COMBINATORIAL ASPECTS OF EXCEPTIONAL SEQUENCES ON (RATIONAL) SURFACES

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Abstract. We investigate combinatorial aspects of exceptional sequences in the derived category of coherent sheaves on smooth and complete algebraic surfaces with $p_g = q = 0$. We show that to any such sequence there is canonically associated a complete toric surface whose torus fixpoints are either smooth or cyclic T-singularities (in the sense of Wahl) of type $\frac{1}{r^2}(1, kr-1)$. We also show that any exceptional sequence be transformed by mutation into an exceptional sequence which consists only of objects of rank one.

Contents

1. Introduction

In this article we want to work out certain combinatorial aspects associated with exceptional sequences in the derived category $D^b(X)$ of coherent sheaves on rational surfaces. In earlier work [HP11] it was found, somewhat surprisingly, that to any exceptional sequence of invertible sheaves on a rational surface there is associated in a canonical way the combinatorial data of a smooth complete toric surface. This finding suggests that in many interesting cases there could be a link between semi-orthogonal decompositions of derived categories and toric geometry. However, this is far from being well-understood so far, even for the case of line bundles. An important development in this direction is recent work by Hacking and Prokhorov [HP10] and Hacking [Hac13]. In [HP10], singular surfaces with ample anticanonical divisor and Picard number one which admit $\mathbb{Q}$-Gorenstein smoothings are classified. These surfaces necessarily have T-singularities in the sense of Wahl [Wah81]. Among such surfaces, there is one family of weighted projective planes $\mathbb{P}(e^2, f^2, g^2)$ such that e, f, g satisfy the Markov equation

$$e^2 + f^2 + g^2 = 3efg.$$

This classification resembles Rudakov's interpretation [Rud89] of the classification of exceptional bundles $\mathbb{P}^2$ by Drezet and Le Potier [DL85]. Rudakov showed that any exceptional sequence $\mathcal{E}, \mathcal{F}, \mathcal{G}$ on $\mathbb{P}^2$ is essentially uniquely determined by the ranks (e, f, g, say) and the possible ranks ranks correspond to solutions of the Markov equation $e^2 + f^2 + g^2 = 3efg$. In [Hac13], Hacking shows that indeed there

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exists a natural bijective correspondence between degenerations of $\mathbb{P}^2$ and exceptional bundles on $\mathbb{P}^2$. More generally, for a rather large class of surfaces (which includes rational surfaces and certain surfaces of general type) Hacking constructs a correspondence between exceptional vector bundles and $\mathbb{Q}$-Gorenstein degenerations.

In this article, we want to follow some ideas of both [HP11] and [Hac13] and show that the relation between exceptional sequences and toric surfaces with T-singularities is a quite general phenomenon. Our main result is the following:

Theorem (11.3)**:** *Let X be a (rational) surface with and let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be a numerically exceptional sequence whose length equals $\operatorname{rk} K_0(X)$ such that $\operatorname{rk} \mathcal{E}_i = e_i \neq 0$ for every i. Then to this sequence there is associated in a canonical way a complete toric surface with n torus fixpoints which are either smooth (if $e_i^2 = 1$) or T-singularities of type $\frac{1}{e_i^2}(1, k_i e_i - 1)$ with $\gcd\{k_i, e_i\} = 1$ (if $e_i^2 > 1$).*

Here, the term "numerical" refers to a weaker version of exceptionality and semi-orthogonality which only requires the vanishing of Euler characteristics rather than Hom-vanishing (see Definition 2.2). By X being a (rational) surface, we mean that X is a smooth and complete algebraic surface which satisfies:

1) $p_g = q = 0$.
2) The Grothendieck group $K_0(X)$ is finitely generated and torsion free.
3) The Chow ring splits degree-wise as $\mathrm{CH}^*(X) \cong \mathbb{Z} \oplus \mathbb{Z}^{n-2} \oplus \mathbb{Z}$.

There does not seem to be a generally agreed upon name for surfaces satisfying these conditions (though, of course, there are the rational surfaces and "fake" rational surfaces of general type), so we settle for "(rational)" for sake of brevity. We will reserve the term "rational" (i.e. without the parentheses) to denote proper rational surfaces. Another interesting result which will be part of our analysis leading to Theorem 11.3 is the following.

Theorem (10.8)**:** *Let X be a (rational) surface. Then any numerically exceptional sequence on X can be transformed by mutation into a numerically exceptional sequence consisting only of objects of rank one.*

Note that both theorems a fortiori apply as well to proper exceptional sequences. The main reason why we formulated them for numerically exceptional sequences is that indeed they are almost purely a result of the surprisingly rich Riemann-Roch arithmetic. The most important consequence of this fact is that the correspondence between exceptional and toric surfaces can be formulated very generally in the sense that it does not depend on any geometric construction or any refined geometric properties of X. This is of particular interest in light of recent work [BGKS12] where exceptional sequences have been constructed on surfaces of general type which are (rational) in our sense. These sequences are almost full — their complements in the derived category are among the first examples of so-called phantom categories, and our results are applicable to these sequences as well. Indeed it is an open question whether the existence of a full exceptional sequence on a variety implies that this variety is rational. So far, no example of a non-rational variety which admits such a sequence is known. We believe that our results in conjunction with Hacking's provide tentative evidence that indeed the existence of an exceptional sequence implies rationality.

Overview. In Section 2 we introduce some basic notions. There is a good chance that the reader will avoid some confusion by paying attention to standing conventions as stated in paragraphs 2.1 and 2.9. Section 3 is devoted to the exploitation of the Riemann-Roch formula for exceptional objects. Sections 4 and 7 deal with some crucial aspects which arise if exceptional objects of rank zero are involved. In Sections 5 and 6 toric systems and their Gale dual are introduced. The latter will be analyzed locally in Sections 8 and 9. The global analysis and the main theorems are contained in Sections 10 and 11. Section 12 then concludes with some observations related to T-singularities. For easier reference, we collect some facts on toric surfaces in an appendix.

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2. Some generalities

2.1 (Standing conventions throughout the rest of this paper). We assume that X is a (rational) surface as defined in the introduction over some algebraically closed base field $\mathbb{K}$. We denote $D^b(X)$ the bounded derived category of coherent sheaves on X. We will always write objects of $D^b(X)$ in calligraphic style, $\mathcal{E}, \mathcal{F}, \dots, \mathcal{Z}$. Then their ranks will be denoted in the corresponding lower case letters $e, f, \dots, z$. By n we will always denote the rank of $K_0(X)$.

For any two objects $\mathcal{E}, \mathcal{F}$ of $D^b(X)$ the Euler Characteristic is defined as:

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \operatorname{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}[k]) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathbf{E}\mathrm{xt}^k_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}).$$

Definition 2.2: (i) We call an object $\mathcal{E}$ of $D^b(X)$ *exceptional* if $\operatorname{End}(\mathcal{E}) \cong \mathbb{K}$ and $\operatorname{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{E}[k]) = 0$ for all $k \neq 0$. We call $\mathcal{E}$ *numerically exceptional* if $\chi(\mathcal{E}, \mathcal{E}) = 1$.

(ii) A sequence of objects $\mathcal{E}_1, \dots, \mathcal{E}_t$ is called an *exceptional sequence* if all $\mathcal{E}_i$ are exceptional and $\operatorname{Hom}_{D^b(X)}(\mathcal{E}_i, \mathcal{E}_j[k]) = 0$ for all $i > j$ and all $k \in \mathbb{Z}$. Similarly, we call it a *numerically exceptional sequence* if all $\mathcal{E}_i$ are numerically exceptional and $\chi(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $i > j$.

(iii) Denote $\omega_X = \mathcal{O}(K_X)$ the canonical sheaf on X. Then we can extend any exceptional sequence $\mathcal{E}_1, \dots, \mathcal{E}_t$ to an infinite sequence $\dots, \mathcal{E}_i, \mathcal{E}_{i+1}, \dots$ by setting $\mathcal{E}_i := \mathcal{E}_r \otimes \omega_X^{-\lfloor \frac{i-1}{t} \rfloor}$ and $i = k\lfloor \frac{i-1}{t} \rfloor + r$ with $k \in \mathbb{Z}$, $1 \leq r \leq t$:

$$\dots, \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_t, \mathcal{E}_{t+1}, \dots = \dots, \mathcal{E}_t \otimes \omega_X, \mathcal{E}_1, \dots, \mathcal{E}_t, \mathcal{E}_1 \otimes \omega_X^{-1}, \dots$$

We call this a *cyclic exceptional sequence*. If the sequence is only numerically exceptional, then we call it *cyclic numerically exceptional sequence*.

(iv) An exceptional sequence is called *strongly* exceptional if $\operatorname{Hom}_{D^b(X)}(\mathcal{E}_i, \mathcal{E}_j[k]) = 0$ for all i, j and all $k \neq 0$. A cyclic exceptional sequence is called cyclic *strongly* exceptional if every winding is strongly exceptional.

(v) Any collection of objects in $D^b(X)$ is called *full* if it generates $D^b(X)$.

As general references for exceptional sequences we refer to [Bon90] and [Rud90].

2.3. Note that any sub-interval of length at most t of a cyclic (numerically) exceptional sequence is a (numerically) exceptional sequence (see [HP11, Proposition 5.1]). By convention, if we are given a fixed exceptional sequence $\mathcal{E}_1, \dots, \mathcal{E}_t$, then we will always implicitly assume that it is extended cyclically, i.e. we consider $\mathcal{E}_i$ for any $i \in \mathbb{Z}$, denoting any element of the original sequence twisted by an appopriate power of ω_X.

2.4. If $\mathcal{E}_1, \dots, \mathcal{E}_t$ is a (numerically) exceptional sequence, then so is $\mathcal{E}_1, \dots, \mathcal{E}_{i-1}, \mathcal{E}_i[j], \mathcal{E}_{i+1}, \dots, \mathcal{E}_t$ for any i and any $j \in \mathbb{Z}$. So, as long as we are not interested in concrete representations for the $\mathcal{E}_i$, we have some flexibility in considering exceptional sequences up to shift. For instance, there usually is no loss of generality to assume $e_i \geq 0$ for all i.

It is a standard fact that a full exceptional sequence $\mathcal{E}_1, \dots, \mathcal{E}_n$ induces a decomposition $K_0(X) \cong \bigoplus_{i=1}^n K(\langle \mathcal{E}_i \rangle)$ and $K(\langle \mathcal{E}_i \rangle) \cong K(\mathbb{K} - \text{vector spaces}) \cong \mathbb{Z}$, hence $K_0(X) \cong \mathbb{Z}^n$. On the other hand, any numerically exceptional sequence induces a semi-orthogonal decomposition of $K_0(X)$. In particular, the length of such a sequence is bounded by $n = \operatorname{rk} K_0(X)$.

2.5. For any pair of objects $\mathcal{E}, \mathcal{F}$ there exist the following two distinguished triangles:

$$L_{\mathcal{E}}\mathcal{F} \longrightarrow R\mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{K}} \mathcal{E} \xrightarrow{\text{can}} \mathcal{F},$$
$$\mathcal{E} \xrightarrow{\text{can}} R\mathcal{H}om(\mathcal{E}, \mathcal{F})^* \otimes_{\mathbb{K}} \mathcal{F} \longrightarrow R_{\mathcal{F}}\mathcal{E},$$

where *can* in both cases denotes the canonical evaluation map. If $\mathcal{E}, \mathcal{F}$ form an exceptional pair, then it follows that both $\mathcal{F}, R_{\mathcal{F}}\mathcal{E}$ and $L_{\mathcal{E}}\mathcal{F}, \mathcal{E}$ form exceptional pairs as well.

Definition 2.6: For an exceptional or numerically exceptional pair $\mathcal{E}, \mathcal{F}$, we call the pairs $\mathcal{F}, R_{\mathcal{F}}\mathcal{E}$ and $L_{\mathcal{E}}\mathcal{F}, \mathcal{E}$ its *right-* and *left-mutation*, respectively.

2.7. The usual invariants for sheaves such as rank and Chern classes extend naturally to $D^b(X)$ (and then factor naturally through $K_0(X)$). In particular, the rank function is additive on triangles and for an exceptional pair $\mathcal{E}, \mathcal{F}$ of ranks e and f, respectively, we obtain

$$\operatorname{rk} L_{\mathcal{E}}\mathcal{F} = \chi(\mathcal{E}, \mathcal{F})e - f \quad \text{and} \quad \operatorname{rk} R_{\mathcal{F}}\mathcal{E} = \chi(\mathcal{E}, \mathcal{F})f - e.$$

The first Chern classes can be written down directly:

$$c_1(L_{\mathcal{E}}\mathcal{F}) = \chi(\mathcal{E}, \mathcal{F})c_1(\mathcal{E}) - c_1(\mathcal{F}) \quad \text{and} \quad c_1(R_{\mathcal{F}}\mathcal{E}) = \chi(\mathcal{E}, \mathcal{F})c_1(\mathcal{F}) - c_1(\mathcal{E}).$$

For the second Chern classes, one can make use of the fact that for any triangle $\mathcal{T}' \to \mathcal{T} \to \mathcal{T}''$, the Chern character satisfies $\operatorname{ch}(\mathcal{T}) = \operatorname{ch}(\mathcal{T}') + \operatorname{ch}(\mathcal{T}'')$. With this, we obtain the following formulas for the second Chern classes of mutations:

$$c_2(L_{\mathcal{E}}\mathcal{F}) = \binom{\chi(\mathcal{E}, \mathcal{F})}{2} c_1(\mathcal{E})^2 - \chi(\mathcal{E}, \mathcal{F})c_1(\mathcal{E})c_1(\mathcal{F}) + c_1(\mathcal{F})^2 + \chi(\mathcal{E}, \mathcal{F})c_2(\mathcal{E}) - c_2(\mathcal{F}),$$

and similarly for $c_2(R_{\mathcal{F}}\mathcal{E})$.

2.8. More generally, for a (numerical or proper) exceptional sequence $\mathbf{E} := \mathcal{E}_1, \ldots, \mathcal{E}_t$, we can consider mutations at the i-th position:

$$\begin{aligned} R_i\mathbf{E} &:= \mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, \mathcal{E}_{i+1}, R_{\mathcal{E}_{i+1}}\mathcal{E}_i, \mathcal{E}_{i+2}, \ldots, \mathcal{E}_t, \\ L_i\mathbf{E} &:= \mathcal{E}_1, \ldots, \mathcal{E}_{i-1},\ L_{\mathcal{E}_i}\mathcal{E}_{i+1},\ \mathcal{E}_i,\ \mathcal{E}_{i+2}, \ldots, \mathcal{E}_t. \end{aligned}$$

Both $R_i\mathbf{E}$ and $L_i\mathbf{E}$ are exceptional sequences again. Moreover, up to natural equivalence, the R_i and L_i satisfy the following properties:

(i) $L_i = R_i^{-1}$;
(ii) the braid relations $R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}$, $L_iL_{i+1}L_i = L_{i+1}L_iL_{i+1}$.

In particular, the operators $L_1, \ldots, L_t, R_1, \ldots, R_t$ establish a braid group action on the exceptional sequences of length t. Note that mutations extend in a natural way to cyclic exceptional sequences.

2.9 (More standing conventions). In the following, mutations will be our main tool for manipulation (numerically) exceptional sequences and we may keep in mind that any mutation of a proper exceptional sequence is proper exceptional again. So, if we like to, we can distinguish between (proper) exceptional and numerically exceptional orbits of the braid group action. In order to avoid cumbersome language or awkward abbreviations, we will throughout sections 3 to 9 use the term "exceptional sequence" for both, proper and numerical exceptional sequences. The reader who does not care about numerical exceptional sequences can safely assume that we are only dealing with proper exceptional sequences. From section 10 on, we will start making the distinction between both cases. Again, the non-numerically inclined reader can safely assume that all results a fortiori apply to proper exceptional sequences.

3. Exceptional pairs and triples on surfaces

In [HP11], for an exceptional sequence of invertible sheaves $\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)$, we have considered the differences of divisor classes $D_{i+1} - D_i$. In our more general setting, we use the following generalization.

Definition: For any objects $\mathcal{E}, \mathcal{F}$ in $D^b(X)$, we denote $c_i(\mathcal{E}, \mathcal{F}) = c_i\big(R\mathcal{H}om(\mathcal{E}, \mathcal{F})\big)$.

3.1. For any two objects $\mathcal{E}$, $\mathcal{F}$ the following formula holds:

$$c_1(\mathcal{E}, \mathcal{F}) = ec_1(\mathcal{F}) - fc_1(\mathcal{E}).$$

This is immediately true for vector bundles, because in this case $R\mathcal{H}om(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^* \otimes \mathcal{F}$. The extension to the general case follows from the fact that, because X is smooth, any object in $D^b(X)$ is quasi-isomorphic to a finite complex of vector bundles.

3.2 (Riemann-Roch formula)**.** For any $\mathcal{E}, \mathcal{F}$ in $D^b(X)$, the Riemann-Roch formula is:

$$\chi(\mathcal{E}, \mathcal{F}) = ef - \frac{1}{2}K_X c_1(\mathcal{E}, \mathcal{F}) + \frac{1}{2}\big(fc_1(\mathcal{E})^2 + ec_1(\mathcal{F})^2 - 2c_1(\mathcal{E})c_1(\mathcal{F})\big) - \big(fc_2(\mathcal{E}) + ec_2(\mathcal{F})\big).$$

We now collect some identities which we obtain from simple inspection of the Riemann-Roch formula.

3.3. Let $\mathcal{E}$ be a numerically exceptional object in $D^b(X)$, i.e. $\chi(\mathcal{E},\mathcal{E})=1$. Then we obtain the following constraints on the Chern classes of $\mathcal{E}$.

