta^*)$ *denotes the Poisson–Dirichlet distribution with parameter* θ^*.

Proof. By virtue of Lemma 5.4, it suffices to show that, for each $m \in \mathbb{N}$,

$$\frac{1}{N\tilde{\nu}}\left(L^{(1)}, \dots, L^{(m)}\right) \xrightarrow{d} \left(X^{(1)}, \dots, X^{(m)}\right), \qquad N \to \infty. \tag{5.39}$$

Let us first verify (5.39) for $m=1$. Fix an integer $n\ge 1$ and observe that, for any $x\in(0,1)$,

$$\tilde{\mathbb{P}}_N\bigl\{(N\tilde{\nu})^{-1}L^{(1)}>x\bigr\}\ge \tilde{\mathbb{P}}_N\Bigl\{(N\tilde{\nu})^{-1}\max_{j\le n}L_j>x\Bigr\}.$$

Hence, by Theorem 5.6 and the continuous mapping theorem, it follows that

$$\begin{aligned}\liminf_{N\to\infty}\tilde{\mathbb{P}}_N\bigl\{(N\tilde{\nu})^{-1}L^{(1)}>x\bigr\}&\ge \lim_{n\to\infty}\mathbb{P}\Bigl\{\max_{j\le n}X_j>x\Bigr\}\\&\ge \mathbb{P}\{X^{(1)}>x\},\end{aligned}\tag{5.40}$$

because $\max_{j\le n}X_j\uparrow X^{(1)}$ as $n\uparrow\infty$. On the other hand, we have an upper bound

$$\tilde{\mathbb{P}}_N\bigl\{(N\tilde{\nu})^{-1}L^{(1)}>x\bigr\}\le \tilde{\mathbb{P}}_N\Bigl\{(N\tilde{\nu})^{-1}\max_{j\le n}L_j>x\Bigr\}+\tilde{\mathbb{P}}_N\Bigl\{N^{-1}\max_{j>n}L_j>\tilde{\nu}x\Bigr\},$$

and by Theorem 5.6 and Lemma 5.8 this yields (cf. (5.40))

$$\begin{aligned}\limsup_{N\to\infty}\tilde{\mathbb{P}}_N\bigl\{(N\tilde{\nu})^{-1}L^{(1)}>x\bigr\}&\le \lim_{n\to\infty}\limsup_{N\to\infty}\tilde{\mathbb{P}}_N\Bigl\{(N\tilde{\nu})^{-1}\max_{j\le n}L_j>x\Bigr\}\\&\le \lim_{n\to\infty}\mathbb{P}\Bigl\{\max_{j\le n}X_j>x\Bigr\}\\&\le \mathbb{P}\bigl\{X^{(1)}\ge x\bigr\}.\end{aligned}\tag{5.41}$$

Combining (5.40) and (5.41) and assuming that $x\in(0,1)$ is a point of continuity of the distribution of $X^{(1)}$ (which is, in fact, automatically true owing to Lemma 5.4), we obtain

$$\lim_{N\to\infty}\tilde{\mathbb{P}}_N\bigl\{(N\tilde{\nu})^{-1}L^{(1)}>x\bigr\}=\mathbb{P}\{X^{(1)}>x\},$$

which proves (5.39) with $m=1$.

The general case $m\ge 2$ is handled in a similar manner, by using lower and upper estimates for the m-dimensional probability $\tilde{\mathbb{P}}_N\bigl\{(N\tilde{\nu})^{-1}L^{(1)}>x_1,\dots,(N\tilde{\nu})^{-1}L^{(m)}>x_m\bigr\}$ through the similar probabilities for the order statistics of the truncated sample $L_1,\dots,L_n$ $(n\ge m)$, where the "discrepancy" term due to the contribution of the tail part $(L_j,\ j>n)$ may be shown to be asymptotically negligible as $n\to\infty$ by virtue of the cut-off Lemma 5.8. ☐

Remark 5.7. As already mentioned in Remark 3.4, the requirement imposed in Section 5.1 that the asymptotic expansion (3.47) of $g_\kappa(z)$ holds with a non-integer index $s>1$ may be extended to allow a *power-logarithmic* term $b_s(1-z/R)^s\log(1-z/R)$ (with *any* $s>1$). With the asymptotic formulas (5.1), (5.2) modified accordingly, the proof of Lemma 5.7 may be adapted as appropriate, implying that Theorems 5.6 and 5.9 remain true.

5.6. **Case $\theta^*=0$.** Let us now turn to studying the asymptotic behaviour of cycles in the case $\theta^*=0$ (see (3.19)). More precisely, throughout this subsection we suppose that, as in Theorem 3.6, the generating function $g_\theta(z)$ is holomorphic in a suitable domain $\varDelta_0$ (see Definition 3.3) and, moreover, is regular at point $z=R$; in particular, the successive (modified) derivatives of $g_\theta(z)$ have Taylor-type asymptotic expansions, as $z\to R$ $(z\in\varDelta_0)$,

$$g_\theta^{\{n\}}(z)\sim\Bigl(\frac{z}{R}\Bigr)^n\sum_{j=n}^{\infty}\frac{(-1)^{j-n}g_\theta^{\{j\}}(R)}{(j-n)!}\,(1-z/R)^{j-n}.$$

We also assume that the generating function $g_\kappa(z)$ admits the asymptotic expansion (3.47) of Theorem 3.6 (with a non-integer $s>1$), which may be differentiated any number of times to yield a nested family of expansions (with some $\delta_n > 0,\ n \in \mathbb{N}_0$),

$$g_\kappa^{\{n\}}(z) = \left(\frac{z}{R}\right)^n \left\{ \sum_{n\le j<s} \frac{(-1)^{j-n} g_\kappa^{\{j\}}(R)}{(j-n)!} \left(1-\frac{z}{R}\right)^{j-n} + \frac{\Gamma(-s+n)\, a_s}{\Gamma(-s)} \left(1-\frac{z}{R}\right)^{s-n} \right\} + O\!\left(\left(1-\frac{z}{R}\right)^{s-n+\delta_n}\right), \tag{5.42}$$

where the first sum is understood to vanish if $n > s$ (cf. (5.1), (5.2)).

Let us now revisit the modified stick-breaking process underpinning our argumentation in the case $\theta^* > 0$ in Section 5.3. Observe that if $\theta^* \downarrow 0$ then the beta distribution $\mathrm{Beta}\,(1,\theta^*)$ of the random variables B_n converges to Dirac delta measure $\delta_1(\mathrm{d}x)$, since for any $x \in [0,1)$

$$\mathbb{P}\{B_n > x\} = \int_x^1 \theta^*(1-u)^{\theta^*-1}\,\mathrm{d}u = (1-x)^{\theta^*} \to 1, \qquad \theta^* \to 0.$$

It is easy to see that, under this limit, equations (5.5) and (5.7) are greatly simplified to the following. Let (ξ_n) be a sequence of i.i.d. Bernoulli random variables with success probability $\mathbb{P}\{\xi_n = 1\} = \tilde\nu > 0$. Let $\tau_1 := \min\{n:\ \xi_n = 1\} < \infty$ (a.s.) be the random time until first success, with geometric distribution

$$\mathbb{P}\{\tau_1 = k\} = (1-\tilde\nu)^{k-1}\tilde\nu, \qquad k \in \mathbb{N}. \tag{5.43}$$

Now, for $n \in \mathbb{N}$ we set

$$X_n := \mathbb{1}_{\{\tau_1 = n\}} = \begin{cases} 0, & n \ne \tau_1, \\ 1, & n = \tau_1. \end{cases} \tag{5.44}$$

Remark 5.8. Formula (5.44) shows that the modified stick-breaking process (X_n) described in Remark 5.2 for $\theta^* > 0$, reduces in the case $\theta^* = 0$ to removing the entire breakable part $[0,\tilde\nu]$ at once after waiting time τ_1.

The following analogue of Lemma 5.5 is formally obtained by substituting $\theta^* = 0$; its proof is elementary by using the definition (5.44) and the distribution of τ_1 (see (5.43)).

Lemma 5.10. *For any* $n_1, n_2 \in \mathbb{N}$,

$$\mathbb{E}[X_1^{n_1}] = \tilde\nu,$$
$$\mathbb{E}\big[X_1^{n_1}X_2^{n_2}(1-\tilde\nu X_1)\big] = 0,$$
$$\mathbb{E}\big[X_2^{n_2}(1-\tilde\nu X_1)\big] = (1-\tilde\nu)\tilde\nu.$$

Next, we prove an analogue of Lemma 5.7, which formally looks as its particular case with $\theta^* = 0$ (cf. (5.22), (5.23) and (5.24)).

