Besov-Type and Triebel–Lizorkin-Type Spaces Associated with Heat Kernels

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Abstract. Let (M, ρ, μ) be an RD-space satisfying the non-collapsing condition. In this paper, the authors introduce Besov-type spaces $B^{s,\tau}_{p,q}(M)$ and Triebel–Lizorkin-type spaces $F^{s,\tau}_{p,q}(M)$ associated to a non-negative self-adjoint operator L whose heat kernels satisfy some Gaussian upper bound estimate, Hölder continuity, and the stochastic completeness property. Characterizations of these spaces via Peetre maximal functions and heat kernels are established for full range of indices. Also, frame characterizations of these spaces are given. When L is the Laplacian operator on $\mathbb{R}^n$, these spaces coincide with the Besov-type and Triebel-Lizorkin-type spaces on $\mathbb{R}^n$ studied in [Lecture Notes in Mathematics 2005, Springer-Verlag, Berlin, 2010]. In the case $\tau = 0$ and the smoothness index s is around zero, comparisons of these spaces with the Besov and Triebel–Lizorkin spaces studied in [Abstr. Appl. Anal. 2008, Art. ID 893409, 250 pp] are also presented.

1 Introduction

The tremendous development of theories of function spaces in the last few decades has resulted in extraordinary accomplishments in several fields of mathematics such as potential theory, partial differential equations, approximation theory and so on. Besov and Triebel–Lizorkin spaces, known so far, are very general scales of functions spaces. They cover various types of function spaces such as Lebesgue spaces, Sobolev spaces, Hardy spaces and BMO (see, for example, [36, 37, 30, 13, 14]). In recent years, due to the applications in partial differential equations such as Navier–Stokes equations, the scale of Besov and Triebel–Lizorkin spaces was further extended to Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces, via replacing the Lebesgue norm in the definition of Besov–Triebel–Lizorkin spaces by the Morrey norm (see, for example, [1, 23, 28, 35, 31, 33, 34]). The classical Morrey spaces and many Morrey-type spaces, like Hardy–Morrey spaces and Sobolev–Morrey spaces, are proved to belong to this scale. Another scale of generalized Besov and Triebel–Lizorkin spaces is the Besov-type and Triebel–Lizorkin-type spaces introduced in [41, 42, 45], which cover Besov and Triebel–Lizorkin spaces, Triebel–Lizorkin–Morrey spaces, and the Q_α spaces (see [16, 15, 38, 39]) as special cases. For more properties on these generalized Besov and Triebel-Lizorkin spaces and their applications in partial differential equations, we also refer to [30, 40, 32, 44, 26, 27, 25, 24] and, especially, to excellent two surveys [33, 34] by Sickel for many unsolved questions on this subject.

In recent years, there is an increasing interest in functions spaces related to operators. It is known that Riesz transforms defined via a general operator L may not be bounded on the classical Hardy spaces. To solve such problems, Auscher, Duong and McIntosh [2] made some prominent contributions, which include a theory of Hardy spaces associated with a general operator L whose heat kernels satisfy pointwise Possion upper bounds; see also Duong and Yan [10, 11, 12]. On metric measure spaces whose measures satisfy a polynomial growth condition, Bui, Duong and

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Yan in [5] studied homogeneous Besov space $\dot{B}^s_{p,q}$ for $|s|<1$ and $p,q\in[1,\infty]$ associated with an operator L whose heat kernels satisfy the upper bound Gauss estimate and the Hölder continuity. A recent work on the theory of Besov and Triebel–Lizorkin spaces associated to an operator L, with full range of the indices s,p and q, is due to Kerkyacharian and Petrushev [22].

Let (M,ρ) be a locally compact complete metric space with a metric ρ. Suppose that μ is a positive regular Borel measure such that the following *doubling condition* holds: there exists a positive constant K such that, for all $x\in M$ and $r\in(0,\infty)$,

$$\mu(B(x,2r))\le K\mu(B(x,r)). \tag{1.1}$$

Then, the triple (M,ρ,μ) is called a *space of homogeneous type* in the sense of Coifman and Weiss [7, 8]. Notice that (1.1) implies that, for all $x\in M$, $r\in(0,\infty)$ and $\lambda\in(1,\infty)$,

$$\mu(B(x,\lambda r))\le K\lambda^d\mu(B(x,r)), \tag{1.2}$$

where $d:=\log_2 K>0$ is the "dimension" constant. Also, we assume that the *reverse doubling condition* is valid: there exists a constant $K_*\in(1,\infty)$ such that, for all $x\in M$ and $0<r\le\frac{\operatorname{diam} M}{3}$,

$$\mu(B(x,2r))\ge K_*\mu(B(x,r)). \tag{1.3}$$

Notice that (1.3) implies that, for all $x\in M$, $\lambda\in[1,\infty)$ and $0<r\le\frac{2\operatorname{diam} M}{3\lambda}$,

$$\mu(B(x,\lambda r))\ge K_*^{-2}\lambda^\kappa\mu(B(x,r)), \tag{1.4}$$

where $\kappa:=\log_2 K_*>0$ also measures the "dimension" of (M,ρ,μ). The doubling condition and the reverse doubling condition make (M,ρ,μ) an RD-space introduced in [20, 21]. Moreover, we require the following *non-collapsing condition*: there exists a positive constant c such that

$$\inf_{x\in M}\mu(B(x,1))\ge c. \tag{1.5}$$

Further, let L be a non-negative self-adjoint operator on $L^2(\mu)$, with domain $D(L)$ dense in $L^2(M)$. Then, L has spectral resolution $\{E_\lambda\}_{\lambda\ge0}$. Given a bounded Borel measurable function f, the theory of functional calculus implies that

$$f(\sqrt{L})=\int_0^\infty f(\sqrt{\lambda})\,dE_\lambda.$$

Assume that the associated heat semigroup $\{e^{-tL}\}_{t>0}$ consists of integral operators with heat kernels $\{p_t\}_{t>0}$ and there exist positive constants c^* and C^* such that $\{p_t\}_{t>0}$ satisfy the following conditions:

(UE) Gaussian upper bound estimate: for all $t\in(0,1]$ and $x,y\in M$,

$$|p_t(x,y)|\le C^*\frac{\exp\left(-c^*\frac{[\rho(x,y)]^2}{t}\right)}{\sqrt{\mu(B(x,\sqrt{t}))\,\mu(B(y,\sqrt{t}))}}. \tag{1.6}$$

(HE) Hölder continuity: there exists a positive constant α_0 such that, for all $t\in(0,1]$ and $x,y,y'\in M$ satisfying that $\rho(y,y')\le\sqrt{t}$,

$$|p_t(x,y)-p_t(x,y')|\le C^*\left[\frac{\rho(y,y')}{\sqrt{t}}\right]^{\alpha_0}\frac{\exp\left(-c^*\frac{[\rho(x,y)]^2}{t}\right)}{\sqrt{\mu(B(x,\sqrt{t}))\,\mu(B(y,\sqrt{t}))}}. \tag{1.7}$$

(SC) Stochastic Completeness: for all $t\in(0,\infty)$ and $x\in M$,

$$\int_M p_t(x,y)\,d\mu(y)=1. \tag{1.8}$$

An interesting example that fits the above frame work arises from the operator $L:=-\mathrm{div}(A\nabla)$, where $A:=\{a_{i,j}(x)\}_{1\le i,j\le d}$ is a uniformly elliptic symmetric matrix-valued function on $\mathbb{R}^d$ so that the heat kernels satisfy **(UE)**, **(HE)** and **(SC)**. For more examples which satisfy the above framework, we refer the reader to [22, 9]. With the previous assumptions, Besov and Triebel-Lizorkin spaces associated with the heat kernels were studied in [22]. Let $\mathbb{R}_+:=[0,\infty)$. Assume that there exists a positive constant c such that $\Phi_0,\Phi\in C^\infty(\mathbb{R}_+)$ satisfy

$$\operatorname{supp}\Phi_0\subset[0,2],\quad \Phi_0^{(2\nu+1)}(0)=1\,\text{for all}\ \nu\ge 0,\qquad |\Phi_0(\lambda)|\ge c\ \ \text{for}\ \lambda\in[0,2^{-3/4}], \tag{1.9}$$

and

$$\operatorname{supp}\Phi\subset[2^{-1},2],\qquad |\Phi(\lambda)|\ge c\ \ \text{for}\ \lambda\in[2^{-3/4},2^{3/4}]. \tag{1.10}$$

For all $j\in\mathbb{N}$, let $\Phi_j(\cdot):=\Phi(2^{-j}\cdot)$. For $s\in\mathbb{R}$ and $p,q\in(0,\infty]$, the *Besov space* $B^s_{p,q}(M)$ is defined to be the collection of all distributions f such that

$$\|f\|_{B^s_{p,q}(M)}:=\left\{\sum_{j=0}^\infty\left\|2^{js}\Phi_j(\sqrt{L})f\right\|^q_{L^p(M)}\right\}^{1/q}<\infty; \tag{1.11}$$

when $p\in(0,\infty)$, the *Triebel–Lizorkin space* $F^s_{p,q}(M)$ is defined to be the collection of all distributions f such that

$$\|f\|_{F^s_{p,q}(M)}:=\left\|\left\{\sum_{j=0}^\infty|2^{js}\Phi_j(\sqrt{L})f|^q\right\}^{1/q}\right\|_{L^p(M)}<\infty. \tag{1.12}$$

Similarly, the spaces $\widetilde{B}^s_{p,q}(M)$ and $\widetilde{F}^s_{p,q}(M)$ are defined with 2^{js} replaced by $[\mu(B(\cdot,2^{-j}))]^{-s/d}$ in the above two (quasi-)norms. One may also equivalently define these Besov and Triebel–Lizorkin spaces by replacing the dilation 2 with a general number $b\in(1,\infty)$ in the definition of $\{\Phi_j\}_{j=1}^\infty$; see [22, Propositions 6.3 and 7.2]. Frame decompositions of these spaces are considered in [22] by using the Calderón reproducing formula.

Inspired by [22] and [41, 42, 45], the main aim of this article is to develop the Besov-type and Triebel–Lizorkin-type spaces on the metric space (M,ρ,μ) satisfying (1.1), (1.3) and (1.5), associated to a non-negative self-adjoint operator L whose heat kernels $\{p_t\}_{t>0}$ satisfy the above stochastic completeness property **(SC)** and the following conditions **(UE′)** and **(HE′)** with positive constants C^*, c^*, $\alpha_0\in(0,\infty)$ and $\beta_0\in(1,\infty)$:

(UE′) Upper bound estimate: for all $t\in(0,1]$ and $x,y\in M$,

$$|p_t(x,y)|\le C^*\frac{\exp\left(-c^*\left[\frac{\rho(x,y)}{t^{1/\beta_0}}\right]^{\frac{\beta_0}{\beta_0-1}}\right)}{\sqrt{\mu(B(x,t^{1/\beta_0}))\,\mu(B(y,t^{1/\beta_0}))}}. \tag{1.13}$$

(HE′) Hölder continuity estimate: for all $t\in(0,1]$ and $x,y,y'\in M$ satisfying that $\rho(y,y')\le t^{1/\beta_0}$,

$$|p_t(x,y)-p_t(x,y')|\le C^*\left[\frac{\rho(y,y')}{t^{1/\beta_0}}\right]^{\alpha_0}\frac{\exp\left(-c^*\left[\frac{\rho(x,y)}{t^{1/\beta_0}}\right]^{\frac{\beta_0}{\beta_0-1}}\right)}{\sqrt{\mu(B(x,t^{1/\beta_0}))\,\mu(B(y,t^{1/\beta_0}))}}. \tag{1.14}$$

Different from the conditions (**UE**) and (**HE**) used in [22], in this article, we consider more general conditions ($\mathbf{UE}'$) and ($\mathbf{HE}'$), which were suggested by Professor Alexander Grigor'yan to us. Obviously, if $\beta_0 = 2$, then the conditions ($\mathbf{UE}'$) and ($\mathbf{HE}'$) go back to (**UE**) and (**HE**), respectively. It is known that a large family of various fractals (e. g., Sierpinski gasket) has a natural diffusion process on these general conditions, and this diffusion process has a transition density p_t with respect to a proper Hausdorff measure μ of the fractals such that

$$p_t(x,y) \asymp \frac{C^*}{t^{d/\beta}} \exp\left(-c_* \left[\frac{\rho(x,y)}{t^{1/\beta}}\right]^{\frac{\beta}{\beta-1}}\right), \tag{1.15}$$

where d is the Hausdorff dimension of the fractals and β the walk dimension which is larger than 2 in many interesting examples; see [18] for a brief introduction. Here the notation $\asymp$ in (1.15) means that both $\le$ and $\ge$ can be used, but the positive constants C^* and c_* may be different in upper and lower bounds. With all metric balls are precompact as a priori, if the heat kernel p_t satisfies (1.15), then it satisfies ($\mathbf{HE}'$); see [3, Theorem 3.1 and Corollary 4.2] and [19, Theorem 7.4]. The topic of heat kernels has been studied intensively in lots of works; see, for example, [3, 18, 19] and the references therein.

In this article, we stick to the philosophy used in [4, 13, 36, 45] (see also [33, 34]) to study the Besov-type and Triebel-Lizorkin-type spaces on the metric measure spaces (M, ρ, μ) which is associated to the heat kernels $\{p_t\}_{t>0}$. Though we are assuming conditions ($\mathbf{UE}'$) and ($\mathbf{HE}'$) in this article, one of the important things is that the smooth functional calculus induced by heat kernels still have fast decay at infinity. This is presented in Proposition 3.4 below, which is a generalized version of [22, Theorem 3.1]. Such smooth functional calculus plays a role of the Schwartz functions as in the Euclidean space. One may also notice this issue from the off-diagonal estimate in Proposition 3.7 below. Furthermore, as in [22], the continuous Calderón reproducing formula (see Section 3.2) keeps valid, which can be used to establish the Peetre maximal function characterizations and heat semigroup characterizations of these spaces (see Section 5). Applying the technique of [22], we build a new discrete Calderón reproducing formula (see Theorem 6.1 below) that is much more parallel to the one used in [4, 13, 36, 45]. Consequently, in Section 6, frames decompositions of such Besov-type and Triebel-Lizorkin-type spaces are considered. The framework we build in this article generalizes the work of [45] (see also [33, 34]) to metric measure spaces, and also generalizes the work of [22] to more general scale of functions spaces.

This article is organized as follows. In Section 2, we introduce the Besov-type spaces $B^{s,\tau}_{p,q}(M)$, $\widetilde{B}^{s,\tau}_{p,q}(M)$, and the Triebel–Lizorkin-type spaces $F^{s,\tau}_{p,q}(M)$, $\widetilde{F}^{s,\tau}_{p,q}(M)$, where $\tau \in (0,\infty)$, $s \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty)$, and q can take ∞ for the spaces $B^{s,\tau}_{p,q}(M)$ and $\widetilde{B}^{s,\tau}_{p,q}(M)$. When $\tau = 0$, these spaces are actually the Besov and Triebel–Lizorkin spaces introduced in [22].

Section 3 is devoted to some auxiliary estimates. In Section 3.1, we present some basic estimates which hold on any metric measure spaces (M, ρ, μ). Then, Section 3.2 gives some estimates related to the smooth functional calculus induced by the heat kernel, including an off-diagonal estimate (see Proposition 3.7 below). The continuous Calderón reproducing formula is given at the end of Section 3.2. Applying these basic estimates and the reproducing formula, we control the Peetre maximal functions (see Proposition 3.11) by the Hardy-Littlewood maximal function, which is a generalized version of [22, Lemma 6.4] and is used elsewhere in this article. It should be remarked that, different from the one in [22, Lemma 6.4], which holds true only on the spectral space, the estimate in Proposition 3.11 holds true for more general distributions.

Using the estimates from Section 3, we in Section 4 prove some embedding properties of the Besov-type and Triebel–Lizorkin-type spaces, and then classify these spaces for the index τ in different ranges.

As an application of the estimates of the Peetre maximal functions and the continuous Calderón reproducing formula, in Section 5, we characterize the Besov-type and Triebel–Lizorkin-type spaces via the Peetre maximal functions (see Theorem 5.2), which also indicates that these spaces are well defined. By this Peetre maximal function characterization, we further establish the heat semigroup characterization of the Besov-type and Triebel–Lizorkin-type spaces in both discrete and continuous versions (see Theorems 5.8 and 5.9). Comparing with the continuous heat semigroup characterization for Besov and Triebel-Lizorkin spaces in [22, Theorems 6.7 and 7.5], wherein $p \in [1,\infty]$, there is no restriction on p in the discrete heat semigroup characterization of the Besov-type and Triebel–Lizorkin-type spaces.

Section 6 is devoted to the frame characterization of these new scales of function spaces. The frame structure we considered here relies on Christ's dyadic cubes in M, which is different from those in [22], and hence we need to establish a new discrete Calderón reproducing formula associated with Christ's dyadic cubes and the functions Φ_0 and Φ in (1.9) and (1.10) (see Theorem 6.1, whose proof is presented in Section 8). As an application, we show that $F^{s,1/p}_{p,q}(M)$ and $\widetilde{F}^{s,1/p}_{p,q}(M)$ are indeed the endpoint case $F^s_{\infty,q}(M)$ and $\widetilde{F}^s_{\infty,q}(M)$ of the Triebel–Lizorkin spaces, where $p \in (0,\infty)$, $q \in (0,\infty]$ and $s \in \mathbb{R}$.

Finally, in Subsection 7.1, we prove that, when (M,ρ,μ) is the Euclidean space and L the Laplacian operator, the Besov-type and Triebel–Lizorkin-type spaces defined in this article coincide with those spaces introduced by Yuan, Sickel and Yang [45], by using their heat semigroup characterizations. Hence, the article here generalized the work of [45]. Further, in Subsection 7.2, when $\tau = 0$ and $\beta_0 = 2$ in (**UE**$'$) and (**HE**$'$), we show that the (quasi-)norms of the Besov and Triebel–Lizorkin spaces on RD-spaces satisfying (1.5) defined in [21] coincide exactly with those in [22] when s is around zero, which answers a question presented in [22]. The proof of this coincidence needs the discrete Calderón reproducing formula obtained in Section 6 and the corresponding one on RD-spaces obtained in [21].

In this article, we use the following notation. Let $\mathbb{N} := \{1,2,\dots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\mathbb{R}_+ := [0,\infty)$. For any numbers $s,t \in \mathbb{R}$, let

$$s \vee t := \max\{s,t\} \quad \text{and} \quad s \wedge t := \min\{s,t\}.$$

If an operator T is bounded from a (quasi)-Banach space $\mathcal{X}$ to a (quasi)-Banach space $\mathcal{Y}$, then we denote by $\|T\|_{\mathcal{X}\to\mathcal{Y}}$ its operator norm. For notational convenience, we let $|E| := \mu(E)$ for any measurable set $E \subset M$. By C we denote a positive constant, independent of the main parameters, whose value may be different on each occasion. Further, $A \lesssim B$ means $A \le CB$ and, similarly, for $A \gtrsim B$. If $B \lesssim A \lesssim B$, then write $A \sim B$. We use $C_{\alpha,\beta,\dots}$ to denote a positive constant depending on $\alpha,\beta,\dots$.

2 Besov-type and Triebel–Lizorkin-type spaces

From now on, we let L be a non-negative self-adjoint operator on $L^2(M)$ such that the associated heat kernels exist and satisfy (**UE**$'$), (**HE**$'$) and (**SC**). When $\mu(M) < \infty$, we use $\mathcal{D}(M)$ to denote the *test function space* which consists of all functions $\phi \in \cap_m D(L^m)$ with topology induced by

$$\mathscr{P}_m(\phi) := \|L^m\phi\|_{L^2(M)}, \qquad m \in \mathbb{Z}_+.$$

Let x_0 be some fixed point in M. When $\mu(M)=\infty$, the *test function space* $\mathcal{D}(M)$ consists of all functions $\phi\in\cap_m D(L^m)$ such that, for all $m,\ \ell\in\mathbb{Z}_+$,

$$\mathscr{P}_{m,\ell}(\phi):=\sup_{x\in M}[1+\rho(x,x_0)]^{\ell}|L^m\phi(x)|<\infty.$$

No matter $\mu(M)$ is finite or not, the distribution space $\mathcal{D}'(M)$ is defined by the set of all continuous linear functionals on $\mathcal{D}(M)$ and the pairing between $f\in\mathcal{D}'(M)$ and $\phi\in\mathcal{D}(M)$ is denoted by $\langle f,\phi\rangle:=f(\bar{\phi})$. See [22] for more details on this aspect.

Before introducing the Besov-type and Triebel–Lizorkin-type spaces associated with the heat kernels of the aforementioned operator L, we recall Christ's dyadic cube construction (see [6]) on the space of homogeneous type. Such Christ dyadic cubes retain most of the properties of the dyadic cubes in the Euclidean space.

Lemma 2.1. *Let (M,ρ,μ) be a space of homogeneous type. Then, there exist a collection $\mathcal{Q}:=\{Q_\alpha^j\subset M:\ j\in\mathbb{Z},\ \alpha\in I_j\}$ of open subsets, where I_j is some index set, and positive constants $\delta\in(0,1)$ and $C_\natural>c_\natural$ such that*

(i) *for each fixed $j\in\mathbb{Z}$, $\mu(M\setminus\bigcup_\alpha Q_\alpha^j)=0$ and $Q_\alpha^j\cap Q_\beta^j=\emptyset$ if $\alpha\neq\beta$;*

(ii) *for any α, β, j, ℓ with $\ell\geq j$, either $Q_\beta^\ell\subset Q_\alpha^j$ or $Q_\beta^\ell\cap Q_\alpha^j=\emptyset$;*

(iii) *for each (j,α) and each $\ell<j$, there is a unique β such that $Q_\alpha^j\subset Q_\beta^\ell$;*

(iv) $\operatorname{diam}Q_\alpha^j\leq C_\natural\delta^j$ *and each Q_α^j contains some ball $B(z_\alpha^j,c_\natural\delta^j)$, where $z_\alpha^j\in M$.*

In what follows, we always use δ to denote the constant appearing in Lemma 2.1. Then, any Christ dyadic cube Q_α^k with $k\in\mathbb{Z}$ and $\alpha\in I_k$ has diameter roughly δ^k.

Definition 2.2. Let $s\in\mathbb{R}$, $\tau\in[0,\infty)$ and $q\in(0,\infty]$. Let $\delta\in(0,1)$ be as in Lemma 2.1. Assume that there exists a positive constant c such that the pair of functions (Φ_0,Φ) in $C^\infty(\mathbb{R}_+)$ satisfies that

$$\operatorname{supp}\Phi_0\subset[0,\delta^{-\beta_0/2}],\quad \Phi_0^{(2\nu+1)}(0)=1\ \text{ for all }\ \nu\in\mathbb{Z}_+,\quad |\Phi_0(\lambda)|\geq c\ \text{ for }\lambda\in[0,\delta^{-3\beta_0/8}]\tag{2.1}$$

and

$$\operatorname{supp}\Phi\subset[\delta^{\beta_0/2},\delta^{-\beta_0/2}],\qquad |\Phi(\lambda)|\geq c\ \text{ for }\lambda\in[\delta^{3\beta_0/8},\delta^{-3\beta_0/8}],\tag{2.2}$$

where β_0 is the constant appearing in $(\mathbf{UE}')$ and $(\mathbf{HE}')$. For all $j\in\mathbb{N}$ and $\lambda\in\mathbb{R}_+$, let

$$\Phi_j(\lambda):=\Phi(\delta^{j\beta_0/2}\lambda).\tag{2.3}$$

(i) For $p\in(0,\infty]$, the *Besov-type space* $B_{p,q}^{s,\tau}(M)$ is defined to be the collection of all $f\in\mathcal{D}'(M)$ such that

$$\|f\|_{B_{p,q}^{s,\tau}(M)}:=\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q_\alpha^k|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\left[\int_{Q_\alpha^k}|\delta^{-js}\Phi_j(\sqrt{L})f(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q}<\infty,$$

with a usual modification made when $p=\infty$ or $q=\infty$. The *Besov-type space* $\widetilde{B}_{p,q}^{s,\tau}(M)$ is defined to be the collection of all $f\in\mathcal{D}'(M)$ such that

$$\|f\|_{\widetilde{B}_{p,q}^{s,\tau}(M)}:=\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q_\alpha^k|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\left[\int_{Q_\alpha^k}|B(\cdot,\delta^j)|^{-sp/d}|\Phi_j(\sqrt{L})f(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q}<\infty,$$

with a usual modification made when $p=\infty$ or $q=\infty$.

(ii) For $p\in(0,\infty)$, the *Triebel–Lizorkin-type space* $F^{s,\tau}_{p,q}(M)$ is defined to be the collection of all $f\in\mathcal{D}'(M)$ such that

$$\|f\|_{F^{s,\tau}_{p,q}(M)}:=\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=k\vee 0}^{\infty}|\delta^{-js}\Phi_j(\sqrt{L})f(x)|^q\right]^{p/q}d\mu(x)\right\}^{1/p}<\infty,$$

with a usual modification made when $q=\infty$. The *Triebel–Lizorkin-type space* $\widetilde{F}^{s,\tau}_{p,q}(M)$ is defined to be the collection of all $f\in\mathcal{D}'(M)$ such that

$$\|f\|_{\widetilde{F}^{s,\tau}_{p,q}(M)}:=\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=k\vee 0}^{\infty}|B(\cdot,\delta^j)|^{-sq/d}|\Phi_j(\sqrt{L})f(x)|^q\right]^{p/q}d\mu(x)\right\}^{1/p}<\infty,$$

with a usual modification made when $q=\infty$.

Remark 2.3. (i) For any Φ_j in Definition 2.2, the operator $\Phi_j(\sqrt{L})$ has an integral kernel, denoted by $\Phi_j(\sqrt{L})(x,y)$, which belongs to $\mathcal{D}(M)$ as a function of the variable y (see Proposition 3.4 below), so that it makes sense to consider $\Phi_j(\sqrt{L})f(x):=\langle f,\,\Phi_j(\sqrt{L})(x,\cdot)\rangle$ for all $f\in\mathcal{D}'(M)$ and $x\in M$.

(ii) When $p=q\in(0,\infty)$, it is easy to see that $B^{s,\tau}_{p,p}(M)=F^{s,\tau}_{p,p}(M)$ and $\widetilde{B}^{s,\tau}_{p,p}(M)=\widetilde{F}^{s,\tau}_{p,p}(M)$.

(iii) In general, the spaces $B^{s,\tau}_{p,q}(M)$ and $\widetilde{B}^{s,\tau}_{p,q}(M)$ may not coincide with each other, unless (M,ρ,μ) is an Ahlfors d-regular metric measure space. Neither do $F^{s,\tau}_{p,q}(M)$ and $\widetilde{F}^{s,\tau}_{p,q}(M)$.

Remark 2.4. Let all the notation be as in Definition 2.2. When $\mu(M)=\infty$ and $\tau\in(-\infty,0)$, it is easy to see that

$$B^{s,\tau}_{p,q}(M)=F^{s,\tau}_{p,q}(M)=\widetilde{B}^{s,\tau}_{p,q}(M)=\widetilde{F}^{s,\tau}_{p,q}(M)=\{0\}.$$

But, when $\mu(M)<\infty$ and $\tau\in(-\infty,0)$, it holds true that

$$B^{s,\tau}_{p,q}(M)=B^{s}_{p,q}(M),\quad \widetilde{B}^{s,\tau}_{p,q}(M)=\widetilde{B}^{s}_{p,q}(M),\quad F^{s,\tau}_{p,q}(M)=F^{s}_{p,q}(M),\quad \widetilde{F}^{s,\tau}_{p,q}(M)=\widetilde{F}^{s}_{p,q}(M).$$

No matter $\mu(M)$ is finite or not, when $\tau=0$, it always holds true that

$$B^{s,0}_{p,q}(M)=B^{s}_{p,q}(M),\quad \widetilde{B}^{s,0}_{p,q}(M)=\widetilde{B}^{s}_{p,q}(M),\quad F^{s,0}_{p,q}(M)=F^{s}_{p,q}(M),\quad \widetilde{F}^{s,0}_{p,q}(M)=\widetilde{F}^{s}_{p,q}(M).$$

When $\tau\in(1/p,\infty)$, in Proposition 4.3 below, we show that

$$B^{s,\tau}_{p,q}(M)=F^{s,\tau}_{p,q}(M)=B^{s+d\tau-d/p}_{\infty,\infty}(M)\quad\text{and}\quad \widetilde{B}^{s,\tau}_{p,q}(M)=\widetilde{F}^{s,\tau}_{p,q}(M)=\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M).$$

Thus, when $\tau\in(0,1/p]$, Definition 2.2 gives new scales of function spaces. Especially, when $\tau=1/p$, it is proved in Theorem 6.7 below that $F^{s,1/p}_{p,q}(M)$ and $\widetilde{F}^{s,1/p}_{p,q}(M)$ are respectively the endpoint cases of the Triebel–Lizorkin spaces $F^{s}_{\infty,q}(M)$ and $\widetilde{F}^{s}_{\infty,q}(M)$.

3 Auxiliary estimates

For any constants $\delta,\sigma\in(0,\infty)$, let

$$D_{\delta,\sigma}(x,y):=\frac{1}{\sqrt{|B(x,\delta)|\,|B(y,\delta)|}}\frac{1}{[1+\delta^{-1}\rho(x,y)]^\sigma},\qquad x,y\in M.$$

For any $\delta, \gamma \in (0, \infty)$ and $\beta \in (0, 1)$, let

$$E_\delta^{\gamma,\beta}(x, y) := \frac{1}{\sqrt{|B(x, \delta)|\,|B(y, \delta)|}} \exp\left\{-\gamma \left[\frac{\rho(x, y)}{\delta}\right]^\beta\right\}, \qquad x, y \in M.$$

Obviously, there exists a positive constant $C_{\gamma,\beta,\sigma}$ such that, for all $\delta \in (0, \infty)$ and $x, y \in M$,

$$E_\delta^{\gamma,\beta}(x, y) \le C_{\gamma,\beta,\sigma} D_{\delta,\sigma}(x, y).$$

Further properties of the kernels $D_{\delta,\sigma}(x, y)$ and $E_\delta^{\gamma,\beta}(x, y)$ are addressed in this section. By using these kernels, in this section, we present smooth functional calculus induced by the heat kernels, off-diagonal estimates, and estimates of Peetre maximal functions.

3.1 Basic properties of $D_{\delta,\sigma}(x, y)$ and $E_\delta^{\gamma,\beta}(x, y)$

The estimates listed in the following lemma follow from (1.1) and (1.3), while their proofs were essentially given in [9, Lemma 2.3] and [22, Lemma 2.1, (3.22)], the details being omitted.

Lemma 3.1. (i) *Let $\sigma \in (d, \infty)$. Then, there exists a constant $C_\sigma^* \in [1, \infty)$ such that, for all $\delta \in (0, \infty)$ and $x \in M$,*

$$\frac{1}{|B(x, \delta)|} \int_M \frac{1}{[1 + \delta^{-1}\rho(x, y)]^\sigma}\, d\mu(y) \le C_\sigma^*, \qquad \int_M \frac{1}{|B(y, \delta)|[1 + \delta^{-1}\rho(x, y)]^\sigma}\, d\mu(y) \le C_\sigma^*$$

and

$$\int_M D_{\delta,\sigma}(x, y)\, d\mu(y) \le C_\sigma^*,$$

where the last inequality indeed holds true only when $\sigma > d/2$.

(ii) *Let $\sigma \in (d, \infty)$. Then, there exists a constant $C_\sigma^* \in [1, \infty)$ such that, for all $s, t \in (0, \infty)$ and $x, y \in M$,*

$$\int_M D_{s,\sigma}(x, z) D_{t,\sigma}(z, y)\, d\mu(z) \le C_\sigma^* \max\{(t^{-1}s)^{d/2},\, (s^{-1}t)^{d/2}\}\, D_{s\vee t,\sigma}(x, y).$$

(iii) *Let $\gamma \in (0, \infty)$ and $\beta \in (0, 1)$. Then, there exists a constant $C_{\gamma,\beta}^* \in [1, \infty)$ such that, for all $\delta \in (0, \infty)$ and $x, y \in M$,*

$$\int_M E_\delta^{\gamma,\beta}(x, u) E_\delta^{\gamma,\beta}(u, y)\, d\mu(u) \le C_{\gamma,\beta}^* E_\delta^{\gamma,\beta}(x, y).$$

Via (1.1), (1.3) and (1.5), the following Lemma 3.2(i) was proved in [9, Proposition 2.9] (see also [22, Proposition 2.4]); while Lemma 3.2(ii) follows from a similar argument.