(i) If $e=0$ then $c_1(\mathcal{E})^2=-1$.
(ii) If $e\neq 0$ then

$$c_2(\mathcal{E})=\frac{1}{2e}(e^2+(e-1)c_1(\mathcal{E})^2-1).$$

3.4. Now for objects $\mathcal{E},\mathcal{F}$ with $\chi(\mathcal{E},\mathcal{E})=\chi(\mathcal{F},\mathcal{F})=1$, we can use 3.3 to simplify the Riemann-Roch formula:

(i) If $e,f\neq 0$, then:

$$\chi(\mathcal{E},\mathcal{F})=-\frac{1}{2}K_Xc_1(\mathcal{E},\mathcal{F})+\frac{1}{2ef}(c_1(\mathcal{E},\mathcal{F})^2+e^2+f^2).$$

(ii) If $e=0$ and $f\neq 0$, then:

$$\chi(\mathcal{E},\mathcal{F})=\frac{f}{2}K_Xc_1(\mathcal{E})-\big(\frac{f}{2}+c_1(\mathcal{E})c_1(\mathcal{F})+fc_2(\mathcal{E})\big).$$

(iii) If $e\neq 0$ and $f=0$, then:

$$\chi(\mathcal{E},\mathcal{F})=-\frac{e}{2}K_Xc_1(\mathcal{F})-\big(\frac{e}{2}+c_1(\mathcal{E})c_1(\mathcal{F})+ec_2(\mathcal{F})\big).$$

(iv) If $e=f=0$, then $\chi(\mathcal{E},\mathcal{F})=-c_1(\mathcal{E})c_1(\mathcal{F})$.

3.5. Symmetrizing and Anti-symmetrizing of the Euler form yields for exceptional objects:

(i) $\chi(\mathcal{E},\mathcal{F})-\chi(\mathcal{F},\mathcal{E})=-K_Xc_1(\mathcal{E},\mathcal{F})$.
(ii) If $e,f\neq 0$, then $\chi(\mathcal{E},\mathcal{F})+\chi(\mathcal{F},\mathcal{E})=\frac{1}{ef}(c_1(\mathcal{E},\mathcal{F})^2+e^2+f^2)$.
(iii) If $e=0$ and $f\neq 0$, then $\chi(\mathcal{E},\mathcal{F})+\chi(\mathcal{F},\mathcal{E})=-\big(f+2c_1(\mathcal{E})c_1(\mathcal{F})+2fc_2(\mathcal{E})\big)$.
(iv) If $e\neq 0$ and $f=0$, then $\chi(\mathcal{E},\mathcal{F})+\chi(\mathcal{F},\mathcal{E})=-\big(e+2c_1(\mathcal{E})c_1(\mathcal{F})+2ec_2(\mathcal{F})\big)$.
(v) If $e=f=0$, then $\chi(\mathcal{E},\mathcal{F})=\chi(\mathcal{F},\mathcal{E})$ and therefore $\chi(\mathcal{E},\mathcal{F})+\chi(\mathcal{F},\mathcal{E})=-2c_1(\mathcal{E})c_1(\mathcal{F})$. In particular, if $\chi(\mathcal{F},\mathcal{E})=0$, then $\chi(\mathcal{E},\mathcal{F})=-2c_1(\mathcal{E})c_1(\mathcal{F})=0$.

Note that for $e=0$ (respectively $f=0$), we have $c_1(\mathcal{E},\mathcal{F})=-fc_1(\mathcal{E})$ (respectively $=ec_1(\mathcal{F})$). In the case that $\chi(\mathcal{F},\mathcal{E})=0$, the formulas of Lemma 3.5 yield particularly nice formulas for the Euler characteristic $\chi(\mathcal{E},\mathcal{F})$.

Lemma 3.6: *If $\mathcal{E},\mathcal{F}$ is an exceptional pair, then also $\mathcal{F},\mathcal{E}\otimes\omega^{-1}$ is and the following equality holds:*

$$\chi(\mathcal{E},\mathcal{F})+\chi(\mathcal{F},\mathcal{E}\otimes\omega^{-1})=efK_X^2$$

Proof. The first assertion follows by Serre duality. For the second assertion, we use 3.5 (i) and $c_1(\mathcal{F},\mathcal{E}\otimes\omega^{-1})=-c_1(\mathcal{E},\mathcal{F})-efK_X$. □

3.7. From 3.1, for any three objects $\mathcal{E},\mathcal{F},\mathcal{G}$ we get:

$$fc_1(\mathcal{E},\mathcal{G})=gc_1(\mathcal{E},\mathcal{F})+ec_1(\mathcal{F},\mathcal{G}).$$

If moreover these objects form an exceptional triple, this equality transfers to the Euler characteristic:

$$f\chi(\mathcal{E},\mathcal{G})=g\chi(\mathcal{E},\mathcal{F})+e\chi(\mathcal{F},\mathcal{G}).$$

In order to obtain an exceptional triple from two exceptional pairs, we have the following compatibility condition.

Proposition 3.8: *Let $\mathcal{E},\mathcal{F},\mathcal{G}$ in $D^b(X)$ with $e,f,g\neq 0$ such that $\mathcal{E},\mathcal{F}$ and $\mathcal{F},\mathcal{G}$ form exceptional pairs. Then $\mathcal{E},\mathcal{F},\mathcal{G}$ forms an exceptional triple (i.e. $\chi(\mathcal{G},\mathcal{E})=0$) if and only if $c_1(\mathcal{E},\mathcal{F})\cdot c_1(\mathcal{F},\mathcal{G})=eg$.*

Proof. By equation 3.4 (i) we have

$$\chi(\mathcal{G},\mathcal{E})=\frac{-K_X}{2}c_1(\mathcal{G},\mathcal{E})+\frac{1}{2eg}(c_1(\mathcal{G},\mathcal{E})^2+e^2+g^2).$$

Using 3.1 and 3.5 we get:

$$\chi(\mathcal{G},\mathcal{E}) =$$
$$\frac{-K_X}{2f}\big(ec_1(\mathcal{G},\mathcal{F}) + gc_1(\mathcal{F},\mathcal{E})\big) + \frac{1}{2ef^2g}\big(e^2c_1(\mathcal{G},\mathcal{F})^2 + g^2c_1(\mathcal{F},\mathcal{E})^2 + 2egc_1(\mathcal{G},\mathcal{F})\cdot c_1(\mathcal{F},\mathcal{E})\big) + \frac{e^2+g^2}{2eg} =$$
$$\frac{e}{f}(\chi(\mathcal{G},\mathcal{F}) - \frac{f^2+g^2}{2fg}) + \frac{g}{f}(\chi(\mathcal{F},\mathcal{E}) - \frac{e^2+f^2}{2ef}) + \frac{1}{f^2}c_1(\mathcal{G},\mathcal{F})\cdot c_1(\mathcal{F},\mathcal{E}) + \frac{e^2+g^2}{2eg} =$$
$$\frac{1}{f^2}(-2e^2g^2 + 2egc_1(\mathcal{G},\mathcal{F})\cdot c_1(\mathcal{F},\mathcal{E})).$$

Hence, we get $\chi(\mathcal{G},\mathcal{E}) = 0$ iff $c_1(\mathcal{G},\mathcal{F})\cdot c_1(\mathcal{F},\mathcal{E}) = c_1(\mathcal{E},\mathcal{F})\cdot c_1(\mathcal{F},\mathcal{G}) = eg$. □

3.9 (Mutations)**.** For an exceptional triple $\mathcal{E},\mathcal{F},\mathcal{G}$, the Chern classes and Euler characteristic transform for right mutation as follows:

$$c_1(\mathcal{F}, R_\mathcal{F}\mathcal{E}) = c_1(\mathcal{E},\mathcal{F}), \quad c_1(R_\mathcal{F}\mathcal{E},\mathcal{G}) = \chi(\mathcal{E},\mathcal{F})c_1(\mathcal{F},\mathcal{G}) - c_1(\mathcal{E},\mathcal{G})$$
$$\chi(\mathcal{F}, R_\mathcal{F}\mathcal{E}) = \chi(\mathcal{E},\mathcal{F}), \quad \chi(R_\mathcal{F}\mathcal{E},\mathcal{G}) = \chi(\mathcal{E},\mathcal{F})\chi(\mathcal{F},\mathcal{G}) - \chi(\mathcal{E},\mathcal{G}).$$

Similarly, for left mutation, we get:

$$c_1(L_\mathcal{F}\mathcal{G}, \mathcal{F}) = c_1(\mathcal{F},\mathcal{G}), \quad c_1(\mathcal{E}, L_\mathcal{F}\mathcal{G}) = \chi(\mathcal{F},\mathcal{G})c_1(\mathcal{E},\mathcal{F}) - c_1(\mathcal{E},\mathcal{G})$$
$$\chi(L_\mathcal{F}\mathcal{G}, \mathcal{F}) = \chi(\mathcal{F},\mathcal{G}), \quad \chi(\mathcal{E}, L_\mathcal{F}\mathcal{G}) = \chi(\mathcal{F},\mathcal{G})\chi(\mathcal{E},\mathcal{F}) - \chi(\mathcal{E},\mathcal{G}).$$

If $e, f \neq 0$, we have by 3.5 (ii) that $\chi(\mathcal{E},\mathcal{F}) = \frac{1}{ef}(a + e^2 + f^2)$ where $a = c_1(\mathcal{E},\mathcal{F})^2$, and the rank formulas of 2.7 specialize as follows:

$$\operatorname{rk} L_\mathcal{E}\mathcal{F} = \frac{a+e^2}{f} \quad \text{and} \quad \operatorname{rk} R_\mathcal{F}\mathcal{E} = \frac{a+f^2}{e}$$

4. Exceptional Sequences containing objects of rank zero

As we have seen in paragraphs 3.3 and 3.5, the Riemann-Roch formula does not allow a uniform treatment of exceptional objects of rank zero as it does for objects of nonzero rank. Indeed, one should consider such objects as associated with exceptional divisors of blow-ups, as indicated by a classical construction due to Orlov.

Example 4.1: Let $b : \tilde{X} \to X$ be a blow-up with exceptional divisor E and let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be a full exceptional sequence on X. Then by [Orl93], $\mathcal{O}_E(E), \mathbf{L}b^*\mathcal{E}_1, \ldots, \mathbf{L}b^*\mathcal{E}_n$ is a full exceptional sequence on $\tilde{X}$. Clearly, $c_2(\mathcal{O}_E(E)) = 0$ and $c_1(\mathcal{E})c_1(\mathbf{L}b^*\mathcal{E}_i) = 0$ for all i. It follows that $\chi(\mathcal{O}_E(E), \mathbf{L}b^*\mathcal{E}_i) = -e_i$ for every i.

4.2. This example motivates that, ideally, the rank zero members of an exceptional sequence should be orthogonal (at least in some appropriate sense) to the rest of the sequence. However, as we can see by formulas 3.5, this is not an immediate consequence of the Riemann-Roch formula.

Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be a numerically exceptional sequence and denote $\mathcal{Z}_1, \ldots, \mathcal{Z}_t$ the maximal sub-sequence consisting of objects of rank zero. The following lemma shows that the $\mathcal{Z}_i$ can never represent a semi-orthogonal basis of $K_0(X)$.

Lemma 4.3: *Under above assumptions we have $t \leq n - 3$.*

Proof. By Lemmas 3.3 (i) and 3.5 (v), the Chern classes $c_1(\mathcal{Z}_i)$ form an orthogonal system of vectors of length -1 in $\mathrm{CH}^1(X)$. Then $t \leq n - 3 = \operatorname{rk}\mathrm{CH}^1(X) - 1$ by the Hodge index theorem. □

Given a divisor D, we can consider the twisted sequence $\mathcal{E}_1(D), \ldots, \mathcal{E}_n(D)$. For the sub-sequence of the $\mathcal{Z}_i$, we observe the following fact:

Lemma 4.4: *With above notation, there exists a divisor D such that $c_2(\mathcal{Z}_i(D)) = 0$ for every $1 \leq i \leq t$.*

Proof. By the multiplicative property of the Chern character, for every i we have $\mathrm{ch}(\mathcal{Z}_i(D)) = \mathrm{ch}(\mathcal{Z}_i) \cdot \mathrm{ch}(\mathcal{O}(D))$. From this we compute that $c_2(\mathcal{Z}_i(D)) = c_2(\mathcal{Z}_i) - c_1(\mathcal{Z}_i)D$. As we have observed in the proof of Lemma 4.3, the $c_1(\mathcal{Z}_i)$ form an orthogonal set of vectors of length -1 with respect to the intersection form. Hence, we can always find a divisor D with $c_1(\mathcal{Z}_i)D = c_2(\mathcal{Z}_i)$ for all i. □

Remark 4.5: Note that $c_1(\mathcal{Z}_i(D)) = c_1(\mathcal{Z}_i)$ for any D.

Now assume that we have an exceptional sequence of the form $\mathcal{Z}, \mathcal{E}_2, \ldots, \mathcal{E}_n$ with $z = 0$. By 3.5 (iii), (v), we know that either $\chi(\mathcal{Z}, \mathcal{E}_i) = 0$ if $e_i = 0$, or $\chi(\mathcal{Z}, \mathcal{E}_i) = -e_i(1 + c_1(\mathcal{Z})s(\mathcal{E}_i) + c_2(\mathcal{Z}))$ otherwise, where we write $s(\mathcal{E}_i)$ for $c_1(\mathcal{E}_i)/e_i \in \mathrm{CH}^1(X)_{\mathbb{Q}}$.

Lemma 4.6: *With above notation, we have* $c_1(\mathcal{Z})s(\mathcal{E}_i) = c_1(\mathcal{Z})s(\mathcal{E}_j)$ *whenever* $e_i, e_j \neq 0$.

Proof. We have $c_1(\mathcal{Z}, \mathcal{E}_i) = -e_i c_1(\mathcal{Z})$ for every i and hence by Lemma 3.5 (i), we get $f\chi(\mathcal{Z}, \mathcal{E}) = e\chi(\mathcal{Z}, \mathcal{F})$. Then the assertion follows from 3.5 (iii). □

So, independent of i, as long as $e_i \neq 0$, we can define:

Definition 4.7: With above notation, for some i with $e_i \neq 0$, we denote $\delta_{\mathcal{Z}} := c_1(\mathcal{Z})s(\mathcal{E}_i) + c_2(\mathcal{Z})$.

In a sense, the presence of $\delta_{\mathcal{Z}}$ represents a "defect" of $\mathcal{Z}$ with respect to the rest of the exceptional sequence which we would like to remove. By Lemma 4.4 we can twist the sequence with an appropriate divisor such that arrange that $c_2(\mathcal{Z}) = 0$ (indeed we can take an appropriate power of $\mathcal{O}(E)$, where E is a divisor representing $c_1(\mathcal{Z})$). However, it is easy to see that $\delta_{\mathcal{Z}}$ is invariant with respect to twist and this way we cannot get rid of it. We will spend the rest of this section to show that indeed $\delta_{\mathcal{Z}} = 0$.

4.8. For this, we will consider the left-orthogonal complements $\mathbf{E}$ and $\mathfrak{E}$ of $\mathcal{Z}$ in $K_0(X)$ and $D^b(X)$, respectively. The former is simply defined as the saturated submodule of corank one of $K_0(X)$ which is generated by the classes of the $\mathcal{E}_i$. Clearly, $\mathbf{E}$ coincides precisely with the classes of all objects $\mathcal{F}$ in $D^b(X)$ such that $\chi(\mathcal{F}, \mathcal{Z}) = 0$. For the latter, we must be somewhat careful, as we are only dealing with numerical exceptionality. By a result of Thomason [Tho97, Theorem 2.1], the objects whose class in $K_0(X)$ are contained in $\mathbf{E}$ do form a strictly full dense triangulated subcategory of $D^b(X)$. We define $\mathfrak{E}$ to be this subcategory. Note that both $\mathfrak{E}$ and $\langle \mathcal{E}_2, \ldots, \mathcal{E}_n \rangle$ generate $\mathbf{E}$, but $\mathfrak{E}$ is a strictly bigger category in general.

Lemma 4.9: *The linear form* $\chi(\mathcal{Z}, -)$ *restricted to* $\mathbf{E}$ *coincides with* $-(1 + \delta_{\mathcal{Z}})$ *times the rank function on* $\mathbf{E}$. *Moreover,* $\chi(\mathcal{Z}, \mathcal{F}) = -(1 + 2\delta_{\mathcal{Z}})\,\mathrm{rk}(\mathcal{F})$ *iff* $\mathcal{F}$ *is an object of* $\mathfrak{E}$.

Proof. For any $\mathcal{E}_i$, we have $\chi(\mathcal{Z}, \mathcal{E}_i) = -e_i(1 + 2\delta_{\mathcal{Z}})$ by 3.5 (iii), hence the first assertion follows. For the second assertion, we can write the class of $\mathcal{F}$ in $K_0(X)$ as $k[\mathcal{Z}] + [\mathcal{F}']$ for some integer k and some object $\mathcal{F}'$ of $\mathfrak{E}$. As $k[\mathcal{Z}] = [\mathcal{Z}^{\oplus k}]$, it follows that $\mathrm{rk}\,\mathcal{F}' = \mathcal{F}$, but $\chi(\mathcal{Z}, \mathcal{F}) = k\chi(\mathcal{Z}, \mathcal{Z}) + \chi(\mathcal{Z}, \mathcal{F}') = k - (1 + 2\delta_{\mathcal{Z}})\,\mathrm{rk}\,\mathcal{F}$. So, the second assertion follows. □

Now we determine, which line bundles $\mathcal{O}(D)$ are contained in $\mathfrak{E}$. We first observe that by the Riemann-Roch formula, we have $\chi(\mathcal{Z}, \mathcal{O}(D)) = -(1 + 2\delta_{\mathcal{Z}}) - c_1(\mathcal{Z})D$ for any divisor D. Hence we have the following consequence of above lemma.

Corollary 4.10: *A line bundle* $\mathcal{O}(D)$ *is contained in* $\mathfrak{E}$ *iff* $c_1(\mathcal{Z})D = 0$.

We can now conclude:

Proposition 4.11: *With* $\mathcal{Z}, \mathcal{E}_2, \ldots, \mathcal{E}_n$ *as above we have* $-K_X c_1(\mathcal{Z}) = 1$ *and* $\delta_{\mathcal{Z}} = 0$. *In particular, if* $c_2(\mathcal{Z}) = 0$, *we get* $c_1(\mathcal{Z})c_1(\mathcal{E}_i) = 0$ *for all* i.

