

GENERATING SEMISIMPLE GROUPS BY TWO ONE-PARAMETER SEMIGROUPS

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ABSTRACT. Let G be a simple Lie group with finite center. We show that G can be generated as a semigroup by a one-parameter subsemigroup X^+ and one additional element g . In fact, given one of the two, a non-constant X^+ or a non-central g there is a g , respectively X^+ , such that the two together generate G as a semigroup. It follows that given a non-constant one-parameter subsemigroup X^+ of G there is another one-parameter subsemigroup Y^+ such that the two generate G as a semigroup. Similarly, given a non-central element g of G there is an element h of G such that the two generate a dense subsemigroup of G . We have analogous results for semisimple Lie groups. For $SL(2, \mathbb{R})$ and its universal cover we spell out when two one-parameter subsemigroups generate the group as a semigroup. On the way we prove results on open subsets of subsemigroups and on exponentiality, which may be of independent interest.

1. INTRODUCTION AND RESULTS

Every real or complex semisimple Lie algebra \mathfrak{g} can be generated by two of its elements. This is an old result of Kuranishi [Ku1, Ku2]. If \mathfrak{g} is simple then given a non-zero element $X \in \mathfrak{g}$ there is an element $Y \in \mathfrak{g}$ such that X and Y generate \mathfrak{g} , [I]. This property is called 1.5-generatedness. In control theory the question arises if a semisimple group G can be generated as a semigroup by two one-parameter subsemigroups. The main result of [AGK] gives a (generic) sufficient condition for a one-parameter subsemigroup and a one-parameter *subgroup* of G to generate G as a semigroup.

Another old result, also proved by Kuranishi [Ku3], is that any semisimple Lie group G has a dense subgroup with two generators. For the case of a compact semisimple Lie group this was already proved in 1935 by Auerbach [Au]. In this case the two group elements actually generate a dense subsemigroup, because the closure of a subsemigroup of a compact group is actually a subgroup. In a – not necessarily compact – semisimple Lie group elements close to the identity generate a dense subgroup if and only if their logarithms generate the Lie algebra,

see [Ku3]. One can ask the corresponding questions for group elements not near the identity and for subsemigroups. Field and Rowley conjectured that 1.5-generatedness holds for compact simple Lie groups without center, see [Fi]. In that paper Field gives an elementary proof of the fact that for a compact semisimple Lie group G the pairs (x, y) in $G \times G$ which generate a dense subgroup form a non-empty Zariski open subset. This is not true for non-compact semisimple groups, see [AV, remark 3.9].

In the joint paper [AV] with Ernest B. Vinberg we show that every semisimple Lie group G with finite center can be generated by two one-dimensional compact tori, as a group or as a semigroup, there is no difference. In fact, we show that given a reasonable element g of G there is a one-dimensional compact torus T such that $T \cup gTg^{-1}$ generates G as a semigroup or as a group. We recall this result below, see theorem 2.1. After hearing about these results M. Kapovich asked if one can generate G as a semigroup by two other one-parameter subsemigroups, e.g. $SL(2, \mathbb{R})$ by two hyperbolic one-parameter subsemigroups. For $SL(2, \mathbb{R})$ the answer was known, see the discussion below in 5.3.

The following main results of this paper give a complete answer to the existence question for semisimple Lie groups with finite center. They are consequences of the main result of [AV], except for theorem 1.2 a), which needs additional considerations. These are contained in sections 3 and 4. Section 3 deals with submanifolds of semigroups, in particular the question when a semigroup contains an open subset of the ambient Lie group. Section 4 deals with exponential elements, namely to find criteria when an element of an algebraic group is contained in a one-parameter subgroup. The results of these two sections may be of independent interest. In the final section 5 we determine precisely when for two elements of the Lie algebra of $SL(2, \mathbb{R})$ the corresponding one parameter semigroups generate $SL(2, \mathbb{R})$ as semigroup and when they generate the universal cover of $SL(2, \mathbb{R})$ as semigroup.

We now state the main results of this paper. For an element X of the Lie algebra \mathfrak{g} of the Lie group G let us denote by X^+ the corresponding one-parameter subsemigroup $X^+ := \{\exp tX, t \geq 0\}$ of G . All the Lie groups we consider will be connected, unless special warning is given.

Theorem 1.1. *Let G be a semisimple Lie group with finite center. Let X be an element of \mathfrak{g} which is contained in no proper ideal of \mathfrak{g} .*

a) There is an element h of G arbitrarily close to $\exp(-X)$ such that h and X^+ generate G as a semigroup.

b) *There is an element Y of \mathfrak{g} , arbitrarily close to $-X$ such that the two one-parameter subsemigroups X^+ and Y^+ generate G as a semigroup.*

This theorem will be deduced from part b) of the following theorem.

Theorem 1.2. *Let G be a semisimple Lie group with finite center. Let g be an element of G not contained in any proper normal subgroup of G .*

a) *There is an element X of \mathfrak{g} such that X^+ and g generate G as a semigroup.*

b) *There is an element h of G such that the subsemigroup of G generated by g and h is actually a group and dense in G . In fact, every neighborhood of g^{-1} contains such an element h .*

If G is a simple Lie group with finite center, the condition on X in theorem 1.1 just means that X is non-zero. The condition on g in theorem 1.2 means for simple G that g is not central in G , and for semisimple G that $Ad(g)$ does not act trivially on any of the simple ideals of \mathfrak{g} . The condition on X in theorem 1.1, respectively on g in theorem 1.2, is obviously necessary for the existence statements.

2. PROOFS

We recall the main result of [AV] , namely theorem 3.1.

Theorem 2.1. *Let G be a semisimple Lie group with finite center and let g be an element of G which is not contained in any proper normal subgroup of G . Then there is a one-dimensional compact torus T in G such that $T \cup gTg^{-1}$ generates G as a semigroup or as a group, there is no difference. For every such T the subsemigroup of G generated by $\{t, gtg^{-1}\}$ is dense in G for every t in T except for a finite set of torsion elements.*

Note the misprint in [AV, theorem 3.1]. The subsemigroup, say S , of G generated by $\{t, gtg^{-1}\}$ is not a subgroup of G , in general, unless t is a torsion element. But the closure of S is a subgroup of G , since $t \in T$. It follows that the closure of S coincides with the closure of the subgroup generated by $\{t, gtg^{-1}\}$.