Lemma 3.2. (i) *Let $\delta \in (0, 1)$ and $\sigma \in [d+1, \infty)$. Assume that the kernels of the integral operators U and V satisfy that*

$$|U(x, y)| \le D_{\delta,\sigma}(x, y) \qquad \text{and} \qquad |V(x, y)| \le D_{\delta,\sigma}(x, y) \tag{3.1}$$

for all $x, y \in M$. Then, for any operator R which is bounded on $L^2(M)$, the operator URV is an integral operator with its kernel satisfying that

$$|URV(x, y)| \le \|U(x, \cdot)\|_{L^2(M)} \|R\|_{L^2(M)\to L^2(M)} \|V(\cdot, y)\|_{L^2(M)} \le C_\sigma^* \frac{\|R\|_{L^2(M)\to L^2(M)}}{\sqrt{|B(x, \delta)|\,|B(y, \delta)|}}$$

for all $x, y \in M$ *and for some constant* $C_\sigma^* \in [1, \infty)$ *independent of* δ.

(ii) *Let* $\gamma \in (0, \infty)$, $\beta \in (0, 1)$ *and* $\delta \in (0, \infty)$. *Instead of* (3.1), *if*

$$|U(x,y)| \le E_\delta^{\gamma,\beta}(x,y) \qquad \text{and} \qquad |V(x,y)| \le E_\delta^{\gamma,\beta}(x,y)$$

hold true for all $x, y \in M$, *then the conclusion of (i) keeps valid, but with the constant* C_σ^* *there replaced by some positive constant* $C_{\gamma,\beta}^*$ *which depends on* γ *and* β, *but not on* δ.

3.2 Smooth functional calculus induced by the heat kernels

According to [29, Theorem 7.3], the doubling condition (1.1) and the assumption (**UE**$'$) imply that the Gaussian upper bound can be extended to the open right half-plane. Indeed, there exist positive constants C and c such that, for all $x, y \in M$ and $z := t + iu$ with $t \in (0, 1]$ and $u \in \mathbb{R}$,

$$|p_z(x,y)| \le C \frac{\exp\big(-c\big[\frac{\rho(x,y)}{|z|^{1/\beta_0}}\big]^{\frac{\beta_0}{\beta_0-1}} \cos\theta\big)}{\sqrt{|B(x, t^{1/\beta_0})|\,|B(y, t^{1/\beta_0})|}}, \tag{3.2}$$

where $\theta := \arg z$.

Lemma 3.3. *Assume that* (1.1) *and* (**UE**$'$) *hold true. Let* $\sigma \in (0, \infty)$ *and* g *be a function in the Schwartz class* $\mathcal{S}(\mathbb{R})$. *Then, there exists a positive constant* C *such that, for all* $\delta \in (0, \infty)$, $u \in \mathbb{R}$ *and* $x, y \in M$,

$$\left| \int_{\mathbb{R}} g(u) p_{\delta^2(1-iu)}(x,y)\, du \right| \le C D_{\delta^{2/\beta_0},\sigma}(x,y).$$

Proof. For $z := \delta^2(1 - iu)$, we have $|z| \sim \delta^2(1 + |u|)$ and $\cos\arg z \sim [1 + |u|]^{-1}$. Thus, for a different positive constant c', we have

$$\begin{aligned}
\exp\left(-c\left[\frac{\rho(x,y)}{|z|^{1/\beta_0}}\right]^{\frac{\beta_0}{\beta_0-1}} \cos\theta\right) &\le \exp\left(-c'\left[\frac{\rho(x,y)}{[\delta^2(1+|u|)]^{1/\beta_0}}\right]^{\frac{\beta_0}{\beta_0-1}} \frac{1}{1+|u|}\right) \\
&\lesssim \left(1 + \left[\frac{\rho(x,y)}{[\delta^2(1+|u|)]^{1/\beta_0}}\right]\left[\frac{1}{1+|u|}\right]^{\frac{\beta_0-1}{\beta_0}}\right)^{-\sigma} \\
&\lesssim \left[1 + \frac{\rho(x,y)}{\delta^{2/\beta_0}}\right]^{-\sigma} [1 + |u|]^{\sigma}.
\end{aligned}$$

From this and (3.2), it follows that

$$\left| \int_{\mathbb{R}} g(u) p_{\delta^2(1-iu)}(x,y)\, du \right| \lesssim \frac{[1 + \frac{\rho(x,y)}{\delta^{2/\beta_0}}]^{-\sigma}}{\sqrt{|B(x, \delta^{2/\beta_0})|\,|B(y, \delta^{2/\beta_0})|}} \int_{\mathbb{R}} |g(u)|[1 + |u|]^{\sigma}\, du \lesssim D_{\delta^{2/\beta_0},\sigma}(x,y),$$

as desired. This finishes the proof of Lemma 3.3. □

Applying Lemma 3.3, we can obtain a result parallel to [9, Theorem 3.1], but invoking the index β_0. Next, slight modifications of the proofs of [9, Theorems 3.7] and [22, Theorem 3.1, Corollary 3.5] show that the following smooth functional calculus holds true, the details being omitted.

Proposition 3.4. *Let $f \in C^\infty(\mathbb{R}_+)$ satisfy that, for all $\nu \in \mathbb{Z}_+$ and $r \in (0,\infty)$, there is a positive constant $C_{\nu,r}$ such that*

$$f^{(2\nu+1)}(0)=0, \qquad |f^{(\nu)}(\lambda)| \le C_{\nu,r}(1+\lambda)^{-r} \qquad \text{for all } \lambda \in (0,\infty). \tag{3.3}$$

Let $\sigma \in (0,\infty)$. Then, for any $m \in \mathbb{Z}_+$ and $\delta \in (0,\infty)$, there exists a constant $C_{\sigma,m} \in [1,\infty)$ such that the operator $L^m f(\delta\sqrt{L})$ is an integral operator with kernel $L^m f(\delta\sqrt{L})(x,y)$ satisfying:

(i) *for all x, $y \in M$,*

$$|L^m f(\delta\sqrt{L})(x,y)| \le C_{\sigma,m}\delta^{-2m} D_{\delta^{2/\beta_0},\sigma}(x,y); \tag{3.4}$$

(ii) *for all x, y, $y' \in M$ such that $\rho(y,y') \le \delta^{2/\beta_0}$,*

$$|L^m f(\delta\sqrt{L})(x,y) - L^m f(\delta\sqrt{L})(x,y')| \le C_{\sigma,m}\delta^{-2m}\left[\frac{\rho(y,y')}{\delta^{2/\beta_0}}\right]^{\alpha_0} D_{\delta^{2/\beta_0},\sigma}(x,y); \tag{3.5}$$

(iii) *for all $x \in M$, $\int_M f(\delta\sqrt{L})(x,y)\,d\mu(y) = f(0)$.*

Here, in (i) *and* (ii), *the constants α_0 and β_0 are as in* (**UE**$'$) *and* (**HE**$'$).

Remark 3.5. In (3.3), the condition $f^{(2\nu+1)}(0)=0$ for all $\nu \in \mathbb{Z}_+$ ensures that the function f has an even extension to $\mathbb{R}$. Conversely, any even function $f \in \mathcal{S}(\mathbb{R})$ satisfies (3.3).

Likewise, one deduces the following conclusion via a slight modification of the proof of [22, Theorem 3.6], the details being omitted.

Proposition 3.6. *With all assumptions as in Proposition 3.4, assume that there exist constants $M \in (0,\infty)$ and $\beta \in (0,1)$ such that, for all $k \in \mathbb{N}$,*

$$\|f^{(k)}\|_{L^\infty(\mathbb{R}_+)} \le (Mk)^{k(2-\beta)}.$$

Then, for any $m \in \mathbb{N}$, there exist positive constants γ, depending on β and m, and $C^\flat_{\gamma,\beta}$, depending on γ and β, such that, for all $\delta \in (0,\infty)$, the operator $L^m f(\delta\sqrt{L})$ is an integral operator with kernel $L^m f(\delta\sqrt{L})(x,y)$ satisfying:

(i) *for all x, $y \in M$,*

$$|L^m f(\delta\sqrt{L})(x,y)| \le C^\flat_{\gamma,\beta,m}\delta^{-2m}\, E^{\gamma,\beta}_{\delta^{2/\beta_0}}(x,y);$$

(ii) *for all x, y, $y' \in M$ such that $\rho(y,y') \le \delta^{2/\beta_0}$,*

$$|f(\delta\sqrt{L})(x,y) - f(\delta\sqrt{L})(x,y')| \le C^\flat_{\gamma,\beta}\left[\frac{\rho(y,y')}{\delta^{2/\beta_0}}\right]^{\alpha} E^{\gamma,\beta}_{\delta^{2/\beta_0}}(x,y).$$

As an application of Proposition 3.4, we have the following off-diagonal estimates.

Proposition 3.7. *Let $\delta \in (0,1)$. Assume that two pairs of functions, (Φ_0,Φ) and (Ψ_0,Ψ), satisfy* (2.1) *and* (2.2). *For any $j \in \mathbb{N}$, define Φ_j and Ψ_j as in* (2.3). *Then, for any $m \in \mathbb{Z}_+$ and $\sigma > d$, there exists a constant $C_{\sigma,m} \in (1,\infty)$ such that, for all $j,k \in \mathbb{N}$ and $x,y \in M$,*

$$\left|\left(\Phi_j(\sqrt{L})\Psi_k(\sqrt{L})\right)(x,y)\right| \le C_{\sigma,m}\delta^{|k-j|(m\beta_0-d/2)} D_{\delta^{k\wedge j},\sigma}(x,y). \tag{3.6}$$

Proof. For $k=j$, notice that (3.6) follows from Proposition 3.4(i) and Lemma 3.1(ii). By symmetry, we only need to show (3.6) for $k>j$. If $j>0$, then, by the functional calculus, we have

$$\begin{aligned}\Phi_j(\sqrt{L})\Psi_k(\sqrt{L}) &= (\Phi_j\Psi_k)(\sqrt{L})\\ &= \int_0^\infty \Phi(\delta^{j\beta_0/2}\sqrt{\lambda})\Psi(\delta^{k\beta_0/2}\sqrt{\lambda})\,dE_\lambda\\ &= \delta^{(k-j)m\beta_0}\int_0^\infty (\delta^{j\beta_0/2}\sqrt{\lambda})^{2m}\Phi(\delta^{j\beta_0/2}\sqrt{\lambda})(\delta^{k\beta_0/2}\sqrt{\lambda})^{-2m}\Psi(\delta^{j\beta_0/2}\sqrt{\lambda})\,dE_\lambda\\ &= \delta^{(k-j)m\beta_0}\phi_j(\sqrt{L})\psi_k(\sqrt{L}),\end{aligned}$$

where $\phi_j(\sqrt{\lambda}):=(\delta^{j\beta_0/2}\sqrt{\lambda})^{2m}\Phi(\delta^{j\beta_0/2}\sqrt{\lambda})$ and $\psi_k(\sqrt{\lambda}):=(\delta^{k\beta_0/2}\sqrt{\lambda})^{-2m}\Psi(\delta^{j\beta_0/2}\sqrt{\lambda})$. By the properties of Φ,Ψ and Proposition 3.4, we see that, for all $x,y\in M$,

$$|\phi_j(\sqrt{L})(x,y)|\lesssim D_{\delta^j,\sigma}(x,y) \quad\text{and}\quad |\psi_k(\sqrt{L})(x,y)|\lesssim D_{\delta^k,\sigma}(x,y).$$

Then, applying $\sigma\in(d,\infty)$ and Lemma 3.1(ii), we conclude that

$$\left|\left(\phi_j(\sqrt{L})\psi_k(\sqrt{L})\right)(x,y)\right|\lesssim\int_M D_{\delta^j,\sigma}(x,u)D_{\delta^k,\sigma}(u,y)\,d\mu(u)\lesssim\delta^{(j-k)d/2}D_{\delta^j,\sigma}(x,y).$$

Consequently, for all $x,y\in M$,

$$|\Phi_j(\sqrt{L})\Psi_k(\sqrt{L})(x,y)|\le\delta^{(k-j)(m\beta_0-d/2)}D_{\delta^j,\sigma}(x,y).$$

A similar argument also shows the desired result for the case $j=0$. This finishes the proof of Proposition 3.7. □

Remark 3.8. From the proof of Proposition 3.7, it follows that, if $k\ge j$ and Φ_0,Φ only satisfy (3.3), then (3.6) keeps valid.

Due to Proposition 3.4, the proofs of [22, Propositions 5.1(b) and 5.3(b)] and [22, Propositions 5.2(a) and 5.4(a)] imply the following conclusions, the details being omitted.

Proposition 3.9. (i) *For any even function $\phi\in\mathcal{S}(\mathbb{R})$, the kernel $\phi(\sqrt{L})(x,y)$ of the operator $\phi(\sqrt{L})$ belongs to $\mathcal{D}(M)$ as a function of x or y.*

(ii) *If $\mu(M)<\infty$, then there exist $m_0\in\mathbb{Z}_+$ and $C_{m_0}\in(0,\infty)$ such that, for all $f\in\mathcal{D}'(M)$ and $\phi\in\mathcal{D}(M)$,*

$$|\langle f,\,\phi\rangle|\le C_{m_0}\max_{0\le m\le m_0}\mathscr{P}_m(\phi).$$

(iii) *If $\mu(M)=\infty$, then there exist $\ell_0,m_0\in\mathbb{Z}_+$ and $C_{m_0,\ell_0}\in(0,\infty)$ such that, for all $f\in\mathcal{D}'(M)$ and $\phi\in\mathcal{D}(M)$,*

$$|\langle f,\,\phi\rangle|\le C_{m_0,\ell_0}\max_{0\le m\le m_0,\,0\le\ell\le\ell_0}\mathscr{P}_{m,\ell}(\phi).$$

Let $\delta\in(0,1)$. Assume that functions (Φ_0,Φ) in $C^\infty(\mathbb{R}_+)$ such that (2.1) and (2.2) hold true. For $j\in\mathbb{N}$, define Φ_j as in (2.3). Then, according to [14, Lemma 6.10] (see also [4, p. 1487, (3.20)]), there exist functions $\widetilde{\Phi}_0,\widetilde{\Phi}\in C^\infty(\mathbb{R}_+)$ satisfying (2.1) and (2.2) such that, for all $\lambda\in\mathbb{R}_+$,

$$\sum_{j=0}^\infty\widetilde{\Phi}_j(\lambda)\Phi_j(\lambda)=1,$$

where $\widetilde{\Phi}_j$ for $j \geq 1$ is defined as in (2.3). Furthermore, by the proof of Lemma 6.10 in [14], if Φ_0 and Φ are required to be nonnegative, then $\widetilde{\Phi}_0$ and $\widetilde{\Phi}$ are also nonnegative. Consequently, for all $f \in \mathcal{D}'(M)$ (or $f \in L^p(M)$ with $p \in [1,\infty)$, or $f \in \mathcal{D}(M)$), by [22, Proposition 5.5(b)], we have

$$f = \sum_{j=0}^{\infty} \widetilde{\Phi}_j(\sqrt{L})\Phi_j(\sqrt{L})f = \sum_{j=0}^{\infty} \Phi_j(\sqrt{L})\widetilde{\Phi}_j(\sqrt{L})f \quad \text{in } \mathcal{D}'(M) \quad (\text{or } L^p(M) \text{ or } \mathcal{D}(M)). \tag{3.7}$$

This is usually called the *continuous Calderón reproducing formula*. As was seen from the works of [13, 14, 4, 45] (see also [33, 34]), the Calderón reproducing formula and the almost orthogonality estimates serve as powerful tools in discussing the theory of function spaces.

3.3 Estimates of Peetre maximal functions

In addition to the aforementioned Calderón reproducing formula and the off-diagonal estimates, another important tool to deal with Besov-type and Triebel–Lizorkin-type spaces is the *Peetre maximal function* (see also [27]).

Definition 3.10. Let $b \in (0,1)$. Let $K_0, K \in C^\infty(\mathbb{R}_+)$ satisfy that $|K_0| \geq c$ on $[0, b^{-\beta_0/2}]$, and $|K| \geq c$ on $[b^{\beta_0/2}, b^{-\beta_0/2}]$, where c is a positive constant. Assume that K_0 and K satisfy (3.3). For each $k \in \mathbb{N}$, let $K_k(\cdot) := K(b^{k\beta_0/2}\cdot)$. Define the *Peetre maximal function*

$$[K_\ell(\sqrt{L})]^*_{a,\gamma} f(x) := \sup_{y\in M} \frac{|B(y,b^\ell)|^\gamma |K_\ell(\sqrt{L})f(y)|}{[1 + b^{-\ell}\rho(x,y)]^a}, \quad x \in M,$$

where $\ell \in \mathbb{Z}_+$, $a \in (0,\infty)$ and $\gamma \in \mathbb{R}$. For $\gamma = 0$, we simply write $[K_\ell(\sqrt{L})]^*_{a,\gamma} f$ as $[K_\ell(\sqrt{L})]^*_a f$.

Recall that, for all $g \in L^1_{\mathrm{loc}}(M)$, its *Hardy-Littlewood maximal function* $\mathcal{M}g$ is defined as follows:

$$\mathcal{M}g(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |g(y)|\, d\mu(y), \qquad x \in M,$$

where the supremum is taken over all balls B containing x. If $r \in (0,\infty)$, then we define $\mathcal{M}_r(g)(x) := [\mathcal{M}(|g|^r)(x)]^{1/r}$ for all $x \in M$.

The following estimate of Peetre maximal functions is used throughout the whole paper. Its proof relies on the Calderón reproducing formula in [22, Proposition 5.5] and the off-diagonal estimates in Proposition 3.7.

Proposition 3.11. *Let $r \in (0,\infty)$, $\nu \in (0,\infty)$ and $\gamma \in \mathbb{R}$. Suppose that $x \in Q^k_\alpha$ with $k \in \mathbb{Z}$ and $\alpha \in I_k$. Then, with all the notation as in Definition 3.10, it holds true that, for any $f \in \mathcal{D}'(M)$ and $\ell \in \mathbb{Z}_+$,*

$$\begin{aligned} &[K_\ell(\sqrt{L})]^*_{\nu+d/r,\gamma} f(x) \\ &\quad \leq C \sum_{j=\ell}^{\infty} \sum_{i=0}^{\infty} b^{(j-\ell)\nu} b^{i\nu} \mathcal{M}_r\left(|B(\cdot,b^j)|^\gamma |K_j(\sqrt{L})f| \chi_{B(z^k_\alpha, b^{\ell-i} + \operatorname{diam} Q^k_\alpha)}\right)(x), \end{aligned} \tag{3.8}$$

where the positive constant C is independent of x, k, α and ℓ.

We remark that, comparing with the estimate in [22, Lemma 6.4], the estimate in Proposition 3.11 holds true for more general set $\mathcal{D}'(M)$ of distributions, and is somehow more delicate due to the existing of characteristic functions in maximal function. To show Proposition 3.11, we begin with two lemmas.

Lemma 3.12. *Let* $b \in (0,1)$, $\sigma \in (d,\infty)$ *and* $g \in L^1_{\mathrm{loc}}(M)$. *Then, for all* $j \in \mathbb{Z}$ *and* $x \in Q^k_\alpha$ *with* $k \in \mathbb{Z}$ *and* $\alpha \in I_k$,

$$\int_M \frac{|g(y)|}{|B(y,b^j)|[1+b^{-j}\rho(x,y)]^\sigma}\,d\mu(y) \le C\sum_{i=0}^\infty b^{i(\sigma-d)}\mathcal{M}(g\chi_{B(z^k_\alpha,b^{j-i}+\operatorname{diam}Q^k_\alpha)})(x), \tag{3.9}$$

where C *is a positive constant independent of* k, j, x *and* g.

Proof. Let J denote the left-hand side of (3.9). Then, obviously,

$$\mathrm{J} \lesssim \sum_{i=0}^\infty \int_{\rho(x,y)\sim b^{j-i}} \frac{1}{|B(y,b^j)|}\frac{|g(y)|}{[1+b^{-j}\rho(x,y)]^\sigma}\,d\mu(y),$$

where the symbol $\rho(x,y)\sim b^{j-i}$ means that $b^{j-i+1}\le\rho(x,y)<b^{j-i}$ for $i\ge1$ and that $\rho(x,y)<b^{j-i}$ for $i=0$. Then, invoking the fact $x\in Q^k_\alpha$, we see that

$$\begin{aligned}&\int_{\rho(x,y)\sim b^{j-i}} \frac{1}{|B(y,b^j)|}\frac{|g(y)|}{[1+b^{-j}\rho(x,y)]^\sigma}\,d\mu(y)\\ &\quad\lesssim b^{i\sigma}\frac{1}{|B(x,b^{j-i})|}\int_{\rho(x,y)<b^{j-i}}\frac{|B(x,b^{j-i})|}{|B(y,b^j)|}|g(y)|\chi_{B(z^k_\alpha,b^{j-i}+\operatorname{diam}Q^k_\alpha)}(y)\,d\mu(y)\\ &\quad\lesssim b^{i(\sigma-d)}\mathcal{M}\left(g\chi_{B(z^k_\alpha,b^{j-i}+\operatorname{diam}Q^k_\alpha)}\right)(x).\end{aligned}$$

Finally, taking the summation over all $i\in\mathbb{Z}_+$, we obtain (3.9), which completes the proof of Lemma 3.12. □

Lemma 3.13. *Let* m_0 *and* ℓ_0 *be as in Proposition 3.9. Let* $r\in(0,\infty)$, $\sigma\ge\ell_0+d$ *and* $N>m_0+d/(2\beta_0)$. *With all the notation as in Definition 3.10, there exists a positive constant* $C_{r,\sigma,N}$ *such that, for all* $f\in\mathcal{D}'(M)$, $k\in\mathbb{Z}_+$ *and* $x\in M$,

$$\begin{aligned}|K_k(\sqrt{L})f(x)|^r &\le C_{r,\sigma,N}\sum_{j=k}^\infty b^{(j-k)[(N\beta_0+d/2)(r\wedge1)-d]}\\ &\qquad\times\int_M\frac{1}{|B(z,b^k)|}\frac{|K_j(\sqrt{L})f(z)|^r}{[1+b^{-k}\rho(x,z)]^{\sigma(r\wedge1)}}\,d\mu(z).\end{aligned}\tag{3.10}$$

Proof. Fix $\ell\in\mathbb{Z}_+$. Choose a non-negative function $\Phi_0\in C^\infty_c(\mathbb{R}_+)$ such that $\operatorname{supp}\Phi_0\subset[0,b^{-\beta_0/2}]$ and $\Phi_0(\lambda)=1$ when $\lambda\in[0,1]$. Define $\Phi(\lambda):=\Phi_0(\lambda)-\Phi_0(b^{-\beta_0/2}\lambda)$ for all $\lambda\in\mathbb{R}_+$. For any $j\in\mathbb{N}$, define Φ_j as in (2.3). Notice that Φ satisfies (2.2). Moreover, for all $\lambda\in\mathbb{R}_+$,

$$\Phi_0(b^{\ell\beta_0/2}\lambda)+\sum_{j=\ell+1}^\infty\Phi_j(\lambda)\equiv1.$$

Define $\widetilde{\Psi}_\ell(\cdot):=\Phi_0(b^{\ell\beta_0/2}\cdot)$ and $\Psi:=\Phi/K$. By the support of K, we know that Ψ is well defined. The assumption on K implies that Ψ satisfies (2.2). Moreover, for all $\lambda\in(0,\infty)$, we have

$$\widetilde{\Psi}_\ell(\lambda)+\sum_{j=\ell+1}^\infty\Psi_j(\lambda)K_j(\lambda)=1,\tag{3.11}$$

where $\Psi_j(\cdot):=\Psi(b^{j\beta_0/2}\cdot)$ for all $j\in\mathbb{N}$. Based on (3.11) and [22, Proposition 5.5(b)], we see that

$$K_\ell(\sqrt{L})f=\widetilde{\Psi}_\ell(\sqrt{L})K_\ell(\sqrt{L})f+\sum_{j=\ell+1}^\infty K_\ell(\sqrt{L})\Psi_j(\sqrt{L})K_j(\sqrt{L})f.$$

Since $\widetilde{\Psi}_\ell(\cdot)=\Phi_0(b^{\ell\beta_0/2}\cdot)$, we apply Proposition 3.4(i) to deduce that, for any $x,y\in M$,

$$|\widetilde{\Psi}_\ell(\sqrt{L})(x,y)|\lesssim D_{b^\ell,\sigma+d/2}(x,y).$$

By Proposition 3.7 and Remark 3.8, we see that, for any $j\ge \ell+1$ and $x,y\in M$,

$$|K_\ell(\sqrt{L})\Psi_j(\sqrt{L})(x,y)|\lesssim b^{(j-\ell)(N\beta_0-d/2)}D_{b^\ell,\sigma+d/2}(x,y).$$

Consequently, for all $x\in M$, we have

$$\begin{aligned}|K_\ell(\sqrt{L})f(x)|&\lesssim\sum_{j=\ell}^\infty b^{(j-\ell)(N\beta_0-d/2)}\int_M D_{b^\ell,\sigma+d/2}(x,y)|K_j(\sqrt{L})f(y)|\,d\mu(y)\\&\lesssim\sum_{j=\ell}^\infty b^{(j-\ell)(N\beta_0-d/2)}\int_M\frac{1}{|B(z,b^\ell)|}\frac{|K_j(\sqrt{L})f(z)|}{[1+b^{-\ell}\rho(x,z)]^\sigma}\,d\mu(z).\end{aligned}\tag{3.12}$$

Thus, the estimate (3.10) for $r=1$ is proved.

To obtain (3.10) for $r\in(1,\infty)$, by Hölder's inequality and Lemma 3.1(i), we find that, for all $x\in M$,

$$\int_M\frac{1}{|B(z,b^\ell)|}\frac{|K_j(\sqrt{L})f(z)|}{[1+b^{-\ell}\rho(x,z)]^\sigma}\,d\mu(z)\lesssim\left\{\int_M\frac{1}{|B(z,b^\ell)|}\frac{|K_j(\sqrt{L})f(z)|^r}{[1+b^{-\ell}\rho(x,z)]^\sigma}\,d\mu(z)\right\}^{1/r}.$$

Inserting this into (3.12) and applying Hölder's inequality again, together with $N\beta_0>d/2$, we see that, for all $x\in M$,

$$|K_\ell(\sqrt{L})f(x)|^r\lesssim\sum_{j=\ell}^\infty b^{(j-\ell)(N\beta_0-d/2)}\int_M\frac{1}{|B(z,b^\ell)|}\frac{|K_j(\sqrt{L})f(z)|^r}{[1+b^{-\ell}\rho(x,z)]^\sigma}\,d\mu(z).$$

Hence, (3.10) holds true for $r\in(1,\infty)$.

To see that (3.10) also holds true for $r\in(0,1)$, we define

$$N(x,k):=\sup_{\ell\ge k}\sup_{y\in M}b^{(\ell-k)(N\beta_0+d/2)}\frac{|K_\ell(\sqrt{L})f(y)|}{[1+b^{-k}\rho(x,y)]^\sigma},\qquad x\in M,\ k\in\mathbb{Z}_+.$$

If $\ell\ge k$, then $|B(z,b^\ell)|\ge b^{(\ell-k)d}|B(z,b^k)|$ and $[1+b^{-k}\rho(x,y)][1+b^{-\ell}\rho(y,z)]\ge 1+b^{-k}\rho(x,z)$ for all $x,y,z\in M$. From this and (3.12), it follows that

$$\begin{aligned}\frac{|K_\ell(\sqrt{L})f(y)|}{[1+b^{-k}\rho(x,y)]^\sigma}&\lesssim\sum_{j=\ell}^\infty b^{(j-\ell)(N\beta_0-d/2)}b^{(k-\ell)d}\int_M\frac{1}{|B(z,b^k)|}\frac{|K_j(\sqrt{L})f(z)|}{[1+b^{-k}\rho(x,z)]^\sigma}\,d\mu(z)\\&\lesssim\sum_{j=\ell}^\infty b^{(j-\ell)(N\beta_0+d/2)}b^{(k-j)d}\int_M\frac{1}{|B(z,b^k)|}\frac{|K_j(\sqrt{L})f(z)|}{[1+b^{-k}\rho(x,z)]^\sigma}\,d\mu(z),\end{aligned}$$

which, together with the definition of $N(x,k)$, further implies that

$$\begin{aligned}N(x,k)&\lesssim\sum_{j=k}^\infty b^{(j-k)(N\beta_0+d/2)}b^{(k-j)d}\int_M\frac{1}{|B(z,b^k)|}\frac{|K_j(\sqrt{L})f(z)|}{[1+b^{-k}\rho(x,z)]^\sigma}\,d\mu(z)\\&\lesssim[N(x,k)]^{1-r}\sum_{j=k}^\infty b^{(j-k)[(N\beta_0+d/2)r-d]}\int_M\frac{1}{|B(z,b^k)|}\frac{|K_j(\sqrt{L})f(z)|^r}{[1+b^{-k}\rho(x,z)]^{\sigma r}}\,d\mu(z).\end{aligned}$$

Therefore, if $N(x,k)<\infty$, we know that

$$|K_k(\sqrt{L})f(x)|^r \le [N(x,k)]^r \lesssim \sum_{j=k}^{\infty} b^{(j-k)[(N\beta_0+d/2)r-d]} \int_M \frac{1}{|B(z,b^k)|}\frac{|K_j(\sqrt{L})f(z)|^r}{[1+b^{-k}\rho(x,z)]^{\sigma r}}\,d\mu(z),$$

as desired.

