ON A THEOREM OF AUSLANDER

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Abstract. We extend a theorem of L.Auslander and give a short proof of it.

1. Introduction and results

The following theorem is due to L.Auslander, see [Ra] Theorem 8.24, and is useful in many contexts.

Theorem 1.1. *Let G be a Lie group and let R be a closed connected solvable normal subgroup of G. Let $\pi : G \to G/R$ be the natural map. Let H be a closed subgroup of G such that H^0, the identity component of H is solvable. Let $U := \overline{\pi(H)}$ be the closure of $\pi(H)$. Then the identity component U^0 of U is solvable.*

We extend this for connected G, namely we can replace π by any continuous homomorphism f with solvable kernel. In particular, the kernel of f need not be connected and the quotient $G/ker(f)$ need not have the quotient topolgy and may be dense in some larger group. We can extend this futher to the case that G has a virtually solvable group of connected components.

To facilitate the language let us call a subgroup H of a topological group G to be CS in G, if the identity component $\overline{H}^0$ of its closure $\overline{H}$ is solvable. Recall that a group is called virtually solvable if it contains a solvable subgroup of finite index.

Theorem 1.2. *Let $G$and G_1 be Lie groups and let $f : G \to G_1$ be a continuous homomorphism. Suppose G is connected and the kernel of f is virtually solvable. If H is a CS subgroup of G then $f(H)$ is a CS subgroup of G_1.*

We will derive the following generalization using lemma 2.6.

Corollary 1.3. *Let G and G_1 be Lie groups and let $f : G \to G_1$ be a continuous homomorphism. Suppose the kernel of the induced map $\pi^0(f) : G/G^0 \to G_1/G_1^0$ of groups of connected components has a virtually solvable kernel. If H is CS in G then $f(H)$ is CS in G_1.*

We derive from Auslander's theorem a handy criterion for when a subgroup of a connected Lie group G is CS, namely we reduce it to the case of the adjoint group of the semisimple quotient of G, see 2.3. Using this criterion we can apply Lie group and algebraic group techniques.

For the convenience of the reader we also give in chapter 3 a short proof of Auslander's theorem.

2. Proofs

The following observation is trivial.

Observation 2.1. *Let $f : G \to G_1$ be a continuous homomorphism of topological groups with solvable kernel. If $f(H)$ is CS in G_1 then H is CS in G.*

The question is: When does the converse hold? Certainly not in general, e.g. if $H = G$ is a free discrete group and f is a dense embedding into a semisimple group G_1. Our main result is that the converse holds if G is connected, and more generally if the induced map of the groups of connected components has a virtually solvable kernel.

We shall need some facts about virtually solvable groups.

Lemma 2.2. *a) Every virtually solvable group A contains a largest solvabel normal subgroup, called the radical of A and denoted $Rad(A)$. Then $Rad(A)$ is of finite index in A and is a characteristic subgroup of A, i.e. every automorphism of A maps $Rad(A)$ to itself automorpically.*
b) Every virtually solvable normal subgroup of a connected Lie group G is actually solvable.
c) Every extension of a virtually solvable group by a virtually solvable group is virtually solvablle, which means in more detail: Every group B which contains a virtually solvable normal subgroup A with virtually solvable quotient B/A is virtually solvable.

Proof a) is clear, since every subgroup of finite index contains a normal subgroup of finite index.
b) Let A be a virtually solvable normal subgroup of the connected Lie group G. We claim that A is solvable. We may assume that A is closed. Then $Rad(A)$ is a closed solvable normal subgroup of G. The group $A/Rad(A)$ is a finite normal subgroup of the connected Lie group $G/Rad(A)$, hence central in $G/Rad(A)$. It follows that A is solvable.
c) We may assume that B/A is solvable. Now B contains the normal solvable subgroup $Rad(A)$ and $B/Rad(A)$ contains the finite normal subgroup $A/Rad(A)$ with solvable quotient B/A. We are thus reduced to the case that A is a finite normal subgroup of the solvable group B. Then B acts on A by inner automorphisms. The kernel C of the

associated homomorphism $B \to Aut(A)$ is of finite index in B and centralizes A. In particular $C \cap A$ is a central subgroup of C. So C is solvable, since $C/(C \cap A)$ is a subgroup of the solvable group B/A. □