Proof. We can assume without loss of generality that $c_2(\mathcal{Z}) = 0$. Let D be a divisor with $c_1(\mathcal{Z})D = 0$. For some $\mathcal{E}_i$ with $e_i \neq 0$ we have $\chi(\mathcal{Z}, \mathcal{E}_i) = -c_1(\mathcal{Z}, \mathcal{E}_i)K_X = e_i c_1(\mathcal{Z})K_X = -e_i(1 + 2\delta_{\mathcal{Z}})$ and hence

$-c_1(\mathcal{Z})K_X = (1 + 2\delta_{\mathcal{Z}})$. Then by the Riemann-Roch formula 3.2 we get:

$$\begin{aligned}\chi(\mathcal{O}(D), \mathcal{Z}) &= -\frac{1}{2}K_X c_1(\mathcal{Z}) - \frac{1}{2} - c_1(\mathcal{Z})D - c_2(\mathcal{Z}) \\ &= \frac{1}{2}(1 + 2\delta_{\mathcal{Z}}) - \frac{1}{2} - c_2(\mathcal{Z}) \\ &= \delta_{\mathcal{Z}}.\end{aligned}$$

But because $\mathcal{O}(D)$ is an object of $\mathfrak{E}$ by above corollary, we necessarily get $\chi(\mathcal{O}(D), \mathcal{Z}) = 0$, hence $\delta_{\mathcal{Z}} = 0$. The second assertion is clear for $e_i = 0$. Otherwise, we observe $c_1(\mathcal{Z})c_1(\mathcal{E}_i) = e_i(\delta_{\mathcal{Z}} - c_2(\mathcal{Z})) = 0$. □

The following Corollary is an immediate consequence of 4.9 and 4.11.

Corollary 4.12: *If $c_2(\mathcal{Z}) = 0$ then $\chi(\mathcal{Z}[1], -)$ coincides with the rank function on* $\mathbf{E}$.

5. Toric systems

In [HP11], exceptional sequences of invertible sheaves $\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)$ have been considered. For such sequences, so-called *toric systems* have been introduced, which represent a normal form for such sequences. More precisely, a toric system is simply given by forming the differences $A_i := D_{i+1} - D_i$ for all $1 \leq i < n$ and $A_n := D_1 - D_n - K_X$. Such a toric system satisfies the following equations:

(i) $A_i \cdot A_{i+1} = 1$ for all i,
(ii) $A_i \cdot A_j = 0$ otherwise,
(iii) $\sum_{i=1}^n A_i = -K_X$.

In [HP11] the peculiar fact was observed that a toric system is equivalent to the data of a smooth complete toric surface, which this way becomes a combinatorial invariant of an exceptional sequence of invertible sheaves.

In this section we will extend the notion of toric systems to the case of general exceptional sequences. This generalization will be straightforward for the most part, with two notable differences:

(1) It is necessary to pass to rational Chern classes, i.e. to an exceptional sequence we will associate elements A_i in a similar fashion, but they are now constructed as elements of $\mathrm{CH}^1(X)_{\mathbb{Q}}$.
(2) Objects of rank zero cannot be treated uniformly together with objects of nonzero rank.

5.1. We start with an exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_t$, where $e_i \neq 0$ for all i, which we assume extended to a cyclic exceptional sequence. This in particular implies that $c_1(\mathcal{E}_i, \mathcal{E}_{i+1}) = c_1(\mathcal{E}_{i+t}, \mathcal{E}_{i+1+t})$ for all i. Then the following are straightforward consequences of 3.8, Serre duality, and the Riemann-Roch formula:

(i) $c_1(\mathcal{E}_{i-1}, \mathcal{E}_i) \cdot c_1(\mathcal{E}_i, \mathcal{E}_{i+1}) = r_{i-1}r_{i+1}$ for every $i \in \mathbb{Z}$.
(ii) $c_1(\mathcal{E}_{i-1}, \mathcal{E}_i) \cdot c_1(\mathcal{E}_{j-1}, \mathcal{E}_j) = 0$ for $1 < |i - j| < t - 1$.

5.2. For any object $\mathcal{E}$ of $D^b(X)$ of nonzero rank, we set $s(\mathcal{E}) := c_1(\mathcal{E})/e \in \mathrm{CH}^1(X)_{\mathbb{Q}}$. For any two such objects $\mathcal{E}, \mathcal{F}$ we set $s(\mathcal{E}, \mathcal{F}) := s(\mathcal{F}) - s(\mathcal{E}) = s(R\mathcal{H}om(\mathcal{E}, \mathcal{F}))$. The intersection product extends in a natural way to a $\mathbb{Q}$-valued bilinear form on $\mathrm{CH}^1(X)_{\mathbb{Q}}$, so that we can reformulate the equalities of 5.1 as follows:

(i) $s(\mathcal{E}_{i-1}, \mathcal{E}_i) \cdot s(\mathcal{E}_i, \mathcal{E}_{i+1}) = 1/r_i^2$,
(ii) $s(\mathcal{E}_{i-1}, \mathcal{E}_i) \cdot s(\mathcal{E}_{j-1}, \mathcal{E}_j) = 0$ for $1 < |i - j| < t - 1$.

Moreover, by $s(\mathcal{E}_t, \mathcal{E}_{t+1}) = s(\mathcal{E}_t, \mathcal{E}_1 \otimes \omega^{-1}) = s(\mathcal{E}_t, \mathcal{E}_1) - K_X$, we have:

(iii) $\sum_{i=1}^s s(\mathcal{E}_i, \mathcal{E}_{i+1}) = -K_X$.

5.3. Assume that we have an exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ and assume that one of the $\mathcal{E}_i$ has rank zero. By choosing the appropriate winding in the cyclic sequence, we can always assume without loss of generality that we have $\mathcal{E}_1 \cong \mathcal{Z}$ with $z = 0$. Then by Lemma 4.6, we have $c_1(\mathcal{Z})s(\mathcal{E}_i, \mathcal{E}_j) = 0$ for every $1 < i < j \leq n$. However, if, say, $1 - n < j < 1 < i \leq n$ and $i - j < n$, then $\mathcal{E}_j = \mathcal{E}_{j+n} \otimes \omega$ and therefore $s(\mathcal{E}_j, \mathcal{E}_i) = s(\mathcal{E}_{j+n}, \mathcal{E}_i) - K_X$, hence $c_1(\mathcal{Z})s(\mathcal{E}_j, \mathcal{E}_i) = -c_1(\mathcal{Z})K_X = 1$ by Proposition 4.11.

5.4. Now consider an arbitrary exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$. Then we can partition $\{1, \ldots, n\} = I \amalg J$, where $I = \{i_1 < \cdots < i_t\}$, $J = \{j_1 < \cdots < j_{n-t}\}$, and such that $e_i = 0$ iff $i \in I$. Then we set:

$$\begin{aligned} E_k &:= c_1(\mathcal{E}_{i_k}) \text{ for } 1 \leq k \leq t, \\ A_k &:= s(\mathcal{E}_{j_k}, \mathcal{E}_{j_{k+1}}) \text{ for } 1 \leq k < n-t, \\ A_{n-t} &:= s(\mathcal{E}_{j_{n-t}}, \mathcal{E}_{j_1} \otimes \omega^{-1}) \end{aligned}$$

Clearly, the A_i satisfy the conditions listed in 5.2 and it follows from from 3.3 (i) and 3.5 (iv) that $E_j^2 = -1$ and $-K_X E_j = 1$ for all j and $E_j \cdot E_k = 0$ for all $j \neq k$. Moreover, by 5.3, there exists for every $i \in J$ precisely one $k \in J$ such that $E_i . A_k \neq 0$.

For easier notation, we give a formal definition for above data.

Definition 5.5: An *abstract toric system* on X is given by the following data:

(1) A collection of integral divisor classes $E_1, \ldots, E_t$ with $E_i^2 = -1$ and $-K_X E_i = 1$ for all i and $E_i \cdot E_j = 0$ for all $i \neq j$.
(2) A sequence of ranks $r_1, \ldots, r_{n-t} \in \mathbb{Z} \setminus \{0\}$.
(3) A sequence of $\mathbb{Q}$-divisor classes $A_1, \ldots, A_{n-t}$ such that
   (i) $r_i r_{i+1} A_i$ is integral for every i,
   (ii) $A_i \cdot A_{i+1} = \frac{1}{r_{i+1}^2}$ for every i,
   (iii) $A_i \cdot A_j = 0$ otherwise,
   (iv) $\sum_{i=1}^{n-t} A_i = -K_X$.
(4) A function $\phi : \{1, \ldots, t\} \to \{1, \ldots, n-t\}$ such that $E_i \cdot A_j \neq 0$ if and only if $j = \phi(i)$ (and thus $E_i \cdot A_{\phi(i)} = 1$ by (1) and (3iv)).

Note that the indices of the r_i and A_i are to be read cyclically; in particular, we have $A_{n-t} \cdot A_1 = 1/r_1^2$.

A *toric system* is an abstract toric system which can be constructed from an exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ by the procedure described in 5.4.

Remark 5.6: In the following we will exclusively with actual (i.e. non-abstract) toric systems. In [HP11, §2], some effort has been devoted to the inverse problem, i.e. the question whether for a given toric system we can check implications such as vanishing of the $\chi(\mathcal{O}(-A_i))$. However, contrary to the case of line bundles, the association of toric systems to exceptional objects is not as straightforward at this stage. Instead, our strategy will in the subsequent sections will be to reduce such question to the case of sequences of rank one objects (see Remark 10.9 below).

Example 5.7: Consider the strongly exceptional sequence $\mathcal{T}, \mathcal{O}(2), \mathcal{O}(4)$ on $\mathbb{P}^2$, where $\mathcal{T}$ denotes the tangent sheaf. If we denote H the class of a line in $\mathrm{CH}^1(\mathbb{P}^2)$, then the toric system associated to this sequence is given by $A_1, A_2, A_3 = \frac{1}{2}H, 2H, \frac{1}{2}H$. Now we take any point $x \in \mathbb{P}^2$ and denote $b : \mathbb{F}_1 \cong \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ the blow-up at x with exceptional curve E. For ease of notation we identify E with its class in $\mathrm{CH}^1(\mathbb{F}_1)$. We also identify H with its pull-back in $\mathrm{CH}^1(\mathbb{F}_1)$. Completing to a full exceptional sequence by adding $\mathcal{O}_E(E)$ we get $\mathcal{O}_E(E), b^*\mathcal{T}, b^*\mathcal{O}(2), b^*\mathcal{O}(4)$. Then the toric system associated to this sequence is given by the -1-divisor $E_1 = E$, the rational classes $A_1, A_2, A_3 = \frac{1}{2}H, 2H, \frac{1}{2}H - E$, and $\phi : \{1\} \to \{1, 2, 3\}$ with $\phi(1) = 3$. Now, by right mutating the pair $\mathcal{O}_E(E), b^*\mathcal{T}$, we obtain $b^*\mathcal{T}, \mathcal{R}, b^*\mathcal{O}(2), b^*\mathcal{O}(4)$. We have $\chi(\mathcal{O}_E(E), b^*\mathcal{T}) = -2$, which implies $\operatorname{rk} \mathcal{R} = -4$. Moreover, we get $s(b^*\mathcal{T}, \mathcal{R}) = \frac{1}{4}E$ and consequently the new toric system consists of four rational divisor classes which are given by

$$\frac{1}{4}E, \frac{1}{2}H - \frac{1}{4}E, 2H, \frac{1}{2}H - E.$$

6. Toric systems and their Gale dual

Let $\mathbf{A} = E_1, \ldots, E_t, A_1, \ldots, A_{n-t}, \phi$ be an abstract toric system.

Definition 6.1: The *contraction* $\tilde{\mathbf{A}}$ of $\mathbf{A}$ is given by $\tilde{A}_1, \ldots, \tilde{A}_{n-t}$, where

$$\tilde{A}_i = A_i + \sum_{j | \phi(j) = i} E_j$$

Both $\mathbf{A}$ and $\tilde{\mathbf{A}}$ give rise to subgroups of $\mathrm{CH}^1(X)_{\mathbb{Q}}$ given by $A := \langle E_1, \ldots, E_t, A_1, \ldots, A_{n-t}\rangle_{\mathbb{Z}}$ and $\tilde{A} := \langle \tilde{A}_1, \ldots, \tilde{A}_{n-t}\rangle_{\mathbb{Z}}$. Clearly, both A and $\tilde{A}$ are finitely generated and torsion free $\mathbb{Z}$-modules of rank at most $n-2$.

Proposition 6.2: $\operatorname{rk} A = n-2$ *and* $\operatorname{rk} \tilde{A} = n-t-2$.

Proof. As the E_i form an orthogonal system of divisors which by construction contain $\tilde{A}$ in their orthogonal complement, it suffices to show that $\operatorname{rk}\tilde{A} = n-t-2$. Starting with the observation that the $\tilde{A}_i$, and in particular $\tilde{A}_1$, are all nonzero we will show by induction that $\tilde{A}_1, \ldots, \tilde{A}_i$ are $\mathbb{Q}$-linearly independent for $1 \leq i \leq n-t-2$. So, for $1 < i \leq n-t-2$ we assume that $\tilde{A}_1, \ldots, \tilde{A}_{i-1}$ are linearly independent. Then, for any $\mathbb{Q}$-linear combination $B := \sum_{j=1}^{i-1} \alpha_j A_j$, we have $B \cdot A_{i+1} = 0$. However, we have $A_i \cdot A_{i+1} = \frac{1}{r_i^2} \neq 0$, hence A_i cannot be contained in the linear span of $A_1, \ldots, A_{i-1}$, hence $A_1, \ldots, A_i$ are linearly independent for all $1 \leq i \leq n-t-2$ and the assertion follows. □

6.3. Consider the structural linear maps $c : \mathbb{Z}^n \twoheadrightarrow A$ and $\tilde{c} : \mathbb{Z}^{n-t} \twoheadrightarrow \tilde{A}$. That is, if we denote $b_1, \ldots, b_n$ of $\mathbb{Z}^n$ the standard basis of $\mathbb{Z}^n$, then we have $c(b_i) = E_i$ for $1 \leq i \leq t$ and $c(b_i) = A_{i-t}$ for $t < i \leq n$. For $\mathbb{Z}^{n-t}$ with standard basis $b'_1, \ldots, b'_{n-t}$, we have $\tilde{c}(b'_i) = \tilde{A}_i$ for every i. We define a linear map $\Phi : \mathbb{Z}^{n-t} \to \mathbb{Z}^n$ by setting $\Phi(b'_i) = b_{i+t} + \sum_{j|\phi(j)=i} b_j$ for every $1 \leq i \leq n-t$. Then Φ induces a linear map $\bar{\Phi} : \tilde{A} \to A$ with $\bar{\Phi}(\tilde{A}_i) = A_i + \sum_{j|\phi(j)=i} E_i$ for $1 \leq i \leq n-t$. Clearly, both Φ and $\bar{\Phi}$ are injective and their image is saturated in $\mathbb{Z}^n$ and A, respectively. We obtain the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\tilde{L}} & \mathbb{Z}^{n-t} & \xrightarrow{\tilde{c}} & \tilde{A} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \bar{\Phi}} & & \\
0 & \longrightarrow & M & \xrightarrow{L} & \mathbb{Z}^{n} & \xrightarrow{c} & A & \longrightarrow & 0,
\end{array}
$$

where $M \cong \mathbb{Z}^2$ by Proposition 6.2. We can represent L and $\tilde{L}$ as row matrices with rows $l_1, \ldots, l_n \in M$ and $\tilde{l}_1, \ldots, \tilde{l}_{n-t} \in M$, respectively. The following proposition justifies the natural identification of the kernels of L and $\tilde{L}$.

Proposition 6.4: *Denote $N := M^*$ and consider the dual maps*

$$
\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{L^T} & N \\
\downarrow{\scriptstyle \Phi^T} & & \downarrow \\
\mathbb{Z}^{n-t} & \xrightarrow{\tilde{L}^T} & N
\end{array}
$$

where we consider the l_i and $\tilde{l}_i$ as column vectors in N. Then the map $N \to N$ in this diagram is an isomorphism and maps l_i to $\tilde{l}_{i-t}$ for $t < i \leq n$ and l_i to $\tilde{l}_{\phi(i)}$ for $1 \leq i \leq t$.

In other words, both L^T and $\tilde{L}^T$ give rise to the same set of column vectors, but some column vectors of L^T can be repeated. That is, every column vector l_{t+i} appears with multiplicity (at least) $1 + |\{j \mid \phi(j) = i\}|$.

Proof. It is clear that the map $N \to N$ is an isomorphism. Now, we choose some E_i and observe that E_i and $A_{\phi(i)}$ are the only elements among $E_1, \ldots, E_t, A_1, \ldots, A_{n-t}$ which are not contained in the orthogonal complement of E_i in A. That is, if we complete E_i to a basis of A by choosing $n-3$ appropriate elements in the orthogonal complement of E_i, we obtain a matrix representation $c = (c_{pq})$, where for the i-th row $(c_{1i}, \ldots, c_{ni})$ we have $c_{ii} = 1$, $c_{i,t+\phi(i)} = -1$ and $c_{ik} = 0$ otherwise. This implies that the i-th and $\phi(i)$-th rows of L satisfy the relation $l_i = l_{\phi(i)}$. Furthermore, if we remove the i-th row from L, it is straightforward to check that we can represent the cokernel by a matrix c', whose columns coincide with that of c, except that the i-th column is left out and the $(t + \phi(i) - 1)$-th column is the sum of the i-th and $(t + \phi(i))$-th column. Applying this argument iteratively to every $1 \leq i \leq t$ proves the proposition. □

Example 6.5: In Example 5.7 a toric system was given with (-1)-divisor E and $A_1, A_2, A_3 = \frac{1}{2}H, 2H, \frac{1}{2}H - E$ such that $E \cdot (\frac{1}{2}H - E) = 1$. For the Gale dual, we obtain vectors l_1, l_2, l_3, l_4 which for a suitable choice of basis can be represented as $l_2 = (1,0)$, $l_3 = (0,1)$, $l_1 = l_4 = (-1,-4)$, i.e. up to latter multiplicity, the l_i generate fan of $\mathbb{P}(1,1,4)$. The mutated toric system $A'_1, A'_2, A'_3, A'_4 = \frac{1}{4}E, \frac{1}{2}H - \frac{1}{4}E, 2H, \frac{1}{2}H - E$ has Gale duals l'_1, l'_2, l'_3, l'_4 which can be represented such that $l'_2 = l_2, l'_3 = l_3, l'_4 = l_4$, and $l'_1 = (3,-4)$. These l'_i can be interpreted to generate the fan of a weighted blow-up of $\mathbb{P}(1,1,4)$ which has two singular points of order 4 and 16, respectively.

