

# The statistical dynamics of a spatial logistic model and the related kinetic equation

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## Abstract

The statistical dynamics are studied of an infinite system of point entities in  $\mathbb{R}^d$ , which reproduce themselves and die, also due to competition. The states of the system are probability measures on the space of configurations of entities. Their evolution is described by means of the corresponding correlation (moment) functions. It is shown that these functions evolve on a bounded time interval and remain sub-Poissonian due to the competition. It is proven that in the Vlasov-type scaling limit the correlation functions converge to those obtained from the Vlasov hierarchy, characterized by ‘chaos preservation’. In such a case, one deals with a time-dependent Poisson random point field, the density function of which solves a (nonlinear) kinetic equation. A number of properties of this density function are also described.

**Keywords:** Individual-based model; evolution; birth-and-death process; random point field.

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## 1 Introduction

### 1.1 Setup

In life sciences, one often deals with large systems of interacting entities distributed over a continuous habitat and evolving in time, cf. [2, 26]. Their collective behavior is observed at a macro-scale, and thus the mathematical theories

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which explain this behavior traditionally describe it by means of phenomenologically deduced nonlinear equations, mostly differential or integro-differential, involving macroscopic characteristics like density, mobility, etc, see, e.g., [35]. However, this kind of macroscopic phenomenology may often be insufficient as it does not take into account individual behavior of constituting entities. Thus, mathematical models and methods are needed for drawing population-level conclusions from individual-level descriptions. The present paper is aimed at contributing to the development of the mathematical modeling of this kind. Its results were announced in [13]. Similar results on jump dynamics were obtained in [3].

Individual-based models of large biological systems attract considerable attention of both mathematicians and theoretical biologists, see, e.g., [17–19, 26, 29] and [4, 6–8, 10, 25, 27], respectively. In this paper, we continue, cf. [17], studying the model introduced and discussed in [6–8, 10, 25, 27]. This model describes a population of entities (e.g., perennial plants) distributed over  $\mathbb{R}^d$ , which reproduce themselves and die, also due to competition. As was suggested already in [6], see also page 1311 in [26], the proper mathematical tool for studying objects of this kind is the theory of random point fields on  $\mathbb{R}^d$ . Herein, populations of entities are modeled as point configurations constituting the set

$$\Gamma := \{ \gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d \}, \quad (1)$$

where  $|A|$  stands for the number of elements in  $A$ . In the language of the theory of dynamical systems,  $\Gamma$  is a phase space. Along with finite configurations it contains also infinite ones, which allows for studying ‘bulk’ properties of large finite systems ignoring boundary and size effects.<sup>1</sup>

In the Hamiltonian mechanics, the motion of  $N$  physical particles in  $\mathbb{R}^d$  is described by a system of  $2dN$  differential equations, subject to initial conditions. For  $N \gg 1$  (Avogadro’s number is  $\simeq 6 \times 10^{23}$ ), the abundance of equations, and hence of the initial conditions, makes the point-wise description meaningless since no observation could indicate at which point of the phase space the system actually is. Moreover, the description in terms of individual trajectories would be ‘too detailed’ to yield understanding the collective behavior of the system. It was realized already in the time of A. Einstein and M. Smoluchowski that the statistical approach can provide the right context in the theory of large systems of interacting particles, including the possibility to link their micro- and macroscopic descriptions to each other. In this approach, one deals with the probabilities with which evolving points of the phase space lie in its subsets. The corresponding probability measures are then considered as the states of the system. However, for interacting particles, the direct study of the evolution of such states encounters serious technical difficulties. In [5], N. N. Bogoliubov suggested to do this by means of the so-called correlation (moment) functions. Their evolution is obtained from an infinite system of equations, called now BBGKY hierarchy or chain, that couples correlation functions of different order,

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<sup>1</sup>A discussion on how infinite systems provide approximations for large finite systems can be found in, e.g., [9].

see [11]. Starting from the late 1990'th, a similar statistical approach is being implemented in studying Markov dynamics of states on the phase space  $\Gamma$ , see [15] and the references therein. Gradually, it has become clear [29] also for theoretical biologists that the theory developed in this framework can provide effective methods for modeling and studying large systems of living entities.

Thus, in this work we assume that the system's states are probability measures on the phase space  $\Gamma$ . Their evolution  $\mu_0 \mapsto \mu_t$  is described by the Fokker-Planck equation

$$\frac{d}{dt}\mu_t = L^*\mu_t, \quad \mu_t|_{t=0} = \mu_0, \quad t > 0, \quad (2)$$

in which the 'generator'  $L^*$  specifies the model. This evolution can also be described by employing *observables* – real valued functions on the phase space. In our case, these are functions  $F : \Gamma \rightarrow \mathbb{R}$  such that the integrals

$$\langle\langle F, \mu \rangle\rangle := \int_{\Gamma} F(\gamma)\mu(d\gamma)$$

make sense for  $\mu$  belonging to the set of states in which the evolution in question is constructed. Then the number  $\langle\langle F, \mu \rangle\rangle$  is interpreted as the mean value of observable  $F$  in state  $\mu$ . By the duality

$$\langle\langle F_0, \mu_t \rangle\rangle = \langle\langle F_t, \mu_0 \rangle\rangle, \quad t > 0, \quad (3)$$

the observed evolution  $\langle\langle F, \mu_0 \rangle\rangle \mapsto \langle\langle F, \mu_t \rangle\rangle$  can also be considered as the evolution  $\langle\langle F_0, \mu \rangle\rangle \mapsto \langle\langle F_t, \mu \rangle\rangle$ , obtained from the Kolmogorov equation

$$\frac{d}{dt}F_t = LF_t, \quad F_t|_{t=0} = F_0, \quad t > 0, \quad (4)$$

where  $L$  and  $L^*$  are dual in the sense of (3). For the model studied in this work,  $L$  in (4) has the form

$$\begin{aligned} (LF)(\gamma) &= \sum_{x \in \gamma} [m + E^-(x, \gamma \setminus x)] [F(\gamma \setminus x) - F(\gamma)] \\ &\quad + \int_{\mathbb{R}^d} E^+(y, \gamma) [F(\gamma \cup y) - F(\gamma)] dy, \end{aligned} \quad (5)$$

where

$$E^\pm(x, \gamma) := \sum_{y \in \gamma} a^\pm(x - y). \quad (6)$$

The first term in (5) describes the death of the particle located at  $x$  occurring independently with rate  $m \geq 0$ , and under the influence of the other particles in  $\gamma$  with rate  $E^-(x, \gamma \setminus x) \geq 0$ . Here and in the sequel in the corresponding context,  $x \in \mathbb{R}^d$  is also treated as a single-point configuration  $\{x\}$ . Note that  $E^-(x, \gamma \setminus x)$  describes the interparticle competition. The second term in (5) describes the birth of a particle at  $y \in \mathbb{R}^d$  given by the whole configuration  $\gamma$

with rate  $E^+(y, \gamma) \geq 0$ . Note that (1) contains also infinite configurations  $\gamma$ , for which the sums in (5), (6) need not converge. Thus, so far such expressions have rather heuristic meaning and will be made precise below. Noteworthy, the birth part of (5) can be rewritten in the following form

$$\sum_{x \in \gamma} \int_{\mathbb{R}^d} a^+(x - y) [F(\gamma \cup y) - F(\gamma)] dy.$$

This interchange of the integration and summation allows one to interpret the birth mechanism as a random (independent) sending by a member of  $\gamma$ , located at  $x$ , a ‘seed’ to point  $y$ , which immediately after that becomes a population member. This process is described by the *dispersal kernel*  $a^+(x - y) \geq 0$ .

## 1.2 Correlation functions

Constructing a Markov process on  $\Gamma$  corresponding to a given generator could be done by solving the Fokker-Planck equation (2) *for all* possible initial states  $\mu_0$ . However, for the model considered here, such a description of the evolution of states in terms of random trajectories could hardly be realized. Instead, we will construct the evolution from (2), (4), (5) for  $\mu_0$  belonging to a *properly chosen class* of probability measures on  $\Gamma$  by employing *correlation functions*, which fully characterize the corresponding states. Namely, given  $n \in \mathbb{N}$  and a probability measure  $\mu$ , the  $n$ -th order correlation function  $k_\mu^{(n)}$  is related to  $\mu$  by the following formula

$$\begin{aligned} \int_{\Gamma} \left( \sum_{\{x_1, \dots, x_n\} \subset \gamma} G^{(n)}(x_1, \dots, x_n) \right) \mu(d\gamma) \\ = \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \dots, x_n) k_\mu^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n, \end{aligned} \quad (7)$$

which has to hold for all symmetric compactly supported Borel functions  $G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ . Thus, each  $k_\mu^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$  is a symmetric function with a number of specific properties, see, e.g., [15–18]. If one puts also  $k^{(0)} \equiv 1$ , then the collection of  $k_\mu^{(n)}$ ,  $n \in \mathbb{N}_0$ , determines a map,  $k_\mu : \Gamma_0 \rightarrow \mathbb{R}$ , defined on the set of finite configurations

$$\Gamma_0 = \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}, \quad (8)$$

which is the disjoint union of the sets of  $n$ -particle configurations:

$$\Gamma^{(0)} = \{\emptyset\}, \quad \Gamma^{(n)} = \{\eta \in \Gamma : |\eta| = n\}, \quad n \in \mathbb{N}. \quad (9)$$

For  $n \geq 2$ ,  $\Gamma^{(n)}$  can be identified with the symmetrization of the set

$$\{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j, \text{ for } i \neq j\}, \quad (10)$$

which allows one to introduce the corresponding (Euclidean) topology on  $\Gamma^{(n)}$ . Then the restriction of  $k_\mu$  to a given  $\Gamma^{(n)}$ ,  $n \in \mathbb{N}$ , is exactly the  $n$ -th order correlation function as in (7). In particular,  $k_\mu^{(1)}$  is the density of the particles in state  $\mu$ . The correlation function  $k_{\pi_\varrho}$  of the inhomogeneous Poisson measure  $\pi_\varrho$  is

$$k_{\pi_\varrho}(\eta) = \prod_{x \in \eta} \varrho(x),$$

or, equivalently,

$$k_{\pi_\varrho}^{(n)}(x_1, \dots, x_n) = \varrho(x_1) \cdots \varrho(x_n), \quad n \in \mathbb{N}, \quad (11)$$

where  $\varrho$  is the density function, which is supposed to be locally integrable. A measure  $\mu$  is said to be *sub-Poissonian* if its correlation function is such that, for all  $n \in \mathbb{N}$ ,

$$k_\mu^{(n)}(x_1, \dots, x_n) \leq C^n \quad (12)$$

holding for some  $C > 0$  and Lebesgue-almost all  $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$ . Then a sub-Poissonian state is, in a way, similar to the Poissonian state in which the particles are independently scattered over  $\mathbb{R}^d$ . At the same time, the increase of  $k_\mu^{(n)}$  with  $n$  as  $n!$ , see (18) below, corresponds to the appearance of *clusters* in state  $\mu$ .