Lemma 5.11. *For any* $n_1, n_2 \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{(N\tilde\nu)^{n_1}}\, \tilde{\mathbb{E}}_N\big[L_1^{n_1}\big] = \tilde\nu, \tag{5.45}$$

$$\lim_{N\to\infty} \frac{1}{(N\tilde\nu)^{n_1+n_2}}\, \tilde{\mathbb{E}}_N\big[L_1^{n_1}L_2^{n_2}(1-L_1/N)\big] = 0, \tag{5.46}$$

$$\lim_{N\to\infty} \frac{1}{(N\tilde\nu)^{n_2}}\, \tilde{\mathbb{E}}_N\big[L_2^{n_2}(1-L_1/N)\big] = (1-\tilde\nu)\tilde\nu. \tag{5.47}$$

Proof. We use similar argumentation as in the proof of Lemma 5.7, but now exploiting Theorem 3.6. First of all, according to (5.25) we have, for any $n_1 \in \mathbb{N}$,

$$\tilde{\mathbb{E}}_N\bigl[(L_1-1)_{n_1}\bigr] = \frac{1}{NH_N}\,[z^N]\Bigl[\bigl(g_\theta^{\{n_1+1\}}(z) + Ng_\kappa^{\{n_1+1\}}(z)\bigr)\,\mathrm{e}^{G_N(z)}\Bigr]. \tag{5.48}$$

Applying Theorem 3.6(i) with $f(z) = g_\theta^{\{n_1+1\}}(z)$ and $\beta = \infty$ (see (3.49) and (3.52)) gives

$$\frac{1}{NH_N}\,[z^N]\bigl[g_\theta^{\{n+1\}}(z)\,\mathrm{e}^{G_N(z)}\bigr] \sim \frac{1}{N} \to 0, \qquad N\to\infty. \tag{5.49}$$

On the other hand, on account of the asymptotic expansion (5.42) (with $n = n_1$), by Theorem 3.6(ii) with $f(z) = g_\kappa^{\{n_1+1\}}(z)$ and $\beta = s - n_1 - 1 < s - 1$ we obtain

$$\frac{1}{NH_N}\,[z^N]\bigl[Ng_\kappa^{\{n_1+1\}}(z)\,\mathrm{e}^{G_N(z)}\}\bigr] \sim (N\tilde{\nu})^{n_1}\,\tilde{\nu}, \qquad N\to\infty, \tag{5.50}$$

recalling that $\tilde{\nu} = 1 - g_\kappa^{\{1\}}(R)$ (see (4.11)). Hence, substituting (5.49) and (5.50) into (5.48) yields $\tilde{\mathbb{E}}_N\bigl[(L_1-1)_{n_1}\bigr] \sim (N\tilde{\nu})^{n_1}\,\tilde{\nu}$, which implies (5.45).

We now turn to (5.46) and (5.47). Again considering the factorial moments, by formula (5.28) we have, for any integers $n_1 \ge 0$, $n_2 \ge 1$,

$$\tilde{\mathbb{E}}_N\bigl[(L_1-1)_{n_1}(L_2-1)_{n_2}(1-L_1/N)\bigr] = \frac{1}{N^2H_N}\,[z^N]\bigl[G_N^{\{n_1+1\}}(z)\,G_N^{\{n_2+1\}}(z)\,\mathrm{e}^{G_N(z)}\bigr], \tag{5.51}$$

where the product $G_N^{\{n_1+1\}}(z)\,G_N^{\{n_2+1\}}(z)$ is expanded in (5.29). Note that, like in (5.49),

$$\frac{1}{N^2H_N}\,[z^N]\bigl[g_\theta^{\{n_1+1\}}(z)\,g_\theta^{\{n_2+1\}}(z)\,\mathrm{e}^{G_N(z)}\}\bigr] = O(N^{-2}), \qquad N\to\infty. \tag{5.52}$$

Suppose that $n_1, n_2 \ge 1$. Then, similarly to (5.50), we have as $N\to\infty$

$$\frac{1}{N^2H_N}\,[z^N]\bigl[Ng_\kappa^{\{n_2+1\}}(z)\,g_\theta^{\{n_1+1\}}(z)\,\mathrm{e}^{G_N(z)}\}\bigr] = O(N^{n_2-1}), \tag{5.53}$$

$$\frac{1}{N^2H_N}\,[z^N]\bigl[Ng_\kappa^{\{n_1+1\}}(z)\,g_\theta^{\{n_2+1\}}(z)\,\mathrm{e}^{G_N(z)}\}\bigr] = O(N^{n_1-1}). \tag{5.54}$$

Furthermore, applying Theorem 3.6(ii) with $f(z) = g_\kappa^{\{n_1+1\}}(z)\,g_\kappa^{\{n_2+1\}}(z)$ and

$$\beta = \min\{s-n_1-1,\, s-n_2-1\} < s-1,$$

and noting that $\beta \ge s - n_1 - n_2$, we obtain

$$\frac{1}{N^2H_N}\,[z^N]\bigl[N^2g_\kappa^{\{n_1+1\}}(z)\,g_\kappa^{\{n_2+1\}}(z)\,\mathrm{e}^{G_N(z)}\bigr] = O(N^{s-\beta-1}) = O(N^{n_1+n_2-1}). \tag{5.55}$$

Hence, collecting the estimates (5.52), (5.53), (5.54) and (5.55), we obtain the claim (5.46).

The same proof shows that the estimates (5.52) and (5.53) are valid with $n_1 = 0$; this is also true for (5.54), which can be proved using Theorem 3.6(iii) with $\beta = s - 1$. Finally, applying Theorem 3.6(ii) with $\beta = s - n_2 - 1 < s - 1$, the estimate (5.55) is sharpened to

$$\frac{1}{N^2H_N}[z^N]\bigl[N^2g_\kappa^{\{1\}}(z)\,g_\kappa^{\{n_2+1\}}(z)\,\mathrm{e}^{G_N(z)}\bigr] \sim (N\tilde{\nu})^{n_2}\,\tilde{\nu},$$

giving the leading term in the asymptotics of (5.51) (with $n_1 = 0$), which proves (5.47). □

Remark 5.9. Lemmas 5.10 and 5.11 can be extended to the m-dimensional case, that is, with $X_1,\dots,X_m$ and $L_1,\dots,L_m$, respectively (cf. Remark 5.6).

The next theorem is the corresponding version of Theorem 5.6.

Theorem 5.12. *For each $m \in \mathbb{N}$,*

$$\frac{1}{N\tilde{\nu}}\,(L_1,\dots,L_m) \xrightarrow{\ d\ } (X_1,\dots,X_m), \qquad N\to\infty,$$

where X_j's are Bernoulli random variables defined in (5.44).

Proof. The proof is precisely the same as that of Theorem 5.6, now using Lemmas 5.10 and 5.11 in place of Lemmas 5.5 and 5.7, respectively. ☐

We finally obtain our main result about convergence of ordered cycles in the case $\theta^* = 0$, which is in sharp contrast with Theorem 5.9.

Theorem 5.13. *In the sense of convergence of finite-dimensional distributions,*

$$\frac{1}{N\tilde{\nu}}\left(L^{(1)}, L^{(2)}, \dots\right) \xrightarrow{\ d\ } (1,0,0,\dots), \qquad N\to\infty. \tag{5.56}$$

Proof. From the definition (5.44) it is clear that rearranging the sequence (X_n) in descending order gives $(1,0,0,\dots)$ (cf. the right-hand side of (5.56)). The rest of the proof proceeds exactly as in Theorem 5.9, again using the general cut-off Lemma 5.8. ☐

Corollary 5.14. *Weak convergence of the first component in* (5.56) *entails the following law of large numbers for the longest cycle in the case $\theta^* = 0$,*

$$\frac{L^{(1)}}{N\tilde{\nu}} \xrightarrow{\ p\ } 1, \qquad N\to\infty.$$

Remark 5.10. The result of Theorem 5.13 means that there is a single giant cycle (of size about $N\tilde{\nu}$) emerging as $N\to\infty$.

Remark 5.11. The condition of regularity of $g_\theta(z)$ at $z = R$ imposed at the beginning of Section 5.6 is not essential; in the spirit of Remark 3.4, Lemmas 5.10 and 5.11 (underlying the proof of Theorem 5.12) may be extended to the case where $g_\theta(z)$ has a power and possibly also a power-logarithmic singularity, $\tilde{b}_{s_1}(1-z/R)^{s_1}\log(1-z/R)$, as long as $s_1 > 0$. Furthermore, again alluding to Remark 3.4 it is not hard to see that all calculations can be adapted for the asymptotic expansion (3.47) of $g_\kappa(z)$ to include a power-logarithmic term $b_s(1-z/R)^s\log(1-z/R)$, in which case the index $s > 1$ is permitted to be integer.

6. Comparison with the spatial model

The aim of this section is to bridge the gap between the surrogate-spatial ($\tilde{\mathbb{P}}_N$) and spatial ($\tilde{\mathbb{P}}_N$) models. First, in Section 6.1.1 we will explore in some detail the Euler–Maclaurin type approximation (1.17) of the Riemann sums arising in the definition of the measure $\mathbb{P}_{N,L}$ (see (1.7)). In Section 6.1.2 we will show how the concept of the system "density" may be interpreted in our model. Finally, in Section 6.1.3 we will compare the asymptotic results for the cycle statistics obtained in the present paper and by Betz and Ueltschi [6].

6.1. Asymptotics of the Riemann sums.

To justify the ansatz (1.17), which has motivated the surrogate-spatial model (1.5), we need to look more carefully at the Riemann integral sums in (1.7). Rather than using Euler–Maclaurin's summation formula as suggested in Section 1.3, we take advantage of the link (1.11) between the functions $\mathrm{e}^{-\varepsilon(\boldsymbol{s})}$ and $\mathrm{e}^{-V(\boldsymbol{x})}$ and deploy the powerful *Poisson summation formula* (see, e.g., [10, §3.12, p. 52] or [13, §XIX.5, p. 630]) yielding the identity

$$\sum_{k\in\mathbb{Z}} \varphi(k/\lambda) = \lambda \sum_{\ell\in\mathbb{Z}} f(\lambda\ell) \qquad (\lambda > 0), \tag{6.1}$$

where the functions $\varphi(s)$ and $f(x)$ are the reciprocal Fourier transforms,

$$\varphi(s) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} xs}\, f(x)\, \mathrm{d}x, \qquad s \in \mathbb{R},$$
$$f(x) = \int_{\mathbb{R}} \mathrm{e}^{2\pi \mathrm{i} xs}\, \varphi(s)\, \mathrm{d}s, \qquad x \in \mathbb{R},$$

such that $f(x)$ is a probability density and, moreover, $\varphi(s)$ is absolutely integrable on $\mathbb{R}$.