So, in both cases the l_i are primitive lattice vectors and generate the fan of a complete toric surface. It is easy to see that the singularities are T-singularities. We will show that this and the observation that the multiplicity $l_1 = l_4$ translates into a weighted blow-ups via mutation are general properties of toric systems.

7. Moving around objects of rank zero

Let $\mathbf{E} = \mathcal{E}_1, \dots, \mathcal{E}_n$ be an exceptional sequence and denote $E_1, \dots, E_t, A_1, \dots, A_{n-t}, \phi$ its associated toric system. As we have seen in the previous section, we can translate this data to a set of lattice vectors $l_1, \dots, l_n$ in $N \cong \mathbb{Z}^2$. Our aim in this and the following sections is to show that these lattice vectors are cyclically ordered and generate the fan associated to a complete toric surface. We start in this section with investigating the "local" constellations of the l_i and their behaviour under mutation by rank zero objects.

7.1. We start with the behaviour of multiplicities of rays as observed in Proposition 6.4 under mutation. For this, consider an exceptional triple $\mathcal{E}, \mathcal{Z}, \mathcal{F}$ with $e, f \neq 0$ and $z = 0$. By moving $\mathcal{Z}$ to the left or right via mutation, we obtain exceptional triples $\mathcal{Z}, R_{\mathcal{Z}}\mathcal{E}, \mathcal{F}$ and $\mathcal{E}, L_{\mathcal{Z}}\mathcal{F}, \mathcal{Z}$, respectively. Using 3.9 and Corollary 4.12, we see that $s(R_{\mathcal{Z}}\mathcal{E}) = s(\mathcal{E}) - c_1(\mathcal{Z})$ and $s(L_{\mathcal{Z}}\mathcal{F}) = s(\mathcal{F}) + c_1(\mathcal{Z})$. Thus we get $s(R_{\mathcal{Z}}\mathcal{E}, \mathcal{F}) = s(\mathcal{E}, L_{\mathcal{Z}}\mathcal{F}) = s(\mathcal{E}, \mathcal{F}) + c_1(\mathcal{Z})$. If we can extend our exceptional triple to the left, say, i.e. we have an exceptional sequence $\mathcal{D}, \mathcal{E}, \mathcal{Z}, \mathcal{F}$ with $d \neq 0$, then we get furthermore that $s(\mathcal{D}, L_{\mathcal{Z}}\mathcal{F}) = s(\mathcal{D}, \mathcal{E}) - c_1(\mathcal{Z})$. The following proposition shows that this simple modification of Chern classes translates in the Gale dual picture to a "hopping" of multiplicities.

Proposition 7.2: *Assume that $\mathcal{E}_k$ has rank zero for some $1 \leq k \leq n$ and let $1 \leq i \leq t$ such that $E_i = c_1(\mathcal{E}_k)$. Consider the mutations $L_k\mathbf{E}$ and $R_{k-1}\mathbf{E}$. Then the corresponding toric systems are given by $E'_1, \dots, E'_t, A'_1, \dots, A'_{n-t}, \phi'$ (for $L_k\mathbf{E}$) and $E''_1, \dots, E''_t, A''_1, \dots, A''_{n-t}, \phi''$ (for $R_{k-1}\mathbf{E}$), where*

- *(i) If $e_{k+1} = 0$ (resp. $e_{k-1} = 0$), then $E'_i = E_{i+1}$, $E'_{i+1} = E_i$ and $E'_j = E_j$ otherwise, $A'_j = A_j$ for all j, and $\phi' = \phi$ (resp. $E''_{i-1} = E_i$, $E''_i = E_{i-1}$ and $E''_j = E_j$ otherwise, $A''_j = A_j$ for all j and $\phi'' = \phi$).*
- *(ii) If $e_{k+1} \neq 0$ then $E'_j = E_j$ for all j, $A'_{\phi(i)} = A_{\phi(i)} + E_i$, $A'_{\phi(i)+1} = A_{\phi(i)+1} - E_i$, and $A'_j = A_j$ otherwise. Moreover, $\phi'(j) = \phi(j)$ for $j \neq i$ and $\phi'(i) = \phi(i) + 1$.*
- *(iii) If $e_{k-1} \neq 0$ then $E'_j = E_j$ for all j, $A'_{\phi(i)-1} = A_{\phi(i)-1} - E_i$, $A'_{\phi(i)} = A_{\phi(i)} + E_i$, and $A'_j = A_j$. Moreover, $\phi'(j) = \phi(j)$ for $j \neq i$ and $\phi'(i) = \phi(i) - 1$.*

Proof. (i) Any Mutation of a pair $\mathcal{E}, \mathcal{F}$ with $e = f = 0$ yields a pair $\mathcal{F}', \mathcal{E}'$, where $[\mathcal{E}'] = [\mathcal{E}]$ and $[\mathcal{F}'] = [\mathcal{F}]$ in $K_0(X)$ so that on the level of toric systems nothing happens except that E_i gets exchanged with its (left or right) neighbour. The assertion correspondingly just reflects the reshuffling of data.

(ii) We have already seen in 7.1 that the A_j behave in the described way (in particular, the A_j for $j \neq \phi(i), \phi(i+1)$ remain the same). Also, as the sequence of E_j's remains constant, the function $\phi(i)$ changes as described.

(iii) follows completely analogously to (ii). □

7.3. In the Gale dual picture, we obtain (for the left mutation, say) $l'_1, \dots, l'_n$, where $l'_j = l_j$ for every $j \neq i$ and l'_i either coincides with $l_i = l_{\phi(i)}$ (if $e_{k-1} = 0$) or with $l_{\phi(i)'}$ (if $e_{k-1} \neq 0$). This can be seen easily by our analysis of Proposition 6.4 and the observation that the contractions $\tilde{A}_1, \dots, \tilde{A}_{n-t}$ and $\tilde{A}'_1, \dots, \tilde{A}'_{n-t}$ coincide. In other words, moving a rank zero object in an exceptional sequence to the left by mutation has the effect that the corresponding multiplicity either remains the same or hops one position to the left. Similarly, moving a rank zero object to the right yields a hop of multiplicity to the right. This is very convenient, as it tells that we can without loss of generality move around the rank zero objects without any effect on a combinatorial level (e.g. we can without loss of generality assume that ϕ is constant, that is, the rank zero objects form an uninterrupted sub-sequence anywhere in the sequence). In particular, we can pretend in many situations that the l_i have no multiplicities at all.

8. Local constellations

In order to simplify notation and to reduce the number of trivial caveats, in this section we will make the assumption that our exceptional sequence $\mathbf{E} = \mathcal{E}_1, \ldots, \mathcal{E}_n$ contains no objects of rank zero (and hence we can identify the l_i with the $\tilde{l}_i$). In the spirit of 7.3, "up to multiplicities" the results extend trivially to the general case.

8.1. We denote $a_i := e_i^2 e_{i+1}^2 A_i^2$. The intersection product in $\mathrm{CH}^1(X)$ induces a bilinear form on A. For any A_i, we denote $A^\perp$ the orthogonal complement of A_i in A with respect to this form. Clearly, $A^\perp$ contains all A_j for $j \notin \{i-1, i, i+1\}$ and therefore the quotient $A/A^\perp$ is isomorphic to $\mathbb{Z}$ and $A/\langle A_j \in A^\perp \rangle_\mathbb{Z} \cong \mathbb{Z} \oplus F$ where $F \cong A^\perp / \langle A_j \in A^\perp \rangle_\mathbb{Z}$ is finite. By restricting the structure map $\mathbb{Z}^n \to A$ to the subspace generated by the $(i-1)$-st, i-th and $(i+1)$-st standard basis vectors, we obtain a surjection $\mathbb{Z}^3 \twoheadrightarrow \mathbb{Z} \oplus F$. Dualizing then yields an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{c^T_{i-1,i,i+i}} \mathbb{Z}^3 \xrightarrow{L^T_{i-1,i,i+1}} N \longrightarrow F \longrightarrow 0,$$

where it is natural to identify $L^T_{i-1,i,i+1}$ with the submatrix of L^T consisting of the $(i-1)$-st, i-th, and $(i+1)$-st row. With this identification, it follows that the vectors l_{i-1}, l_i, l_{i+1} generate a sublattice of order $|F|$ of N. Moreover, we have $c_{i-1,i,i+1} = (r, s, t)$, where $r = \det(l_i, l_{i+1}), s = \det(l_{i+1}, l_{i-1})$, $t = \det(l_{i-1}, l_i)$. Note that $\gcd\{r, s, t\} = |F|$. The values r, s, t are determined in the following lemma.

Proposition 8.2: *We have* $(\det(l_i, l_{i+1}), \det(l_{i+1}, l_{i-1}), \det(l_{i-1}, l_i)) = (e_i^2, a_i, e_{i+1}^2)$.

Proof. The projection $A \to \mathbb{Z} \oplus F$ induces a $\mathbb{Q}$ valued bilinear form on $\mathbb{Z} \oplus F$. This form vanishes on the torsion part and, projecting further to $\mathbb{Z}$, we can represent this form by a rational number q. We denote $\bar{A}_{i-1}, \bar{A}_i, \bar{A}_{i+1}$ the images of A_{i-1}, A_i, A_{i+1} in $\mathbb{Z}$. We distinguish the cases $a_i \neq 0$ and $a_i = 0$.

In the case $a_i \neq 0$, we denote $g := \gcd\{r, s, t\}$. Then we have $\bar{A}_{i-1} q \bar{A}_i = 1/e_i^2$, $\bar{A}_i q \bar{A}_i = a_i / e_i^2, e_{i+1}^2$, and $\bar{A}_i q \bar{A}_{i+1} = 1/e_{i+1}^2$. Without loss of generality, we can write $q = x/a e_1^2 e_2^2$ for some $x \in \mathbb{Q}$ and evaluate $\bar{A}_{i-1} = e_i^2 y$, $\bar{A}_i = a_i y$, and $\bar{A}_{i+1} = e_{i+1}^2 y$ with $x = y^2$. But, as the $\bar{A}$'s are integral and generate $\mathbb{Z}$, we get $x = 1/g$. For the dual map this yields $c^T_{i-1,i,i+1} = (e_i^2, a_i, e_{i+1}^2)$.

The case $a_i = 0$ follows analogously, where we set $g = \gcd\{r, t\}$ and $q = 1/e_i^2 e_{i+1}^2$. ☐

Corollary 8.3: *For every i the pair l_i, l_{i+1} is positively oriented (i.e.* $\det(l_i, l_{i+1}) > 0$*) and generates a strictly convex polyhedral cone in $N_\mathbb{Q}$. Every triple l_{i-1}, l_i, l_{i+1} generates a fan which contains two 2-dimensional cones which intersect in the common facet $\mathbb{Q}_{\geq 0} l_i$.*

Proof. The first assertion follows from $\det(l_i, l_{i+1}) = e_i^2 > 0$ for every i. This also implies that l_{i-1} and l_{i+1} lie in opposite half spaces with respect to the line $\mathbb{Q} l_i$, hence the cones generated by l_{i-1}, l_i and l_i, l_{i+1}, respectively, intersect at the common facet $\mathbb{Q}_{\geq 0} l_i$ and therefore form a fan. ☐

As the following lemma shows, the case $a_i = 0$ corresponds to a very special configuration.

Lemma 8.4: *If $a_i = 0$ then $e_i^2 = e_{i+1}^2$ and $l_{i-1} = -l_{i+1}$.*

Proof. By 3.5 (ii) we have $\chi(\mathcal{E}_i, \mathcal{E}_{i+1}) = \frac{e_i^2 + e_{i+1}^2}{e_i e_{i+1}}$. If we denote $g := \gcd\{e_i, e_{i+1}\}$ and $e_i' := e_i / g$, $e_{i+1}' := e_{i+1}/g$, then we get immediately $\chi(\mathcal{E}_i, \mathcal{E}_{i+1}) = \frac{(e_i')^2 + (e_{i+1}')^2}{e_i' e_{i+1}'}$. But then e_i' divides $(e_{i+1}')^2$ and e_{i+1}' divides $(e_i')^2$. But because $\gcd\{e_i', e_{i+1}'\} = 1$, this implies $(e_i')^2 = (e_{i+1}')^2$, hence $e_i^2 = e_{i+1}^2$. For the second assertion, observe that the relation $e_{i+1}^2 l_{i-1} + e_i^2 l_{i+1} = 0$ holds. ☐

We want now describe what happens to the vectors l_i if we perform a mutation of the sequence $\mathbf{E}$. If we apply a mutation $L_i \mathbf{E}$ or $R_i \mathbf{E}$, then we can construct by above procedure a sequence of vectors $l_1', \ldots, l_n' \in N'$, where N' is the dual of the structural morphism c' corresponding to the new toric system. The following lemma shows that in terms of the l_i, the effect of mutation is local.

Proposition 8.5: *With above notation, we can naturally identify N with N' such that $l'_j = l_j$ for all $j \neq i$.*

Proof. Without loss of generality, we only consider left mutations $L_i\mathbf{E}$; the case of right mutations then follows analogously. On the level of toric systems, the effects of such a mutation can be described by the formulas of 3.7 and 3.9. For this, we distinguish two cases, depending on whether $R := \operatorname{rk} L_{\mathcal{E}_i}\mathcal{E}_{i+1}$ is nonzero or not.

In the first case, we obtain a toric system $A'_1, \dots, A'_n$, where $A'_{i-1} = A_{i-1} - \frac{e_{i+1}}{R}A_i$, $A'_i = \frac{e_{i+1}}{R}A_i$, $A'_{i+1} = A_i + A_{i+1}$, and $A'_j = A_j$ otherwise. In particular, $\langle A'_1, \dots, A'_n\rangle_{\mathbb{Z}}$ contains all A_j with $j \neq i, i+1$. As we have seen in Proposition 6.2, these A_j are linearly independent and therefore form a $\mathbb{Q}$-basis of $\mathrm{CH}^1(X)_{\mathbb{Q}}$. By rescaling these basis vectors by a factor $\frac{1}{e_{i+1}^2}$, we can represent the $\mathbb{Q}$-linear extension of the maps c and c' in 6.3 by the following matrices:

$$c = \begin{pmatrix} e_2^2 & a_2 & e_1^2 & 0 & \cdots & 0 \\ 0 & x_2 & y_2 & e_2^2 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & x_{n-2} & y_{n-2} & 0 & & e_2^2 \end{pmatrix} \text{ and } c' = \begin{pmatrix} e_2^2 - \frac{e_2}{R}a_2 & \frac{e_2}{R}a_2 & a_2 + e_1^2 & 0 & \cdots & 0 \\ -\frac{e_2}{R}x_2 & \frac{e_2}{R}x_2 & x_2 + y_2 & e_2^2 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ -\frac{e_2}{R}x_{n-2} & \frac{e_2}{R}x_{n-2} & x_{n-2} + y_{n-2} & 0 & & e_2^2 \end{pmatrix}$$

where by cyclic renumbering we assume without loss of generality that $i = 2$. The entries x_j, y_j are determined by the relations $x_j l_2 + y_j l_3 + e_2^2 l_{j+2} = 0$, in particular we have $x_j = \det(l_3, l_{j+2})$ and $y_j = \det(l_{j+1}, l_2)$ for every j. Consider first the case $a_2 \neq 0$. Then $l'_2 = \frac{-1}{a_2}(e_1^2 l'_1 + R^2 l'_3)$. Now it follows from a direct calculation that the we can represent the Gale transforms $l'_1, \dots, l'_n$ by $l_1, l'_2, l_3, \dots, l_n$, in particular, we have $l'_2 = \frac{-1}{a_2}(e_1^2 l_1 + R^2 l_3)$. In the case $a_2 = 0$, we use any row with $x_j \neq 0$ in order to find the representation $l'_2 = l_2 + 2l_1$. As before, we check that we can represent $l'_1, \dots, l'_n$ by $l_1, l'_2, l_3, \dots, l_n$.

In the second case, our toric system is given by $E_1, A'_1, \dots, A'_{n-1}, \phi$, where $A'_j = A_j$ for $j < i$, $A'_i = A_i + A_{i+1}$ $A'_j = A_{j-1}$ for $j > i$, and $\phi(1) = i - 1$. By similar arguments as in the proof of Proposition 6.4, we can con conclude that effect of the mutation is that the vector l_i "hops" onto l_{i-1}. □

We can now prove that the l_i indeed are primitive vectors.

Proposition 8.6: *The l_i are primitive lattice vectors.*

Proof. Assume there is one l_i which is not primitive. Without loss of generality, we cyclically renumber the sequence such that $i = 1$. Then $l_1 = p\tilde{l_1}$ for some $t > 1$ and $\tilde{l_1}$ is a primitive lattice vector. Then p divides both $e_1^2 = \det(l_n l_1)$ and $e_2^2 = \det(l_1 l_2)$. Now for every $i > 3$ with $e_i \neq 0$, we can perform right-mutations in order to move $\mathcal{E}_i$ to the left:

$$\mathcal{E}_1, \mathcal{E}_i, R_{\mathcal{E}_i}\mathcal{E}_2, \dots, R_{\mathcal{E}_i}\mathcal{E}_{i-1}, \mathcal{E}_{i+1}, \dots, \mathcal{E}_n.$$

As these mutations do not alter l_n and l_1, the exceptional pair $\mathcal{E}_1, \mathcal{E}_i$ corresponds to rays l_n, l_1, l'_2 and $e_i^2 = \det(l_1, l'_2)$. Therefore p divides e_i^2 as well, and hence we get $\gcd\{e_1^2, \dots, e_n^2\} \neq 1$ and thus $\gcd\{e_1, \dots, e_n\} \neq 1$. But this contradicts the fact that the rank morphism from $K_0(X)$ to $\mathbb{Z}$ is surjective. □

Another special configuration arises if $L_{\mathcal{E}_i}\mathcal{E}_{i+1}$ or $R_{\mathcal{E}_{i+1}}\mathcal{E}_i$ has rank zero.