Now we show that $N(x,k)<\infty$ when $\mu(M)<\infty$. Indeed, by (i) and (ii) of Proposition 3.9, we see that, for all $y\in M$,

$$|K_\ell(\sqrt{L})f(y)| \lesssim \max_{0\le m\le m_0}\|L^m K_\ell(\cdot,y)\|_{L^2(M)}.$$

From Proposition 3.4(i) and Lemma 3.1(i), it follows that, for any given $\widetilde{\sigma}>d/2$,

$$\|L^m K_\ell(\cdot,y)\|_{L^2(M)} \lesssim b^{-\beta_0 m\ell}\left\{\int_M \frac{1}{|B(y,b^\ell)||B(z,b^\ell)|}\frac{1}{[1+b^{-\ell}\rho(z,y)]^{2\widetilde{\sigma}}}\,d\mu(z)\right\}^{1/2} \lesssim \frac{b^{-\beta_0 m\ell}}{|B(y,b^\ell)|^{1/2}},$$

which further implies that, when $\sigma\ge d/2$,

$$\begin{aligned} N(x,k) &\lesssim \sup_{\ell\ge k}\sup_{y\in M}\max_{0\le m\le m_0}\frac{b^{(\ell-k)(N\beta_0+d/2)-\beta_0 m\ell}}{[1+b^{-k}\rho(x,y)]^{\sigma}|B(y,b^\ell)|^{1/2}}\\ &\lesssim \frac{b^{-\beta_0 m_0 k}}{|B(x,b^k)|^{1/2}}\sup_{\ell\ge k}\sup_{y\in M}\frac{b^{(\ell-k)N\beta_0}}{[1+b^{-k}\rho(x,y)]^{\sigma-d/2}} \lesssim \frac{b^{-\beta_0 m_0 k}}{|B(x,b^k)|}<\infty. \end{aligned}$$

Finally, we show that, if $\mu(M)=\infty$, then $N(x,k)<\infty$. For any $0\le m\le m_0$, it follows from Proposition 3.4(i) that, for all $\ell\in\mathbb{Z}_+$ and $z,y\in M$,

$$|L^m K_\ell(z,y)| \lesssim \frac{b^{-\beta_0 m\ell}}{|B(y,b^\ell)|[1+b^{-\ell}\rho(z,y)]^{\ell_0}} \lesssim \frac{b^{-\ell(\beta_0 m_0+d)}}{|B(y,1)|[1+\rho(z,y)]^{\ell_0}}.$$

By this, (i) and (ii) of Proposition 3.9, we see that, for all $y\in M$,

$$|K_\ell(\sqrt{L})f(y)| \lesssim \max_{0\le m\le m_0,\,0\le\nu\le\ell_0}\sup_{z\in M}[1+\rho(z,x_0)]^{\nu}|L^m K_\ell(z,y)| \lesssim \frac{b^{-\ell(\beta_0 m_0+d)}[1+\rho(y,x_0)]^{\ell_0}}{|B(y,1)|}.$$

Therefore, if $\sigma\ge\ell_0+d$ and $N\ge m_0+d/(2\beta_0)$, we have

$$\begin{aligned} N(x,k) &\lesssim \sup_{\ell\ge k}\sup_{y\in M}\frac{b^{(\ell-k)(N\beta_0+d/2)-\ell(\beta_0 m_0+d)}[1+\rho(y,x_0)]^{\ell_0}}{[1+\rho(x,y)]^{\sigma}|B(y,1)|}\\ &\lesssim \frac{b^{-k(\beta_0 m_0+d)}[1+\rho(x,x_0)]^{\ell_0}}{|B(x,1)|}\sup_{\ell\ge k}\sup_{y,z\in M}\frac{b^{(\ell-k)(N\beta_0-m_0\beta_0-d/2)}}{[1+\rho(x,y)]^{\sigma-\ell_0-d}}\\ &\lesssim \frac{b^{-k(\beta_0 m_0+d)}[1+\rho(x,x_0)]^{\ell_0}}{|B(x,1)|}, \end{aligned}$$

which is finite. This finishes the proof of Lemma 3.13. □

Proof of Proposition 3.11. Since $\mathcal{M}g\le\mathcal{M}_r g$ for all $r\in[1,\infty)$ and $g\in L^1_{\mathrm{loc}}(M)$, it suffices to show that (3.8) holds true for $r\in(0,1]$. Fix $r\in(0,1]$. Let σ and N be large numbers satisfying the assumptions of Lemma 3.13, $\sigma>\nu+d/r+d|\gamma|$ and $N\beta_0+d/2>d/r-d|\gamma|+\nu/r$. Then, applying Lemma 3.13, we conclude that, for all $\ell\in\mathbb{Z}_+$ and $y\in M$,

$$(3.13)\quad \frac{|B(y,b^\ell)|^{\gamma}|K_\ell(\sqrt{L})f(y)|}{[1+b^{-\ell}\rho(x,y)]^{\nu+d/r}} \lesssim \left\{\sum_{j=\ell}^{\infty} b^{(j-\ell)[(N\beta_0+d/2)r-d]}\int_M \frac{|B(y,b^\ell)|^{r\gamma}}{|B(z,b^j)|^{r\gamma}}\right.$$

$$\left.\times\frac{|B(z,b^j)|^{r\gamma}|K_j(\sqrt{L})f(z)|^r}{|B(z,b^\ell)|[1+b^{-\ell}\rho(x,y)]^{\nu r+d}[1+b^{-\ell}\rho(y,z)]^{\sigma r}}\,d\mu(z)\right\}^{1/r}.$$

Since

$$|B(y,b^\ell)|\lesssim|B(z,b^\ell)|[1+b^{-\ell}\rho(y,z)]^d\lesssim b^{(\ell-j)d}|B(z,b^j)|[1+b^{-\ell}\rho(y,z)]^d$$

and

$$|B(z,b^j)|\le|B(z,b^\ell)|\lesssim|B(y,b^\ell)|[1+b^{-\ell}\rho(y,z)]^d,$$

it follows that

$$\frac{|B(y,b^\ell)|^{r\gamma}}{|B(z,b^j)|^{r\gamma}}\lesssim b^{(\ell-j)dr|\gamma|}[1+b^{-\ell}\rho(y,z)]^{dr|\gamma|}.$$

By this and the fact $\sigma r>\nu r+d+dr|\gamma|$, we see that

$$\begin{aligned}(3.14)\qquad&\frac{|B(y,b^\ell)|^\gamma|K_\ell(\sqrt{L})f(y)|}{[1+b^{-\ell}\rho(x,y)]^{\nu+d/r}}\\&\lesssim\left\{\sum_{j=\ell}^\infty b^{(j-\ell)[(N\beta_0+d/2)r-d-dr|\gamma|]}\int_M\frac{|B(z,b^j)|^{r\gamma}|K_j(\sqrt{L})f(z)|^r}{|B(z,b^\ell)|[1+b^{-\ell}\rho(x,z)]^{\nu r+d}}\,d\mu(z)\right\}^{1/r}\\&\lesssim\sum_{j=\ell}^\infty b^{(j-\ell)[(N\beta_0+d/2)r-d-dr|\gamma|]}\left\{\int_M\frac{|B(z,b^j)|^{r\gamma}|K_j(\sqrt{L})f(z)|^r}{|B(z,b^\ell)|[1+b^{-\ell}\rho(x,z)]^{\nu r+d}}\,d\mu(z)\right\}^{1/r},\end{aligned}$$

where the second step is by Hölder's inequality and $(N\beta_0+d/2)r-d-dr|\gamma|>\nu>0$. Furthermore, applying Lemma 3.12, we conclude that, for all $y\in M$,

$$\frac{|B(y,b^\ell)|^\gamma|K_\ell(\sqrt{L})f(y)|}{[1+b^{-\ell}\rho(x,y)]^{\nu+d/r}}\lesssim\sum_{j=\ell}^\infty\sum_{i=0}^\infty b^{(j-\ell)\nu}b^{i\nu}\mathcal{M}_r(|B(\cdot,b^j)|^\gamma|K_j(\sqrt{L})f|\chi_{B(z_\alpha^k,b^{\ell-i}+\operatorname{diam}Q_\alpha^k)})(x).$$

Then, taking the supremum of all $y\in M$ and using the definition of $[K_\ell(\sqrt{L})]^*_{\nu+d/r,\gamma}f(x)$, we see that (3.8) holds, which completes the proof of Proposition 3.11. $\square$

By the proof of Proposition 3.11, especially (3.14), one easily sees that the following holds true, the details being omitted.

Corollary 3.14. *Let* $r\in(0,\infty)$, $\nu\in(0,\infty)$ *and* $\gamma\in\mathbb{R}$. *Then, with all the notation as in Definition 3.10, there exists a positive constant* C *such that, for all* $f\in\mathcal{D}'(M)$, $\ell\in\mathbb{Z}_+$ *and* $x\in M$,

$$[K_\ell(\sqrt{L})]^*_{\nu+d/r,\gamma}f(x)\le C\sum_{j=\ell}^\infty b^{(j-\ell)\nu}\mathcal{M}_r\left(|B(\cdot,b^j)|^\gamma|K_j(\sqrt{L})f|\right)(x).$$

Remark 3.15. It should be remarked that the conclusion of Corollary 3.14 was obtained in [22, Lemma 6.4], but only for functions in the *spectral space* Σ_λ^p defined as follows:

$$\Sigma_\lambda^p(M):=\{f\in L^p(M):\ \theta(\sqrt{L})f=f\text{ for all }\theta\in C_c^\infty(\mathbb{R}_+),\ \theta\equiv1\text{ on }[0,\lambda]\},$$

where $p\in[1,\infty]$, $\lambda\in(0,\infty)$ and $C_c^\infty(\mathbb{R}_+)$ denotes the *space of all functions in* $C^\infty(\mathbb{R}_+)$ *with compact support*. Clearly, the result in Corollary 3.14 improves [22, Lemma 6.4], since it holds true for general distributions.

4 Properties of Besov–Triebel–Lizorkin-type spaces

In this section, we establish some embedding properties of Besov-type and Triebel–Lizorkin-type spaces, and then we identify Besov-type and Triebel–Lizorkin-type spaces when the smooth index is in different ranges.

4.1 Embeddings

For $s \in \mathbb{R}$, notice that $B^s_{\infty,\infty}(M) = B^{s,0}_{\infty,\infty}(M)$ and $\widetilde{B}^s_{\infty,\infty}(M) = \widetilde{B}^{s,0}_{\infty,\infty}(M)$. Thus, for any $f \in \mathcal{D}'(M)$,

$$\|f\|_{B^s_{\infty,\infty}(M)} = \|f\|_{B^{s,0}_{\infty,\infty}(M)} = \sup_{j\in\mathbb{Z}_+} \sup_{x\in M} \delta^{-js} |\Phi_j(\sqrt{L})f(x)|$$

and

$$\|f\|_{\widetilde{B}^s_{\infty,\infty}(M)} = \|f\|_{\widetilde{B}^{0,s}_{\infty,\infty}(M)} = \sup_{j\in\mathbb{Z}_+} \sup_{x\in M} |B(x,\delta^j)|^{-s/d} |\Phi_j(\sqrt{L})f(x)|.$$

In what follows, if the function space $\mathcal{X}$ is continuously embedded into $\mathcal{Y}$, then we write $\mathcal{X} \hookrightarrow \mathcal{Y}$. We start by the following embedding property. The corresponding result for Besov-type and Triebel–Lizorkin-type spaces on $\mathbb{R}^n$ were obtained in [34, Proposition 4.2].

Proposition 4.1. *Let $s \in \mathbb{R}$, $\tau \in [0,\infty)$ and $q \in (0,\infty]$.*

(i) *If $p \in (0,\infty]$, then $B^{s,\tau}_{p,q}(M) \hookrightarrow B^{s+d\tau-d/p}_{\infty,\infty}(M)$ and $\widetilde{B}^{s,\tau}_{p,q}(M) \hookrightarrow \widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)$.*

(ii) *If $p \in (0,\infty)$, then $F^{s,\tau}_{p,q}(M) \hookrightarrow B^{s+d\tau-d/p}_{\infty,\infty}(M)$ and $\widetilde{F}^{s,\tau}_{p,q}(M) \hookrightarrow \widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)$.*

Proof. Due to the similarity, we only consider embeddings of $\widetilde{B}^{s,\tau}_{p,q}(M)$ and $\widetilde{F}^{s,\tau}_{p,q}(M)$. By the Minkowski inequality, we know that

$$\widetilde{B}^{s,\tau}_{p,\min\{p,q\}}(M) \hookrightarrow \widetilde{F}^{s,\tau}_{p,q}(M) \hookrightarrow \widetilde{B}^{s,\tau}_{p,\max\{p,q\}}(M), \tag{4.1}$$

which, together with the monotonicity of $\widetilde{B}^{s,\tau}_{p,r}(M)$ on $r \in (0,\infty]$, further implies that

$$\widetilde{B}^{s,\tau}_{p,q}(M),\ \widetilde{F}^{s,\tau}_{p,q}(M) \hookrightarrow \widetilde{B}^{s,\tau}_{p,\infty}(M).$$

Thus, to complete the proof, it suffices to show that $\widetilde{B}^{s,\tau}_{p,\infty}(M) \hookrightarrow \widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)$. To see this, we let (Φ_0,Φ) be a pair of functions satisfying (2.1) and (2.2), and $f \in \widetilde{B}^{s,\tau}_{p,\infty}(M)$. Then,

$$\|f\|_{\widetilde{B}^{s,\tau}_{p,\infty}(M)} = \sup_{k\in\mathbb{Z},\,\alpha\in I_k} \sup_{j\ge k\vee 0} \frac{1}{|Q^k_\alpha|^\tau} \left[\int_{Q^k_\alpha} |B(x,\delta^j)|^{-sp/d} |\Phi_j(\sqrt{L})f(x)|^p \, d\mu(x)\right]^{1/p} < \infty,$$

where Φ_j for $j \in \mathbb{N}$ is defined as in (2.3). By Lemma 3.13, for all $k \in \mathbb{Z}_+$ and $x \in M$, we have

$$\begin{aligned} &|B(x,\delta^k)|^{-(s+d\tau-d/p)/d} |\Phi_k(\sqrt{L})f(x)| \\ &\lesssim \left[\sum_{j=k}^{\infty} \delta^{(j-k)[(N\beta_0+d/2)(p\wedge 1)-d]} \int_M \frac{|B(x,\delta^k)|^{-sp/d-p\tau+1}}{|B(z,\delta^k)|} \frac{|\Phi_j(\sqrt{L})f(z)|^p}{[1+\delta^{-k}\rho(x,z)]^{\sigma(p\wedge 1)}} \, d\mu(z)\right]^{1/p}, \end{aligned}$$

where σ and N are large numbers satisfying the hypothesis of Lemma 3.13, $\sigma(p\wedge 1) - |s|p - p\tau d > 2d+1$ and $(N\beta_0 + d/2)(p\wedge 1) - d > |s|p + 1$. For all $k \in \mathbb{Z}_+$, $j \ge k$ and $x, z \in M$, we find that

$$\frac{|B(x,\delta^k)|^{-sp/d-p\tau+1}}{|B(z,\delta^k)|[1+\delta^{-k}\rho(x,z)]^{\sigma(p\wedge 1)}} \lesssim \frac{|B(z,\delta^k)|^{-sp/d-p\tau}}{[1+\delta^{-k}\rho(x,z)]^{\sigma(p\wedge 1)-|s|p-p\tau d-d}}$$

$$\lesssim \delta^{(k-j)|s|p} \frac{|B(z,\delta^j)|^{-sp/d}}{|B(z,\delta^k)|^{\tau p}[1+\delta^{-k}\rho(x,z)]^{d+1}},$$

and hence

$$\begin{aligned}
&|B(x,\delta^k)|^{-(s+d\tau-d/p)/d}|\Phi_k(\sqrt{L})f(x)|\\
&\quad\lesssim \left[\sum_{j=k}^{\infty}\delta^{j-k}\sum_{\alpha\in I_k}\int_{Q_\alpha^k}\frac{1}{|B(z,\delta^k)|^{\tau p}}\frac{|B(z,\delta^j)|^{-sp/d}|\Phi_j(\sqrt{L})f(z)|^p}{[1+\delta^{-k}\rho(x,z)]^{d+1}}\,d\mu(z)\right]^{1/p}\\
&\quad\sim \left[\sum_{j=k}^{\infty}\delta^{j-k}\sum_{\alpha\in I_k}\inf_{u\in Q_\alpha^k}\frac{1}{[1+\delta^{-k}\rho(x,u)]^{d+1}}\frac{1}{|Q_\alpha^k|^{\tau p}}\int_{Q_\alpha^k}|B(z,\delta^j)|^{-sp/d}|\Phi_j(\sqrt{L})f(z)|^p\,d\mu(z)\right]^{1/p}\\
&\quad\lesssim \left[\sum_{j=k}^{\infty}\delta^{j-k}\sum_{\alpha\in I_k}\inf_{u\in Q_\alpha^k}\frac{1}{[1+\delta^{-k}\rho(x,u)]^{d+1}}\right]^{1/p}\|f\|_{\widetilde{B}^{s,\tau}_{p,\infty}(M)},
\end{aligned}$$

which implies that $\|f\|_{\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)} \lesssim \|f\|_{\widetilde{B}^{s,\tau}_{p,\infty}(M)}$ if we observe that $\sum_{j=k}^{\infty}\delta^{j-k}\lesssim 1$ and

$$\sum_{\alpha\in I_k}\inf_{u\in Q_\alpha^k}\frac{1}{[1+\delta^{-k}\rho(x,u)]^{d+1}}\lesssim\sum_{\alpha\in I_k}\int_{Q_\alpha^k}\frac{1}{|B(z,\delta^k)|[1+\delta^{-k}\rho(x,z)]^{d+1}}\,d\mu(z)\lesssim 1.$$

This proves that $\widetilde{B}^{s,\tau}_{p,\infty}(M)\hookrightarrow\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)$ and hence finishes the proof of Proposition 4.1. □

Theorem 4.2. *Let $s\in\mathbb{R}$, $\tau\in[0,\infty)$ and $q\in(0,\infty]$.*

(i) *If $p\in(0,\infty]$, then $\mathcal{D}(M)\hookrightarrow B^{s,\tau}_{p,q}(M)\hookrightarrow\mathcal{D}'(M)$ and $\mathcal{D}(M)\hookrightarrow\widetilde{B}^{s,\tau}_{p,q}(M)\hookrightarrow\mathcal{D}'(M)$.*

(ii) *If $p\in(0,\infty)$, then $\mathcal{D}(M)\hookrightarrow F^{s,\tau}_{p,q}(M)\hookrightarrow\mathcal{D}'(M)$ and $\mathcal{D}(M)\hookrightarrow\widetilde{F}^{s,\tau}_{p,q}(M)\hookrightarrow\mathcal{D}'(M)$.*

Proof. We only consider the case $\mu(M)=\infty$, the proof for the case $\mu(M)<\infty$ being similar. Let Φ_0,Φ satisfy (2.1) and (2.2). Fix $f\in\mathcal{D}(M)$. Assume that $m>(|s|+\tau d+2/d)\beta_0$ and $\ell>|s|+d\tau+d/p$. As in the proof of [22, (6.19)], it holds true that, for all $j\in\mathbb{Z}_+$ and $x\in M$,

$$\begin{aligned}
(4.2)\qquad |\Phi_j(\sqrt{L})f(x)| &= \delta^{j\beta_0 m}\left|\int_M(\delta^{-j\beta_0 m}L^{-m}\Phi_j(\sqrt{L}))(x,y)L^m f(y)\,d\mu(y)\right|\\
&\lesssim \delta^{j\beta_0 m}\mathscr{P}_{m,\ell}(f)\int_M D_{\delta^j,\ell}(x,y)D_{1,\ell}(y,x_0)\,d\mu(y)\\
&\lesssim \delta^{j(\beta_0 m-d/2)}[1+\rho(x,x_0)]^{-\ell}\mathscr{P}_{m,\ell}(f),
\end{aligned}$$

where we recall that x_0 is some fixed point of M. Then, for any cube Q_α^k with $k\in\mathbb{Z}$ and $\alpha\in I_k$, we see that

$$\begin{aligned}
(4.3)\ \mathrm{J} &:= \frac{1}{|Q_\alpha^k|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\left[\int_{Q_\alpha^k}|B(x,\delta^j)|^{-sp/d}|\Phi_j(\sqrt{L})f(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q}\\
&\lesssim \mathscr{P}_{m,\ell}(f)\left\{\sum_{j=k\vee 0}^{\infty}\delta^{j(\beta_0 m-d/2)q}\left[\frac{1}{|Q_\alpha^k|^{\tau p}}\int_{Q_\alpha^k}|B(x,\delta^j)|^{-sp/d}[1+\rho(x,x_0)]^{-\ell p}\,d\mu(x)\right]^{q/p}\right\}^{1/q}.
\end{aligned}$$

Then, notice that, for all $x\in Q_\alpha^k$, we have $|Q_\alpha^k|\sim|B(x,\delta^k)|\gtrsim|B(x,\delta^j)|$ and

$$\delta^{jd}[1+\rho(x,x_0)]^{-d}\lesssim\frac{|B(x,\delta^j)|}{|B(x_0,1)|}\lesssim[1+\rho(x,x_0)]^d,$$

which implies that

$$|Q_\alpha^k|^{-\tau p}|B(x,\delta^j)|^{-sp/d} \lesssim |B(x,\delta^j)|^{-\tau p-sp/d} \lesssim \delta^{-jd\tau p-j|s|p}[1+\rho(x,x_0)]^{|s|p+d\tau p}.$$

Inserting this estimate into (4.3), we know that

$$\begin{aligned}\mathrm{J} &\lesssim \mathscr{P}_{m,\ell}(f)\Bigg\{\sum_{j=k\vee 0}^{\infty}\delta^{j(\beta_0 m-d/2)q-j\tau dq-j|s|q}\\ &\quad\times\left[\int_M[1+\rho(x,x_0)]^{-\ell p+|s|p+d\tau p}\,d\mu(x)\right]^{q/p}\Bigg\}^{1/q} \lesssim \mathscr{P}_{m,\ell}(f),\end{aligned}$$

by using the fact that

$$\frac{1}{|B(x_0,1)|}\int_M[1+\rho(x,x_0)]^{-\ell p+|s|p+d\tau p}\,d\mu(x) \lesssim 1$$

when $\ell p-|s|p-d\tau p>d$. Hence, we obtain $f\in\widetilde{B}_{p,q}^{s,\tau}(M)$ and $\|f\|_{\widetilde{B}_{p,q}^{s,\tau}(M)} \lesssim \mathscr{P}_{m,\ell}(f)$, which implies that $\mathcal{D}(M)\hookrightarrow\widetilde{B}_{p,q}^{s,\tau}(M)$. In a similar way, one deduces that $\mathcal{D}(M)\hookrightarrow B_{p,q}^{s,\tau}(M)$. From these and (4.1), it follows that $\mathcal{D}(M)\hookrightarrow F_{p,q}^{s,\tau}(M)$ and $\mathcal{D}(M)\hookrightarrow\widetilde{F}_{p,q}^{s,\tau}(M)$.

Based on Proposition 4.1, we still need to show that $B_{\infty,\infty}^{s}(M)$, $\widetilde{B}_{\infty,\infty}^{s}(M)\hookrightarrow\mathcal{D}'(M)$ for all $s\in\mathbb{R}$. Indeed, for all $f\in\mathcal{D}'(M)$ and $\phi\in\mathcal{D}(M)$, by the Calderón reproducing formula (3.7), we have

$$\langle f,\phi\rangle=\sum_{j=0}^{\infty}\langle\Phi_j(\sqrt{L})f,\,\widetilde{\Phi}_j(\sqrt{L})\phi\rangle,$$

where $(\widetilde{\Phi}_0,\widetilde{\Phi})$ satisfy (2.1) and (2.2), and $\widetilde{\Phi}_j$ with $j\in\mathbb{Z}$ is defined as in (2.3). Given any $m,\ell\in\mathbb{N}$ such that $m\beta_0>d/2+|s|$ and $\ell>|s|+d$, applying (4.2) to $\widetilde{\Phi}_j(\sqrt{L})\phi$, we obtain

$$\begin{aligned}|\langle f,\phi\rangle| &\lesssim \mathscr{P}_{m,\ell}(\phi)\|f\|_{\widetilde{B}_{\infty,\infty}^{s}(M)}\sum_{j=0}^{\infty}\delta^{j(m\beta_0-d/2)}\int_M\frac{|B(x,\delta^j)|^{s/d}}{|B(x_0,1)|[1+\rho(x,x_0)]^{\ell}}\,d\mu(x)\\ &\lesssim \mathscr{P}_{m,\ell}(\phi)\|f\|_{\widetilde{B}_{\infty,\infty}^{s}(M)}\sum_{j=0}^{\infty}\delta^{j(m\beta_0-d/2-|s|)}\int_M\frac{1}{|B(x_0,1)|[1+\rho(x,x_0)]^{\ell-|s|}}\,d\mu(x)\\ &\lesssim \mathscr{P}_{m,\ell}(\phi)\|f\|_{\widetilde{B}_{\infty,\infty}^{s}(M)},\end{aligned}$$

which implies that $\widetilde{B}_{\infty,\infty}^{s}(M)\hookrightarrow\mathcal{D}'(M)$. Similarly, $B_{\infty,\infty}^{s}(M)\hookrightarrow\mathcal{D}'(M)$. This finishes the proof of Theorem 4.2. □

4.2 Classifications of Besov-type and Triebel–Lizorkin-type spaces

We begin with the case that τ is lager than $1/p$, and show that in this case both Besov-type and Triebel–Lizorkin-type spaces go back to the Besov spaces. Different from the corresponding result on $\mathbb{R}^n$ in [44], which was obtained via the coincidences between the related sequence spaces and the frame characterizations, here we give a more direct proof.

Proposition 4.3. *Let* $\tau\in(1/p,\infty)$, $s\in\mathbb{R}$ *and* $p,q\in(0,\infty]$ ($p<\infty$ *for* $F_{p,q}^{s,\tau}(M)$ *and* $\widetilde{F}_{p,q}^{s,\tau}(M)$). *Then,* $B_{p,q}^{s,\tau}(M)=F_{p,q}^{s,\tau}(M)=B_{\infty,\infty}^{s+d\tau-d/p}(M)$ *and* $\widetilde{B}_{p,q}^{s,\tau}(M)=\widetilde{F}_{p,q}^{s,\tau}(M)=\widetilde{B}_{\infty,\infty}^{s+d\tau-d/p}(M)$ *with equivalent (quasi-)norms.*

Proof. By similarity, we only show $\widetilde{B}^{s,\tau}_{p,q}(M)=\widetilde{F}^{s,\tau}_{p,q}(M)=\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)$. By (4.1) and Proposition 4.1, it suffices to prove $\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)\hookrightarrow\widetilde{B}^{s,\tau}_{p,q}(M)$. To this end, suppose that $f\in\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)$ with norm 1, then $|B(x,\delta^j)|^{-s/d-\tau+1/p}|\Phi_j(\sqrt{L})f(x)|\le 1$ for all $j\in\mathbb{Z}_+$ and $x\in M$. Then, applying the reverse doubling condition (1.4), we obtain

$$\begin{aligned}
&\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^\infty\left[\int_{Q^k_\alpha}|B(x,\delta^j)|^{-sp/d}|\Phi_j(\sqrt{L})f(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q}\\
&\quad\le\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^\infty\left[\int_{Q^k_\alpha}|B(x,\delta^j)|^{\tau p-1}\,d\mu(x)\right]^{q/p}\right\}^{1/q}\\
&\quad\lesssim\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^\infty\left[\delta^{(j-k)\kappa(\tau p-1)}\int_{Q^k_\alpha}|B(x,\delta^k)|^{\tau p-1}\,d\mu(x)\right]^{q/p}\right\}^{1/q}\lesssim 1,
\end{aligned}$$

where, in the last step, we used the fact $|B(x,\delta^k)|\sim|Q^k_\alpha|$ for all $x\in Q^k_\alpha$. This proves that $\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)\hookrightarrow\widetilde{B}^{s,\tau}_{p,q}(M)$ and hence finishes the proof of Proposition 4.3. ☐

The proofs of the following two propositions are similar to those proofs presented in [33, Section 3.2].

Proposition 4.4. *Let $\tau\in[1/p,\infty)$ and all other notation be as in Definition 2.2. Then, the supremum $\sup_{k\in\mathbb{Z},\,\alpha\in I_k}$ in Definition 2.2 can be equivalently replaced by $\sup_{k\in\mathbb{Z}_+,\,\alpha\in I_k}$.*

Proof. For any $k\in\mathbb{Z}$, let $g_k:=\left[\sum_{j=k\vee 0}^\infty|B(\cdot,\delta^j)|^{-sq/d}|\Phi_j(\sqrt{L})f|^q\right]^{p/q}$. When $k<0$, we have

$$\begin{aligned}
\frac{1}{|Q^k_\alpha|^{\tau p}}\int_{Q^k_\alpha}|g_k(x)|\,d\mu(x)&=\sum_{\{\beta\in I_0:\,Q^0_\beta\cap Q^k_\alpha\neq\emptyset\}}\left(\frac{|Q^0_\beta|}{|Q^\alpha_k|}\right)^{\tau p}\frac{1}{|Q^0_\beta|^{\tau p}}\int_{Q^0_\beta}|g_k(x)|\,d\mu(x)\\
&\le\left(\sum_{\{\beta\in I_0:\,Q^0_\beta\cap Q^k_\alpha\neq\emptyset\}}\frac{|Q^0_\beta|}{|Q^\alpha_k|}\right)^{\tau p}\sup_{k\in\mathbb{Z}_+,\alpha\in I_k}\frac{1}{|Q^k_\alpha|^{\tau p}}\int_{Q^k_\alpha}|g_k(x)|\,d\mu(x)\\
&\le\sup_{k\in\mathbb{Z}_+,\alpha\in I_k}\frac{1}{|Q^k_\alpha|^{\tau p}}\int_{Q^k_\alpha}|g_k(x)|\,d\mu(x),
\end{aligned}$$

where in the second step we used the fact $\tau p\ge 1$. This implies that

$$\|f\|_{\widetilde{F}^{s,\tau}_{p,q}(M)}\lesssim\sup_{\substack{k\in\mathbb{Z}_+\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha)|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=k}^\infty|B(\cdot,\delta^j)|^{-sq/d}|\Phi_j(\sqrt{L})f(x)|^q\right]^{p/q}d\mu(x)\right\}^{1/p}.$$

The converse of this inequality is obvious. The equivalence for the spaces $F^{s,\tau}_{p,q}(M)$, $\widetilde{B}^{s,\tau}_{p,q}(M)$ and $\widetilde{B}^{s,\tau}_{p,q}(M)$ follow from the same manner. This finishes the proof of Proposition 4.4. ☐

Proposition 4.5. *Let $\tau\in[0,1/p)$ and all other notation be as in Definition 2.2. Then, the summation $\sum_{j=k\vee 0}^\infty$ in Definition 2.2 can be equivalently replaced by $\sum_{j=0}^\infty$.*

Proof. By similarity, we only consider the space $\widetilde{F}^{s,\tau}_{p,q}(M)$. It suffices to show that, when $k>0$,

$$\mathrm{J}:=\frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=0}^{k-1}|B(x,\delta^j)|^{-sq/d}|\Phi_j(\sqrt{L})f(x)|^q\right]^{p/q}d\mu(x)\right\}^{1/p}\lesssim\|f\|_{\widetilde{F}^{s,\tau}_{p,q}(M)}.\tag{4.4}$$

By Proposition 4.1, we have $\widetilde{F}^{s,\tau}_{p,q}(M) \hookrightarrow \widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)$, so it suffices to show that J is bounded by a positive constant multiple of $\|f\|_{\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)}$. Indeed, using the reverse doubling condition (1.4), we see that, for all $x \in Q^k_\alpha$ and $0 \le j \le k-1$,

$$|B(x,\delta^j)| \gtrsim \delta^{(j-k)\kappa}|B(x,\delta^k)| \sim \delta^{(j-k)\kappa}|Q^k_\alpha|,$$

which, combined with the fact $\tau < 1/p$, further implies that

$$\begin{aligned}
\mathrm{J} &\le \|f\|_{\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)} \frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=0}^{k-1}|B(x,\delta^j)|^{q(\tau-1/p)}\right]^{p/q} d\mu(x)\right\}^{1/p} \\
&\lesssim \|f\|_{\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)} \frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=0}^{k-1}\delta^{q(\tau-1/p)(j-k)\kappa}|Q^k_\alpha|^{q(\tau-1/p)}\right]^{p/q} d\mu(x)\right\}^{1/p} \\
&\lesssim \|f\|_{\widetilde{B}^{s+d\tau-d/p}_{\infty,\infty}(M)}.
\end{aligned}$$

This finishes the proof of (4.4) and hence Proposition 4.5. $\square$

5 Equivalent characterizations

In this section, we characterize the Besov-type and the Triebel–Lizorkin-type spaces via the Peetre maximal functions and the heat semigroups; see Theorems 5.2, 5.8 and 5.9 below.

5.1 Peetre maximal function characterizations

For notation convenience, we introduce the following (quasi)-norms.

Definition 5.1. Let $p, q \in (0,\infty]$ and $\tau \in [0,\infty)$. The *space* $\ell^q(L^p_\tau)$ is defined to be the set of all sequences $\{g_j\}_{j\in\mathbb{Z}_+}$ of measurable functions on M such that

$$\|\{g_j\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} := \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\left[\int_{Q^k_\alpha}|g_j(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q} < \infty,$$

with a suitable modification made when $p=\infty$ or $q=\infty$. The *space* $L^p_\tau(\ell^q)$ (only for $p\in(0,\infty)$) is defined to be the set of all sequences $\{g_j\}_{j\in\mathbb{Z}_+}$ of measurable functions on M such that

$$\|\{g_j\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)} := \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=k\vee 0}^{\infty}|g_j(x)|^q\right]^{p/q} dx\right\}^{1/p} < \infty,$$

with a suitable modification made when $q=\infty$.