By an appropriate procedure [18], the Cauchy problem in (4) is transformed into the following one

$$\frac{d}{dt} k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_0, \quad (13)$$

where the ‘generator’

$$\begin{aligned} (L^\Delta k)(\eta) &= - \left[ \sum_{x \in \eta} (m + E^-(x, \eta \setminus x)) \right] k(\eta) + \sum_{x \in \eta} E^+(x, \eta \setminus x) k(\eta \setminus x) \\ &+ \int_{\mathbb{R}^d} \sum_{y \in \eta} a^+(x - y) k((\eta \setminus y) \cup x) dx + \int_{\mathbb{R}^d} E^-(x, \eta) k(\eta \cup x) dx \end{aligned} \quad (14)$$

is calculated from that of (5), and  $k_0$  is the correlation function of the initial state  $\mu_0$ . In terms of the ‘components’  $k_t^{(n)}$ , the equation in (13) is an infinite chain of coupled linear equations, analogous to the BBGKY hierarchy mentioned above. In particular, the first equations in (14) have the following form:  $dk_t^{(0)}/dt = 0$ , and

$$\frac{d}{dt} k_t^{(1)}(x) = -m k_t^{(1)}(x) - \int_{\mathbb{R}^d} a^-(x - y) k_t^{(2)}(x, y) dy + \int_{\mathbb{R}^d} a^+(x - y) k_t^{(1)}(y) dy. \quad (15)$$

The right-hand side of the equation with  $dk_t^{(2)}/dt$  contains  $k_t^{(n)}$  with  $n = 1, 2, 3$ , etc. Theoretical biologists try to solve chains like (14) by decoupling them; cf. [25]. In the simplest version of such a decoupling, one sets

$$k_t^{(2)}(x, y) \simeq k_t^{(1)}(x) k_t^{(1)}(y), \quad (16)$$

which amounts to neglecting spatial pair correlations (so called *mean field* approximation). This turns (15) into the following nonlinear equation

$$\begin{aligned} \frac{d}{dt}k_t^{(1)}(x) &= -mk^{(1)}(x) - k_t^{(1)}(x) \int_{\mathbb{R}^d} a^-(x-y)k_t^{(1)}(y)dy \\ &\quad + \int_{\mathbb{R}^d} a^+(x-y)k_t^{(1)}(y)dy, \end{aligned} \quad (17)$$

which, as we will see in Section 4 below, is a kinetic equation for our model. Its possible solution provides an approximate description of the evolution of the particle density. Note that the question of whether the evolving states remain sub-Poissonian, and hence clusters do not appear, can be answered only by studying the whole chain  $k_\mu^{(n)}$ ,  $n \in \mathbb{N}_0$ , cf. (12). For the contact model, which one obtains by setting  $a^- = 0$  in (5) and (14), it is known [17] that

$$\text{const} \cdot n!c_t^n \leq k_t^{(n)}(x_1, \dots, x_n) \leq \text{const} \cdot n!C_t^n, \quad (18)$$

where the left-hand inequality holds if all  $x_i$  belong to a ball of small enough radius. Hence,  $k_t$  are definitely not sub-Poissonian in this case.

### 1.3 Mesoscopic description

Along with the microscopic theory based on (4), (13), we develop the mesoscopic description of the system's evolution obtained from (13) by means of a Vlasov-type scaling procedure, see [16] and also Section 6 in [11] and [32], where the general aspects of the scaling of interacting particle systems are presented. In the 'physical language', the Vlasov scaling can be outlined as follows. One considers the system at the scale where the particle density is big, and hence the interaction should be made respectively small in order to preserve the intermediate values of the total interaction energy. In the scaling limit, the corpuscular structure disappears and the system turns into a medium described by the density alone, cf. (11), whereas the interactions between the particles are taken into account in a 'mean-field-like' way. In this scaling limit, the ansatz in (16) becomes exact and thus the evolution of the density is obtained by solving (17). An important problem here is to control this passage in a mathematically rigorous way, which includes controlling the convergence of the 'rescaled' correlation functions.

In the scaling procedure that we use in this work, the scale is described by a single parameter,  $\varepsilon \in (0, 1]$ , tending to zero in the scaling limit and with value  $\varepsilon = 1$  corresponding to the initial system described by (13). In order to have high densities for small  $\varepsilon$ , we assume that the correlation function  $k_{0,\varepsilon}$  for small  $\varepsilon$  behaves like  $k_{0,\varepsilon}(\eta) \sim \varepsilon^{-|\eta|}$ ,  $\eta \in \Gamma_0$ , and thus the rescaled correlation function  $r_{0,\varepsilon}(\eta) = \varepsilon^{|\eta|}k_{0,\varepsilon}(\eta)$ , or equivalently

$$r_{0,\varepsilon}^{(n)}(x_1, \dots, x_n) = \varepsilon^n k_{0,\varepsilon}^{(n)}(x_1, \dots, x_n), \quad n \in \mathbb{N}, \quad (19)$$

converges as  $\varepsilon \rightarrow 0$  to the correlation function of a certain state. Namely, we assume that  $r_{0,\varepsilon}^{(n)} \rightarrow r_0^{(n)}$  in  $L^\infty((\mathbb{R}^d)^n)$  for each  $n \in \mathbb{N}$ . Next, in order to describe the evolution of such a ‘dense’ system, we rescale the interactions in (14) by multiplying  $a^-$  by  $\varepsilon$ , which yields  $L_\varepsilon^\Delta$  from  $L^\Delta$  given in (14). This means that the evolution of the ‘dense’ system  $k_{0,\varepsilon} \mapsto k_{t,\varepsilon}$ ,  $t > 0$ , is now governed by (13) with  $L_\varepsilon^\Delta$  in the right-hand side. Next, we expect that the evolving system remains ‘dense’, and thus introduce the rescaled correlation functions

$$r_{t,\varepsilon}^{(n)}(x, \dots, x_n) = \varepsilon^n k_{t,\varepsilon}^{(n)}(x, \dots, x_n), \quad n \in \mathbb{N}, \quad (20)$$

that solve the following Cauchy problem

$$\frac{d}{dt} r_{t,\varepsilon} = L_{\varepsilon,\text{ren}}^\Delta r_{t,\varepsilon}, \quad r_{t,\varepsilon}|_{t=0} = r_{0,\varepsilon}, \quad (21)$$

which one derives from (13) by means of (19) and (20). Herein,

$$L_{\varepsilon,\text{ren}}^\Delta = R_\varepsilon L_\varepsilon^\Delta R_{\varepsilon^{-1}}, \quad (22)$$

and  $(R_c k)(\eta) := c^{|\eta|} k(\eta)$  for  $c > 0$ . The operator introduced in (22) can be written in the form, cf. (14),

$$L_{\varepsilon,\text{ren}}^\Delta = V + \varepsilon C, \quad (23)$$

where  $V$  and  $C$  are given in (39) below. Along with (21) it is natural to consider the Cauchy problem

$$\frac{d}{dt} r_t = V r_t, \quad r_t|_{t=0} = r_0, \quad (24)$$

where  $r_0$  is the same as in (21), and to expect that the solution of (21) converges to that of (24) as  $\varepsilon \rightarrow 0$ . This would give interpolation between the case of  $\varepsilon = 1$  and that of  $\varepsilon = 0$ , i.e., between (13) and (24). The main peculiarity of (24) is that the evolution  $r_0 \mapsto r_t$  obtained therefrom ‘preserves chaos’. That is, if  $r_0$  is the correlation function of the Poisson measure  $\pi_{\varrho_0}$ , see (11), then, for all  $t > 0$  for which one can solve (24), the product form of (11) is preserved, i.e., the solution is such that

$$r_t^{(n)}(x_1, \dots, x_n) = \varrho_t(x_1) \cdots \varrho_t(x_n), \quad n \in \mathbb{N},$$

where  $\varrho_t$  is obtained as the solution of the *kinetic equation*, cf. (17),

$$\begin{aligned} \frac{d}{dt} \varrho_t(x) &= -m \varrho_t(x) - \varrho_t(x) \int_{\mathbb{R}^d} a^-(x-y) \varrho_t(y) dy \\ &+ \int_{\mathbb{R}^d} a^+(x-y) \varrho_t(y) dy, \quad \varrho_t|_{t=0} = \varrho_0, \end{aligned} \quad (25)$$

Thus, for the model described by (5) we aim at:

- proving that due to the presence of the competition in (5) the Cauchy problems in (21) and in (24) have sub-Poissonian solutions on the same time interval  $[0, T_*)$ ;

- proving that the solution of (24) has the product form of (11) with  $\varrho_t$  which solves (25);
- proving that the solutions of (21) converge to that of (24), i.e.,  $r_{t,\varepsilon} \rightarrow r_t$  as  $\varepsilon \rightarrow 0$ ;
- studying the solvability and the solutions of the kinetic equation (25).

## 2 The Basic Notions and the Model

All the details of the mathematical framework of this paper can be found in [1, 17, 18, 21–24, 28]. By  $\mathcal{B}(\mathbb{R}^d)$  and  $\mathcal{B}_b(\mathbb{R}^d)$  we denote the set of all Borel and all bounded Borel subsets of  $\mathbb{R}^d$ , respectively.

### 2.1 The configuration spaces

The configuration space (1) is endowed with the vague topology – the weakest topology such that all the maps

$$\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle = \int_{\mathbb{R}^d} f(x) \gamma(dx) = \sum_{x \in \gamma} f(x), \quad f \in C_0(\mathbb{R}^d),$$

are continuous. Here  $C_0(\mathbb{R}^d)$  stands for the set of all continuous functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  with compact support. The vague topology is metrizable in such a way that turns  $\Gamma$  into a complete and separable metric (Polish) space, see, e.g., Theorem 3.5 in [22]. By  $\mathcal{B}(\Gamma)$  we denote the corresponding Borel  $\sigma$ -algebra. The metric properties of  $\Gamma$ , as well as the main aspects of the analysis on configuration spaces, were studied in [1, 21, 22].