Lemma 8.7: *If $\operatorname{rk} L_{\mathcal{E}_i}\mathcal{E}_{i+1} = 0$ then $e_i^4 l_{i-1} - e_i^2 l_i + e_i^2 l_{i+1} = 0$. If $\operatorname{rk} R_{\mathcal{E}_{i+1}}\mathcal{E}_i = 0$ then $e_i^2 l_{i-1} - e_i^2 l_i + e_i^4 l_{i+1} = 0$.*

Proof. We only prove the first equation. Observe that $0 = \operatorname{rk} L_{\mathcal{E}_i}\mathcal{E}_{i+1} = \chi(\mathcal{E}_i, \mathcal{E}_{i+1})e_i - e_{i+1}$ by 2.7 and $\chi(\mathcal{E}_i, \mathcal{E}_{i+1}) = \chi(L_{\mathcal{E}_i}\mathcal{E}_{i+1}, \mathcal{E}_i) = -e_i$ by Corollary 4.12, hence $e_{i+1} = -e_i^2$ and $a_i = -e_i^2$ and the statement follows. □

Remark 8.8: As remarked at the beginning of this section, in order to simplify the presentation we considered only the case where our exceptional sequence contains only objects of nonzero rank. Given an arbitrary sequence, we can always produce such a sequence by mutation. More precisely, by 7.3, we can assume that $e_i = 0$ for $1 \leq i \leq t$ for some $0 \leq t \leq n-3$ and $e_i \neq 0$ for $t < i \leq n$. Then by right mutation, we can produce a sequence

$$\mathcal{E}_{t+1}, R_{\mathcal{E}_{t+1}}\mathcal{E}_1, \ldots, R_{\mathcal{E}_{t+1}}\mathcal{E}_t, \mathcal{E}_{t+2}, \ldots, \mathcal{E}_n$$

with $\operatorname{rk} R_{\mathcal{E}_{t+1}}\mathcal{E}_i = -e_{t+1}^2 \neq 0$ for $1 \leq i \leq t$. The local configuration of the new l_i then arises iteratively from Lemma 8.7. In the case $e_i = \pm 1$ this corresponds (at least locally, so far) to a series of smooth toric blow-ups (see also Remark 11.4)

For any exceptional pair $\mathcal{E}_0, \mathcal{E}_1$, we define inductively for $i \geq 0$ $\mathcal{E}_{i+2} = R_{\mathcal{E}_{i+1}}\mathcal{E}_i$. The Chern classes $s(\mathcal{E}_i, \mathcal{E}_{i+1})$ are all collinear to $c_1(\mathcal{E}_0, \mathcal{E}_1)$ in $\mathrm{CH}^1(X)_{\mathbb{Q}}$, where the proportionality is successively given by the quotients of ranks e_i/e_{i+1}. These ranks are determined in the following proposition which generalizes a similar statement by Rudakov [Rud90, §4]. It is not needed in the rest of this paper but it might be of some general interest.

Proposition 8.9: *Let $\mathcal{E}_0, \mathcal{E}_1$ be an exceptional pair with $\chi(\mathcal{E}_0, \mathcal{E}_1)^2 > 4$ and for $i \geq 0$ define inductively $\mathcal{E}_{i+2} = R_{\mathcal{E}_{i+1}}\mathcal{E}_i$. Moreover, denote $\alpha_\pm = \frac{1}{2}(\chi(\mathcal{E}_0, \mathcal{E}_1) \pm \sqrt{\chi(\mathcal{E}_0, \mathcal{E}_1)^2 - 4})$ the roots of the polynomial $x^2 - \chi(\mathcal{E}_0, \mathcal{E}_1)x + 1$. Then for $i \geq 2$ we get:*

$$e_i = \frac{\alpha_+^{i+1} - \alpha_-^{i+1}}{\alpha_+ - \alpha_-} e_0 - \frac{\alpha_+^i - \alpha_-^i}{\alpha_+ - \alpha_-}(\chi(\mathcal{E}_0, \mathcal{E}_1)e_0 - e_1).$$

Proof. We set $\chi(\mathcal{E}_0, \mathcal{E}_1) =: \chi$. Then we have the following recurrence relation for $i \geq 0$:

$$e_{i+2} = \chi e_{i+1} - e_i.$$

By forming formal power series in a variable x on both sides and rearranging terms we obtain:

$$\sum_{i \geq 0} e_i x^i = \frac{(e_1 - \chi e_0)x + e_0}{x^2 - \chi x + 1}$$

The denominator on the right hand side has two zeros at

$$\alpha_\pm := \frac{1}{2}(\chi \pm \sqrt{\chi^2 - 4}), \text{ where } \alpha_- = \frac{1}{\alpha_+},$$

which gives a partial fractions expansion:

$$\frac{(e_1 - \chi e_0)x + e_0}{x^2 - \chi x + 1} = \frac{A}{1 - \alpha_+ x} + \frac{B}{1 - \alpha_- x},$$

with

$$A = \frac{(e_1 - \chi e_0) + \alpha_+ e_0}{\alpha_+ - \alpha_-}, \qquad B = \frac{(e_1 - \chi e_0) + \alpha_- e_0}{\alpha_- - \alpha_+}.$$

Now, the assertion follows from expanding the geometric series for $(1 - \alpha_\pm x)^{-1}$. □

Remark 8.10: For the five remaining cases $\chi(\mathcal{E}_0, \mathcal{E}_1)^2 \leq 4$, we can directly use the first equation in the proof of Proposition 8.9 end get the following results by induction.

(i) If $\chi(\mathcal{E}_0, \mathcal{E}_1) = 0$ then $e_i = \begin{cases} (-1)^{i/2} e_0 & \text{for } i \text{ even,} \\ (-1)^{(i-1)/2} e_1 & \text{for } i \text{ odd} \end{cases}$

(ii) If $\chi(\mathcal{E}_0, \mathcal{E}_1)^2 = 1$ then $e_i = \begin{cases} (-\chi(\mathcal{E}_0, \mathcal{E}_1))^{i/3} e_0 & \text{for } i \equiv 0(3) \\ (-\chi(\mathcal{E}_0, \mathcal{E}_1))^{(i-1)/3} e_1 & \text{for } i \equiv 1(3) \\ (-\chi(\mathcal{E}_0, \mathcal{E}_1))^{(i-2)/3}(\chi(\mathcal{E}_0, \mathcal{E}_1)e_1 - e_0) & \text{for } i \equiv 2(3) \end{cases}$

(iii) For $\chi(\mathcal{E}_0, \mathcal{E}_1) = 2$ we get $e_i = ie_1 + e_0(1-i)$ and for $\chi(\mathcal{E}_0, \mathcal{E}_1) = -2$ we get $e_i = (-1)^{i+1}(ie_1 - e_0(1-i))$.

Funnily, we have a periodic behaviour for $\chi(\mathcal{E}_0, \mathcal{E}_1)^2 \leq 1$.

9. Mutations

In this section we consider an exceptional pair $\mathcal{E}, \mathcal{F}$ with $e, f \neq 0$ and $a := c_1(\mathcal{E}, \mathcal{F})^2$. We assume that this pair can be extended to an exceptional sequence of length n, $\mathcal{E}, \mathcal{F}, \mathcal{E}_3, \ldots, \mathcal{E}_n$. Then the set of Gale duals of the associated toric system contains primitive lattice vectors $l_e, l, l_f \in N$ such that the following relation holds:

$$f^2 l_e + al + e^2 l_f = 0.$$

Note that by Proposition 8.6 and Lemma A.5, l_e, l, l_f are essentially uniquely determined by the integers e^2, a, f^2.

9.1. As in Definition A.2, we have circumference segments $p_e := l - l_e$, $p_f := l_f - l$ and reduced circumference segments $q_e := \frac{p_e}{e^2}$, $q_f := \frac{p_f}{f^2}$. It is also convenient to define

$$w_e := eq_e, \quad \text{and} \quad w_f := fq_f.$$

Using 3.5 and 3.9, we immediately get the following formulas:

$$\begin{aligned} \det(w_e, w_f) &= \frac{1}{ef}(a + e^2 + f^2) = \chi(\mathcal{E}, \mathcal{F}), \\ \det(w_f, l_e) &= \frac{a + e^2}{f} = \operatorname{rk} L_{\mathcal{E}}\mathcal{F}, \\ \det(w_e, l_f) &= \frac{a + f^2}{e} = \operatorname{rk} R_{\mathcal{F}}\mathcal{E}. \end{aligned}$$

Note that mutation can change the orientation of the w's with respect to the p's and q's.

In our subsequent analysis, the circumference segments will play a crucial role. We start with the following observation.

Lemma 9.2: *Both w_e and w_f are integral.*

Proof. After a choice of coordinates we can assume without loss of generality that $l_e = (1, 0)$, $l = (x, e^2)$, $l_f = (y, -a)$ for some integers x, y. Then $v_e = (x - 1, e^2)$ and $v_f = (y - x, -a - e^2)$ and $ef\chi(\mathcal{E}, \mathcal{F}) = \det(v_e, v_f) = e^2(1 - y) - a(x - 1)$. So, in particular, e divides $a(x - 1)$ and therefore e divides $a \cdot v_e$. Moreover, as e divides $a + f^2$, it also divides $(a + f^2)v_e$ and therefore also $f^2 \cdot v_e$. Hence, e divides $\gcd\{a, e^2, f^2\} \cdot v_e$. Now, via mutation, we can replace $\mathcal{F}$ by any $\mathcal{E}_i$ with $e_i \neq 0$. In the fan, this leaves l_e, l and v_e unchanged, and we obtain analogously that e divides $\gcd\{c_1(\mathcal{E}, \mathcal{E}_i)^2, e^2, e_i^2\} \cdot v_e$. So we get that e divides $\gcd\{e^2, f^2, e_i^2 \mid e_i \neq 0\} \cdot v_e$. But $\gcd\{e^2, f^2, e_i \mid e_i \neq 0\} = 1$ and the assertion follows for w_e and, by exchanging the roles of $\mathcal{E}$ and $\mathcal{F}$, also for w_f. □

As we have seen in Proposition 8.5, the effect of mutation is local in the sense that if we apply a mutation to the pair $\mathcal{E}, \mathcal{F}$, say, then the triple l_e, l, l_f gets transformed to a triple l_e, l', l_f and the other l_i remain constant. The transformation of l to l' can be described nicely with help of the circumference segments. We consider the mutated pairs $\mathcal{F}, R_{\mathcal{F}}\mathcal{E}$ and $L_{\mathcal{E}}\mathcal{F}, \mathcal{E}$, by which l gets transformed to some l' and l'', respectively, which satisfy the following relations:

$$\left(\frac{a + f^2}{e}\right)^2 l_e + al' + f^2 l_f = 0 \quad \text{and} \quad e^2 l_e + al'' + \left(\frac{a + e^2}{f}\right)^2 l_f = 0.$$

We then get the following transformation formulas.

Lemma 9.3: *We have $l' = l_f + \frac{a+f^2}{e} w_e = l_e + f(\det(w_e, w_f)w_e + w_f)$ and $l'' = l_e - \frac{a+e^2}{f} w_f = l_f - e(\det(w_e, w_f)w_f + w_e)$.*

Proof. We only prove the formulas for l'; the case l'' then follows analogously. By A.5, l' is completely determined by a and the volumes f^2 and $\left(\frac{a+f^2}{e}\right)^2$ relative to l_f and l_e, respectively. Then the first equality follows from

$$\det(l_f + \frac{a + f^2}{e} w_e, l_f) = \left(\frac{a + e^2}{f}\right)^2 \quad \text{and} \quad \det(l_e, l_f + \frac{a + f^2}{e} w_e) = f^2,$$

which both are immediate consequences of formulas 9.1. The second equality follows from a simple rearrangement of terms. $\square$

For $\chi(\mathcal{E},\mathcal{F})^2 < 1$ we see that the transformations of w_e and w_f reflect the periodic behaviour we have observed in Remark 8.10. For $\chi(\mathcal{E},\mathcal{F})^2 > 4$ we observe the following.

Corollary 9.4: *With above notation, consider the pair $\mathcal{F}, R_{\mathcal{F}}\mathcal{E}$ and assume that $\chi(\mathcal{E},\mathcal{F}) \neq 0$. If $\operatorname{rk} R_{\mathcal{F}}\mathcal{E} \neq 0$, then w_e and w_f transform in the sublattice which they generate by the matrix $\left(\begin{smallmatrix} \chi(\mathcal{E},\mathcal{F}) & -1 \\ 1 & 0 \end{smallmatrix}\right)$. If $\operatorname{rk} L_{\mathcal{E}}\mathcal{F} \neq 0$, w_e and w_f transform as and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \chi(\mathcal{E},\mathcal{F}) \end{smallmatrix}\right)$*

Remark 9.5: For $\chi(\mathcal{E},\mathcal{F})^2 > 4$, Corollary gives us a nice method in order to compute the terms $\frac{\alpha_+^i - \alpha_-^i}{\alpha_+ - \alpha_-}$ of Corollary 9.4 for $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{E}_1 = \mathcal{F}$. It follows by induction that for $i \geq 1$, $\frac{\alpha_+^i - \alpha_-^i}{\alpha_+ - \alpha_-}$ coincides with the upper left entry of the $(i-1)$-st power of the matrix $\left(\begin{smallmatrix} \chi(\mathcal{E},\mathcal{F}) & -1 \\ 1 & 0 \end{smallmatrix}\right)$

The following statement is less nice but stronger, as it in particular implies that w_e and w_f do not change their lattice length in N under mutation.

Lemma 9.6: *We have $\det(w_e, w_f)w_e + w_f = Gw_f$ and $\det(w_e, w_f)w_f + w_e = G'w_e$ for $G, G' \in \mathrm{GL}_{\mathbb{Z}}(N)$. In the case that $\chi(\mathcal{E},\mathcal{F}) \neq 0$ and $\operatorname{rk} R_{\mathcal{F}}\mathcal{E} \neq 0$, .*

Proof. We consider only the first equality; the second equality follows analogously. By construction, $\det(w_e, wf_f)w_e + w_f = \frac{1}{f}(l' - l_e)$ which is a lattice vector by Lemma 9.2. We show that its lattice length coincides with that of w_f, which then implies the assertion. For this, we consider the exceptional triples $\mathcal{E}_0, \mathcal{F}, R_{\mathcal{F}}\mathcal{E}$ and $L_{\mathcal{E}_0}\mathcal{E}, \mathcal{E}_0, \mathcal{F}$ (for simplicity, we assume that $e_0 \neq 0$, which can always be arranged). In the first case, the exceptional pair $\mathcal{E}_0, \mathcal{F}$ gives rise to a relation $f^2 l_{n-1} + bl_e + e_0^2 l' = 0$, and in the second case we get $f^2 l'_e + bl + e_0^2 l_f = 0$, where l'_e is the mutation of l_e corresponding to $L_{\mathcal{E}_0}\mathcal{E}$. As the triples l_{n-1}, l_e, l' and l'_e, l, l_f correspond to the same triple of volumes f^2, b, e_0^2, it follows by Lemma A.5 that $l' - l_e = G \cdot (l_f - l)$ for some $G \in \mathrm{GL}_{\mathbb{Z}}(N)$, which implies the assertion. $\square$

9.7. We have seen that the value $\frac{1}{ef}\chi(\mathcal{E},\mathcal{F}) = \det(q_e, q_f) = \frac{a+e^2+f^2}{e^2 f^2}$ measures the convexity of the configuration of lattice vectors l_e, l, l_f. More precisely, we have three possibilities:

$$\begin{aligned} a + e^2 + f^2 &> 0 && \text{(convex)}, \\ a + e^2 + f^2 &< 0 && \text{(concave)}, \\ a + e^2 + f^2 &= 0 && \text{(flat)}. \end{aligned}$$

Definition 9.8: According to above inequalities, we call an exceptional pair $\mathcal{E}, \mathcal{F}$ with $e, f \neq 0$ either *convex*, *concave*, or *flat*. We call the value $a + e^2 + f^2$ the *convexity* of the pair $\mathcal{E}, \mathcal{F}$.

The following lemma shows that very often we can reduce the convexity of an exceptional pair by mutation.

Lemma 9.9: *Let $\mathcal{E}, \mathcal{F}$ be a convex exceptional pair and assume that $a < 0$. Then either $(\operatorname{rk} L_{\mathcal{E}}\mathcal{F})^2 < \max\{e^2, f^2\}$ or $(\operatorname{rk} R_{\mathcal{F}}\mathcal{E})^2 < \max\{e^2, f^2\}$.*

Proof. We can assume without loss of generality that $0 < e^2 \leq f^2$. Then we have to show that both $\frac{a+e^2}{f} < f$ and $-\frac{a+e^2}{f} < f$, which both follow trivially from the assumptions. $\square$

9.10. In particular, the lemma implies that for any convex pair with $a < 0$, we can obtain by mutation a sequence of exceptional pairs where both the maximal rank as well as the convexity strictly decrease until either we arrive at a concave pair or one of the exceptional objects acquires ranks zero.

9.11. Moreover, observe that flatness implies $a < 0$ and that mutation preserves flatness. In particular, we have $\operatorname{rk} L_{\mathcal{E}}\mathcal{F} = -f$ and $\operatorname{rk} R_{\mathcal{F}}\mathcal{E} = -e$ and it can never occur that iterative mutations of a non-flat pair can result in a flat pair.