With all the notation as in Definition 5.1, we rewrite the (quasi-)norms defined in Definition 2.2 as follows:

$$\|f\|_{B^{s,\tau}_{p,q}(M)} = \|\{\delta^{-js}\Phi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}, \quad \|f\|_{\widetilde{B}^{s,\tau}_{p,q}(M)} = \|\{|B(\cdot,\delta^j)|^{-s/d}\Phi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)},$$

and

$$\|f\|_{F^{s,\tau}_{p,q}(M)} = \|\{\delta^{-js}\Phi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)}, \quad \|f\|_{\widetilde{F}^{s,\tau}_{p,q}(M)} = \|\{|B(\cdot,\delta^j)|^{-s/d}\Phi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)}.$$

Theorem 5.2. *Let* (Ψ_0, Ψ) *satisfy* (2.1) *and* (2.2). *For* $j \in \mathbb{N}$, *define* Ψ_j *as in* (2.3).

(i) *Let all the notation be as in Definition 2.2(i). If* $a > d(\tau + 1/p)$, *then there exists a constant* $C \in (1, \infty)$ *such that, for all* $f \in \mathcal{D}'(M)$,

$$C^{-1}\|f\|_{B^{s,\tau}_{p,q}(M)} \le \|\{\delta^{-js}[\Psi_j(\sqrt{L})]^*_a f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} \le C\|f\|_{B^{s,\tau}_{p,q}(M)} \tag{5.1}$$

and

$$C^{-1}\|f\|_{\widetilde{B}^{s,\tau}_{p,q}(M)} \le \|\{[\Psi_j(\sqrt{L})]^*_{a,-s/d} f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} \le C\|f\|_{\widetilde{B}^{s,\tau}_{p,q}(M)}. \tag{5.2}$$

(ii) *Let all the notation be as in Definition 2.2(ii). If* $a > d[\tau + 1/(p \wedge q)]$, *then there exists a constant* $C \in (1, \infty)$ *such that, for all* $f \in \mathcal{D}'(M)$,

$$C^{-1}\|f\|_{F^{s,\tau}_{p,q}(M)} \le \|\{\delta^{-js}[\Psi_j(\sqrt{L})]^*_a f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)} \le C\|f\|_{F^{s,\tau}_{p,q}(M)} \tag{5.3}$$

and

$$C^{-1}\|f\|_{\widetilde{F}^{s,\tau}_{p,q}(M)} \le \|\{[\Psi_j(\sqrt{L})]^*_{a,-s/d} f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)} \le C\|f\|_{\widetilde{F}^{s,\tau}_{p,q}(M)}. \tag{5.4}$$

(iii) *The spaces* $B^{s,\tau}_{p,q}(M)$, $\widetilde{B}^{s,\tau}_{p,q}(M)$, $F^{s,\tau}_{p,q}(M)$ *and* $\widetilde{F}^{s,\tau}_{p,q}(M)$ *are independent of the choices of the functions* (Φ_0, Φ) *satisfying* (2.1) *and* (2.2).

We recall that the corresponding results for Besov-type and Triebel–Lizorkin-type spaces on $\mathbb{R}^n$ can be found in [43, 26]. As an immediate consequence of Theorem 5.2, we have the following conclusion.

Corollary 5.3. *The spaces* $B^{s,\tau}_{p,q}(M)$, $\widetilde{B}^{s,\tau}_{p,q}(M)$, $F^{s,\tau}_{p,q}(M)$ *and* $\widetilde{F}^{s,\tau}_{p,q}(M)$ *are independent of the choices of* (Φ_0, Φ) *satisfying* (2.1) *and* (2.2).

To prove Theorem 5.2, we need the following estimate, which is simple but very useful.

Lemma 5.4. *Let* $\varepsilon \in (0, 1)$ *and* $\sigma \in (0, \infty)$. *Assume that* $\{\varepsilon_j\}_{j=0}^\infty \subset [0, 1]$ *satisfies that*

$$\left(\sum_{i=0}^\infty |\varepsilon_j|^{\varepsilon\sigma'}\right)^{1/\sigma'} =: B < \infty$$

when $\sigma \in (1, \infty)$, *where* σ' *denotes the conjugate index of* σ. *Then, for any sequence* $\{a_j\}_{j=0}^\infty \subset \mathbb{C}$,

$$\left(\sum_{j=0}^\infty |\varepsilon_j a_j|\right)^\sigma \le (\max\{B^\sigma, 1\}) \sum_{j=0}^\infty |\varepsilon_j|^{\sigma(1-\varepsilon)} |a_j|^\sigma. \tag{5.5}$$

Proof. If $\sigma \in (0, 1]$, then (5.5) follows from the facts that every $|\varepsilon_j| < 1$ and that

$$\left(\sum_{j=0}^\infty |\varepsilon_j||a_j|\right)^\sigma \le \sum_{j=0}^\infty |\varepsilon_j|^\sigma |a_j|^\sigma \le \sum_{j=0}^\infty |\varepsilon_j|^{\sigma(1-\varepsilon)} |a_j|^\sigma.$$

When $\sigma \in (1, \infty)$, Hölder's inequality implies that

$$\left(\sum_{j=0}^\infty |\varepsilon_j a_j|\right)^\sigma \le \left(\sum_{j=0}^\infty |\varepsilon_j|^{\varepsilon\sigma'}\right)^{\sigma/\sigma'} \left(\sum_{j=0}^\infty |\varepsilon_j|^{\sigma(1-\varepsilon)} |a_j|^\sigma\right) \le B^\sigma \sum_{j=0}^\infty |\varepsilon_j|^{\sigma(1-\varepsilon)} |a_j|^\sigma.$$

This finishes the proof of Lemma 5.4. □

The following estimate was essentially proved in [43, Lemma 2.3]; however we give a much simpler proof here by using Lemma 5.4.

Lemma 5.5. *Let* $b \in (0,1)$, $q \in (0,\infty]$, $\tau \in [0,\infty)$ *and* $\theta \in (d\tau,\infty)$. *Suppose that* $\{g_m\}_{m\in\mathbb{Z}_+}$ *are measurable functions on* M. *For all* $j \in \mathbb{Z}_+$, *let* $G_j := \sum_{m\in\mathbb{Z}_+} b^{|m-j|\theta} g_m$.

(i) *If* $p \in (0,\infty]$, *then there exists a positive constant* C, *independent of* $\{g_m\}_{m\in\mathbb{Z}_+}$, *such that*

$$\|\{G_j\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} \le C\|\{g_m\}_{m\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}.$$

(ii) *If* $p \in (0,\infty)$, *then there exists a positive constant* C, *independent of* $\{g_m\}_{m\in\mathbb{Z}_+}$, *such that*

$$\|\{G_j\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)} \le C\|\{g_m\}_{m\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)}.$$

Proof. By similarity, we only prove (i). Let $k \in \mathbb{Z}$ and $\alpha \in I_k$. Fix $\epsilon \in (0,1)$. Applying Lemma 5.4 twice, we find that

$$\begin{aligned}\mathrm{J}^k_\alpha &:= \frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\left[\int_{Q^k_\alpha}\left|\sum_{m=0}^{\infty} b^{|m-j|\theta} g_m(x)\right|^p d\mu(x)\right]^{q/p}\right\}^{1/q}\\ &\lesssim \frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\sum_{m=0}^{\infty} b^{|m-j|\theta q(1-\epsilon)^2}\left[\int_{Q^k_\alpha}|g_m(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q}.\end{aligned}$$

Clearly,

$$\mathrm{J}^{k,1}_\alpha := \frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\sum_{m=k\vee 0}^{\infty} b^{|m-j|\theta q(1-\epsilon)^2}\left[\int_{Q^k_\alpha}|g_m(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q} \lesssim \|\{g_m\}_{m\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}.$$

Thus, it suffices to show that

$$\mathrm{J}^{k,2}_\alpha := \frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\sum_{m=0}^{(k\vee 0)-1} b^{|m-j|\theta q(1-\epsilon)^2}\left[\int_{Q^k_\alpha}|g_m(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q} \lesssim \|\{g_m\}_{m\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}.$$

Notice that $\mathrm{J}^{k,2}_\alpha$ is void when $k \le 0$. If $k > 0$ and $0 \le m \le k-1$, then Q^k_α is covered by a finite number of dyadic cubes $\{Q^m_\beta\}_{\beta\in I}$, with $\#I \lesssim 1$ uniformly in m. For such cubes Q^m_β, they must intersect Q^k_α, so that $|Q^m_\beta| \lesssim \delta^{(m-k)d}|Q^k_\alpha|$. Since both $b, \delta \in (0,1)$, there exists a unique $k_0 \in \mathbb{N}$ such that $\delta^{k_0} \le b < \delta^{k_0-1}$. Hence,

$$\begin{aligned}\mathrm{J}^{k,2}_\alpha &\lesssim \sum_{\beta\in I}\frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k}^{\infty}\sum_{m=0}^{k-1} b^{(j-m)\theta q(1-\epsilon)^2}\left[\int_{Q^m_\beta}|g_m(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{\frac{1}{q}}\\ &\lesssim \sum_{\beta\in I}\left\{\sum_{j=k}^{\infty}\sum_{m=0}^{k-1} b^{(j-m)\theta q(1-\epsilon)^2}\delta^{(m-k)d\tau q}\right\}^{\frac{1}{q}}\|\{g_m\}_{m\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}\\ &\lesssim \sum_{\beta\in I}\left\{\sum_{j=k}^{\infty}\sum_{m=0}^{k-1} \delta^{k_0(j-k)\theta q(1-\epsilon)^2}\delta^{k_0(k-m)\theta q(1-\epsilon)^2}\delta^{(m-k)d\tau q}\right\}^{\frac{1}{q}}\|\{g_m\}_{m\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}\\ &\lesssim \|\{g_m\}_{m\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)},\end{aligned}$$

provided that we choose $\epsilon \in (0,1)$ such that $k_0\theta(1-\epsilon)^2 > d\tau$, while the latter is possible since $\theta > d\tau$ and $k_0 \ge 1$. This finishes the proof of Lemma 5.5. □

Proof of Theorem 5.2. Notice that (iii) follows directly from (i), (ii) and the fact that

$$|B(x,\delta^j)|^{\gamma}|\Psi_j(\sqrt{L})f(x)|\le[\Psi_j(\sqrt{L})]^*_{a,\gamma}f(x)$$

for all $j\in\mathbb{Z}_+$, $x\in M$, $a\in(0,\infty)$ and $\gamma\in\mathbb{R}$. Thus, it suffices to show (i) and (ii). Due to similarity, we only show (5.2) and (5.4).

Let $k\in\mathbb{Z}$ and $j\ge(k\vee 0)$. Assume that (Φ_0,Φ) satisfy (2.1) and (2.2), and Φ_j is defined as in (2.3) when $j\in\mathbb{N}$. With (Ψ_0,Ψ) as in Theorem 5.2, there exists $(\widetilde{\Psi}_0,\widetilde{\Psi})$ such that the Calderón reproducing formula (3.7) holds. Then, for all $x\in M$,

$$\Phi_j(\sqrt{L})f(x)=\sum_{\ell=0}^{\infty}\int_M\Phi_j(\sqrt{L})\widetilde{\Psi}_\ell(\sqrt{L})(x,y)\Psi_\ell(\sqrt{L})f(y)\,d\mu(y). \tag{5.6}$$

Here the dilated functions $\widetilde{\Psi}_\ell$ are defined as in (2.3). Notice that the kernel $\Phi_j(\sqrt{L})\widetilde{\Psi}_\ell(\sqrt{L})$ is non-zero only when $\operatorname{supp}\Phi_j\cap\operatorname{supp}\widetilde{\Psi}_\ell\neq\emptyset$, which implies that the summation in (5.6) is only valid for these ℓ's satisfying $|\ell-j|\le 2$. For such ℓ, applying Proposition 3.4(i) and Lemma 3.1(i), we see that, for any $\sigma>|s|+a+d$ and $x,y\in M$,

$$|\Phi_j(\sqrt{L})\widetilde{\Phi}_\ell(\sqrt{L})(x,y)|\lesssim\int_M D_{\delta^j,\sigma}(x,z)D_{\delta^\ell,\sigma}(z,y)\,d\mu(z)\lesssim D_{\delta^\ell,\sigma}(x,y).$$

By this and Lemma 3.1(ii), we know that, for all $x\in M$,

$$\begin{aligned}
&|B(x,\delta^j)|^{-s/d}|\Phi_j(\sqrt{L})f(x)|\\
&\quad\lesssim|B(x,\delta^j)|^{-s/d}\sum_{\substack{\ell\ge0\\|\ell-j|\le2}}\int_M D_{\delta^\ell,\sigma}(x,y)|\Psi_\ell(\sqrt{L})f(y)|\,d\mu(y)\\
&\quad\lesssim\sum_{\substack{\ell\ge0\\|\ell-j|\le2}}\sup_{y\in M}\frac{|B(y,\delta^\ell)|^{-s/d}|\Psi_\ell(\sqrt{L})f(y)|}{[1+\delta^{-\ell}\rho(x,y)]^a}\int_M\frac{[1+\delta^{-\ell}\rho(x,y)]^{-(\sigma-|s|-a)}}{\sqrt{|B(x,\delta^\ell)||B(y,\delta^\ell)|}}\,d\mu(y)\\
&\quad\lesssim\sum_{\substack{\ell\ge0\\|\ell-j|\le2}}[\Psi_\ell(\sqrt{L})]^*_{a,-s/d}f(x).
\end{aligned}\tag{5.7}$$

Noticing that the sum in ℓ has at most 5 terms, we then apply Lemma 5.5 to conclude that

$$\|f\|_{\widetilde{B}^{s,\tau}_{p,q}(M)}=\|\{|B(\cdot,\delta^j)|^{-s/d}\Phi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}\lesssim\|\{[\Psi_j(\sqrt{L})]^*_{a,-s/d}f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} \tag{5.8}$$

and

$$\|f\|_{\widetilde{F}^{s,\tau}_{p,q}(M)}=\|\{|B(\cdot,\delta^j)|^{-s/d}\Phi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)}\lesssim\|\{[\Psi_j(\sqrt{L})]^*_{a,-s/d}f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)}. \tag{5.9}$$

This proves the first inequalities of (5.2) and (5.4).

To finish the proof of (5.2), it remains to show the converse of (5.8). To this end, it suffices to prove that

$$\|\{[\Psi_j(\sqrt{L})]^*_{a,-s/d}f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}\lesssim\|\{|B(\cdot,\delta^j)|^{-s/d}\Psi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}. \tag{5.10}$$

Indeed, if (5.10) holds, then

$$\|\{[\Psi_j(\sqrt{L})]^*_{a,-s/d}f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}\lesssim\|\{|B(\cdot,\delta^j)|^{-s/d}\Psi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}$$

$$
\begin{aligned}
&\lesssim \|\{[\Phi_j(\sqrt{L})]^*_{a,-s/d} f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}\\
&\lesssim \|\{|B(\cdot,\delta^j)|^{-s/d}\Phi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)},
\end{aligned}
$$

where the second inequality is deduced from reversing the roles of (Φ_0,Φ) and (Ψ_0,Ψ) in (5.8), and the third inequality is deduced from reversing the roles of (Φ_0,Φ) and (Ψ_0,Ψ) in (5.10).

Now we show (5.10). Since $a > d(\tau+1/p)$, we can find $\nu\in(0,\infty)$ and $r\in(0,p)$ such that $a\geq \nu+d/r$ and $\nu > d\tau$. Applying Proposition 3.11, we know that, for any $x\in Q^k_\alpha$, with $k\in\mathbb{Z}$ and $\alpha\in I_k$, and for all $\ell\geq(k\vee 0)$,

$$
[\Psi_\ell(\sqrt{L})]^*_{a,-s/d}f(x)\lesssim \sum_{j=\ell}^\infty\sum_{i=0}^\infty \delta^{(j-\ell)\nu}\delta^{i\nu}\mathcal{M}_r\left(|B(\cdot,\delta^j)|^{-s/d}|\Psi_j(\sqrt{L})f|\chi_{B(z^k_\alpha,\delta^{\ell-i}+C_\natural\delta^k)}\right)(x).
$$

For notation convenience, we let $g_j:=|B(\cdot,\delta^j)|^{-s/d}|\Psi_j(\sqrt{L})f|$. Fix $\epsilon\in(0,1)$ small enough such that $\nu(1-\epsilon)>d\tau$. Then, Lemma 5.4 implies that

$$
\left\{[\Psi_\ell(\sqrt{L})]^*_{a,-s/d}f(x)\right\}^p\lesssim \sum_{j=\ell}^\infty\sum_{i=0}^\infty\left[\delta^{(j-\ell)\nu}\delta^{i\nu}\right]^{p(1-\epsilon)}\left[\mathcal{M}_r\left(g_j\chi_{B(z^k_\alpha,\delta^{k-i}+C_\natural\delta^k)}\right)(x)\right]^p.
$$

Since $0<r<p$, by the boundedness of $\mathcal{M}$ on $L^{p/r}(M)$, we have

$$
\int_{Q^k_\alpha}\left\{[\Psi_\ell(\sqrt{L})]^*_{a,-s/d}f(x)\right\}^p d\mu(x)\lesssim \sum_{j=\ell}^\infty\sum_{i=0}^\infty\left[\delta^{(j-\ell)\nu}\delta^{i\nu}\right]^{p(1-\epsilon)}\int_{B(z^k_\alpha,\delta^{k-i}+C_\natural\delta^k)}|g_j(x)|^p\,d\mu(x).
$$

For every $i\in\mathbb{N}$, define

$$
\mathcal{J}_{k,i}:=\left\{\beta\in I_{k-i}:\ Q^{k-i}_\beta\cap B(z^k_\alpha,\delta^{k-i}+C_\natural\delta^k)\neq\emptyset\right\}. \tag{5.11}
$$

Clearly, the union of all Q^{k-i}_β with $\beta\in\mathcal{J}_{k,i}$ covers the ball $B(z^k_\alpha,\delta^{k-i}+C_\natural\delta^k)$. Since $\{Q^{k-i}_\beta\}_{\beta\in\mathcal{J}_{k,i}}$ are mutually disjoint, it follows that

$$
\sharp\mathcal{J}_{k,i}\leq C, \tag{5.12}
$$

where C is a positive constant depending only on δ and K. Also, notice that, for any $\beta\in\mathcal{J}_{k,i}$,

$$
|Q^{k-i}_\beta|\lesssim|B(z^k_\alpha,(C_\natural+1)\delta^{k-i}+C_\natural\delta^k)|\lesssim\delta^{-id}|Q^k_\alpha|. \tag{5.13}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{|Q^k_\alpha|^{\tau p}}\int_{Q^k_\alpha}\left\{[\Psi_\ell(\sqrt{L})]^*_{a,-s/d}f(x)\right\}^p d\mu(x)\\
&\quad\lesssim\sum_{j=\ell}^\infty\sum_{i=0}^\infty\sum_{\beta\in\mathcal{J}_{k,i}}\frac{\left[\delta^{(j-\ell)\nu}\delta^{i\nu}\right]^{p(1-\epsilon)}\delta^{-id\tau p}}{|Q^{k-i}_\beta|^{\tau p}}\int_{Q^{k-i}_\beta}|g_j(x)|^p\,d\mu(x).
\end{aligned}
$$

Again, using Lemma 5.4, we see that

$$
\frac{1}{|Q^k_\alpha|^{\tau q}}\sum_{\ell=k\vee 0}^\infty\left[\int_{Q^k_\alpha}\left\{[\Psi_\ell(\sqrt{L})]^*_{a,-s/d}f(x)\right\}^p d\mu(x)\right]^{q/p}
$$

$$
\begin{aligned}
&\lesssim \sum_{\ell=k\vee 0}^{\infty}\left\{\sum_{j=\ell}^{\infty}\sum_{i=0}^{\infty}\sum_{\beta\in\mathcal{J}_{k,i}}\frac{\left[\delta^{(j-\ell)\nu}\delta^{i\nu}\right]^{p(1-\epsilon)}\delta^{-id\tau p}}{|Q_\beta^{k-i}|^{\tau p}}\int_{Q_\beta^{k-i}}|g_j(x)|^p\,d\mu(x)\right\}^{q/p}\\
&\lesssim \sum_{\ell=k\vee 0}^{\infty}\sum_{j=\ell}^{\infty}\sum_{i=0}^{\infty}\sum_{\beta\in\mathcal{J}_{k,i}}\left[\delta^{(j-\ell)\nu}\delta^{i\nu}\right]^{q(1-\epsilon)^2}\delta^{-id\tau q(1-\epsilon)}\left[\frac{1}{|Q_\beta^{k}|^{\tau p}}\int_{Q_\beta^{k}}|g_j(x)|^p\,d\mu(x)\right]^{q/p}\\
&\lesssim \|\{|B(\cdot,\delta^j)|^{-s/d}\Psi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)}^q,
\end{aligned}
$$

where the last step follows from interchanging the summations over j and ℓ and then using the inequality

$$
\sum_{\ell=k\vee 0}^{j}\sum_{i=0}^{\infty}\sum_{\beta\in\mathcal{J}_{k,i}}\left[\delta^{(j-\ell)\nu}\delta^{i\nu}\right]^{q(1-\epsilon)^2}\delta^{-id\tau q(1-\epsilon)}\lesssim 1.
$$

Thus, (5.10) holds. This finishes the proof of (5.2).

To obtain (5.4), it suffices to show the converse of (5.9), which follows from

$$
\|\{[\Psi_j(\sqrt{L})]^*_{a,-s/d}f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)}\lesssim\|\{|B(\cdot,\delta^j)|^{-s/d}\Psi_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)}. \tag{5.14}
$$

To see (5.14), we again use the notation $g_j:=|B(\cdot,\delta^j)|^{-s/d}|\Psi_j(\sqrt{L})f|$. Since $a>d[\tau+1/(p\wedge q)]$, we choose r,ν,ϵ satisfying that $r\in(0,p\wedge q)$, $d\tau<\nu<a-d/r$ and $\nu(1-\epsilon)^3>d\tau$. Applying Proposition 3.11 and Lemma 5.4, we find that, for any $x\in Q_\alpha^k$ with $k\in\mathbb{Z}$ and $\alpha\in I_k$, and for all $\ell\ge k\vee 0$,

$$
\left\{[\Psi_\ell(\sqrt{L})]^*_{a,-s/d}f(x)\right\}^q\lesssim\sum_{j=\ell}^{\infty}\sum_{i=0}^{\infty}\left[\delta^{(j-\ell)\nu}\delta^{i\nu}\right]^{q(1-\epsilon)}\left[\mathcal{M}_r\left(g_j\chi_{B(z_\alpha^k,\delta^{k-i}+C_\natural\delta^k)}\right)(x)\right]^q.
$$

Then, applying the vector-valued Fefferman-Stein maximal inequality for spaces of homogeneous type (see [17]), we have

$$
\begin{aligned}
\mathrm{Z}_\alpha^k&:=\frac{1}{|Q_\alpha^k|^\tau}\left\{\int_{Q_\alpha^k}\left(\sum_{\ell=k\vee 0}^{\infty}\left\{[\Psi_\ell(\sqrt{L})]^*_{a,-s/d}f(x)\right\}^q\right)^{p/q}d\mu(x)\right\}^{1/p}\\
&\lesssim\frac{1}{|Q_\alpha^k|^\tau}\left\{\int_M\left(\sum_{\ell=k\vee 0}^{\infty}\sum_{j=\ell}^{\infty}\sum_{i=0}^{\infty}\left[\delta^{(j-\ell)\nu}\delta^{i\nu}\right]^{q(1-\epsilon)}|g_j(x)|^q\chi_{B(z_\alpha^k,\delta^{k-i}+C_\natural\delta^k)}(x)\right)^{p/q}d\mu(x)\right\}^{1/p}\\
&\lesssim\sum_{i=0}^{\infty}\delta^{i\nu(1-\epsilon)^3}\frac{1}{|Q_\alpha^k|^\tau}\left\{\int_{B(z_\alpha^k,\delta^{k-i}+C_\natural\delta^k)}\left[\sum_{\ell=k\vee 0}^{\infty}\sum_{j=\ell}^{\infty}\delta^{(j-\ell)\nu q(1-\epsilon)}|g_j(x)|^q\right]^{p/q}d\mu(x)\right\}^{1/p}\\
&\lesssim\sum_{i=0}^{\infty}\delta^{i\nu(1-\epsilon)^3}\frac{1}{|Q_\alpha^k|^\tau}\left\{\int_{B(z_\alpha^k,\delta^{k-i}+C_\natural\delta^k)}\left[\sum_{j=k\vee 0}^{\infty}|g_j(x)|^q\right]^{p/q}d\mu(x)\right\}^{1/p},
\end{aligned}
$$

where in the second inequality we used Lemma 5.4 twice and in the third inequality we interchanged the summations over j and ℓ. For every $i\in\mathbb{N}$, define $\mathcal{J}_{k,i}$ as in (5.11). From the estimates (5.12) and (5.13), we deduce that

$$
\begin{aligned}
\mathrm{Z}_\alpha^k&\lesssim\sum_{i=0}^{\infty}\delta^{i\nu(1-\epsilon)^3}\left\{\sum_{\beta\in\mathcal{J}_{k,i}}\frac{\delta^{-id\tau p}}{|Q_\beta^{k-i}|^{\tau p}}\int_{Q_\beta^{k-i}}\left[\sum_{j=k\vee 0}^{\infty}|g_j(x)|^q\right]^{p/q}d\mu(x)\right\}^{1/p}\\
&\lesssim\sum_{i=0}^{\infty}\delta^{i\nu(1-\epsilon)^3}\delta^{-id\tau}\|\{g_j\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)}\lesssim\|\{g_j\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)},
\end{aligned}
$$

by $\sum_{i=0}^{\infty}\delta^{i\nu(1-\epsilon)^3}\delta^{-id\tau}\lesssim 1$. Thus, (5.14) holds. This shows (5.4) and hence Theorem 5.2. □

5.2 Heat kernel characterizations

Applying the Peetre maximal function characterizations in Theorem 5.2, we now characterize Besov-type and Triebel–Lizorkin-type spaces via the heat semigroup.

Definition 5.6. Let $\tau \in [0,\infty)$, $s \in \mathbb{R}$ and $m \in \mathbb{N}$ such that $m > s/\beta_0$. For any $\lambda \in (0,\infty)$ and $j \in \mathbb{N}$, define

$$(5.15) \quad h_0(\lambda) := e^{-\lambda^2}, \qquad h(\lambda) := \lambda^{2m} e^{-\lambda^2} \quad \text{and} \quad h_j(\lambda) := h(\delta^{j\beta_0/2}\lambda) = (\delta^{j\beta_0/2}\lambda)^{2m} e^{-\delta^{j\beta_0}\lambda^2}.$$

If $p, q \in (0,\infty]$, then define $B^{s,\tau}_{p,q}(H)$ to be the collection of all $f \in \mathcal{D}'(M)$ such that

$$\|f\|_{B^{s,\tau}_{p,q}(H)} := \|\{\delta^{-js} h_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} < \infty,$$

and define $\widetilde{B}^{s,\tau}_{p,q}(H)$ to be the collection of all $f \in \mathcal{D}'(M)$ such that

$$\|f\|_{\widetilde{B}^{s,\tau}_{p,q}(H)} := \|\{|B(\cdot,\delta^j)|^{-s/d} h_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} < \infty.$$

If $p \in (0,\infty)$ and $q \in (0,\infty]$, then define $F^{s,\tau}_{p,q}(H)$ to be the collection of all $f \in \mathcal{D}'(M)$ such that

$$\|f\|_{F^{s,\tau}_{p,q}(H)} := \|\{\delta^{-js} h_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)} < \infty,$$

and define $\widetilde{F}^{s,\tau}_{p,q}(H)$ to be the collection of all $f \in \mathcal{D}'(M)$ such that

$$\|f\|_{\widetilde{F}^{s,\tau}_{p,q}(H)} := \|\{|B(\cdot,\delta^j)|^{-s/d} h_j(\sqrt{L})f\}_{j\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)} < \infty.$$

Remark 5.7. Notice that, for any given $\sigma > 0$ and $k \in \mathbb{Z}_+$, it follows from Proposition 3.4 that

$$|L^k e^{-L}(x,y)| \lesssim D_{1,\sigma}(x,y) \qquad \text{and} \qquad |L^k h_j(\sqrt{L})(x,y)| \lesssim \delta^{-2k} D_{\delta^j,\sigma}(x,y)$$

for all $x, y \in M$. This implies that $e^{-L}(x,y), h_j(\sqrt{L})(x,y) \in \mathcal{D}(M)$ as a function of variable x or as a function of variable y. Thus, $e^{-L}f$ and $h_j(\sqrt{L})f$ make sense for any $f \in \mathcal{D}'(M)$.

Now we have the following discrete heat kernel characterizations. Comparing with the continuous versions for Besov and Triebel-Lizorkin spaces in [22, Theorems 6.7 and 7.5], wherein $p \in [1,\infty]$, there is no restriction on p in Theorem 5.8.

Theorem 5.8. *Let $q \in (0,\infty]$, $\tau \in [0,\infty)$, $s \in \mathbb{R}$ and $m \in \mathbb{N}$ such that $m > s/\beta_0$.*

(i) *If $p \in (0,\infty]$, then $B^{s,\tau}_{p,q}(M) = B^{s,\tau}_{p,q}(H)$ and $\widetilde{B}^{s,\tau}_{p,q}(M) = \widetilde{B}^{s,\tau}_{p,q}(H)$ with equivalent (quasi)-norms.*

(ii) *If $p \in (0,\infty)$, then $F^{s,\tau}_{p,q}(M) = F^{s,\tau}_{p,q}(H)$ and $\widetilde{F}^{s,\tau}_{p,q}(M) = \widetilde{F}^{s,\tau}_{p,q}(H)$ with equivalent (quasi)-norms.*

Proof. Due to similarity, we only show $\widetilde{B}^{s,\tau}_{p,q}(M) = \widetilde{B}^{s,\tau}_{p,q}(H)$ and $\widetilde{F}^{s,\tau}_{p,q}(M) = \widetilde{F}^{s,\tau}_{p,q}(H)$. With (Φ_0, Φ) as in Definition 2.2, there exist $(\widetilde{\Phi}_0, \widetilde{\Phi})$ satisfying (2.1) and (2.2) such that the Calderón reproducing formula (3.7) holds. Hence, for all $j \in \mathbb{Z}_+$ and $x \in M$,

$$(5.16) \qquad h_j(\sqrt{L})f(x) = \sum_{\ell=0}^{\infty} h_j(\sqrt{L})\widetilde{\Phi}_\ell(\sqrt{L})\Phi_\ell(\sqrt{L})f(x).$$

Given any $\sigma > 0$, we claim that, for all $j, \ell \in \mathbb{Z}_+$ and $x, y \in M$,

$$|h_j(\sqrt{L})\widetilde{\Phi}_\ell(\sqrt{L})(x,y)| \lesssim \delta^{(j-\ell)\beta_0 m} e^{-\delta^{(j-\ell+1)\beta_0/2}} D_{\delta^\ell,\sigma}(x,y). \tag{5.17}$$

Now we show (5.17). Consider first the case $j \in \mathbb{N}$. Write $h_j(\sqrt{L})\widetilde{\Phi}_0(\sqrt{L}) = \Psi(\sqrt{L})$, where $\Psi(\lambda) := \delta^{j\beta_0 m}\lambda^{2m}e^{-\delta^{j\beta_0}\lambda^2}\widetilde{\Phi}_0(\lambda)$ for all $\lambda \in \mathbb{R}_+$. One easily verifies that Ψ has compact support and that, for any $\nu, i \in \mathbb{Z}_+$,

$$|\Psi^{(2\nu+1)}(0)| = 0, \qquad |\Psi^{(\nu)}(\lambda)| \le C_{i,\nu}\delta^{j\beta_0 m}(1+\lambda)^{-i},$$

where $C_{i,\nu} \in (0,\infty)$ is independent of λ and j. Hence, Proposition 3.4 implies that, for all $x, y \in M$,

$$|h_j(\sqrt{L})\widetilde{\Phi}_0(\sqrt{L})(x,y)| = |\Psi(\sqrt{L})(x,y)| \lesssim \delta^{j\beta_0 m} D_{1,\sigma}(x,y),$$

and hence (5.17) holds for $j \in \mathbb{N}$ and $\ell = 0$. If $\ell \in \mathbb{N}$, to prove (5.17), we write $h_j(\sqrt{L})\widetilde{\Phi}_\ell(\sqrt{L}) = \omega(\delta^{\ell\beta_0/2}\sqrt{L})$ with $\omega(\lambda) := \delta^{(j-\ell)m\beta_0}\lambda^{2m}e^{-\delta^{(j-\ell)\beta_0}\lambda^2}\widetilde{\Phi}(\lambda)$ for all $\lambda \in \mathbb{R}_+$. By the properties of $\widetilde{\Phi}$, we know that $\operatorname{supp}\omega \subset [\delta^{\beta_0/2}, \delta^{-\beta_0/2}]$ and that, for any $\nu, i \in \mathbb{Z}_+$,

$$|\omega^{(\nu)}(\lambda)| \le C_{i,\nu}\delta^{(j-\ell)\beta_0 m}e^{-\delta^{(j-\ell+1)\beta_0/2}}(1+\lambda)^{-i},$$

where $C_{i,\nu} \in (0,\infty)$ is independent of λ, j and ℓ. Then, applying Proposition 3.4, we find that

$$|h_j(\sqrt{L})\widetilde{\Phi}_\ell(\sqrt{L})(x,y)| = |\omega(\sqrt{L})(x,y)| \lesssim \delta^{(j-\ell)\beta_0 m}e^{-\frac{1}{2}\delta^{(j-\ell+1)\beta_0}} D_{\delta^\ell,\sigma}(x,y),$$

which proves (5.17) for $j, \ell \in \mathbb{N}$. Similar argument shows (5.17) for $j = 0$ and $\ell \in \mathbb{Z}_+$.