For simplicity, in higher dimensions we restrict ourselves to *isotropic* (radially symmetric) potentials,

$$V(\boldsymbol{x}) = \sum_{i=1}^{d} V_0(x_i), \qquad \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

leading to the decomposition

$$\varepsilon(\boldsymbol{s}) = \sum_{i=1}^{d} \varepsilon_0(s_i), \qquad \boldsymbol{s} = (s_1, \dots, s_d) \in \mathbb{R}^d, \tag{6.2}$$

with the function $\varepsilon_0(\cdot)$ defined by

$$\mathrm{e}^{-\varepsilon_0(s)} = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} xs} f_0(x)\, \mathrm{d}x, \qquad f_0(x) := \mathrm{e}^{-V_0(x)}. \tag{6.3}$$

Hence, the d-dimensional sum in (1.7) (with $j = 1, \dots, N$) is decomposable as a product,

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)} = \prod_{i=1}^{d} \sum_{k_i\in\mathbb{Z}} \mathrm{e}^{-j\varepsilon_0(k_i/L)} = \Biggl(\sum_{k\in\mathbb{Z}} \mathrm{e}^{-j\varepsilon_0(k/L)}\Biggr)^{d}, \tag{6.4}$$

and likewise

$$\int_{\mathbb{R}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\, \mathrm{d}\boldsymbol{s} = \prod_{i=1}^{d} \int_{\mathbb{R}} \mathrm{e}^{-j\varepsilon_0(s_i)}\, \mathrm{d}s_i = \biggl(\int_{\mathbb{R}} \mathrm{e}^{-j\varepsilon_0(s)}\, \mathrm{d}s\biggr)^{d}. \tag{6.5}$$

The relation (6.3) implies that the function $\exp\{-j\varepsilon_0(s)\}$ is the Fourier transform of the j-fold convolution $f_0^{\star j}(x) = f_0\star\cdots\star f_0(x)$ $(j \in \mathbb{N})$. Furthermore, from the Fourier inversion formula for $f_0^{\star j}(x)$ (with $x = 0$) we get

$$f_0^{\star j}(0) = \int_{\mathbb{R}} \mathrm{e}^{-j\varepsilon_0(s)}\, \mathrm{d}s. \tag{6.6}$$

Hence, the Poisson formula (6.1) (with $\lambda = L$) together with the product decompositions (6.4) and (6.5) yields the key representation

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)} = L^d \Biggl(\sum_{\ell\in\mathbb{Z}} f_0^{\star j}(\ell L)\Biggr)^{d} \tag{6.7}$$

$$= \rho^{-1} N \int_{\mathbb{R}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\, \mathrm{d}\boldsymbol{s} \cdot \left(1 + \frac{\sum_{\ell\neq 0} f_0^{\star j}(\ell L)}{f_0^{\star j}(0)}\right)^{d}, \tag{6.8}$$

on account of the thermodynamic calibration $L^d = \rho^{-1}N$ (see Section 1.2). On comparison with the conjectural approximation (1.17), we see that the expression (6.8) may be quite useful in providing information about the Riemann sums asymptotics.

Let us prove one general result in this direction. Recall that the probability density $f_0(x)$ is always assumed to be symmetric, $f_0(-x) = f_0(x)$ $(x \in \mathbb{R})$. A symmetric function is called *unimodal* if it is non-increasing for $x \geq 0$.

Lemma 6.1. *Assume that $f_0(x)$ is unimodal, then for each $j \in \mathbb{N}$ and any $L > 0$*

$$\sum_{\ell \neq 0} f_0^{\star j}(\ell L) \leq \frac{4}{L} \int_{L/2}^{\infty} f_0^{\star j}(x)\, \mathrm{d}x. \tag{6.9}$$

Proof. Observe that the convolutions $f_0^{\star j}(x)$ are also symmetric and unimodal (see [17, Theorem 2.5.2, p. 67]). Hence, for any $\ell \geq 1$ and $y > 0$

$$f_0^{\star j}(\ell y) \leq \int_{\ell-1}^{\ell} f_0^{\star j}(uy)\, \mathrm{d}u. \tag{6.10}$$

On the other hand, inserting extra "mid-terms" into the sum in (6.9) and then using the bound (6.10) with $y = L/2$, we obtain

$$\begin{aligned}\sum_{\ell \neq 0} f_0^{\star j}(\ell L) = 2 \sum_{\ell=1}^{\infty} f_0^{\star j}(\ell L) &\leq 2 \sum_{\ell=2}^{\infty} f_0^{\star j}(\ell L/2) \\ &\leq 2 \int_1^{\infty} f_0^{\star j}(uL/2)\, \mathrm{d}u = \frac{4}{L} \int_{L/2}^{\infty} f_0^{\star j}(x)\, \mathrm{d}x,\end{aligned}$$

and the estimate (6.9) is proved. □

Remark 6.1. The bound (6.9), together with the Poisson formula (6.8), ensures that the Riemann sum on the left-hand side of (6.8) is finite for all $L > 0$, and so the series convergence condition (1.13) is automatically satisfied.

By Lemma 6.1, from (6.6) and (6.8) we get for each $j \in \mathbb{N}$ the asymptotic equivalence

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)} \sim \rho^{-1} N \int_{\mathbb{R}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\, \mathrm{d}\boldsymbol{s} \qquad (N \to \infty),$$

with the absolute error

$$\begin{aligned}\Delta_N^{(j)} &:= \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)} - \rho^{-1} N \int_{\mathbb{R}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\, \mathrm{d}\boldsymbol{s} \\ &\sim \rho^{-1} N d \left\{ f_0^{\star j}(0) \right\}^{d-1} \sum_{\ell \neq 0} f_0^{\star j}(\ell L) = o(N^{1-1/d}), \qquad N \to \infty.\end{aligned} \tag{6.11}$$

Obtaining more accurate asymptotics of $\Delta_N^{(j)}$ (in particular, investigating if it converges to a *constant* as suggested by the term θ_j in (1.17)), as well as treating the case of j growing with $N \to \infty$, requires more information about the tail of the density $f_0(x)$ and its convolutions $f_0^{\star j}(x)$ as $x \to \infty$, which will also translate into the bahaviour of the function $\varepsilon_0(s)$ for $s \approx 0$ needed for the asymptotics of the integral in (6.8), according to the Laplace method.

Remark 6.2. Note that the integral in (6.9) can be written as the probability $\mathbb{P}\{S_j \geq L/2\}$, where $S_j := X_1 + \cdots + X_j$ and (X_i) are i.i.d. random variables each with density $f_0(x)$; hence, one can use suitable results from the large deviations theory (see, e.g., [26]) to get further bounds on (6.9). We will use this idea below to treat the example in Section 6.1.3.

Rather then attempting to develop any further general results, we will illustrate some typical asymptotic effects by considering a few "exactly solvable" examples classified according to the type of the probability density $f_0(x) = \mathrm{e}^{-V_0(x)}$: (i) $\varepsilon_0(s) = s^2$ (*Gaussian*); (ii) $\varepsilon_0(s) = |s|^\gamma$, $0<\gamma<2$ (*stable*); (iii) $f_0(x) = \mu_0\,\mathrm{e}^{-|x|^\gamma}$, $0<\gamma<2$ (*exponential-power*). In what follows, we write $a_N \asymp b_N$ if $0 < \liminf_{N\to\infty} b_N/a_N \le \limsup_{N\to\infty} b_N/a_N < \infty$.

6.1.1. *Gaussian case.* Here we have $\varepsilon_0(s) = s^2$, $f_0(x) = \sqrt{\pi}\,\mathrm{e}^{-\pi^2x^2}$, and the convolutions $f_0^{\star j}$ ($j\in\mathbb{N}$) are easily found,

$$f_0^{\star j}(x) = \sqrt{\frac{\pi}{j}}\,\mathrm{e}^{-\pi^2x^2j^{-1}}, \qquad x\in\mathbb{R}. \tag{6.12}$$

Hence, the representation (6.8) specializes to

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-j\,\|\boldsymbol{k}\|^2/L^2} = \rho^{-1}N\int_{\mathbb{R}^d}\mathrm{e}^{-j\,\|\boldsymbol{s}\|^2}\,\mathrm{d}\boldsymbol{s}\cdot\Biggl(1+\sum_{\ell\ne0}\mathrm{e}^{-\pi^2\ell^2L^2/j}\Biggr)^{d}, \tag{6.13}$$

where $N=\rho L^d$ and (see (6.6) and (6.12))

$$\int_{\mathbb{R}^d}\mathrm{e}^{-j\,\|\boldsymbol{s}\|^2}\,\mathrm{d}\boldsymbol{s} = \biggl(\int_{\mathbb{R}}\mathrm{e}^{-js^2}\,\mathrm{d}s\biggr)^{d} = \bigl\{f_0^{\star j}(0)\bigr\}^d = \biggl(\frac{\pi}{j}\biggr)^{d/2}. \tag{6.14}$$

The analysis of the expression (6.13) is straightforward. Recall that the index j ranges from 1 to N (see (1.7)). As long as $jL^{-2}=o(1)$ (which is always true in dimension $d=1$), from (6.11) and (6.12) we obtain

$$\Delta_N^{(j)} = \biggl(\frac{L^2}{j}\biggr)^{d/2} O\bigl(\mathrm{e}^{-\mathrm{const}\cdot L^2/j}\bigr) = o(1), \qquad N\to\infty,$$

which means that the approximation (1.17), (1.18) is valid in this range of j's, with $\theta_j\equiv0$. However, if $j\asymp L^2$ (when $d\ge2$) then the sum on the left-hand side of (6.13) is of order of a constant; more specifically, if $jL^{-2}\to c>0$ then

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\mathrm{e}^{-j\,\|\boldsymbol{k}\|^2/L^2} = \Biggl(\sum_{k\in\mathbb{Z}}\mathrm{e}^{-(j/L^2)\,k^2}\Biggr)^{d} \to \Biggl(\sum_{k\in\mathbb{Z}}\mathrm{e}^{-ck^2}\Biggr)^{d}, \qquad L\to\infty. \tag{6.15}$$

Note that the integral part in (1.17), (1.18) contributes to the limit (6.15) the amount

$$L^d\biggl(\int_{\mathbb{R}}\mathrm{e}^{-js^2}\,\mathrm{d}s\biggr)^{d} = L^d\biggl(\frac{\pi}{j}\biggr)^{d/2}\to\biggl(\frac{\pi}{c}\biggr)^{d/2}, \tag{6.16}$$

whilst the rest of it must come from additional *positive* constants (see (6.13)), leading accordingly to the coefficients $\theta_j>0$ in (1.17).

