

UNIQUENESS RESULT FOR NON-NEGATIVE SOLUTIONS OF SEMI-LINEAR INEQUALITIES ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We consider certain semi-linear partial differential inequalities on complete connected Riemannian manifolds and provide a simple condition in terms of volume growth for the uniqueness of a non-negative solution. We also show the sharpness of this condition.

1. INTRODUCTION

Consider a geodesically complete connected manifold M and the following differential inequality on M

$$\operatorname{div}(A(x)\nabla u) + V(x)u^\sigma \leq 0, \quad (1.1)$$

where div and ∇ are the Riemannian divergence and gradient respectively, $A(x)$ is a non-negative definite symmetric operator in the tangent space T_xM , such that $x \mapsto A(x)$ is measurable, V is a given locally integrable positive measurable function, and $\sigma > 1$ is a given constant.

Our purpose is to provide simple geometric condition on M to ensure that the only non-negative solution u of (1.1) is identical zero.

This problem has a long history. For the inequality

$$\Delta u + u^\sigma \leq 0, \quad (1.2)$$

in \mathbb{R}^n , the following result was proved by Ni and Serrin, Caristi and Mitidieri(cf.[4, 23, 24]): In the case $n \geq 3$, the only non-negative solution of (1.2) is identical zero if and only if $\sigma \leq \frac{n}{n-2}$, while in the case $n \leq 2$, the same is true for any σ .

Note for exact equality $\Delta u + u^\sigma = 0$, the critical value of parameter σ is different and is equal to $\frac{n+2}{n-2}$, when $n > 2$ (cf. [9]).

A more general inequality of (1.1) in \mathbb{R}^n and even more complicated inequalities and equations have been thoroughly studied in a series of papers by Mitidieri, Pohozaev [17, 18, 19, 20, 21, 22], D'Ambrosio, Mitidieri, Pohozaev [6, 7, 8], and Caristi, D'Ambrosio, Mitidieri [3, 4, 5]. They have developed a universal method of proving uniqueness for non-negative solutions, which is based on capacity estimates, which in turn rely on suitable choice of test functions.

In this paper we also use the method of test functions. In fact our proof up to (2.15) follows the same argument as in [20] and other papers cited in the above paragraph. However, after that we make a different choice of test function that enables us to work with minimal geometric assumption about the underlying manifold, namely, with the restriction on the volume growth of geodesic balls.

One of the difficulties that arises in the setting of manifold is that it is not possible to produce test function φ with suitable estimate of $L\varphi$, where $L = \operatorname{div}(A\nabla\cdot)$. More precisely, estimate of this kind would require restrictions of the curvature of the manifold,

¹ Supported by IGK of University Bielefeld.

Keywords and phrases. semilinear inequalities; critical exponents; Riemannian manifolds; volume growth;

2010 *Mathematics Subject Classification.* Primary: 35J61, Secondary: 58J05.

which we avoid. In fact, the only geometric assumption that we impose on manifold is the volume growth restriction. Note that under this mild assumption the commonly used in PDEs estimates of the fundamental solution, Harnack inequalities etc. are not available.

Denote by μ the Riemannian measure on M and by $B(x, r)$ the geodesic ball on M of radius r centered at $x \in M$. Given that $d(\cdot, \cdot)$ is geodesic distance, and μ is the Riemannian measure. Assume that $V(x) \in L_{loc}^1(M, \mu)$ throughout the paper.

Cheng and Yau proved that if for some $x_0 \in M$ and all large enough r

$$\mu(B(x_0, r)) \leq Cr^2, \quad (1.3)$$

then any positive solution to $\Delta u \leq 0$ is identical constant(cf. [2]).

Grigor'yan and the author proved in [15] that when

$$\mu(B(x_0, r)) \leq Cr^p \ln^q r, \quad (1.4)$$

where

$$p = \frac{2\sigma}{\sigma-1}, \quad q = \frac{1}{\sigma-1}. \quad (1.5)$$

then the only non-negative solution of (1.2) is identical zero. Note that (1.2) is a particular case of (1.1) for $A(x) = Id$ and $V(x) = 1$. Moreover, they constructed an example to show the sharpness of the exponents p and q , that is, if either $p > \frac{2\sigma}{\sigma-1}$, or $p = \frac{2\sigma}{\sigma-1}$ and $q > \frac{1}{\sigma-1}$, then there is a manifold satisfying the volume estimate (1.4), and (1.2) has a non-trivial non-negative solution.

For general A, V in (1.1), Grigor'yan and Kondratiev in [14] used measure ν_ϵ defined for any $\epsilon > 0$ by

$$d\nu_\epsilon = \|A\|^{\frac{\sigma}{\sigma-1}-\epsilon} V^{-\frac{1}{\sigma-1}+\epsilon} d\mu, \quad \text{for } \epsilon > 0,$$

and proved that if

$$\nu_\epsilon(B(x_0, r)) \leq Cr^{p+C\epsilon} \ln^\kappa r, \quad (1.6)$$

holds for some $\kappa < q$ and all small enough $\epsilon > 0$, where p, q are given by (1.5). Then any non-negative solution of (1.1) is identically equal to zero. Some conditions for uniqueness of non-negative solutions in terms of capacities were proved in [14, 16].

In the present paper, we improve the result of [14, Theorem 1.3] by letting $\epsilon = 0$. Namely, consider the measure ν , defined by

$$d\nu = \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu. \quad (1.7)$$

We also need the following assumption on A, V : There exists a non-negative pair (δ_1, δ_2) and positive constants c_0, C_0 such that, for almost all $x \in M$

$$c_0(1+r(x))^{-\delta_1} \leq \frac{V(x)}{\|A(x)\|} \leq C_0(1+r(x))^{\delta_2}, \quad (\mathbf{VA})$$

holds for large enough $r(x) := d(x, x_0)$. In particular, we assume $V(x) > 0$ and $\|A(x)\| > 0$ for almost all $x \in M$. Let us emphasize that the operator $A(x)$ is only assumed to be non-negative definite, so it can be degenerate.

Here is our main result.

Theorem 1.1. *Assume that (VA) holds with some $\delta_1, \delta_2 \geq 0$. If for some $x_0 \in M$, the following inequality*

$$\nu(B(x_0, r)) \leq Cr^p \ln^q r, \quad (1.8)$$

holds for all large enough r , where ν is defined as in (1.7), p and q are defined by (1.5). Then the only non-negative weak solution of (1.1) is identical zero.