Fix $k \in \mathbb{Z}$, $j \ge (k \vee 0)$, and $x \in Q_\alpha^k$ for some $\alpha \in I_k$. Let a be a sufficiently large number satisfying the condition of Theorem 5.2. By (5.16) and (5.17), we see that

$$\begin{aligned}
&|B(x,\delta^j)|^{-s/d}|h_j(\sqrt{L})f(x)| \\
&\quad\lesssim \sum_{\ell=0}^{\infty} \delta^{(j-\ell)\beta_0 m}e^{-\frac{1}{2}\delta^{(j-\ell+1)\beta_0}}|B(x,\delta^j)|^{-s/d}\int_M D_{\delta^\ell,a+|s|}(x,y)|\Phi_\ell(\sqrt{L})f(y)|\,d\mu(y) \\
&\quad\lesssim \sum_{\ell=0}^{\infty} \delta^{(j-\ell)\beta_0 m}e^{-\frac{1}{2}\delta^{(j-\ell+1)\beta_0}}\left(\max\{1,\,\delta^{(\ell-j)s}\}\right)[\Phi_\ell(\sqrt{L})]_{a,-s/d}^{*}f(x).
\end{aligned} \tag{5.18}$$

Notice that $\sum_{\ell=0}^{\infty}\delta^{(j-\ell)\beta_0 m}e^{-\frac{1}{2}\delta^{(j-\ell+1)\beta_0}}\left(\max\{1,\,\delta^{(\ell-j)s}\}\right) < \infty$ when $m > s/\beta_0$. From this, Lemma 5.5 and Theorem 5.2, we deduce that

$$\|f\|_{\widetilde{B}_{p,q}^{s,\tau}(H)} \lesssim \|\{[\Phi_j(\sqrt{L})]_{a,-s/d}^{*}f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L_\tau^p)} \sim \|f\|_{\widetilde{B}_{p,q}^{s,\tau}(M)}$$

and

$$\|f\|_{\widetilde{F}_{p,q}^{s,\tau}(H)} \lesssim \|\{[\Phi_j(\sqrt{L})]_{a,-s/d}^{*}f\}_{j\in\mathbb{Z}_+}\|_{L_\tau^p(\ell^q)} \sim \|f\|_{\widetilde{F}_{p,q}^{s,\tau}(M)}.$$

It remains to prove the converse of these two inequalities. Let $r \in (0, \min\{1,p,q\})$ and a be sufficiently large. Define $\phi_0(\lambda) := e^{\lambda^2}\Phi_0(\lambda)$ and $\phi(\lambda) := \lambda^{-m}e^{\lambda^2}\Phi(\lambda)$, where $\lambda \in \mathbb{R}_+$. For any $k \in \mathbb{Z}$, $\ell \in \mathbb{N}$ such that $\ell \ge k \vee 0$, and $x \in Q_\alpha^k$ for some $\alpha \in I_k$, the functional calculus gives that

$$|B(x,\delta^\ell)|^{-s/d}\Phi_\ell(\sqrt{L})f(x) = |B(x,\delta^\ell)|^{-s/d}\phi_\ell(\sqrt{L})h_\ell(\sqrt{L})f(x). \tag{5.19}$$

Given any $\sigma \in (0,\infty)$, according to Proposition 3.4, it holds true that, for all $x, y \in M$,

$$|\phi_\ell(\sqrt{L})(x,y)| \lesssim D_{\delta^\ell,\sigma}(x,y). \tag{5.20}$$

From (5.19) and (5.20), it follows that, when $\sigma > a + 2d + |s|$,

$$(5.21)\quad |B(x,\delta^\ell)|^{-s/d}|\Phi_\ell(\sqrt{L})f(x)| \lesssim \int_M D_{\delta^\ell,\sigma-|s|}(x,y)|B(y,\delta^\ell)|^{-s/d}h_\ell(\sqrt{L})f(y)\,d\mu(y)$$
$$\lesssim [h_\ell(\sqrt{L})]^*_{a,-s/d}f(x).$$

Furthermore, Proposition 3.11 implies that, for all $x \in M$,

$$(5.22)\quad [h_\ell(\sqrt{L})]^*_{a,-s/d}f(x) \lesssim \sum_{j=\ell}^{\infty}\sum_{i=0}^{\infty} \delta^{(j-\ell)\nu}\delta^{i\nu r}\mathcal{M}_r\left(|B(\cdot,\delta^j)|^\gamma|h_j(\sqrt{L})f|\chi_{B(z^k_\alpha,\delta^{j-i}+C_\natural\delta^k)}\right)(x).$$

Invoking this and (5.21), we proceed the same lines as in the proof of (5.10) to obtain

$$\|f\|_{\widetilde{B}^{s,\tau}_{p,q}(M)} = \|\{|B(\cdot,\delta^\ell)|^{-s/d}|\Phi_\ell(\sqrt{L})f|\}_{\ell\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} \lesssim \|f\|_{\widetilde{B}^{s,\tau}_{p,q}(H)}.$$

Likewise, applying (5.21) and (5.22), we follow the same procedure as in the proof of (5.14) to see that

$$\|f\|_{\widetilde{F}^{s,\tau}_{p,q}(M)} = \|\{|B(\cdot,\delta^\ell)|^{-s/d}|\Phi_\ell(\sqrt{L})f|\}_{\ell\in\mathbb{Z}_+}\|_{L^p_\tau(\ell^q)} \lesssim \|f\|_{\widetilde{F}^{s,\tau}_{p,q}(H)},$$

which completes the proof of Theorem 5.8. □

As in [22, Theorems 6.7 and 7.5], continuous versions of the heat semigroup characterizations of the Besov-type and Triebel-Lizorkin type spaces also hold true. In what follows, for all $p \in (0,\infty]$, $\tau \in [0,\infty)$ and $f \in \mathcal{D}'(M)$, we let

$$\|f\|_{p,\tau} := \sup_{k\le 0,\ \alpha\in I_k}\left[\frac{1}{|Q^k_\alpha|^\tau}\int_{Q^k_\alpha}|e^{-L}f(x)|^p\,d\mu(x)\right]^{1/p}$$

and

$$\widetilde{\|f\|}_{p,\tau} := \sup_{k\le 0,\ \alpha\in I_k}\left[\frac{1}{|Q^k_\alpha|^\tau}\int_{Q^k_\alpha}|B(x,1)|^{-sp/d}|e^{-L}f(x)|^p\,d\mu(x)\right]^{1/p}$$

with the usual modification made when $p = \infty$.

Theorem 5.9. *Let $\tau \in [0,\infty)$, $s \in \mathbb{R}$ and $m \in \mathbb{N}$ such that $m > s/\beta_0$.*

(i) *If $p \in [1,\infty]$ and $q \in (0,\infty]$, then $\|f\|_{B^{s,\tau}_{p,q}(M)}$ is equivalent to*

$$(5.23)\quad \|f\|_{p,\tau} + \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\int_0^{\min\{1,\delta^k\}}\left[\int_{Q^k_\alpha}t^{-sp}|(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f(x)|^p\,d\mu(x)\right]^{q/p}\frac{dt}{t}\right\}^{1/q}$$

for all $f \in \mathcal{D}'(M)$, and a similar result also holds true for $\|\cdot\|_{\widetilde{B}^{s,\tau}_{p,q}(M)}$, but with $\|f\|_{p,\tau}$ and t^{-sp} in (5.23) replaced by $\widetilde{\|f\|}_{p,\tau}$ and $|B(x,t)|^{-sp/d}$, respectively.

(ii) *If $p \in [1,\infty)$ and $q \in [1,\infty]$, then $\|f\|_{F^{s,\tau}_{p,q}(M)}$ is equivalent to*

$$(5.24)\quad \|f\|_{p,\tau} + \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\int_0^{\min\{1,\delta^k\}}t^{-sq}|(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f(x)|^q\,\frac{dt}{t}\right]^{p/q}d\mu(x)\right\}^{1/p}$$

for all $f \in \mathcal{D}'(M)$, and a similar result also holds true for $\|\cdot\|_{\widetilde{F}^{s,\tau}_{p,q}(M)}$, but with $\|f\|_{p,\tau}$ and t^{-sq} in (5.24) replaced by $\widetilde{\|f\|}_{p,\tau}$ and $|B(x,t)|^{-sq/d}$, respectively.

Proof. By similarity, we only consider $\|\cdot\|_{\widetilde{B}^{s,\tau}_{p,q}(M)}$ in (i). Via writing the integral in (5.23) as

$$\int_0^{\min\{1,\delta^k\}}\cdots\frac{dt}{t}=\sum_{j=k\vee 0}^{\infty}\int_{\delta^{j+1}}^{\delta^j}\cdots\frac{dt}{t}$$

and noticing that $t\sim\delta^j$ when $t\in[\delta^{j+1},\,\delta^j]$, we know that

$$\widetilde{\|f\|}_{p,\tau}+\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\int_0^{\min\{1,\delta^k\}}\left[\int_{Q^k_\alpha}t^{-sp}|(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f(x)|^p\,d\mu(x)\right]^{q/p}\frac{dt}{t}\right\}^{1/q}\lesssim\|f\|_{\widetilde{B}^{s,\tau}_{p,q}(M)},$$

by following the same procedure as the first part of the proof of Theorem 5.8. The proof of this direction works for all $p\in(0,\infty]$.

Now we consider the inverse inequality. This time we assume that $p\in[1,\infty]$. Fix $x\in Q^k_\alpha$, with $k\in\mathbb{Z}$ and $\alpha\in I_k$, and let $\sigma>d\tau q+|s|+d+1$. By an argument similar to that used in the proof of (5.21), we see that, for all $t\sim\delta^\ell$ with $\ell\geq(k\vee 1)$ and $k\in\mathbb{Z}$,

$$\begin{aligned}|B(x,\delta^\ell)|^{-s/d}|\Phi_\ell(\sqrt{L})f(x)|&\lesssim\int_M D_{\delta^\ell,\sigma-|s|}(x,y)|B(y,t)|^{-s/d}(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f(y)\,d\mu(y)\\&\lesssim\left[\int_M D_{\delta^\ell,\sigma-|s|}(x,y)|B(y,t)|^{-sp/d}|(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f(y)|^p\,d\mu(y)\right]^{1/p},\end{aligned}$$

where the second inequality is due to Hölder's inequality and Lemma 3.1(i). Splitting the integral over M into annuals, when $\ell\in\mathbb{N}$, we write

$$\begin{aligned}&|B(x,\delta^\ell)|^{-sp/d}|\Phi_\ell(\sqrt{L})f(x)|^p\\&\quad\lesssim\int_{B(z^k_\alpha,C^\natural\delta^k)}D_{\delta^\ell,\sigma-|s|}(x,y)|B(y,t)|^{-sp/d}|(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f(y)|^p\,d\mu(y)\\&\qquad+\sum_{j\in\mathbb{N}}\int_{C^\natural\delta^{k-j+1}\leq\rho(y,z^k_\alpha)<C^\natural\delta^{k-j}}\cdots,\end{aligned}$$

where $C^\natural$ is as in Lemma 2.1 and z^k_α is the "center" of Q^k_α. For notational convenience, let $g_t(\cdot):=|B(\cdot,t)|^{-sp/d}|(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f(\cdot)|^p$. Notice that, if $x\in Q^k_\alpha$ and $C^\natural\delta^{k-j+1}\leq\rho(y,z^k_\alpha)<C^\natural\delta^{k-j}$, then $D_{\delta^\ell,\sigma-|s|}(x,y)\lesssim\delta^{j(\sigma-|s|-d-1)}D_{\delta^\ell,d+1}(x,y)$. Therefore, by Fubini's theorem, we find that, for all $k\in\mathbb{Z}$,

$$\begin{aligned}&\sum_{\ell=k\vee 1}^{\infty}\left[\int_{Q^k_\alpha}|B(x,\delta^\ell)|^{-sp/d}|\Phi_\ell(\sqrt{L})f(x)|^p\,d\mu(x)\right]^{q/p}\\&\quad\lesssim\sum_{\ell=k\vee 1}^{\infty}\int_{\delta^{\ell+1}}^{\delta^\ell}\left[\int_{Q^k_\alpha}\int_{B(z^k_\alpha,C^\natural\delta^k)}D_{\delta^\ell,\sigma-|s|}(x,y)g_t(y)\,d\mu(y)\,d\mu(x)\right]^{q/p}\frac{dt}{t}\\&\qquad+\sum_{\ell=k\vee 1}^{\infty}\int_{\delta^{\ell+1}}^{\delta^\ell}\left[\sum_{j\in\mathbb{N}}\delta^{j(\sigma-|s|-d-1)}\int_{Q^k_\alpha}\int_{\rho(y,z^k_\alpha)<C^\natural\delta^{k-j}}D_{\delta^\ell,d+1}(x,y)g_t(y)\,d\mu(y)\,d\mu(x)\right]^{q/p}\frac{dt}{t}\\&\quad\lesssim\int_0^{\min\{1,\delta^k\}}\left[\int_{B(z^k_\alpha,C^\natural\delta^k)}g_t(y)\,d\mu(y)\right]^{q/p}\frac{dt}{t}\\&\qquad+\sum_{j\in\mathbb{N}}\delta^{j(\sigma-|s|-d-1)[1-\epsilon]}\int_0^{\min\{1,\delta^k\}}\left[\int_{B(z^k_\alpha,C^\natural\delta^{k-j})}g_t(y)\,d\mu(y)\right]^{q/p}\frac{dt}{t},\end{aligned}$$

where, in the last inequality, we used Lemma 5.5 with ϵ being any number in $(0,1)$. Notice that the ball $B(z_\alpha^k, C^\natural\delta^{k-j})$ can be covered with N Christ cubes Q_β^{k-j}, where $N\in(0,\infty)$ is independent of $k\in\mathbb{Z}, j\in\mathbb{N}$ and $\alpha\in I_k$. Moreover, $|Q_\beta^{k-j}|^{\tau p}\le\delta^{-jd\tau p}|Q_\alpha^k|^{\tau p}$. Therefore, choosing $\epsilon\in(0,1)$ satisfying that $(\sigma-|s|-d-1)(1-\epsilon)>d\tau q$, we conclude that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z},\ \alpha\in I_k}\frac{1}{|Q_\alpha^k|}\left\{\sum_{\ell=k\vee 1}^{\infty}\left[\int_{Q_\alpha^k}|B(x,\delta^\ell)|^{-sp/d}|\Phi_\ell(\sqrt{L})f(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q}\\
&\quad\lesssim\sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q_\alpha^k|^\tau}\left\{\int_0^{\min\{1,\delta^k\}}\left[\int_{Q_\alpha^k}|B(x,t)|^{-sp/d}|(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f(x)|^p\,d\mu(x)\right]^{q/p}\frac{dt}{t}\right\}^{1/q}.
\end{aligned}$$

In a similar way, when $\ell=0$, we have

$$\begin{aligned}
|B(x,1)|^{-sp/d}|\Phi_0(\sqrt{L})f(x)|^p\lesssim&\int_{B(z_\alpha^k,C^\natural\delta^k)}D_{1,\sigma-|s|}(x,y)|B(y,1)|^{-sp/d}|e^{-L}f(y)|^p\,d\mu(y)\\
&+\sum_{j\in\mathbb{N}}\int_{C^\natural\delta^{k-j+1}\le\rho(y,z_\alpha^k)<C^\natural\delta^{k-j}}\cdots.
\end{aligned}$$

Then, repeating the above arguments, we see that

$$\begin{aligned}
&\sup_{k\in\mathbb{Z},\ \alpha\in I_k}\frac{1}{|Q_\alpha^k|}\left\{\sum_{\ell=k\vee 0}^{(k\vee 1)-1}\left[\int_{Q_\alpha^k}|B(x,\delta^\ell)|^{-sp/d}|\Phi_\ell(\sqrt{L})f(x)|^p\,d\mu(x)\right]^{q/p}\right\}^{1/q}\\
&\quad=\sup_{k\le 0,\ \alpha\in I_k}\frac{1}{|Q_\alpha^k|}\left\{\int_{Q_\alpha^k}|B(x,1)|^{-sp/d}|\Phi_0(\sqrt{L})f(x)|^p\,d\mu(x)\right\}^{1/p}\\
&\quad\lesssim\sup_{k\le 0,\ \alpha\in I_k}\frac{1}{|Q_\alpha^k|^\tau}\left\{\int_0^1\left[\int_{Q_\alpha^k}|B(x,1)|^{-sp/d}|e^{-L}f(x)|^p\,d\mu(x)\right]^{q/p}\frac{dt}{t}\right\}^{1/q}\\
&\quad\lesssim\widetilde{\|f\|}_{p,\tau}.
\end{aligned}$$

Combining these two estimates, we obtain the desired conclusion. This finishes the proof of (i).

The proof for (ii) follows from a similar method, the details being omitted. This finishes the proof of Theorem 5.9. □

Taking $\tau=0$ in Theorem 5.9, we easily obtain the following corollary; see [22, Theorems 6.7 and 7.5] for the case $\beta_0=2$.

Corollary 5.10. *Let $s\in\mathbb{R}$ and $m\in\mathbb{N}$ such that $m>s/\beta_0$.*

(i) *If $p\in[1,\infty]$ and $q\in(0,\infty]$, then*

$$\|f\|_{B^s_{p,q}(M)}\sim\|e^{-L}f\|_{L^p(M)}+\left\{\int_0^1 t^{-sp}\|(t^{\beta_0}L)^m e^{-t^{\beta_0}L}f\|^q_{L^p(M)}\frac{dt}{t}\right\}^{1/q}$$

for all $f\in\mathcal{D}'(M)$, and a similar result also holds true for $\|\cdot\|_{\widetilde{B}^s_{p,q}(M)}$, but with $\|e^{-L}f\|_{L^p(M)}$ and t^{-sp} in (5.23) replaced by $\||B(\cdot,1)|e^{-L}f\|_{L^p(M)}$ and $|B(\cdot,t)|^{-sp/d}$, respectively.

(ii) *If $p \in [1,\infty)$ and $q \in [1,\infty]$, then*

$$\|f\|_{F^s_{p,q}(M)} \sim \|e^{-L}f\|_{L^p(M)} + \left\| \left[\int_0^1 t^{-sq} |(t^{\beta_0}L)^m e^{-t^{\beta_0}L} f|^q \, \frac{dt}{t} \right]^{1/q} \right\|_{L^p(M)}$$

for all $f \in \mathcal{D}'(M)$, and a similar result also holds true for $\|\cdot\|_{\widetilde{F}^s_{p,q}(M)}$, but with $\|e^{-L}f\|_{L^p(M)}$ and t^{-sq} in (5.23) *replaced by $\||B(\cdot,1)|e^{-L}f\|_{L^p(M)}$ and $|B(\cdot,t)|^{-sq/d}$, respectively.*

6 Frame decompositions

In this section, we establish the frame characterization of the Besov-type and Triebel–Lizorkin-type spaces. Using this, we further show that $F^{s,1/p}_{p,q}(M)$ and $\widetilde{F}^{s,1/p}_{p,q}(M)$ are indeed the endpoint spaces $F^s_{\infty,q}(M)$ and $\widetilde{F}^s_{\infty,q}(M)$ of the Triebel–Lizorkin spaces, where $p \in (0,\infty)$, $q \in (0,\infty]$ and $s \in \mathbb{R}$.

6.1 Frame decompositions

Different from those in [22], the frame structure we considered in this section is more specific and relies on Christ's dyadic cubes in M, due to the definition of the Besov-type and Triebel–Lizorkin-type spaces. Therefore, we need to establish a new discrete Calderón reproducing formula associated with Christ's dyadic cubes and the functions Φ_0 and Φ in (1.9) and (1.10), which is different from the one obtained in [22, Theorem 4.3 and Proposition 5.5].

In what follows, we shall use the following notation. For $j \in \mathbb{Z}$ and $\tau \in I_j$, we denote by $Q^{j,\nu}_\tau$, $\nu \in \{1,\ldots,N^j_\tau\}$, the set of all cubes $Q^{j+j_0}_{\tau'} \subset Q^j_\tau$, where $Q^{j+j_0}_{\tau'}$ is the Christ dyadic cube as in Lemma 2.1 and j_0 a positive large integer such that

$$(C_\natural \delta^{j_0})^{\alpha_0} \le \min\{1,\, \epsilon_0\,(8C_{\sigma,0}C^*_\alpha)^{-1}\}, \tag{6.1}$$

where $\epsilon_0 \in (0,1)$ is a small number to be determined in (8.11) below, $\sigma > 3d/2$, $C_{\sigma,0}$ is the constant from Proposition 3.4, C^*_σ the constant from Lemma 3.1, and α_0 the constant from **(HE′)**. Denote by $z^{j,\nu}_\tau$ the "center" of $Q^{j,\nu}_\tau$, and by $\xi^{j,\nu}_\tau$ a point in $Q^{j,\nu}_\tau$.

The discrete Calderón reproducing formula we need is as follows, which is different from the one used in [22]. For the completeness of the paper, we present its proof at the end of this article (see Section 8 Appendix), though a majority of the skills used comes from [9, 22].

Theorem 6.1. *Let $\delta \in (0,1)$ be as in Lemma 2.1. Suppose that $\Phi_0, \Phi \in C^\infty(\mathbb{R}_+)$ satisfy* (2.1) *and* (2.2). *For any $j \in \mathbb{N}$, let $\Phi_j(\cdot) := \Phi(\delta^{j\beta_0/2}\cdot)$. Then, there exists a sequence $\{\Psi_j(\sqrt{L})\}_{j=0}^\infty$ of operators such that, for any $f \in \mathcal{D}'(M)$ and all $\xi^{j,\nu}_\tau \in Q^{j,\nu}_\tau$ with $\tau \in I_j$ and $\nu \in \{1,\ldots,N^j_\tau\}$,*

$$f(\cdot) = \sum_{j=0}^\infty \sum_{\tau \in I_j} \sum_{\nu=1}^{N^j_\tau} |Q^{j,\nu}_\tau| (\Phi_j(\sqrt{L})f)(\xi^{j,\nu}_\tau)\, \Psi_j(\xi^{j,\nu}_\tau, \cdot), \tag{6.2}$$

where the series converge in $\mathcal{D}'(M)$. Moreover, $\{\Psi_j\}_{j=0}^\infty$ satisfy the following: for any $\sigma \in (0,\infty)$, there exists a constant $C_{\sigma,m} \in [1,\infty)$ such that the following hold:

(i) *for all x, $y \in M$,*

$$|L^m \Psi_j(\sqrt{L})(x,y)| \le C_{\sigma,m} \delta^{-2mj} D_{\delta^j,\sigma}(x,y);$$

(ii) *with* α_0 *as in condition* ($\mathbf{HE}'$), *then, for all* x, y, $y' \in M$ *such that* $\rho(y,y') \le \delta^j$,

$$\begin{aligned}&|\Psi_j(\sqrt{L})(x,y) - \Psi_j(\sqrt{L})(x,y')| + |\Psi_j(\sqrt{L})(y,x) - \Psi_j(\sqrt{L})(y',x)|\\ &\quad \le C_{\sigma,m}\left[\delta^{-j}\rho(y,y')\right]^{\alpha_0} D_{\delta,\sigma}(x,y);\end{aligned}$$

(iii) *letting* $(\widetilde{\Psi}_0, \widetilde{\Psi})$ *satisfy* (2.1) *and* (2.2), *then, for any* $m \in \mathbb{N}$ *and* $\sigma \in (d,\infty)$, *there exists a positive constant* $C_{\sigma,m}$ *such that, for all* j, $k \in \mathbb{Z}_+$ *and* x, $y \in M$,

$$\left|\left(\widetilde{\Psi}_k(\sqrt{L})\Psi_j(\sqrt{L})\right)(x,y)\right| \le C_{\sigma,m}\delta^{|k-j|(m\beta_0-2d)}D_{\delta^{k\wedge j},\sigma}(x,y).$$

Now we introduce the related sequence spaces.

Definition 6.2. Let $s \in \mathbb{R}$, $\tau \in [0,\infty)$ and $p,q \in (0,\infty]$. The *sequence space* $b^{s,\tau}_{p,q}(M)$ is defined to be the collection of all sequences $a := \{a^{j,\nu}_t\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N^j_t} \subset \mathbb{C}$ such that

$$\|a\|_{b^{s,\tau}_{p,q}(M)} := \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\delta^{-jsq}\left[\int_{Q^k_\alpha}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|a^{j,\nu}_t|\chi_{Q^{j,\nu}_t}(x)\right)^p d\mu(x)\right]^{q/p}\right\}^{1/q} < \infty.$$

The *sequence space* $\widetilde{b}^{s,\tau}_{p,q}(M)$ is defined to be the collection of all $a := \{a^{j,\nu}_t\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N^j_t} \subset \mathbb{C}$ such that

$$\|a\|_{\widetilde{b}^{s,\tau}_{p,q}(M)} := \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^{\infty}\left[\int_{Q^k_\alpha}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|Q^{j,\nu}_t|^{-s/d}|a^{j,\nu}_t|\chi_{Q^{j,\nu}_t}(x)\right)^p d\mu(x)\right]^{q/p}\right\}^{1/q} < \infty.$$

Definition 6.3. Let $s \in \mathbb{R}$, $\tau \in [0,\infty)$, $p \in (0,\infty)$ and $q \in (0,\infty]$. The *sequence space* $f^{s,\tau}_{p,q}(M)$ is defined to be the collection of all sequences $a := \{a^{j,\nu}_t\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N^j_t} \subset \mathbb{C}$ such that

$$\|a\|_{f^{s,\tau}_{p,q}(M)} := \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=k\vee 0}^{\infty}\delta^{-jsq}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|a^{j,\nu}_t|\chi_{Q^{j,\nu}_t}(x)\right)^q d\mu(x)\right]^{p/q}\right\}^{1/p} < \infty.$$

The *sequence space* $\widetilde{f}^{s,\tau}_{p,q}(M)$ is defined to be the collection of all $a := \{a^{j,\nu}_t\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N^j_t} \subset \mathbb{C}$ such that

$$\|a\|_{\widetilde{f}^{s,\tau}_{p,q}(M)} := \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q^k_\alpha|^\tau}\left\{\int_{Q^k_\alpha}\left[\sum_{j=k\vee 0}^{\infty}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|Q^{j,\nu}_t|^{-s/d}|a^{j,\nu}_t|\chi_{Q^{j,\nu}_t}(x)\right)^q d\mu(x)\right]^{p/q}\right\}^{1/p} < \infty.$$

Suppose that Φ_0 and Φ satisfy (2.1) and (2.2), and Φ_j with $j \in \mathbb{N}$ is defined as in (2.3). By (6.2), there exist $\{\Psi_j\}_{j=0}^\infty$ satisfying (i)–(iii) of Theorem 6.1 such that, for any $f \in \mathcal{D}'(M)$,

$$f(\cdot) = \sum_{j=0}^{\infty}\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|Q^{j,\nu}_t|(\Phi_j(\sqrt{L})f)(\xi^{j,\nu}_t)\,\Psi_j(\sqrt{L})(\xi^{j,\nu}_t,\cdot) \qquad \text{in } \mathcal{D}'(M),$$

where $\xi^{j,\nu}_t \in Q^{j,\nu}_t$, $t \in I_j$ and $\nu \in \{1,\dots,N^j_t\}$. Define the "analysis" and "synthesis" operators as follows:

$$S_\Phi:\ f \to \{(\Phi_j(\sqrt{L})f)(\xi^{j,\nu}_t)\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N^j_t} \tag{6.3}$$

and

$$T_\Psi:\ \{a_t^{j,\nu}\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N_t^j}\to\sum_{j=0}^\infty\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|\Psi_j(\sqrt{L})(\xi_t^{j,\nu},\cdot)a_t^{j,\nu}. \tag{6.4}$$

These operators S_Φ and T_Ψ are generalizations of the φ-transform and the inverse φ-transform of Frazier and Jawerth [13]. Notice that the Calderón reproducing formula (6.2) implies that $T_\Psi\circ S_\Phi=\mathrm{Id}$ on $\mathcal{D}'(M)$, here and hereafter, we use Id to denote the identity operator. Then we have the following frame characterizations.

Theorem 6.4. *Let $\tau\in[0,\infty)$, $s\in\mathbb{R}$ and $q\in(0,\infty]$.*

(i) *Let $p\in(0,\infty]$. Then, the operators $S_\Phi:\widetilde{B}_{p,q}^{s,\tau}(M)\to\widetilde{b}_{p,q}^{s,\tau}(M)$ and $T_\Psi:\widetilde{b}_{p,q}^{s,\tau}(M)\to\widetilde{B}_{p,q}^{s,\tau}(M)$ are bounded, and $T_\Psi\circ S_\Phi=\mathrm{Id}$ on $\widetilde{B}_{p,q}^{s,\tau}(M)$. Moreover, $f\in\widetilde{B}_{p,q}^{s,\tau}(M)$ if and only if $S_\Phi f\in\widetilde{b}_{p,q}^{s,\tau}(M)$, and there exists a constant $C\in(1,\infty)$ such that, for any $f\in\widetilde{B}_{p,q}^{s,\tau}(M)$,*

$$\frac{1}{C}\|f\|_{\widetilde{B}_{p,q}^{s,\tau}(M)}\le\|S_\Phi f\|_{\widetilde{b}_{p,q}^{s,\tau}(M)}\le C\|f\|_{\widetilde{B}_{p,q}^{s,\tau}(M)}.$$

(ii) *Item* (i) *keeps valid if $\widetilde{B}_{p,q}^{s,\tau}(M)$ and $\widetilde{b}_{p,q}^{s,\tau}(M)$ therein are replaced by $B_{p,q}^{s,\tau}(M)$ and $b_{p,q}^{s,\tau}(M)$, respectively.*

(iii) *Let $p\in(0,\infty)$. Then, the operators $S_\Phi:\widetilde{F}_{p,q}^{s,\tau}(M)\to\widetilde{f}_{p,q}^{s,\tau}(M)$ and $T_\Psi:\widetilde{f}_{p,q}^{s,\tau}(M)\to\widetilde{F}_{p,q}^{s,\tau}(M)$ are bounded, and $T_\Psi\circ S_\Phi=\mathrm{Id}$ on $\widetilde{F}_{p,q}^{s,\tau}(M)$. Moreover, $f\in\widetilde{B}_{p,q}^{s,\tau}(M)$ if and only if $S_\Phi f\in\widetilde{b}_{p,q}^{s,\tau}(M)$, and there exists a constant $C\in(1,\infty)$ such that, for any $f\in\widetilde{F}_{p,q}^{s,\tau}(M)$,*

$$\frac{1}{C}\|f\|_{\widetilde{F}_{p,q}^{s,\tau}(M)}\le\|S_\Phi f\|_{\widetilde{f}_{p,q}^{s,\tau}(M)}\le C\|f\|_{\widetilde{F}_{p,q}^{s,\tau}(M)}.$$

(iv) *Item* (iii) *keeps valid if $\widetilde{F}_{p,q}^{s,\tau}(M)$ and $\widetilde{f}_{p,q}^{s,\tau}(M)$ therein are replaced by $F_{p,q}^{s,\tau}(M)$ and $f_{p,q}^{s,\tau}(M)$, respectively.*

Theorem 6.4 generalizes the φ-transform characterization for Besov-type and Triebel–Lizorkin-type spaces on $\mathbb{R}^n$ in [45]. To prove Theorem 6.4, we need the following two lemmas. In what follows, for any $s\in\mathbb{R}$, we write $b_{\infty,\infty}^s(M):=b_{\infty,\infty}^{s,0}(M)$ and $\widetilde{b}_{\infty,\infty}^s(M):=\widetilde{b}_{\infty,\infty}^{s,0}(M)$. Then, for any sequence $a:=\{a_t^{j,\nu}\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N_t^j}\subset\mathbb{C}$, let

$$\|a\|_{\widetilde{b}_{\infty,\infty}^s(M)}:=\|a\|_{\widetilde{b}_{\infty,\infty}^{s,0}(M)}:=\sup_{j\in\mathbb{Z}_+}\sup_{x\in M}\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{Q_t^{j,\nu}}(x). \tag{6.5}$$

Similarly, we define the norm $\|a\|_{b_{\infty,\infty}^s(M)}$ via replacing $|Q_t^{j,\nu}|^{-s/d}$ in (6.5) by δ^{-js}.