The proofs of theorems 1.2, b) and 1.1 are immediate consequences of theorem 2.1. For the proof of 1.2, a) we need some auxiliary results which we postpone to the next sections.

Proof of theorem 1.2.

b) Let g be an element of G not contained in any proper normal subgroup of G and let T be a torus as in theorem 2.1. Let then t be a

torsion element of T close enough to the identity element so that the statement of theorem 2.1 holds for t . Put $h = tg^{-1}$. Let H be the subsemigroup of G generated by g and h . Then H contains t , hence t^{-1} , since t is a torsion element and consequently H also contains g^{-1} . It follows that H is the subgroup of G generated by g and t . So H is dense in G by theorem 2.1.

a) First of all, it suffices to prove the theorem for the case that $G = Ad(G)$, for the following reason. Suppose $Ad(g)$ and $X^+ \subset Ad(G)$ generate $Ad(G)$ as a semigroup. We claim that then g and $X^+ \subset G$ generate G as a semigroup. The subsemigroup S of G generated by g and X^+ contains an open subset U of G , by section 3, corollary 3.3, say $h \in U \subset S$, and $Ad(S) = Ad(G)$ contains $Ad(h^{-1})$, say $Ad(s) = Ad(h^{-1})$ with $s \in S$. Then S contains the neighborhood sU of the element sh of the finite center of G , hence $S = G$, since then the semigroup S contains a neighborhood of the identity element and thus $S = G$. So we may suppose that $G = Ad(G)$. Let again g and T be as in the proof of b). Then there is a positive natural number d such that g^{-d} is exponential and so are all elements of the form $g^{-d}t$ with $t \in T$ close enough to the identity element, except for a countable set of $t \in T$, see section 4, corollary 4.5. In particular, there are non-identity elements $t \in T$ arbitrarily close to the identity with the property that tg^{-d} is contained in a one-parameter (semi-)group X^+ . Let S be the subsemigroup of G generated by g and X^+ . We claim that $S = G$. First of all, S is dense in G , since S contains t , so the closure \bar{S} of S contains t^{-1} , hence contains g^{-d} , hence contains g^{-1} , and thus is dense in G . The corresponding element X of \mathfrak{g} cannot be contained in any proper ideal of \mathfrak{g} , since otherwise X^+ would be contained in the corresponding proper normal subgroup H of G and g would generate a dense subsemigroup of the factor group G/H . But a semisimple group has no abelian non-trivial factor group. It follows from section 3 corollary 3.3 that S contains a non-empty open subset of G . So the semigroup S is dense in G and contains a non-empty open subset of G , hence $S = G$. □

Proof of theorem 1.1.

a) Let g be an element of the one-parameter subsemigroup $X^+ = \{\exp tX, t \geq 0\}$ not contained in any proper normal subgroup of G . Note that we can choose $g = \exp(X)$, unless the image of $\exp(X)$ is central in some non-identity factor group of G . Then there is arbitrarily close to g^{-1} an element h of G such that the subsemigroup H of G generated by g and h is a group and dense in G , by theorem 1.2 b). Let S be the subsemigroup of G generated by X^+ and h . We claim

that $S = G$. The semigroup S contains the group H , hence is dense in G . From this point on the proof can be completed in two ways, either by invoking the corollary 3.3 of the next section, as in the proof of 1.2 a) or, avoiding this, as follows: The semigroup S is actually a group for the following reasons. S contains the one-parameter subgroup $\{\exp tX, t \in \mathbb{R}\}$, since S contains the one-parameter semigroup X^+ and the element $g^{-1} \in (X^+)^{-1}$. And S contains the group H , in particular h and h^{-1} . So S contains the inverses of its generators X^+ and $h \in H$ and is hence a subgroup of G . It follows that S contains the one-parameter subgroup with infinitesimal generator X and all its conjugates under the dense subgroup S of G . This implies that $S = G$, since the subgroup S_1 of S of all those elements of S which can be connected to the identity element by a piecewise analytic arc in S form a Lie subgroup of G (see e.g. [KoNo1], Appendix 4) whose Lie algebra \mathfrak{s}_1 is an ideal in \mathfrak{g} and contains X , which is contained in no proper ideal of \mathfrak{g} , hence $\mathfrak{s}_1 = \mathfrak{g}$ and thus $S_1 = G$.

b) We may assume that X is small, namely contained in an open convex symmetric neighborhood U of 0 such that the exponential map $\exp : U \rightarrow G$ is a diffeomorphism from U onto an open neighborhood of the identity in G . Let h be as in a) and let $Y \in U$ be such that $\exp(Y) = h$. Then Y fulfills our claim. \square

3. SUBMANIFOLDS OF SEMIGROUPS

In this section we determine the smooth submanifolds of maximal dimension contained in a subsemigroup S of a Lie group G . They are local cosets of a certain Lie subgroup of G , which we call H . As an application we give a criterion when S contains an open subset of G . This is used in the proof of theorem 1.2 a).