Similarly, if $jL^{-2}\to+\infty$ (which is possible in dimensions $d\ge3$) then

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\mathrm{e}^{-j\,\|\boldsymbol{k}\|^2/L^2} = \Biggl(1+2\sum_{k=1}^{\infty}\mathrm{e}^{-(j/L^2)\,k^2}\Biggr)^{d}\to1, \qquad L\to\infty. \tag{6.17}$$

Note that here the integral contribution is asymptotically vanishing (cf. (6.16)), so in this range of j's we must have $\theta_j\sim\mathrm{e}^{-\alpha_j}$, according to (1.17) and (6.17).

6.1.2. *Stable case.* Here $\varepsilon_0(s)=|s|^\gamma$ with $0<\gamma<2$, which implies that the (symmetric) density $f_0(x)$ is *stable* (see, e.g., [17, Theorem 2.2.2, p. 43]) and therefore unimodal [17, Theorem 2.5.3, p. 67]. The particular case $\gamma=1$ corresponds to the Cauchy distribution, $f_0(x)=\pi^{-1}(1+x^2)^{-1}$. The convolutions $f_0^{\star j}(x)$ are easily found by rescaling,

$$f_0^{\star j}(x)=j^{-1/\gamma} f_0\bigl(j^{-1/\gamma}x\bigr),\qquad x\in\mathbb{R}.$$

In particular, note that (cf. (6.6))

$$f_0^{\star j}(0)=j^{-1/\gamma} f_0(0)=j^{-1/\gamma}\int_{\mathbb{R}} \mathrm{e}^{-|s|^\gamma}\,\mathrm{d}s=\frac{2\,\Gamma(1+1/\gamma)}{j^{1/\gamma}}\,,\qquad j\in\mathbb{N}. \tag{6.18}$$

Hence, the representation (6.8) takes the form

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)}=\rho^{-1}N\int_{\mathbb{R}^d}\mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\,\mathrm{d}\boldsymbol{s}\cdot\left(1+\frac{1}{f_0(0)}\sum_{\ell\ne0} f_0(j^{-1/\gamma}\ell L)\right)^{d},$$

where (see (6.6) and (6.18))

$$\int_{\mathbb{R}^d}\mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\,\mathrm{d}\boldsymbol{s}=\left(\int_{\mathbb{R}}\mathrm{e}^{-j|s|^\gamma}\,\mathrm{d}s\right)^{d}=\bigl\{f_0^{\star j}(0)\bigr\}^d=\bigl\{2\,\Gamma(1+1/\gamma)\bigr\}^d j^{-d/\gamma}. \tag{6.19}$$

Furthermore, the tail asymptotics of the stable density $f_0(x)$ are given by (see, e.g., [17, Theorem 2.4.1, p. 54, for $0<\gamma<1$ and Theorem 2.4.2, p. 55, for $1<\gamma<2$])

$$f_0(x)\sim\frac{1}{\pi|x|^{1+\gamma}}\,\Gamma(1+\gamma)\,\sin\frac{\pi\gamma}{2}\,,\qquad x\to\infty.$$

(The case $\gamma=1$ is automatic in view of the explicit form of the Cauchy density $f_0(x)$, as mentioned above.) Therefore, in the range $j=o(L^\gamma)$ the error term (6.11) is estimated as

$$\Delta_N^{(j)}\asymp N j^{-d/\gamma}\sum_{\ell=1}^{\infty}\frac{j^{1+1/\gamma}}{\ell^{1+\gamma}L^{1+\gamma}}\asymp\left(\frac{L}{j^{1/\gamma}}\right)^{d-1-\gamma},\qquad N\to\infty. \tag{6.20}$$

Since $Lj^{-1/\gamma}\to\infty$, the right-hand side of (6.20) tends to zero only if $d<1+\gamma$, which is always true for $d=1$ but false for $d\ge3$ (where in fact $\Delta_N^{(j)}\to\infty$); for $d=2$ we have $\Delta_N^{(j)}=o(1)$ if $\gamma>1$, while $\Delta_N^{(j)}\asymp1$ if $\gamma=1$ and $\Delta_N^{(j)}\to\infty$ if $0<\gamma<1$. More precisely, if $j\asymp L^{\gamma-\epsilon}$ ($0<\epsilon\le\gamma$) then $\Delta_N^{(j)}\asymp L^{\epsilon(d-1-\gamma)/\gamma}$, which identifies the scale of $\Delta_N^{(j)}$ in the "moderate" range from $j=1$ (with $\Delta_N^{(1)}\asymp L^{d-1-\gamma}$ up to $j\asymp L^\gamma$, when (6.20) is formally reduced to $\Delta_N^{(j)}\asymp1$.

On the other hand, if $jL^{-\gamma}\to\infty$ then, using the product formula (6.4) and the expression $\varepsilon_0(s)=|s|^\gamma$, we see directly that

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)}=\left(1+2\sum_{k=1}^{\infty}\mathrm{e}^{-(j/L^\gamma)\,k^\gamma}\right)^{d}\to1,\qquad L\to\infty. \tag{6.21}$$

6.1.3. *Exponential-power case.* In this example, the density is specified as $f_0(x)=\mu_0\,\mathrm{e}^{-|x|^\gamma}$ ($x\in\mathbb{R}$), with $0<\gamma<2$ and the normalization constant (cf. (6.18))

$$\mu_0\equiv\mu_0(\gamma)=\left(2\int_0^\infty\mathrm{e}^{-x^\gamma}\,\mathrm{d}x\right)^{-1}=\frac{1}{2\,\Gamma(1+1/\gamma)}\,. \tag{6.22}$$

By the duality, from Section 6.1.2 we see that the Fourier transform of $f_0(x)$ is given by $\varphi_0(s)=\tilde f(s)/\tilde f(0)\ge0$, where $\tilde f(\cdot)$ is the (stable) density with Fourier transform $\mathrm{e}^{-|s|^\gamma}$.

Remark 6.3. The case $\gamma = 2$, which corresponds to a Gaussian density (see Section 6.1.1), is easily included in the analysis below.

To estimate the error term (6.11), let us use Lemma 6.1 together with Remark 6.2, giving

$$\Delta_N^{(j)} \le L^d \left(\int_{\mathbb{R}} \mathrm{e}^{-j\varepsilon_0(s)}\,\mathrm{d}s \right)^d \left\{ \left(1 + \frac{\mathbb{P}\{S_j \ge L/2\}}{(L/4)\, f_0^{\star j}(0)} \right)^d - 1 \right\}. \tag{6.23}$$

We have to distinguish two cases, (i) $1 \le \gamma \le 2$ and (ii) $0 < \gamma < 1$.

(i) Consider the absolute moments of order $r \ge 0$ (cf. (6.22))

$$\mu_r := \int_{\mathbb{R}} |x|^r f_0(x)\,\mathrm{d}x = 2\mu_0 \int_0^\infty x^r \mathrm{e}^{-x^\gamma}\,\mathrm{d}x = \frac{\Gamma(1 + (r+1)/\gamma)}{(r+1)\,\Gamma(1+1/\gamma)}\,. \tag{6.24}$$

We will need a simple lemma about the gamma function.

Lemma 6.2. *For any $\gamma \in [1, 2]$ and all integers $r \ge 2$, the following inequality holds*

$$\Gamma\!\left(1 + \frac{r+1}{\gamma}\right) \le \frac{(r+1)!}{6}\, \Gamma\!\left(1 + \frac{3}{\gamma}\right). \tag{6.25}$$

Proof. We argue by induction in r. For $r = 2$ the claim (6.25) is obvious (with the equality sign). Now, assume that (6.25) holds for some $r \ge 2$. Note that the gamma function $\Gamma(t)$ is convex on $(0, \infty)$, because

$$\Gamma''(t) = \int_0^\infty (\log x)^2\, x^{t-1}\, \mathrm{e}^{-x}\,\mathrm{d}x > 0, \qquad t > 0.$$

Since $\Gamma(1) = \Gamma(2) = 1$, the convexity implies that $\Gamma(t)$ is monotone increasing on $[2, \infty)$. On the other hand, it is easy to check that if $\gamma \in [1, 2]$ and $r \ge 2$ then

$$2 \le \frac{r+2}{\gamma} \le 1 + \frac{r+1}{\gamma}. \tag{6.26}$$

Hence, by the monotonicity (using (6.26)) and the induction hypothesis we obtain

$$\begin{aligned} \Gamma\!\left(1 + \frac{r+2}{\gamma}\right) = \frac{r+2}{\gamma}\,\Gamma\!\left(\frac{r+2}{\gamma}\right) &\le (r+2)\,\Gamma\!\left(1 + \frac{r+1}{\gamma}\right) \\ &\le \frac{(r+2)!}{6}\,\Gamma\!\left(1 + \frac{3}{\gamma}\right), \end{aligned}$$

which proves (6.25) for $r + 1$. Thus, the lemma is valid for all integer $r \ge 2$. □