Recall that any connected Lie group G contains a largest connected solvable normal subgroup, called the radical $R := rad(G)$ of G. The semisimple group $S := G/R$ contains a largest solvable normal subgroup, namely its center Z. It follows that G contains a largest solvable normal subgroup, namely the inverse image of Z under the natural map $\pi : G \to G/R$. Let us call it $\pi^{-1}(Z) =: Rad(G)$, the big radical of G. Thus the group $Rad(G)$ is the kernel of the composed homomorphism of $\varphi := Ad_S \circ \pi$ of π and the adjoint homomorphism of $S = G/R$. The group $G/Rad(G)$ is hence algebraically isomorphic to $Ad(S)$. The natural isomorhism $G/Rad(G) \to Ad(S)$ is actually an isomorphism of Lie groups, as follows from lemma 2.5. In fact every virtually solvable normal subgroup of G is contained in $Rad(G)$, by the preceding lemma.

Theorem 2.3. *Let G be a connected Lie group. A subgroup H of G is CS in G if and only if its image in $G/Rad(G) = Ad(S)$ under the natural map φ is CS in $G/Rad(G)$. Hence if $\varphi(H)$ is Zariski dense in $Ad(S)$ then H is CS in G if and only if $\varphi(H)$ is a discrete subgroup of $Ad(S)$.*

Proof Sufficency follows from observation 2.1, necessity from Auslander's theorem and the following lemma 2.4. In case $H_1 := \varphi(H)$ is Zariski dense in $Ad(S)$ the connected Lie subgroup $\overline{H_1}^0$ is normalized by H_1, hence by its Zariski closure $Ad(S)$. It follows that $\overline{H_1}^0$ is trivial, if H_1 is CS in $Ad(S)$, since then $\overline{H_1}^0$ is a connected normal solvable subgroup of the semisimple group $Ad(S)$. □

Lemma 2.4. *Let G be a topological group and let Z be a closed central subgroup of G. A subgroup H of G is CS in G (if and) only if its natural image in G/Z is CS in G/Z.*

Proof The commutator map of G

$$c : G \times G \to G$$
$$c(x, y) := xyx^{-1}y^{-1}$$

descends to a well defined continuous map

$$\overline{c} : \overline{G} \times \overline{G} \to G$$
$$\overline{c}(\pi(x), \pi(y)) = c(x, y),$$

where we set $\overline{G} = G/Z$ and let π be the natural map from G to $\overline{G}$. It follows that

$$\overline{c}(\pi(H),\pi(H)) = c(H,H) \subset [H,H],$$

hence

$$\overline{c}(\overline{\pi(H)},\overline{\pi(H)}) \subset \overline{[H,H]} \subset \overline{H}$$

and

$$\overline{c}(\,(\overline{\pi(H)})^0,(\overline{\pi(H)})^0) \subset (\overline{H})^0.$$

This implies that $H_1 := \pi^{-1}((\overline{\pi(H)})^0)$ is solvable, since

$$c(H_1,H_1) = \overline{c}((\overline{\pi(H)})^0,(\overline{\pi(H)})^0) \subset (\overline{H})^0.$$

Hence also $(\overline{\pi(H)})^0$ is solvable, as claimed. □

We can now prove theorem 1.2 .