In the following, we will explain in which sense the weak solution of (1.1) means. Let us introduce the following notations. If v, w are the vectors in the tangent space $T_x M$, denote

$$(v, w)_A := (A(x)v, w), \quad (1.9)$$

with the corresponding semi-norm

$$\|v\|_A = (A(x)v, v)^{1/2}.$$

Then for the operator norm $\|A(x)\|$, we have for any $v \in T_x M$

$$\|v\|_A^2 \leq \|A(x)\| \cdot \|v\|^2, \quad (1.10)$$

where $\|v\|$ is the Riemannian length of v .

Define

$$d\omega = \|A(x)\| d\mu,$$

and denote by

$$W_{loc}^{1,2}(M, \omega) := \{f \mid f \in L_{loc}^2(M, \omega), \nabla f \in L_{loc}^2(M, \omega)\}, \quad (1.11)$$

and denote by $W_c^{1,2}(M, \omega)$ the subspace of $W_{loc}^{1,2}(M, \omega)$ of functions with compact support.

Solutions of (1.1) are understood in the following weak sense

Definition 1.2. A function u on M is called a weak solution of the inequality (1.1) if u is a non-negative function from $W_{loc}^{1,2}(M, \omega)$, and for any non-negative function $\psi \in W_c^{1,2}(M, \omega)$, the following inequality holds:

$$- \int_M (\nabla u, \nabla \psi)_A d\mu + \int_M V(x) u^\sigma \psi d\mu \leq 0, \quad (1.12)$$

where $(\cdot, \cdot)_A$ is defined as in (1.9).

Remark 1.3. Notice here if u is the solution to (1.1), the first integral term is bounded. Furthermore, the finiteness of the first integral on the left-hand side will lead to the finiteness of the second one, since this is derived from (1.12) automatically.

The paper is organized as follows: In Section 2, we give the first proof of Theorem 1.1. In Section 3, we provide the second proof of Theorem 1.1, and we use quite different test function from the one in Section 2. In Section 4, we give two examples in \mathbb{R}^n to show the sharpness of the parameters p, q in (1.8).

NOTATION. The letters C, C', C_0, C_1, \dots denote positive constants whose values are unimportant and may vary at different occurrences.

2. FIRST PROOF OF THEOREM 1.1

Let u be a non-negative solution of (1.1). Fix some ball $B_R := B(x_0, R)$, where x_0 is the reference point from the hypothesis (1.8), and $R > 0$ to be chosen later. Take a Lipschitz function φ on M with compact support, such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in a neighborhood of \bar{B}_R . Particularly, $\varphi \in W_c^{1,2}(M, \omega)$. We use the following test function for (1.12):

$$\psi_\rho(x) = \varphi(x)^s (u + \rho)^{-t}, \quad (2.1)$$

where $\rho > 0$ is a parameter near zero, and the constants t, s satisfy the conditions

$$\begin{cases} 0 < t < \min\left(1, \frac{\sigma-1}{2}\right), \\ s > \max\left\{\frac{4\sigma}{\sigma-1}, 1 + \frac{2+\delta_2}{\sigma-1}, 1 + \frac{2\sigma(\delta_1-2)}{(\sigma-1)^2}\right\}. \end{cases} \quad (2.2)$$

In fact, in what follows s will be chosen to be a large enough fixed constant, and t will take arbitrarily small positive values.

Since $\frac{1}{u+\rho}$ is bounded, hence, ψ_ρ has compact support and is bounded. The identity

$$\nabla\psi_\rho = -t\varphi^s(u+\rho)^{-t-1}\nabla u + s\varphi^{s-1}(u+\rho)^{-t}\nabla\varphi,$$

implies that $\nabla\psi_\rho \in L^2(M, \omega)$, hence, $\psi_\rho \in W_c^{1,2}(M, \omega)$. We obtain from (1.12) that

$$t \int_M \varphi^s(u+\rho)^{-t-1} \|\nabla u\|_A^2 d\mu + \int_M \varphi^s V u^\sigma (u+\rho)^{-t} d\mu \leq s \int_M \varphi^{s-1}(u+\rho)^{-t} (\nabla u, \nabla \varphi)_A d\mu. \quad (2.3)$$

Applying Cauchy-Schwarz inequality, let us estimate the right hand side of (2.3) as follows

$$\begin{aligned} & s \int_M \varphi^{s-1}(u+\rho)^{-t} (\nabla u, \nabla \varphi)_A d\mu \\ &= \int_M \left(\sqrt{t}\varphi^{\frac{s}{2}}(u+\rho)^{-\frac{t+1}{2}} \nabla u, \frac{s}{\sqrt{t}}\varphi^{\frac{s}{2}-1}(u+\rho)^{-\frac{t-1}{2}} \nabla \varphi \right)_A d\mu \\ &\leq \frac{t}{2} \int_M \varphi^s(u+\rho)^{-t-1} \|\nabla u\|_A^2 d\mu \\ &\quad + \frac{s^2}{2t} \int_M \varphi^{s-2}(u+\rho)^{1-t} \|\nabla \varphi\|_A^2 d\mu. \end{aligned}$$

Substituting the above into (2.3), and cancelling out the half of the first term in (2.3), we obtain

$$\frac{t}{2} \int_M \varphi^s(u+\rho)^{-t-1} \|\nabla u\|_A^2 d\mu + \int_M \varphi^s V u^\sigma (u+\rho)^{-t} d\mu \leq \frac{s^2}{2t} \int_M \varphi^{s-2}(u+\rho)^{1-t} \|\nabla \varphi\|_A^2 d\mu. \quad (2.4)$$

Using the following Young inequality

$$\int_M fg d\mu \leq \varepsilon \int_M |f|^{p_1} d\mu + C_\varepsilon \int_M |g|^{p'_1} d\mu,$$

where $\varepsilon > 0$ is arbitrary, and (p_1, p'_1) is Hölder conjugate such that

$$p_1 = \frac{\sigma-t}{1-t}, \quad p'_1 = \frac{\sigma-t}{\sigma-1}.$$

Let us estimate the right hand side of (2.4) as follows

$$\begin{aligned} & \frac{s^2}{2t} \int_M \varphi^{s-2}(u+\rho)^{1-t} \|\nabla \varphi\|_A^2 d\mu \\ &= \int_M [\varphi^{\frac{s}{p_1}} V^{\frac{1}{p_1}} (u+\rho)^{1-t}] \cdot [\frac{s^2}{2t} \varphi^{\frac{s}{p'_1}-2} V^{-\frac{1}{p_1}} \|\nabla \varphi\|_A^2] d\mu \\ &\leq \varepsilon \int_M \varphi^s V (u+\rho)^{\sigma-t} d\mu \\ &\quad + C_\varepsilon \left(\frac{s^2}{2t} \right)^{\frac{\sigma-t}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla \varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu. \end{aligned} \quad (2.5)$$

Choosing $\varepsilon = \frac{1}{2}$ and using in the right hand side of (2.5) the simple inequality

$$\left(\frac{s^2}{t} \right)^{\frac{\sigma-t}{\sigma-1}} \leq \left(\frac{s^2}{t} \right)^{\frac{\sigma}{\sigma-1}}.$$

and combining (2.5) with (2.4), we obtain that

$$\begin{aligned} & \frac{t}{2} \int_M \varphi^s (u + \rho)^{-t-1} \|\nabla u\|_A^2 d\mu + \int_M \varphi^s V u^\sigma (u + \rho)^{-t} d\mu \\ & \leq \frac{1}{2} \int_M \varphi^s V (u + \rho)^{\sigma-t} d\mu + C t^{-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla \varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu, \end{aligned} \quad (2.6)$$

where the value of s is absorbed into constant C .