Proposition 3.1. *Let S be a non-empty subsemigroup of the Lie group G . Then there is a connected, not necessarily closed, Lie subgroup H of G with the following properties.*

- a) H is normalized by the closure of the subgroup generated by S .
- b) S contains a (say left) translate of an open subset of H .
- c) Given a smooth map $f : M \rightarrow G$ of a connected smooth manifold M to G such that $f(M) \subset S$. Then $f(m_1)^{-1}f(m_2) \in H$ for every pair of points m_1, m_2 in M .
- d) There is only one connected Lie subgroup H of G with the properties b) and c). The group H is generated by the following set of elements

$$\{c(t_1)^{-1}c(t_2); t_1, t_2 \text{ in } I \text{ where } c : I \rightarrow G \text{ is a smooth curve with } c(I) \subset S\}.$$

Proof Let \mathcal{M} be the set of pairs (s, \mathfrak{m}) , where $s \in S$, \mathfrak{m} is a vector subspace of the Lie algebra \mathfrak{g} of G and there is a smooth submanifold M of G contained in S such that $s \in M$ and the tangent space $T_s M$ is the left translate of \mathfrak{m} , so $T_s M = dL_s(\mathfrak{m})$, where $dL_s : \mathfrak{m} \rightarrow T_s M$ is the tangent map at the identity of the left translation $L_s : G \rightarrow G, x \mapsto sx$. Let now d be the maximal dimension of subspaces \mathfrak{m} occurring as second component in \mathcal{M} . We claim that for every $(s, \mathfrak{m}) \in \mathcal{M}$ with $\dim(\mathfrak{m}) = d$ the subspace \mathfrak{m} of \mathfrak{g} is the same and is a Lie algebra \mathfrak{h} and that the corresponding Lie group H will have the properties of the proposition. To prove this we start with

(i) If $(s_i, \mathfrak{m}_i) \in \mathcal{M}$ for $i = 1, 2$ then

$$(s_1 s_2, Ad(s_2^{-1})\mathfrak{m}_1 + \mathfrak{m}_2) \in \mathcal{M}.$$

To see this, let M_i be smooth submanifolds of G such that $s_i \in M_i \subset S$ and $dL_{s_i} \mathfrak{m}_i = T_{s_i} M_i$ for $i = 1, 2$. The tangent map at (e, e) of the multiplication map $G \times G \rightarrow G, (x, y) \mapsto xy$ is the addition map of the Lie algebra $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}, (X, Y) \mapsto X + Y$. It follows that the tangent map at (s_1, s_2) of the multiplication map $M_1 \times M_2 \rightarrow M_1 M_2, (m_1, m_2) \mapsto m_1 m_2 = s_1 s_2 (s_2^{-1} (s_1^{-1} m_1) s_2) (s_2^{-1} m_2)$ sends $(dL_{s_1} X, dL_{s_2} Y)$ to $dL_{s_1 s_2} (Ad(s_2^{-1})X + Y)$ for X, Y in \mathfrak{g} . Hence if we take a submanifold M'_2 of M_2 such that $dL_{s_2}^{-1} T_{s_2} M'_2 =: \mathfrak{m}'_2 \subset \mathfrak{m}_2$ is a complementary vector subspace of $Ad(s_2^{-1})\mathfrak{m}_1$ in $Ad(s_2^{-1})\mathfrak{m}_1 + \mathfrak{m}_2$, then the image of $M_1 M'_2$ contains a smooth submanifold whose tangent space at $s_1 s_2$ is $dL_{s_1 s_2} (Ad(s_2^{-1})\mathfrak{m}_1 + \mathfrak{m}_2)$. This proves (i).

Let now $(s_i, \mathfrak{m}_i) \in \mathcal{M}$ be such that \mathfrak{m}_i have the maximal dimension d for $i = 1, 2$. Then $Ad(s_2^{-1})\mathfrak{m}_1 + \mathfrak{m}_2$ is also a second component of some element of \mathcal{M} by (i), hence also has dimension d , and consequently

(ii) $\mathfrak{m}_1 = Ad(s_2)\mathfrak{m}_2$.

If we take in particular $(s_1, \mathfrak{m}_1) = (s_2, \mathfrak{m}_2)$ we see that $\mathfrak{m}_1 = Ad(s_1)\mathfrak{m}_1$, whenever \mathfrak{m}_1 has dimension d . It follows then from (ii) that $\mathfrak{m}_1 = \mathfrak{m}_2$, whenever both \mathfrak{m}_1 and \mathfrak{m}_2 are second components of elements of \mathcal{M} and both have dimension d . So any two \mathfrak{m} that occur as second component in \mathcal{M} and have maximal dimension d are equal. Let us denote this vector subspace of \mathfrak{g} by \mathfrak{h} .

If in (i) we take $\mathfrak{m}_1 = \mathfrak{h}$ and $(s_2, \mathfrak{m}_2) \in \mathcal{M}$ arbitrary, we see that

(iii) $Ad(s^{-1})\mathfrak{h} + \mathfrak{m} = \mathfrak{h}$, whenever $(s, \mathfrak{m}) \in \mathcal{M}$,

since \mathfrak{h} has maximal dimension. This implies two facts, namely

(iv) $\mathfrak{m} \subset \mathfrak{h}$, whenever $(s, \mathfrak{m}) \in \mathcal{M}$

and

(v) $Ad(s)\mathfrak{h} = \mathfrak{h}$ for every $s \in S$.

So \mathfrak{h} is not only of maximal dimension but also contains $dL_s^{-1} T_s M$ for every submanifold M of G such that $s \in M \subset S$. And (v) says

that S normalizes \mathfrak{h} , hence so does the subgroup generated by S and hence also its closure. In fact

(vi) the Zariski closure of $Ad(S)$ in $GL(\mathfrak{g})$ normalizes \mathfrak{h} .

We next show that \mathfrak{h} is a Lie algebra. For every $X \in \mathfrak{h}$ there is a smooth curve $c : I \rightarrow G$ which is defined on an open interval I that contains 0, has image in S and whose tangent at 0 is $dL_{c(0)}(X)$. Then $Ad(c(t))(Y) \in \mathfrak{h}$ for every $t \in I$ and $Y \in \mathfrak{h}$, by (v). Set $s = c(0)$. It follows again from the $Ad(S)$ -invariance of \mathfrak{h} that \mathfrak{h} also contains $Ad(s^{-1}c(t))(Y)$. This implies that \mathfrak{h} also contains the tangent vector at 0 of the curve $t \mapsto Ad(s^{-1}c(t))(Y)$, which is $[X, Y]$, since the tangent map of $Ad : G \rightarrow GL(\mathfrak{g})$ at the identity is $ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$.

