Volume mean densities for the heat equation

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Abstract

It is shown that, for solid caps D of heat balls in $\mathbb{R}^{d+1}$ with center $z_0 = (0,0)$, there exist Borel measurable functions w on D such that $\inf w(D) > 0$ and $\int v(z)w(z)\,dz \le v(z_0)$, for every supertemperature v on a neighborhood of $\overline{D}$. This disproves a conjecture by N. Suzuki and N.A. Watson. On the other hand, it turns out that there is no such volume mean density, if the bounded domain D in $\mathbb{R}^d \times (-\infty, 0)$ is only slightly wider at z_0 than a heat ball.

1 Introduction

Let D be a bounded domain in $\mathbb{R}^{d+1}$, $d \ge 1$, such that $z_0 := (0,0) \in \overline{D}$. This paper is devoted to the question if there exists a Borel measurable function w on D such that $\inf w(D) > 0$ and

$$\int_D h(z)w(z)\,dz = h(z_0), \tag{1.1}$$

for every temperature h (solution to the heat equation $\Delta u - \partial u/\partial t = 0$) on a neighborhood of $\overline{D}$ (cf. [13, Corollary 3.3] and [1] for classical harmonic functions). N. Suzuki and N.A. Watson [15, Remark 8] conjectured that this is impossible, whatever the choice of D is, and the open question is mentioned again in [17, p. 32].

Clearly, there is no chance for such a density unless $D \subset \mathbb{R}^d \times (-\infty, 0)$, since, choosing $z = (y, 0) \notin \overline{D}$, the Green function for the heat equation with pole at z, that is, the function $(x,t) \mapsto 1_{(0,\infty)}(t)(4\pi t)^{-d/2} \exp(-|x-y|^2/(4t))$, is a temperature on the neighborhood $\mathbb{R}^{d+1} \setminus \{z\}$ of $\overline{D}$, which vanishes at z_0, but is strictly positive on $\mathbb{R}^d \times (0,\infty)$.

Let us recall that, for heat balls, volume mean densities (not bounded away from zero) are explicitly known. Indeed, for $r > 0$, let Ω_r denote the *heat ball* of "radius" r and "center" $z_0 := (0,0)$, that is,

$$\Omega_r := \{(x,-t) \in \mathbb{R}^{d+1} \colon |x| < \sqrt{2dt \log(r/t)},\ 0 < t < r\}.$$

It is the set, where the Green function for the coheat equation $\Delta u + \partial u/\partial t = 0$ with pole at z_0 has a value which is larger than $(4\pi r)^{-d/2}$.

*Research supported by CRC-701, Bielefeld.

Let μ_{r,z_0} denote the *Fulks measure for* Ω_r, that is, μ_{r,z_0} is the (probability) measure on $\partial\Omega_r$ having the continuous density

$$\varphi_r(x,t) := \begin{cases} (4\pi r)^{-d/2}|x|^2\big[4|x|^2t^2 + (|x|^2-2dt)^2\big]^{-1/2}, & (x,t)\in\partial\Omega_r\setminus\{z_0\},\\ (4\pi r)^{-d/2}, & (x,t)=z_0,\end{cases}$$

with respect to the surface measure σ_r on $\partial\Omega_r$. It is known that $\mu_{r,z_0} = \varepsilon_{z_0}^{\Omega_r^c}$, that is, that μ_{r,z_0} is obtained sweeping the Dirac mass at z_0 on the complement of Ω_r (see [3, Theorem 2.2] and [14, p. 20]) and that $\int h\,d\mu_{r,z_0} = h(z_0)$ for every temperature h on a neighborhood of $\overline{\Omega}_r$ (see, for example, [17, Theorem 1.6] or [11], [5, VII.9.5], and [3, Corollary 2.3]).

The measure

$$\mu_{z_0} := (d/2)\int_0^1 r^{d/2-1}\mu_{r,z_0}\,dr$$

has the (continuous) density

$$K(x,t) := (4\pi)^{-d/2}\,|x|^2/(4t^2) \tag{1.2}$$

with respect to Lebesgue measure on Ω_1 and (1.1) holds with $w := K$ (see [17, p. 15]). In fact, we (even) have $\int_{\Omega_1} v(z)K(z)\,dz \le v(z_0)$, for every supertemperature v on a neighborhood of $\overline{\Omega}_1$ (see [17, Theorem 3.51]; a temperature h is a function such that both h and $-h$ are supertemperatures). However, K vanishes on the "axis" $\{0\}\times(-1,0)$ of Ω_1.

Nevertheless, we claim the following (disproving the conjecture by N. Suzuki and N.A. Watson).

THEOREM 1.1. *Let* $0 < a < 1/e$ *and* $D := \{(x,s)\in\Omega_1 \colon s > -a\}$. *Then there exists a Borel measurable function* w *on* D *such that* $\inf w(D) > 0$ *and*

$$\int_D v(z)w(z)\,dz \le v(z_0), \tag{1.3}$$

for every supertemperature v *on a neighborhood of* $\overline{D}$.

But we shall see that such a result is impossible, if D is only slightly wider at z_0 than a heat ball.

THEOREM 1.2. *Let* D *be a bounded open subset of* $\mathbb{R}^d\times(-\infty,0)$ *and suppose that there exist* $t_0, r\in(0,\infty)$ *such that*

$$\Big\{(x,-t)\in\mathbb{R}^{d+1}\colon 0<t<t_0,\ |x| < \frac{d+1}{d}\sqrt{2dt\log(r/t)}\Big\} \subset D.$$

Then there is no *Borel measurable function* w *on* D *such that* $\inf w(D) > 0$ *and* (1.3) *holds, for every supertemperature* v *on a neighborhood of* $\overline{D}$.

Finally, let us recall the following. If D is a bounded domain in $\mathbb{R}^d\times(-\infty,0)$ such that

$$D\cap(\mathbb{R}^d\times(-t_0,0)) \subset \big\{(x,-t)\in\mathbb{R}^{d+1}\colon t>0,\ |x|<c\sqrt{t}\big\},$$

for some $t_0 > 0$ and $c > 0$, or, more generally, if z_0 is a stable point of $\overline{D}$ (that is, the complement of $\overline{D}$ is not thin at z_0), then ε_{z_0} is the only measure μ on $\overline{D}$ such that $\int h d\mu = h(z_0)$ for every temperature h on a neighborhood of $\overline{D}$ (see [4, Corollaries 3.14 and 2.7], for a more direct proof for the non-existence of a volume mean density see Proposition 6.6; cf. also [15, Theorem 4]).

The proof of Theorem 1.1 requires some preliminaries (Section 2) and the construction of two representing measures having densities which are bounded away from 0 on certain subsets of Ω_1 (Sections 3 and 4). A combination of these two measures then yields Theorem 1.1 (Section 5). A proof of Theorem 6.1 and a discussion of the case, where z_0 is a stable point of $\overline{D}$, finish the paper (Section 6).

2 Preliminaries

Again and again, we shall use swept measures. So let us briefly recall their definition and some properties we shall need (most of them hold in much more general settings). Given a set A in $\mathbb{R}^{d+1}$ and a supertemperature $u \geq 0$ on $\mathbb{R}^{d+1}$, let R_u^A denote the infimum of all supertemperatures $v \geq 0$ on $\mathbb{R}^{d+1}$ such that $v \geq u$ on A and define $\hat{R}_u^A(z) := \liminf_{z' \to z} R_u^A(z')$, $z \in \mathbb{R}^{d+1}$ (see [2, Section III.2] or [17, Section 7.4]). Clearly, $\hat{R}_u^A = R_u^A$ on the interior of A. Moreover, $\hat{R}_u^A = R_u^A$ on the complement of A (see [5, VI.2.3]). The set A is polar if and only if $\hat{R}_u^A = 0$, for every supertemperature $u \geq 0$ on $\mathbb{R}^{d+1}$. It is thin at a point z, if there exists a supertemperature $u \geq 0$ on $\mathbb{R}^{d+1}$ such that $\hat{R}_u^A(z) < u(z)$. Thinness is a local property, that is, A is thin at a point z if only if the intersection of A with some neighborhood of z is thin at z.