Lemma 6.5. *Let $s\in\mathbb{R}$, $\tau\in[0,\infty)$ and $q\in(0,\infty]$.*

(i) *If $p\in(0,\infty]$, then $b_{p,q}^{s,\tau}(M)\hookrightarrow b_{\infty,\infty}^{s+d\tau-d/p}(M)$ and $\widetilde{b}_{p,q}^{s,\tau}(M)\hookrightarrow\widetilde{b}_{\infty,\infty}^{s+d\tau-d/p}(M)$.*

(ii) *If $p\in(0,\infty)$, then $f_{p,q}^{s,\tau}(M)\hookrightarrow b_{\infty,\infty}^{s+d\tau-d/p}(M)$ and $\widetilde{f}_{p,q}^{s,\tau}(M)\hookrightarrow\widetilde{b}_{\infty,\infty}^{s+d\tau-d/p}(M)$.*

Proof. By Minkowski's inequality, we see that

$$b^{s,\tau}_{p,\min(p,q)}(M)\subset f^{s,\tau}_{p,q}(M)\subset b^{s,\tau}_{p,\max(p,q)}(M)\quad\text{and}\quad \widetilde{b}^{s,\tau}_{p,\min(p,q)}(M)\subset \widetilde{f}^{s,\tau}_{p,q}(M)\subset \widetilde{b}^{s,\tau}_{p,\max(p,q)}(M).$$

Hence, it suffices to show (i). Notice that $\{Q^{j,\nu}_t:\ t\in I_j,\ \nu\in\{1,\ldots,N^j_t\}\}$ are disjoint with each other. Then, for each $j\in\mathbb{Z}_+$, there exists a unique cube in $\{Q^{j,\nu}_t:\ t\in I_j,\ \nu\in\{1,\ldots,N^j_t\}\}$, denoted by $Q^{j,\nu}_{t,x}$, such that $x\in Q^{j,\nu}_{t,x}$ and hence

$$\begin{aligned}
\|a\|_{\widetilde{b}^{s+d\tau-d/p}_{\infty,\infty}(M)} &= \sup_{j\in\mathbb{Z}_+}\sup_{x\in M}|Q^{j,\nu}_{t,x}|^{-s/d-\tau+1/p}|a^{j,\nu}_{t,x}|\\
&\lesssim \sup_{k\in\mathbb{Z}_+}\sup_{\alpha\in I_k}\sup_{x\in Q^k_\alpha}\frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k}^\infty\left[|Q^{j,\nu}_{t,x}|^{1-sp/d}|a^{j,\nu}_{t,x}|^p\right]^{q/p}\right\}^{1/q}\\
&\lesssim \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}}\frac{1}{|Q^k_\alpha|^\tau}\left\{\sum_{j=k\vee 0}^\infty\left[\int_{Q^k_\alpha}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|Q^{j,\nu}_t|^{-s/d}|a^{j,\nu}_t|\chi_{Q^{j,\nu}_t}(x)\right)^p d\mu(x)\right]^{q/p}\right\}^{1/q}\\
&\sim \|a\|_{\widetilde{b}^{s,\tau}_{p,q}(M)}.
\end{aligned}$$

This proves that $\widetilde{b}^{s,\tau}_{p,q}(M)\hookrightarrow \widetilde{b}^{s+d\tau-d/p}_{\infty,\infty}(M)$. Similarly, it holds that $b^{s,\tau}_{p,q}(M)\hookrightarrow b^{s+d\tau-d/p}_{\infty,\infty}(M)$, the details being omitted. This finishes the proof of Lemma 6.5. □

Lemma 6.6. *Let $s\in\mathbb{R}$, $\tau\in[0,\infty)$ and $p,q\in(0,\infty]$. If the sequence $a:=\{a^{j,\nu}_t\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N^j_t}$ belongs to any of the sequence spaces $b^{s,\tau}_{p,q}(M)$, $\widetilde{b}^{s,\tau}_{p,q}(M)$, $f^{s,\tau}_{p,q}(M)$ or $\widetilde{f}^{s,\tau}_{p,q}(M)$, then the series*

$$T_\Psi(a)(\cdot)=\sum_{j=0}^\infty\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|Q^{j,\nu}_t|\Psi_j(\sqrt{L})(\xi^{j,\nu}_t,\cdot)a^{j,\nu}_t \tag{6.6}$$

converges in $\mathcal{D}'(M)$.

Proof. Due to Lemma 6.5, it suffices to show that the series in (6.6) converges in $\mathcal{D}'(M)$ when $a\in b^s_{\infty,\infty}(M)$ or $a\in\widetilde{b}^s_{\infty,\infty}(M)$. We only show the conclusion for $a\in\widetilde{b}^s_{\infty,\infty}(M)$ and $\mu(M)=\infty$, the proofs for the rest cases being similar. If $a\in\widetilde{b}^s_{\infty,\infty}(M)$ with norm 1, then $|a^{j,\nu}_t|\le|Q^{j,\nu}_t|^{s/d}$ for any $j\in\mathbb{Z}_+$, $t\in I_j$ and $1\le\nu\le N^j_t$. Thus, for any $\phi\in\mathcal{D}(M)$, we have

$$\sum_{j=0}^\infty\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|Q^{j,\nu}_t||\langle\Psi_j(\sqrt{L})(\xi^{j,\nu}_t,\cdot),\,\phi\rangle||a^{j,\nu}_t|\le\sum_{j=0}^\infty\sum_{t\in I_j}\sum_{\nu=1}^{N^j_t}|Q^{j,\nu}_t|^{1+s/d}|\langle\Psi_j(\sqrt{L})(\xi^{j,\nu}_t,\cdot),\,\phi\rangle|. \tag{6.7}$$

Let Φ_0,Φ satisfy (2.1) and (2.2), respectively. Define $\{\Phi_j\}_{j\in\mathbb{N}}$ as in (2.3). Using the Calderón reproducing formula (3.7), we know that there exist $\widetilde{\Phi}_0$ and $\widetilde{\Phi}$ satisfying (2.1) and (2.2) such that $\phi=\sum_{\ell=0}^\infty\widetilde{\Phi}_\ell(\sqrt{L})\Phi_\ell(\sqrt{L})\phi$, where the series converges in both $\mathcal{D}(M)$ and $L^2(M)$. Notice that $\Psi_j(\sqrt{L})(\xi^{j,\nu}_t,\cdot)\in L^2(M)$. Hence,

$$\langle\Psi_j(\sqrt{L})(\xi^{j,\nu}_t,\cdot),\,\phi\rangle=\sum_{\ell=0}^\infty\left\langle\left(\widetilde{\Phi}_\ell(\sqrt{L})\Psi_j(\sqrt{L})\right)(\xi^{j,\nu}_t,\cdot),\ \Phi_\ell(\sqrt{L})\phi\right\rangle. \tag{6.8}$$

Choose η,N,m,σ such that $\eta>|s|+d$, $\beta_0N>3d/2+\sigma+|s|$, $\beta_0m>2d+|s|$ and $\sigma>3d/2+\eta$. According to the proof of [22, (6.19)] (see also (4.2)), it holds true that, for all $\ell\in\mathbb{Z}_+$ and $z\in M$,

$$|\Phi_\ell(\sqrt{L})\phi(z)|\lesssim\delta^{\ell(\beta_0N-d/2)}[1+\rho(z,x_0)]^{-\eta}\mathscr{P}_{N,\eta}(\phi),$$

where we recall that x_0 is some fixed point of M. By Theorem 6.1(iii), we see that, for all $j,\ell\in\mathbb{Z}_+$, $z\in M$ and $\xi\in Q_t^{j,\nu}$,

$$\left|\left(\widetilde{\Phi}_\ell(\sqrt{L})\Psi_j(\sqrt{L})\right)(\xi_t^{j,\nu},z)\right| \lesssim \delta^{|\ell-j|(m\beta_0-2d)}D_{\delta^{\ell\wedge j},\sigma}(\xi_t^{j,\nu},z)$$
$$\lesssim \delta^{|\ell-j|(m\beta_0-2d)}\delta^{-(\ell\wedge j)(d+\sigma)}D_{1,\sigma}(\xi,z).$$

From these two estimates and (6.8), we deduce that, for any $\xi\in Q_t^{j,\nu}$,

$$|\langle\Psi_j(\sqrt{L})(\xi_t^{j,\nu},\cdot),\,\phi\rangle| \lesssim \mathscr{P}_{N,\eta}(\phi)\sum_{\ell=0}^{\infty}\int_M \frac{\delta^{|\ell-j|(m\beta_0-2d)}\delta^{(\beta_0N-3d/2-\sigma)\ell}}{|B(\xi,1)|[1+\rho(\xi,z)]^{\sigma-d/2}[1+\rho(z,x_0)]^{\eta}}\,d\mu(z)$$
$$\lesssim \mathscr{P}_{N,\eta}(\phi)\sum_{\ell=0}^{\infty}\frac{\delta^{|\ell-j|(m\beta_0-2d)}\delta^{(\beta_0N-3d/2-\sigma)\ell}}{[1+\rho(\xi,x_0)]^{\eta}}. \tag{6.9}$$

Observe that, for any $\xi\in Q_t^{j,\nu}$, it holds true that

$$|Q_t^{j,\nu}|^{s/d}\lesssim \delta^{-j|s|}[1+\rho(\xi,x_0)]^{|s|}\lesssim \delta^{-|\ell-j||s|}\delta^{-\ell|s|}[1+\rho(\xi,x_0)]^{|s|}.$$

Inserting this and (6.9) into (6.7), we have

$$\sum_{j=0}^{\infty}\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}||\langle\Psi_j(\sqrt{L})(\xi_t^{j,\nu},\cdot),\,\phi\rangle||a_t^{j,\nu}|$$
$$\lesssim \mathscr{P}_{N,\eta}(\phi)\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{1+s/d}\inf_{\xi\in Q_t^{j,\nu}}\frac{\delta^{|\ell-j|(m\beta_0-2d)}\delta^{(\beta_0N-3d/2-\sigma)\ell}}{[1+\rho(\xi,x_0)]^{\eta}}$$
$$\lesssim \mathscr{P}_{N,\eta}(\phi)\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\int_M\frac{\delta^{|\ell-j|(m\beta_0-2d-|s|)}\delta^{(\beta_0N-3d/2-\sigma-|s|)\ell}}{[1+\rho(\xi,x_0)]^{\eta-|s|}}\,d\mu(\xi)$$
$$\lesssim \mathscr{P}_{N,\eta}(\phi),$$

which implies that (6.6) converges in $\mathcal{D}'(M)$ and hence completes the proof of Lemma 6.6. ☐

Now we are ready to prove Theorem 6.4.

Proof of Theorem 6.4. We only show (i) and (iii), the proofs for (ii) and (iv) being similar. Let $f\in\mathcal{D}'(M)$ and $a>d[\tau+1/(p\wedge q)]$. By (3.7), we know that there exist $\widetilde{\Phi}_0$ and $\widetilde{\Phi}$ satisfying (2.1) and (2.2) such that $f=\sum_{\ell=0}^{\infty}\widetilde{\Phi}_\ell(\sqrt{L})\Phi_\ell(\sqrt{L})f$. Then, given any $m>(d+|s|+a)/\beta_0$ and $\sigma>a+|s|+d$, applying Proposition 3.7, we see that, for any $j\in\mathbb{Z}_+, t\in I_j$ and $\nu\in\{1,\ldots,N_t^j\}$,

$$|Q_t^{j,\nu}|^{-s/d}|\Phi_j(\sqrt{L})f(\xi_t^{j,\nu})| \le \sum_{\ell=0}^{\infty}|Q_t^{j,\nu}|^{-s/d}|\Phi_j(\sqrt{L})\widetilde{\Phi}_\ell(\sqrt{L})\Phi_\ell(\sqrt{L})f(\xi_t^{j,\nu})|$$
$$\lesssim \sum_{\ell=0}^{\infty}\delta^{|j-\ell|(m\beta_0-d/2)}|Q_t^{j,\nu}|^{-s/d}\int_M D_{\delta^{j\wedge\ell},\sigma}(\xi_t^{j,\nu},\,y)|\Phi_\ell(\sqrt{L})f(y)|\,d\mu(y).$$

If $x\in Q_t^{j,\nu}$ and $y\in M$, then, by $\xi_t^{j,\nu}\in Q_t^{j,\nu}$, we see that $D_{\delta^{j\wedge\ell},\sigma}(\xi_t^{j,\nu},\,y)\sim D_{\delta^{j\wedge\ell},\sigma}(x,\,y)$ and

$$|Q_t^{j,\nu}|^{-s/d}|B(y,\delta^\ell)|^{s/d}\sim|B(x,\delta^j)|^{-s/d}|B(y,\delta^\ell)|^{s/d}\lesssim\delta^{-|j-\ell||s|}[1+\delta^{-(j\wedge\ell)}\rho(x,y)]^{|s|}.$$

Hence, for any $x \in Q_t^{j,\nu}$,

$$
\begin{aligned}
|Q_t^{j,\nu}|^{-s/d}|\Phi_j(\sqrt{L})f(\xi_t^{j,\nu})| &\lesssim \sum_{\ell=0}^{\infty} \delta^{|j-\ell|(m\beta_0-d/2-|s|)}[\Phi_\ell(\sqrt{L})f]^*_{a,-s/d}(x) \\
&\quad \times \int_M \frac{1}{\sqrt{|B(x,\delta^{j\wedge\ell})||B(y,\delta^{j\wedge\ell})|}} \frac{[1+\delta^{-\ell}\rho(x,y)]^a}{[1+\delta^{-(j\wedge\ell)}\rho(x,y)]^{\sigma-|s|}}\,d\mu(y) \\
&\lesssim \sum_{\ell=0}^{\infty} \delta^{|j-\ell|(m\beta_0-d/2-|s|-a)}[\Phi_\ell(\sqrt{L})f]^*_{a,-s/d}(x).
\end{aligned}
$$

Consequently, for any $x \in Q_\alpha^k$ with $k \in \mathbb{Z}$ and $\alpha \in I_k$, and for any $j \geq (k \vee 0)$,

$$
\begin{aligned}
g_j(x) &:= \sum_{t\in I_j} \sum_{\nu=1}^{N_t^j} |Q_t^{j,\nu}|^{-s/d}|\Phi_j(\sqrt{L})f(\xi_t^{j,\nu})|\chi_{Q_t^{j,\nu}}(x) \\
&\lesssim \sum_{\ell=0}^{\infty} \delta^{|j-\ell|(m\beta_0-d/2-|s|-a)}[\Phi_\ell(\sqrt{L})f]^*_{a,-s/d}(x).
\end{aligned}
$$

By this and the definition of $\widetilde{b}_{p,q}^{s,\tau}(M)$, applying Lemma 5.5 and Theorem 5.2, we see that

$$
\|S_\Phi f\|_{\widetilde{b}_{p,q}^{s,\tau}(M)} = \|\{g_j\}_{j=0}^\infty\|_{\ell^q(L^p_\tau)} \lesssim \|\{[\Phi_\ell(\sqrt{L})f]^*_{a,-s/d}\}_{\ell=0}^\infty\|_{\ell^q(L^p_\tau)} \lesssim \|f\|_{\widetilde{B}_{p,q}^{s,\tau}(M)},
$$

which implies the boundedness of S_Φ from $\widetilde{B}_{p,q}^{s,\tau}(M)$ to $\widetilde{b}_{p,q}^{s,\tau}(M)$. Due to the same reasons, we also see that S_Φ is bounded from $\widetilde{F}_{p,q}^{s,\tau}(M)$ to $\widetilde{f}_{p,q}^{s,\tau}(M)$.

Now we consider the boundedness of T_Ψ. Fix $r \in (0, \min\{1,p,q\})$. For any $a \in \widetilde{b}_{p,q}^{s,\tau}(M)$ or $\widetilde{f}_{p,q}^{s,\tau}(M)$, let

$$
f(\cdot) := T_\Psi a(\cdot) = \sum_{j=0}^{\infty} \sum_{t\in I_j} \sum_{\nu=1}^{N_t^j} |Q_t^{j,\nu}|\Psi_j(\sqrt{L})(\xi_t^{j,\nu},\cdot)a_t^{j,\nu}.
$$

Then, $f \in \mathcal{D}'(M)$ by Lemma 6.6. Hence, for any $x \in Q_\alpha^k$, with $k \in \mathbb{Z}$ and $\alpha \in I_k$, and any $j \geq (k \vee 0)$,

$$
\Phi_\ell(\sqrt{L})f(x) = \sum_{j=0}^{\infty} \sum_{t\in I_j} \sum_{\nu=1}^{N_t^j} |Q_t^{j,\nu}| \left(\Phi_\ell(\sqrt{L})\Psi_j(\sqrt{L})\right)(\xi_t^{j,\nu},x)a_t^{j,\nu}.
$$

From this and Theorem 6.1(iii), it follows that

$$
|B(x,\delta^\ell)|^{-s/d}|\Phi_\ell(\sqrt{L})f(x)| \lesssim \sum_{j=0}^{\infty} \sum_{t\in I_j} \sum_{\nu=1}^{N_t^j} \delta^{|\ell-j|(m\beta_0-2d)}|a_t^{j,\nu}||B(x,\delta^\ell)|^{-s/d}|Q_t^{j,\nu}|D_{\delta^{j\wedge\ell},\sigma}(\xi_t^{j,\nu},x),
$$

where we chose $\sigma \geq 2d/r + |s| + 1$ and $m > (3d + |s| + \sigma)/2$. Notice that

$$
|Q_t^{j,\nu}|D_{\delta^{j\wedge\ell},\sigma}(\xi_t^{j,\nu},x) \lesssim \delta^{-|j-\ell|d}[1+\delta^{-(j\wedge\ell)}\rho(\xi_t^{j,\nu},x)]^{-\sigma} \lesssim \delta^{-|j-\ell|(d+\sigma)}[1+\delta^{-j}\rho(\xi_t^{j,\nu},x)]^{-\sigma}
$$

and

$$
|B(x,\delta^\ell)|^{-s/d} \lesssim |Q_t^{j,\nu}|^{-s/d}\delta^{-|j-\ell||s|}[1+\delta^{-j}\rho(\xi_t^{j,\nu},x)]^{|s|}.
$$

Therefore,

$$|B(x,\delta^\ell)|^{-s/d}|\Phi_\ell(\sqrt{L})f(x)| \lesssim \sum_{j=0}^{\infty} \delta^{|\ell-j|(m\beta_0-3d-\sigma-|s|)} \sum_{t\in I_j}\sum_{\nu=1}^{N_t^j} \frac{|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|}{[1+\delta^{-j}\rho(\xi_t^{j,\nu},x)]^{\sigma-|s|}}. \tag{6.10}$$

Observe that

$$\begin{aligned}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j} \frac{|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|}{[1+\delta^{-j}\rho(\xi_t^{j,\nu},x)]^{\sigma-|s|}}\right)^r &\le \int_M \sum_{t\in I_j}\sum_{\nu=1}^{N_t^j} \frac{|Q_t^{j,\nu}|^{-sr/d}|a_t^{j,\nu}|^r\chi_{Q_t^{j,\nu}}(z)}{|Q_t^{j,\nu}|[1+\delta^{-j}\rho(\xi_t^{j,\nu},x)]^{(\sigma-|s|)r}}\,d\mu(z)\\ &\sim \int_M \frac{\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{Q_t^{j,\nu}}(z)\right)^r}{|B(z,\delta^j)|[1+\delta^{-j}\rho(z,x)]^{(\sigma-|s|)r}}\,d\mu(z).\end{aligned} \tag{6.11}$$

For notational convenience, set $f_j := \sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{Q_t^{j,\nu}}$. Then, applying Lemma 3.12 to (6.11) further gives that

$$\begin{aligned}\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j} \frac{|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|}{[1+\delta^{-j}\rho(\xi_t^{j,\nu},x)]^{\sigma-|s|}} &\lesssim \left\{\sum_{i=0}^{\infty}\delta^{i(\sigma r-|s|r-d)}\mathcal{M}(|f_j|^r\chi_{B(z_\alpha^k,\delta^{k-i}+C_\natural\delta^k)})(x)\right\}^{1/r}\\ &\lesssim \sum_{i=0}^{\infty}\delta^{i(\sigma r-|s|r-d)}\mathcal{M}_r(|f_j|\chi_{B(z_\alpha^k,\delta^{k-i}+C_\natural\delta^k)})(x),\end{aligned}$$

where the second inequality is due to Hölder's inequality. Combining this with (6.10) implies that

$$\begin{aligned}&|B(x,\delta^\ell)|^{-s/d}|\Phi_\ell(\sqrt{L})f(x)|\\ &\quad\lesssim \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\delta^{|\ell-j|(m\beta_0-3d-|s|-\sigma)}\delta^{i(\sigma r-|s|r-d)}\mathcal{M}_r(|f_j|\chi_{B(z_\alpha^k,\delta^{k-i}+C_\natural\delta^k)})(x).\end{aligned} \tag{6.12}$$

Then, repeating the proof for (5.10), we obtain $\|f\|_{\widetilde{B}_{p,q}^{s,\tau}(M)} \lesssim \|\{f_j\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L_\tau^p)} \sim \|a\|_{\widetilde{b}_{p,q}^{s,\tau}(M)}$, which proves that $T_\Psi : \widetilde{b}_{p,q}^{s,\tau}(M) \to \widetilde{B}_{p,q}^{s,\tau}(M)$ is bounded. Again, applying (6.12) and repeating the proof for (5.14) gives that $\|f\|_{\widetilde{F}_{p,q}^{s,\tau}(M)} \lesssim \|\{f_j\}_{j\in\mathbb{Z}_+}\|_{L_\tau^p(\ell^q)} \sim \|a\|_{\widetilde{f}_{p,q}^{s,\tau}(M)}$, which implies that $T_\Psi : \widetilde{f}_{p,q}^{s,\tau}(M) \to \widetilde{F}_{p,q}^{s,\tau}(M)$ is bounded.

By (6.2), we have $T_\Psi\circ S_\Phi = \mathrm{Id}$ on $\widetilde{B}_{p,q}^{s,\tau}(M)$ or $\widetilde{F}_{p,q}^{s,\tau}(M)$. If $S_\Phi f \in \widetilde{b}_{p,q}^{s,\tau}(M)$, then the boundedness of T_Ψ implies that $f = T_\Psi(S_\Phi f) \in \widetilde{B}_{p,q}^{s,\tau}(M)$ and $\|f\|_{\widetilde{B}_{p,q}^{s,\tau}(M)} = \|T_\Psi(S_\Phi f)\|_{\widetilde{B}_{p,q}^{s,\tau}(M)} \lesssim \|S_\Phi f\|_{\widetilde{b}_{p,q}^{s,\tau}(M)}$. Likewise, $\|f\|_{\widetilde{F}_{p,q}^{s,\tau}(M)} \lesssim \|S_\Phi f\|_{\widetilde{f}_{p,q}^{s,\tau}(M)}$ holds. This finishes the proof of Theorem 6.4. $\square$

6.2 The endpoint Triebel-Lizorkin spaces $F_{\infty,q}^s(M)$ and $\widetilde{F}_{\infty,q}^s(M)$

For $s\in\mathbb{R}$ and $q\in(0,\infty]$, inspired by the definition of $F_{\infty,q}^s(\mathbb{R}^n)$ on $\mathbb{R}^n$ in [13], we define the *endpoint Triebel-Lizorkin spaces* $F_{\infty,q}^s(M) := F_{q,q}^{s,1/q}(M)$ and $\widetilde{F}_{\infty,q}^s(M) := \widetilde{F}_{q,q}^{s,1/q}(M)$. Then we have the following coincidence.

Theorem 6.7. *Let $s\in\mathbb{R}$, $p\in(0,\infty)$ and $q\in(0,\infty]$. Then $F_{\infty,q}^s(M) = F_{p,q}^{s,1/p}(M)$ and $\widetilde{F}_{\infty,q}^s(M) = \widetilde{F}_{p,q}^{s,1/p}(M)$ with equivalent (quasi-)norms.*

For the corresponding result on Triebel-Lizorkin spaces on $\mathbb{R}^n$, we refer to [13, Corollary 5.7]. Due to Theorem 6.4, to show Theorem 6.7, it suffices to prove the following fact.

Proposition 6.8. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Then $f_{q,q}^{s,1/q}(M) = f_{p,q}^{s,1/p}(M)$ and $\widetilde{f}_{q,q}^{s,1/q}(M) = \widetilde{f}_{p,q}^{s,1/p}(M)$ with equivalent (quasi-)norms.*

To prove Proposition 6.8, we follow the proof of [13, Corollary 5.7] and need the following lemmas. Lemma 6.9 follows from an argument similar to that used in the proof of Proposition 4.4 (see also [45, Lemma 2.2]), the details being omitted.

Lemma 6.9. *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. If $\tau \in [1/p,\infty)$, then $\sup_{k\in\mathbb{Z},\,\alpha\in I_k}$ in the definitions of $f_{p,q}^{s,\tau}(M)$ and $\widetilde{f}_{p,q}^{s,\tau}(M)$ can be equivalently replaced by $\sup_{k\in\mathbb{Z}_+,\,\alpha\in I_k}$.*

Lemma 6.10. *Let $\varepsilon \in (0,1]$, $s \in \mathbb{R}$, $p \in (0,\infty), q \in (0,\infty]$ and $\tau \in [0,\infty)$. For all $j \in \mathbb{Z}_+, t \in I_j, 1 \le \nu \le N_t^j$, let $S_t^{j,\nu}$ be a set contained in $Q_t^{j,\nu}$ such that $|S_t^{j,\nu}|/|Q_t^{j,\nu}| \ge \varepsilon$. Then, there exists a positive constant C, depending on ϵ, s, p, q, τ, such that, for all $a := \{a_t^{j,\nu}\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N_t^j} \subset \mathbb{C}$,*

$$\begin{aligned}\frac{1}{C}\|a\|_{f_{p,q}^{s,\tau}(M)} &\le \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q_\alpha^k|^\tau}\left\{\int_{Q_\alpha^k}\left[\sum_{j=k\vee 0}^{\infty}\delta^{-jsq}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|a_t^{j,\nu}|\chi_{S_t^{j,\nu}}(x)\right)^q\right]^{p/q} d\mu(x)\right\}^{1/p}\\ &\le C\|a\|_{f_{p,q}^{s,\tau}(M)}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{C}\|a\|_{\widetilde{f}_{p,q}^{s,\tau}(M)} &\le \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q_\alpha^k|^\tau}\left\{\int_{Q_\alpha^k}\left[\sum_{j=k\vee 0}^{\infty}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{S_t^{j,\nu}}(x)\right)^q\right]^{p/q} d\mu(x)\right\}^{1/p}\\ &\le C\|a\|_{\widetilde{f}_{p,q}^{s,\tau}(M)}.\end{aligned}$$

Proof. By similarity, we only consider the second formula regarding $\widetilde{f}_{p,q}^{s,\tau}(M)$. Obviously, the right-hand side is dominated by the left-hand side. To see the inverse, notice that, for all $j \in \mathbb{Z}_+, t \in I_j, 1 \le \nu \le N_t^j$, $A \in (0,\infty)$ and $x \in M$, $\chi_{Q_t^{j,\nu}}(x) \le \varepsilon^{-1/A}[\mathcal{M}(\chi_{S_t^{j,\nu}})(x)]^{1/A}$. We choose A such that $0 < A < \min\{p,q\}$. Then, by the Fefferman-Stein vector-valued inequality, we see that

$$\begin{aligned}&\|a\|_{\widetilde{f}_{p,q}^{s,\tau}(M)}\\ &\quad\lesssim \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q_\alpha^k|^\tau}\left\{\int_{Q_\alpha^k}\left[\sum_{j=k\vee 0}^{\infty}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|[\mathcal{M}(\chi_{S_t^{j,\nu}}\chi_{Q_\alpha^k})(x)]^{1/A}\right)^q\right]^{p/q} d\mu(x)\right\}^{1/p}\\ &\quad\lesssim \sup_{\substack{k\in\mathbb{Z}\\ \alpha\in I_k}} \frac{1}{|Q_\alpha^k|^\tau}\left\{\int_{Q_\alpha^k}\left[\sum_{j=k\vee 0}^{\infty}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{S_t^{j,\nu}}(x)\right)^q\right]^{p/q} d\mu(x)\right\}^{1/p}.\end{aligned}$$

This finishes the proof of Lemma 6.10. □

Combining Lemmas 6.9 and 6.10, we can show that, if $\tau \in [1/p,\infty)$, then the supremum $\sup_{k\in\mathbb{Z},\,\alpha\in I_k}$ and $Q_t^{j,\nu}$ in the definitions of $f_{p,q}^{s,\tau}(M)$ and $\widetilde{f}_{p,q}^{s,\tau}(M)$ can be equivalently replaced by $\sup_{k\in\mathbb{Z}_+,\,\alpha\in I_k}$ and $S_t^{j,\nu}$, respectively.

For any sequence $a := \{a_t^{j,\nu}\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N_t^j} \subset \mathbb{C}$, define

$$\widetilde{G}_{k,\alpha,u}^{s,q}(a)(x) := \left[\sum_{j=k}^{\infty}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{Q_t^{j,\nu}}(x)\chi_{Q_\alpha^{k,u}}(x)\right)^q\right]^{1/q}, \qquad x \in M,$$

$$\widetilde{m}^{s,q}_{k,\alpha,u}(a) := \inf\left\{\lambda>0:\ |\{x\in Q_\alpha^{k,u}:\ \widetilde{G}^{s,q}_{k,\alpha,u}(a)(x)>\lambda\}|<|Q_\alpha^{k,u}|/4\right\}$$

and

$$\widetilde{m}^{s,q}(a)(x) := \sup_{k\in\mathbb{Z}_+,\alpha\in I_k,1\le u\le N_\alpha^k} \widetilde{m}^{s,q}_{k,\alpha,u}(a)\chi_{Q_\alpha^{k,u}}(x), \qquad x\in M.$$

By the same way as above with $|Q_t^{j,\nu}|^{-s/d}$ replaced by δ^{-jsq}, we also define $G^{s,q}_{k,\alpha,u}(a)$, $m^{s,q}_{k,\alpha,u}(a)$ and $m^{s,q}(a)$, respectively.