As mentioned above,  $\Gamma$  contains both finite and infinite configurations. That is,  $\Gamma_0$ , defined in (8), (9), is a proper subset of  $\Gamma$ . We recall that each  $\Gamma^{(n)}$  is endowed with the topology related to the Euclidean topology of the sets in (10). Then the topology of  $\Gamma_0$  is defined as the topology of the disjoint union, which allows one to introduce the corresponding Borel  $\sigma$ -algebra  $\mathcal{B}(\Gamma_0)$ . Note that, as a subset of  $\Gamma$ , (8) can also be equipped with the topology induced thereon by the vague topology of  $\Gamma$ . These two topologies clearly differ from each other. However, it can be proven, see Lemma 1.1 and Proposition 1.3 in [28], that the corresponding Borel  $\sigma$ -algebras coincide. In particular, this means that

$$\mathcal{B}(\Gamma_0) = \{A \in \mathcal{B}(\Gamma) : A \subset \Gamma_0\}. \quad (26)$$

Therefore, a probability measure  $\mu$  on  $(\Gamma, \mathcal{B}(\Gamma))$  such that  $\mu(\Gamma_0) = 1$  can be redefined as a measure on  $(\Gamma_0, \mathcal{B}(\Gamma_0))$ .

For  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ , we set

$$\Gamma_\Lambda = \{\gamma \in \Gamma : \gamma \subset \Lambda\}.$$

Clearly,  $\Gamma_\Lambda \subset \Gamma_0$ , which by (26) yields that

$$\mathcal{B}(\Gamma_\Lambda) := \{A \subset \Gamma_\Lambda : A \in \mathcal{B}(\Gamma_0)\} = \{A \subset \Gamma_\Lambda : A \in \mathcal{B}(\Gamma)\}. \quad (27)$$

Consider

$$\Gamma \ni \gamma \mapsto p_\Lambda(\gamma) = \gamma_\Lambda := \gamma \cap \Lambda, \quad \Lambda \in \mathcal{B}_b(\mathbb{R}^d), \quad (28)$$

and let  $\mathcal{B}(\Gamma_\Lambda)$  be defined by the first equality in (27). Then, cf. page 451 in [1],  $\mathcal{B}(\Gamma)$  is the smallest  $\sigma$ -algebra of subsets of  $\Gamma$  such that the maps  $p_\Lambda$  with all  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$  are  $\mathcal{B}(\Gamma)/\mathcal{B}(\Gamma_\Lambda)$  measurable. This means that  $(\Gamma, \mathcal{B}(\Gamma))$  is the projective limit of the measurable spaces  $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ ,  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ .

**Definition 2.1.** A set  $\Upsilon \in \mathcal{B}(\Gamma_0)$  is said to be bounded if

$$\Upsilon \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$$

for some  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$  and  $N \in \mathbb{N}$ .

## 2.2 Measures on configuration spaces

The basic examples of measures on  $\Gamma$  and  $\Gamma_0$  are the Poisson and the Lebesgue–Poisson measures, respectively, cf. Section 2.2 in [1].

Let  $\varrho : \mathbb{R}^d \rightarrow [0, +\infty)$  be measurable and essentially bounded. The latter means that, for any  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ ,

$$\ell_\varrho(\Lambda) := \int_\Lambda \varrho(x) dx < \infty. \quad (29)$$

The image of the product measure  $\varrho(x_1)dx_1 \cdots \varrho(x_n)dx_n$  in  $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$  is denoted by  $\sigma_\varrho^{(n)}$ . Then the (inhomogeneous) Lebesgue–Poisson measure on  $(\Gamma_0, \mathcal{B}(\Gamma_0))$  is set to be

$$\lambda_\varrho := \delta_\emptyset + \sum_{n=1}^{\infty} \frac{1}{n!} \sigma_\varrho^{(n)}.$$

Given  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ , the restriction of  $\lambda_\varrho$  to  $\Gamma_\Lambda$  will be denoted by  $\lambda_\varrho^\Lambda$ . Clearly,  $\lambda_\varrho^\Lambda$  is a finite measure on  $\mathcal{B}(\Gamma_\Lambda)$  such that  $\lambda_\varrho^\Lambda(\Gamma_\Lambda) = \exp(\ell_\varrho(\Lambda))$ . Then

$$\pi_\varrho^\Lambda := \exp(-\ell_\varrho(\Lambda)) \lambda_\varrho^\Lambda \quad (30)$$

is a probability measure on  $\mathcal{B}(\Gamma_\Lambda)$ . It can be shown [1] that the family  $\{\pi_\varrho^\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$  is consistent, and hence there exists a unique probability measure,  $\pi_\varrho$ , on  $\mathcal{B}(\Gamma)$  such that

$$\pi_\varrho^\Lambda = \pi_\varrho \circ p_\Lambda^{-1}, \quad \Lambda \in \mathcal{B}_b(\mathbb{R}^d),$$

$p_\Lambda$  being the same as in (28). This  $\pi_\varrho$  is called the (inhomogeneous) Poisson measure. For  $\varrho \equiv 1$  we obtain the homogeneous Lebesgue–Poisson and Poisson measures, which we denote by  $\lambda$  and  $\pi$ , respectively.

Let  $\mathcal{M}^1(\Gamma)$  be the set of all probability measures on  $(\Gamma, \mathcal{B}(\Gamma))$ . For  $\mu \in \mathcal{M}^1(\Gamma)$  and  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ , we set

$$\mu^\Lambda = \mu \circ p_\Lambda^{-1}. \quad (31)$$

By  $\mathcal{M}_{\text{fm}}^1(\Gamma)$  we denote the set of all  $\mu \in \mathcal{M}^1(\Gamma)$  which have finite local moments, that is, for which

$$\int_{\Gamma} |\gamma_{\Lambda}|^n \mu(d\gamma) < \infty \quad \text{for all } n \in \mathbb{N} \text{ and } \Lambda \in \mathcal{B}_b(\mathbb{R}^d).$$

A measure  $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$  is said to be *locally absolutely continuous* with respect to the Poisson measure  $\pi$  if, for every  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ ,  $\mu^{\Lambda}$  is absolutely continuous with respect to  $\pi^{\Lambda}$ , cf. (30) and (31). A measure  $\chi$  on  $(\Gamma_0, \mathcal{B}(\Gamma_0))$  is said to be *locally finite* if  $\rho(\Upsilon) < \infty$  for every bounded  $\Upsilon \subset \Gamma_0$ , cf. Definition 2.1. For a bounded  $\Upsilon \subset \Gamma_0$ , let  $\mathbb{I}_{\Upsilon}$  be its indicator function. For  $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ , the representation

$$\chi_{\mu}(\Upsilon) = \int_{\Gamma} \left( \sum_{\eta \subset \gamma} \mathbb{I}_{\Upsilon}(\eta) \right) \mu(d\gamma)$$

uniquely determines a locally finite measure  $\chi_{\mu}$  – the *correlation measure* for  $\mu$ . It is known, see Proposition 4.14 in [21], that  $\chi_{\mu}$  is absolutely continuous with respect to  $\lambda$  if  $\mu$  is locally absolutely continuous with respect to  $\pi$ . In this case, we have that, for any  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$  and  $\lambda^{\Lambda}$ -almost all  $\eta \in \Gamma_{\Lambda}$ ,

$$\begin{aligned} k_{\mu}(\eta) = \frac{d\chi_{\mu}}{d\lambda}(\eta) &= \int_{\Gamma_{\Lambda}} \frac{d\mu^{\Lambda}}{d\pi^{\Lambda}}(\eta \cup \gamma) \pi^{\Lambda}(d\gamma) \\ &= \int_{\Gamma_{\Lambda}} \frac{d\mu^{\Lambda}}{d\lambda}(\eta \cup \gamma) \lambda(d\gamma). \end{aligned} \quad (32)$$

The Radon–Nikodym derivative  $k_{\mu}$  is called the *correlation function* corresponding to the measure  $\mu$ . In the sequel,  $\lambda$  will be the basic measure on  $(\Gamma_0, \mathcal{B}(\Gamma_0))$ , and we shall tacitly assume that the formulas like (32) hold for  $\lambda$ -almost all  $\eta \in \Gamma_0$ .

Finally, we mention the following integration rule, see, e.g., [17],

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, \eta) \lambda(d\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi) \lambda(d\xi) \lambda(d\eta), \quad (33)$$

which holds for all such  $H$  for which both sides are finite.

### 2.3 The model

Regarding the kernels in (6) we suppose that

$$a^{\pm} \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d), \quad a^{\pm}(x) = a^{\pm}(-x) \geq 0. \quad (34)$$

Then we set

$$\langle a^{\pm} \rangle = \int_{\mathbb{R}^d} a^{\pm}(x) dx, \quad \|a^{\pm}\| = \text{ess sup}_{x \in \mathbb{R}^d} a^{\pm}(x), \quad (35)$$

and

$$E^\pm(\eta) = \sum_{x \in \eta} E^\pm(x, \eta \setminus x) = \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^\pm(x - y). \quad (36)$$

By (34), we have

$$E^\pm(\eta) \leq \|a^\pm\| |\eta|^2. \quad (37)$$

The operator  $L_{\varepsilon, \text{ren}}^\Delta$  in (21) and (22) has the following form, cf. (23) and [14],

$$L_{\varepsilon, \text{ren}}^\Delta = A_0 + B + \varepsilon C = V + \varepsilon C \quad (38)$$

where

$$(A_0 k)(\eta) = -m|\eta|k(\eta), \quad (39)$$

$$(Bk)(\eta) = - \int_{\mathbb{R}^d} E^-(y, \eta) k(\eta \cup y) dy + \int_{\mathbb{R}^d} \sum_{x \in \eta} a^+(x - y) k(\eta \setminus x \cup y) dy,$$

$$(Ck)(\eta) = -E^-(\eta)k(\eta) + \sum_{x \in \eta} E^+(x, \eta \setminus x)k(\eta \setminus x).$$

Note that  $L_{1, \text{ren}}^\Delta$  is exactly the operator  $L^\Delta$  defined in (14).

### 3 The Evolution of Correlation Functions

#### 3.1 The setting

For  $\alpha \in \mathbb{R}$ , we set

$$\mathcal{K}_\alpha = \{k : \Gamma_0 \rightarrow \mathbb{R} : \|k\|_\alpha < \infty\}, \quad (40)$$

where

$$\|k\|_\alpha = \text{ess sup}_{\eta \in \Gamma_0} |k(\eta)| \exp(\alpha|\eta|). \quad (41)$$

As mentioned above, each  $k : \Gamma_0 \rightarrow \mathbb{R}$  is a family of symmetric functions  $k^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}_0$ . Then (41) can be rewritten as

$$\|k\|_\alpha = \sup_{n \in \mathbb{N}_0} e^{n\alpha} \|k^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)}. \quad (42)$$

Thus,  $\mathcal{K}_\alpha$  is a Banach space. For  $\alpha'' < \alpha'$ , we have that  $\|k\|_{\alpha''} \leq \|k\|_{\alpha'}$ ; and hence,

$$\mathcal{K}_{\alpha'} \hookrightarrow \mathcal{K}_{\alpha''}, \quad \text{for } \alpha'' < \alpha'. \quad (43)$$

This embedding is continuous but not dense.