Given a finite measure μ on $\mathbb{R}^{d+1}$, there exists a unique measure μ^A on $\mathbb{R}^d$ (obtained by *sweeping* μ *on* A) such that

$$\int u \, d\mu^A = \int \hat{R}_u^A \, d\mu, \qquad \text{for every supertemperature } u \geq 0 \text{ on } \mathbb{R}^{d+1}$$

(see [2, III.4] or [5, VI.2.1]); for the case of open sets A, the crucial additivity of $u \mapsto R_u^A$ is part of [17, Theorem 7.31]). The measure μ^A is supported by the closure of A. It is supported by the boundary of A, if μ does not charge the interior of A. Clearly,

$$\mu^A = \int \varepsilon_z^A \, d\mu(z). \tag{2.1}$$

The set A is thin at a point z if and only if $\varepsilon_z^A \neq \varepsilon_z$ (true if z is not in the closure of A, not true if z is an interior point of A).

The following holds for iterated sweeping. If $z \in \mathbb{R}^{d+1}$ and B is a Borel measurable set contained in A such that $z \notin B$ or B is not thin at z or A is thin at z, then (and only then)

$$\varepsilon_z^B = \varepsilon_z^A|_B + \left(\varepsilon_z^A|_{B^c}\right)^B \tag{2.2}$$

(see [5, VI.9.4]). This implies, in particular, the fact that

$$\varepsilon_z^{\partial U} = \varepsilon_z^{U^c}, \qquad \text{whenever } U \text{ is open and } U^c \text{ is thin at } z. \tag{2.3}$$

It is easy to understand (2.2) (and several of its consequences) in terms of the associated *space-time Brownian motion*, that is, the diffusion (Y_t) in $\mathbb{R}^{d+1}$ which, starting at $z = (x, s)$, is obtained from Brownian motion (X_t) in $\mathbb{R}^d$ starting at x by

$$Y_t(\omega) := (X_t(\omega), s - t).$$

The measure ε_z^A (for Borel measurable A) is the distributions of the process (Y_t) at the time T_A of the first hitting A (defined by $T_A(\omega) := \inf\{t > 0\colon Y_t(\omega) \in A\}$), and (2.2) is a consequence of the strong Markov property and the trivial fact that $T_A \le T_B$.

The parabolic character of the heat equation implies the following. If $u \ge 0$ is a supertemperature on $\mathbb{R}^{d+1}$ and $s \in \mathbb{R}$, then

$$\hat{R}_u^{\mathbb{R}^d \times [s,\infty)} = 0 \quad \text{ on } \mathbb{R}^d \times (-\infty, s]\,. \tag{2.4}$$

Indeed $\hat{R}_u^{\mathbb{R}^d\times[s,\infty)} \le \hat{R}_u^{\mathbb{R}^d\times(s,\infty)} + \hat{R}_u^{\mathbb{R}^d\times\{s\}}$, and both functions on the right side of the inequality vanish on $(-\infty, s]$ (see [5, V.1.2, V.6.16, VI.5.8.3]). It has the following consequence for swept measures.

LEMMA 2.1. (i) *If $A \subset \mathbb{R}^{d+1}$ and $H := \mathbb{R}^d \times (-\infty, 0)$, then $\varepsilon_{z_0}^A = \varepsilon_{z_0}^{A\cap H}$.*

(ii) *If $A \subset \mathbb{R}^{d+1}$ is closed, $t > 0$, and $S := \mathbb{R}^d \times (-t, 0)$, then $\varepsilon_{z_0}^{A\cap S} = 1_{\mathbb{R}^d\times(-t,\infty)}\varepsilon_{z_0}^A$.*

Proof. (i) Consequence of (2.4), since, for every supertemperature $u \ge 0$ on $\mathbb{R}^{d+1}$,

$$\hat{R}_u^A(z_0) \le \hat{R}_u^{A\cap H}(z_0) + \hat{R}_u^{H^c}(z_0) = \hat{R}_u^{A\cap H}(z_0) \le \hat{R}_u^A(z_0).$$

(ii) By (i), the statement holds trivially, if A is not thin at z_0, since then $A \cap H$ and hence $A \cap S$ are not thin at z_0. So let us suppose that the closed set A is thin at z_0, and let $S' := \mathbb{R}^d \times (-t, 0]$. By (i), $\varepsilon_{z_0}^{A\cap S'} = \varepsilon_{z_0}^{A\cap S}$. The measure $\varepsilon_{z_0}^{A\cap H}$ is supported by $A \cap \overline{H}$ and $\hat{R}_1^{A\cap S'} \le \hat{R}_1^{\mathbb{R}^d\times[-t,\infty)} = 0$ on $\mathbb{R}^d \times (-\infty, -t]$, by (2.4). Thus, by (2.2),

$$\varepsilon_{z_0}^{A\cap S'} = 1_{A\cap S'}\varepsilon_{z_0}^A + \left(\varepsilon_{z_0}^A|_{\mathbb{R}^d\times(-\infty,-t]}\right)^{A\cap S'} = 1_{A\cap S'}\varepsilon_{z_0}^A = 1_{\mathbb{R}^d\times(-t,\infty)}\varepsilon_{z_0}^A.$$

□

Let us define, for $t, \rho \in (0, \infty)$,

$$H_t := \mathbb{R}^d \times \{-t\}, \qquad H_t^+ := \mathbb{R}^d \times (-t, \infty), \qquad H_t^- := \mathbb{R}^{d+1} \setminus H_t^+,$$

$$B_\rho := \{x \in \mathbb{R}^d\colon |x| < \rho\}, \qquad Z_\rho := B_\rho \times \mathbb{R}, \qquad Z_\rho(t) := B_\rho \times (-t, 0).$$

Let λ_t denote d-dimensional Lebesgue measure on H_t. We note that

$$\varepsilon_{z_0}^{H_t^-} = g_t\lambda_t = \varepsilon_{z_0}^{H_t}, \tag{2.5}$$

where $g_t(x, -t) = (4\pi t)^{-d/2} \exp\bigl(-|x|^2/(4t)\bigr)$ (see [5, V.6.1, VII.2.9, VII.2.3] and (2.3); cf. also [17, Theorem 2.2]).

For $\rho, t \in (0, \infty)$, let

$$c_\rho(t) := \sqrt{\frac{d}{4\pi}}\,\rho\, \sup\nolimits_{0<u\le t} u^{-3/2} \exp\bigl(-\frac{\rho^2}{4du}\bigr). \tag{2.6}$$

If $f(u) := u^{-3/2}\exp\big(-\rho^2/(4du)\big)$, $u>0$, then $f'(u) = f(u)(-3/(2u)+\rho^2/(4du^2)) > 0$, if and only if $u < \rho^2/(6d)$. In particular, the supremum in (2.6) is $t^{-3/2}\exp(-\rho^2/(4dt))$, if $t \le \rho^2/(6d)$, and $\lim_{t\to 0} c_\rho(t) = 0$.

LEMMA 2.2. *Let $\rho, t \in (0,\infty)$ and $A \subset \mathbb{R}^{d+1}\setminus Z_\rho(t)$. Then $\varepsilon_{z_0}^A(H_t^+) \le t c_\rho(t)$.*

Proof. Let $Q(t) := \{(y_1,\dots,y_d,s) \in H_t^+ \colon\ \max|y_j| < \rho/\sqrt{d}\}$ and, for $z \in Q(t)$,

$$g(z) := \varepsilon_z^A(H_t^+), \qquad h(z) := \varepsilon_z^{Q(t)^c}(H_t^+).$$

Then $A \subset Q(t)^c$, both g and h are temperatures on $Q(t)$, and $g \le h$, by the minimum principle. Furthermore, let

$$D_{j,l} := \{(y,s) \in H_t^+ \colon y_j < \rho/\sqrt{d}\} \quad \text{and} \quad D_{j,r} := \{(y,s) \in H_t^+ \colon y_j > -\rho/\sqrt{d}\}.$$

By the minimum principle, for all $z \in Q(t)$,

$$h(z) \le \sum_{j=1}^d \big(\varepsilon_z^{D_{j,l}}(H_t^+) + \varepsilon_z^{D_{j,r}}(H_t^+)\big).$$

In particular, using the symmetries of the heat equation,

$$\varepsilon_{z_0}^A(H_t^+) = g(z_0) \le h(z_0) \le 2d\, \varepsilon_{z_0}^{D_{1,l}}(H_t^+).$$

Since

$$\varepsilon_{z_0}^{D_{1,l}}(H_t^+) \le \frac{t}{\sqrt{4\pi}}\frac{\rho}{2\sqrt{d}} \cdot \sup\nolimits_{0<u\le t} u^{-3/2}\exp\big(-\frac{\rho^2}{4du}\big)$$