Before moving to the next step, let us specify the boundedness of the above integrals. It is easy to obtain from the definition of the solution the boundedness of the following three integral terms

$$\int_M \varphi^s (u + \rho)^{-t-1} \|\nabla u\|_A^2 d\mu,$$

and

$$\int_M \varphi^s V u^\sigma (u + \rho)^{-t} d\mu,$$

and

$$\int_M \varphi^{s-1} (u + \rho)^{-t} (\nabla u, \nabla \varphi)_A d\mu.$$

The boundedness of $\int_M \varphi^s V (u + \rho)^{\sigma-t} d\mu$ follows by the boundedness of

$$\int_M \varphi^s V u^\sigma (u + \rho)^{-t} d\mu,$$

and $V \in L_{loc}^1(M, \mu)$.

By Dominated Convergence theorem, we know

$$\lim_{\rho \downarrow 0} \int_M \varphi^s V (u + \rho)^{\sigma-t} d\mu = \int_M \varphi^s V u^{\sigma-t} d\mu,$$

Letting $\rho \downarrow 0$ in (2.6), applying Monotone Convergence theorem, we have

$$\begin{aligned} & \frac{t}{2} \int_M \varphi^s u^{-t-1} \|\nabla u\|_A^2 d\mu + \int_M \varphi^s V u^{\sigma-t} d\mu \\ & \leq \lim_{\rho \downarrow 0} \frac{1}{2} \int_M \varphi^s V (u + \rho)^{\sigma-t} d\mu + C t^{-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla \varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu, \end{aligned}$$

which is

$$\begin{aligned} & \frac{t}{2} \int_M \varphi^s u^{-t-1} \|\nabla u\|_A^2 d\mu + \frac{1}{2} \int_M \varphi^s V u^{\sigma-t} d\mu \\ & \leq C t^{-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla \varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu. \end{aligned} \quad (2.7)$$

We apply (1.12) once more, using another test function $\psi = \varphi^s$, which yields

$$\begin{aligned} \int_M \varphi^s V u^\sigma d\mu & \leq s \int_M \varphi^{s-1} (\nabla u, \nabla \varphi)_A d\mu \\ & \leq s \left(\int_M \varphi^s u^{-t-1} \|\nabla u\|_A^2 d\mu \right)^{\frac{1}{2}} \left(\int_M \varphi^{s-2} u^{t+1} \|\nabla \varphi\|_A^2 d\mu \right)^{\frac{1}{2}}. \end{aligned} \quad (2.8)$$

On the other hand, we obtain from (2.7) that

$$\int_M \varphi^s u^{-t-1} \|\nabla u\|_A^2 d\mu \leq C t^{-1-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla \varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu.$$

Substituting into (2.8) yields

$$\begin{aligned} \int_M \varphi^s V u^\sigma d\mu &\leq C \left[t^{-1-\frac{\sigma}{\sigma-1}} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla\varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu \right]^{\frac{1}{2}} \\ &\quad \times \left[\int_M \varphi^{s-2} u^{t+1} \|\nabla\varphi\|_A^2 d\mu \right]^{\frac{1}{2}}. \end{aligned} \quad (2.9)$$

Recalling that $\nabla\varphi = 0$ on B_R , and applying Hölder inequality to the last term of (2.9) with the Hölder couple

$$p_3 = \frac{\sigma}{t+1}, \quad p'_3 = \frac{\sigma}{\sigma-t-1},$$

we obtain

$$\begin{aligned} &\int_M \varphi^{s-2} u^{t+1} \|\nabla\varphi\|_A^2 d\mu \\ &= \int_{M \setminus B_R} \left(\varphi^{\frac{s}{p_3}} V^{\frac{1}{p_3}} u^{t+1} \right) \left(\varphi^{\frac{s}{p'_3}-2} V^{-\frac{1}{p'_3}} \|\nabla\varphi\|_A^2 \right) d\mu \\ &\leq \left(\int_{M \setminus B_R} \varphi^s V u^\sigma d\mu \right)^{\frac{t+1}{\sigma}} \left(\int_{M \setminus B_R} \varphi^{s-\frac{2\sigma}{\sigma-t-1}} V^{-\frac{t+1}{\sigma-t-1}} \|\nabla\varphi\|_A^{\frac{2\sigma}{\sigma-t-1}} d\mu \right)^{\frac{\sigma-t-1}{\sigma}} \end{aligned} \quad (2.10)$$

Substituting (2.10) into (2.9), we obtain

$$\begin{aligned} &\int_M \varphi^s V u^\sigma d\mu \\ &\leq C t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla\varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} V^{-\frac{t+1}{\sigma-t-1}} \|\nabla\varphi\|_A^{\frac{2\sigma}{\sigma-t-1}} d\mu \right)^{\frac{\sigma-t-1}{2\sigma}} \left(\int_{M \setminus B_R} \varphi^s V u^\sigma d\mu \right)^{\frac{t+1}{2\sigma}}. \end{aligned} \quad (2.11)$$

Since $\int_M \varphi^s V u^\sigma d\mu$ is finite due to Remark 1.3, it follows from (2.11) that

$$\begin{aligned} &\left(\int_M \varphi^s V u^\sigma d\mu \right)^{1-\frac{t+1}{2\sigma}} \\ &\leq C t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla\varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} V^{-\frac{t+1}{\sigma-t-1}} \|\nabla\varphi\|_A^{\frac{2\sigma}{\sigma-t-1}} d\mu \right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned} \quad (2.12)$$