Our next aim is to define  $L_{\varepsilon, \text{ren}}^\Delta$ , given in (38), as a linear operator in  $\mathcal{K}_\alpha$ . Put

$$\mathcal{D}_\alpha(A_0) = \{k \in \mathcal{K}_\alpha : A_0 k \in \mathcal{K}_\alpha\}. \quad (44)$$

The sets  $\mathcal{D}_\alpha(B)$  and  $\mathcal{D}_\alpha(C)$  are defined analogously. Then

$$\mathcal{D}_\alpha(L_{\varepsilon, \text{ren}}^\Delta) := \mathcal{D}_\alpha(A_0) \cap \mathcal{D}_\alpha(B) \cap \mathcal{D}_\alpha(C) \quad (45)$$

is the domain of  $L_{\varepsilon, \text{ren}}^\Delta$  in  $\mathcal{K}_\alpha$ . Note that  $\mathcal{D}_\alpha(L_{\varepsilon, \text{ren}}^\Delta)$  in (45) is one and the same for all for  $\varepsilon > 0$ . Furthermore,

$$\mathcal{D}_\alpha(L_{0, \text{ren}}^\Delta) = \mathcal{D}_\alpha(V) = \mathcal{D}_\alpha(A_0) \cap \mathcal{D}_\alpha(B) \supset \mathcal{D}_\alpha(L_{\varepsilon, \text{ren}}^\Delta), \quad \varepsilon \in (0, 1]. \quad (46)$$

Let us show that

$$\forall \alpha' > \alpha \quad \mathcal{K}_{\alpha'} \subset \mathcal{D}_\alpha(L_{\varepsilon, \text{ren}}^\Delta). \quad (47)$$

By (42), we have

$$|k(\eta)| \leq \|k\|_{\alpha'} \exp(-\alpha'|\eta|), \quad \eta \in \Gamma_0.$$

Applying this in (39), by (35) and (37) we then get

$$|(Ck)(\eta)| \leq |\eta|^2 \exp(-(\alpha' - \alpha)|\eta|) \left[ \|a^-\| + \|a^+\| e^{\alpha'} \right] \|k\|_{\alpha'} \exp(-\alpha|\eta|), \quad (48)$$

and then also

$$\begin{aligned} |(Bk)(\eta)| &\leq \|k\|_{\alpha'} \exp(-\alpha'|\eta| - \alpha') \int_{\mathbb{R}^d} E^-(y, \eta) dy \\ &+ \|k\|_{\alpha'} \exp(-\alpha'|\eta|) \int_{\mathbb{R}^d} \sum_{x \in \eta} a^+(x - y) dy \\ &\leq |\eta| \exp(-(\alpha' - \alpha)|\eta|) \left[ \langle a^- \rangle e^{-\alpha'} + \langle a^+ \rangle \right] \|k\|_{\alpha'} \exp(-\alpha|\eta|). \end{aligned} \quad (49)$$

In a similar way, one estimates  $|(A_0k)(\eta)|$ . These three estimates readily yield (47).

### 3.2 The statements

Now we turn to solving the problems in (21) and (24).

**Definition 3.1.** By a classical solution of the problem (21), in the space  $\mathcal{K}_\alpha$  and on the time interval  $[0, T)$ , we understand a map  $[0, T) \ni t \mapsto r_{t, \varepsilon} \in \mathcal{D}_\alpha(L_{\varepsilon, \text{ren}}^\Delta)$ , cf. (45), continuously differentiable on  $[0, T)$ , such that (21) is satisfied for  $t \in [0, T)$ . A classical solution of (24), cf. (46), is defined in the same way.

*Remark 3.2.* In view of (47), we have that  $r_{t, \varepsilon} \in \mathcal{D}_\alpha(L_{\varepsilon, \text{ren}}^\Delta)$  whenever  $r_{t, \varepsilon} \in \mathcal{K}_{\alpha_t}$  for some  $\alpha_t > \alpha$ .

The main assumption under which we are going to solve (21) is the following: there exists  $\theta > 0$  such that, for almost all  $x \in \mathbb{R}^d$ , the following holds

$$a^+(x) \leq \theta a^-(x). \quad (50)$$

For  $\alpha^* \in \mathbb{R}$  and  $\alpha < \alpha^*$ , we set, cf. (35),

$$T(\alpha) = \frac{\alpha^* - \alpha}{\langle a^+ \rangle + \langle a^- \rangle e^{-\alpha}}. \quad (51)$$

**Theorem 3.3.** *Let (50) be satisfied, and let  $\alpha^* \in \mathbb{R}$  be such that*

$$e^{\alpha^*} \theta < 1. \quad (52)$$

*Then, for each  $\alpha < \alpha^*$ , the problem in (21) with  $r_0 \in \mathcal{K}_{\alpha^*}$  has a unique classical solution in  $\mathcal{K}_\alpha$  on  $[0, T(\alpha))$ .*

**Theorem 3.4.** *For each  $\alpha^* \in \mathbb{R}$  and  $\alpha < \alpha^*$ , the problem in (24) with  $r_0 \in \mathcal{K}_{\alpha^*}$  has a unique classical solution in  $\mathcal{K}_\alpha$  on  $[0, T(\alpha))$ .*

**Theorem 3.5.** *Let the assumptions of Theorems 3.3 and 3.4 be satisfied, and  $r_{t,\varepsilon}$  and  $r_t$  be the solution of (21) and (24), respectively. Then, for each  $\alpha < \alpha^*$  and  $t \in (0, T(\alpha))$ , it follows that*

$$\sup_{s \in [0, t]} \|r_{s,\varepsilon} - r_s\|_\alpha \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0. \quad (53)$$

*Remark 3.6.* (i) The condition in (50) can certainly be satisfied if the dispersal kernel decays faster than the competition kernel. The magnitude parameter  $\theta$  determines the large  $n$  asymptotics of the initial correlation function, see (42) and (52). Note that in Theorem 3.4 we do not require (50).

(ii) The main characteristic feature of the solutions mentioned in Theorems 3.3 and 3.4 is that, at a given  $t$ , they lie in a space,  $\mathcal{K}_\alpha$ , ‘bigger’ than the initial  $r_0$  does, cf. (43). The bigger  $t$ , the bigger should be the space  $\mathcal{K}_\alpha$ .

(iii) The function  $(-\infty, \alpha^*) \ni \alpha \mapsto T(\alpha)$  defined in (51) is bounded from above by a certain  $T^*(\langle a^+ \rangle, \langle a^- \rangle, \alpha^*)$  beyond which the solutions of both problems could not be extended, see, however, Remark 4.5 below.

### 3.3 The proof of the statements

In the sequel, for  $\alpha' > \alpha$  and a bounded linear operator  $Q : \mathcal{K}_{\alpha'} \rightarrow \mathcal{K}_\alpha$ , cf. (43), by  $\|Q\|_{\alpha'\alpha}$  we denote the corresponding operator norm.

The proof of the three theorems above is based on the following results, proven in Subsection 3.4 below.

**Lemma 3.7.** *Let  $\theta$  be as in (50) and  $\alpha \in \mathbb{R}$  be such that  $e^\alpha \theta < 1$ , cf. (52). Then, for arbitrary  $\varepsilon \in (0, 1]$ , there exists a closed subspace,  $\mathcal{A}_{\alpha,\varepsilon} \subset \mathcal{K}_\alpha$ , and a  $C_0$ -semigroup of linear contractions,  $S_{\alpha,\varepsilon}(t) : \mathcal{A}_{\alpha,\varepsilon} \rightarrow \mathcal{A}_{\alpha,\varepsilon}$ ,  $t \geq 0$ , with generator  $A_{\alpha,\varepsilon}$ , such that, for each  $\alpha' > \alpha$ , cf. (43), the following holds:*

- (a)  $\mathcal{K}_{\alpha'} \subset \text{Dom}(A_{\alpha,\varepsilon}) \subset \mathcal{A}_{\alpha,\varepsilon}$ ;
- (b) for each  $k \in \mathcal{K}_{\alpha'}$ ,  $A_{\alpha,\varepsilon}k = (A_0 + \varepsilon C)k$ , where the latter two operators are defined in (39) and (44), (45);
- (c) for each  $\alpha'' < \alpha$ , cf. (43), the restriction of  $S_{\alpha'',\varepsilon}$  to  $\mathcal{A}_{\alpha,\varepsilon}$  coincides with  $S_{\alpha,\varepsilon}$ .

*Remark 3.8.* By claim (c) of Lemma 3.7, for  $\alpha' > \alpha$  and  $\alpha'' \leq \alpha$ , each  $S_{\alpha'',\varepsilon}(t)$  can be considered as a bounded linear contraction from  $\mathcal{K}_{\alpha'}$  to  $\mathcal{K}_\alpha$ .

**Lemma 3.9.** *For  $\varepsilon = 0$ , all the statements of Lemma 3.7 hold true without any restrictions on  $\alpha$ . Furthermore, for each  $\alpha' > \alpha$ ,  $\alpha'' \leq \alpha$ ,  $\varepsilon \in (0, 1]$ , and any  $t > 0$ , it follows that*

$$\sup_{s \in [0, t]} \|(S_{\alpha'',\varepsilon}(s) - S_{\alpha'',0}(s))\|_{\alpha'\alpha} \leq \varepsilon t M(\alpha' - \alpha), \quad (54)$$

where, cf. (35),

$$M(\varkappa) := \left(\frac{2}{e\varkappa}\right)^2 \left(\|a^-\| + \|a^+\|e^{\alpha^*}\right), \quad (55)$$

which is independent of  $\varepsilon$ ,  $\alpha''$ , and  $t$ .

From these lemmas we immediately conclude that

**Corollary 3.10.** *For each  $\alpha' > \alpha$ ,  $\varepsilon \in [0, 1]$ , and any  $k \in \mathcal{K}_{\alpha'}$ , the map*

$$[0, +\infty) \ni t \mapsto S_{\alpha,\varepsilon}(t)k \in \mathcal{K}_\alpha$$

is continuous.