Let now H be the connected Lie subgroup of G corresponding to \mathfrak{h} . We claim that H has all the properties stated in the proposition. Clearly, H is normalized by the closure of the subgroup of G generated by S , since \mathfrak{h} is so. Let $c : I \rightarrow G$ be a smooth curve defined on some open interval I containing 0. If $c(t) \in S$ for every $t \in I$, then $dL_{c(t)}^{-1}\dot{c}(t) \in \mathfrak{h}$ for every $t \in I$ by (iv) and hence $c(I)$ is contained in one integral manifold of the distribution which associates with every $g \in G$ the vector subspace $dL_g\mathfrak{h}$, and the corresponding integral manifolds are precisely the left cosets of H . So $c(t_1)^{-1}c(t_2) \in H$ for every two numbers t_1, t_2 in I . This implies c), since M is connected by smooth curves.

Conversely, there is a smooth submanifold M of G contained in S whose tangent space at some point s has dimension d . Then the connected component of s in M has dimension d everywhere, is contained in sH by the arguments of the last paragraph and has the same dimension as sH , hence is an open subset of this coset.

We thus have proved a) through c). Concerning d): Let H_1 be a connected Lie subgroup of G with the properties b) and c). Then H_1 contains $f(M)^{-1}f(M) := \{f(m_1)^{-1}f(m_2); m_1, m_2 \text{ in } M\}$ for every smooth map f from a connected manifold M to G such that $f(M) \subset S$, by c) for H_1 . In particular, H_1 contains $(sU)^{-1}sU$ for some $s \in S$ and some open connected neighborhood U of e in H , by b) for H , hence $H \subset H_1$, since every connected Lie group is generated as a group by every neighborhood of the identity. On the other hand S contains sU for some $s \in S$ and some open connected neighborhood U of e in H_1 , by b) for H_1 . Hence $(sU)^{-1}sU$ is contained in H , by c) for H , and thus $H_1 \subset H$. This also shows the second claim of d), namely to generate H it suffices to use smooth curves, since every connected smooth manifold is connected by smooth curves. □

The main application of our proposition we had in mind were criteria for when a subsemigroup of a Lie group G contains an open subset of G . Easiest to state is the following striking corollary for simple groups.

Corollary 3.2. *Let S be a subsemigroup of a simple Lie group G . Suppose that (the subgroup generated by) $Ad(S)$ is Zariski dense in $Ad(G)$. If there is a non-constant smooth curve contained in S then S contains a non empty open subset of G .*

Proof The Lie algebra \mathfrak{h} of the group H of the proposition is normalized by S , hence by $Ad(S)$, hence by its Zariski closure, which is supposed to contain $Ad(G)$. So \mathfrak{h} is an ideal in \mathfrak{g} which by hypothesis is non-zero, so $\mathfrak{h} = \mathfrak{g}$ and thus $H = G$. □

For semisimple groups we obtain the following result with virtually the same proof.

Corollary 3.3. *Let S be a subsemigroup of a semisimple Lie group G . Suppose that (the subgroup generated by) $Ad(S)$ is Zariski dense in $Ad(G)$. Then S contains a non empty open subset of G if the set of $dL_{c(t)}^{-1}\dot{c}(t)$ for smooth curves c with values in S is not contained in any ideal of the Lie algebra of G .*

For general Lie groups we have:

Corollary 3.4. *Let G be a Lie group. Let E be a subset of the Lie algebra \mathfrak{g} of G which generates \mathfrak{g} as a Lie algebra. Let S be the subsemigroup of G generated by the one-parameter subsemigroups X^+ corresponding to the elements $X \in E$. Then S contains a non empty open subset of G .*

Proof The Lie algebra \mathfrak{h} of the Lie group H of the proposition contains E , hence $\mathfrak{h} = \mathfrak{g}$ and thus $H = G$. □

Remark 3.5. A key step in the proof of proposition 3.1 is assertion (i). In the proof we used only the information about the second component \mathfrak{m} . If we also use the information about the first component we obtain that under the hypotheses of the last corollary the interior of S is dense in S . This is also a consequence of the results of [HR].

4. EXPONENTIAL ELEMENTS

We shall give a sufficient condition for when an element of a linear algebraic group is exponential. The main result of this section is the stability result proposition 4.3. It is applied in the proof of theorem 1.2 a).

Recall that an element of a Lie group G is called *exponential* if it is contained in a one-parameter sub(semi)group of G . We shall use the following sufficient condition for an element to be exponential. Let G be a linear algebraic subgroup of $GL(m, \mathbb{R})$ over \mathbb{R} . For an element $g \in G$ let \widehat{g} be the Zariski closure of the sub(semi)group generated by g . Let us say that g has property (E) if g belongs to the (Euclidean) identity component of \widehat{g} . Every element with property (E) is exponential since every element of a connected abelian Lie group is exponential.

We shall now give a necessary and sufficient condition for g to have property (E) in terms of its eigenvalues, see lemma 4.1. Let $g = su$ be the multiplicative Jordan decomposition of g . So s is semisimple, u is unipotent and $su = us = g$, and u and s are uniquely determined by these conditions. Then the one-parameter group $\exp(t \log u)$ is contained in \widehat{g} . Furthermore one can find unique semisimple elements p and k in $GL(m, \mathbb{R})$ which satisfy $s = kp = pk$, and where the eigenvalues of p are positive and the eigenvalues of k have modulus 1. Then the one-parameter group $\exp(t \log p)$ is contained in \widehat{g} . Let us call k the *compact component* of g . Recall that an element of $GL(m, \mathbb{R})$ is called *elliptic* if it is semisimple and all its eigenvalues are of modulus 1. For an elliptic element k the Zariski closure \widehat{k} is compact and coincides with the Euclidean closure of the (semi)group generated by k . We have $\widehat{g} = \widehat{k}\widehat{p}\widehat{u}$ and the Euclidean identity component \widehat{g}^0 of \widehat{g} is $\widehat{g}^0 = \widehat{k}^0\widehat{p}^0\widehat{u}$. So $g \in \widehat{g}^0$ if and only if $k \in \widehat{k}^0$, since $u \in \widehat{u} = \widehat{u}^0$ and $p \in \widehat{p}^0$.

Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be the eigenvalues of g , counted with algebraic multiplicities. The eigenvalues of k are obtained by replacing each of these numbers λ_i by their argument $\arg(\lambda_i)$, where $\arg(z) = z/|z|$ for every non-zero $z \in \mathbb{C}$. For every $n = (n_1, n_2, \dots, n_m) \in \mathbb{Z}^m$ set $\lambda^n = \lambda_1^{n_1} \lambda_2^{n_2} \dots \lambda_m^{n_m}$. For a given $n \in \mathbb{Z}^m$ there is a unique continuous homomorphism f from \widehat{k} to the group S^1 of complex numbers of modulus 1 satisfying $f(k) = (\arg(\lambda))^n$, and all continuous homomorphisms $f : \widehat{k} \rightarrow S^1$ are of this form. So Pontryagin duality gives the equivalence of c) and d) in the following lemma: We have $k \notin \widehat{k}^0$ if and only if there is a continuous homomorphism $f : \widehat{k} \rightarrow S^1$ such that $f(k)$ is a root of unity different from 1. The remaining properties have already been established.

Lemma 4.1. *Let $g \in GL(m, \mathbb{R})$. Then the following statements are equivalent:*

- a) g has property (E)
- b) $g \in \widehat{g}^0$
- c) $k \in \widehat{k}^0$

d) For every $n \in \mathbb{Z}^m$ the complex number $(\arg(\lambda))^n$ is either equal to 1 or not a root of unity.

e) g is exponential in every linear algebraic \mathbb{R} -subgroup of $GL(m, \mathbb{R})$ that contains g .

Corollary 4.2. For every element $g \in GL(m, \mathbb{R})$ there is a positive natural number n such that g^n has property (E).

Proof For every $n \in \mathbb{Z}$ we have the decomposition $g^n = k^n p^n u^n$, if $g = kpu$ is the decomposition of g as above. Furthermore, $\widehat{k}/\widehat{k}^0$ is a finite cyclic group, of order t , say. Then $\widehat{k}^t = \widehat{k}^0 = \widehat{k}^0$, hence g^t has property (E). \square

The main result of this section is the following proposition.

Proposition 4.3. Let $c : I \rightarrow GL(m, \mathbb{R})$ be an analytic map defined on an open interval $I \subset \mathbb{R}$ containing 0. There is positive real number ϵ , such that for the closed interval $J = [0, \epsilon]$ we have the following dichotomy. The element $c(t)$ has property (E) for either no number $t \in J$ or for all $t \in J$ but a countable subset of J . In particular, if $c(0)$ has property (E) then $c(t)$ has property (E) for all t in some (two-sided!) interval $(-\epsilon, \epsilon)$ except for a countable subset.

Proof. We may assume that there is a holomorphic map of some neighborhood D of 0 in \mathbb{C} to $GL(m, \mathbb{C})$ which extends c . We call it c as well. Let Λ be the set of pairs $(z, \lambda) \in D \times \mathbb{C}$ where λ is an eigenvalue of $c(z)$. Then, after again passing to a smaller D , if necessary, Λ is a union of its irreducible components Λ_j , counted with multiplicities, for each of which the projection π to the first component z is a holomorphic branched covering whose only branching point is the point over 0, which is $(0, \lambda_j) \in \Lambda_j$. For the results on complex analysis used here see e.g. [GR] chapter III. If the branching order of $\pi|_{\Lambda_j}$ at $(0, \lambda_j)$ is m_j , then there are m_j holomorphic sections $\lambda_{j,i}, i = 1, \dots, m_j$ for $\pi|_{\Lambda_j}$ defined on some open neighborhood D' of $I^+ = \{t > 0; t \in I\}$ in D . The set D' does not contain 0, but the section $\lambda_{j,i}$ can be continuously extended to $D' \cup \{0\}$ by the continuity of eigenvalues. We call the extended section $\lambda_{j,i}$ as well. For a given $t \in I^+ \cup 0$ the set of all the images of $\lambda_{j,i}(t)$ is the set of all eigenvalues of $c(t)$, counted with multiplicities. We have thus obtained a parametrization of the eigenvalues of $c(t)$ for $t \in I^+ \cup \{0\}$, which is real analytic over I^+ and continuous at 0. Define for every $n \in \mathbb{Z}^m$ the function f_n from $I^+ \cup 0$ to S^1 by putting $f_n(t) = (\arg(\lambda))^n$. Then f_n is real analytic on I^+ and continuous at 0. Now the first case is that for one $n \in \mathbb{Z}^m$ the function f_n is constant with value a root of unity $\neq 1$, then, for every $t \in I^+ \cup 0$ the element $c(t)$ does not have property (E). The second case is that

for every $n \in \mathbb{Z}^m$ the function f_n is constant with value 1 or constant with value an element of S^1 of infinite order or f_n is non-constant. In the latter case the set of $t \in I^+ \cup \{0\}$, for which f_n has a given value is countable. Let F_n be the set of $t \in I^+ \cup \{0\}$ for which $f_n(t)$ is a root of unity $\neq 1$. In the second case the set F_n is at most countable, so the union $F = \cup F_n$ is at most countable as well, and $c(t)$ has property (E) for all $t \in I^+ \cup \{0\}$ not in F . □

Corollary 4.4. *Let G be a linear algebraic group defined over \mathbb{R} and let $c : I \rightarrow G$ be an analytic map defined on some open interval $I \subset \mathbb{R}$. Then for every $t_0 \in I$ there is a neighborhood $J \subset I$ and a natural number $d > 0$ such that $(c(t))^d$ has property (E) – and in particular is exponential – for every $t \in J$ except for a countable subset of t in J .*

Proof. The element $(c(t_0))^d$ has property (E) for some natural number $d > 0$, by corollary 4.2. Now apply proposition 4.3 to the analytic map $c^d : I \rightarrow G$ with $c^d(t) = (c(t))^d$. We are in the second case of the dichotomy since $(c(t_0))^d$ has property (E). □

Remark 4.5. One might conjecture several refinements and generalizations of the preceding proposition. The group $G = SL(2, \mathbb{R})$ furnishes counterexamples to some of them. An element $g \in G$ has property (E) if and only if the trace of g is greater than -2 and not of the form $2 \cos(2\pi/n)$ for some positive natural number n . Thus an analytic curve c in G such that $\text{trace}(c(t)) \leq -2$ for $t \leq 0$ and $\text{trace}(c(t)) > -2$ for $t > 0$ exhibits the phenomenon that the behavior on the right and on the left of 0 are different, namely according to the dichotomy.