Proof of theorem 1.2. Let G and G_1 be Lie groups and let $f : G \to G_1$ be a continuous homomorphism with virtually solvable kernel. By hypothesis, G is connected. We may furthermore assume that $f(G)$ is dense in G_1, hence that G_1 is connected as well. We have $\overline{f(Rad(G)} \subset Rad(G_1)$, since $f(Rad(G)$ is a normal solvable subgroup of the dense subgroup $f(G)$ of G_1, so its closure $\overline{f(Rad(G)}$ is contained in the big radical $Rad(G_1)$ of G_1. On the other hand, the inverse image $f^{-1}(Rad(G_1)$ of the big radical of G_1 is a normal subgroup of G which is virtually solvable by lemma 2.2 c), hence solvable by lemma 2.2 b) and thus $f^{-1}(Rad(G_1) \subset Rad(G)$. So f induces a continuous injective homomorphism $G/Rad(G) \to G_1/Rad(G_1)$. To prove our theorem we may thus assume, by theorem 2.3, that G and G_1 are both connected semisimple groups without center, that f is injective and has dense image. But then f is an isomorphism of Lie groups by the following lemma. □

Lemma 2.5. *Let G and H be connected Lie groups and let $f : G \to H$ be a continuous injective homomorphism with dense image. Suppose H is semisimple. Then f is an isomorphism of Lie groups.*

Proof A dense connected Lie subgroup of a semisimple Lie group H must be equal to H, see [Ho] XVI Theorem 2.1. So f is surjective. Then a Baire category argument implies that f is actually an open map, see [Ho] I Theorem 2.5. □

Corollary 1.3 is implied by the the following lemma for the following reason. We may assume that G_1 is connected and hence that G/G^0 is virtually solvable. Set $H_1 := f(H)$. Then take $N := f(H \cap G^0)$ in the following lemma and use that H_1/N is virtually solvable since it is the image of $H/(H \cap G^0)$ under the homomorphism induced by f and $H/(H \cap G^0)$ is virtually solvable since isomorphic to a subgroup of G/G^0.

Lemma 2.6. *Let G be a Lie group and let H and N be subgroups of G. Suppose N is a normal subgroup of H with H/N virtually solvable. If N is CS in G then also H is CS in G.*

Proof This is not trivial, since in the chain of subgroups $\overline{H} \rhd \overline{N} \rhd \overline{N}^0$ we have no information about $\overline{N}/\overline{N}^0$. To prove the lemma we may assume that G is connected. Then H is CS in G if its image in $G/Rad(G)$ is so. This is the trivial observation 2.1. We may assume that the image of N in $G/Rad(G)$ is CS in $G/Rad(G)$. This is the non-trivial part of theorem 2.3. We may thus assume that G is a connected semisimple adjoint group. It suffices to show that H is CS in the identity component of its Zariski closure. Using the same arguments as before for the Zariski closure of H in G, we may thus assume that G is a connected semisimple adjoint group and H is furthermore Zariski dense in G. Now $\overline{N}^0$ is a normal solvable connected Lie subgroup of $\overline{H}$, hence is normalized by G, since H is Zariski dense in G. But a semisimple group has no connected solvable normal subgroup, so $\overline{N}^0$ is trivial and thus N is a discrete subgroup of G, in particular a closed subgroup of G. Now $\overline{H}^0$ is a connected Lie subgroup of G, normal in $\overline{H}$, hence normal in G. So $\overline{H}^0$ is a semisimple subgroup of G or trivial. We claim that $\overline{H}^0$ is trivial. The group N is a normal subgroup of H, so the connected group $\overline{H}^0$ normalizes the discrete group N, hence centralizes it. So $\overline{H}^0/(N \cap \overline{H}^0)$ is a semisimple group or trivial, but is on the other hand contained in $\overline{H}/N$, a virtually solvable group. So $\overline{H}^0$ must be trivial. □

3. Auslander's Theorem

For the convenience of the reader we give a short proof of Auslander's theorem. The method of proof is similar to the construction of Zassenhaus neighborhoods. We repeat the statement of the theorem.