In view of the expressions (6.24), the inequality (6.25) yields the following estimate on the growth of successive moments in the case $1 \le \gamma \le 2$,

$$\mu_r \le \tfrac{1}{2}\,\mu_2\, r!\,, \qquad r \ge 2.$$

Thus, one can apply Bernstein's inequality (in its enhanced modern form, see [2, Eq. (7), p. 38]), which gives for all $j \in \mathbb{N}$

$$\mathbb{P}\{S_j \ge L/2\} \le \exp\!\left(-\frac{L^2/4}{2j\,\mu_2 + L} \right), \qquad L > 0. \tag{6.27}$$

Noting that $2j\,\mu_2 + L \le (2\mu_2 + 1)\max\{j, L\}$, from (6.27) we easily deduce a more convenient estimate

$$\mathbb{P}\{S_j \ge L/2\} \le \exp\!\left(-\frac{L^2/j}{4(2\mu_2 + 1)} \right) + \exp\!\left(-\frac{L}{4(2\mu_2+1)} \right). \tag{6.28}$$

(ii) If $0<\gamma<1$, we utilize a different suitable bound from the large deviations theory (see [26, Eq. (2.32), p. 764]) yielding

$$\mathbb{P}\{S_j\ge L/2\}\le c\left(\exp\biggl(-\frac{L^2}{80j}\biggr)+j\,\mathbb{P}\{X_1\ge L/4\}\right),\qquad L>0,\tag{6.29}$$

where $c>0$ is a constant depending only on γ. Integrating by parts it is easy to find

$$\mathbb{P}\{X_1\ge y\}=\mu_0\int_y^\infty \mathrm{e}^{-u^\gamma}\,\mathrm{d}u\sim\frac{\mu_0}{\gamma}\,y^{1-\gamma}\mathrm{e}^{-y^\gamma},\qquad y\to+\infty.$$

Hence, for $L\to\infty$ the bound (6.29) becomes

$$\mathbb{P}\{S_j\ge L/2\}=O(1)\exp\biggl(-\frac{L^2}{80j}\biggr)+O(jL^{1-\gamma})\exp\biggl(-\frac{L^\gamma}{4^\gamma}\biggr).\tag{6.30}$$

Returning to (6.23), from the estimates (6.28) and (6.30) we get, for any fixed $j\in\mathbb{N}$,

$$\begin{aligned}\Delta_N^{(j)}&=O(L^{d-1})\exp(-c_1L),&&1\le\gamma\le2,\\ \Delta_N^{(j)}&=O(L^{d-\gamma})\exp(-c_2L^\gamma),&&0<\gamma<1,\end{aligned}$$

with some constants c_1, $c_2>0$. In particular, $\Delta_N^{(j)}$ is (exponentially) small as $L\to\infty$.

For $j\to\infty$, the asymptotics of $f_0^{\star j}(0)$ represented as the integral (6.6) can be found using the Laplace method [10, Ch. 4]. Specifically, $\varepsilon_0(0)=0$ is the unique minimum of $\varepsilon_0(s)=-\log\varphi_0(s)$; noting that $\varphi_0'(0)=0$, $\varphi_0''(0)=-4\pi^2\mu_2$ (see (6.24)), we have $\varepsilon_0''(0)=4\pi^2\mu_2$ and hence

$$f_0^{\star j}(0)=\int_{\mathbb{R}}\mathrm{e}^{-j\varepsilon_0(s)}\,\mathrm{d}s\sim\sqrt{\frac{2\pi}{j\,\varepsilon_0''(0)}}=\frac{1}{\sqrt{2\pi\mu_2 j}}\,,\qquad j\to\infty.\tag{6.31}$$

Consequently, the estimates (6.23), (6.28) and (6.30) give

$$\begin{aligned}\Delta_N^{(j)}&=O\bigl((Lj^{-1/2})^{d-1}\bigr)\Bigl\{\exp(-\tilde c_1L^2j^{-1})+\exp(-\tilde c_2L)\Bigr\},&&1\le\gamma\le2,\\ \Delta_N^{(j)}&=O\bigl((Lj^{-1/2})^{d-1}\bigr)\Bigl\{\exp(-\tilde c_3L^2j^{-1})+jL^{1-\gamma}\exp(-\tilde c_4L^\gamma)\Bigr\},&&0<\gamma<1,\end{aligned}\tag{6.32}$$

where $\tilde c_i>0$ are some constants. Again, it is easy to see from (6.32) that in all cases $\Delta_N^{(j)}=o(1)$ as $L\to\infty$, provided that $L^2/j\to\infty$.

Consider now the opposite case where the index j grows as L^2 or faster. If $j\asymp L^2$ then the Poisson summation formula (6.7), by virtue of Lemma 6.1 and formula (6.31), yields

$$\begin{aligned}1\le\sum_{\boldsymbol{k}\in\mathbb{Z}^d}^\infty \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)}&\le L^d\left(f_0^{\star j}(0)+\frac{4}{L}\int_0^\infty f_0^{\star j}(x)\,\mathrm{d}x\right)^d\\ &=\left(L\int_{\mathbb{R}}\mathrm{e}^{-j\varepsilon_0(s)}\,\mathrm{d}s+2\right)^d\\ &=\bigl\{O(Lj^{-1/2})+2\bigr\}^d=O(1),\end{aligned}\tag{6.33}$$

that is, $\Delta_N^{(j)}\asymp1$ as $L\to\infty$.

In the remaining case where $j/L^2\to\infty$, observe that the function $\varepsilon_0(s)$ is *strictly increasing* in the right neighbourhood of $s=0$, and moreover $\varepsilon_0(s)\to+\infty$ as $s\to\infty$. This implies that $0 < \varepsilon_0(1/L) \le \varepsilon_0(k/L)$ for all $k \ge 1$ (at least for L large enough). Hence,

$$\sum_{k=1}^{\infty} \mathrm{e}^{-j\varepsilon_0(k/L)} = \sum_{k=1}^{\infty} \mathrm{e}^{-(j-L^2)\,\varepsilon_0(k/L)}\,\mathrm{e}^{-L^2\varepsilon_0(k/L)}$$
$$\le \mathrm{e}^{-(j-L^2)\,\varepsilon_0(1/L)} \sum_{k=1}^{\infty} \mathrm{e}^{-L^2\varepsilon_0(k/L)} = \mathrm{e}^{-(j-L^2)\,\varepsilon_0(1/L)} \cdot O(1), \qquad (6.34)$$

where the $O(1)$-term appears according to the estimate (6.33) with $j = L^2$. Furthermore, using the expansion $\varepsilon_0(s) = 4\pi^2 s^2 + O(s^4)$ as $s \to 0$ and recalling that $j/L^2 \to +\infty$, we see that $(j - L^2)\,\varepsilon_0(1/L) \asymp jL^{-2} \to +\infty$. Hence, the right-hand side of (6.34) tends to zero and therefore we get (cf. (6.17) and (6.21))

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)} = \left(1 + 2\sum_{k=1}^{\infty} \mathrm{e}^{-j\varepsilon_0(k/L)}\right)^d \to 1, \qquad L \to \infty.$$

6.1.4. *Behaviour of the function $\varepsilon(\boldsymbol{s})$ at the origin.* Betz and Ueltschi [6, p. 1176] work under the condition that, with some $a > 0$, $\delta > 0$ and $0 < \eta < d$,

$$\varepsilon(\boldsymbol{s}) \ge a\,\|\boldsymbol{s}\|^{\eta}, \qquad \|\boldsymbol{s}\| \le \delta, \qquad (6.35)$$

which guarantees that the critical density (1.16) is finite, $\rho_{\mathrm{c}} < \infty$.

For the examples considered in Sections 6.1.1 and 6.1.3 we have $\varepsilon(\boldsymbol{s}) \sim \mathrm{const}\,\|\boldsymbol{s}\|^2$ as $\boldsymbol{s} \to \boldsymbol{0}$, and so the condition (6.35) is fulfilled (with $\eta = 2$) in dimensions $d \ge 3$.

For the "stable" example in Section 6.1.2, the function $\varepsilon(\boldsymbol{s}) = \sum_{i=1}^{d} |s_i|^{\gamma}$ $(0 < \gamma < 2)$ is comparable with $\|\boldsymbol{s}\|^{\gamma}$ due to the well-known fact that any two norms in $\mathbb{R}^d$ are equivalent; more explicitly, this follows from the elementary inequalities (see [16, Theorem 16, p. 26, and Theorem 19, p. 28])

$$\|\boldsymbol{s}\|^{\gamma} \le \sum_{i=1}^{d} |s_i|^{\gamma} \le d^{1-\gamma/2}\,\|\boldsymbol{s}\|^{\gamma}, \qquad \boldsymbol{s} = (s_1, \dots, s_d) \in \mathbb{R}^d.$$

Thus, here we have $\eta = \gamma < d$ unless $d = 1$, $\gamma \ge 1$.

Under the condition (6.35) it is easy to justify the universal behaviour of the Riemann sums in (1.7) for large indices j (even without the condition of radial symmetry, see (6.2)).