Lemma 6.11. *Let $s\in\mathbb{R}$ and $q\in(0,\infty]$. Then, there exists a positive constant C such that, for all sequences $a:=\{a_t^{j,\nu}\}_{j\in\mathbb{Z}_+,t\in I_j,1\le\nu\le N_t^j}\subset\mathbb{C}$,*

$$\frac{1}{C}\|a\|_{f^{s,1/q}_{q,q}(M)} \le \|m^{s,q}(a)\|_{L^\infty(M)} \le C\|a\|_{f^{s,1/q}_{q,q}(M)}$$

and

$$\frac{1}{C}\|a\|_{\widetilde{f}^{s,1/q}_{q,q}(M)} \le \|\widetilde{m}^{s,q}(a)\|_{L^\infty(M)} \le C\|a\|_{\widetilde{f}^{s,1/q}_{q,q}(M)}.$$

Proof. If we choose $\lambda\in(4^{1/q}\|a\|_{\widetilde{f}^{s,1/q}_{q,q}(M)},\infty)$, then, by the Chebyshev inequality, we have

$$\left|\left\{x\in Q_\alpha^{k,u}:\ \widetilde{G}^{s,q}_{k,\alpha,u}(a)(x)>\lambda\right\}\right| \le \frac{1}{\lambda^q}\int_{Q_\alpha^{k,u}}[\widetilde{G}^{s,q}_{k,\alpha,u}(a)(x)]^q\,d\mu(x) \le \frac{|Q_\alpha^{k,u}|\|a\|^q_{\widetilde{f}^{s,1/q}_{q,q}(M)}}{\lambda^q} < \frac{1}{4}|Q_\alpha^{k,u}|.$$

Therefore, $\|\widetilde{m}^{s,q}(a)\|_{L^\infty(M)}\lesssim\|a\|_{\widetilde{f}^{s,1/q}_{q,q}(M)}$. Conversely, for any $x\in M$, we define a stopping time function

$$v(x) := \inf\left\{v\in\mathbb{Z}_+:\ \left[\sum_{j=v}^\infty\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{Q_t^{j,\nu}}(x)\right)^q\right]^{1/q}\le\widetilde{m}^{s,q}(a)(x)\right\},$$

and let

$$S_t^{j,\nu} := \left\{x\in Q_t^{j,\nu}:\ v(x)\le j\right\} = \left\{x\in Q_t^{j,\nu}:\ \widetilde{G}^{s,q}_{j,t,\nu}(a)(x)\le\widetilde{m}^{s,q}(a)(x)\right\}.$$

Moreover, $|S_t^{j,\nu}|/|Q_t^{j,\nu}|\ge 3/4$ and, for all $x\in M$,

$$\left[\sum_{j\in\mathbb{Z}_+}\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{S_t^{j,\nu}}(x)\right)^q\right]^{1/q}\le\widetilde{m}^{s,q}(a)(x).$$

By this and Lemma 6.10, we obtain $\|a\|_{\widetilde{f}^{s,1/q}_{q,q}(M)}\lesssim\|\widetilde{m}^{s,q}(a)\|_{L^\infty(M)}$, which completes the proof of Lemma 6.11. □

Proof of Proposition 6.8. By similarity, we only prove $\widetilde{f}^{s,1/p}_{p,q}(M)=\widetilde{f}^{s,1/q}_{q,q}(M)$. If $p\ge q$, then $\|a\|_{\widetilde{f}^{s,1/q}_{q,q}(M)}\lesssim\|a\|_{\widetilde{f}^{s,1/p}_{p,q}(M)}$ follows immediately from Hölder's inequality. Conversely, let $S_t^{j,\nu}$ be as in the proof of Lemma 6.11. Then, by Lemmas 6.9, 6.10 and 6.11, we know that

$$\begin{aligned}\|a\|_{\widetilde{f}^{s,1/p}_{p,q}(M)} &\sim \sup_{\substack{k\in\mathbb{Z}_+\\ \alpha\in I_k}}\left\{\frac{1}{|Q_\alpha^k|}\int_{Q_\alpha^k}\left[\sum_{j=k}^\infty\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{S_t^{j,\nu}}(x)\right)^q\right]^{p/q}d\mu(x)\right\}^{1/p}\\ &\lesssim \left\|\left[\sum_{j=0}^\infty\left(\sum_{t\in I_j}\sum_{\nu=1}^{N_t^j}|Q_t^{j,\nu}|^{-s/d}|a_t^{j,\nu}|\chi_{S_t^{j,\nu}}\right)^q\right]^{1/q}\right\|_{L^\infty(M)} \lesssim \|a\|_{\widetilde{f}^{s,1/q}_{q,q}(M)}.\end{aligned}$$

Now we consider $p < q$. By Hölder's inequality, we have $\|a\|_{\widetilde{f}^{s,1/p}_{p,q}(M)} \lesssim \|a\|_{\widetilde{f}^{s,1/q}_{q,q}(M)}$. To prove the converse, we only need to repeat the proof of Lemma 6.11 involving the Chebyshev inequality. The only difference is that we need to replace q therein by p now, which completes the proof of Proposition 6.8. $\square$

7 Further remarks

In this section, we first prove that, on the Euclidean spaces, when L is the Laplacian operator, the Besov-type and Triebel–Lizorkin-type spaces coincide with those introduced in [45]. Furthermore, when $\tau = 0$ and $\beta_0 = 2$ in ($\mathbf{UE}'$) and ($\mathbf{HE}'$), we show that the Besov and Triebel–Lizorkin spaces on RD-spaces satisfying (1.5) defined in [21] coincide with the Besov and Triebel–Lizorkin spaces in [22], which gives a positive answer to a question presented in [22].

7.1 Go back to the Euclidean setting

In this section, we consider the case that $M = \mathbb{R}^n$, ρ is the Euclidean distance, and the measure μ is the Lebesgue measure. In this case, the collection of Christ cubes is chosen to be the classical dyadic cubes

$$\mathcal{Q} := \left\{ Q^j_m := 2^{-j}([0,1)^n + m) : \, j \in \mathbb{Z}, \, m \in \mathbb{Z}^n \right\},$$

and the constant δ in Lemma 2.1 is exactly 1/2. Assume that L is the Laplacian operator $\Delta := -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. Then, L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ such that the associated heat semigroup $\{e^{-tL}\}_{t>0}$ consists of integral operators with heat kernels p_t satisfying the conditions (1.6) with $\beta_0 = 2$, (1.7) with $\alpha_0 = 1$, and (1.8).

Let $\mathcal{S}(\mathbb{R}^n)$ denote the class of all Schwartz functions on $\mathbb{R}^n$. By the Newton-Leibniz formula and the mathematical induction, an easy calculation leads to that the test function space $\mathcal{D}(\mathbb{R}^n)$ in the beginning of Section 2 is exactly the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. As a consequence, the distribution space $\mathcal{D}'(\mathbb{R}^n)$ coincides with the space $\mathcal{S}'(\mathbb{R}^n)$ of Schwartz distributions.

With the previous discussions, the previous discussed Besov-type and Triebel–Lizorkin-type spaces on $\mathbb{R}^n$ read as follows.

Definition 7.1. Let $\Phi_0, \Phi \in C^\infty(\mathbb{R}_+)$ such that

$$\operatorname{supp} \Phi_0 \subset [0,2], \quad \Phi_0^{(2\nu+1)}(0) = 1 \text{ for all } \nu \in \mathbb{N}, \qquad |\Phi_0(\lambda)| \geq c \text{ for } \lambda \in [0, 2^{3/4}],$$

and

$$\operatorname{supp} \Phi \subset [2^{-1}, 2], \qquad |\Phi(\lambda)| \geq c \text{ for } \lambda \in [2^{-3/4}, 2^{3/4}],$$

where c is a positive constant. Let $\Phi_j(\cdot) := \Phi(2^{-j}\cdot)$ for all $j \in \mathbb{N}$. Let $\tau \in [0,\infty)$, $s \in \mathbb{R}$ and $q \in (0,\infty]$. For $p \in (0,\infty]$, the *Besov-type space* $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}, m\in\mathbb{Z}^n} \frac{1}{|Q^k_m|^\tau} \left\{ \sum_{j=k\vee 0}^\infty \left[\int_{Q^k_m} 2^{jsp} |\Phi_j(\sqrt{\Delta}) f(x)| \, dx \right]^{q/p} \right\}^{1/q} < \infty.$$

For $p \in (0,\infty)$, the *Triebel–Lizorkin-type space* $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}, m\in\mathbb{Z}^n} \frac{1}{|Q^k_m|^\tau} \left\{ \int_{Q^k_m} \left[\sum_{j=k\vee 0}^\infty 2^{jsq} |\Phi_j(\sqrt{\Delta}) f(x)|^q \right]^{p/q} dx \right\}^{1/p} < \infty.$$

Recall that in [45] Besov-type and Triebel–Lizorkin-type spaces were introduced as follows.

Definition 7.2. Let ϕ_0, $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \widehat{\phi_0} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}, \qquad |\widehat{\phi_0}(\xi)| \ge c \quad \text{if} \quad |\xi| \le 5/3, \tag{7.1}$$

and

$$\operatorname{supp} \widehat{\phi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}, \qquad |\widehat{\phi}(\xi)| \ge c \quad \text{if} \quad 3/5 \le |\xi| \le 5/3, \tag{7.2}$$

where c is a positive constant. For $j \in \mathbb{N}$, define $\phi_j(\cdot) := 2^{jn}\phi(2^j\cdot)$. Let $\tau \in [0,\infty)$, $s \in \mathbb{R}$ and $q \in (0,\infty]$. For $p \in (0,\infty]$, the *Besov-type space* $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}, m\in\mathbb{Z}^n} \frac{1}{|Q^k_m|^\tau} \left\{ \sum_{j=k\vee 0}^{\infty} \left[\int_{Q^k_m} 2^{jsp} |\phi_j * f(x)|\, dx \right]^{q/p} \right\}^{1/q} < \infty.$$

For $p \in (0,\infty)$, the *Triebel–Lizorkin-type space* $\mathcal{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{F}^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}, m\in\mathbb{Z}^n} \frac{1}{|Q^k_m|^\tau} \left\{ \int_{Q^k_m} \left[\sum_{j=k\vee 0}^{\infty} 2^{jsq} |\phi_j * f(x)|^q \right]^{p/q} dx \right\}^{1/p} < \infty.$$

Theorem 7.3. *Let all the notation be as in Definitions 7.1 and 7.2. Then* $B^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ *and* $F^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathcal{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ *with equivalent (quasi-)norms.*

Proof. Let $m > s/2$ and h_0, h and h_j be as in (5.15). That is, $h_0(\lambda) := e^{-\lambda^2}$, $h(\lambda) := \lambda^{2m}e^{-\lambda^2}$ and $h_j(\lambda) := h(2^j\lambda) = (2^j\lambda)^{2m}e^{-2^{2j}\lambda^2}$ for all $j \in \mathbb{N}$ and $\lambda \in (0,\infty)$. By Theorem 5.8, we know that $f \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and

$$\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n,H)} := \|\{2^{-js}h_j(\sqrt{\Delta})f\}_{j\in\mathbb{Z}_+}\|_{\ell^q(L^p_\tau)} < \infty.$$

Moreover, $\|\cdot\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n,H)} \sim \|\cdot\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}$.

On the other hand, let $H_0(x) := e^{-|x|^2}$ and $H(x) := (|\cdot|^{2m}e^{-|\cdot|^2})^\vee(x)$ for all $x \in \mathbb{R}^n$. Then, it is well known that $h_0(\sqrt{\Delta})f = e^{-\Delta}f = C_0H_0 * f$ and $h(\sqrt{\Delta})f = \Delta^m e^{-\Delta}f = C_1H * f$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and some positive constants C_0 and C_1. Notice that $\widehat{H}_0$ and $\widehat{H}$ are positive on $B(0,2)$ and $B(0,2) \setminus B(0,1/2)$, respectively, and $\partial^\alpha\widehat{H}(0) = 0$ for all $|\alpha| \le 2m$. Thus, by the local mean characterization of $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ (see [26]), we know that $\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n) = B^{s,\tau}_{p,q}(H)$ with equivalent (quasi-)norms. Thus, $B^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with equivalent (quasi-)norms.

The proof for the Triebel–Lizorkin-type case is similar, the details being omitted. This finishes the proof of Theorem 7.3. $\square$

7.2 Go back to the RD-space

A systemic treatment of the theory of (in)homogeneous Besov and Triebel–Lizorkin spaces on RD-spaces was due to the work [21]. Here, in this subsection, we compare the Besov and Triebel–Lizorkin spaces introduced in [21] with those in [22]. Throughout this subsection, for all x, $y \in M$ and $\delta > 0$, let $V_\delta(x) := \mu(B(x,\delta))$ and $V(x,y) := \mu(B(x,d(x,y)))$.

Recall that the definitions of Besov and Triebel–Lizorkin spaces in [21] rely on the existence of the so-called *approximation of the identity*; see [21, Definition 2.2].

Definition 7.4. Let $\epsilon_1 \in (0,1]$ and ϵ_2, $\epsilon_3 \in (0,\infty)$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(M)$ is called an *approximation of the identity of order* $(\epsilon_1,\epsilon_2,\epsilon_3)$ (in short, $(\epsilon_1,\epsilon_2,\epsilon_3)$ - ATI), if there exists a positive constant C such that, for all $k\in\mathbb{Z}$, x, x', y and $y'\in M$, $S_k(x,y)$, the integral kernel of S_k, is a measurable function from $M\times M$ into $\mathbb{C}$ satisfying that

(i) $|S_k(x,y)| \le C\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}[\frac{2^{-k}}{2^{-k}+\rho(x,y)}]^{\epsilon_2}$;

(ii) for $\rho(x,x') \le [2^{-k}+\rho(x,y)]/2$,

$$|S_k(x,y)-S_k(x',y)| \le C\left[\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right]^{\epsilon_1}\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}\left[\frac{2^{-k}}{2^{-k}+\rho(x,y)}\right]^{\epsilon_2};$$

(iii) S_k satisfies (ii) with x and y interchanged;

(iv) for $\rho(x,x') \le [2^{-k}+\rho(x,y)]/3$ and $\rho(y,y') \le [2^{-k}+\rho(x,y)]/3$,

$$\begin{aligned}&|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\\&\quad\le C\left[\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right]^{\epsilon_1}\left[\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right]^{\epsilon_1}\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}\left[\frac{2^{-k}}{2^{-k}+\rho(x,y)}\right]^{\epsilon_3};\end{aligned}$$

(v) $\int_M S_k(x,w)\,d\mu(w) = 1 = \int_M S_k(w,y)\,d\mu(w)$.

Definition 7.5. Let $x_1\in M$, $r\in(0,\infty)$, $\beta\in(0,1]$ and $\gamma\in(0,\infty)$. A function φ on M is said to belong to the *space* $\mathcal{G}(x_1,r,\beta,\gamma)$ *of test functions*, if there exists a positive constant C such that, for all $x,y\in M$,

(i) $|\varphi(x)| \le C\frac{1}{V_r(x_1)+V_r(x)+V(x_1,x)}[\frac{r}{r+\rho(x_1,x)}]^{\gamma}$;

(ii) $|\varphi(x)-\varphi(y)| \le C[\frac{\rho(x,y)}{r+\rho(x_1,x)}]^{\beta}\frac{1}{V_r(x_1)+V_r(x)+V(x_1,x)}[\frac{r}{r+\rho(x_1,x)}]^{\gamma}$ when $\rho(x,y)\le[r+\rho(x_1,x)]/2$.

If $\varphi\in\mathcal{G}(x_1,r,\beta,\gamma)$, then its norm is defined by $\|\varphi\|_{\mathcal{G}(x_1,\,r,\,\beta,\,\gamma)} := \inf\{C:\ \text{(i) and (ii) hold}\}$.

Fix $x_1\in M$ and let $\mathcal{G}(\beta,\gamma) := \mathcal{G}(x_1,1,\beta,\gamma)$. For any $x_2\in M$ and $r\in(0,\infty)$, it is easy to see that $\mathcal{G}(x_2,r,\beta,\gamma)=\mathcal{G}(\beta,\gamma)$ with equivalent norms. Also, the space $\mathcal{G}(\beta,\gamma)$ is a Banach space. Let $\epsilon\in(0,1]$ and β, $\gamma\in(0,\epsilon]$. Denote by $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$ the completion of $\mathcal{G}(\epsilon,\epsilon)$ in $\mathcal{G}(\beta,\gamma)$. Then, $\varphi\in\mathcal{G}_0^{\epsilon}(\beta,\gamma)$ if and only if $\varphi\in\mathcal{G}(\beta,\gamma)$ and there exist functions $\{\phi_j\}_{j\in\mathbb{N}}$ converging to φ in $\mathcal{G}(\epsilon,\epsilon)$. For any $\varphi\in\mathcal{G}_0^{\epsilon}(\beta,\gamma)$, define $\|\varphi\|_{\mathcal{G}_0^{\epsilon}(\beta,\gamma)} := \|\varphi\|_{\mathcal{G}(\beta,\gamma)}$. For the above chosen $\{\phi_j\}_{j\in\mathbb{N}}$, we have $\|\varphi\|_{\mathcal{G}_0^{\epsilon}(\beta,\gamma)} = \lim_{j\to\infty}\|\phi_j\|_{\mathcal{G}(\beta,\gamma)}$. Notice that $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$ is also a Banach space. Denote by $(\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$ the *set of all bounded linear functionals* on $\mathcal{G}_0^{\epsilon}(\beta,\gamma)$. Define $\langle f,\varphi\rangle$ to be the natural pairing of elements $f\in(\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$ and $\varphi\in\mathcal{G}_0^{\epsilon}(\beta,\gamma)$.

We now recall the definition of Besov and Triebel–Lizorkin spaces on RD-spaces in [21, Definition 5.29]. In what follows, for all $\epsilon\in(0,1)$ and $|s|<\epsilon$, let $p(s,\epsilon) := \max\{d/(d+\epsilon), d/(d+\epsilon+s)\}$. We also write $m_Q(g) := \frac{1}{|Q|}\int_Q g(y)\,d\mu(y)$ for all $g\in L^1_{\rm loc}(M)$ and Christ dyadic cubes Q.

Definition 7.6. Let $\epsilon_1\in(0,1]$, ϵ_2, $\epsilon_3\in(0,\infty)$, $\epsilon\in(0,\epsilon_1\wedge\epsilon_2)$, β, $\gamma\in(0,\epsilon)$, $|s|<\epsilon$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$- ATI. Define $D_0 := S_0$, and $D_k := S_k - S_{k-1}$ for all $k\in\mathbb{N}$. Let $\{Q_\tau^{0,v}\}_{\tau\in I_0, v\in\{1,\dots,N_\tau^0\}}$ be dyadic cubes in Section 6.

(i) Let $p\in(p(s,\epsilon),\infty]$ and $q\in(0,\infty]$. The *Besov space* $\mathcal{B}^s_{p,q}(M)$ is defined to be the set of all $f\in(\mathcal{G}_0^{\epsilon}(\beta,\gamma))'$, for some β, γ satisfying

$$\max\{s,0,-s+d(1/p-1)_+\}<\beta<\epsilon,\quad d(1/p-1)_+<\gamma<\epsilon \tag{7.3}$$

such that

$$\|f\|_{\mathcal{B}^s_{p,q}(M)} := \left[\sum_{\tau\in I_0}\sum_{v=1}^{N^0_\tau}|Q^{0,v}_\tau|[m_{Q^{0,v}_\tau}(|D_0 f|)]^p\right]^{1/p} + \left[\sum_{k=1}^{\infty}2^{ksq}\|D_k f\|^q_{L^p(M)}\right]^{1/q} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

(ii) Let $p \in (p(s,\epsilon),\infty)$ and $q \in (p(s,\epsilon),\infty]$. The *Triebel–Lizorkin space* $\mathcal{F}^s_{p,q}(M)$ is defined to be the set of all $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$ for some β, γ satisfying (7.3) such that

$$\|f\|_{\mathcal{F}^s_{p,q}(M)} := \left[\sum_{\tau\in I_0}\sum_{v=1}^{N^0_\tau}|Q^{0,v}_\tau|[m_{Q^{0,v}_\tau}(|D_0 f|)]^p\right]^{1/p} + \left\|\left[\sum_{k=1}^{\infty}2^{ksq}|D_k f|^q\right]^{1/q}\right\|_{L^p(M)} < \infty$$

with the usual modification made when $q = \infty$.

For all $\epsilon_1 \in (0,1]$, ϵ_2, $\epsilon_3 \in (0,\infty)$, $\epsilon \in (0,\epsilon_1\wedge\epsilon_2)$, β, $\gamma \in (0,\epsilon)$ and $|s| < \epsilon$, it was proved in [21, Proposition 5.32] that, if $p \in [1,\infty]$ and $q \in (0,\infty]$, then, for all $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$,

$$\|f\|_{\mathcal{B}^s_{p,q}(M)} \sim \left[\sum_{k=0}^{\infty}2^{ksq}\|D_k f\|^q_{L^p(M)}\right]^{1/q}$$

and, if $p \in [1,\infty)$ and $q \in (p(s,\epsilon),\infty]$, then for all $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$,

$$\|f\|_{\mathcal{F}^s_{p,q}(M)} \sim \left\|\left[\sum_{k=0}^{\infty}2^{ksq}|D_k f|^q\right]^{1/q}\right\|_{L^p(M)},$$

where the implicit equivalent positive constants are independent of f.

Applying the discrete Calderón reproducing formula in Theorem 6.1, by an argument similar to that used the proof of [21, Proposition 5.32], we obtain the following main result of this subsection, which answers a question presented in [22].

Theorem 7.7. *Let* $\epsilon_1 \in (0,1]$*,* ϵ_2*,* $\epsilon_3 \in (0,\infty)$*,* $\epsilon \in (0,\alpha_0\wedge\epsilon_1\wedge\epsilon_2)$*,* β*,* $\gamma \in (0,\epsilon)$ *and* $|s| < \epsilon$*, where* α_0 *is as in* (**HE**′)*.*

(i) *If* $p \in (p(s,\epsilon),\infty]$*,* $q \in (0,\infty]$ *and* β*,* γ *satisfy* (7.3)*, then* $\|f\|_{\mathcal{B}^s_{p,q}(M)} \sim \|f\|_{B^s_{p,q}(M)}$ *for any* $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))' \cap \mathcal{D}'(M)$*.*

(ii) *If* $p \in (p(s,\epsilon),\infty)$*,* $q \in (p(s,\epsilon),\infty]$ *and* β*,* γ *satisfy* (7.3)*, then* $\|f\|_{\mathcal{F}^s_{p,q}(M)} \sim \|f\|_{F^s_{p,q}(M)}$ *for any* $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))' \cap \mathcal{D}'(M)$*.*

Here, in (i) and (ii), the implicit positive equivalent constants are independent of f*.*

Proof. First, we show (i). Let $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-ATI. Set $D_0 := S_0$, and $D_k := S_k - S_{k-1}$ for $k \in \mathbb{N}$. It was proved in [21, Lemma 3.2, (3.2)] that, for any $\epsilon_1' \in (0,\epsilon_1\wedge\epsilon_2)$ and all $x, y \in M$,

$$|D_k D_j(x,y)| \lesssim 2^{-|j-k|\epsilon_1'}\frac{1}{V_{2^{-(k\wedge j)}}(x)+V_{2^{-(k\wedge j)}}(y)+V(x,y)}\frac{2^{-(k\wedge j)\epsilon_2}}{[2^{-(k\wedge j)}+\rho(x,y)]^{\epsilon_2}}. \tag{7.4}$$

For $\beta,\gamma \in (0,\epsilon)$, we recall that the discrete inhomogeneous Calderón reproducing formula in [21, Theorems 4.14 and 4.16]: for all $f \in (\mathcal{G}^\epsilon_0(\beta,\gamma))'$,

$$f = \sum_{\tau\in I_0}\sum_{v=1}^{N^0_\tau}\int_{Q^{0,v}_\tau}\widetilde{D}_0(\cdot,y)\,d\mu(y)D^{0,v}_{\tau,1}f + \sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{v=1}^{N^k_\tau}|Q^{k,v}_\tau|\widetilde{D}_k(\cdot,y^{k,v}_\tau)D_k f(y^{k,v}_\tau) \tag{7.5}$$

converges in $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$, where $y_\tau^{k,v}$ is an arbitrary point in $Q_\tau^{k,v}$, $D_{\tau,1}^{0,v}$ denotes the integral operator with kernel $D_{\tau,1}^{0,v}(z) := \frac{1}{|Q_\tau^{0,v}|}\int_{Q_\tau^{0,v}} D_0(z,u)\,d\mu(u)$, and the kernels of the operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}_+}$ satisfy the conditions (i) and (iii) of Definition 7.4 with ϵ_1 and ϵ_2 replaced by $\epsilon' \in (\epsilon, \epsilon_1\wedge\epsilon_2)$, and $\int_M \widetilde{D}_k(x,y)\,d\mu(y) = 0$ when $k\in\mathbb{N}$ and $=1$ when $k=0$.

Let $\Phi_0 \in C_c^\infty(\mathbb{R}_+)$ such that $\operatorname{supp}\Phi_0 \subset [0, 2^{\beta_0/2}]$ and $\Phi_0 \equiv 1$ on $[0,1]$. Define $\Phi(\lambda) := \Phi_0(\lambda) - \Phi_0(2^{\beta_0/2}\lambda)$ for $\lambda\in\mathbb{R}_+$. For $j\in\mathbb{N}$, define $\Phi_j(\cdot) := \Phi(2^{-j\beta_0/2}\cdot)$. By (i) and (ii) of Proposition 3.4, we see that $\Phi_j(L)(x,\cdot) \in \mathcal{G}_0^\epsilon(\alpha_0,\gamma) \subset \mathcal{G}_0^\epsilon(\beta,\gamma)$. Moreover, by Proposition 3.4(iii), we see that $\int_M \Phi_j(\sqrt{L})(x,y)\,d\mu(y) = 0$ for $j\in\mathbb{N}$ and $=1$ for $j=0$. From these and the proof of [21, Lemma 3.2, (3.2)], it follows that an orthogonal estimate similar to (7.4) holds, namely, for any given $\epsilon_1'' \in (0, \alpha_0\wedge\epsilon_1\wedge\epsilon_2)$,

$$(7.6)\quad |\Phi_j(\sqrt{L})\widetilde{D}_k(x,y)| \lesssim 2^{-|j-k|\epsilon_1''}\frac{1}{V_{2^{-(k\wedge j)}}(x)+V_{2^{-(k\wedge j)}}(y)+V(x,y)}\frac{1}{[1+2^{(k\wedge j)}\rho(x,y)]^{\epsilon_2}}$$

holds for all $j,k\in\mathbb{Z}_+$ and $x,y\in M$. Further, applying (7.5), we know that, for all $x\in M$,

$$\begin{aligned}
(7.7)\ &|\Phi_j(\sqrt{L})f(x)|\\
&= \Bigg|\sum_{\tau\in I_0}\sum_{v=1}^{N_\tau^0}\int_{Q_\tau^{0,v}}\Phi_j(\sqrt{L})\widetilde{D}_0(x,y)\,d\mu(y)D_{\tau,1}^{0,v}f\\
&\quad+\sum_{k=1}^\infty\sum_{\tau\in I_k}\sum_{v=1}^{N_\tau^k}|Q_\tau^{k,v}|\Phi_j(\sqrt{L})\widetilde{D}_k(x,y_\tau^{k,v})D_kf(y_\tau^{k,v})\Bigg|\\
&\lesssim \sum_{\tau\in I_0}\sum_{v=1}^{N_\tau^0}\int_{Q_\tau^{0,v}}\frac{2^{-j\epsilon_1''}}{V_1(x)+V_1(y)+V(x,y)}\frac{1}{[1+\rho(x,y)]^{\epsilon_2}}\,d\mu(y)|D_{\tau,1}^{0,v}f|\\
&\quad+\sum_{k=1}^\infty\sum_{\tau\in I_k}\sum_{v=1}^{N_\tau^k}|Q_\tau^{k,v}|\frac{2^{-|k-j|\epsilon_1''}}{V_{2^{-(k\wedge j)}}(x)+V_{2^{-(k\wedge j)}}(y_\tau^{k,v})+V(x,y_\tau^{k,v})}\frac{|D_kf(y_\tau^{k,v})|}{[1+2^{(k\wedge j)}\rho(x,y_\tau^{k,v})]^{\epsilon_2}}.
\end{aligned}$$

Since both $\epsilon, \epsilon_1'' \in (0,\alpha_0\wedge\epsilon_1\wedge\epsilon_2)$, $|s|<\epsilon$ and β, $\gamma\in(0,\epsilon)$, we may choose ϵ_1'' such that $|s|<\epsilon_1''$, $p(s,\epsilon_1'')<p$ and β, $\gamma\in(0,\epsilon_1'')$ satisfy (7.3) with ϵ replaced by ϵ_1''. On the other hand, since $V_1(x)+V_1(y)+V(x,y)\sim V_1(x)+V_1(y_\tau^{0,v})+V(x,y_\tau^{0,v})$ and $1+\rho(x,y)\sim 1+\rho(x,y_\tau^{0,v})$ for any $y\in Q_\tau^{0,v}$, applying [21, Lemma 5.3], we see that, for all $x\in M$,

$$\begin{aligned}
(7.8)\ &|2^{js}\Phi_j(\sqrt{L})f(x)|\\
&\lesssim \sum_{\tau\in I_0}\sum_{v=1}^{N_\tau^0}|Q_\tau^{0,v}|\frac{2^{-j(\epsilon_1''-s)}}{V_1(x)+V_1(y_\tau^{0,v})+V(x,y_\tau^{0,v})}\frac{1}{[1+\rho(x,y_\tau^{0,v})]^{\epsilon_2}}|D_{\tau,1}^{0,v}f|\\
&\quad+\sum_{k=1}^\infty\sum_{\tau\in I_k}\sum_{v=1}^{N_\tau^k}|Q_\tau^{k,v}|\frac{2^{-|k-j|(\epsilon_1''-s)}}{V_{2^{-(k\wedge j)}}(x)+V_{2^{-(k\wedge j)}}(y_\tau^{k,v})+V(x,y_\tau^{k,v})}\frac{2^{ks}|D_kf(y_\tau^{k,v})|}{[1+2^{(k\wedge j)}\rho(x,y_\tau^{k,v})]^{\epsilon_2}}\\
&\lesssim 2^{-j(\epsilon_1''-s)}\left[\mathcal{M}\left(\sum_{\tau\in I_0}\sum_{v=1}^{N_\tau^0}|D_{\tau,1}^{0,v}f|^r\chi_{Q_\tau^{0,v}}\right)(x)\right]^{1/r}\\
&\quad+\sum_{k=1}^\infty 2^{-|k-j|\epsilon_1''}2^{(j-k)s}2^{[(k\wedge j)-k]d(1-1/r)}\left[\mathcal{M}\left(\sum_{\tau\in I_k}\sum_{v=1}^{N_\tau^k}2^{ksr}|D_kf(y_\tau^{k,v})|^r\chi_{Q_\tau^{k,v}}\right)(x)\right]^{1/r},
\end{aligned}$$

where we chose $r \in (p(s,\epsilon_1''), p)$ if $p \le 1$ or $r = 1$ if $p \in (1,\infty]$. Therefore, letting $\sigma \in (0, \epsilon_1'' - \max\{s, 0, -s + d(1/r-1)\})$, by Lemma 5.4 and the boundedness of $\mathcal{M}$ on $L^{p/r}(M)$, we see that

$$\begin{aligned}
(7.9)\qquad \|f\|_{B^s_{p,q}(M)} &\lesssim \left\{\sum_{j=0}^{\infty} 2^{-j(\epsilon_1''-s-\sigma)q} \left\| \left[\mathcal{M}\left(\sum_{\tau\in I_0}\sum_{v=1}^{N_\tau^0} |D^{0,v}_{\tau,1} f|^r \chi_{Q_\tau^{0,v}}\right)\right]^{1/r} \right\|^q_{L^p(M)}\right.\\
&\quad + \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} 2^{-|k-j|(\epsilon_1''-\sigma)q} 2^{(j-k)sq} 2^{[(k\wedge j)-k]d(1-1/r)q}\\
&\quad \left.\times \left\| \left[\mathcal{M}\left(\sum_{\tau\in I_k}\sum_{v=1}^{N_\tau^k} 2^{ksr} |D_k f(y_\tau^{k,v})|^r \chi_{Q_\tau^{k,v}}\right)\right]^{1/r} \right\|^q_{L^p(M)}\right\}^{1/q}\\
&\lesssim \left\| \sum_{\tau\in I_0}\sum_{v=1}^{N_\tau^0} |D^{0,v}_{\tau,1} f| \chi_{Q_\tau^{0,v}} \right\|_{L^p(M)}\\
&\quad + \left\{\sum_{k=1}^{\infty} \left\| \sum_{\tau\in I_k}\sum_{v=1}^{N_\tau^k} 2^{ks} |D_k f(y_\tau^{k,v})| \chi_{Q_\tau^{k,v}} \right\|^q_{L^p(M)}\right\}^{1/q} \sim \|f\|_{\mathcal{B}^s_{p,q}(M)},
\end{aligned}$$

where the last inequality follows from the frame characterization of $\mathcal{B}^s_{p,q}(M)$ in [21, Theorem 7.4].