(see [5, V.6.2]; cf. [8, 2.VII.8.3]), the proof is finished. □

LEMMA 2.3. *There exist $\eta, \delta \in (0,1)$ such that, for every set A in $\mathbb{R}^{d+1}\setminus Z_1(\eta)$,*

$$\varepsilon_{z_0}^{A\cup H_\eta^-} \ge (1/2)\lambda_\eta \quad \textit{on } H_\eta \cap Z_\delta. \tag{2.7}$$

Proof. There exists $0 < \eta < (4\pi)^{-1}$ such that $c_1(\eta) < 1/2$ and

$$(4\pi t)^{-d/2}\exp\big(-\frac{1}{16\,t}\big) \le 1, \quad \text{whenever } 0 < t \le \eta. \tag{2.8}$$

Since obviously $g_\eta(0,-\eta)) > 1$, we may choose $\delta \in (0,1/2)$ such that

$$g_\eta > 1 \quad \text{on } H_\eta \cap Z_\delta. \tag{2.9}$$

Now let $A \subset \mathbb{R}^{d+1}\setminus Z_1(\eta)$ and $\nu := \varepsilon_{z_0}^{A\cup H_\eta^-}|_{H_\eta^+}$. Then $\|\nu\| \le \eta c_1(\eta) < 1/2$, by Lemma 2.2. Clearly $\varepsilon_{z_0}^{A\cup H_\eta^-}$ does not charge the interior of H_η^-. So, by (2.2),

$$g_\eta\lambda_\eta = \varepsilon_{z_0}^{H_\eta} = \varepsilon_{z_0}^{A\cup H_\eta^-}|_{H_\eta} + \nu^{H_\eta}. \tag{2.10}$$

Finally, let B be a Borel measurable subset of $H_\eta \cap Z_\delta$. By (2.9), $\int_B g_\eta\, d\lambda_\eta \ge \lambda_\eta(B)$, whereas, for every $z = (y,s) \in (\mathbb{R}^d \times (-\eta,0))\setminus Z_1$,

$$\varepsilon_z^{H_\eta}(B) = (4\pi(\eta+s))^{-d/2}\int_B \exp\big(-\frac{|x-y|^2}{4(\eta+s)}\big)\, d\lambda_\eta(x) \le \lambda_\eta(B),$$

by (2.8) (observe that $0 < \eta+s < \eta$ and $|x-y| \ge 1-\delta \ge 1/2$, for every $(x,-\eta) \in B$). Hence $\nu^{H_\eta}(B) \le \|\nu\|\lambda_\eta(B) \le (1/2)\lambda_\eta(B)$. Thus, by (2.10),

$$\varepsilon_{z_0}^{A\cup H_\eta^-}(B) \ge (1/2)\lambda_\eta(B).$$

□

Let us recall that the heat equation is *scaling invariant*, that is, invariant under the transformations T_α, $\alpha > 0$, defined by

$$T_\alpha(x,s) := (\sqrt{\alpha}x, \alpha s), \qquad (x,s) \in \mathbb{R}^{d+1}.$$

The following result can presumably be derived from estimates of hitting densities for the d-dimensional Bessel process. We show that it can easily be deduced from Lemma 2.2 using Harnack inequalities on cylinders in $\mathbb{R}^{d+1}$.

PROPOSITION 2.4. *Let $\rho > 0$, $\beta \in (0,1)$, and $\varepsilon > 0$. Then there exists $\delta \in (0,1)$ such that, for all $x \in \overline{B}_{(1-\beta)\rho}$ and $-\delta\rho^2 \le s_1 < s_2 \le 0$,*

$$\varepsilon_{(x,0)}^{Z_\rho^c}(\mathbb{R}^d \times (s_1, s_2)) \le \varepsilon \rho^{-2}(s_2 - s_1). \tag{2.11}$$

Proof. By scaling invariance, it suffices to consider the case $\rho = 1$. We define $z' := (0, -1)$ (0 being the origin in $\mathbb{R}^d$). By [2, Satz 1.4.4], there exists $C > 0$ such that $h(z') \le Ch(z_0)$, for every temperature $h \ge 0$ on $Z_{1-\beta}$. By translation and scaling invariance, we obtain that, for all temperatures $h \ge 0$ on Z_1 and $x \in \overline{B}_{1-\beta}$,

$$h(x,s) \le Ch(x,s'), \qquad \text{whenever } s \in \mathbb{R} \text{ and } s \le s' \le s+1. \tag{2.12}$$

We now fix $\delta \in (0,1)$ such that $c_{1-\beta}(2t) < \varepsilon/(2C)$, for all $0 < t \le \delta$. Let $x \in \overline{B}_{1-\beta}$. Then, by Lemma 2.2 and translation invariance,

$$\varepsilon_{(x,t)}^{Z_1^c}\big(\mathbb{R}^d \times (-t,t)\big) \le t\varepsilon/C \qquad \text{for all } 0 < t \le \delta. \tag{2.13}$$

We next fix $0 < t \le \delta$, $n \in \mathbb{N}$, define $\gamma := t/n$,

$$z_i := (x, (i-1)\gamma), \quad J := (-t, -t+\gamma), \quad \text{and} \quad A_i := \mathbb{R}^d \times ((i-1)\gamma + J)$$

so that $z_1 = (x,0)$ and $A_1 = \mathbb{R}^d \times J$. The functions

$$h_i \colon z \mapsto \varepsilon_z^{Z_1^c}(A_i), \qquad 1 \le i \le n,$$

are positive temperatures on Z_1. Therefore, by translation invariance and (2.12),

$$h_1(z_1) = h_i(z_i) \le Ch_i(x,t), \qquad 1 \le i \le n.$$

Consequently, by (2.13),

$$nh_1(z_1) \le C \sum\nolimits_{i=1}^n h_i(x,t) = C\, \varepsilon_{(x,t)}^{Z_1^c}\big(\mathbb{R}^d \times (-t,0)\big) \le t\varepsilon.$$

Thus $\varepsilon_{(x,0)}^{Z_1^c}(\mathbb{R}^d \times J) = h_1(z_1) \le \gamma\varepsilon$, that is, (2.11) holds for $s_1 = -t$ and $s_2 = -t+\gamma$. Now the proof is easily finished approximating arbitrary intervals (s_1, s_2) in $(-\delta, 0)$ by rational intervals. □

We recall from the Introduction that, for every $r > 0$, the Fulks measure μ_{r,z_0}, having the density φ_r with respect to σ_r on $\partial\Omega_r$, is the swept measure $\varepsilon_{z_0}^{\Omega_r^c}$. In particular, Ω_r^c is thin at z_0 and, by (2.3),

$$\varepsilon_{z_0}^{\Omega_r^c} = \varepsilon_{z_0}^{\partial\Omega_r}. \tag{2.14}$$

The set of heat balls with center z_0 is scaling invariant, that is, for all $r > 0$,

$$T_\alpha(\Omega_r) = \Omega_{\alpha r}. \tag{2.15}$$

For a proof of (2.15), let

$$F_r(t) := \sqrt{2dt \log(r/t)}, \qquad 0 < t < r,$$

(we observe that $\lim_{t\to 0} F_r(t) = \lim_{t\to r} F_r(t) = 0$ and F_r has its maximum at r/e) so that Ω_r is the set of all $(x, s) \in \mathbb{R}^{d+1}$ such that $-r < s < 0$ and $|x| < F_r(-s)$. If $(x, s) \in \Omega_r$, then $-\alpha r < \alpha s < 0$ and $|\sqrt{\alpha} x| < F_{\alpha r}(-\alpha s)$, hence $T_\alpha((x, s)) \in \Omega_{\alpha r}$. So $T_\alpha(\Omega_r) \subset \Omega_{\alpha r}$. Replacing α by α^{-1} and r by αr we see that as well $(T_\alpha)^{-1}(\Omega_{\alpha r}) = T_{\alpha^{-1}}(\Omega_{\alpha r}) \subset \Omega_r$, and hence $\Omega_{\alpha r} \subset T_\alpha(\Omega_r)$.

Given $0 < t < r < \infty$, we define

$$\Omega_r(t) := \Omega_r \cap H_t^+, \qquad L_r(t) := \partial\Omega_r \cap H_t^+, \qquad S_r(t) := \Omega_r \cap H_t.$$

Then $\partial\Omega_r(t)$ is the union of $L_r(t)$, $S_r(t)$, and the λ_t-null set $\partial\Omega_r \cap H_t$.