Lemma 6.3. *Assume that* (6.35) *is satisfied, and suppose that $jL^{-\eta} \to \infty$. Then*

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)} \to 1, \qquad L \to \infty. \qquad (6.36)$$

Proof. Note that $j \le N \asymp L^d$ and, since $\eta < d$, the range of j's considered in the lemma is non-empty. Recalling that $\varepsilon(\boldsymbol{0}) = 0$, for the proof of (6.36) it suffices to show that the sum over $\boldsymbol{k} \ne \boldsymbol{0}$ is asymptotically small. With $\delta > 0$ as in the condition (6.35) we have

$$0 \le \sum_{\boldsymbol{k}\ne\boldsymbol{0}} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)} \le \sum_{0<\|\boldsymbol{k}\|\le\delta L} \mathrm{e}^{-ja\|\boldsymbol{k}/L\|^{\eta}} + \sum_{\|\boldsymbol{k}\|>\delta L} \mathrm{e}^{-j\varepsilon(\boldsymbol{k}/L)}. \qquad (6.37)$$

Using (6.35) and the condition $j/L^{\eta} \to +\infty$, we obtain by dominated convergence

$$\sum_{0<\|\boldsymbol{k}\|\le\delta L} \mathrm{e}^{-ja\|\boldsymbol{k}/L\|^{\eta}} \le \left(\sum_{0<k\le\delta L} \mathrm{e}^{-a(j/L^{\eta})\,k^{\eta}}\right)^d \to 0, \qquad L \to \infty, \qquad (6.38)$$

since $j \ge aL^\eta$ (for L large enough) and $\sum_{k=1}^{\infty} \mathrm{e}^{-ak^\eta} < \infty$. To estimate the second sum in (6.37), we can assume that $\varepsilon(\boldsymbol{s}) \ge c_0 > 0$ for $\|\boldsymbol{s}\| > \delta$, hence, owing to the bound (1.12),

$$\begin{aligned}\sum_{\|\boldsymbol{k}\|>\delta L} \mathrm{e}^{-j\,\varepsilon(\boldsymbol{k}/L)} &\le \mathrm{e}^{-(j-1)\,c_0} \sum_{\|\boldsymbol{k}\|>\delta L} \mathrm{e}^{-\varepsilon(\boldsymbol{k}/L)}\\ &\le \mathrm{e}^{-c_0 L^\eta} L^d \sum_{\|\boldsymbol{k}\|>\delta L} \mathrm{e}^{-\varepsilon(\boldsymbol{k}/L)}\, L^{-d}\\ &\sim \mathrm{e}^{-c_0 L^\eta} L^d \int_{\|\boldsymbol{s}\|>\delta} \mathrm{e}^{-\varepsilon(\boldsymbol{s})}\,\mathrm{d}\boldsymbol{s}\\ &= O\bigl(\mathrm{e}^{-c_0 L^\eta} L^d\bigr) \to 0, \qquad L \to \infty.\end{aligned}$$

By the estimates (6.38) and (6.38) the right-hand side of (6.37) vanishes as $L \to \infty$, which completes the proof. □

6.1.5. *Some heuristic conclusions.* Empirical evidence provided by the examples in Sections 6.1.1–6.1.3 suggests that the approximation picture is qualitatively universal in the class of probability densities $f_0(x)$ with fast decaying tails (more precisely, Gaussian as in Section 6.1.1 or exponential-power as in Section 6.1.3). Namely, here the error $\Delta_N^{(j)}$ is asymptotically small if j is fixed or growing slower than $L^2 \asymp N^{2/d}$; in the transition zone $j \asymp L^2$, a bounded correction $\Delta_N^{(j)} \asymp 1$ emerges (comparable with the contribution of the "main" integral term $N\kappa_j$ as defined in (1.18), cf. (6.16) and (6.31)), whereas with a faster growth of j's (possible in dimensions $d \ge 3$) this is transformed into the flat asymptotics $\Delta_N^{(j)} = 1 + o(1)$ (in accordance with Lemma 6.3), but now with a polynomially small error arising entirely due to the integral contribution, $N\kappa_j \asymp (Lj^{-1/2})^d$ (see (6.31)).

For polynomially decaying distribution tails as exemplified in Section 6.1.2, the situation is more complex: here, the range $j = o(L^\gamma)$ produces an extended scale of the power asymptotics $\Delta_N^{(j)} \asymp (Lj^{-1/\gamma})^{d-1-\gamma}$, which is unbounded in sharp contrast with the ansatz (1.17) unless $d = 1$ or $d = 2$, $\gamma \ge 1$. In the transition zone $j \asymp L^\gamma$ this is naturally transformed into $\Delta_N^{(j)} \asymp 1$, and furthermore, if $Lj^{-1/\gamma} = o(1)$ then we have the universal (distribution-free) asymptotics $\Delta_N(j) = 1 + o(1)$, similarly to the exponential tails (and again in line with Lemma 6.3).

It should be clear from this discussion that the naïve use of any specific surrogate-spatial model $\tilde{\mathbb{P}}_N$ as a proxy to the spatial model $\mathbb{P}_{N,L}$ cannot be correct in the entire range of the cycle lengths $j = 1, \dots, N$. For instance, in the simplest Gaussian case, taking $\theta_j \equiv 0$ works well for moderate values of j (i.e., asymptotically smaller than $N^{2/d}$) but fails above this threshold; on the other hand, choosing $\theta_j \equiv \mathrm{e}^{-\alpha_j}$ is adequate for cycles of size j above $N^{2/d}$ but would incorrectly enhance the weighting of shorter cycles.

One might attempt to achieve a better approximation to $\mathbb{P}_{N,L}$ by choosing the coefficients θ_j in formula (1.17) so as to emulate the different asymptotics of the correction term $\Delta_N^{(j)}$ (in particular, allowing θ_j to depend on L). For instance, noting that the Gaussian case is essentially characterized in terms of the natural *order parameter* $\eta_{j,L} := Lj^{-1/2}$ (see Section 6.1.1), the following phenomenological formula may be suggested,

$$\mathrm{e}^{\alpha_j}\,\theta_{j,L} \propto \Theta_{j,L}^{d-1} \exp\bigl(-\Theta_{j,L}^2\bigr), \qquad j \in \mathbb{N}, \tag{6.39}$$

where

$$\Theta_{j,L} := \frac{1}{1 - \mathrm{e}^{-1/\eta_{j,L}}}, \qquad j \in \mathbb{N}. \tag{6.40}$$

Similarly, in the stable case (see Section 6.1.2) a plausible approximation is given by

$$\mathrm{e}^{\alpha_j}\,\theta_{j,L}\propto\Theta_{j,L}^{d-1-\gamma},\qquad j\in\mathbb{N},\tag{6.41}$$

where the parameter $\eta_{j,L}$ in (6.40) is now re-defined as $\eta_{j,L}:=Lj^{-1/\gamma}$ (see (6.20)).

The corresponding generating function $g_\theta(t)$ (see (2.2)) for the coefficients (6.39) or (6.41) may be too complicated to deal with, but if we opt to ignore the transitional details in the narrow zone $j\asymp L^2$ or $j\asymp L^\gamma$, respectively, then we get much simpler heuristic formulas

$$\mathrm{e}^{\alpha_j}\,\theta_{j,L}\propto\begin{cases}0, & j\le L^2,\\ 1, & j>L^2,\end{cases}\qquad\text{and}\qquad \mathrm{e}^{\alpha_j}\,\theta_{j,L}\propto\begin{cases}(Lj^{-1/\gamma})^{d-1-\gamma}, & j\le L^\gamma,\\ 1, & j\ge L^\gamma.\end{cases}$$

We intend to study such modifications of the surrogate-spatial model in another paper.

6.2. **Density dependence in the surrogate-spatial model.**

6.2.1. *Introducing an analogue of the particle density.* Although the surrogate-spatial model (1.5), (1.6) is defined with no reference to any underlying spatial structure, an analogue of the density ρ (cf. Section 1.2) can be incorporated in the system using the expression (1.18), which provides a heuristic link between the surrogate-spatial and spatial models. Namely, by analogy with formula (1.18), let us write the coefficients κ_j in the form

$$\kappa_j=\tilde\rho^{\,-1}\,\breve\kappa_j,\qquad j\in\mathbb{N},\tag{6.42}$$

where the parameter $\tilde\rho>0$ is interpreted as "density" and the constants

$$\breve\kappa_j=\mathrm{e}^{-\alpha_j}\int_{\mathbb{R}^d}\mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\,\mathrm{d}\boldsymbol{s},\qquad j\in\mathbb{N},\tag{6.43}$$

are treated as the baseline (density-free) coefficients that define a specific subclass of the models (1.5). For the corresponding generating function this gives

$$g_\kappa(z)=\tilde\rho^{\,-1}\sum_{j=1}^{\infty}\frac{\breve\kappa_j}{j}\,z^j=:\tilde\rho^{\,-1}\,\breve g_\kappa(z),\tag{6.44}$$

hence

$$g_\kappa^{\{1\}}(z)=\tilde\rho^{\,-1}\sum_{j=1}^{\infty}\breve\kappa_j\,z^j=\tilde\rho^{\,-1}\,\breve g_\kappa^{\{1\}}(z).\tag{6.45}$$

6.2.2. *Critical density.* At the singularity point $z=R$, formula (6.45) specializes to

$$g_\kappa^{\{1\}}(R)=\tilde\rho^{\,-1}\sum_{j=1}^{\infty}\breve\kappa_j\,R^j=\tilde\rho^{\,-1}\,\breve g_\kappa^{\{1\}}(R).\tag{6.46}$$

According to Definition 3.2 (see also Section 4), the critical case is determined by the condition $g_\kappa^{\{1\}}(R)=1$; therefore, (6.43) and (6.46) imply that the critical density is given by

$$\tilde\rho_{\mathrm{c}}=\sum_{j=1}^{\infty}\breve\kappa_j\,R^j=\sum_{j=1}^{\infty}R^j\,\mathrm{e}^{-\alpha_j}\int_{\mathbb{R}^d}\mathrm{e}^{-j\varepsilon(\boldsymbol{s})}\,\mathrm{d}\boldsymbol{s}.\tag{6.47}$$

This is consistent with the sub- and supercritical regimes as introduced in Definition 3.2:

$$g_\kappa^{\{1\}}(R)>1\quad\Leftrightarrow\quad\tilde\rho<\tilde\rho_{\mathrm{c}}\,.$$

Hence, we can express the expected fraction of points in infinite cycles (see (4.11)) as

$$\tilde{\nu} = \begin{cases} 0, & \tilde{\rho} \le \tilde{\rho}_{\rm c}, \\ 1 - \dfrac{\tilde{\rho}_{\rm c}}{\tilde{\rho}}, & \tilde{\rho} > \tilde{\rho}_{\rm c}, \end{cases}$$

which exactly reproduces the formula (1.15) for the spatial model.