Corollary 9.12: *Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be an exceptional sequence, then we can produce by mutation an exceptional sequence $\mathcal{Z}_1, \ldots, \mathcal{Z}_t$, $\mathcal{F}_1, \ldots, \mathcal{F}_{n-t}$ such that the $\mathcal{Z}_i$ have rank zero and for any $1 \leq i \leq n-t$, if $\mathcal{F}_i, \mathcal{F}_{i+1}$ is not a concave exceptional pair then $\tilde{A}_i^2 > 0$, where $\tilde{A}_i$ is the corresponding element of the reduced toric system.*

Proof. We iterate the following procedure.

1) Choose any pair $\mathcal{E}_i, \mathcal{E}_j$ with $i < j$ and $e_i, e_j \neq 0$ such that $e_k = 0$ for all $i < k < j$. If $\mathcal{E}_i, \mathcal{E}_j$ is a convex pair and $c_1(\mathcal{E}_i, \mathcal{E}_{i+1})^2 < 0$ then continue with steps 2) and 3).
2) Move all $\mathcal{E}_k$ for $i < k < j$, to the position left of $\mathcal{E}_i$ via right mutation. We denote $R_{\mathcal{E}_{j-1}} \cdots R_{\mathcal{E}_{i+1}} \mathcal{E}_i =: \mathcal{E}_i'$. The resulting pair $\mathcal{E}_i', \mathcal{E}_j$ is still convex and $(\operatorname{rk} \mathcal{E}_i')^2 = e_i^2$.
3) We iterate mutations as in Lemma 9.9 and successively minimize the maximal rank of the resulting mutated pairs until either the pair becomes concave or a mutation results in an object of rank zero.

We repeat these steps until the resulting sequence does not contain any concave pair $\mathcal{E}_i, \mathcal{E}_j$ with $c_1(\mathcal{E}_i, \mathcal{E}_{i+1})^2 < 0$. As we minimize the absolute value of ranks it is guaranteed that this iteration will terminate. We then finalize this procedure by moving all rank zero objects to the leftmost range. □

10. The global picture

So far, we have established that the Gale transforms of a toric system at least locally represent the data of a complete toric surface. That is, by Corollary 8.3 and Proposition 8.6, every triple l_{i-1}, l_i, l_{i+1} generates a fan with two maximal cones which intersect in a facet such that l_{i-1}, l_i, l_{i+1} are the primitive vectors which generate the 1-dimensional cones. In this section we want to show that all the l_i indeed form the set of primitive vectors which generate the fan of a complete toric surface. For this, we can start with the two cones σ_1, σ_2, generated by, say, l_1, l_2, l_3. Then clearly, we can try to add a third cone σ_3 generated by l_3 and l_4. By construction, this cone lies in counterclockwise direction from the first two cones, and we know that σ_2 and σ_3 again form a fan with two maximal cones. However, so far we do not have any information on whether $\sigma_1, \sigma_2, \sigma_3$ fit together to form a proper fan. For this, we would have to prove that either $\sigma_1 \cap \sigma_3 = \{0\}$ or $\sigma_1 \cap \sigma_3 = \mathbb{Q}_{\geq 0} l_1$ (which implies $l_1 = l_4$). Similarly, if we successively add $\sigma_i = \langle l_i, l_{i+1} \rangle_{\mathbb{Q}_{\geq 0}}$ in counterclockwise fashion, we have to show that σ_i obeys the correct intersection properties with the previously added fans.

The only possibility that this construction can violate these intersection properties is that for some $3 \leq i < n$, the intersection $\sigma_1 \cap \sigma_i$ is a two-dimensional cone by itself. Then $\sigma_1, \ldots, \sigma_i$ cover all of $N_{\mathbb{Q}}$ for some $i < n$ without closing up to a proper fan. Continuing this way, we would end up with a sequence of cones which in counter-clockwise order cover $N_{\mathbb{Q}}$ several times by "winding" around the origin until finally σ_n and σ_1 close up these windings via the triple l_n, l_1, l_2. We are going to show that there indeed can only be one winding.

10.1. We first observe that the number of windings does not depend on the existence of objects of rank zero among the $\mathcal{E}_i$. Indeed, if $e_i = 0$ for some i, we can use mutation to produce a sequence which only contains objects of nonzero rank. As we have seen in Lemma 8.7, this is a completely local operation which cannot change the number of windings. Therefore we may start without loss of generality with the case that $e_i \neq 0$ for all i and our toric system therefore consists only of $A_1, \ldots, A_n$.

10.2. With notation from the previous section, we consider now the case $n = 3$. Then we have one single relation $e_3^2 l_1 + e_2^2 l_2 + e_1^2 l_3 = 0$. The problem of additional windings does not occur in this case and we obtain the fan of a weighted projective space $\mathbb{P}(e_1^2, e_2^2, e_3^2)$. In that case, we have $\langle A_1, A_2, A_3 \rangle_{\mathbb{Q}} \cong \mathbb{Q}$ and $\langle c_1(\mathcal{E}_i, \mathcal{E}_{i+1} \mid i = 1, 2, 3 \rangle_{\mathbb{Z}} \cong \mathbb{Z}$, where under a suitable identification (and possibly shifting the $\mathcal{E}_i$ in $D^b(X)$), we can assume that $-K_X = 3$ and $c_1(\mathcal{E}_i, \mathcal{E}_{i+1}) > 0$ for all i. Then from

$$c_1(\mathcal{E}_{i-1}, \mathcal{E}_i) \cdot c_1(\mathcal{E}_i, \mathcal{E}_{i+1}) = e_{i-1} e_{i+1}$$

for $i = 1, 2, 3$ we get

$$c_1(\mathcal{E}_{i-1}, \mathcal{E}_i) = e_{i+1}$$

for any i. Now, via the identity

$$\chi(\mathcal{E}_{i-1}, \mathcal{E}_i) = \frac{1}{e_{i-1} e_i} (c_1(\mathcal{E}_{i-1}, \mathcal{E}_i)^2 + e_{i-1}^2 + e_i^2) = -K_X c_1(\mathcal{E}_{i-1}, \mathcal{E}_i)$$

with $c_1(\mathcal{E}_{i-1}, \mathcal{E}_i) = e_{i+1}$ and $-K_X = 3$, we see that the e_i satisfy the *Markov equation*:

$$e_1^2 + e_2^2 + e_3^2 = 3e_1e_2e_3.$$

This reproduces a well-known result of Rudakov [Rud89] for $\mathbb{P}^2$ from a purely combinatorial perspective. It is also a combinatorial variant of results of Hacking [HP10, Hac13] which gives a correspondence of the construction of exceptional sequences on $\mathbb{P}^2$ to $\mathbb{Q}$-Gorenstein degenerations of $\mathbb{P}^2$ whose exceptional fiber is a weighted projective spaces $\mathbb{P}(e_1^2, e_2^2, e_3^2)$, where the e_i satisfy the Markov equation.

We have seen in Section 8 that the self-intersection numbers A_i^2 (respectively $c_1(\mathcal{E}_i, \mathcal{E}_{i+1})^2$) encode convexity properties among the l_i. Indeed, the case $n = 3$ is special, because $A_i^2 > 0$ for $i = 1, 2, 3$. More generally, we have the following lemma.

Lemma 10.3: *Let $n \geq 4$.*

(i) Assume that $A_i^2 \geq 0$ for some i, then there exists at most one other A_j such that $A_j^2 > 0$. Then j is either $i - 1$ or $i + 1$.
(ii) If $A_i^2 = 0$, $A_j^2 = 0$, and $A_i \cdot A_j = 0$ then $j = i + 1$ and $n = 4$.
(iii) If $A_i^2 = A_{i+1}^2 = 0$ for some i, then either $n = 4$ and $A_j^2 = 0$ for all j, or $n > 4$ and $A_j^2 < 0$ for all $j \neq i, i + 1$.

Proof. We first remark that $\mathrm{CH}^1(X)_\mathbb{Q}$ is a metric space with respect to the intersection product which has an orthogonal decomposition $\mathrm{CH}^1(X)_\mathbb{Q} \cong C^+ \perp C^-$ into positive and negative definitive part. By the Hodge Index Theorem we have $\dim_\mathbb{Q} C^+ = 1$ and $\dim_\mathbb{Q} C^- = n - 3$. Up to cyclic renumbering let us now assume that $A_1^2 \geq 0$. Then clearly A_1 and C^- generate $\mathrm{CH}^1(X)_\mathbb{Q}$ as a $\mathbb{Q}$-vector space and for every i we can write $A_i = \alpha_i A_1 + n_i$ for some $\alpha_i \in \mathbb{Q}$ and $n_i \in C^-$. Now, for any $2 < i < n$ we have $0 = A_1 \cdot A_i = A_1 \cdot (\alpha_i A_1 + n_i) = \alpha_i A_1^2 + A_1 n_i$, hence $\alpha_i A_i^2 = -A_1 n_i$. Now, for $2 < i < n$ we compute $A_i^2 = (\alpha_i A_1 + n_i)^2 = \alpha_i^2 A_1^2 + 2\alpha_i A_i \cdot n_i + n_i^2 = \alpha_i A_1 \cdot (\alpha_i A_1 + 2n_i) + n_i^2 = \alpha_i A_1 A_i + \alpha_i A_1 n_i + n_i^2 = \alpha_i A_1 n_i + n_i^2 = -\alpha_i^2 A_i^2 + n_i^2 \leq 0$, as both $-\alpha_i^2 A_i^2 \leq 0$ and $n_i^2 \leq 0$. If $A_1^2 > 0$ then this inequality is strict for every i. This leaves only A_2 and A_n which can have positive self-intersection number. But because $n \geq 4$, we also have $A_2 \cdot A_n = 0$. So if one of A_2, A_n has positive self-intersection, then we can argue as for A_1 and conclude that the other must have negative self-intersection and (i) follows.

(ii) The same computation leads to $A_i^2 = n_i^2 = 0$, hence $n_i = 0$ and thus $A_i = \alpha_i A_1$. Then $A_i \cdot A_2 \neq 0$ and $A_i \cdot A_n \neq 0$, which is only possible if $n = 4$ and $i = 3$.

(iii) We know from (i) that there cannot be any A_j with $A_j^2 > 0$. If there exists a third A_j with $A_j^2 = 0$, then $A_j \cdot A_i = 0$ or $A_j \cdot A_{i+1} = 0$ and by (ii) this implies $n = 4$ and $l_1 = -l_3$, $l_2 = -l_4$ by Lemma 8.4. □

Remark 10.4: Note that the arguments of Lemma 10.3 also apply to reduced toric systems $\tilde{A}_1, \ldots, \tilde{A}_{n-t}$, because the positive definite part of $\langle \tilde{A}_1, \ldots, \tilde{A}_{n-t} \rangle$ is at most one-dimensional. That is, whenever we will create rank zero objects by mutation of our original sequence, above estimates on the number and configuration of objects non-negative self-intersection apply also to the resulting $\tilde{A}_i$.

10.5. The case $n = 4$ still exhibits some nice symmetry. Clearly, we have $c_1(\mathcal{E}_i, \mathcal{E}_{i+1})^2 = \det(l_{i+2}, l_i) = -\det(l_i, l_{i+2}) = -c_1(\mathcal{E}_{i+2}, \mathcal{E}_{i+3})^2$ for every i. In the case that $l_i = -l_{i+2}$ for some i, we get by Lemma 8.4 that $e_{i-1}^2 = e_i^2$ and $e_{i+1}^2 = e_{i+2}^2$. Then $8 = K_X^2 = \sum_{j=1}^4 A_j^2 = 4(\frac{1}{e_i^2} + \frac{1}{e_{i+1}^2})$ and thus $e_i^2 = 1$ for all i. That is, the l_i generate the fan of a Hirzebruch surface and we recover a special case of the corresponding result for exceptional sequences of line bundles as was proven in [HP11, Theorem 3.5]

If no two opposing l_i exist, then it is elementary to see that the l_i can produce only one winding and for some i we have $A_{i-2}^2, A_{i-1}^2 > 0$ and $A_i^2, A_{i+1}^2 < 0$. The two pairs $\mathcal{E}_i, \mathcal{E}_{i+1}$ and $\mathcal{E}_{i+1}, \mathcal{E}_{i+2}$ also cannot be both concave, because otherwise the positive span of the l_i could not generate $N_\mathbb{Q}$. This means that whenever there is no opposing pair l_i's, we can use Lemma 9.9 in order to decrease the maximal rank of a concave pair. By iteration we will eventually end up in one of two cases:

1) We produce a pair of opposite l_i and therefore obtain an exceptional sequence of objects of rank one.
2) We produce a sequence of the form $\mathcal{Z}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ with $z = 0$ and $f_i^2 > 0$ for $i = 1, 2, 3$. For the corresponding reduced toric system $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ it follows analogously to 10.2 that the f_i must satisfy the Markov equation $f_1^2 + f_2^2 + f_3^2 = 3f_1f_2f_3$.

As for the case of $\mathbb{P}^2$ (and with an eye on 7.3), it follows that we can mutate the $\mathcal{F}_i$ further such that we end up with an exceptional sequence $\mathcal{Z}', \mathcal{F}_1', \mathcal{F}_2', \mathcal{F}_3'$ with $z' = 0$ and ${f_1'}^2 = {f_2'}^2 = {f_3'}^2 = 1$. By another right-mutation we obtain $\mathcal{F}_1', R_{\mathcal{F}_1'}\mathcal{Z}', \mathcal{F}_2, \mathcal{F}_3'$ such that every element has rank ($\pm$) one. The corresponding smooth toric surface can easily be seen to correspond with the first Hirzebruch surface.

So we have reproduced a result of Nogin [Nog91, §3] which states that every exceptional sequence on a Hirzebruch surface can be mutated to a sequence consisting of objects of rank one (see also Example 10.10).

Proposition 10.6: *Let $\mathcal{E}_1, \dots, \mathcal{E}_n$ be a numerically exceptional sequence with associated reduced toric system $\tilde{A}_1, \dots, \tilde{A}_{n-t}$. Then the Gale duals $\tilde{l}_1, \dots, \tilde{l}_{n-t}$ generate the fan corresponding to a complete toric surface such that the $\tilde{l}_i$ span the rays in this fan and the maximal cones are generated by $\tilde{l}_i, \tilde{l}_{i+1}$ for $1 \leq i < n-t$ and $\tilde{l}_{n-t}, \tilde{l}_1$.*

Proof. By Corollary 9.12, we can produce by mutation an exceptional sequence $\mathcal{Z}_1, \dots, \mathcal{Z}_s, \mathcal{F}_1, \dots, \mathcal{F}_{n-s}$ with $s \geq t$ and associated reduced toric system $B_1, \dots, B_{n-s}$, such that any exceptional pair $\mathcal{F}_i, \mathcal{F}_{i+1}$ (where we read the indices cyclically modulo $n-s$) with $B_i^2 < 0$ is concave. We denote $k_1, \dots, k_{n-s}$ the Gale duals of the B_i. The cases $n - s \leq 4$ have been covered in 10.2 and 10.5, so we assume $n \geq 5$. It follows from Lemma 10.3 that there exist at most two B_i with $B_i^2 > 0$ which then must be adjacent.

Now consider any subsequent $B_j, B_{j+1}, \dots, B_{j+r}$ such that $B_{j+j'}^2 < 0$ for all $0 \leq j' \leq r$. Then the sequence of circumference segments $p_j, \dots, p_{j+r}$ is non-convex with respect to the origin, i.e. for any $0 \leq j' < r$, the vector $k_{j+j'}$ is contained in the convex hull of $0, k_{j+j'-1}, k_{j+j'+1}$. But this implies that all the $k_{j+j'}$ for $-1 \leq j' \leq r$ are contained in the same half space whose boundary is given by $\mathbb{Q}k_{j-1}$. In particular, k_{j-1} is the only one which is contained in the boundary.

With this observation it follows that there must be at least one B_i with $B_i^2 > 0$, say B_1, and $B_j, \dots, B_n$ is the maximal sequence with $B_i^2 < 0$, where $j = 2$ or $j = 3$. This implies that all cones except for possibly two (if $j = 2$) or three (if $j = 3$) are contained in a half space and it follows from elementary geometric arguments that we cannot produce more than one winding this way.

Now, as we have argued 10.1, we can conclude that also the original reduced toric system does not produce more than one winding and the assertion follows. □

Corollary 10.7: *Let $\mathcal{E}_1, \dots, \mathcal{E}_n$ and $\mathcal{Z}_1, \dots, \mathcal{Z}_t$, $\mathcal{F}_1, \dots, \mathcal{F}_{n-t}$ be numerically exceptional sequences as in Corollary 9.12. Then either $t = n-3$ and f_1, f_2, f_3 satisfy the Markov equation of 10.2 or $t = n-4$ and the $\mathcal{F}_i$ are objects of ranks ± 1. In particular, any exceptional sequence can by mutation be transformed into an exceptional sequence consisting only of objects of ranks ± 1 and 0.*

Proof. The assertions follow from the proof of Proposition 10.6 and the elementary observation that for $n - t > 4$ our exceptional sequence always contains a convex pair $\mathcal{E}_i, \mathcal{E}_{i+1}$ with $c_1(\mathcal{E}_i, \mathcal{E}_{i+1})^2 < 0$. Hence, the procedure of Corollary 9.12 always results in an exceptional sequence with $n - t \leq 4$. Then we can conclude as in the last part of 10.5. □

The following theorem can also be considered as a corollary of Proposition 10.6 and Corollary 10.7.