The following lemma shows that, for every $z \in \Omega_r$, the measure $\varepsilon_z^{\Omega_r^c}$ is absolutely continuous with respect to σ_r.

LEMMA 2.5. *Let $r > 0$ and let L be a compact in Ω_r. Then there exists $C > 0$ such that, for every $z \in L$, $\varepsilon_z^{\Omega_r^c} \le C\varepsilon_{z_0}^{\Omega_r^c}$.*

Proof. There exists $t \in (0, r)$ with $L \subset \mathbb{R}^d \times (-\infty, -t)$. Let $\nu := 1_{S_r(t)}\varepsilon_{z_0}^{\Omega_r(t)^c}$. Then

$$\varepsilon_{z_0}^{\Omega_r^c} = \varepsilon_{z_0}^{\Omega_r(t)^c}|_{\overline{L_r(t)}} + \nu^{\Omega_r^c}, \tag{2.16}$$

by (2.2) and (2.14). In particular, $\nu \neq 0$. By [2, Satz 1.4.4], there exists $C > 0$ such that, for every $z \in L$, $\varepsilon_z^{\Omega_r^c} \le C\nu^{\Omega_r^c}$ (if A is a Borel set in $\partial\Omega_r$ and h denotes the temperature $z \mapsto \varepsilon_z^{\Omega_r^c}(A)$, then $\int h\, d\nu = \nu^{\Omega_r^c}(A)$). By (2.16), the proof is finished. □

LEMMA 2.6. *For all $r > 0$ and $t \in (0, r)$ the following holds:*

(i) $\varepsilon_{z_0}^{\Omega_r(t)^c} = \varphi_r\sigma_r$ *on* $L_r(t)$.

(ii) *The measure* $\varepsilon_{z_0}^{\Omega_r(t)^c}|_{H_t}$ *admits a density* $\psi_{r,t} \le g_t$ *with respect to* λ_t *which locally is bounded away from* 0 *on* $S_r(t)$.

In particular, the measure $\varepsilon_{z_0}^{\Omega_r(t)^c}$ is supported by the union of $L_r(t)$ and $S_r(t)$.

Proof. (i) Consequence of Lemma 2.1.

(ii) By (2.2) and (2.5), $\varepsilon_{z_0}^{\Omega_r(t)^c}|_{H_t} \le \varepsilon_{z_0}^{H_t} = g_t\lambda_t$. Let us choose $\eta, \delta \in (0, 1)$ according to Lemma 2.3 and fix $z = (x, -t) \in S_r(t)$. There exists $\rho \in (0, 1)$ such that the closure of the cylinder

$$\{y \in \mathbb{R}^d \colon |y - x| < 2\rho\} \times (-t, -t + \eta\rho^2)$$

is contained in Ω_r. Let

$$S := \{y \in \mathbb{R}^d \colon |y - x| < \rho\delta/2\} \times \{-t + \eta\rho^2\} \quad \text{and} \quad \nu := \varepsilon_{z_0}^{\Omega_r(t-\eta\rho^2)^c}|_S.$$

Then $\nu \neq 0$ and $\varepsilon_{z_0}^{\Omega_r(t)^c} \geq 1_{S_r(t)} \nu^{\Omega_r(t)^c}$, by (2.2). Let B' be a Borel measurable set in $B := \{y \in \mathbb{R}^d \colon |y - x| < \rho\delta/2\} \times \{-t\}$. Then, by Lemma 2.3 (applied to $A := \Omega_r^c$) and by translation and scaling invariance, there exists $c > 0$ such that

$$\varepsilon_z^{\Omega_r(t)^c}(B') \geq c\lambda_t(B') \qquad \text{for every } z \in S.$$

By integration with respect to ν, we obtain that $\nu^{\Omega_r(t)^c}(B') \geq c\|\nu\|\lambda_t(B')$. Thus $\varepsilon_{z_0}^{\Omega_r(t)^c} \geq c\|\nu\|\lambda_t$ on A. □

3 Construction of a first representing measure

For the moment, we shall assume that D is an arbitrary bounded open set in $\mathbb{R}^{d+1}$. Let $S_0(\overline{D})$, $H_0(\overline{D})$, respectively, denote the set of restrictions on $\overline{D}$ of functions which are continuous supertemperatures, temperatures, respectively, on a neighborhood of the compact $\overline{D}$. Given $z \in \overline{D}$, a positive Radon measure μ on $\overline{D}$ is a *representing measure for z with respect to $S_0(\overline{D})$* provided

$$\int u \, d\mu \leq u(z), \qquad \text{for all } u \in S_0(\overline{D}). \tag{3.1}$$

The set of these measures is denoted by $\mathcal{M}_z(S_0(\overline{D}))$. The set $\mathcal{M}_z(H_0(\overline{D}))$ of all *representing measures for z with respect to $H_0(\overline{D})$*) is defined in an analogous way (where, of course, the inequality corresponding to (3.1) can be replaced by an equality, since $H_0(\overline{D})$ is a linear space). Clearly, $\mathcal{M}_z(S_0(\overline{D})) \subset \mathcal{M}_z(H_0(\overline{D}))$ because of $H_0(\overline{D}) \subset S_0(\overline{D})$), and both $\mathcal{M}_z(S_0(\overline{D}))$ and $\mathcal{M}_z(H_0(\overline{D}))$ are compact convex sets with respect to weak convergence.

Such sets of representing measures have been studied in much more generality (see [4], [5, VII.9]). In particular, it is known that

$$\{\varepsilon_z^{A \cup \overline{D}^c} \colon A \subset \mathbb{R}^{d+1}\} \tag{3.2}$$

is the set of all extreme points of $\mathcal{M}_z(S_0(\overline{D}))$ and that a positive Radon measure μ on $\overline{D}$ is contained in $\mathcal{M}_z(H_0(\overline{D}))$ if and only if

$$\mu^{\overline{D}^c} = \varepsilon_z^{\overline{D}^c} \tag{3.3}$$

(which, for example, implies that $\varepsilon_z^{\overline{D}^c} + \alpha(\varepsilon_{z'} - \varepsilon_{z'}^{\overline{D}^c}) \in \mathcal{M}_z(H_0(\overline{D})) \setminus \mathcal{M}_z(S_0(\overline{D}))$, whenever $\alpha > 0$ and $z' \in \overline{D} \setminus \{z\}$ satisfy $\alpha\varepsilon_{z'}^{\overline{D}^c} \leq \varepsilon_z^{\overline{D}^c}$).

For our purpose, it is sufficient to know that $\varepsilon_z^{U^c} \in \mathcal{M}_z(S_0(\overline{D}))$, for every open set U in D (the non-trivial case $z \in \partial U$ follows from the simple fact that there exists a sequence (x_n) in U such that $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} \varepsilon_{x_n}^{U^c} = \varepsilon_z^{U^c}$; see, for example, [5, VI.11.6]).

Let us now fix $a \in (0,1)$ and define

$$D := \Omega_1(a) = \left\{(x, -t) \in \mathbb{R}^{d+1} \colon 0 < t < a,\ |x| < \sqrt{2dt \log(1/t)}\,\right\}.$$

For every open set U in $\mathbb{R}^{d+1}$, let λ_U denote Lebesgue measure on U. Sweeping ε_{z_0} on the complement of scaled versions $D_r := T_r(D) = \Omega_r(ra)$ of D, $0 < r \leq 1$, and then integrating with respect to r, we shall obtain a representing measure for z_0 having a density with respect to λ_D which is bounded away from zero on many paraboloids containing the axis $\{0\} \times (-a, 0)$. Indeed, let

$$\nu_r := \varepsilon_{z_0}^{D_r^c}, \quad 0 < r \leq 1, \qquad \text{and} \qquad \nu := (d/2) \int_0^1 r^{d/2-1} \nu_r \, dr. \tag{3.4}$$

Moreover, we define paraboloids P_β, $a < \beta \leq 1$, by

$$P_\beta := \left\{ (x, -t) \in \mathbb{R}^{d+1} : 0 < t < a, \, |x| < \sqrt{2dt \log(\beta/a)} \right\}$$

(see Figure 1).