Note that an element g of G is exponential in G if and only if g has trace greater than -2 or $g = -1$. Thus, there are analytic curves $c : I \rightarrow G$ such that $c(0) = -1$ is exponential but $c(t)$ is exponential for no other $t \in I$, for instance $c(t) = -1 + tn$ for a nilpotent matrix $n \neq 0$. So the proposition does not hold if we replace property (E) by exponentiality.

5. $SL(2, \mathbb{R})$ AND ITS UNIVERSAL COVER

An interesting question is for which pairs $\{X, Y\}$ of elements of a simple Lie algebra \mathfrak{g} the corresponding one-parameter subsemigroups X^+, Y^+ generate G as a semigroup. If G has finite center, the question is easily reduced to the adjoint group, as in the beginning of the proof of theorem 1.2 a). If G has infinite center, the answer may be quite

different, as we will show now by the example of $SL(2, \mathbb{R})$ and its universal cover.

An essential tool is the action of $G = SL(2, \mathbb{R})$ on the upper half plane and on its boundary $\mathbb{R} \cup \{\infty\}$, a circle. We will think of this circle as the one-dimensional real projective space $\mathbb{P}^1\mathbb{R}$ with the action of $SL(2, \mathbb{R})$ by projective transformations induced from the natural action of $SL(2, \mathbb{R})$ on \mathbb{R}^2 . A hyperbolic element $g \in G$ has exactly two fixed points in $\mathbb{P}^1\mathbb{R}$, one attracting, which we call $Fix^+(g)$, and one repelling, which we call $Fix^-(g)$. A subset of $\mathbb{P}^1\mathbb{R}$ is called an interval if it is homeomorphic to an interval in the real line, equivalently, if it is connected, not the whole circle $\mathbb{P}^1\mathbb{R}$ and contains at least two points. A parabolic element $g \in G$ has exactly one fixed point in $\mathbb{P}^1\mathbb{R}$, called $Fix(g)$. One side of $Fix(g)$ is attracting and the other side is repelling. For parabolic g an interval I in $\mathbb{P}^1\mathbb{R}$ is g -invariant, if and only if one of its endpoints is $Fix(g)$ and it contains an interval near $Fix(g)$ on the attracting side. Finally, an elliptic element in G has no fixed point and no invariant interval. All the nonidentity elements of a one-parameter subsemigroup X^+ of G have the same fixed points and the same behavior concerning attracting and repelling properties and the same invariant intervals. We call the corresponding non zero element $X \in \mathfrak{g}$ accordingly hyperbolic, parabolic or elliptic.

Theorem 5.1. *Let X and Y be two linearly independent elements of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Then the following conditions are equivalent:*

- a) *The corresponding one-parameter subsemigroups X^+ and Y^+ of $G = SL(2, \mathbb{R})$ generate G as a semigroup.*
- b) *There is no interval contained in $\mathbb{P}^1\mathbb{R}$ which is mapped to itself by both X^+ and Y^+ .*
- c) *The segment from X to Y in \mathfrak{g} contains an elliptic element.*

Proof a) implies b), since G acts transitively on $\mathbb{P}^1\mathbb{R}$.

c) implies a). Suppose the segment from X to Y in \mathfrak{g} contains an elliptic element. Then the Lie algebra generated by $\{X, Y\}$ is either 3-dimensional and hence equals $\mathfrak{sl}(2, \mathbb{R})$ or two-dimensional and hence conjugate to the Lie algebra of the Borel subgroup of upper triangular matrices and hence does not contain an elliptic element. Thus $\{X, Y\}$ generates \mathfrak{g} as a Lie algebra. Let S be the subsemigroup of G generated $X^+ \cup Y^+$. Then S contains a non-empty open subset of G , by corollary 3.4. It thus suffices to show that S is dense in G . The closure \bar{S} contains Z^+ for every element Z of the positive hull of $\{X, Y\}$, as follows from the Campbell-Hausdorff formula, see [HL]. The set of elliptic elements of \mathfrak{g} is open in \mathfrak{g} . So \bar{S} contains the group generated by two different one-dimensional compact tori, which is G .

b) implies c). Let X and Y be two linearly independent elements of \mathfrak{g} and suppose that b) holds. So there is no interval $I \subset \mathbb{P}^1\mathbb{R}$ which is mapped to itself by both X^+ and Y^+ . If X^+ and Y^+ have a common fixed point in $\mathbb{P}^1\mathbb{R}$ then the complement in $\mathbb{P}^1\mathbb{R}$ of this fixed point is an invariant interval. So we may assume that X^+ and Y^+ have no common fixed point. For $Z \in \mathfrak{g}$ let $\varphi(Z)$ be the vector field on $\mathbb{P}^1\mathbb{R}$ which associates to every point $x \in \mathbb{P}^1\mathbb{R}$ the tangent vector at 0 of the curve $t \mapsto X^+(t)(x)$. We denote this vector by $\varphi(Z)_x$. A non-zero element $Z \in \mathfrak{g}$ is elliptic if and only if $\varphi(Z)$ has a zero. Let C be the positive hull $C = \{aX + bY; a \geq 0, b \geq 0\}$ of $\{X, Y\}$.