Note that the first integral in the right-hand side of (2.12) has the following estimate

$$\begin{aligned} \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} \|\nabla\varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} d\mu &\leq \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} \|\nabla\varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} \|A\|_{\sigma-1}^{\frac{\sigma-t}{\sigma-1}} V^{-\frac{1-t}{\sigma-1}} d\mu \\ &= \int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} \|\nabla\varphi\|_A^{\frac{2(\sigma-t)}{\sigma-1}} \left(\frac{V}{\|A\|} \right)^{\frac{t}{\sigma-1}} d\nu, \end{aligned} \quad (2.13)$$

where we have used that $d\nu = \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu$. Similarly, the second integral in the right-hand side of (2.12) can be estimated as follows

$$\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} V^{-\frac{t+1}{\sigma-t-1}} \|\nabla\varphi\|_A^{\frac{2\sigma}{\sigma-t-1}} d\mu \leq \int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} \|\nabla\varphi\|_{\sigma-t-1}^{\frac{2\sigma}{\sigma-t-1}} \left(\frac{V}{\|A\|}\right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}} d\nu. \quad (2.14)$$

Substituting that (2.13) and (2.14) into (2.11), we have

$$\begin{aligned} \int_M \varphi^s V u^\sigma d\mu &\leq C t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} \|\nabla\varphi\|_{\sigma-1}^{\frac{2(\sigma-t)}{\sigma-1}} \left(\frac{V}{\|A\|}\right)^{\frac{t}{\sigma-1}} d\nu \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} \|\nabla\varphi\|_{\sigma-t-1}^{\frac{2\sigma}{\sigma-t-1}} \left(\frac{V}{\|A\|}\right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}} d\nu \right)^{\frac{\sigma-t-1}{2\sigma}} \\ &\quad \times \left(\int_{M \setminus B_R} \varphi^s V u^\sigma d\mu \right)^{\frac{t+1}{2\sigma}}. \end{aligned} \quad (2.15)$$

Substituting that (2.13) and (2.14) into (2.12), we obtain

$$\begin{aligned} &\left(\int_M \varphi^s V u^\sigma d\mu \right)^{1-\frac{t+1}{2\sigma}} \\ &\leq C t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(\int_M \varphi^{s-\frac{2(\sigma-t)}{\sigma-1}} \|\nabla\varphi\|_{\sigma-1}^{\frac{2(\sigma-t)}{\sigma-1}} \left(\frac{V}{\|A\|}\right)^{\frac{t}{\sigma-1}} d\nu \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_M \varphi^{s-\frac{2\sigma}{\sigma-t-1}} \|\nabla\varphi\|_{\sigma-t-1}^{\frac{2\sigma}{\sigma-t-1}} \left(\frac{V}{\|A\|}\right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}} d\nu \right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned} \quad (2.16)$$

Fix $R > 1$ large enough, and set $t = \frac{1}{\ln R}$ to satisfy (2.2), and consider the function

$$\varphi(x) = \begin{cases} 1, & r(x) < R, \\ \left(\frac{r(x)}{R}\right)^{-t}, & r(x) \geq R. \end{cases} \quad (2.17)$$

Here $r(x) = d(x, x_0)$.

We would like to use (2.16) with this function $\varphi(x)$. But, notice that $\text{supp } \varphi$ here is not compact, we transfer to consider a sequence $\{\varphi_n\}$ of functions with compact supports, which is constructed as follows. For any $n \in \mathbb{N}$, define a sequence of cut-off functions $\{\eta_n\}$ by

$$\eta_n(x) = \begin{cases} 1, & 0 \leq r(x) \leq nR, \\ 2 - \frac{r(x)}{nR}, & nR \leq r(x) \leq 2nR, \\ 0, & r(x) \geq 2nR. \end{cases} \quad (2.18)$$

Consider the sequence of functions

$$\varphi_n(x) := \varphi(x)\eta_n(x), \quad (2.19)$$

so that $\varphi_n(x) \uparrow \varphi(x)$ as $n \rightarrow \infty$. Notice that for any $a \geq 2$,

$$\|\nabla\varphi_n\|^a \leq C_a (\varphi^a \|\nabla\eta_n\|^a + \eta_n^a \|\nabla\varphi\|^a). \quad (2.20)$$

Here in our case the values of a satisfies $a \leq \frac{6\sigma}{\sigma-1}$, so that the constant C_a can be regarded as uniformly bounded.

Consider the integral

$$I_n(a, b) := \int_M \varphi_n^{s-a} \|\nabla \varphi_n\|^a \left(\frac{V}{\|A\|} \right)^b d\nu. \quad (2.21)$$

Here (a, b) are taking values from the couples $\left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1} \right)$ and $\left(\frac{2\sigma}{\sigma-t-1}, -\frac{\sigma t}{(\sigma-t-1)(\sigma-1)} \right)$. By (2.20), we have

$$\begin{aligned} I_n(a, b) &\leq C \int_{M \setminus B_R} \varphi^{s-a} \eta_n^s \|\nabla \varphi\|^a \left(\frac{V}{\|A\|} \right)^b d\nu \\ &\quad + C \int_{B_{2nR} \setminus B_{nR}} \varphi^s \eta_n^{s-a} \|\nabla \eta_n\|^a \left(\frac{V}{\|A\|} \right)^b d\nu \\ &\leq C \int_{M \setminus B_R} \varphi^{s-a} \|\nabla \varphi\|^a \left(\frac{V}{\|A\|} \right)^b d\nu \\ &\quad + C \int_{B_{2nR} \setminus B_{nR}} \varphi^s \|\nabla \eta_n\|^a \left(\frac{V}{\|A\|} \right)^b d\nu. \end{aligned} \quad (2.22)$$

Here $\nabla \varphi = 0$ in B_R , and $\nabla \eta_n = 0$ outside $B_{2nR} \setminus B_{nR}$ are used, and s will be chosen bigger enough than a .

Since $\|\nabla \eta_n\| \leq \frac{1}{nR}$, and using **(VA)**, when $b > 0$, the last integral in (2.22) can be estimated as follows

$$\begin{aligned} &\int_{B_{2nR} \setminus B_{nR}} \varphi^s \|\nabla \eta_n\|^a \left(\frac{V}{\|A\|} \right)^b d\nu \\ &\leq \frac{C}{(nR)^a} \int_{B_{2nR} \setminus B_{nR}} \varphi^s (1+r)^{\delta_2 b} d\nu \\ &\leq \frac{C}{(nR)^a} \left(\sup_{B_{2nR} \setminus B_{nR}} \varphi^s \right) (2nR)^{\delta_2 b} \nu(B_{2nR}) \\ &\leq \frac{C}{(nR)^a} \left(\frac{nR}{R} \right)^{-st} (2nR)^{\delta_2 b} (2nR)^p \ln^q(2nR) \\ &= C' n^{\delta_2 b + p - a - st} R^{\delta_2 b + p - a} \ln^q(2nR), \end{aligned} \quad (2.23)$$

where we have used the definition (2.17) of φ and the volume estimate (1.8).