*Proof of Theorem 3.3.* For  $\varkappa > 0$  and  $\eta \in \Gamma_0$ , we have that

$$|\eta|e^{-\varkappa|\eta|} \leq \frac{1}{e\varkappa}, \quad |\eta|^2 e^{-\varkappa|\eta|} \leq \left(\frac{2}{e\varkappa}\right)^2. \quad (56)$$

Let us fix some  $\alpha_* < \alpha^*$  and then set  $T_* = T(\alpha_*)$ . For  $\alpha', \alpha \in [\alpha_*, \alpha^*]$  such that  $\alpha' > \alpha$ , the expression for  $B$  given in (39) can be used to define a bounded linear operator  $B : \mathcal{K}_{\alpha'} \rightarrow \mathcal{K}_\alpha$ . We shall keep the notation  $B$  for this operator if it is clear between which spaces it acts. However, additional labels will be used if we deal with more than one such  $B$  acting between different spaces. The norm of  $B : \mathcal{K}_{\alpha'} \rightarrow \mathcal{K}_\alpha$  can be estimated by means of (49) and (56) in the following way, cf. (51),

$$\|B\|_{\alpha'\alpha} \leq \frac{\alpha^* - \alpha_*}{e(\alpha' - \alpha)T_*}. \quad (57)$$

In a similar way, we define also bounded operators  $A_0, C : \mathcal{K}_{\alpha'} \rightarrow \mathcal{K}_\alpha$ , for which we get, cf. (39), (48), (55), and (56),

$$\|A_0\|_{\alpha'\alpha} \leq \frac{m}{e(\alpha' - \alpha)}, \quad \|C\|_{\alpha'\alpha} \leq \left(\frac{2}{e(\alpha' - \alpha)}\right)^2 \left(\|a^-\| + \|a^+\|e^{\alpha^*}\right). \quad (58)$$

Now we fix  $\alpha \in (\alpha_*, \alpha^*)$  and recall that  $T(\alpha)$  is defined in (51). Then, for  $r_0 \in \mathcal{K}_{\alpha^*}$  as in (21),  $n \in \mathbb{N}_0$ , and  $t \in [0, T(\alpha))$ , we define:  $u_{t,0} = 0$ , and, for  $n \in \mathbb{N}$ ,

$$u_{t,n} = S_{\alpha,\varepsilon}(t)r_0 + \sum_{l=1}^{n-1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} U_{\alpha,\varepsilon}^{(l)}(t, t_1, \dots, t_l) r_0 dt_1 \cdots dt_l, \quad (59)$$

where

$$U_{\alpha,\varepsilon}^{(l)}(t, t_1, \dots, t_l) := S_{\alpha,\varepsilon}(t - t_1)B_1S_{\alpha,\varepsilon}(t_1 - t_2)B_2 \quad (60)$$

$$\times \cdots \times S_{\alpha,\varepsilon}(t_{l-1} - t_l)B_lS_{\alpha,\varepsilon}(t_l).$$

The latter operator is suppose to act from  $\mathcal{K}_{\alpha^*}$  to  $\mathcal{K}_\alpha$ . In order to define it, we first fix  $q > 1$  such that also  $qt < T(\alpha)$ . Then introduce

$$\alpha_{2p} = \alpha_0 - p\epsilon_1 - p\epsilon_2, \quad \alpha_{2p+1} = \alpha_0 - (p+1)\epsilon_1 - p\epsilon_2, \quad p = 0, \dots, l, \quad (61)$$

where  $\alpha_0 = \alpha^*$ ,  $\alpha_{2l+1} = \alpha$ , and

$$\epsilon_1 = \frac{(q-1)(\alpha^* - \alpha)}{q(l+1)}, \quad \epsilon_2 = \frac{\alpha^* - \alpha}{ql}. \quad (62)$$

Then the operators in the product in (60) act as follows, cf. Lemma 3.7, (39), and (57),

$$S_{\alpha,\varepsilon}(t_l) : \mathcal{K}_{\alpha_0} \rightarrow \mathcal{K}_{\alpha_1}, \quad (63)$$

$$S_{\alpha,\varepsilon}(t_{l-p} - t_{l-p+1}) : \mathcal{K}_{\alpha_{2p}} \rightarrow \mathcal{K}_{\alpha_{2p+1}},$$

$$B_{l-p+1} : \mathcal{K}_{\alpha_{2p-1}} \rightarrow \mathcal{K}_{\alpha_{2p}}, \quad p = 1, \dots, l.$$

Since all  $S_{\alpha,\varepsilon}(t)$  are contractions, from (59) we have

$$\|u_{t,n+1} - u_{t,n}\|_\alpha \leq \frac{t^n}{n!} \|B_1 B_2 \cdots B_n\|_{\alpha^* \alpha} \|r_0\|_{\alpha^*}, \quad n \in \mathbb{N}. \quad (64)$$

Note that the right-hand side of (64) is independent of  $\varepsilon$ . To estimate it we use (57), then (62) and the last line in (63) with  $l = n$ , which yields, cf. (62),

$$\|B_{n-p+1}\|_{\alpha_{2p-1} \alpha_{2p}} \leq \frac{\alpha^* - \alpha}{e\epsilon_2 T(\alpha)} = \frac{qn}{eT(\alpha)}, \quad p = 1, \dots, n. \quad (65)$$

Applying this in (64) we obtain

$$\sup_{s \in [0, t]} \|u_{s,n+1} - u_{s,n}\|_\alpha \leq \frac{1}{n!} \left(\frac{n}{e}\right)^n \left(\frac{qt}{T(\alpha)}\right)^n, \quad (66)$$

which means that  $\{u_{s,n}\}_{n \in \mathbb{N}_0}$  is a Cauchy sequence in each of  $\mathcal{K}_{\alpha'}$ ,  $\alpha' \in [\alpha_*, \alpha]$ , uniformly in  $\varepsilon \in (0, 1]$  and in  $s \in [0, t]$ . Let  $u_s \in \mathcal{K}_\alpha$  be its limit. By Corollary 3.10,  $U_{\alpha,\varepsilon}^{(l)}$  is integrable in  $t_1, \dots, t_l$ , and hence  $u_{t,n} \in \mathcal{K}_\alpha$  is continuously differentiable in  $\mathcal{K}_\alpha$  on  $t \in [0, T(\alpha))$ . Then, see claim (b) of Lemma 3.7,

$$\frac{d}{dt} u_{t,n} = A_{\alpha,\varepsilon} u_{t,n} + B u_{t,n-1} = (A_0 + \varepsilon C) u_{t,n} + B u_{t,n-1}, \quad (67)$$

where both  $(A_0 + \varepsilon C)$  and  $B$  act from  $\mathcal{K}_\alpha$  to  $\mathcal{K}_{\alpha^*}$ . On the other hand,

$$\begin{aligned} \sup_{s \in [0, t]} \|(A_0 + \varepsilon C)(u_{s, n+1} - u_{s, n})\|_{\alpha^*} &\leq \left( \|A_0\|_{\alpha\alpha^*} + \|C\|_{\alpha\alpha^*} \right) \sup_{s \in [0, t]} \|u_{s, n+1} - u_{s, n}\|_\alpha, \\ \sup_{s \in [0, t]} \|B(u_{s, n+1} - u_{s, n})\|_{\alpha^*} &\leq \|B\|_{\alpha\alpha^*} \sup_{s \in [0, t]} \|u_{s, n+1} - u_{s, n}\|_\alpha, \end{aligned}$$

where the operator norms can be estimated as in (58). Hence, by (66)  $\{du_{s, n}/ds\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{K}_{\alpha^*}$ , uniformly in  $s \in [0, t]$ . Therefore, the limiting  $u_s \in \mathcal{K}_\alpha \subset \mathcal{K}_{\alpha^*}$  is continuously differentiable on  $[0, t]$ , and

$$du_{s, n}/ds \rightarrow du_s/ds, \quad n \rightarrow +\infty.$$

On the other hand, the right-hand side of (67) converges in  $\mathcal{K}_{\alpha^*}$  to  $L_{\varepsilon, \text{ren}}^\Delta u_s$ , see (38). Hence,  $u_s$  is the classical solution on  $[0, t]$ , see Definition 3.1 and Remark 3.2. Since such  $t$  can be arbitrary in  $(0, T_*)$ , this proves the existence of the solution in question.

Let us now prove the uniqueness. In view of the linearity of the problem (21), it is enough to prove that the zero function is its only solution corresponding to the zero initial condition. Let  $\alpha \in (\alpha_*, \alpha^*)$  be fixed, and let  $t \in (0, T_*)$  be such that  $t < T(\alpha)$ , see (51). Then the problem in (21) with the zero initial condition has classical solutions in  $\mathcal{K}_\alpha$  on  $[0, t]$ . Each of them solves also the following integral equation, cf. (59),

$$u_t = \int_0^t S_{\alpha, \varepsilon}(t-s) B u_s ds, \quad (68)$$

see the proof of Theorem IX.1.19, page 486 in [20]. Since  $t < T(\alpha)$ , there exists  $\alpha' \in (\alpha_*, \alpha)$  such that also  $t < T(\alpha')$ . Note that  $u_t$  in the left-hand side of (68) lies in  $\mathcal{K}_{\alpha'}$ , cf. (43). Then the meaning of (68) is the following. As a bounded operator, cf (57),  $B$  maps  $u_s \in \mathcal{K}_\alpha$  to  $B u_s \in \mathcal{K}_{\alpha'}$ ,  $\alpha' \in (\alpha', \alpha)$ , continuously in  $s$ . Furthermore, in view of claim (c) of Lemma 3.7,  $S_{\alpha, \varepsilon}$  in (68) can be replaced by  $S_{\alpha', \varepsilon}$ . Then, by Corollary 3.10, the integrand in (68) is continuous in  $s$ , and hence the integral makes sense. Now we iterate (68)  $n$  times and get, cf. (59) and (60),

$$\begin{aligned} u_t &= \int_0^t S_{\alpha', \varepsilon}(t-t_1) B_1 \int_0^{t_1} S_{\alpha', \varepsilon}(t_1-t_2) B_2 \times \\ &\times \cdots \times \int_0^{t_n} S_{\alpha', \varepsilon}(t_{n-1}-t_n) B_n u_{t_n} dt_1 \cdots dt_n, \end{aligned}$$

from which we deduce the following estimate, cf (64),

$$\|u_t\|_{\alpha'} \leq \frac{t^n}{n!} \|B_1 B_2 \cdots B_n\|_{\alpha\alpha'} \sup_{s \in [0, t]} \|u_s\|_\alpha. \quad (69)$$

Recall that  $[0, t] \ni s \mapsto u_s \in \mathcal{K}_\alpha$  is continuous. Similarly as in the case of (64) we then get

$$\|B_1 B_2 \cdots B_n\|_{\alpha\alpha'} \leq \left(\frac{n}{e}\right)^n \left[ \frac{\alpha^* - \alpha}{T(\alpha)(\alpha - \alpha')} \right]^n.$$

Hence, by picking large enough  $n$  the right-hand side of (69) can be made arbitrarily small whenever

$$t < \left( \frac{\alpha - \alpha'}{\alpha^* - \alpha} \right) T(\alpha).$$

This proves that  $u_t$  is the zero element of  $\mathcal{K}_{\alpha'}$  for small enough  $t > 0$ . Since  $u_t$  lies in  $\mathcal{K}_\alpha \subset \mathcal{K}_{\alpha'}$ , then it is also the zero element of  $\mathcal{K}_\alpha$ . The extension of this result to each  $t < T_*$  can be made in the same way.  $\square$

*Remark 3.11.* The proof of the uniqueness just presented is a modification of the corresponding result in the so called Ovcyannikov theorem, see pages 16 and 17 in [34]. Here, however, we deal with operators which cannot be directly accommodated to the Ovcyannikov scheme, see the second estimate in (58). We managed to take these operators into account through the semigroup as in Lemma 3.7.