If for some $x \in \mathbb{P}^1\mathbb{R}$ the tangent vectors $\varphi(X)_x$ and $\varphi(Y)_x$ point in the same direction then x cannot be a zero of $\varphi(Z)$ for any $Z \in C$. Even if exactly one of the two vectors $\varphi(X)_x$ or $\varphi(Y)_x$ is non-zero then x cannot be a zero of $\varphi(Z)$ for any $Z \in C$ outside the rays spanned by X and Y . Thus if X and Y are parabolic and $\varphi(X)_x$ and $\varphi(Y)_x$ point in the same direction for one $x \in \mathbb{P}^1\mathbb{R}$ and hence for every x except for the zeroes of $\varphi(X)$ and $\varphi(Y)$, then $\varphi(Z)$ has no zero for any $Z \in C$ outside the rays spanned by X and Y , and hence is elliptic. If on the other hand, X and Y are parabolic and $\varphi(X)_x$ and $\varphi(Y)_x$ point in opposite directions for one $x \in \mathbb{P}^1\mathbb{R}$ and hence for every x except for the zero $Fix(X^+)$ of $\varphi(X)$ and the zero $Fix(Y^+)$ of $\varphi(Y)$, then one of the two intervals with end points $Fix(X^+)$ and $Fix(Y^+)$ is X^+ - and Y^+ -invariant, so b) does not hold for $\{X, Y\}$. Now suppose X is parabolic and Y is hyperbolic. If $\varphi(Y)$ points in the opposite direction of $\varphi(X)$ at points near $Fix(X^+)$ then the interval with end points $Fix(X^+)$ and $Fix^+(Y^+)$, which does not contain $Fix^-(Y^+)$ is X^+ - and Y^+ -invariant, so b) does not hold for $\{X, Y\}$. If $\varphi(Y)$ points in the same direction as $\varphi(X)$ at points near $Fix(X^+)$ then $\varphi(Z)$ has no zero on the interval where $\varphi(Y)$ points in the direction of $\varphi(X)$ including $Fix(X^+)$ except on the ray spanned by X . But if every such Z^+ has a fixed point, this contradicts the continuous dependence of fixed points of Z , since $Fix(X^+)$ is the fixed point of X^+ , the fixed points of Y^+ are both different from $Fix(X^+)$ and there is no fixed point of Z^+ for any non-zero $Z \in C$ near $Fix(X^+)$ other than $Fix(X^+)$ itself. So the segment from X to Y must contain an elliptic element. Finally, suppose both X and Y are hyperbolic. If there is an interval with end points $Fix^+(X^+)$ and $Fix^+(Y^+)$ which contains no other fixed point of X^+ or Y^+ then this interval is X^+ - and Y^+ -invariant. So the remaining case is that for some orientation of $\mathbb{P}^1\mathbb{R}$ the fixed points are in the order $x^+ := Fix^+(X^+)$, $y^- := Fix^-(Y^+)$, $y^+ := Fix^+(Y^+)$, $x^- := Fix^-(X^+)$. Then on the oriented open interval from x^+ to y^- the vector fields $\varphi(X)$ and $\varphi(Y)$ point in the same direction, namely

against the orientation of $\mathbb{P}^1\mathbb{R}$, and the same holds on the oriented open interval from y^+ to x^- . So zeroes of $\varphi(Z)$ for non zero elements $Z \in C$ can only exist in the two remaining closed intervals $[x^-, x^+]$ and $[y^-, y^+]$. But for $Z = X$ the zeroes are in the first interval, for $Z = Y$ they are in the second interval. If $\varphi(Z)$ has a zero for every non zero $Z \in C$ this cannot happen, since the zeroes of $\varphi(Z)$ depend continuously on Z . So there must be an elliptic element on the segment from X to Y . □

The proof yields explicit conditions for when two one-parameter semigroups generate $SL(2, \mathbb{R})$, as follows.

Corollary 5.2. *Let X and Y be two elements of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. The corresponding one-parameter subsemigroups X^+ and Y^+ of $SL(2, \mathbb{R})$ generate $SL(2, \mathbb{R})$ as a semigroup if and only if X and Y are linearly independent, X^+ and Y^+ have no common fixed point and we have one of the following cases:*

- a) X or Y is elliptic.
- b) X and Y are parabolic and $\varphi(X)$ and $\varphi(Y)$ point in the same direction for one point of $\mathbb{P}^1\mathbb{R}$ (equivalently, for every point of $\mathbb{P}^1\mathbb{R}$, where both $\varphi(X)$ and $\varphi(Y)$ don't vanish).
- c) One of the two is parabolic, say X , and the other one, Y , is hyperbolic and $\varphi(X)$ and $\varphi(Y)$ point in the same direction near the zero of $\varphi(X)$.
- d) X and Y are hyperbolic and every interval with endpoints $Fix^+(X^+)$ and $Fix^+(Y^+)$ contains $Fix^-(X^+)$ or $Fix^-(Y^+)$.

Remark 5.3. We do not claim that the results on $SL(2, \mathbb{R})$ are new. The equivalences of 5.1 can be concluded from [HH] Proposition 4.3 and its proof, a result we will now state in a more geometric language.

Theorem 5.4. *[Hilgert and Hofmann] A subsemigroup of $SL(2, \mathbb{R})$ that is generated by one-parameter subsemigroups is either contained in a proper closed connected subgroup of $SL(2, \mathbb{R})$, or maps an interval in $\mathbb{P}^1\mathbb{R}$ to itself or is dense in $SL(2, \mathbb{R})$.*

In [HH] a prominent role is played by the subsemigroup $SL(2, \mathbb{R})^+$ of those matrices in $SL(2, \mathbb{R})$ all of whose entries are ≥ 0 . Geometrically speaking, this is the subsemigroup of those elements of $SL(2, \mathbb{R})$ which map a certain interval I in $\mathbb{P}^1\mathbb{R}$ to itself, namely $I = [0, \infty]$. Let us remark that in the statement of [HH] Proposition 4.3 one case is missing, namely that the semigroup is contained in a Borel subgroup.

For the universal covering group G of $SL(2, \mathbb{R})$ the situation is quite different, as follows.

Theorem 5.5. *Let X and Y be two elements of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Then the corresponding subsemigroups X^+ and Y^+ of G generate G as a semigroup if and only if X and Y are linearly independent, elliptic and belong to different connected components of the set of elliptic elements.*

So G is two-generated by one-parameter semigroups, but not 1.5-generated. Also, G cannot be generated as a semigroup by two conjugate one-parameter semigroups.