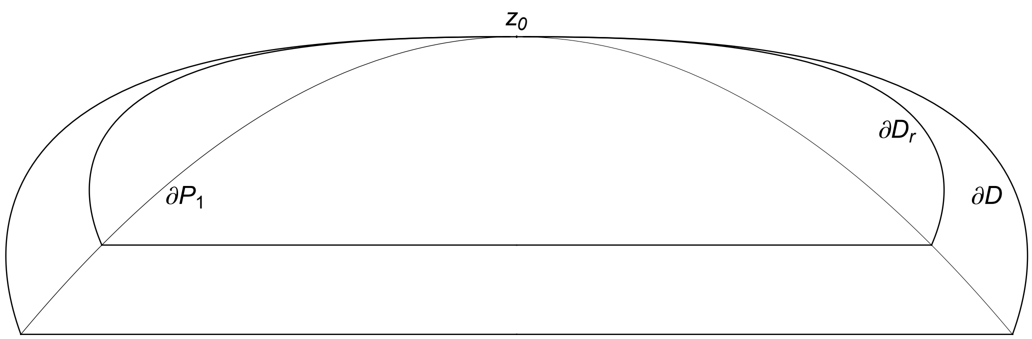

Figure 1. D, D_r, and P_1 ($d = 1$, $a = 0.5$, $r = 0.7$)

Then $\overline{P}_1 \cap H_a^+ \subset \Omega_1 \cup \{z_0\}$. Moreover, for all $\beta, r \in (0, 1]$, $T_r(P_\beta) = P_\beta \cap H_{ra}^+$ and $\inf K(D \setminus P_\beta) \geq (4\pi)^{-d/2} d/(2a) \log(\beta/a) > 0$.

PROPOSITION 3.1. *The measure ν is contained in $\mathcal{M}_{z_0}(S_0(\overline{D}))$ and there exists a density ψ with respect to λ_{P_1} such that $\nu = K\lambda_{D \setminus P_1} + \psi \lambda_{P_1}$ and $\inf \psi(P_\beta) > 0$, for every $\beta \in (a, 1)$.*

Proof. Since $\nu_r \in \mathcal{M}_{z_0}(S_0(\overline{D}))$, for every $0 < r \leq 1$, we obtain that $\nu \in \mathcal{M}_{z_0}(S_0(\overline{D}))$. By Lemma 2.1, for every $0 < r \leq 1$,

$$\nu_r|_{L_r(ar)} = 1_{L_r(ar)} \mu_{z_0, r} + \kappa_r, \qquad \text{where } \kappa_r := \nu_r|_{S_r(ar)}. \tag{3.5}$$

Given $0 < t < r$, we have $F_r(t) > \sqrt{2dt \log(1/a)}$ if and only if $t < ra$. Therefore

$$\bigcup\nolimits_{0<r<1} L_r(ar) = D \setminus P_1. \tag{3.6}$$

Moreover, by Lemma 2.6, there exists a density ψ_1 for κ_1 with respect to λ_a such that, for every $\beta \in (a, 1)$,

$$\inf \psi_1(P_\beta \cap H_a) > 0. \tag{3.7}$$

By scaling invariance,

$$(x, -ra) \mapsto r^{-d/2} \psi_1(x/\sqrt{r}, -a) \tag{3.8}$$

is a density for κ_r with respect to λ_{ra}, $0 < r < 1$. Defining

$$\kappa := (d/2) \int_0^1 r^{d/2-1} \kappa_r \, dr$$

we thus obtain, using (1.2), (3.5), and (3.6), that

$$\nu = K\lambda_{D\setminus P_1} + \kappa = K\lambda_{D\setminus P_1} + \psi\lambda_{P_1} \tag{3.9}$$

with

$$\psi(x,s) := (d/2)s^{-1}\psi_1(\sqrt{-a/s}\,x, -a).$$

By (3.7), for every $\beta \in (a,1)$, $\inf \psi(P_\beta) > 0$ (even $\lim_{z\to z_0, z\in P_\beta} \psi(z) = \infty$). □

4 Construction of a second representing measure

In this section we assume, more restrictively, that

$$D := \Omega_1(a), \qquad \text{where } 0 < a < 1/e.$$

Moreover, we define $R := ea$ and fix $\beta' \in (R,1)$. Then $F_R(a) = \sqrt{2da} < \sqrt{2ad\log(\beta'/a)}$. So there exists $a' \in (0,a)$ such that

$$F_R(t) < \sqrt{2dt\log(\beta'/a)}, \qquad \text{for every } t \in [a', a].$$

By definition of $P_{\beta'}$, this implies that

$$D = P_{\beta'} \cup \Omega_1(a') \cup (D \setminus \Omega_R) \tag{4.1}$$

(see Figure 2).

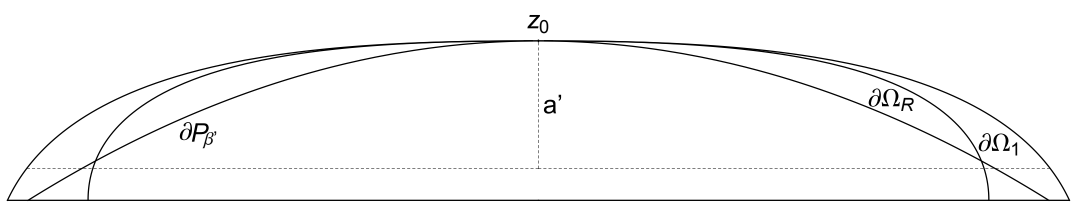

Figure 2. $P_{\beta'}$, $\Omega_1(a')$, and $\Omega_R(a)$ ($d = 1$, $a = 0.25$, $a' = 0.2$, $\beta' = 0.9$)

We shall prove the following, and observe already now that, by (4.1), a convex combination of ν (defined by (3.4)) and ν' will then immediately lead to Theorem 1.1 (see Section 5).

PROPOSITION 4.1. *There exists $\nu' \in \mathcal{M}_{z_0}(S_0(\overline{D}))$ such that ν' admits a density with respect to λ_D which is equal to K on $\Omega_1(a')$ and larger than $K/2$ on $D \setminus \Omega_R$.*

Similarly as in Section 3, the measure ν' is obtained integrating with respect to r,

$$\nu' := (d/2)\int_0^1 r^{d/2-1}\nu'_r\,dr. \tag{4.2}$$

Here we define, for $0 < r < R$,

$$\nu'_r := \varepsilon_{z_0}^{\Omega_r(a'+r(a-a'))^c} \tag{4.3}$$

so that $\nu'_r = \varphi_r\sigma_r$ on $\partial\Omega_r \cap H^+_{a'}$. For every $R \le r < 1$, we shall "hide" the bottom $\Omega_r \cap H_a$ of $\Omega_r(a)$ by a countable union I_r of subsets of hyperplanes H_t such that

$$\nu'_r := \varepsilon_{z_0}^{I_r \cup \Omega_r(a)^c} \tag{4.4}$$

does not charge H_a and satisfies $\nu'_r \ge (1/2)\varphi_r\sigma_r$ on $\partial\Omega_r \cap H^+_a$ (see Figure 3).

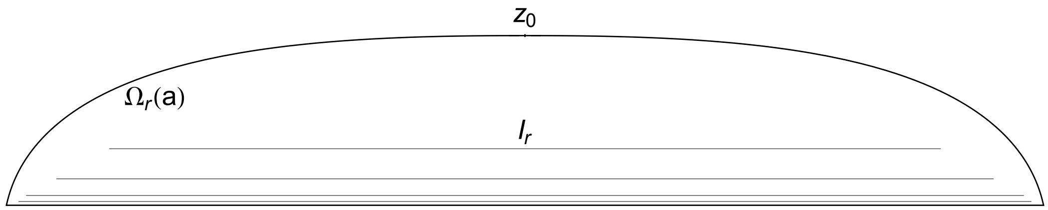

Figure 3. $\Omega_r(a)$ and I_r ($d = 1$, $a = 0.25$, $r = 0.8$)

Then, of course, $\nu' \in \mathcal{M}_{z_0}(S_0(\overline{D}))$, since $\nu'_r \in \mathcal{M}_{z_0}(S_0(\overline{D}))$, for every $r \in (0,1)$. To construct the sets I_r, we recall that, for $\rho > 0$, $Z_\rho = B_\rho \times \mathbb{R}$, and define

$$I_\rho(t) := \overline{Z}_\rho \cap H_t, \qquad t \in \mathbb{R}.$$

Let us say that a set A in $\mathbb{R}^{d+1}$ is *rotationally invariant*, if A is invariant under the mappings $(x,s) \mapsto (Tx,s)$, T being a rotation on $\mathbb{R}^d$. Similarly for measures on rotationally invariant sets. The following lemma is crucial for the proof of Proposition 4.1.