When $b < 0$, the second integral on the right-hand side of (2.22) can be estimated as follows

$$\int_{B_{2nR} \setminus B_{nR}} \varphi^s \|\nabla \eta_n\|^a \left(\frac{V}{\|A\|} \right)^b d\nu \leq C n^{-\delta_1 b + p - a - st} R^{-\delta_1 b + p - a} \ln^q(2nR). \quad (2.24)$$

Before we give the estimate of the first integral in (2.22), using the following estimate from [15]: if f is a non-negative decreasing function on \mathbb{R}_+ , then, for large enough R ,

$$\int_{M \setminus B_R} f(r(x)) d\nu(x) \leq C \int_{R/2}^{\infty} f(r) r^{p-1} \ln^q r dr, \quad (2.25)$$

Thus, using $\|\nabla\varphi\| \leq R^t t r^{-t-1}$, (2.25), and $R/2 > 1$, when $b > 0$, we obtain

$$\begin{aligned} \int_{M \setminus B_R} \varphi^{s-a} \|\nabla\varphi\|^a \left(\frac{V}{\|A\|} \right)^b d\nu &\leq C \int_{R/2}^{\infty} \left(\frac{r}{R} \right)^{-t(s-a)} R^{at} t^a r^{-at-a} r^{\delta_2 b} r^{p-1} \ln^q r dr \\ &\leq C R^{st} t^a \int_1^{\infty} r^{-st-a+\delta_2 b+p} \ln^q r \frac{dr}{r} \\ &= C R^{st} t^a \int_0^{\infty} e^{-h_1 \xi} \xi^q d\xi, \end{aligned} \quad (2.26)$$

where we have used the change $\xi = \ln r$ and set

$$h_1 := st + a - \delta_2 b - p. \quad (2.27)$$

Assuming that $h_1 > 0$ and making one more change $\tau = h_1 \xi$, we obtain

$$\int_{M \setminus B_R} \varphi^{s-a} \|\nabla\varphi\|^a \left(\frac{V}{\|A\|} \right)^b d\nu \leq C R^{st} t^a h_1^{-q-1} \int_0^{\infty} e^{-\tau} \tau^q d\tau \leq C' R^{st} t^a h_1^{-q-1}, \quad (2.28)$$

where the value $\Gamma(q+1)$ of the integral is absorbed into the constant C' .

When $b > 0$, substituting (2.23) and (2.28) into (2.22) yields

$$I_n(a, b) \leq C R^{st} t^a h_1^{-q-1} + C n^{-h_1} R^{\delta_2 b + p - a} \ln^q(2nR). \quad (2.29)$$

Similarly, when $b < 0$, we have

$$I_n(a, b) \leq C R^{st} t^a h_2^{-q-1} + C n^{-h_2} R^{-\delta_1 b + p - a} \ln^q(2nR). \quad (2.30)$$

where

$$h_2 = st + a + \delta_1 b - p. \quad (2.31)$$

We will use (2.29) for which $h_1 > t$. Noticing also that $R^t = \exp(t \ln R) = e$, we obtain

$$I_n(a, b) \leq C e^{st} t^{a-q-1} + C n^{-t} R^{\delta_2 b + p - a} \ln^q(2nR).$$

As we have remarked above, we will consider only the values of a in the bounded range $a \leq 3p$. Hence, the term e^a in the above is uniformly bounded. Letting $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} I_n(a, b) \leq C t^{a-q-1}. \quad (2.32)$$

Similarly, when $b < 0$, we have

$$\limsup_{n \rightarrow \infty} I_n(a, b) \leq C t^{a-q-1}. \quad (2.33)$$

Let us first use (2.32) with $(a, b) = \left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1} \right)$. Note that $a < p$, and $b > 0$, for this value of a and for t as in (2.2), we should testify that $h_1 > t$, that is

$$\begin{aligned} h_1 &= st + a - \delta_2 b - p \\ &= st + \frac{2(\sigma-t)}{\sigma-1} - \delta_2 \frac{t}{\sigma-1} - \frac{2\sigma}{\sigma-1} \\ &= \left(s - \frac{2+\delta_2}{\sigma-1} \right) t > t, \end{aligned}$$

Since

$$a - q - 1 = \frac{2(\sigma-t)}{\sigma-1} - \frac{1}{\sigma-1} - 1 = \frac{\sigma-2t}{\sigma-1}.$$

Hence, we use (2.32) to obtain that

$$\limsup_{n \rightarrow \infty} I_n \left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1} \right) \leq C t^{\frac{\sigma-2t}{\sigma-1}}. \quad (2.34)$$

While, for $(a, b) = \left(\frac{2\sigma}{\sigma-t-1}, -\frac{\sigma t}{(\sigma-t-1)(\sigma-1)} \right)$, note that $b < 0$, and from (2.2) to get that $a < 3p$, we should testify that $h_2 > t$, that is

$$\begin{aligned} h_2 &= st + a + \delta_1 b - p \\ &= st + \frac{2\sigma}{\sigma-t-1} - \delta_1 \frac{\sigma t}{(\sigma-t-1)(\sigma-1)} - \frac{2\sigma}{\sigma-1} \\ &= \left(s - \frac{(\delta_1 - 2)\sigma}{(\sigma-t-1)(\sigma-1)} \right) t > t. \end{aligned}$$

Since

$$a - q - 1 = \frac{2\sigma}{\sigma-t-1} - \frac{1}{\sigma-1} - 1 = \frac{\sigma^2 - \sigma + \sigma t}{(\sigma-t-1)(\sigma-1)}.$$

whence (2.33) yields

$$\limsup_{n \rightarrow \infty} I_n \left(\frac{2\sigma}{\sigma-t-1}, -\frac{\sigma t}{(\sigma-t-1)(\sigma-1)} \right) \leq Ct^{\frac{\sigma^2 - \sigma + \sigma t}{(\sigma-t-1)(\sigma-1)}}. \quad (2.35)$$

The inequality (2.16) with function φ_n implies that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \left(\int_M \varphi_n^s V u^\sigma d\mu \right)^{1 - \frac{t+1}{2\sigma}} \\ &\leq \limsup_{n \rightarrow \infty} Ct^{-\frac{1}{2} - \frac{\sigma}{2(\sigma-1)}} I_n \left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1} \right)^{\frac{1}{2}} \\ &\quad \times I_n \left(\frac{2\sigma}{\sigma-t-1}, -\frac{\sigma t}{(\sigma-t-1)(\sigma-1)} \right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned} \quad (2.36)$$

Combining with (2.34) and (2.35), noting that $\varphi_n \uparrow \varphi$, we have

$$\left(\int_M \varphi^s V u^\sigma d\mu \right)^{1 - \frac{t+1}{2\sigma}} \leq Ct^{-\frac{t}{2(\sigma-1)}}. \quad (2.37)$$