Theorem 10.8: *Let X be a (rational) surface. Then any numerically exceptional sequence on X can be transformed by mutation into a numerically exceptional sequence consisting only of objects of rank one.*

Proof. By Corollary 10.7, every sequence can be transformed into an exceptional sequence of objects of ranks one and zero. To this sequence is associated the fan of a toric surface which is given by the Gale duals $l_1, \dots, l_{n-t}$ of the associated reduced toric system. By assumption, we have $\det(l_i l_{i+1}) = 1$ for all i, hence the toric surface is smooth. If we instead consider the Gale duals of the unreduced toric system, we obtain by Proposition 6.4 the same fan, but where the rays come with possible multiplicities, i.e. if $\mathcal{E}_i, \mathcal{E}_{i+1}, \dots, \mathcal{E}_j$ is a sub-sequence for $i < j$ such that $e_k = 0$ for all $i < k < j$, then the multiplicity of the Gale dual of the reduced element $s(\mathcal{E}_j) - s(\mathcal{E}_i) + \sum_{k=i+1}^{j-1} c_1(\mathcal{E}_k)$ is $j - i$. For any pair $\mathcal{E}_i, \mathcal{E}_{i+1}$ with $e_i = 0$, we have $\operatorname{rk} L_{\mathcal{E}_i}\mathcal{E}_{i+1} = -e_i$ and $\operatorname{rk} R_{\mathcal{E}_{i+1}}\mathcal{E}_i = -e_{i+1}$ (and similarly if $e_{i+1} = 0$). We have seen in Section 7 the mutation from $\mathcal{E}_i, \mathcal{E}_{i+1}$ to $L_{\mathcal{E}_i}\mathcal{E}_{i+1}, \mathcal{E}_i$ leaves the associated fan unchanged, except for a possible exchange of multiplicities. On the other hand, if $e_{i+1} \neq 0$, then the mutation to $\mathcal{E}_{i+1}, R_{\mathcal{E}_{i+1}}\mathcal{E}_i$ yields a new object of rank ± 1, and the associated fan obtains a new primitive vector l', whose position is given by the relation $e_{i+1}^2 l_k - e_{i+1}^2 l' + e_{i+1}^4 l_{k+1}$ for the corresponding $1 \leq k \leq n-t$, and with $e_{i+1}^2 = 1$ we get

$l' = l_k + l_{k+1}$. That is, the right mutation yields a bigger fan which corresponds to a toric blow-up of the original fan. Similarly, if $e_i \neq 0$ and $e_{i+1} = 0$, then left mutation results in a blow-up as well. Now we can iterate the following two steps of mutations $\mathcal{E}_i, \mathcal{E}_{i+1}$ to $L_{\mathcal{E}_i}\mathcal{E}_{i+1}, \mathcal{E}_i$ (respectively $\mathcal{E}_{i+1}, R_{\mathcal{E}_{i+1}}\mathcal{E}_i$):

1) If $e_i = 0$ (respectively $e_{i+1} = 0$), which only affect multiplicities of the primitive vectors.
2) If $(e_i^2, e_{i+1}^2) = (1, 0)$ (respectively $(e_i^2, e_{i+1}^2) = (0, 1)$), which correspond to smooth blow-ups.

Iterating these steps at will we can realize every smooth toric surface which can be obtained from the original $l_1, \ldots, l_{n-t}$ by at most t blow-ups. □

Remark 10.9: By Theorem 10.8, we have now justified the statements we made in Remark 5.6. Indeed, any toric system coming from a numerically exceptional sequence indeed can be constructed by mutation from a toric system associated to rank one objects as were considered in [HP11]. Note that the relevant analysis of [HP11, §2] is strictly on the numerical level and therefore also applies to any numerical exceptional sequence whose elements' classes in $K_0(X)$ coincide with that of invertible sheaves. Note that any numerically exceptional object $\mathcal{L}$ of rank one has the same class in $K_0(X)$ as an invertible sheaf $\mathcal{O}(D)$, where the class of D in $\mathrm{CH}^1(X)$ coincides with $c_1(\mathcal{L})$. This is easy to see from $c_2(\mathcal{L}) = 0$ by 3.3 (ii) and by the injectivity of the Chern-isomorphism with $\mathrm{ch}(\mathcal{L}) = \mathrm{ch}(\mathcal{O}(D)) \in \mathrm{CH}^1(X)_{\mathbb{Q}}$.

Example 10.10: We construct examples of exceptional objects of rank one which are not isomorphic to invertible sheaves. Consider an even Hirzebruch surface $\mathbb{F}_a$, where $a = 2b$ for $b >= 0$ and denote P, Q the generators of its nef cone, where $P^2 = 0, Q^2 = a$ and $P \cdot Q = 1$. In [HP11, Proposition 5.2] it was shown that $\mathbb{F}_a$ admits two families of toric systems for exceptional sequences consisting of objects of rank one, which are of the following form:

$$F_1(s): \quad (-s-b)P + Q, P, (s-b)P + Q, P \text{ for } s \in \mathbb{Z},$$
$$F_2(s): \quad -bP + Q, P + s(-bP + Q), -bP + Q, P - s(-bP + Q) \text{ for } s \in \mathbb{Z}.$$

Note that both families meet at $s = 0$. Any toric system of type $F_1(s)$ corresponds to an exceptional sequence of invertible sheaves $\mathbf{L} = \mathcal{L}_1, \ldots, \mathcal{L}_4$. Assume that, say, $c_1(\mathcal{L}_2, \mathcal{L}_3) = P$, then $\mathrm{rk}\, L_{\mathcal{L}_2}\mathcal{L}_3 = \mathrm{rk}\, R_{\mathcal{L}_3}\mathcal{L}_2 = 1$ and both mutations are isomorphic to invertible sheaves, and $c_1(L_{\mathcal{L}_2}\mathcal{L}_3, \mathcal{L}_2) = c_1(\mathcal{L}_3, R_{\mathcal{L}_3}\mathcal{L}_2) = P$. Moreover, $c_1(\mathcal{L}_1, L_{\mathcal{L}_2}\mathcal{L}_3) = c_1(\mathcal{L}_1, \mathcal{L}_2) - 2P$ and $c_1(R_{\mathcal{L}_3}\mathcal{L}_2) = c_1(\mathcal{L}_3, \mathcal{L}_4) - 2P$. That is, from a sequence of type $F_1(s)$ we obtain by left mutation of the pair $\mathcal{L}_2, \mathcal{L}_3$ a sequence of type $F_1(s-2)$ and by right mutation a sequence of type $F_1(s+2)$ and we can conclude that the exceptional sequences of invertible sheaves of type $F_1(s)$ can be transformed into each other via mutations (note 2 facts: 1. that this is strictly true only up to overall twist by an invertible sheaf; 2. $Y(\mathbf{L}) \cong \mathbf{F}_{2s}$, which due to the type of the intersection form is the only allowed case). We can argue similarly that exceptional sequences of type $F_2(s)$ can be transformed into each other via mutations.

Both families intersect in the particular case $s = 0$, where $Y(\mathbf{L}) \cong \mathbf{P}^1 \times \mathbf{P}^1$. Then, starting with an exceptional sequence of invertible sheaves representing $F_1(0) = F_2(0)$, we can produce both families by either mutate the pair $\mathcal{L}_2, \mathcal{L}_3$ (to obtain $F_1(s)$) or $\mathcal{L}_1, \mathcal{L}_2$ (for $F_2(s)$). However, by [HP11, Proposition 5.2], for $b > 0$ and $s \neq 0$, the resulting sequences of type $F_2(s)$ cannot consist entirely of invertible sheaves.

11. The main theorem

We are now in possession of everything we need in order to show our main theorem.

Theorem 11.1: *Let X be a (rational) surface with $\mathrm{rk}\, K_0(X) = n$ and $\mathbf{E} = \mathcal{E}_1, \ldots, \mathcal{E}_n$ a numerically exceptional sequence. Then to the maximal sub-sequence of objects of nonzero rank $\mathcal{E}_{i_1}, \ldots, \mathcal{E}_{i_{n-t}}$ there is associated in a canonical way a complete toric surface $Y(\mathbf{E})$ with $n - t$ torus fixpoints. These fixpoints are either smooth (if $e_i^2 = (\mathrm{rk}\, \mathcal{E}_{i_j})^2 = 1$) or T-singularities of type $\frac{1}{e_{i_j}^2}(1, k_{i_j}e_{i_j} - 1)$ with $\gcd\{k_{i_j}, e_{i_j}\} = 1$ (if $e_{i_j}^2 \neq 1$).*

Remark 11.2: To justify "canonical", we should, strictly speaking, also incorporate the multiplicities of the rays of $Y(\mathbf{E})$ coming from the rank zero objects. However, in light of our discussion in Section 7, at least on the combinatorial level there seems not much to be gained from this.

In absence of rank zero objects, we can state Theorem 11.1 in a more convenient form.

Theorem 11.3: *Let X be a (rational) surface with and let $\mathbf{E} = \mathcal{E}_1, \ldots, \mathcal{E}_n$ be a numerically exceptional sequence of maximal length such that $\operatorname{rk} \mathcal{E}_i = e_i > 0$ for every i. Then to this sequence there is associated in a canonical way a complete toric surface $Y(\mathbf{E})$ with n torus fixpoints which are either smooth (if $e_i = 1$) or T-singularities of type $\frac{1}{e_1^2}(1, k_i e_i - 1)$, where $\gcd\{k_i, e_i\} = 1$ (if $e_i^2 \neq 1$).*

Proof of Theorem 11.1. By Proposition 10.6, we have constructed our toric variety as the Gale duals of the associated reduced toric system. It only remains to show that this surface indeed can only have at most T-singularities. Consider the vectors $l_1, \ldots, l_{n-t}$ generating the fan of $Y(\mathbf{E})$ and their corresponding $n - t$ circumference segments $p_i = l_i - l_{i-1}$. For any given i, with a convenient choice of coordinates we can represent the vectors l_i, l_{i+1} as $l_1 = (1, 0)$ and $l_{i+1} = (-x, e^2)$ for some $0 < x < e^2$. We have seen in Lemma 9.2 that the vector $w_i = \frac{1}{e_i} p_i$ is integral, hence we get $x = k_i e_i - 1$ for some $1 \leq k_i \leq e$. By Lemma A.3, $\gcd\{k_i, e_i\} = 1$ if and only if w_i is primitive. By Lemma 9.6 we know that any left or right mutation of a pair $\mathcal{E}_i, \mathcal{E}_{i+1}$ with $e_i, e'_{i+1} \neq 0$ which results in a pair $\mathcal{E}'_i, \mathcal{E}'_{i+1}$ such that $e'_i, e'_{i+1} \neq 0$ leaves the lattice lengths of the involved w_i invariant. By Theorem 10.8 we can transform any sequence by mutation into a sequence of length n of objects of rank one, where obviously the w_j are primitive for every $1 \leq j \leq n$. However, in this process we might destroy or create new circumference segments by creating or destroying objects of rank zero, and it remains to check whether we obtain non-primitive w_j this way. By Lemma 8.7, for every pair $\mathcal{E}_i, \mathcal{E}_{i+1}$ with $\operatorname{rk} L_{\mathcal{E}_i} \mathcal{E}_{i+1} = 0$ we have the relation $e_i^4 l_{i-1} - e_i^2 l_i + e_i^2 l_{i+1} = 0$, respectively we have $l_i = e_i^2 l_{i-1} + l_{i+1}$. Then mapping l_i to l_{i-1} extends to a linear map on N which leaves l_{i-1} invariant and thus the pairs l_{i-1}, l_i and l_{i-1}, l_{i+1} both correspond to cones whose associated affine toric surfaces are isomorphic, hence $l_{i+1} - l_{i-1}$ has lattice length e_i iff $l_i - l_{i-1}$ has. The difference $l_{i+1} - l_i = e^2 l_{i-1}$ clearly corresponds to the circumference segment of a T-singularity of order e_i^4. This concludes the proof of the theorem. □

Remark 11.4: As we have already seen in the proof of Theorem 10.8, passing between sequences which consist of objects of rank one and zero is much akin to lifting the toric minimal model program for smooth toric surfaces to the level of exceptional sequences. Likewise, we can regard a toric minimal model program implicit in the proof of Theorem 11.1 for the toric surfaces with T-singularities which can be realized via exceptional sequences and their associated toric systems.

11.5. We conclude this section with an observation on cyclic strongly exceptional sequences. In [HP11, Theorems 5.13, 5.14] it was shown that any rational surface which admits a cyclic strongly exceptional sequence necessarily has Picard-rank at most 7 and that every del Pezzo surface meeting this conditions indeed does admit a cyclic strongly exceptional sequence of invertible sheaves. More generally, in [van09] van den Bergh constructed cyclic strongly exceptional sequences for all del Pezzo surfaces, which for for the Picard-ranks 8 and 9 necessarily cannot consist of line bundles only. The crucial observation in [HP11] was the fact that for the rank one case the associated toric surface must be a weak del Pezzo surface or, equivalently, that the fan must be convex. This is easily seen to be true also in the general case, as by 9.1, $\chi(\mathcal{E}_i, \mathcal{E}_{i+1}) \geq 0$ implies the convexity of the triple l_{i-1}, l_i, l_{i+1}. The following implication for the Picard-rank then is quite straightforward. Note that $K_X^2 > 0$ implies that $\operatorname{rk} \operatorname{Pic}(X) < 10$.

Theorem 11.6: *The length of a cyclic strongly exceptional sequence is at most* 11*. In particular, if X admits a cyclic strongly exceptional sequence which generates $K_0(X)$, then $K_X^2 > 0$.*

Proof. By Proposition 4.3, we can choose $\mathcal{E}_i, \mathcal{E}_j$ with $e_i e_j > 0$ and $i < j$. From strong cyclicity it follows that $\chi(\mathcal{E}_i, \mathcal{E}_j), \chi(\mathcal{E}_j, \mathcal{E}_i \otimes \omega_X^{-1}) \geq 0$. From the fact that the fan for $Y(\mathbf{E})$ necessarily has a convex configuration of primitive vectors it follows that at least one inequality is strict, where up to cyclic renumbering we can assume without loss of generality that $\chi(\mathcal{E}_i, \mathcal{E}_j) > 0$. By Lemma 3.6, we have $\chi(\mathcal{E}_j, \mathcal{E}_i \otimes \omega_X^{-1}) = e_i e_j K_X^2 - \chi(\mathcal{E}_i, \mathcal{E}_j) \geq 0$, hence

$$e_i e_j K_X^2 \geq \chi(\mathcal{E}_i, \mathcal{E}_j) > 0.$$

from which our assertion follows. □

12. Some remarks on the singularities

Consider a numerically exceptional sequence $\mathbf{E} = \mathcal{E}_1, \ldots, \mathcal{E}_n$, where for simplicity we will assume that $e_i > 0$ for all i. Then by the classification of Section 11, $Y(\mathbf{E})$ belongs to a very nice class of toric surfaces with the following properties:

1) $\det(l_i, l_{i+1}) = e_i^2$ for every i.
2) The circumference segments $p_i = l_i - l_{i-1}$ have lattice length e_i for every i.
3) $K_{Y(\mathbf{E})}^2 = K_X^2 = 12 - n$.

For $K_{Y(\mathbf{E})}^2$, we have by A.7, 3.5, and 9.1 the formulas:

$$K_{Y(\mathbf{E})}^2 = \sum_{i=1}^n \det(q_i q_{i+1}) = \sum_{i=1}^n \frac{1}{e_i e_{i+1}} \chi(\mathcal{E}_i, \mathcal{E}_{i+1}) = \sum_{i=1}^n \frac{c_1(\mathcal{E}_i, \mathcal{E}_{i+1})^2 + e_i^2 + e_{i+1}^2}{e_i^2 e_{i+1}^2}.$$

As we have already stated in Remark 11.4, this kind of surface can be classified in terms of mutations, for instance by starting from the fan of a smooth toric surface. However, a classification of such surfaces independent of exceptional sequences might be of interest as well (e.g. one could relax property 3) above and just require that $K_{Y(\mathbf{E})}^2$ be integral). This is beyond the scope of this article, but with this perspective we want conclude with some general remarks on the singularities which occur in our setting.

Cyclic T-singularities of type $\frac{de^2}{kde-1}$ with $\gcd\{k, e\} = 1$ have been classified Kollar and Shepherd-Barron. Their statement is:

Proposition 12.1: [KS88, Proposition 3.11] *A cyclic singularity is of class T if and only if its continued fraction expansion $[b_1, \ldots, b_r]$ is of one of the following forms:*

(i) $[4]$ *and* $[3, 2, \ldots, 2, 3]$ *are of class T,*
(ii) If $[b_1, \ldots, b_r]$ *is of class T then so are* $[2, b_1, \ldots, b_r + 1]$ *and* $[b_1 + 1, \ldots, b_r, 2]$.
(iii) Every singularity of class T that is not a rational double point can be obtained by starting with one of the singularities described in (i) and iterating the steps described in (ii).

12.2. It follows from the analysis in the proof of [KS88, Proposition 3.11] that $d = 1$ implies that the T-singularities of type $\frac{e^2}{ke-1}$ are precisely those obtained from [4] and iterating step (ii). Consider the minimal desingularization of $Z \to Y = Y(\mathbf{E})$. Denote $I := \{1 \leq k \leq n \mid e_k^2 \neq 1\}$, then the fan of Z is obtained from Y by inserting t_k new primitive vectors corresponding to the continued fractions $[b_1^k, \ldots, b_{t_k}^i]$. We have

$$K_Z^2 = \sum_{i=1}^n a_i + 2n - \sum_{k \in I} (\sum_{j=1}^{t_k} b_j^k - 2t_i) = 12 - n - \sum_{i \in I} t_i,$$

where the a_i are the self-intersection of the pullbacks of the original n divisors to the surface Z. A simple induction using Proposition 12.1 shows that $\sum_{j=1}^{t_k} b_j^k = 1 + 3t_k$ for every $k \in I$, hence

$$\sum_{i=1}^n a_i = 12 - 3n + |I|.$$

This is almost the same formula as for the sum of self-intersection numbers of the toric prime divisors on a smooth toric surface with n rays, except that we obtain the number of singularities as an extra term.