LEMMA 4.2. *Let $\eta > 0$ and $\alpha \in (0,1)$. There exists $\gamma' > 0$ such that, for all $R \le r \le 1$ and $0 < \gamma \le \gamma'$, the compact $I_{(1-\alpha)F_r(a)}(a - r\gamma)$ is contained in $\Omega_r(a)$ and, for every rotationally invariant measure τ on this set, the restriction of $\tau^{\Omega_r(a)^c}$ on $\partial\Omega_r$ admits a density with respect to σ_r which is bounded by $\eta\|\tau\|$.*

Proof. By Dini's lemma, there exists $a'' \in (a,a')$ such that, for every $R \le r \le 1$, $(1-\alpha/2)F_r(a) < F_r(a'')$, and hence

$$B_{(1-\alpha/2)F_r(a)} \times (-a, a''] \subset \Omega_r(a), \qquad \text{for every } R \le r \le 1. \tag{4.5}$$

Let π_d denote the surface of the unit ball in $\mathbb{R}^d$ and let $\eta' := \pi_d(ad/2)^{(d-1)/2}\eta$. Taking $\varepsilon := (ad/4)\eta'$ and $\beta := (1-\alpha)/(1-\alpha/2)$ let us choose $\delta > 0$ according to Proposition 2.4 and define

$$\gamma' := \min\{a'' - a, (ad/2)\delta\}.$$

Now let us fix $r \in [R,1]$, $0 < \gamma \le \gamma'$, and let τ be a rotationally invariant measure on the compact $I_{(1-\alpha)F_r(a)}(a - r\gamma)$, which, by (4.5), is contained in $\Omega_r(a)$. We claim that, for all $(1-\alpha/2)F_r(a) \le \rho \le F_r(a)$ and $-a < s_1 < s_2 < -a + \gamma$,

$$\tau^{Z_\rho^c}(\mathbb{R}^d \times (s_1, s_2)) \le (\eta'/2)\|\tau\|(s_2 - s_1). \tag{4.6}$$

Indeed, we have $(1-\alpha)F_r(a) = \beta(1-\alpha/2)F_r(a) \le \beta\rho$. So the measure τ is supported by $I_{\beta\rho}(a - r\gamma)$. Moreover,

$$\sqrt{ad/2} = F_R(a)/2 \le (1-\alpha/2)F_R(a) \le \rho. \tag{4.7}$$

Therefore $\gamma \le (ad/2)\delta < \rho^2\delta$ and $\varepsilon\rho^{-2} \le \eta'/2$. Hence (4.6) follows by Proposition 2.4, translation invariance, and integration with respect to τ.

Let $\kappa := \tau^{\Omega_r(a)^c}|_{\partial\Omega_r}$. By Lemma 2.5 and rotational invariance of both τ and Ω_r, there exists a bounded Borel measurable function $f \ge 0$ on $(-a, 0)$ such that f vanishes outside $(-a, -a+\gamma)$ and $(x,s) \mapsto f(s)$ is a density for κ with respect to σ_r.

For the moment, let us fix a Lebesgue point $s \in (-a, -a+\gamma)$ of f such that $f(s) > 0$. Then there exist $-a < s_1 < s < s_2 < -a+\gamma$ such that

$$\frac{1}{s_2 - s_1} \int_{s_1}^{s_2} f(\xi)\, d\xi > f(s)/2. \tag{4.8}$$

Let

$$\rho := F_r(-s_2) \quad \text{ and } \quad A := \mathbb{R}^d \times (s_1, s_2).$$

Considering the harmonic functions $g\colon z \mapsto \varepsilon_z^{\Omega_r(a)^c}(A)$ on $\Omega_r(a)$ and $h\colon z \mapsto \varepsilon_z^{Z_\rho^c}(A)$ on Z_ρ, we see, by the minimum principle, that $g \le h$ on $\Omega_r(a) \cap Z_\rho$. Integration with respect to τ yields that

$$\kappa(A) \le \tau^{Z_\rho^c}(A).$$

Of course, $F_r(a) \ge F_r(-s_2) \ge F_r(a-\gamma) \ge F_r(a'') > (1-\alpha/2)F_r(a)$. So, by (4.6),

$$\tau^{Z_\rho^c}(A) \le (\eta'/2)\|\tau\|(s_2 - s_1).$$

On the other hand, by (4.7) and (4.8),

$$\kappa(A) = \int_A f(s)\, d\sigma_r(x,s) \ge \pi_d \rho^{d-1} \int_{s_1}^{s_2} f(\xi)\, d\xi > \pi_d (ad/2)^{(d-1)/2}(s_2 - s_1) f(s)/2.$$

So $f(s) < \eta\|\tau\|$.

Thus the function $(x,s) \mapsto \min\{f(s), \eta\|\tau\|\}$ is a density for κ with respect to σ_r. □

Proof of Proposition 4.1. Clearly,

$$\eta := \frac{1}{2} \inf\{\varphi_r(x,s)\colon (x,s) \in \partial\Omega_r,\ R \le r \le 1,\ -a \le s \le -a'\} > 0. \tag{4.9}$$

Let $\alpha_n \in (0,1)$, $n \in \mathbb{N}$, such that $\alpha_{n+1} < \alpha_n$ and $\lim_{n\to\infty} \alpha_n = 0$. We may choose $a - a' > \gamma_1 > \gamma_2 > \cdots > 0$ such that $\lim_{n\to\infty} \gamma_n = 0$ and, for every $n \in \mathbb{N}$, the statements of Lemma 4.2 hold with α_n and γ_n in place of α and γ'. Let

$$I_{r,n} := I_{(1-\alpha_n)F_r(a)}(a - r\gamma_n), \qquad R \le r \le 1,\ n \in \mathbb{N}.$$

For every $R \le r \le 1$, let

$$I_r := \bigcup\nolimits_{n\in\mathbb{N}} I_{r,n}, \qquad \nu_r' := \varepsilon_{z_0}^{I_r \cup \Omega_r(a)^c}, \qquad \tau_r := 1_{I_r}\nu_r'$$

(see (4.4)). Of course, $\|\tau_r\| \le 1$ and τ_r is the sum of rotationally invariant measures $\tau_{r,n}$ on $I_{r,n}$, $n \in \mathbb{N}$. Hence, by Lemma 4.2, $\tau_r^{\Omega_r(a)^c}|_{\partial\Omega_r}$ admits a density with respect to σ_r which is bounded by η. By (2.2), $\varepsilon_{z_0}^{\Omega_r(a)^c} = \nu_r'|_{\partial\Omega_r} + \tau_r^{\Omega_r(a)^c}$. Since $\varepsilon_{z_0}^{\Omega_r(a)^c} = \varphi_r\sigma_r$ on H_a^+, we conclude that

$$\nu_r' \ge (1/2)\varphi_r\sigma_r \quad \text{ on } \mathbb{R}^d \times (-a, -a'). \tag{4.10}$$

Obviously, the measure ν_r' is supported by $\partial\Omega_r(a) \cup I_r$. However, ν_r' does not charge $S_r(a)$. This is obvious from a probabilistic point of view, since every continuous arc from z_0 to some point in $S_r(a)$ has to intersect I_r. For an analytic proof, let us fix $\delta > 0$, $0 < \rho < F_r(a)$, and let $A := B_\rho \times \{a\}$. There exist $m \in \mathbb{N}$ and $\beta, t \in (0,1)$ such that $\rho + \beta < (1-\alpha_m)F_r(a)$ and $\int_{B_\rho} g_s(y-x)\,dx < \delta$, whenever $y \in \mathbb{R}^d \setminus B_\beta$ and $s \in (0,t)$. We now consider $n \in \mathbb{N}$ such that $n \ge m$ and $\gamma_n r < t$. Then $\varepsilon_z^{H_a}(A) < \delta$ for every point $z = (y, -a + \gamma_n r) \in F := S_r(-a+\gamma_n r) \setminus I_{r,n})$. Hence, by (2.2), $\nu_r'(A) \le \big(\varepsilon_{z_0}^{\Omega_r(-a+\gamma_n r)^c}|_F\big)^{H_a}(A) < \delta$.