To show the converse, for $f \in B^s_{p,q}(M)$, we use (6.2) to write every $D_j f$ as

$$D_j f(\cdot) = \sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N_\tau^k} |Q_\tau^{k,\nu}| (\Phi_k(\sqrt{L})f)(\xi_\tau^{k,\nu})\, D_j \Psi_k(\xi_\tau^{k,\nu}, \cdot),$$

where Φ_k and Ψ_k are as in (6.2). Notice that $D_j \Psi_k(\sqrt{L})(x,y)$ has the same estimate as in (7.6). Thus, repeating the previous proofs of (7.7)-(7.9), but with the roles of D_j and $\Phi_j(\sqrt{L})$ exchanged, and applying Theorem 6.4, we obtain $\|f\|_{\mathcal{B}^s_{p,q}(M)} \lesssim \|f\|_{B^s_{p,q}(M)}$. This finishes the proof of (i).

Now we show (ii). This time we choose ϵ_1'' such that $|s| < \epsilon_1''$, $p(s,\epsilon_1'') < \min\{p,q\}$ and β, $\gamma \in (0, \epsilon_1'')$ satisfy (7.3) with ϵ replaced by ϵ_1''. In this case, (7.7) and (7.8) keep valid, but this time we choose $r \in (p(s,\epsilon_1''), \min\{p,q\})$ if $\min\{p,q\} \le 1$ or $r = 1$ if $\min\{p,q\} \in (1,\infty]$. Then, by Lemma 5.4, the frame characterization of $\mathcal{F}^s_{p,q}(M)$ in [21, Theorem 7.4], and repeating the above proof for Besov spaces, with the boundedness of $\mathcal{M}$ on $L^{p/r}(M)$ replaced by the Fefferman-Stein vector-valued maximal inequality (see [17]), we see that

$$\begin{aligned}
\|f\|_{F^s_{p,q}(M)} &\lesssim \left\| \left\{ \sum_{j=0}^{\infty} 2^{-j(\epsilon_1''-s)q} \left[\mathcal{M}\left(\sum_{\tau\in I_0}\sum_{v=1}^{N_\tau^0} |D^{0,v}_{\tau,1} f|^r \chi_{Q_\tau^{0,v}}\right)\right]^{q/r}\right.\right.\\
&\quad + \sum_{k=1}^{\infty} 2^{-|k-j|(\epsilon_1''-\sigma)q} 2^{(j-k)sq} 2^{[(k\wedge j)-k]d(1-1/r)q}\\
&\quad \left.\left.\times \left[\mathcal{M}\left(\sum_{\tau\in I_k}\sum_{v=1}^{N_\tau^k} 2^{ksr} |D_k f(y_\tau^{k,v})|^r \chi_{Q_\tau^{k,v}}\right)\right]^{q/r}\right\}^{1/q} \right\|_{L^p(M)} \lesssim \|f\|_{\mathcal{F}^s_{p,q}(M)}.
\end{aligned}$$

The converse of this inequality follows from an argument similar to that used in the proof of $\|f\|_{\mathcal{B}^s_{p,q}(M)} \lesssim \|f\|_{B^s_{p,q}(M)}$, with the boundedness of $\mathcal{M}$ on $L^{p/r}(M)$ replaced by the Fefferman-Stein vector-valued inequality. Thus, (ii) holds, and the proof of Theorem 7.7 is then completed. $\Box$

8 Appendix

The main aim of this section is to show Theorem 6.1. Using the language of Christ cubes, we restate the Marcinkiewicz–Zygmund inequality, whose proof was essentially given in [9, Proposition 4.1], here we re-present its proof for convenience as the constants involved is very subtle.

Lemma 8.1. *Let* $\lambda, p \in [1,\infty)$. *Then, for any* $f \in \Sigma_\lambda^p(M)$, *all* $j \in \mathbb{Z}$ *such that* $j \geq -\frac{2}{\beta_0}\log_\delta \lambda$, *and* $\xi_\tau^{j,\nu} \in Q_\tau^{j,\nu}$ *with* $\tau \in I_j$ *and* $\nu \in \{1,\ldots,N_\tau^j\}$,

$$\left\{\sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j}\int_{Q_\tau^{j,\nu}} |f(x)-f(\xi_\tau^{j,\nu})|^p\,d\mu(x)\right\}^{1/p} \leq \frac{\epsilon_0}{8}\left(\lambda^{2/\beta_0}\delta^j\right)^{\alpha_0}\|f\|_{L^p(M)}. \tag{8.1}$$

When $p=\infty$, *it holds true that*

$$\sup_{\tau\in I_j}\ \sup_{1\leq\nu\leq N_\tau^j}\ \sup_{x\in Q_\tau^{j,\nu}} |f(x)-f(\xi_\tau^{j,\nu})| \leq \frac{\epsilon_0}{8}(\lambda^{2/\beta_0}\delta^j)^{\alpha_0}\|f\|_{L^\infty(M)}. \tag{8.2}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}_+)$ be such that $\operatorname{supp}\phi \subset [0,2]$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $[0,1]$. For any $\sigma > 3d/2$, by Proposition 3.4 and the self-adjoint property of L, there exists a positive constant $C_{\sigma,0}$ such that, for all x, x', $y \in M$ satisfying that $\rho(x,x') \leq \lambda^{-2/\beta_0}$,

$$|\phi(\lambda^{-1}\sqrt{L})(x,y)-\phi(\lambda^{-1}\sqrt{L})(x',y)| \leq C_{\sigma,0}\left[\lambda^{2/\beta_0}\rho(x,x')\right]^{\alpha_0} D_{\lambda^{-2/\beta_0},\sigma}(x,y).$$

Also, $\phi(\lambda^{-1}\sqrt{L})f = f$ for all $f \in \Sigma_\lambda^p(M)$. For all $x \in Q_\tau^{j,\nu}$, by (6.1), we have $\rho(x,\xi_\tau^{j,\nu}) \leq \operatorname{diam} Q_\tau^{j,\nu} \leq \delta^j \leq \lambda^{-2/\beta_0}$ and hence

$$\begin{aligned}|f(x)-f(\xi_\tau^{j,\nu})| &= \left|\int_M [\phi(\lambda^{-1}\sqrt{L})(x,y)-\phi(\lambda^{-1}\sqrt{L})(\xi_\tau^{j,\nu},y)]f(y)\,d\mu(y)\right|\\ &\leq C_{\sigma,0}(C_\natural\lambda^{2/\beta_0}\delta^{j+j_0})^{\alpha_0}\int_M D_{\lambda^{-2/\beta_0},\sigma}(x,y)|f(y)|\,d\mu(y).\end{aligned}$$

Let J denote the left-hand side of (8.1). Applying Hölder's inequality and Lemma 3.1(iii), and then using Fubini's theorem and (6.1), we see that

$$\begin{aligned}\mathrm{J} &\leq C_{\sigma,0}(C_\natural\lambda^{2/\beta_0}\delta^{j+j_0})^{\alpha_0}\left\{\int_M\left|\int_M D_{\lambda^{-2/\beta_0},\sigma}(x,y)|f(y)|\,d\mu(y)\right|^p d\mu(x)\right\}^{1/p}\\ &\leq (C_\sigma^*)^{1/p'}C_{\sigma,0}(C_\natural\lambda^{2/\beta_0}\delta^{j+j_0})^{\alpha_0}\left\{\int_M\int_M D_{\lambda^{-2/\beta_0},\sigma}(x,y)|f(y)|^p\,d\mu(y)\,d\mu(x)\right\}^{1/p}\\ &\leq C_\sigma^* C_{\sigma,0}(C_\natural\lambda^{2/\beta_0}\delta^{j+j_0})^{\alpha_0}\|f\|_{L^p(M)} \leq \frac{\epsilon_0}{8}(\lambda^{2/\beta_0}\delta^j)^{\alpha_0}\|f\|_{L^\infty(M)}.\end{aligned}$$

This proves (8.1). A modification of the above proof also shows (8.2), which completes the proof of Lemma 8.1. □

Applying Lemma 8.1 and (6.1), and arguing as in the proof of [9, Theorem 4.2], we obtain the following sampling theorem, the details being omitted.

Lemma 8.2. *Let* $\lambda \in [1,\infty)$ *and* $p \in [1,2]$. *Then, for any* $f \in \Sigma_\lambda^p(M)$ *and* $j \in \mathbb{Z}$ *such that* $j \geq -\frac{2}{\beta_0}\log_\delta\lambda$, *and all* $\xi_\tau^{j,\nu} \in Q_\tau^{j,\nu}$ *with* $\tau \in I_j$ *and* $\nu \in \{1,\ldots,N_\tau^j\}$,

$$(1-\epsilon_0)\|f\|_{L^p(M)} \leq \left\{\sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j}|Q_\tau^{j,\nu}||f(\xi_\tau^{j,\nu})|^p\right\}^{1/p} \leq (1+\epsilon_0)\|f\|_{L^p(M)}.$$

The following Cubature formula (see [9, Theorem 4.4]) follows directly from Lemma 8.1 and [9, Proposition 4.6], the details being omitted.

Corollary 8.3. *Let* $\lambda \in [1,\infty)$ *and* $j \in \mathbb{Z}$ *such that* $j \ge -\frac{2}{\beta_0}\log_\delta \lambda$. *Then, there exists a sequence of positive constants,* $\{\varepsilon_\tau^{j,\nu} : \tau \in I_j,\ 1 \le \nu \le N_\tau^j\}$ *with* $\frac{2}{3} \le \varepsilon_\tau^{j,\nu} \le 2$, *such that, for all* $f \in \Sigma_\lambda^1(M)$ *and* $\xi_\tau^{j,\nu} \in Q_\tau^{j,\nu}$ *with* $\tau \in I_j$ *and* $\nu \in \{1,\ldots,N_\tau^j\}$,

$$\int_M f(x)\,d\mu(x) = \sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j} \varepsilon_\tau^{j,\nu}|Q_\tau^{j,\nu}| f(\xi_\tau^{j,\nu}).$$

The following lemma follows from the same manner as that used in the proof of [22, Lemma 3.10], the details being also omitted.

Lemma 8.4. *Let* $\gamma,\beta \in (0,\infty)$ *and* $\delta \in (0,1)$ *be as in Lemma 2.1. Then, there exists a positive constant* $C^\diamond_{\gamma,\beta}$ *such that, for all* $j \in \mathbb{Z}$, $x, y \in M$, *and* $\xi_\tau^{j,\nu} \in Q_\tau^{j,\nu}$ *with* $\tau \in I_j$ *and* $1 \le \nu \le N_\tau^j$,

$$\sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j} |Q_\tau^{j,\nu}| E_{\delta^j}^{\gamma,\beta}(x,\xi_\tau^{j,\nu}) E_{\delta^j}^{\gamma,\beta}(\xi_\tau^{j,\nu},y) \le C^\diamond_{\gamma,\beta} E_{\delta^j}^{\gamma,\beta}(x,y).$$

Applying Lemma 8.4 and Corollary 8.3, and invoking the ideas used in the proof of [22, Lemma 4.2], we have the following conclusion.

Lemma 8.5. *Suppose that* $\Gamma_0, \Gamma \in C_c^\infty(\mathbb{R}_+)$ *satisfy that*

$$\operatorname{supp}\Gamma_0 \subset [0,\delta^{-\beta_0}], \quad \Gamma_0(\lambda) = 1 \text{ for } \lambda \in [0,\delta^{-\beta_0/2}],$$

and

$$\operatorname{supp}\Gamma \subset [\delta^{\beta_0},\delta^{-\beta_0}], \qquad \Gamma_0(\lambda) = 1 \text{ for } \lambda \in [\delta^{\beta_0/2},\delta^{-\beta_0/2}], \quad \text{and } 0 \le \Gamma_0,\ \Gamma \le 1.$$

Assume that there exist constants $A \in (0,\infty)$ *and* $\beta \in (0,1)$ *such that, for any* $i \in \{0,1\}$ *and* $k \in \mathbb{N}$,

$$\|\Gamma_0^{(k)}\|_{L^\infty(\mathbb{R}_+)} \le (Ak)^{k(2-\beta)} \quad \text{and} \quad \|\Gamma^{(k)}\|_{L^\infty(\mathbb{R}_+)} \le (Ak)^{k(2-\beta)}.$$

Let $\Gamma_j := \Gamma(\delta^{j\beta_0/2}\cdot)$ *for* $j \in \mathbb{N}$. *Assume that* $f \in L^2(M)$ *such that* $\Gamma_j(\sqrt{L})f = f$ *for some* $j \in \mathbb{Z}_+$. *Then, there exist operators* $\{\varphi_j(\sqrt{L})\}_{j\in\mathbb{Z}_+}$ *such that, for all* $x \in M$ *and* $\xi_\tau^{j,\nu} \in Q_\tau^{j,\nu}$ *with* $\tau \in I_j$ *and* $\nu \in \{1,\ldots,N_\tau^j\}$,

$$f(x) = \sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j} |Q_\tau^{j,\nu}| f(\xi_\tau^{j,\nu}) \widetilde{\varphi}_j(\sqrt{L})(\xi_\tau^{j,\nu},x), \tag{8.3}$$

where $\widetilde{\varphi}_j(\sqrt{L})(\xi_\tau^{j,\nu},x)$ *is the integral kernel of an operator* $\widetilde{\varphi}_j(\sqrt{L})$. *Moreover, for any* $m \in \mathbb{Z}_+$, *there exist positive constants* C *and* γ, *depending only on* β, M *and* m, *such that the following hold:*

(i) *For all* $x, y \in M$, $|L^m\widetilde{\varphi}_j(\sqrt{L})(x,y)| \le C\delta^{-2mj} E_{\delta^j}^{\frac{\gamma}{2},\beta}(x,y)$;

(ii) *For any* $x, y, y' \in \mathcal{X}$ *satisfying* $\rho(x,y) \le \delta^j$,

$$|\widetilde{\varphi}_j(\sqrt{L})(x,y) - \widetilde{\varphi}_j(\sqrt{L})(x,y')| + |\widetilde{\varphi}_j(\sqrt{L})(y,x) - \widetilde{\varphi}_j(\sqrt{L})(y',x)| \le C[\delta^{-j}\rho(y,y')]^{\alpha_0} E_{\delta^j}^{\frac{\gamma}{2},\beta}(x,y);$$

(iii) *Let* (Φ_0, Φ) *satisfy* (2.1) *and* (2.2). *Then, for any* $m \in \mathbb{N}$ *and* $\sigma \in (d, \infty)$, *there exists a positive constant* $C_{\sigma,m}$ *such that, for all* $j, k \in \mathbb{N}$ *and* $x, y \in M$,

$$\left|\left(\Phi_k(\sqrt{L})\widetilde{\varphi}_j(\sqrt{L})\right)(x, y)\right| \le C_{\sigma,m}\delta^{|k-j|(m\beta_0-d)}D_{\delta^{k\wedge j},\sigma}(x, y).$$

Proof. Suppose that $\Theta \in C_c^\infty(\mathbb{R}_+)$ satisfies that $\operatorname{supp}\Theta \subset [0, \delta^{-2\beta_0}]$ and $\Theta \equiv 1$ on $[0, \delta^{-3\beta_0/2}]$. Define $\Theta_j := \Theta(\delta^{j\beta_0/2}\sqrt{L})$. For simplicity, we use $\Gamma_j(x, y)$ and $\Theta_j(x, y)$ to denote the integral kernels of the operators $\Gamma_j(\sqrt{L})$ and $\Theta_j(\sqrt{L})$, respectively. Define the operator U_j which is associated with the kernel

$$U_j(x, y) := \sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j}\frac{1}{(1+\epsilon_0)^2}|Q_\tau^{j,\nu}|\Theta_j(x, \xi_\tau^{j,\nu})\Theta_j(\xi_\tau^{j,\nu}, y), \qquad x,\ y \in M.$$

Define $V_j := \Gamma_j U_j \Gamma_j$ and $R_j := \Gamma_j(\mathrm{Id} - U_j)\Gamma_j = \Gamma_j^2 - V_j$. Since $\Theta_j\Gamma_j = \Gamma_j$, we use the expression of U_j to deduce that the operator V_j has a kernel as follows:

$$V_j(x, y) = \sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j}\frac{1}{(1+\epsilon_0)^2}|Q_\tau^{j,\nu}|\Gamma_j(x, \xi_\tau^{j,\nu})\Gamma_j(\xi_\tau^{j,\nu}, y), \qquad x,\ y \in M.$$

By Lemma 8.2, for any $f \in \Sigma^2_{\delta^{-j\beta_0/2}}(M)$ and all $\xi_\tau^{j,\nu} \in Q_\tau^{j,\nu}$ with $\tau \in I_j$ and $\nu \in \{1, \dots, N_\tau^j\}$, we have

$$\frac{(1-\epsilon_0)^2}{(1+\epsilon_0)^2}\|f\|^2_{L^2(M)} \le \sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j}\frac{1}{(1+\epsilon_0)^2}|Q_\tau^{j,\nu}||f(\xi_\tau^{j,\nu})|^2 \le \|f\|^2_{L^2(M)}.$$

Notice that $\frac{1-\epsilon_0}{1+\epsilon_0} \ge 1 - 2\epsilon_0$. Also, for any $f \in \Sigma^2_{\delta^{-(j+3)\beta_0/2}}(M)$, it holds true that $\Theta_j f = f$, and hence

$$(1-2\epsilon_0)^2\|f\|^2_{L^2(M)} \le \langle U_j f,\, f\rangle = \sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j}\frac{1}{(1+\epsilon_0)^2}|Q_\tau^{j,\nu}||f(\xi_\tau^{j,\nu})|^2 \le \|f\|^2_{L^2(M)}.$$

Notice that $\Gamma_j(L^2(M)) \subset \Sigma^2_{\delta^{-(j+3)\beta_0/2}}(M)$. As $0 \le \Gamma_j \le 1$, the functional calculus implies that $\|\Gamma_j f\|_{L^2(M)} \le \|f\|_{L^2(M)}$. Thus, for any $f \in L^2(M)$, by $R_j = \Gamma_j^2 - V_j = \Gamma_j^2 - \Gamma_j U_j \Gamma_j$, we have

$$0 \le \langle R_j f,\, f\rangle = \langle \Gamma_j f,\, \Gamma_j f\rangle - \langle U_j\Gamma_j f,\, \Gamma_j f\rangle \le [1 - (1-2\epsilon_0)^2]\|\Gamma_j f\|^2_{L^2(M)} \le 4\epsilon_0\|f\|^2_{L^2(M)},$$

which implies that R_j is bounded on $L^2(M)$ with operator norm at most $2\sqrt{\epsilon_0}$. Hence, it makes sense to define $T_j := (I - R_j)^{-1} = I + \sum_{k=1}^\infty R_j^k$. Further, if $f \in L^2(M)$ such that $\Gamma_j(\sqrt{L})f = f$, then $f = T_j(f - R_j f) = T_j(f - \Gamma_j^2 f + V_j f) = T_j V_j f$, so that

$$f(x) = \sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j}\frac{1}{(1+\epsilon_0)^2}|Q_\tau^{j,\nu}|\Gamma_j f(\xi_\tau^{j,\nu})T_j\left(\Gamma_j(\cdot, \xi_\tau^{j,\nu})\right)(x) = \sum_{\tau\in I_j}\sum_{\nu=1}^{N_\tau^j}|Q_\tau^{j,\nu}|\, f(\xi_\tau^{j,\nu})\,\widetilde{\varphi}_j(\sqrt{L})(\xi_\tau^{j,\nu}, x),$$

by setting $\widetilde{\varphi}_j(\sqrt{L})$ to be the operator whose associate integral kernel

$$\widetilde{\varphi}_j(\sqrt{L})(x, y) := \frac{1}{(1+\epsilon_0)^2}T_j\left(\Gamma_j(\cdot, y)\right)(x), \quad x, y \in M.$$

This proves (8.3).

For simplicity, we write $\widetilde{\varphi}_j(x,y)$ instead of the kernel $\widetilde{\varphi}_j(\sqrt{L})(x,y)$. It remains to show that $\widetilde{\varphi}_j(x,y)$ satisfies the properties (i)-(iii). By Proposition 3.6, there exist positive constants γ and $C^\flat_{\gamma,\beta,0}$ such that, for all x, $y\in M$,

$$|\Gamma_j(x,y)| \le C^\flat_{\gamma,\beta,0} E^{\gamma,\beta}_{\delta^j}(x,y),$$

which, together with Lemma 8.4, implies that

$$|V_j(x,y)| \le \frac{(C^\flat_{\gamma,\beta,0})^2}{(1+\epsilon_0)^2} \sum_{\tau\in I_j}\sum_{\nu=1}^{N^j_\tau} |Q^{j,\nu}_\tau| E^{\gamma,\beta}_{\delta^j}(x,\xi^{j,\nu}_\tau) E^{\gamma,\beta}_{\delta^j}(\xi^{j,\nu}_\tau,y) \le \frac{(C^\flat_{\gamma,\beta,0})^2 C^\diamond_{\gamma,\beta}}{(1+\epsilon_0)^2} E^{\gamma,\beta}_{\delta^j}(x,y) \tag{8.4}$$

and also, together with Lemma 3.1(iii), implies that

$$|\Gamma_j^2(x,y)| \le (C^\flat_{\gamma,\beta,0})^2 \int_M E^{\gamma,\beta}_{\delta^j}(x,u) E^{\gamma,\beta}_{\delta^j}(u,y)\,d\mu(u) \le (C^\flat_{\gamma,\beta,0})^2 C^*_{\gamma,\beta} E^{\gamma,\beta}_{\delta^j}(x,y). \tag{8.5}$$

Combining (8.4) and (8.5), we find that

$$|R_j(x,y)| = |\Gamma_j^2(x,y) - V_j(x,y)| \le (C^\diamond_{\gamma,\beta} + C^*_{\gamma,\beta})(C^\flat_{\gamma,\beta,0})^2 E^{\gamma,\beta}_{\delta^j}(x,y) =: \mathbb{A} E^{\gamma,\beta}_{\delta^j}(x,y). \tag{8.6}$$

Applying this estimate and Lemma 3.1(iii) repeatedly, we see that, for all $k\in\mathbb{N}$ and $x, y\in M$,

$$|R_j^k(x,y)| \le \mathbb{A}^k (C^*_{\gamma,\beta})^{k-1} E^{\gamma,\beta}_{\delta^j}(x,y). \tag{8.7}$$

Consequently, for all $k\in\mathbb{N}$,

$$|R_j^k\Gamma_j(x,y)| \le (\mathbb{A}C^*_{\gamma,\beta})^k C^\flat_{\gamma,\beta,0} E^{\gamma,\beta}_{\delta^j}(x,y). \tag{8.8}$$

By Lemma 3.2(ii), we see that, for $k\ge 2$,

$$\begin{aligned} |R_j^k\Gamma_j(x,y)| = |R_j R_j^{k-1}\Gamma_j(x,y)| &\le \mathbb{A} C^\flat_{\gamma,\beta,0} C^*_{\gamma,\beta} \frac{\|R_j^{k-1}\|_{L^2(M)\to L^2(M)}}{\sqrt{|B(x,\delta^j)|\,|B(y,\delta^j)|}} \\ &\le \mathbb{A} C^\flat_{\gamma,\beta,0} C^*_{\gamma,\beta} \frac{(2\sqrt{\epsilon_0})^{k-1}}{\sqrt{|B(x,\delta^j)|\,|B(y,\delta^j)|}}. \end{aligned} \tag{8.9}$$

By taking geometric means between the above two inequalities, we know that

$$|R_j^k\Gamma_j(x,y)| \lesssim \sqrt{(C^*_{\gamma,\beta}\mathbb{A})^{k-1}(2\sqrt{\epsilon_0})^{k-1}}\, E^{\gamma/2,\beta}_{\delta^j}(x,y) \lesssim 2^{-(k-1)}\, E^{\gamma/2,\beta}_{\delta^j}(x,y), \tag{8.10}$$

provided that we choose ϵ_0 small enough satisfying that

$$0 < \sqrt{\epsilon_0} \le \frac{1}{4\mathbb{A}C^*_{\gamma,\beta}} = \frac{1}{4(C^\diamond_{\gamma,\beta} + C^*_{\gamma,\beta})(C^\flat_{\gamma,\beta,0})^2 C^*_{\gamma,\beta}}. \tag{8.11}$$

Therefore, by writing

$$(1+\epsilon_0)^2\widetilde{\varphi}_j(\sqrt{L})(x,y) = T_j\left(\Gamma_j(\cdot,y)\right)(x) = \Gamma_j(x,y) + R_j\left(\Gamma_j(\cdot,y)\right)(x) + \sum_{k=2}^{\infty} R_j^k\left(\Gamma_j(\cdot,y)\right)(x),$$

we use (8.10) to obtain that $|\widetilde{\varphi}_j(\sqrt{L})(x,y)| \lesssim E_{\delta^j}^{\gamma/2,\beta}(x,y)$, which proves (i) for $m=0$. If $m \in \mathbb{N}$, then, by writing

$$(1+\epsilon_0)^2 L^m \widetilde{\varphi}_j(x,y) = T_j\left(L^m\Gamma_j(\cdot,y)\right)(x) = L^m\Gamma_j(x,y) + \sum_{k=1}^{\infty} R_j^k\left(L^m\Gamma_j(\cdot,y)\right)(x),$$

and following the previous argument, we also obtain the desired result of (i). To obtain (ii), write

$$\begin{aligned}
&(1+\epsilon_0)^2|\widetilde{\varphi}_j(\sqrt{L})(x,y) - \widetilde{\varphi}_j(\sqrt{L})(x,y')| \\
&\quad\le |\Gamma_j(x,y)-\Gamma_j(x,y')| + \left|\int_M R_j(x,u)[\Gamma_j(u,y)-\Gamma_j(u,y')]\,d\mu(u)\right| \\
&\qquad + \sum_{k=2}^{\infty}\left|\int_M R_j^k(x,u)[\Gamma_j(u,y)-\Gamma_j(u,y')]\,d\mu(u)\right|.
\end{aligned}$$

Then, applying Proposition 3.6, we see that, when $\rho(y,y') \le \delta^j$,

$$|\Gamma_j(x,y) - \Gamma(x,y')| \le C^*_{\gamma,\beta}[\delta^{-j}\rho(y,y')]^{\alpha_0} E_{\delta^j}^{\gamma,\beta}(x,y).$$

From this estimate and following the previous argument, one deduces (ii), the details being omitted.

To show (iii), for $j \in \mathbb{Z}_+$, we use $\Phi_j\Gamma_j(x,y)$ to denote the kernel of $\Phi_j(\sqrt{L})\Gamma_j(\sqrt{L})$. Notice that Proposition 3.7 implies that

$$|\Phi_k\Gamma_j(x,y)| \lesssim \delta^{|k-j|(m\beta_0-d/2)} D_{\delta^{k\wedge j},2\sigma}(x,y), \qquad x,\ y \in M. \tag{8.12}$$

Then, instead of (8.8), we apply (8.7) to conclude that, for all $i \in \mathbb{N}$,

$$|R_j^i\Phi_k\Gamma_j(x,y)| \lesssim (C^*_{\gamma,\beta}\mathbb{A})^i \delta^{|k-j|(m\beta_0-d/2)} D_{\delta^{k\wedge j},2\sigma}(x,y), \qquad x,\ y \in M.$$

Similar to the estimate of (8.9), we use Lemma 3.2(i) and (8.12) to find that, for all $i \ge 2$,

$$\begin{aligned}
|R_j^i\Phi_k\Gamma_j(x,y)| &\le \|R_j(x,\cdot)\|_{L^2(M)}\|R_j^{i-1}\|_{L^2(M)\to L^2(M)}\|\Phi_k\Gamma_j(\cdot,y)\|_{L^2(M)} \\
&\lesssim \frac{\delta^{|k-j|(m\beta_0-d/2)}(4\epsilon_0)^{i-1}}{\sqrt{|B(x,\delta^j)|\,|B(y,\delta^{j\wedge k})|}} \lesssim \frac{\delta^{|k-j|(m\beta_0-d)}(4\epsilon_0)^{i-1}}{\sqrt{|B(x,\delta^{j\wedge k})|\,|B(y,\delta^{j\wedge k})|}}, \qquad x,\ y \in M.
\end{aligned}$$

By taking geometric mean of the above inequalities, we see that, for all $x,\ y \in M$,

$$|R_j^i\Phi_k\Gamma_j(x,y)| \lesssim (\epsilon_0 C^*_{\gamma,\beta}\mathbb{A})^i \delta^{|k-j|(m\beta_0-d)} D_{\delta^{k\wedge j},\sigma}(x,y) \lesssim 2^{-i}\delta^{|k-j|(m\beta_0-d)} D_{\delta^{k\wedge j},\sigma}(x,y),$$

if we choose ϵ_0 small enough. Then, summing over all $i \ge 2$ and using the fact that

$$(1+\epsilon_0)^2\left(\Phi_k(\sqrt{L})\widetilde{\varphi}_j(\sqrt{L})\right)(x,y) = \Phi_k\Gamma_j(x,y) + \sum_{i=1}^{\infty} R_j^i\Phi_k\Gamma_j(x,y), \qquad x,\ y \in M,$$

we argue as before and obtain the desired estimate in (iii), which completes the proof of Lemma 8.5. □

Now we are ready to deduce the discrete Calderón reproducing formula.

Proof of Theorem 6.1. By similarity, we only prove the case $f \in \mathcal{D}'(M)$. Starting from the continuous Calderón reproducing formula (3.7), we write

$$f = \sum_{j=0}^{\infty} \widetilde{\Phi}_j(\sqrt{L})\Phi_j(\sqrt{L})f = \sum_{j=0}^{\infty} \int_M \int_M \widetilde{\Phi}_j(\sqrt{L})(\cdot, y)\Phi_j(\sqrt{L})(y,z)f(z)\,d\mu(y)\,d\mu(z).$$

For any $z \in M$, applying Lemma 8.5 to the function $\Phi_j(\sqrt{L})(\cdot, z)$, we know that, for all $y \in M$,

$$\Phi_j(\sqrt{L})(y,z) = \sum_{\tau \in I_j} \sum_{\nu=1}^{N_\tau^j} |Q_\tau^{j,\nu}| \Phi_j(\sqrt{L})(\xi_\tau^{j,\nu}, z)\widetilde{\varphi}_j(\sqrt{L})(\xi_\tau^{j,\nu}, y),$$

where $\widetilde{\varphi}_j(\sqrt{L})$ satisfies (i), (ii) and (iii) of Lemma 8.5. Consequently,

$$f(\cdot) = \sum_{j=0}^{\infty} \sum_{\tau \in I_j} \sum_{\nu=1}^{N_\tau^j} |Q_\tau^{j,\nu}|(\Phi_j(\sqrt{L})f)(\xi_\tau^{j,\nu}) \int_M \widetilde{\Phi}_j(\sqrt{L})(\cdot, y)\widetilde{\varphi}_j(\sqrt{L})(\xi_\tau^{j,\nu}, y)\,d\mu(y),$$

which implies (6.2), provided that the operator $\Psi_j(\sqrt{L})$ is defined with an associated kernel as follows: for all $x, z \in M$,

$$\Psi_j(\sqrt{L})(x,z) := \int_M \widetilde{\Phi}_j(\sqrt{L})(x,y)\widetilde{\varphi}_j(\sqrt{L})(z,y)\,d\mu(y).$$

Noticing that (i) and (ii) of Lemma 8.5 and the fact that Proposition 3.4 implies that $\widetilde{\Phi}_j(\sqrt{L})$ satisfies (3.4) and (3.5), we apply Lemma 3.1 to obtain (i) and (ii). Moreover, (iii) of the theorem follows from Lemma 8.5(iii), Proposition 3.4(i) and Lemma 3.1(ii). This finishes the proof of Theorem 6.1. ☐

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