12.3. Consider $e, f > 1$ and a triple of primitive lattice vectors l_e, l, l_f such that the cones generated by l_e, l and l, l_f, respectively, correspond to T-singularities of types $\frac{1}{e^2}(1, \alpha e - 1)$ and $\frac{1}{f^2}(1, \beta f - 1)$, respectively. We can choose coordinates such that $l = (0, 1)$ and $l_e = (e^2, -\alpha e + 1 + \lambda_1 e^2)$, $l_f = (-f^2, -\beta f + 1 + \lambda_2 f^2)$. It is easy to verify that the term $\lambda := \lambda_1 + \lambda_2$ does not depend on any choice of coordinates with $l = (0, 1)$. Using 9.1, we compute:

$$\det(l_f, l_e) = e^2 f^2 (\frac{\alpha}{e} + \frac{\beta}{f} + \lambda) - e^2 - f^2 = e^2 f^2 \frac{\chi(\mathcal{E}, \mathcal{F})}{ef} - e^2 - f^2,$$

hence $\frac{1}{ef}\chi(\mathcal{E}, \mathcal{F}) = \frac{\alpha}{e} + \frac{\beta}{f} + \lambda$. If $e = 1, f > 1$ (or $e > 1, f = 1$ or $e = f = 1$, respectively), then the same calculation yields $\frac{1}{f}\chi(\mathcal{E}, \mathcal{F}) = 1 + \frac{\beta}{f} + \lambda$ (respectively $\frac{1}{e}\chi(\mathcal{E}, \mathcal{F}) = \frac{\alpha}{e} + 1 + \lambda$ and $\chi(\mathcal{E}, \mathcal{F}) = 2 + \lambda$). In the case $e = f = 1$, λ coincides with the self-intersection number of the toric divisor associated to l.

12.4. Now, if we transfer above considerations to $Y(\mathbf{E})$, we have in terms of reduced circumference segments for every $1 \le i \le n$ that $\frac{1}{e_i e_{i+1}}\chi(\mathcal{E}_i, \mathcal{E}_{i+1}) = \det(q_i, q_{i+1})$, where

$$\det(q_i, q_{i+1}) = \begin{cases} \frac{\alpha_i}{e_i} + \frac{\beta_i}{e_{i+1}} + \lambda_i & \text{if } e_i, e_{i+1} > 1 \\ 1 + \frac{\beta_i}{e_{i+1}} + \lambda_i & \text{if } e_i = 1, e_{i+1} > 1 \\ \frac{\alpha_i}{e_i} + 1 + \lambda_i & \text{if } e_i > 1, e_{i+1} = 1 \\ 2 + \lambda_i \text{ if } e_i^2, e_{i+1}^2 = 1. \end{cases}$$

It is a consequence of local change of coordinates that $\alpha_i = e_i - \beta_{i-1}$ for every i and in particular, for any i with $e_i > 1$ we have $\frac{\alpha_i}{e_i} + \frac{\beta_{i-1}}{e_i} = 1$. Thus we get:

$$K^2_{Y(\mathbf{E})} = 12 - n = \sum_{i=1}^{n} \det(q_i, q_{i+1}) = \sum_{i=1}^{n} \lambda_i + 2n - |I|,$$

where $I = \{1 \le i \le n \mid e_i > 1\}$ as in 12.2, hence

$$\sum_{i=1}^{n} \lambda_i = 12 - 3n + |I|.$$

That is, the sum of the λ_i coincides with the sum of the a_i from 12.2. Using mutation, one can trace the a_i and the λ_i to see that indeed $a_i = \lambda_i$ for every i, though we will leave the verification to the reader.

Appendix: Toric surfaces

A toric surface is a normal algebraic surface X on which an algebraic torus $T \cong (\mathbb{K}^*)^2$ acts such that T embeds into X as an open dense orbit and the group action extends the multiplication on T. In this appendix we want to remind the reader on standard material on toric surfaces as it can be found in standard textbooks such as [CLS11]. However, we will need to paraphrase some of the material in order to suit its use in the main text.

Let us denote $M \cong \mathbb{Z}^2$ the character group of X and N its dual module. We denote $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. The complement of T in X (if nonempty) is given by a union of divisors $D_1 \cup \cdots \cup D_n$. We are interested in three cases:

(1) X is affine and has a fixpoint with respect to the torus action.
(2) X can be covered by two affine toric varieties and has two fixpoints.
(3) X is complete.

Each of these cases is completely determined by a collection of primitive lattice vectors $l_1, \dots, l_n$ in N. In every case, we will write L for the matrix whose rows are given by the l_i which we will interpret as $\mathbb{Z}$-linear map from M to $\mathbb{Z}^n$.

A.1 (The affine case). In the affine case, we have $n = 2$ and the T-fixpoint is $D_1 \cap D_2$, where D_1 and D_2 both are isomorphic to $\mathbb{K}$. The vectors l_1 and l_2 generate over $\mathbb{Q}_{\ge 0}$ a strictly convex polyhedral cone in $N_{\mathbb{Q}}$. In general, the fixpoint of X is a quotient singularity, i.e. $X \cong \mathbb{K}^2/\mu_v$, where $\mu_v = \operatorname{spec} \mathbb{K}[G]$ the abelian group scheme over $\mathbb{K}$ corresponding to the cyclic group $G \cong N/(\mathbb{Z}l_1 + \mathbb{Z}l_2) \cong \mathbb{Z}/v\mathbb{Z}$.

As l_1 and l_2 are primitive, one can choose a suitable basis for N such that $l_1 = (1, 0)$ and $l_2 = (-k, v)$, where $\gcd\{k, v\} = 1$ and $0 < k < v$. In the case that $\operatorname{char} \mathbb{K}$ and v are coprime, this yields a customary representation for the action of μ_v on $\mathbb{K}^2$ as $\operatorname{diag}(\eta, \eta^k)$, where $\eta \in \mathbb{K}$ is a v-th root of unity. We also use the notation $\frac{1}{v}(1, k)$ to denote a toric singularity which up to choice of coordinates can be represented in this way. Note that $v = \det(l_1, l_2)$, which equals the lattice volume of the parallelogram spanned by l_1 and l_2. We will therefore very often refer to $\det(l_1, l_2)$ as the *volume* of l_1 and l_2. A useful invariant for us will the lattice vector $l_2 - l_1$:

Definition A.2: We call $l_2 - l_1$ the *circumference segment* of the pair l_1, l_2. We call $\frac{1}{\det(l_1, l_2)}(l_2 - l_1)$ the *reduced* circumference segment.

Note that the circumference segment $l_2 - l_1$ in general is not primitive and the reduced segment in general is not integral.

Lemma A.3: *Assume that μ_v acts on $\mathbb{K}^2$ with weights $\frac{1}{v}(1, k)$. Then the lattice length of $l_2 - l_1$ is* $\gcd\{v, k+1\}$.

Proof. With above coordinate representation, we get $l_2 - l_1 = (-k-1, v) = g((-k-1)/g, v/g) =: gP$, where $g = \gcd\{k+1, v\}$ and P is a primitive lattice vector. □

A.4 (The minimal linearly dependent case). Let l_1, l, l_2 be three primitive lattice vectors such that l_1 and l_2 lie in opposite half spaces with respect to the line $\mathbb{Q}l$. Then l_1, l and l, l_2 generate strictly convex polyhedral cones in $N_\mathbb{Q}$ which intersect in the common facet $\mathbb{Q}_{\geq 0} l$ and therefore generated a fan corresponding to a two-dimensional toric surface which is covered by two affine toric surface each of which contains a single torus fixpoint. We choose an orientation on $N_\mathbb{Q}$ such that l_1, l, l_2 are ordered counter-clockwise. Then they satisfy a relation

$$wl_1 + al + vl_2 = 0,$$

where $v = \det(l_1, l)$, $w = \det(l, l_2)$, and $a = \det(l_2, l_1)$. This relation is unique up to a common scalar multiple of the coefficients. If $a = 0$, then $v_1 = w_1$ and $l_1 = -l_2$. We have the two circumference segments $p_1 = l - l_1$ and $p_2 = l_2 - l$ and the corresponding reduced circumference segments $q_1 = \frac{1}{v} p_1$ and $q_2 = \frac{1}{w} p_2$. We observe:

$$\det(p_1, p_2) = \det(l - l_1, l_2 - l) = a + v + w, \qquad \det(q_1, q_2) = \frac{1}{vw}(a + v + w).$$

It follows that the sum $a + v + w$ determines the convexity of the configuration of lattice vectors. That is, l is not contained in the convex hull of l_1, l_2 and the origin if and only if $a + v + w > 0$. The vectors l_1, l, l_2 lie on a line in $N_\mathbb{Q}$ if and only if $a + v + w = 0$.

Lemma A.5: *Let $a_1, a_3 > 0$ and a_2 be integers and $l_1, l_2, l_3 \in N$ primitive such that $a_{\pi(1)} = (\operatorname{sgn} \pi) \cdot \det(l_{\pi(2)}, l_{\pi(3)})$ for any permutation $\pi \in S_3$. Then*

(i) $a_1 l_1 + a_2 l_2 + a_3 l_3 = 0$,
(ii) $\gcd\{a_i, a_j\} = \gcd\{a_1, a_2, a_3\}$ *for any* $1 \leq i \neq j \leq 3$,
(iii) The l_i are uniquely determined up to a transformation by $\mathrm{GL}_\mathbb{Z}(N)$.

Proof. We have seen (i) already in A.4.

(ii) Without loss of generality, we assume that $i = 1$, $j = 2$. Clearly, $\gcd\{a_1, a_2, a_3\}$ divides $\gcd\{a_1, a_2\} =: g$. Now we write $a_1 l_1 + a_2 l_2 = -a_3 l_3$. Clearly, g divides the left hand side and hence the right hand side. But l_3 is primitive, so g cannot divide l_3, hence g divides a_3 and thus g divides $\gcd\{a_1, a_2, a_3\}$.

(iii) Up to a choice of basis for N, we can write $l_1 = (1, 0)$, $l_2 = (x, a_3)$, $l_3 = (y, -a_2)$, where $-a_3 < x < 0$ and y is determined by the relation $a_1 + a_2 x + a_3 y = 0$. We set $a_i =: a_i' g$, where $g := \gcd\{a_1, a_2, a_3\}$. Then the a_i' are pairwise coprime by (ii) and equation $a_1' + a_2' x + a_3' y = 0$ holds as well. Then the set of solutions (x, y) is given by the set $(x_0', y_0') + k(-a_3', a_2')$, where $k \in \mathbb{Z}$ and (x_0, y_0) is some special solution. Multiplying by by g, the solutions are given by the set $(x_0, y_0) + k(-a_3, a_2)$ or $k \in \mathbb{Z}$ and (x_0, y_0) some special solution. It follows that the condition $-a_3 < x < 0$ determines x (and therefore y) uniquely. □

A.6 (The complete case). In this case the D_i form a cycle of $\mathbb{P}^1$'s and there are n torus fixpoints which are given by the intersections $D_i \cap D_{i+1}$. Here it is customary to consider in this case the integers $1, \ldots, n$ as a system of representatives for $\mathbb{Z}/n\mathbb{Z}$, i.e. we read the indices modulo n. The Chow group of X is determined by the following standard short exact sequence

$$0 \longrightarrow M \xrightarrow{L} \mathbb{Z}^n \longrightarrow \mathrm{CH}^1(X) \longrightarrow 0,$$

such that the i-th standard basis vector of $\mathbb{Z}^n$ maps to the class of D_i in $\mathrm{CH}^1(X)$. The canonical divisor on X can be represented by $K_X = -\sum_{i=1}^n D_i$. On $\mathrm{CH}^1(X)$, there exists a $\mathbb{Q}$-valued intersection product which is completely determined by triple relations as in A.4. That is, if for every i we denote $v_i := \det(l_{i-1}, l_i)$ and $a_i := \det(l_{i+1}, l_{i-1})$, then we have for every i the following relation:

$$v_{i+1} l_{i-1} + a_i l_i + v_i l_{i+1} = 0, \qquad \text{respectively} \qquad \frac{1}{v_i} l_{i-1} + \frac{a_i}{v_i v_{i+1}} l_i + \frac{1}{v_{i+1}} l_{i+1} = 0,$$

which translates to intersection products:

$$D_{i-1}D_i = \frac{1}{v_i}, \quad D_i^2 = \frac{a_i}{v_i v_{i+1}}, \quad D_i D_{i+1} = \frac{1}{v_{i+1}} \qquad \text{for every } i$$

and $D_i D_j = 0$ else.

For any i, we have a circumference segments $p_i := l_i - l_{i-1}$ and its reduction $q_i := \frac{p_i}{v_i}$. As in A.4, we have equations

$$\det(p_i, p_{i+1}) = a_i + v_i + v_{i+1} \quad \text{and} \quad \det(q_i, q_{i+1}) = \frac{1}{v_i v_{i+1}}(a_i + v_i + v_{i+1}) = D_i(D_{i-1} + D_i + D_{i+1})$$

for every i.

Lemma A.7: *We have* $K_X^2 = \sum_{i=1}^n \det(q_i, q_{i+1}) = \sum_{i=1}^n \frac{a_i + v_i + v_{i+1}}{v_i v_{i+1}}$.

Proof. By above discussion and $K_X = -\sum_{i=1}^n D_i$. □

Note that in the case that X is smooth, we have $v_i = 1$ for every i and this formula specializes to $K_X^2 = 12 - n = 2n + \sum_{i=1}^n a_i$ or equivalently, $\sum_{i=1}^n a_i = 12 - 3n$.

A.8. The minimal resolution of a toric surface singularity can be described with help of Hirzebruch-Jung continued fractions. That is, in the situation of A.1, we have $l_1 = (1, 0)$ and $l_2 = (-k, v)$ and we write the quotient $\frac{v}{k} = [b_1, \ldots, b_r] := b_1 - 1/(b_2 - 1/(b_3 - \cdots /(b_{r-1} - 1/b_r) \cdots))$ and the b_i are integers ≥ 2. Now, in the first step of the resolution, we introduce a new primitive vector $(0, 1)$ which subdivides the cone into smaller cones of volumes 1 and k, respectively. Ultimately, this new vector will correspond to a component with self-intersection number $-b_1$ of the the exceptional divisor in the minimal resolution. After this first step we see that the fraction $\frac{v}{k}$ can be interpreted as the ratio of the volume of the original cone and the volume of the (possibly still) singular cone after the first resolution step. Iterating this, we see that the continued fraction $[b_i, \ldots, b_r]$ equals the ratio of volumes V_{i-1}/V_i, where V_i is the volume of the remaining singular cone after the i-th resolution step (we set $V_0 := v$ and $V_1 := k$, and it follows that $V_r = 1$).

We now are interested in the behaviour of the canonical self-intersection number under minimal resolutions. That is, we consider a complete toric surface X which has a singular point and $Y \to X$ a toric morphism which is the result of $s \leq r$ steps of the minimal desingularization procedure for this point as described in A.8.

Proposition A.9: *With notation of A.8, we have* $K_X^2 - K_Y^2 = \sum_{i=1}^s \frac{(V_i - V_{i-1} + 1)^2}{V_{i-1} V_i}$.

Proof. We consider the very first step of of the resolution, i.e. we factor $Y = Z_s \to \cdots \to Z_1 \to Z_0 = X$, where every map $Z_i \to Z_{i-1}$ is a partial resolution step which precisely adds one more toric divisor according to the procedure of A.8. For the first step, we use the presentation of K_X^2 and $K_{Z_1}^2$ of Lemma A.7. Up to cyclic renumbering and a choice of basis of N, we can assume that the singular point on X being resolved is represented by the cone generated by $l_2 = (1, 0)$ and $l_3 = (-k, v_2)$ and one primitive vector $l = (0, 1)$ is added as described in A.8. Using the relations $v_{i+1} l_{i-1} + a_i l_i + v_i l_{i+1} = 0$, we compute $l_1 = (\frac{v_2 k - a_2}{v_3}, -v_2)$ and $l_4 = (\frac{a_3 k - v_4}{v_3}, -a_3)$ (note that these will coincide if $n = 3$). Then we immediately obtain the relations

$$l_1 + a l_2 + v_2 l = 0 \text{ with } a = \frac{a_2 - v_2 k}{v_3} \quad \text{and} \quad v_4 l + b l_3 + k l_4 = 0 \text{ with } b = \frac{a_3 k - v_4}{v_3}.$$

Now, the effect of the canonical self-intersection number is local in the sense that, if we write $K_X^2 =: R + \frac{a_2 + v_2 + v_3}{v_2 v_3} + \frac{a_3 + v_3 + v_4}{v_3 v_4}$, then

$$\begin{aligned} K_{Z_1}^2 - R &= \frac{a + v_2 + 1}{v_2} + \frac{-v_2 + 1 + k}{k} + \frac{b + k + v_4}{k v_4} \\ &= \frac{a_2 + v_2 + v_3}{v_2 v_3} + \frac{a_3 + v_3 + v_4}{v_3 v_4} + 2 + \frac{2}{k} - \frac{k}{v_3} - \frac{v_3}{k} - \frac{1}{v_3 k} - \frac{2}{v_3} \\ &= K_X^2 - R - \frac{(k - v_3 + 1)^2}{k v_3}. \end{aligned}$$

Iterating this procedure implies the assertion. □

It is useful to understand also how the terms $\det(q_i, q_{i+1})$ behave under resolution of singularities.

Corollary A.10: *Let l_1, l_2, l_3 and w, a, v as in A.4 such that l_1 and l_2 generate a singular cone and assume that we have a minimal resolution corresponding to a continued fraction $[b_1, \ldots, b_r]$. If we denote $v =: V_0, V_1, \ldots V_s$ for $s \leq r$ the associated volumes and $\xi_1, \ldots, \xi_s$ the successively added primitive vectors, then we have a relation*

$$w\xi_s + bl_2 + V_s l_3 = 0 \quad \text{such that} \quad \frac{a+v+w}{vw} - \frac{b+V_s+w}{V_s w} = \sum_{i=1}^{s} \frac{V_i - V_{i-1} + 1}{V_{i-1}V_i}.$$

Proof. As in the proof of A.9 it is enough to consider one iteration step. We have seen that we can write $l_1 = (1, 0)$, $l_2 = (-V_1, w)$, $l_3 = (y, -a)$ and thus we obtain the equation $y = \frac{1}{v}(aV_1 - w)$. It follows that for $\xi_1 = (0, 1)$ we have a relation $w\xi_1 + bl_2 + V_1 l_3$, where $b := y$. By plugging in b, we get $\frac{a+V_0+w}{V_0 w} - \frac{b+V_1+w}{V_1 w} = \frac{V_1 - V_0 + 1}{V_0 V_1}$. □

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