Under natural assumptions on the coefficients α_j's, the expression (6.47) recovers the formula for the critical density $\rho_{\rm c}$ obtained in [6, Eq. (2.9), p. 1177].

Lemma 6.4. *Suppose that α_j's satisfy the bounds*

$$c_1 j^{-\delta} \le {\rm e}^{-\alpha_j} \le c_2, \qquad j \in \mathbb{N}, \tag{6.48}$$

with some positive constants δ, c_1 and c_2. Then $R = 1$ and formula (6.47) *is reduced to*

$$\tilde{\rho}_{\rm c} = \sum_{j=1}^{\infty} {\rm e}^{-\alpha_j} \int_{\mathbb{R}^d} {\rm e}^{-j\varepsilon(\boldsymbol{s})}\, {\rm d}\boldsymbol{s}.$$

Proof. Using the upper bound in (6.48), for any real $r \in (0, R)$ we have

$$\tilde{\rho}\, g_\kappa(r) \le c_2 \int_{\mathbb{R}^d} \sum_{j=1}^{\infty} \frac{{\rm e}^{-j\varepsilon(\boldsymbol{s})}}{j}\, r^j \,{\rm d}\boldsymbol{s} = -c_2 \int_{\mathbb{R}^d} \log\bigl(1 - r{\rm e}^{-\varepsilon(\boldsymbol{s})}\bigr)\, {\rm d}\boldsymbol{s}. \tag{6.49}$$

Note that the right-hand side of (6.49) is finite due to the bound (1.12),

$$\int_{\mathbb{R}^d} \sum_{j=1}^{\infty} \frac{{\rm e}^{-j\varepsilon(\boldsymbol{s})}}{j}\, r^j \,{\rm d}\boldsymbol{s} \le \int_{\mathbb{R}^d} {\rm e}^{-\varepsilon(\boldsymbol{s})}\,{\rm d}\boldsymbol{s} \cdot \sum_{j=1}^{\infty} \frac{r^j}{j} < \infty.$$

Since $\varepsilon(\mathbf{0}) = 0$, the right-hand side of (6.49) has singularity as $r \uparrow 1$. Therefore, by Pringsheim's Theorem (see Lemma 2.1) the estimate (6.49) implies that $R \ge 1$.

Similarly, by the lower bound in (6.48) we get

$$\tilde{\rho}\, g_\kappa(r) \ge c_1 \int_{\mathbb{R}^d} \sum_{j=1}^{\infty} \frac{{\rm e}^{-j\varepsilon(\boldsymbol{s})}}{j^{1+\delta}}\, r^j \,{\rm d}\boldsymbol{s} = c_1 \int_{\mathbb{R}^d} {\rm Li}_{1+\delta}\bigl(r{\rm e}^{-\varepsilon(\boldsymbol{s})}\bigr)\, {\rm d}\boldsymbol{s}. \tag{6.50}$$

By the known asymptotics of polylogarithm (see Lemma 3.14) the right-hand side of (6.50) has singularity at $r = 1$ and it follows that $R \le 1$. Thus $R = 1$ and the lemma is proven. $\square$

Remark 6.4. Assumption (6.48) covers the cases considered in [6] (see Section 6.3 below).

6.2.3. *Total number of cycles.* Ansatz (6.42) also enables us to investigate the $\tilde{\rho}$-dependence of the asymptotic statistics of cycles. For instance, it is easy to check that the total number of cycles, T_N, stochastically decreases with the growth of the density $\tilde{\rho}$, as one would expect. Indeed, according to Corollary 4.5 and formula (4.1), T_N/N converges to $g_\kappa(r_1)$ if $g_\kappa^{\{1\}}(R) \ge 1$ (i.e., $\tilde{\rho} \le \tilde{\rho}_{\rm c}$) or $g_\kappa(R)$ if $g_\kappa^{\{1\}}(R) \le 1$ (i.e., $\tilde{\rho} \ge \tilde{\rho}_{\rm c}$), and to verify the claim it suffices to show that the $\tilde{\rho}$-derivative of the limit is negative.

Differentiating with respect to $\tilde{\rho}$ and using the representation (6.44), we readily obtain

$$\frac{\partial g_\kappa(R)}{\partial \tilde{\rho}} = -\tilde{\rho}^{-2} \breve{g}_\kappa(R) = -\tilde{\rho}^{-1} g_\kappa(R) < 0.$$

Similarly,

$$\frac{\partial g_\kappa(r_1)}{\partial \tilde{\rho}} = -\tilde{\rho}^{-1} g_\kappa(r_1) + g_\kappa'(r_1)\, \frac{\partial r_1}{\partial \tilde{\rho}}\,. \tag{6.51}$$

On the other hand, differentiation of the equation $g_\kappa^{\{1\}}(r_1)=1$, rewritten for convenience as $r_1\breve g_\kappa'(r_1)=\tilde\rho$, gives

$$\frac{\partial r_1}{\partial\tilde\rho}\,\tilde\rho\, g_\kappa'(r_1)+r_1\tilde\rho\, g_\kappa''(r_1)\,\frac{\partial r_1}{\partial\tilde\rho}=1,$$

whence we find

$$\frac{\partial r_1}{\partial\tilde\rho}=\frac{\tilde\rho^{-1}}{g_\kappa'(r_1)+r_1 g_\kappa''(r_1)}=\frac{\tilde\rho^{-1}r_1}{1+g_\kappa^{\{2\}}(r_1)}>0. \tag{6.52}$$

Hence, returning to (6.51) and again using the identity $r_1 g_\kappa'(r_1)=g_\kappa^{\{1\}}(r_1)\equiv 1$, we get

$$\frac{\partial g_\kappa(r_1)}{\partial\tilde\rho}=-\tilde\rho^{-1}\left(g_\kappa(r_1)-\frac{1}{1+g_\kappa^{\{2\}}(r_1)}\right)<0,$$

where the inequality follows from Lemma 3.1.

6.2.4. *Cycle counts.* Let us now investigate the asymptotic trend of the individual cycle counts C_j (for each $j\in\mathbb{N}$) with the growth of the density $\tilde\rho$. Assuming that all $\kappa_j>0$, by Theorem 4.1 and formula (4.1) we know that C_j/N converges to $\kappa_j r_1^j/j$ (for $\tilde\rho\le\tilde\rho_{\rm c}$) or $\kappa_j R^j/j$ (for $\tilde\rho\ge\tilde\rho_{\rm c}$).

First, consider the supercritical domain, $\tilde\rho>\tilde\rho_{\rm c}$. Using the representation (6.42) we obtain

$$\frac{\partial(\kappa_j R^j)}{\partial\tilde\rho}=-\tilde\rho^{-2}\breve\kappa_j R^j=-\tilde\rho^{-1}\kappa_j R^j<0,\qquad j\in\mathbb{N},$$

which means that the asymptotic proportion of cycles of any finite length has the tendency to decrease with the growth of $\tilde\rho$ (whilst the infinite cycle stays infinite).

In the subcritical domain ($\tilde\rho>\tilde\rho_{\rm c}$), again invoking (6.42) and also using formula (6.52) for the derivative $\partial r_1/\partial\tilde\rho$, we get

$$\frac{\partial(\kappa_j r_1^j)}{\partial\tilde\rho}=-\tilde\rho^{-2}\breve\kappa_j r_1^j+\kappa_j j\, r_1^{j-1}\frac{\partial r_1}{\partial\tilde\rho}=-\tilde\rho^{-1}\kappa_j r_1^j\left(1-\frac{j}{1+g_\kappa^{\{2\}}(r_1)}\right).$$

Thus, with the growth of the density $\tilde\rho$, as long as $\tilde\rho<\tilde\rho_{\rm c}$, the limiting proportions of short cycles (with lengths $j<1+g_\kappa^{\{2\}}(r_1)$) decrease whereas those of longer cycles (with lengths $j>1+g_\kappa^{\{2\}}(r_1)$) increase.

Note, however, that the threshold $1+g_\kappa^{\{2\}}(r_1)$ varies itself, and it is natural to expect that it is *increasing* with $\tilde\rho$, which is corroborated heuristically by the limiting case $\tilde\rho\uparrow\tilde\rho_{\rm c}$, with $r_1\uparrow R$ and $g_\kappa^{\{2\}}(r_1)\uparrow g_\kappa^{\{2\}}(R)=\max_{0\le r\le R} g_\kappa^{\{2\}}(r)$. More precisely, observing that

$$\sum_{j=1}^\infty\breve\kappa_j r_1^j=\tilde\rho\, g_\kappa^{\{1\}}(r_1)=\tilde\rho$$

and

$$1+g_\kappa^{\{2\}}(r_1)=g_\kappa^{\{1\}}(r_1)+g_\kappa^{\{2\}}(r_1)=\tilde\rho^{-1}\sum_{j=1}^\infty j\,\breve\kappa_j r_1^j=\frac{\sum_{j=1}^\infty j\,\breve\kappa_j r_1^j}{\sum_{j=1}^\infty\breve\kappa_j r_1^j}, \tag{6.53}$$

we differentiate the right-hand side of (6.53) to obtain

$$\frac{\partial\left(1+g_\kappa^{\{2\}}(r_1)\right)}{\partial\tilde\rho}=\tilde\rho^{-2}r_1^{-1}\frac{\partial r_1}{\partial\tilde\rho}\cdot\left\{\sum_{j=1}^\infty j^2\breve\kappa_j r_1^j\sum_{j=1}^\infty\breve\kappa_j r_1^j-\left(\sum_{j=1}^\infty j\,\breve\kappa_j r_1^j\right)^2\right\}\ge 0,$$

according to (6.52) and the Cauchy–Schwarz inequality (cf. Lemma 3.1).