By definition of ν_r', $0 < r < R$, and ν' (see (4.3) and (4.2)), it is now easily seen that ν' is absolutely continuous with respect to λ_D. By Lemma 2.1, for every $r \in (0,1)$, $\nu_r' = \varphi_r \sigma_r$ on $L_r(a')$. Therefore $1_{\Omega_1(a')}\nu' = K\lambda_{\Omega_1(a')}$. This and (4.10) imply that $\nu' \ge (1/2)K\lambda_{\Omega_1(a)\setminus\Omega_R}$. $\square$

5 Proof of Theorem 1.1

Let ν and ν' be measures according to Propositions 3.1 and 4.1, respectively. Then $\mu := (1/2)(\nu + \nu')$ is contained in $\mathcal{M}_{z_0}(S_0(\overline{D}))$ and admits a density φ with respect to λ_D such that $\inf \varphi(P_{\beta'}) \ge (1/2)\inf \psi(P_{\beta'}) > 0$ and $\varphi \ge (1/4)K$ on the union of $(D \setminus \Omega_R)$ and $\Omega_1(a')$. Since

$$D = P_{\beta'} \cup (D \setminus \Omega_R) \cup \Omega_1(a'),$$

by (4.1), and $\inf K(D \setminus P_{\beta'}) > 0$, the proof is finished.

6 Counterexample

In contrast to Theorem 1.1 we have the following, if D is only slightly wider at z_0 than a heat ball.

THEOREM 6.1. *Let D be a bounded open subset of $\mathbb{R}^d \times (-\infty, 0)$ and suppose that there exist $t_0, r \in (0,\infty)$ such that*

$$\left\{(x,-t) \in \mathbb{R}^{d+1} \colon 0 < t \le t_0,\ |x| < \frac{d+1}{d}\sqrt{2dt\log(r/t)}\right\} \subset D.$$

Then there is no *$\mu \in \mathcal{M}_{z_0}(S_0(\overline{D}))$ such that $\mu \ge c\lambda_D$, $c \in (0,\infty)$.*

REMARK 6.2. *Let us note that the assumption of Theorem 6.1 is satisfied, if there exist $0 < \beta < 1/2$ and $t_0, c_0 \in (0,\infty)$ such that $(x,-t) \in D$, whenever $t \in (0,t_0)$ and $|x| < c_0 t^\beta$.*

In [15, Corollary 7, part (d)] it is stated that, for a (smaller) class of domains

$$D(\varphi) := \{(x,-t) \colon |x| < \varphi(t), 0 < t < 1\},$$

where $C^{-1}t^\beta \le \varphi(t) \le Ct^\beta$ for some $C > 0$, $0 < \beta < 1/2$, and all $0 < t < t_0$, there is (even) no $\mu \in \mathcal{M}_{z_0}(H_0(\overline{D}))$ admitting a density which is bounded away from zero. In its proof, however, the integral in front of the sum at the bottom of page 95 clearly has to be taken (more restrictively) over all $x \in \mathbb{R}^d$ satisfying $|x| < \varphi(t)$, and then the subsequent estimates seem to break down.

To prove Theorem 6.1 we first observe the following, where $\mathcal{S}^+(\mathbb{R}^{d+1})$ is the set of all positive supertemperatures on $\mathbb{R}^{d+1}$ and $\mathcal{M}_{z_0}(\mathcal{S}^+(\mathbb{R}^{d+1}))$ denotes the set of all positive Radon measures μ on $\mathbb{R}^{d+1}$ such that $\int u\,d\mu \le u(z_0)$ for all $u \in \mathcal{S}^+(\mathbb{R}^{d+1})$. Of course, $\mathcal{M}_{z_0}(S_0(\overline{D})) \subset \mathcal{M}_{z_0}(\mathcal{S}^+(\mathbb{R}^{d+1}))$ for every bounded open set D in $\mathbb{R}^{d+1}$.

LEMMA 6.3. *For all $t>0$ and $\mu \in \mathcal{M}_{z_0}(S^+(\mathbb{R}^{d+1}))$, $1_{H_t}\mu \le g_t\lambda_t$.*

Proof. It suffices to prove this for each extremal measure in $\mathcal{M}_{z_0}(\mathcal{S}^+(\mathbb{R}^{d+1}))$, that is, for $\mu = \varepsilon_{z_0}^A$, where A is a finely closed Borel set in $\mathbb{R}^{d+1}$ (see [5, III.6.10,VI.12.5, VI.4.6]). By (2.2), $\mu = \varepsilon_{z_0}^{A\cup H_t}|_A + \big(\varepsilon_{z_0}^{A\cup H_t}|_{H_t\setminus A}\big)^A$. Since $\varepsilon_z^A(H_t)=0$, for every $z \in H_t \setminus A$, we see that $1_{H_t}\mu = 1_{A\cap H_t}\varepsilon_{z_0}^{A\cup H_t} \le 1_{H_t}\varepsilon_{z_0}^{A\cup H_t}$. On the other hand, by (2.5) and (2.2), $g_t\lambda_t = \varepsilon_{z_0}^{H_t} = 1_{H_t}\varepsilon_{z_0}^{A\cup H_t} + \big(1_{A\setminus H_t}\varepsilon_{z_0}^{A\cup H_t}\big)^{H_t} \ge 1_{H_t}\varepsilon_{z_0}^{A\cup H_t}$, finishing the proof. □

Defining

$$\gamma := \frac{d+1}{d} \quad \text{and} \quad \varphi(t) := \gamma F_1(t) = \gamma\sqrt{2dt\log(1/t)}, \qquad 0<t\le 1,$$

we also need yet another consequence of Lemma 2.2.

LEMMA 6.4. *Given $\delta>0$, there exists $t_1 \in (0,1)$ such that $\varepsilon_{z_0}^{Z_{\delta\varphi(t)}(u)^c}(H_u) > 1/2$, for all $0<t\le t_1$ and $0<u\le \delta^2 t$.*

Proof. Let $t\in(0,1)$, $0<u\le\delta^2 t$, $\rho := \delta\varphi(t)$, and $V := Z_\rho(u)$. Since $\|\varepsilon_{z_0}^{V^c}\| = 1$ and $\varepsilon_{z_0}^{V^c}$ is supported by the subset ∂V of $H_u \cup H_u^+$, it suffices to know that $\varepsilon_{z_0}^{V^c}(H_u^+) < 1/2$.

If t is sufficiently small, then

$$\frac{\rho^2}{6d} = \frac{2d(\delta\gamma)^2 t\log(1/t)}{6d} > \delta^2 t,$$

and hence (see the lines following (2.6)),

$$uc_\rho(u) \le \delta^2 t\sqrt{\frac{d}{4\pi}}\,\rho\,(\delta^2 t)^{-3/2}\exp\big(-\frac{\rho^2}{4d\delta^2 t}\big),$$

where

$$\frac{\rho^2}{4d\delta^2 t} = \frac{(\delta\gamma)^2 2dt\log(1/t)}{4d\delta^2 t} > \frac{1}{2}\log(1/t), \qquad \exp\big(-\frac{\rho^2}{4d\delta^2 t}\big) < t^{1/2},$$

and thus $uc_\rho(u) < \sqrt{d}\,\varphi(t)$. Since $\lim_{t\to 0}\varphi(t) = 0$, the proof is finished by Lemma 2.2. □

Proof of Theorem 6.1. By scaling invariance, it suffices to consider the case $r=1$. Let $\mu \in \mathcal{M}_{z_0}(S_0(\overline{D}))$ and $c\in[0,1]$ with $\mu \ge c\lambda_D$. We fix $0<\delta<1/5$ such that

$$b := (1-5\delta)^2\gamma^2 > 1 + \frac{2}{d}. \tag{6.1}$$

We choose t_1 according to Lemma 6.4, take $0<t\le\min\{t_0,t_1,1/e\}$, and define

$$x := ((1-3\delta)\varphi(t),0,\dots,0) \in \mathbb{R}^d, \qquad B := x + B_{2\delta\varphi(t)},$$

$$Z := B \times (-t, -t+\delta t), \quad \text{and} \quad S := \overline{B} \times \{-t\} = \partial Z \cap H_t.$$

If $(y,s) \in Z$, then $|y| < (1-\delta)\varphi(t) \le \varphi((1-\delta)t) \le \varphi(|s|)$, and hence $Z \subset D$. We next define

$$U := (x + B_{\delta\varphi(t)}) \times (-t, -t+\delta^2 t), \qquad \nu := c\lambda_U, \qquad \mu' := (\mu - \nu) + \nu^{Z^c}.$$

Then $U \subset Z$ and $\mu' \in \mathcal{M}_{z_0}(S_0(\overline{D}))$. So $1_{H_t}\mu' \le g_t\lambda_t$, by Lemma 6.3, and hence

$$1_S \nu^{Z^c} \le g_t \lambda_t. \tag{6.2}$$

For the moment, let us fix $z = (y,s) \in U$ and consider $V := (y + B_{\delta\varphi(t)}) \times (-t, s)$. By Lemma 6.4 and translation invariance, $\varepsilon_z^{V^c}(S) = \varepsilon_z^{V^c}(H_t) > 1/2$. Since $V \subset Z$, we know that $\varepsilon_z^{Z^c}(S) \ge \varepsilon_z^{V^c}(S)$, by (2.2).