The remaining term $t^{-\frac{t}{2(\sigma-1)}}$ on the right-hand side of (2.37) tends to 1 as $t \rightarrow 0$, which implies that the right-hand side of (2.37) is a *bounded* function of t . Hence, there is a constant C_1 such that

$$\left(\int_M \varphi^s V u^\sigma d\mu \right)^{1 - \frac{t+1}{2\sigma}} \leq C_1 < \infty, \quad \text{for all small enough } t. \quad (2.38)$$

It follows that also

$$\int_M \varphi^s V u^\sigma d\mu \leq C < \infty, \quad (2.39)$$

Since $\varphi = 1$ on B_R , it follows that

$$\int_{B_R} V u^\sigma d\mu \leq C < \infty,$$

which implies for $R \rightarrow \infty$ that

$$\int_M V u^\sigma d\mu \leq C < \infty. \quad (2.40)$$

Applying the same argument, inequality (2.15) with function φ_n implies that

$$\int_M \varphi_n^s V u^\sigma d\mu \leq C_1 \left(\int_{M \setminus B_R} \varphi_n^s V u^\sigma d\mu \right)^{\frac{t+1}{2\sigma}}. \quad (2.41)$$

Letting $n \rightarrow \infty$ and applying that $\varphi_n \uparrow \varphi$, we obtain

$$\int_M \varphi^s V u^\sigma d\mu \leq C_1 \left(\int_{M \setminus B_R} \varphi^s V u^\sigma d\mu \right)^{\frac{t+1}{2\sigma}},$$

whence

$$\int_{B_R} V u^\sigma d\mu \leq C_1 \left(\int_{M \setminus B_R} V u^\sigma d\mu \right)^{\frac{t+1}{2\sigma}}, \quad (2.42)$$

Since by (2.40), letting $R \rightarrow \infty$, we have

$$\int_{M \setminus B_R} V u^\sigma d\mu \rightarrow 0,$$

letting in (2.42) $R \rightarrow \infty$, we obtain

$$\int_M V u^\sigma d\mu = 0.$$

Since $V > 0$ a.e. on M , thus $u \equiv 0$.

3. SECOND PROOF OF THEOREM 1.1

Here we present a modification of the above proof of Theorem 1.1. We use the first proof up to (2.16). Then letting $s > \frac{4\sigma}{\sigma-1}$, and $t < \frac{\sigma-1}{2}$, and noting that $0 \leq \varphi \leq 1$, from (2.15), we obtain

$$\begin{aligned} \int_M \varphi^s V u^\sigma d\mu &\leq C t^{-\frac{1}{2} - \frac{\sigma}{2(\sigma-1)}} \left(\int_M \|\nabla \varphi\|^{\frac{2(\sigma-t)}{\sigma-1}} \left(\frac{V}{\|A\|} \right)^{\frac{t}{\sigma-1}} d\nu \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_M \|\nabla \varphi\|^{\frac{2\sigma}{\sigma-t-1}} \left(\frac{V}{\|A\|} \right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}} d\nu \right)^{\frac{\sigma-t-1}{2\sigma}} \\ &\quad \times \left(\int_{M \setminus B_R} \varphi^s V u^\sigma d\mu \right)^{\frac{t+1}{2\sigma}}, \end{aligned} \quad (3.1)$$

and from (2.16), we obtain

$$\begin{aligned} \left(\int_M \varphi^s V u^\sigma d\mu \right)^{1 - \frac{t+1}{2\sigma}} &\leq C t^{-\frac{1}{2} - \frac{\sigma}{2(\sigma-1)}} \left(\int_M \|\nabla \varphi\|^{\frac{2(\sigma-t)}{\sigma-1}} \left(\frac{V}{\|A\|} \right)^{\frac{t}{\sigma-1}} d\nu \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_M \|\nabla \varphi\|^{\frac{2\sigma}{\sigma-t-1}} \left(\frac{V}{\|A\|} \right)^{-\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}} d\nu \right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned} \quad (3.2)$$

We see, that all integral terms in the right hand side of (3.1) and (3.2) has the form

$$\int_M \|\nabla \varphi\|^a \left(\frac{V}{\|A\|} \right)^b d\nu,$$

with the following two pairs of (a, b)

$$(a, b) = \left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1} \right), \quad (a, b) = \left(\frac{2\sigma}{\sigma-t-1}, -\frac{\sigma t}{(\sigma-t-1)(\sigma-1)} \right). \quad (3.3)$$

Consequently, we could write a in the following way

$$a = p + lt, \quad (3.4)$$

with the corresponding two values of l

$$l = -\frac{2}{\sigma-1} \quad \text{and} \quad l = \frac{2\sigma}{(\sigma-t-1)(\sigma-1)}. \quad (3.5)$$

where $p = \frac{2\sigma}{\sigma-1}$ is defined as before in (1.5). It is very clear to obtain that the values of a, b, l are uniformly bounded, when t is near zero. Let $\{\tilde{\varphi}_k\}_{k \in \mathbb{N}}$ be a sequence satisfying that each $\tilde{\varphi}_k$ is a Lipschitz function such that $\text{supp}(\tilde{\varphi}_k) \subset B_{2^k}$, $\tilde{\varphi}_k = 1$ in a neighborhood of $B_{2^{k-1}}$, and

$$|\nabla \tilde{\varphi}_k| \begin{cases} \leq \frac{C}{2^{k-1}} & \text{for } x \in B_{2^k} \setminus B_{2^{k-1}}, \\ = 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

where C does not depend on k .

Fix some $n \in \mathbb{N}$ and set

$$t = \frac{1}{n}, \quad (3.7)$$

and

$$\varphi_n = \frac{\sum_{k=n+1}^{2n} \tilde{\varphi}_k}{n}, \quad (3.8)$$

Note that $\varphi_n = 1$ on B_{2^n} , $\varphi_n = 0$ outside $B_{2^{2n}}$, $0 \leq \varphi_n \leq 1$ on M . Note that for any $a \geq 1$, using that $\text{supp}(\nabla \tilde{\varphi}_k)$ are disjoint, we have

$$\|\nabla \varphi_n\|^a = \frac{\sum_{k=n+1}^{2n} \|\nabla \tilde{\varphi}_k\|^a}{n^a}. \quad (3.9)$$

It is easy to see that

$$\varphi_n \in W_{loc}^{1,2}(M, \omega).$$

Consider the integral

$$J_n(a, b) = \int_M \|\nabla \varphi_n\|^a \left(\frac{V}{\|A\|} \right)^b d\nu, \quad (3.10)$$