*Proof of Theorem 3.4.* By repetition of the construction used in the proof of Theorem 3.3 we obtain that the sequence defined recursively, cf. (59), by

$$\begin{aligned} v_{t,n} = S_{\alpha_*,0}(t)r_0 &+ \sum_{l=1}^{n-1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} S_{\alpha_*,0}(t-t_1)B_1 \\ &\times S_{\alpha_*,0}(t_1-t_2)B_2 \cdots S_{\alpha_*,0}(t_{l-1}-t_l)B_l S_{\alpha_*,0}(t_l)r_0 dt_1 \cdots dt_l, \end{aligned} \quad (70)$$

converges in  $\mathcal{K}_{\alpha_*}$  to a classical solution of (24). In contrast to the case of  $\varepsilon > 0$ , the semigroup  $S_{\alpha,0}$  can be constructed explicitly. Indeed, cf. claim (b) of Lemma 3.7,

$$(S_{\alpha,0}(t)v)(\eta) = \exp(-tm|\eta|)v(\eta), \quad v \in \mathcal{K}_\alpha, \quad (71)$$

which allows for dropping (52) in this case. The rest of the proof goes exactly as in Theorem 3.3.  $\square$

*Proof of Theorem 3.5.* Let  $r_{t,\varepsilon}$  be the limit of the sequence  $\{u_{t,n}\}_{n \in \mathbb{N}}$  defined in (59). We prove that this  $r_{t,\varepsilon}$  converges to  $r_t$ , as stated in the theorem. Given  $\delta > 0$  and  $t \in (0, T_*)$ , let  $n_\delta \in \mathbb{N}$  be such that, for all  $n > n_\delta$ , both following estimates hold

$$\sup_{s \in [0,t]} \|u_{s,n} - r_{s,\varepsilon}\|_{\alpha_*} < \delta/3, \quad \sup_{s \in [0,t]} \|v_{s,n} - r_s\|_{\alpha_*} < \delta/3, \quad (72)$$

where  $v_{s,n}$  is defined in (70). Note that the first estimate in (72) is uniform in  $\varepsilon \in (0, 1]$  as the right-hand side of (64) is  $\varepsilon$ -independent. By (59) and (70), we get

$$\begin{aligned} u_{s,n} - v_{s,n} &= Q_\varepsilon(s)r_0 \\ &+ \sum_{l=1}^{n-1} \int_0^s \int_0^{s_1} \cdots \int_0^{s_{l-1}} \sum_{p=1}^{l+1} R_\varepsilon^{(p,l)}(s, s_1, \dots, s_l)r_0 ds_1 \cdots ds_l, \end{aligned} \quad (73)$$

where

$$\begin{aligned} Q_\varepsilon(s) &:= S_{\alpha_*,\varepsilon}(s) - S_{\alpha_*,0}(s), \\ R_\varepsilon^{(0,l)}(s, s_1, \dots, s_l) &:= S_{\alpha_*,\varepsilon}(s - s_1)B_1 \cdots S_{\alpha_*,\varepsilon}(s_{l-1} - s_l)B_l Q_\varepsilon(s_l), \end{aligned} \quad (74)$$

and, for  $p = 1, \dots, l$ ,

$$\begin{aligned} R_\varepsilon^{(p,l)}(s, s_1, \dots, s_l) &:= S_{\alpha_*,\varepsilon}(s - s_1)B_1 \cdots S_{\alpha_*,\varepsilon}(s_{l-p-1} - s_{l-p})B_{l-p} \\ &\times Q_\varepsilon(s_{l-p} - s_{l-p+1})B_{l-p+1}S_{\alpha_*,0}(s_{l-p+1} - s_{l-p+2})B_{l-p+1} \\ &\times \cdots \times S_{\alpha_*,0}(s_{l-1} - s_l)B_l S_{\alpha_*,0}(s_l). \end{aligned} \quad (75)$$

In (75), the operators  $S_{\alpha_*,\varepsilon}$  and  $B_p$  act as in (63). Then taking into account that  $\|S_{\alpha_*,\varepsilon}(s)\|_{\alpha'\alpha} \leq 1$ , see Lemma 3.7, and likewise  $\|S_{\alpha_*,0}(s)\|_{\alpha'\alpha} \leq 1$ , cf. (71), for a fixed  $s \in (0, T_*)$  and  $q > 1$  such that  $qs < T_*$ , we obtain from (74), (75) and from (54), (55), (61), (62), (63), and (65) that

$$\begin{aligned} \|R_\varepsilon^{(0,l)}(s, s_1, \dots, s_l)\|_{\alpha^*\alpha_*} &\leq \varepsilon K(l+1)^2 s_l \|B_1 \cdots B_l\|_{\alpha^*\alpha_*} \\ &\leq \varepsilon K(l+1)^2 s_l \left(\frac{ql}{eT_*}\right)^l, \end{aligned}$$

and likewise for  $p = 1, \dots, l$ ,

$$\|R_\varepsilon^{(p,l)}(s, s_1, \dots, s_l)\|_{\alpha^*\alpha_*} \leq \varepsilon K(l+1)^2 (s_{l-p} - s_{l-p+1}) \left(\frac{ql}{eT_*}\right)^l,$$

where

$$K := \left[ \frac{2q}{e(q-1)(\alpha_* - \alpha_*)} \right]^2.$$

Applying both latter estimates in (73) we finally get that, for  $t \in [0, T_*)$  and  $q > 1$  such that  $qt < T_*$ , the following holds

$$\sup_{s \in [0, t]} \|u_{s,n} - v_{s,n}\|_{\alpha_*} \leq \varepsilon \|r_0\|_{\alpha^*} \varphi_q(t), \quad (76)$$

where

$$\varphi_q(t) := tK \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{l}{e}\right)^l (l+1)^2 \left(\frac{qt}{T_*}\right)^l.$$

Now we fix  $n > n_\delta$  such that both estimates in (72) hold, independently of  $\varepsilon$ . Next, for this fixed  $n$ , we pick  $\varepsilon$  such that also the left-hand side of (76) is less than  $\delta/3$ , which by the triangle inequality yields (53).  $\square$

### 3.4 The proof of Lemmas 3.7 and 3.9

The semigroups in question will be obtained with the help of the corresponding semigroups constructed in the pre-dual spaces.

For a given  $\alpha \in \mathbb{R}$ , the space  $\mathcal{K}_\alpha$  defined in (40), (41) is dual to the following Banach space

$$\mathcal{G}_\alpha := L^1(\Gamma_0, e^{-\alpha|\cdot|} d\lambda),$$

in which the norm is given by

$$\|G\|_\alpha = \int_{\Gamma_0} |G(\eta)| \exp(-\alpha|\eta|) \lambda(d\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{-n\alpha} \|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}.$$

Recall that  $G : \Gamma_0 \rightarrow \mathbb{R}$  is a sequence of symmetric  $G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ . The duality is defined by the pairing

$$\begin{aligned} \langle\langle G, k \rangle\rangle &= \int_{\Gamma_0} G(\eta) k(\eta) \lambda(d\eta) \\ &= G^{(0)} k^{(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \dots, x_n) k^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n. \end{aligned} \quad (77)$$

For  $G : \Gamma_0 \rightarrow \mathbb{R}$ , we set

$$(\widehat{A}_\varepsilon^{(1)} G)(\eta) = -(|\eta|m + \varepsilon E^-(\eta)) G(\eta) := -E(\eta) G(\eta), \quad (78)$$

$$(\widehat{A}_\varepsilon^{(2)} G)(\eta) = \varepsilon \int_{\mathbb{R}^d} E^+(y, \eta) G(\eta \cup y) dy,$$

and

$$\widehat{A}_\varepsilon = \widehat{A}_\varepsilon^{(1)} + \widehat{A}_\varepsilon^{(2)}. \quad (79)$$

For  $\alpha \in \mathbb{R}$  as in Lemma 3.7, we define

$$\begin{aligned} \mathcal{D}_\alpha(\widehat{A}_\varepsilon^{(1)}) &= \{G \in \mathcal{G}_\alpha : E(\cdot)G(\cdot) \in \mathcal{G}_\alpha\}, \\ \mathcal{D}_\alpha(\widehat{A}_\varepsilon^{(2)}) &= \{G \in \mathcal{G}_\alpha : E^+(\cdot)G(\cdot) \in \mathcal{G}_\alpha\}, \end{aligned} \quad (80)$$

where  $E^\pm(\eta)$  are given in (36). Now we use (78) to define the corresponding operators in  $\mathcal{G}_\alpha$ . As a multiplication operator,  $\widehat{A}_\varepsilon^{(1)} : \mathcal{G}_\alpha \rightarrow \mathcal{G}_\alpha$  with  $\text{Dom}(\widehat{A}_\varepsilon^{(1)}) = \mathcal{D}_\alpha(\widehat{A}_\varepsilon^{(1)})$  is closed. By (33), for  $G \in \mathcal{D}_\alpha(\widehat{A}_\varepsilon^{(2)})$ , we get

$$\begin{aligned} \|\widehat{A}_\varepsilon^{(2)} G\|_\alpha &\leq \varepsilon \int_{\Gamma_0} \int_{\mathbb{R}^d} E^+(y, \eta) |G(\eta \cup y)| e^{-\alpha|\eta|} dy \lambda(d\eta) \\ &= e^\alpha \varepsilon \int_{\Gamma_0} |G(\eta)| e^{-\alpha|\eta|} \left( \sum_{x \in \eta} E^+(x, \eta \setminus x) \right) \lambda(d\eta) \\ &= e^\alpha \varepsilon \|E^+(\cdot)G(\cdot)\|_\alpha. \end{aligned} \quad (81)$$

Hence,  $\widehat{A}_\varepsilon^{(2)} : \mathcal{G}_\alpha \rightarrow \mathcal{G}_\alpha$  with  $\text{Dom}(\widehat{A}_\varepsilon^{(2)}) = \mathcal{D}_\alpha(\widehat{A}_\varepsilon^{(2)})$  is well-defined.