Theorem 3.1. *Let G be a Lie group and let R be a closed connected solvable normal subgroup of G. Let $\pi : G \to G/R$ be the natural map. Let H be a closed subgroup of G such that H^0, the identity component of H, is solvable. Let $U := \overline{\pi(H)}$ be the closure of $\pi(H)$. Then the identity component U^0 of U is solvable.*

Proof By induction we may assume that $R = A$ is abelian. The case that A is compact is easy since then H^0A is the identity component $(HA)^0$ of HA for the following reason. H^0A is a connected subgroup of HA and open in HA since the complement of H^0A in HA is the closed set $H'A$ where H' is the union of those connected components

H^0h of H which do not intersect A. It follows that $\pi(H)$ is a closed subgroup of $\pi(G)$ and $\pi(H^0)$ is the identity component of $\pi(H)$, which implies the theorem in this case. It thus suffices to consider the case that A is a vector group. Here is the key lemma.

Lemma 3.2. *Let G be a connected Lie group and let A be a closed normal vector subgroup of G. Then there is a neighborhood U of the identity in G with the following property. For every closed subgroup H of G every subgroup of HA generated by a finite subset of $HA \cap U$ is nilpotent modulo H^0A.*

Note that H^0A is a normal subgroup of HA.

Using this lemma we can complete the proof of Auslander's theorem as follows. We use the hypotheses and notations of the lemma. Then the connected group $H_1 := \overline{HA}^0$ is generated by $U \cap H_1$. So the subgroup H_2 of H_1 generated by $U \cap H_1 \cap HA$ is dense in H_1. For every finite subset Σ of H_2 the group generated by Σ is nilpotent modulo H^0A, since this holds for finite subsets Σ of the generating set $U \cap H_1 \cap HA$, by lemma 3.2. This implies that H_1/H^0A is nilpotent by lemma 3.3 and hence the theorem, since $\overline{\pi(H)}^0 = \pi(H_1)$, because $\pi(H_1)$ is connected and open in $\overline{\pi(H)} = \pi(\overline{HA})$, since H_1 is generated by a neighborhood of the identity in $\overline{HA}$.

Proof of lemma 3.2. We identify A with its Lie algebra $\mathfrak{a}$ via the exponential map. We compute the commutator $[x, y] := xyx^{-1}y^{-1}$ for $x = ga$ and $y = hb$ with $g, h \in G$ and $a, b \in \mathfrak{a}$. We have

$$[ga, hb] = [g, h]((\alpha(hgh^{-1})(\mathbf{1} - \alpha(h))a + \alpha(h)(\alpha(g) - \mathbf{1})b),$$

where $\alpha(g) = Ad(g)|\mathfrak{a}$ for $g \in G$. Note that $\alpha(ga) = \alpha(g)$ for $a \in A$. We thus define a commutator map $c : (G \times A) \times (G \times A) \to G \times A$ by

$$c((g, a), (h, b)) = ([g, h], (\alpha(hgh^{-1})(\mathbf{1} - \alpha(h))a + \alpha(h)(\alpha(g) - \mathbf{1})b).$$

We shall study the iterates of this commutator map for ga close to the identity. Choose a norm $||.||$ on the vector space $\mathfrak{g}$. Let $B(0, \epsilon) := \{X \in \mathfrak{g}; ||X|| < \epsilon\}$. We may suppose that ϵ is small enough so that the exponential map induces a diffeomorphism of $B(0, \epsilon)$ to a neighborhood of the identity in G. We denote by log the inverse of this diffeomorphism. The commutator map $\overline{c} : G \times G \to G, \overline{c}(x, y) = [x, y]$ of G is analytic and $\overline{c}(x, e) = e = \overline{c}(e, y)$ for every $x, y \in G$. It follows that

$$|| \log \overline{c}(x, y)|| \leq K \cdot || \log x|| \cdot || \log y||$$

for ϵ sufficiently small where K is a constant depending on $||.||$ and ϵ, by the following argument. Let V and W be finite dimensional normed real vector spaces and let f be a smooth function defined on

$B(0,\epsilon) \times B(0,\epsilon)$. If $f(v,0) = 0 = f(0,w)$ for every $v \in B(0,\epsilon)$ and $w \in B(0,\epsilon)$ then

$$|f(v,w)| \leq ||v|| \cdot ||w|| \cdot sup\,||\partial_w \partial_v f||$$

by the mean value theorem. Define

$$B(\delta, L) := \{(g,a) \in G \times A;\ ga \in \exp B(0,\epsilon), ||\log(ga)|| \leq \delta, ||a|| \leq L\}.$$