Assume that $b > 0$. Substituting (3.8) into (3.10), applying (3.9), (3.6), and **(VA)**, we obtain

$$\begin{aligned} J_n(a, b) &= \int_M \|\nabla \varphi_n\|^a \left(\frac{V}{\|A\|} \right)^b d\nu \\ &= \int_M \frac{\sum_{k=n+1}^{2n} \|\nabla \tilde{\varphi}_k\|^a}{n^a} \left(\frac{V}{\|A\|} \right)^b d\nu \\ &\leq \sum_{k=n+1}^{2n} \int_{B_{2^k} \setminus B_{2^{k-1}}} \frac{\|\nabla \tilde{\varphi}_k\|^a}{n^a} \left(\frac{V}{\|A\|} \right)^b d\nu \\ &\leq C \sum_{k=n+1}^{2n} \int_{B_{2^k} \setminus B_{2^{k-1}}} \left(\frac{2^{1-k}}{n} \right)^a (1+r)^{\delta_2 b} d\nu \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{1-k}}{n} \right)^a (2^k)^{\delta_2 b} \nu(B_{2^k}) \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{1-k}}{n} \right)^a (2^k)^{\delta_2 b} \nu(B_{2^k}). \end{aligned} \quad (3.11)$$

Noting that $a = p + lt$, $n + 1 \leq k \leq 2n$,

$$\begin{aligned} \left(\frac{2^{1-k}}{n}\right)^a (2^k)^{\delta_2 b} &= \left(\frac{2^{-k}}{n}\right)^p \left(\frac{2^{-k}}{n}\right)^{lt} (2^k)^{\delta_2 b} \\ &\leq \left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b} \sup_{n+1 \leq k \leq 2n, t=\frac{1}{n}} \left(\frac{2^{-k}}{n}\right)^{lt} \\ &\leq C \left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b}. \end{aligned} \quad (3.12)$$

Using (3.12) and (1.8), recalling that by (1.5) $p = \frac{2\sigma}{\sigma-1}$, $q = \frac{1}{\sigma-1}$, when $b \geq 0$, we obtain

$$\begin{aligned} J_n(a, b) &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b} \nu(B_{2^k}) \\ &\leq C \sum_{k=n+1}^{2n} \left(\frac{2^{-k}}{n}\right)^p (2^k)^{\delta_2 b} (2^k)^p \ln^q(2^k) \\ &\leq C \frac{1}{n^p} \sum_{k=n+1}^{2n} k^q 2^{k\delta_2 b} \\ &\leq C n^{q+1-p} 2^{2n\delta_2 b} \\ &\leq C n^{-\frac{\sigma}{\sigma-1}} 2^{2n\delta_2 b}. \end{aligned} \quad (3.13)$$

Similarly, when $b \leq 0$, we have

$$J_n(a, b) \leq C n^{-\frac{\sigma}{\sigma-1}} 2^{-2n\delta_1 b}. \quad (3.14)$$

Setting $\varphi = \varphi_n$ in (3.2), we obtain

$$\begin{aligned} \left(\int_M \varphi_n^s V u^\sigma d\mu\right)^{1-\frac{t+1}{2\sigma}} &\leq C t^{-\frac{1}{2}-\frac{\sigma}{2(\sigma-1)}} \left(J_n\left(\frac{2(\sigma-t)}{\sigma-1}, \frac{t}{\sigma-1}\right)\right)^{\frac{1}{2}} \\ &\quad \times \left(J_n\left(\frac{2\sigma}{\sigma-t-1}, -\frac{\sigma t}{(\sigma-t-1)(\sigma-1)}\right)\right)^{\frac{\sigma-t-1}{2\sigma}}. \end{aligned} \quad (3.15)$$

Substituting (3.13) and (3.14) into (3.15), recalling $t = \frac{1}{n}$, we have

$$\begin{aligned} \left(\int_M \varphi_n^s V u^\sigma d\mu\right)^{1-\frac{\frac{1}{n}+1}{2\sigma}} &\leq C n^{\frac{1}{2}+\frac{\sigma}{2(\sigma-1)}} \left(n^{-\frac{\sigma}{\sigma-1}} 2^{2n\delta_2 \frac{1}{\sigma-1}}\right)^{\frac{1}{2}} \\ &\quad \times \left(n^{-\frac{\sigma}{\sigma-1}} 2^{2n\delta_1 \frac{\frac{1}{n}}{(\sigma-\frac{1}{n}-1)(\sigma-1)}}\right)^{\frac{\sigma-\frac{1}{n}-1}{2\sigma}} \\ &\leq C n^{\frac{1}{2n(\sigma-1)}} 2^{\frac{\delta_1+\delta_2}{\sigma-1}} \\ &\leq C' n^{\frac{1}{2n(\sigma-1)}}. \end{aligned} \quad (3.16)$$

Recalling that $\varphi_n = 1$ on B_{2^n} , and taking the limsup of both sides in (3.16) as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \int_M V u^\sigma d\mu &\leq C \limsup_{n \rightarrow \infty} n^{\frac{1}{2n(\sigma-1)}/[1-\frac{\frac{1}{n}+1}{2\sigma}]} \\ &< \infty. \end{aligned} \quad (3.17)$$

Applying the same argument as in the first proof, we obtain that $u \equiv 0$.

4. SHARPNESS OF p, q

In this section, we will construct examples to show the parameters of p and q in (1.8) are sharp and can not be relaxed.

The sharpness of p is already known in \mathbb{R}^n , which is given by Mitidieri and Pohozaev in [21]: Let μ be the classical Lebesgue measure, if $\sigma > \frac{n-\gamma_2}{n-2+\gamma_1}$, and $2-n < \gamma_1 < 2-\gamma_2$, then the function

$$u(x) := \epsilon[1 + |x|^{2-\gamma_1-\gamma_2}]^{-\frac{1}{\sigma-1}}$$

is a solution to (1.1) with $A(x) = |x|^{\gamma_1}, V(x) = |x|^{-\gamma_2}$, where ϵ is a suitable small positive constant. In this case, we know **(VA)** holds, $V \in L^1_{loc}(\mathbb{R}^n)$, and

$$\begin{aligned} \nu(B(0, r)) &= \int_{B(0, r)} \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu \\ &= \int_{B(0, r)} |x|^{\frac{\sigma\gamma_1}{\sigma-1}} |x|^{\frac{\gamma_2}{\sigma-1}} d\mu \\ &= O(r^p), \quad \text{as } r \rightarrow \infty, \end{aligned} \tag{4.1}$$

where $p = \frac{\sigma\gamma_1 + \gamma_2}{\sigma-1} + n$. From the assumption $\sigma > \frac{n-\gamma_2}{n-2+\gamma_1}$, we know it is equivalent to

$$p > \frac{2\sigma}{\sigma-1}, \tag{4.2}$$

hence, by carefully choosing parameters of γ_1 and γ_2 , p could be close to $\frac{2\sigma}{\sigma-1}$.