We say that  $G \in \mathcal{G}_\alpha$  is positive if  $G(\eta) \geq 0$  for  $\lambda$ -almost all  $\eta \in \Gamma_0$ . Let  $\mathcal{G}_\alpha^+$  be the cone of all positive  $G \in \mathcal{G}_\alpha$ . An operator  $(Q, \text{Dom}Q)$  on  $\mathcal{G}_\alpha$  is said to be positive if  $Q : \text{Dom}(Q) \cap \mathcal{G}_\alpha^+ \rightarrow \mathcal{G}_\alpha^+$ . A semigroup of operators  $S(t) : \mathcal{G}_\alpha \rightarrow \mathcal{G}_\alpha$ ,  $t \geq 0$ , is called *sub-stochastic* if each  $S(t)$  is positive and  $\|S(t)G\|_\alpha \leq \|G\|_\alpha$  for all  $G \in \mathcal{G}_\alpha$ . The proof of Lemma 3.7 is based on Theorem 2.2 of [33] which we formulate here in the following form.

**Proposition 3.12.** *Let  $(Q_0, \text{Dom}(Q_0))$  be the generator of a  $C_0$ -semigroup of positive operators on  $\mathcal{G}_\alpha$ , and let  $Q_1 : \text{Dom}(Q_0) \rightarrow \mathcal{G}_\alpha$  be positive and such that, for all  $G \in \text{Dom}(Q_0) \cap \mathcal{G}_\alpha^+$ ,*

$$\int_{\Gamma_0} ((Q_0 + Q_1)G)(\eta) \exp(-\alpha|\eta|) \lambda(d\eta) \leq 0.$$

*Then, for all  $\varkappa \in (0, 1)$ , the operator  $(Q_0 + \varkappa Q_1, \text{Dom}(Q_0))$  is the generator of a sub-stochastic semigroup on  $\mathcal{G}_\alpha$ .*

*Proof of Lemma 3.7.* The operator  $\widehat{A}_\varepsilon^{(1)}$  defined in (78) and (80) generates a positive semigroup on  $\mathcal{G}_\alpha$ . For a  $\varkappa \in (0, 1)$ , by (33) we have, cf. (81),

$$\begin{aligned} & \int_{\Gamma_0} \left( (\widehat{A}_\varepsilon^{(1)} + \varkappa^{-1} \widehat{A}_\varepsilon^{(2)})G(\eta) \right) \exp(-\alpha|\eta|) \lambda(d\eta) \tag{82} \\ &= -m \int_{\Gamma_0} |\eta|G(\eta) \exp(-\alpha|\eta|) \lambda(d\eta) - \varepsilon \int_{\Gamma_0} E^-(\eta)G(\eta) \exp(-\alpha|\eta|) \lambda(d\eta) \\ &+ \frac{\varepsilon}{\varkappa} \int_{\Gamma_0} \int_{\mathbb{R}^d} E^+(y, \eta)G(\eta \cup y) \exp(-\alpha|\eta|) \lambda(d\eta) \\ &= -m \int_{\Gamma_0} |\eta|G(\eta) \exp(-\alpha|\eta|) \lambda(d\eta) \\ &- \varepsilon \int_{\Gamma_0} [E^-(\eta) - \varkappa^{-1} e^\alpha E^+(\eta)] \exp(-\alpha|\eta|) \lambda(d\eta). \end{aligned}$$

Since  $e^\alpha \theta < 1$ , we can pick  $\varkappa \in (0, 1)$  such that also  $e^\alpha \theta \varkappa^{-1} < 1$ . For this  $\varkappa$ , by (50) we have  $[E^-(\eta) - \varkappa^{-1} e^\alpha E^+(\eta)] \geq 0$  for  $\lambda$ -almost all  $\eta$ , which means that the left-hand side of (82) is non-positive for such  $\varkappa$ . By (81), we also have that  $\varkappa^{-1} \widehat{A}_\varepsilon^{(2)}$  is positive and defined on the domain of  $\widehat{A}_\varepsilon^{(1)}$ . Then, by Proposition 3.12, the operator  $\widehat{A}_\varepsilon = \widehat{A}_\varepsilon^{(1)} + \varkappa(\varkappa^{-1} \widehat{A}_\varepsilon^{(2)})$ , cf. (79), is the generator of a sub-stochastic semigroup on  $\mathcal{G}_\alpha$ , which we denote by  $\widehat{S}_{\alpha, \varepsilon}(t)$ ,  $t \geq 0$ . Note that this in particular means

$$\|\widehat{S}_{\alpha, \varepsilon}(t)G\|_\alpha \leq \|G\|_\alpha, \quad t > 0, \quad G \in \mathcal{G}_\alpha. \tag{83}$$

Now for  $t > 0$ , let  $\widehat{S}_{\alpha, \varepsilon}^*(t)$  be the operator adjoint to  $\widehat{S}_{\alpha, \varepsilon}(t)$ . All such operators constitute a semigroup on  $\mathcal{K}_\alpha$ , which, however is not strongly continuous. Let

$\widehat{A}_\varepsilon^*$  be the adjoint to  $\widehat{A}_\varepsilon$ . Its domain is, cf. (77),

$$\text{Dom}(\widehat{A}_\varepsilon^*) = \{k \in \mathcal{K}_\alpha : \exists \tilde{k} \in \mathcal{K}_\alpha \quad \forall G \in \text{Dom}(\widehat{A}_\varepsilon) \quad \langle \widehat{A}_\varepsilon G, k \rangle = \langle G, \tilde{k} \rangle\}, \quad (84)$$

Let  $\mathcal{A}_{\alpha,\varepsilon}$  denote the closure of (84) in  $\mathcal{K}_\alpha$ . By (33), from the very definition in (84) it follows that

$$\forall \alpha' > \alpha \quad \mathcal{K}_{\alpha'} \subset \text{Dom}(\widehat{A}_\varepsilon^*) \subset \overline{\text{Dom}(\widehat{A}_\varepsilon^*)} =: \mathcal{A}_{\alpha,\varepsilon}, \quad (85)$$

and

$$\forall k \in \mathcal{K}_{\alpha'} \quad \widehat{A}_\varepsilon^* k = (A_0 + \varepsilon C)k. \quad (86)$$

Note that  $\mathcal{A}_{\alpha,\varepsilon}$  is a proper subspace of  $\mathcal{K}_\alpha$ . Now, for  $t > 0$ , we set

$$S_{\alpha,\varepsilon}(t) = \widehat{S}_{\alpha,\varepsilon}^*(t) \big|_{\mathcal{A}_{\alpha,\varepsilon}}. \quad (87)$$

By Theorem 10.4, page 39 in [30], the collection  $\{S(t)_{\alpha,\varepsilon}\}_{t \geq 0}$  constitutes a  $C_0$ -semigroup on  $\mathcal{A}_{\alpha,\varepsilon}$ . Its generator  $A_{\alpha,\varepsilon}$  is the part of  $\widehat{A}_\varepsilon^*$  in  $\mathcal{A}_{\alpha,\varepsilon}$ , that is, the restriction of  $\widehat{A}_{\alpha,\varepsilon}^*$  to the set

$$\text{Dom}(A_{\alpha,\varepsilon}) := \{k \in \text{Dom}(\widehat{A}_\varepsilon^*) : \widehat{A}_\varepsilon^* k \in \mathcal{A}_{\alpha,\varepsilon}\}. \quad (88)$$

By (48) and (49), it can be shown that, for any  $\alpha'' \in (\alpha, \alpha')$ , both  $A_0$  and  $C$  act as bounded operators from  $\mathcal{K}_{\alpha'}$  to  $\mathcal{K}_{\alpha''}$ . Therefore,  $(A_0 + \varepsilon C)k \in \mathcal{K}_{\alpha''} \subset \mathcal{A}_{\alpha,\varepsilon}$ , and hence

$$\mathcal{K}_{\alpha'} \subset \text{Dom}(A_{\alpha,\varepsilon}). \quad (89)$$

Thus, the objects introduced in (86), (87), and (88) have the properties stated in the lemma, cf. (85) and (89).  $\square$

*Proof of Lemma 3.9.* For  $\varepsilon = 0$ , we have, cf. (78) and (79),  $\widehat{A}_0 = \widehat{A}_0^{(1)}$ , where the latter is the multiplication operator by  $|\cdot|$ . Hence, the operator, cf. (39),

$$(A_{\alpha,0}k)(\eta) = -m|\eta|k(\eta), \quad \mathcal{A}_{\alpha,0} = \text{Dom}(A_{\alpha,0}) := \{k \in \mathcal{K}_\alpha : |\cdot|k \in \mathcal{K}_\alpha\} \quad (90)$$

is the generator of the semigroup of  $S_{\alpha,0}(t)$ ,  $t \geq 0$ , defined by

$$(S_{\alpha,0}(t)k)(\eta) = \exp(-tm|\eta|)k(\eta).$$

Clearly, for any  $\varepsilon \in (0, 1)$ ,

$$\mathcal{A}_{\alpha,\varepsilon} \subset \mathcal{A}_{\alpha,0}. \quad (91)$$

Let us shown now (54). For  $k \in \mathcal{K}_{\alpha'}$ , by (86) and (90), we have

$$(A_{\alpha,\varepsilon} - A_{\alpha,0})k = \varepsilon Ck. \quad (92)$$

For such  $k$ , we set

$$u_t = (S_{\alpha,\varepsilon}(t) - S_{\alpha,0}(t))k. \quad (93)$$

Then  $u_0 = 0$  and, cf. (92),

$$\begin{aligned} \frac{d}{dt}u_t &= S_{\alpha,\varepsilon}(t)A_{\alpha,\varepsilon}k - S_{\alpha,0}(t)A_{\alpha,0}k \\ &= \varepsilon S_{\alpha,\varepsilon}(t)Ck + A_{\alpha,0}u_t. \end{aligned} \quad (94)$$

In the latter line, we have taken into account also (91). By (48), one can define  $C$  as a bounded linear operator  $C : \mathcal{K}_{\alpha'} \rightarrow \mathcal{K}_{\alpha''}$  for  $\alpha'' \in (\alpha, \alpha')$ . Then  $Ck \in \text{Dom}(A_{\alpha,\varepsilon})$ , cf. (89), and hence

$$[0, +\infty) \ni t \mapsto \varphi_t := S_{\alpha,\varepsilon}(t)Ck \in \mathcal{K}_\alpha$$

is continuously differentiable in  $\mathcal{K}_\alpha$  on  $[0, +\infty)$ . In view of (83),

$$\|S_{\alpha,\varepsilon}(t)\| \leq 1, \quad \text{for all } t \geq 0 \text{ and } \varepsilon \in [0, 1].$$