Proof The set of elliptic elements of \mathfrak{g} has two connected components. Let us call one of them the positive time cone E^+ and let us set $E^- := -E^+$. Let π be the universal covering projection of G onto $SL(2, \mathbb{R})$. The center Z of G is infinite cyclic and the kernel of π is the subgroup of Z of index 2. Clearly, a subsemigroup S of G is all of G if and only if $\pi(S) = SL(2, \mathbb{R})$ and S contains the kernel of π . For every elliptic X the circle $\exp(tX)$ in $SL(2, \mathbb{R})$ for $0 \leq t \leq \tau$ represents a generator of $\ker \pi$, where τ is the smallest positive real number such that $\exp(\tau X) = \mathbb{1}$. Here \exp denotes the exponential map of $SL(2, \mathbb{R})$. Let us call an element of $\ker \pi$ positive if it is represented by a curve of the form $\exp(tX)$ for $0 \leq t \leq n\tau$ for some positive $n \in \mathbb{N}$ and $X \in E^+$ and negative if $X \in E^-$. It follows that our condition on X and Y is sufficient, since the two elliptic (semi-)groups $\pi(X^+)$ and $\pi(Y^+)$ generate $SL(2, \mathbb{R})$ and the subsemigroup S of G generated by $X^+ \cup Y^+$ contains the kernel of π , since X and Y belong to the different components of the time cone.

For the proof of necessity we use the following action of G on the universal covering space of $\mathbb{P}^1\mathbb{R}$. The action of $SL(2, \mathbb{R})$ on $\mathbb{P}^1\mathbb{R}$ lifts to a unique action of G , the universal covering group of $SL(2, \mathbb{R})$, on the universal covering space $\widetilde{\mathbb{P}^1\mathbb{R}}$ of $\mathbb{P}^1\mathbb{R}$. The different types of elements of G act on $\widetilde{\mathbb{P}^1\mathbb{R}}$ as follows. For elliptic X the semigroup X^+ acts on $\widetilde{\mathbb{P}^1\mathbb{R}}$ without fixed points. We identify $\widetilde{\mathbb{P}^1\mathbb{R}}$ with \mathbb{R} in such a way that $\exp(tX)r \geq r$ for $t \geq 0$ and $X \in E^+$ and $\exp(1/2\tau X)r = r + 1$ for $X \in E^+$. So the center of G acts on \mathbb{R} via translations by integers and the inverse image $\pi^{-1}(x)$ of a point $x \in \mathbb{P}^1\mathbb{R}$ is a \mathbb{Z} -coset in \mathbb{R} . For parabolic X the one-parameter semigroup X^+ fixes every point of one \mathbb{Z} -coset and X^+ maps every interval I between two neighboring fixed points to itself. This map is increasing for $X \in P^+$, i.e. $\exp(tX)r > r$ for $t > 0$ and $r \in \overset{\circ}{I}$, and decreasing for $X \in P^-$. Here we denote by P^+ the set of non zero boundary points of E^+ and put $E^- := -E^+$. For hyperbolic X the semigroup X^+ has two \mathbb{Z} -cosets of fixed points. One of these cosets consists of attracting fixed points, the other one of repelling ones. On the open interval between a repelling fixed point and the next larger attracting fixed point the non-identity maps of X^+

are increasing, and decreasing on the remaining open intervals. Note that the elements of G described here are not all the elements of G . We listed only the exponential ones. Every element of G is of the form hz where z is an element of the center of G and h is of the form we described, hence exponential, since $PSL(2, \mathbb{R})$ is exponential.

Let us now turn to the proof of necessity of our condition. So suppose X^+ and Y^+ generate G as a semigroup. Then X and Y must be linearly independent and cannot have a common fixed point in \mathbb{R} . The discussion above shows that for $X \in \mathfrak{g}$, $t > 0$ and $r \in \mathbb{R}$ if $\exp(tX)$ is weakly increasing at r , i.e. $\exp(tX)r \geq r$, then $\exp(tX)$ maps the interval (r, ∞) into itself for every $t \geq 0$, and similarly for weakly decreasing at r and the interval $(r, -\infty, r)$. Thus if both X^+ and Y^+ are weakly increasing at the same point or weakly decreasing at the same point, then X^+ and Y^+ cannot generate G as a semigroup. In particular, if one of the two semigroups, say X^+ has a fixed point, say r , then Y^+ cannot have a fixed point at r , so one of the two intervals (r, ∞) or $(r, -\infty, r)$ is invariant under the semigroup generated by X^+ and Y^+ . It follows that if $X^+ \cup Y^+$ generates G as a semigroup then both X^+ and Y^+ have no fixed point, so X and Y are elliptic, and they must move in opposite directions. \square

In the beginning of this section we asked the more general question. Given a simple Lie group G . For which pairs $\{X, Y\}$ of elements of its Lie algebra \mathfrak{g} the corresponding one-parameter subsemigroups X^+, Y^+ generate G as a semigroup. We see from the example of $SL(2, \mathbb{R})$ and its universal cover that the answer depends not only on the Lie algebra but on the groups and can be very different for the cases of finite center and of infinite center. The condition c) in 5.1 is certainly not a necessary condition in general, because one can show that the set \mathcal{S}_m of m -tuples $(X_1, \dots, X_m) \in \mathfrak{g}^m$ for which $X_1^+ \cup \dots \cup X_m^+$ generates G as a semigroup is an open subset of \mathfrak{g}^m , whereas if for the pair $(X, Y) \in \mathcal{S}_2$ the segment from X to Y in \mathfrak{g} contains an elliptic element, the same will not be true for a neighborhood of (X, Y) , in general. In the direction of the equivalence of a) and b) in 5.1 there is the following general and very beautiful result of Luiz San-Martin, see [SM] Theorem 4.2.

Theorem 5.6. *Let G be a semisimple Lie group with finite center and let P be a minimal parabolic subgroup of G . Then G is the only subsemigroup of G with non-empty interior which acts transitively on G/P .*

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