Regarding to the sharpness of q , setting as before $p = \frac{2\sigma}{\sigma-1}$, and choosing $q = \frac{1}{\sigma-1} + \epsilon$ for small $\epsilon > 0$, we will construct a positive solution in \mathbb{R}^n with (1.8) holding with these values p, q .

Let $M = \mathbb{R}^n$, μ is the classical Lebesgue measure, and x_0 is the origin point, take

$$A = a(r)Id, \quad V = r^{\beta_1} \ln^{\beta_2} r, \quad \text{for large enough } r > 0. \tag{4.3}$$

where $a(r) = r^{\alpha_1} \ln^{\alpha_2} r$ for large enough r , and for small r near zero, $a(r)$ and V are constants. Moreover, parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ are chosen to satisfy the following condition

$$\begin{cases} \frac{\alpha_1\sigma - \beta_1}{\sigma-1} + n = \frac{2\sigma}{\sigma-1}, \\ \frac{\alpha_2\sigma - \beta_2}{\sigma-1} = \frac{1}{\sigma-1} + \epsilon, \\ \alpha_1 + n > 2. \end{cases} \tag{4.4}$$

It is easily to check that **(VA)** is satisfied with $\delta_1 = \delta_2 = |\beta_1 - \alpha_1| + 1$. Moreover, under the measure $d\nu = \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu$, for large enough r , the volume $\nu(B(0, r))$ could be estimated from (4.3) as follows

$$\begin{aligned} \nu(B(0, r)) &= \int_{B(0, r)} \|A\|^{\frac{\sigma}{\sigma-1}} V^{-\frac{1}{\sigma-1}} d\mu \\ &\leq \int^r (s^{\alpha_1} \ln^{\alpha_2} s)^{\frac{\sigma}{\sigma-1}} (s^{\beta_1} \ln^{\beta_2} s)^{-\frac{1}{\sigma-1}} \omega_n s^{n-1} ds \\ &\leq C \int^r s^{\frac{\alpha_1\sigma - \beta_1}{\sigma-1} + n - 1} \ln^{\frac{\alpha_2\sigma - \beta_2}{\sigma-1}} s ds \\ &\leq Cr^{\frac{\alpha_1\sigma - \beta_1}{\sigma-1} + n} \ln^{\frac{\alpha_2\sigma - \beta_2}{\sigma-1}} r, \end{aligned} \tag{4.5}$$

where ω_n is the surface area of the unit ball in \mathbb{R}^n . By (4.4), we obtain

$$\nu(B(0, r)) \leq Cr^p \ln^q r, \quad \text{for large enough } r > 0. \tag{4.6}$$

Hence, \mathbb{R}^n satisfies the volume growth condition (1.8) with A, V from (4.3).

In fact, since A, V are radially defined, thus the solution u to (1.1) actually depends on polar radius r , we can write $u = u(r)$. Hence, (1.1) could be wrote in the following form

$$(Sau')' + SVu^\sigma \leq 0, \quad (4.7)$$

where $S(r) = \omega_n r^{n-1}$.

By applying the following result in [25]

Proposition 4.1. *Let $\alpha(r)$ be a positive C^1 -function on $(r_0, +\infty)$ satisfying*

$$\int_{r_0}^{\infty} \frac{dr}{\alpha(r)} < \infty. \quad (4.8)$$

Define the function $\gamma(r)$ on (r_0, ∞) by

$$\gamma(r) = \int_r^{\infty} \frac{ds}{\alpha(s)}. \quad (4.9)$$

Let $\beta(r)$ be a continuous function on (r_0, ∞) such that

$$\int_{r_0}^{\infty} \gamma(r)^\sigma |\beta(r)| dr < \infty. \quad (4.10)$$

Then the differential equation

$$(\alpha(r)y')' + \beta(r)y^\sigma = 0, \quad (4.11)$$

has a positive solution $y(r)$ in an interval $[R_0, +\infty)$ for large enough $R_0 > r_0$, such that

$$y(r) \sim \gamma(r), \quad \text{as } r \rightarrow \infty. \quad (4.12)$$

Applying Proposition 4.1 with

$$\alpha(r) = S(r)a(r) = \omega_n r^{\alpha_1+n-1} \ln^{\alpha_2} r,$$

and

$$\beta(r) = S(r)V = \omega_n r^{\beta_1+n-1} \ln^{\beta_2} r,$$

we know from (4.4)

$$\int_{r_0}^{\infty} \frac{dr}{\alpha(r)} < \infty, \quad (4.13)$$

and for $r \gg 1$

$$\gamma(r) = \int_r^{\infty} \frac{ds}{\alpha(s)} = \int_r^{\infty} \frac{ds}{\omega_n s^{\alpha_1+n-1} \ln^{\alpha_2} s} \leq \frac{C_0}{r^{\alpha_1+n-2} \ln^{\alpha_2} r}, \quad (4.14)$$

and by (4.4), we obtain

$$\begin{aligned} \int_{r_0}^{\infty} \gamma(r)^\sigma \beta(r) dr &\leq C_1 \int_{r_0}^{\infty} \frac{1}{(r^{\alpha_1+n-2} \ln^{\alpha_2} r)^\sigma} r^{\beta_1+n-1} \ln^{\beta_2} r dr \\ &\leq C_2 \int_{r_0}^{\infty} \frac{1}{r^{\sigma(\alpha_1+n-2) - (\beta_1+n-1)} \ln^{\sigma\alpha_2 - \beta_2} r} dr \\ &= C_2 \int_{r_0}^{\infty} \frac{1}{r^{\sigma\alpha_1 + \sigma(n-2) - \beta_1 - n} \ln^{\sigma\alpha_2 - \beta_2} r} \frac{dr}{r} \\ &= C_2 \int_{r_0}^{\infty} \frac{1}{\ln^{1+(\sigma-1)\epsilon} r} \frac{dr}{r} < \infty. \end{aligned} \quad (4.15)$$

Applying Proposition 4.1, we know there exists a solution on $[r_0, \infty)$

$$u(r) \sim \gamma(r) \simeq r^{2-n-\alpha_1} \ln^{-\alpha_2} r, \quad \text{as } r \rightarrow \infty, \quad (4.16)$$

one could apply similar argument as in [15] to extend u to be a positive solution of (1.1) in \mathbb{R}^n .

Acknowledgements The author would like to thank his supervisor, Prof. Alexander Grigor'yan from Bielefeld University for leading him to this topic, and for his continuously encouragement and patience. The author would also like to thank Xueping Huang from Tohoku University for many valuable suggestions. The author also kindly thanks the anonymous referee for the valuable comments and suggestions to improve the quality of the paper.

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