6.3. **Comparison of the asymptotic results for long cycles.**

6.3.1. *Choosing a suitable surrogate-spatial model.* As was stressed in Section 6.1.5, the surrogate-spatial model $\tilde{\mathbb{P}}_N$ defined by (1.5)–(1.6) cannot approximate correctly the spatial model $\mathbb{P}_{N,L}$ (1.7) in the entire range of the cycle lengths $j = 1, \dots, N$. However, if we focus on the asymptotics of *long cycles* only (i.e., with lengths $j \asymp N$), then the discussion in Section 6.1 suggests the following choice of the coefficients in the surrogate-spatial model,

$$\kappa_j = \mathrm{e}^{-\alpha_j} \kappa^* j^{-s}, \qquad \theta_j = \mathrm{e}^{-\alpha_j}, \qquad j \in \mathbb{N}, \tag{6.54}$$

with some index $s > 0$. Here we suppress the dependence on the density $\tilde{\rho}$ (cf. (6.42), (6.43)), which is not essential for the comparison. The expression (6.54) for κ_j's bears on the asymptotics of the integral (6.5) as $j \to \infty$, exemplified by the Gaussian and the exponential-power cases (both with $s = d/2$, see (6.14) and (6.31), respectively) and the stable case (with $s = d/\gamma$, see (6.19)). The expression for θ_j in (6.54) picks up on the universal behaviour of the correction term to the integral approximation of the Riemann sum in (1.7) (for j large enough, see Lemma 6.3).

Remark 6.5. It would be interesting to compare the measures $\mathbb{P}_{N,L}$ and $\tilde{\mathbb{P}}_N$ with regard to the asymptotic statistics of *short cycles* (say, with fixed lengths $j = 1, 2, \dots$ as in Theorem 4.1). According to the discussion in Section 6.1, a better match between the two models may be expected when the expression for θ_j in (6.54) is replaced by $\theta_j \equiv 0$. The asymptotics of the total number of cycles T_N is also of significant interest, especially in the critical case (cf. Theorem 4.4). However, such information is currently not available under the spatial measure $\mathbb{P}_{N,L}$ (see [6] and further references therein).

6.3.2. *Convergence to the Poisson–Dirichlet distribution.* For the coefficients α_j's entering the definition of the spatial measure $\mathbb{P}_{N,L}$ (see (1.7)), Betz and Ueltschi [6, p. 1176] have considered *inter alia* the following two classes,

$$\text{(i)} \quad \lim_{j\to\infty} \alpha_j = \alpha > 0, \quad \sum_j |\alpha_j - \alpha| < \infty; \tag{6.55}$$

$$\text{(ii)} \quad \lim_{j\to\infty} \alpha_j = \alpha \le 0, \quad \sum_j \frac{|\alpha_j - \alpha|}{j} < \infty. \tag{6.56}$$

With either of these assumptions, they prove for $\mathbb{P}_{N,L}$ [6, Theorem 2.1 (b), p. 1177] that in the supercritical regime (i.e., $\rho > \rho_{\mathrm{c}}$), the ordered cycle lengths $L^{(1)}, L^{(2)}, \dots$ (see Definition 5.1) converge to the Poisson–Dirichlet distribution with parameter $\mathrm{e}^{-\alpha}$,

$$\frac{1}{N\nu} \bigl((L^{(1)}, L^{(2)}, \dots\bigr) \xrightarrow{d} \mathrm{PD}(\mathrm{e}^{-\alpha}), \qquad N, L \to \infty. \tag{6.57}$$

This resonates well with our Theorem 5.9. Indeed, let the coefficients α_j have the form

$$\alpha_j = \alpha - \log(1 + \xi(j)), \qquad j \in \mathbb{N}, \tag{6.58}$$

where the function $\xi(z)$ satisfies the analyticity conditions of Section 3.5.3, together with the estimate $\xi(z) = O(z^{-\epsilon})$ ($\epsilon > 0$). The simplest example is $\xi(j) = j^{-\epsilon}$, leading to $\alpha - \alpha_j = \log(1 + j^{-\epsilon}) \sim j^{-\epsilon} \to 0$ as $j \to \infty$. Let us stress that, as opposed to (6.55)–(6.56), the sign of α in (6.58) is not important, and also that the difference $\alpha_j - \alpha$ satisfies the series convergence condition (6.56), but not necessarily (6.55) (which only holds for $\epsilon > 1$).

With (6.58), the coefficients (6.54) take the form

$$\kappa_j = \mathrm{e}^{-\alpha}\kappa^*\,\frac{1+\xi(j)}{j^s}\,, \qquad \theta_j = \mathrm{e}^{-\alpha}(1+\xi(j))\,, \qquad j\in\mathbb{N}. \tag{6.59}$$

Let $s>1$, which ensures the existence of the supercritical regime (see (3.116)), and suppose first that s is non-integer, $q<s<q+1$ ($q\in\mathbb{N}$); without loss of generality (by reducing $\epsilon>0$ if necessary) we can assume that $s+\epsilon<q+1$. Then, using Lemma 3.16 (more precisely, its part (a) for $g_\kappa(z)$ and part (b) with $q=0$ and $\kappa^*=\mathrm{e}^{-\alpha}$ for $g_\theta(z)$), it is not hard to see that the generating functions $g_\kappa(z)$, $g_\theta(z)$ satisfy all the conditions of Theorem 3.5, including the asymptotic formulas (3.19) and (3.20) with $\theta^*=\mathrm{e}^{-\alpha}>0$. Hence, Theorem 5.9 may be applied, thus replicating the convergence (6.57) for the surrogate-spatial measure $\widetilde{\mathbb{P}}_N$ (of course, with $\tilde{\nu}$ in place of ν).

The case of integer $s=q>1$ may be handled similarly, using the suitable versions of Theorems 3.5 and 5.9 as indicated in Remark 5.7.

6.3.3. *Emergence of a giant cycle.* The third and last specific class of the coefficients α_j's considered by Betz and Ueltschi [6, p. 1177] is given by

$$\alpha_j = \gamma_0 \log j, \qquad j\in\mathbb{N}, \tag{6.60}$$

where $\gamma_0>0$. Then it is proven [6, Theorem 2.2(b), p. 1178] that, under the supercritical spatial measure $\mathbb{P}_{N,L}$, there is asymptotically a single giant cycle,

$$\frac{1}{N\nu}\,L^{(1)} \xrightarrow{d} 1, \qquad N,L\to\infty. \tag{6.61}$$

This is directly analogous to the convergence (5.56) in the statement of Theorem 5.13. Indeed, substituting the expression (6.60) into (6.54) we get

$$\kappa_j = \frac{\kappa^*}{j^{s+\gamma_0}}\,, \qquad \theta_j = \frac{1}{j^{\gamma_0}}\,, \qquad j\in\mathbb{N}, \tag{6.62}$$

so that the corresponding generating functions are given by

$$g_\kappa(z) = \kappa^*\,\mathrm{Li}_{s+\gamma_0+1}(z), \qquad g_\theta(z) = \mathrm{Li}_{\gamma_0+1}(z). \tag{6.63}$$

Although Theorem 5.13 is not immediately applicable here (because the function $g_\theta(z)$ has singularity at $z=1$), the result (5.56) is valid for (6.63) in view of Remark 5.11.

Moreover, we can go further and generalize the narrow class (6.60) to

$$\alpha_j = \gamma_0\log j - \log\,(1+\xi(j)), \qquad j\in\mathbb{N},$$

thus replacing (6.62) by (cf. (6.59))

$$\kappa_j = \kappa^*\,\frac{1+\xi(j)}{j^{s+\gamma_0}}\,, \qquad \theta_j = \frac{1+\xi(j)}{j^{\gamma_0}}\,, \qquad j\in\mathbb{N}.$$

Then, using Lemmas 3.15 and 3.16, and deploying Remark 5.11, we see that the convergence (5.56) holds true.

6.4. **Summary of the comparison.** To wrap up the discussion in Section 6, we have demonstrated that, under natural conditions on the coefficients κ_j and θ_j, the surrogate-spatial model successfully reproduces the main features of the spatial model, including the formulas for the critical density and the limiting fraction of points in infinite clusters, as well as the asymptotic convergence of the descending cycle lengths either to the Poisson–Dirichlet distribution or to the degenerate distribution (with a single giant cycle), depending on the asymptotic behaviour of the modulating coefficients $\mathrm{e}^{-\alpha_j}$ (i.e., convergent *vs.* divergent α_j's); in our terms, this is translated into the distinction between the type of singularity of the generating function $g_\theta(z)$ (i.e., purely logarithmic or power-logarithmic, respectively). Overall, our analysis shows that the surrogate-spatial model, being of significant interest in its own right, proves to be a flexible and efficient approximation of the spatial model, providing at the same time a much greater analytical tractability thus making it a useful exploratory tool.

Acknowledgements. Partial support from SFB 701 and ZiF (Bielefeld) is gratefully acknowledged. L.V. B. was also partially supported by a Leverhulme Research Fellowship and by the Hausdorff Research Institute for Mathematics (Bonn). The authors would like to thank Joseph Najnudel, Ashkan Nikeghbali and Anatoly Vershik for stimulating discussions on the topic of random permutations. D. Z. is also grateful to Volker Betz and Daniel Ueltschi for enlightening comments on the model of spatial permutations.

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