It then follows from the commutator formula that for δ_i sufficiently small we have

$$c(B(\delta_0, L_0), B(\delta_1, L_1)) \subset B(\delta_2, L_2),$$

where

$$\delta_2 \leq K\delta_0\delta_1$$

and

$$L_2 \leq K(\delta_0 L_1 + \delta_1 L_0)$$

for some (conceivably larger) constant $K \geq 1$. Choose δ_0 so small that $\delta_0 K < 1/2$. Define inductively $c^n(B(\delta_0, L_0))$ by $c^0 B(\delta_0, L_0) = B(\delta_0, L_0)$ and

$$c^{n+1} B(\delta_0, L_0) = c(B(\delta_0, L_0), c^n(B(\delta_0, L_0)).$$

We thus obtain

$$c^n(B(\delta_0, L_0)) \subset B(\delta_n, L_n),$$

where

$$\delta_n \leq K\delta_0\delta_{n-1} \leq (1/2)^n$$

and

$$L_n \leq K\delta_0 L_{n-1} + \delta_{n-1} L_0 \leq 1/2 L_{n-1} + (1/2)^{n-1} L_0$$

and then by induction

$$L_n \leq (1/2)^n L_0 + n(1/2)^{n-1} L_0.$$

Let now H be a closed subgroup of G. Then

$$c^n((H \times A) \cap B(\delta_0, L_0)) \subset (H \cap A) \cap B(\delta_n, L_n).$$

Hence if (h,a) is an element of $(H \cap A) \cap B(\delta_n, L_n)$ then for n large both $||\log(ha)||$ and $||a||$ are close to 0 and hence $h \in H$ is contained in the open subgroup H^0 of H. In particular, if Σ is a finite subset of $HA \cap B(0, \delta_0)$ then, for some L_0 we can write every $g \in \Sigma$ in the form ha with $(h,a) \in B(\delta_0, L_0)$. It follows that for n large every iterated commutator $\overline{c}^n(\Sigma)$ of Σ is contained in $H^0 A$, hence the subgroup of HA generated by Σ is nilpotent modulo $H^0 A$. □

Lemma 3.3. *Let G be a connected Lie group which contains a dense subgroup all of whose finitely generated subgroups are nilpotent. Then G is nilpotent.*

Proof It suffices to show that $Ad(G)$ is nilpotent. Thus, replacing G by the Zariski closure of $Ad(G)$, we may assume that G is the identity component of a linear algebraic group which contains a Zariski dense subgroup H all of whose finitely generated subgroups are nilpotent. So for every finite subset Σ of H the subgroup $\langle\Sigma\rangle$ of H generated by Σ has a non-trivial center. In particular the centralizer $C(\Sigma)$ of Σ in G is a non-trivial finite index subgroup of a linear algebraic group. It follows by a descending chain argument over first the dimension and then the number of connected components that there is finite subset Σ of H such that $C(\Sigma) = C(\Sigma')$ for every finite subset Σ' of H that contains Σ. Hence $C(\Sigma) = C(H)$. Now $C(H)$ is the center of G, since H is Zariski dense in G. It follows that G has a non-trivial center $C(G)$. This implies the lemma by induction on the dimension of G in view of the following two facts. The property of G is inherited by the quotient group $G/C(G)$. It cannot happen that the center of G is finite and non-trivial and the center of $G/C(G)$ is finite and non-trivial, since then the inverse image of the center of $G/C(G)$ under the natural homomorphism $G \to G/C(G)$ is a finite normal subgroup in the connected group G, hence central. □

References

[Ho] Hochschild, G., *The structure of Lie groups*, Holden-Day Inc., San Francisco (1965), ix+230

[Ra] Raghunathan, M. S. , *Discrete subgroups of Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg, 1972. ix+227

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