Integrating with respect to ν we hence conclude that

$$\nu^{Z^c}(S) > \frac{c}{2}\|\lambda_U\| = \frac{\omega_d c}{2}(\delta\varphi(t))^d\,\delta^2 t, \tag{6.3}$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. On the other hand,

$$\int_S g_t\,d\lambda_t \le \sup g_t(S)\cdot\lambda_t(S) \le (4\pi t)^{-d/2}\exp\Big(-\frac{((1-5\delta)\varphi(t))^2}{4t}\Big)\cdot\omega_d(2\delta\varphi(t))^d,$$

where

$$\frac{((1-5\delta)\varphi(t))^2}{4t} = \frac{b\cdot 2dt\log(1/t)}{4t} = \frac{bd}{2}\log(1/t),$$

and hence

$$\exp\Big(-\frac{((1-5\delta)\varphi(t))^2}{4t}\Big) = t^{bd/2}.$$

Having (6.2) and (6.3) we therefore obtain that

$$\frac{c}{2}\,\delta^2 t < (\pi t)^{-d/2} t^{bd/2} = \pi^{-d/2} t^{(b-1)d/2}. \tag{6.4}$$

By (6.1), $(b-1)d/2 > 1$. Thus (6.4) can only hold for arbitrarily small $t > 0$, if $c = 0$. □

A special case of the following result is known for harmonic spaces (see [10]; for classical potential theory and $d = 2$, see [12]). The proof shows that the lemma holds for general balayage spaces.

LEMMA 6.5. *Let A be a Borel set in $\mathbb{R}^{d+1}$ which is not thin at a point $z \in \mathbb{R}^{d+1}$. Then there exists a compact set L in $\mathbb{R}^{d+1}$ such that $L \setminus \{z\} \subset A$ and L is not thin at z.*

Proof. Let p be a continuous strict potential on $\mathbb{R}^{d+1}$ and let V_n, $n \in \mathbb{N}$, be open neighborhoods of z such that $\overline{V}_{n+1} \subset V_n$ and $\bigcap_{n\in\mathbb{N}} V_n = \{z\}$. Then the sets $A \cap V_n$, $n \in \mathbb{N}$, are not thin at z, and hence, by [5, VI.1.9], there exist compact sets L_n in $A \cap V_n$ such that $\hat{R}_p^{L_n}(z) > p(z) - 1/n$. Clearly, the set $L := \{z\} \cup \bigcup_{n\in\mathbb{N}} L_n$ is compact, $L \setminus \{z\} \subset A$, and $\hat{R}_p^L(z) = p(z)$, that is, L is not thin at z. □

The next result is usually proved using Choquet's theory (see [12, 9, 10, 6, 7, 4]).

PROPOSITION 6.6. *Let D be a bounded open set in $\mathbb{R}^{d+1}$ and z a stable point for $\overline{D}$, that is, that $\overline{D}^c$ is not thin at z. Then $\mathcal{M}_z(H_0(\overline{D})) = \{\varepsilon_z\}$.*

Proof. Let $\mu \in \mathcal{M}_z(H_0(\overline{D}))$. Of course, μ is a probability measure, since the constant function 1 is a temperature on $\mathbb{R}^{d+1}$. By Lemma 6.5, there exists a compact L in $\mathbb{R}^{d+1}$ such that $\overline{D} \cap (L \setminus \{z\}) = \emptyset$ and L is not thin at z. For every $n \in \mathbb{N}$, let $V_n := \{z' \in \mathbb{R}^{d+1} \colon |z' - z| < 1/n\}$. Let L' be a compact in $\mathbb{R}^{d+1} \setminus \{z\}$ and $\delta > 0$.

Since z is polar, there exists $m \in \mathbb{N}$ such that $R_1^{V_m} < \delta$ on L'. Since $(L \setminus \{z\}) \cap V_m$ is not thin at z, there exists $n > m$ such that the function $u := R_1^{L \cap (V_m \setminus V_n)}$ satisfies $u(z) > 1 - \delta$. Of course, $u \leq 1$ on $\mathbb{R}^{d+1}$ and $u \leq R_1^{V_m} < \delta$ on L'. Moreover, u is a temperature on the open neighborhood $(\mathbb{R}^{d+1} \setminus L) \cup V_n$ of $\overline{D}$. Therefore

$$1 - \delta < u(z) = \int u \, d\mu \leq (1 - \mu(L')) + \delta\mu(L'),$$

that is, $(1 - \delta)\mu(L') < \delta$. Since this holds for every $\delta > 0$, we see that $\mu(L') = 0$. So μ does not charge the complement of $\{z\}$, and hence $\mu = \varepsilon_z$. □

References

[1] H. Aikawa. Densities with the mean value property for harmonic functions in a Lipschitz domain. *Proc. Amer. Math. Soc.*, 125(1):229–234, 1997.

[2] H. Bauer. *Harmonische Räume und ihre Potentialtheorie.* Lecture Notes in Mathematics 22. Springer, Berlin, 1966.

[3] H. Bauer. Heat balls and Fulks measures. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 10:67–82, 1985.

[4] J. Bliedtner and W. Hansen. Simplicial cones in potential theory. *Invent. Math.*, 29(2):83–110, 1975.

[5] J. Bliedtner and W. Hansen. *Potential Theory – An Analytic and Probabilistic Approach to Balayage.* Universitext. Springer, Berlin-Heidelberg-New York-Tokyo, 1986.

[6] A. de la Pradelle. Approximation des fonctions harmoniques à l'aide d'un théorème de G. F. Vincent-Smith. In *Séminaire de Théorie du Potentiel, dirigé par M. Brelot, G. Choquet et J. Deny (1969/70), Exp. 3*, page 16. Secrétariat Mathématique, Paris, 1971.

[7] M. Brelot. *On topologies and boundaries in potential theory.* Enlarged edition of a course of lectures delivered in 1966. Lecture Notes in Mathematics, Vol. 175. Springer-Verlag, Berlin, 1971.

[8] J.L. Doob. *Classical potential theory and its probabilistic counterpart.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1984 edition.

[9] D.A. Edwards. Minimum-stable wedges of semicontinuous functions. *Math. Scand.*, 19:15–26, 1966.

[10] E.G. Effros and J.L. Kazdan. Applications of Choquet simplexes to elliptic and parabolic boundary value problems. *J. Differential Equations*, 8:95–134, 1970.

[11] W. Fulks. A mean value theorem for the heat equation. *Proc. Amer. Math. Soc.*, 17:6–11, 1966.

[12] T.W. Gamelin and H. Rossi. Jensen measures and algebras of analytic functions. In *Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965)*, pages 15–35. Scott-Foresman, Chicago, Ill., 1966.

[13] W. Hansen and I. Netuka. Volume densities with the mean value property for harmonic functions. *Proc. Amer. Math. Soc.*, 123(1):135–140, 1995.

[14] I. Netuka. Fine behaviour of solutions of the Dirichlet problem near an irregular point. *Bull. Sci. Math.*, 114(1):1–22, 1990.

[15] N. Suzuki and N. A. Watson. Mean value densities for temperatures. *Colloq. Math.*, 98(1):87–96, 2003.

[16] G.F. Vincent-Smith. Uniform approximation of harmonic functions. *Ann. Inst. Fourier (Grenoble)*, 19(fasc. 2):339–353 (1970), 1969.

[17] N. A. Watson. *Introduction to heat potential theory*, volume 182 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2012.

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