Then

$$\|\varphi_t\|_\alpha \leq \|C\|_{\alpha'\alpha} \|k\|_{\alpha'}. \quad (95)$$

Then, by Theorem 1.19, page 486 in [20], we have from the second line in (94)

$$u_t = \varepsilon \int_0^t S_{\alpha,0}(t-s)\varphi_s ds,$$

which by (95) yields

$$\sup_{t \in [0, T]} \|u_t\|_\alpha \leq \varepsilon T \|C\|_{\alpha'\alpha} \|k\|_{\alpha'}.$$

Then (54) and (55) follow from the latter by (93) and (58).  $\square$

## 4 The Kinetic Equation

### 4.1 Solving the equation

For the model which we consider, the kinetic equation is the following Cauchy problem in  $L^\infty(\mathbb{R}^d)$ , cf. (25),

$$\frac{d}{dt}\varrho_t = -m\varrho_t - (a^- * \varrho_t)\varrho_t + (a^+ * \varrho_t), \quad \varrho_t|_{t=0} = \varrho_0. \quad (96)$$

Here, for an appropriate function  $\varrho : \mathbb{R}^d \rightarrow \mathbb{R}$ , we write

$$(a^\pm * \varrho)(x) = \int_{\mathbb{R}^d} a^\pm(x-y)\varrho(y)dy = \int_{\mathbb{R}^d} \varrho(x-y)a^\pm(y)dy, \quad (97)$$

where  $a^\pm$  are the kernels as in (6). The main peculiarity of (96) is that the solution of (24) can be sought in the form

$$r_t(\eta) = e(\varrho_t, \eta) := \prod_{x \in \eta} \varrho_t(x), \quad (98)$$

where  $\varrho_t \in L^\infty(\mathbb{R}^d)$  is a solution of (96). Denote

$$\begin{aligned}\Delta^+ &= \{\varrho \in L^\infty(\mathbb{R}^d) : \varrho(x) \geq 0 \text{ for a.a. } x\}, \\ \Delta_b &= \{\varrho \in L^\infty(\mathbb{R}^d) : \|\varrho\|_{L^\infty(\mathbb{R}^d)} \leq b\}, \quad b > 0, \\ \Delta_b^+ &= \Delta^+ \cap \Delta_b.\end{aligned}\tag{99}$$

**Lemma 4.1.** *Let  $\alpha^*$ ,  $\alpha < \alpha^*$ , and  $T(\alpha)$  be as in Theorem 3.3. Set  $b_0 = \exp(-\alpha^*)$  and  $b = \exp(-\alpha)$ . Suppose that, for  $t \in [0, T(\alpha))$ , the problem in (96) with  $\varrho_0 \in \Delta_{b_0}^+$ , has a unique classical solution  $\varrho_s \in \Delta_b^+$  on  $[0, t]$ . Then the unique solution  $r_s \in \mathcal{K}_\alpha$  of (24) with  $r_0(\eta) = e(\varrho_0, \eta)$ , see Theorem 3.4, is given by (98).*

*Proof.* First of all we note that, for a given  $\alpha$ ,  $e(\varrho, \cdot) \in \mathcal{K}_\alpha$  if and only if  $\varrho \in \Delta_b$  with  $b = e^{-\alpha}$ , see (40) and (41). Now set  $\tilde{r}_t = e(\varrho_t, \cdot)$  with  $\varrho_t$  solving (96). This  $\tilde{r}_t$  solves (24), which can easily be checked by computing  $d/dt$  and employing the equation in (96). In view of the uniqueness as in Theorem 3.4, we then have  $\tilde{r}_s = r_s$  on  $[0, t]$ , from which it can be continued to  $[0, T(\alpha))$ .  $\square$

*Remark 4.2.* As (98) is the correlation function of the Poisson measure  $\pi_{\varrho_t}$ , see (29) and (30), the above lemma establishes the so called *chaos preservation* or *chaos propagation* in time as the most chaotic states are those corresponding to Poisson measures.

Now let us turn to solving (96).

**Theorem 4.3.** *For arbitrary  $\varrho_0 \in \Delta^+$ , the problem (96) has a unique classical solution  $\varrho_t \in \Delta^+$  on  $[0, +\infty)$ .*

*Proof.* For a certain  $\epsilon \geq 0$ , let us consider

$$u_t(x) = e^{-\epsilon t} \varrho_t(x), \quad t \geq 0, \quad x \in \mathbb{R}^d.\tag{100}$$

Then  $\varrho_t$  solves (96) if and only if  $u_t$  solves the following problem

$$\frac{d}{dt} u_t = -(m + \epsilon)u_t - e^{\epsilon t} u_t (a^- * u_t) + (a^+ * u_t), \quad u_t|_{t=0} = \varrho_0.$$

This differential problem is equivalent to the integral equation

$$\begin{aligned}u_t &= \varrho_0 \exp\left(- (m + \epsilon)t - \int_0^t e^{\epsilon \tau} (a^- * u_\tau) d\tau\right) \\ &+ \int_0^t (a^+ * u_\tau) \exp\left(- (m + \epsilon)\tau - \int_\tau^t e^{\epsilon s} (a^- * u_s) ds\right) d\tau,\end{aligned}\tag{101}$$

which we will consider in the Banach space  $\mathcal{C}_T$  of all continuous maps  $u : [0, T] \rightarrow L^\infty(\mathbb{R}^d)$  with norm

$$\|u\|_T := \sup_{t \in [0, T]} \|u_t\|_{L^\infty(\mathbb{R}^d)}.$$

Here  $T > 0$  is a fixed parameter, which we choose later together with  $\epsilon$ . Then (101) can be written in the form

$$u = F(u),$$

and hence the solution of (101) is a fixed point of  $F$  defined by the right-hand side of this equation. Set  $\mathcal{C}_T^+ = \{u \in \mathcal{C}_T : \forall t \in [0, T] \ u_t \in \Delta^+\}$ . Then  $F : \mathcal{C}_T^+ \rightarrow \mathcal{C}_T^+$ . By (101) we have that, for each  $u \in \mathcal{C}_T^+$  and for all  $t \in [0, T]$ , the following holds

$$\|F(u)_t\|_{L^\infty(\mathbb{R}^d)} \leq \|\varrho_0\|_T \exp[-t(m + \epsilon)] + \|u\|_T \frac{\langle a^+ \rangle}{m + \epsilon} \left(1 - \exp[-t(m + \epsilon)]\right),$$

where we consider  $\varrho_0$  as a constant map from  $[0, T]$  to  $L^\infty(\mathbb{R}^d)$ . Now we set  $\epsilon = 0$  if  $\langle a^+ \rangle \leq m$ , and  $\epsilon = \langle a^+ \rangle - m$  otherwise. Then  $\|F(u)_t\|_{L^\infty(\mathbb{R}^d)} \leq b$ , and hence  $\|F(u)\|_T \leq b$  whenever  $\max\{\|\varrho_0\|_T; \|u\|_T\} \leq b$ . Therefore,  $F$  maps the positive part of each ball in  $\mathcal{C}_T$  centered at zero into itself. Let us now show that  $F$  is a contraction on such sets whenever  $T$  is small enough. For  $\alpha, \beta \geq 0$ , one easily checks that

$$|e^{-\alpha} - e^{-\beta}| \leq |\alpha - \beta|.$$

By means of this inequality, for fixed  $b > 0$  and  $t \in [0, T]$ , and for  $\varrho_0, u, \tilde{u} \in \mathcal{C}_T^+$  such that  $\|\varrho_0\|_T, \|u\|_T, \|\tilde{u}\|_T \leq b$ , we obtain from (101)

$$\begin{aligned} \|F(u)_t - F(\tilde{u})_t\|_{L^\infty(\mathbb{R}^d)} &\leq b e^{-(m+\epsilon)t} \int_0^t e^{\epsilon\tau} \|U_\tau^- - \tilde{U}_\tau^-\|_{L^\infty(\mathbb{R}^d)} d\tau \quad (102) \\ &\quad + \int_0^t e^{-(m+\epsilon)\tau} \|U_\tau^+ - \tilde{U}_\tau^+\|_{L^\infty(\mathbb{R}^d)} d\tau \\ &\quad + \int_0^t e^{-(m+\epsilon)\tau} \|\tilde{U}_\tau^+\|_{L^\infty(\mathbb{R}^d)} \left( \int_\tau^t e^{\epsilon s} \|U_s^- - \tilde{U}_s^-\|_{L^\infty(\mathbb{R}^d)} ds \right) d\tau, \end{aligned}$$

where  $U_s^\pm := (a^\pm * u_s)$ ,  $\tilde{U}_s^\pm := (a^\pm * \tilde{u}_s)$ , and hence, for all  $s \in [0, T]$ , we have that

$$\max\{\|U_s^\pm\|_{L^\infty(\mathbb{R}^d)}; \|\tilde{U}_s^\pm\|_{L^\infty(\mathbb{R}^d)}\} \leq b \langle a^\pm \rangle.$$

Now we use this in (102) and obtain

$$\begin{aligned} \|F(u)_t - F(\tilde{u})_t\|_{L^\infty(\mathbb{R}^d)} &\leq b \left[ e^{-(m+\epsilon)t} + \frac{\langle a^+ \rangle}{m + \epsilon} \left(1 - e^{-(m+\epsilon)t}\right) \right] \\ &\quad \times \int_0^t e^{\epsilon\tau} \|U_\tau^- - \tilde{U}_\tau^-\|_{L^\infty(\mathbb{R}^d)} d\tau + \frac{\langle a^+ \rangle}{m + \epsilon} \left(1 - e^{-(m+\epsilon)t}\right) \|u - \tilde{u}\|_T. \end{aligned}$$

The latter estimate yields

$$\begin{aligned} \|F(u) - F(\tilde{u})\|_T &\leq q(T) \|u - \tilde{u}\|_T, \\ q(T) &:= \langle a^- \rangle \int_0^T e^{\epsilon s} ds + \left(1 - e^{-(m+\epsilon)T}\right), \end{aligned}$$

where we have also taken into account that  $\langle a^+ \rangle \leq m + \epsilon$  due to our choice of  $\epsilon$ . Thus, for small enough  $T$ ,  $F$  is a contraction, which yields that (101) has a unique solution,  $u$ , such that  $\|u\|_T \leq b$ . Then we can repeat the same arguments and obtain that (101) has a unique solution,  $u \in \mathcal{C}_T^+$ , lying in the same ball. Now by means of (100) we return to the problem in (96) and obtain that it has a positive solution  $\varrho_t \in L^\infty(\mathbb{R}^d)$ ,  $t > 0$  such that

$$\|\varrho_s\|_{L^\infty(\mathbb{R}^d)} \leq \|\varrho_0\|_{L^\infty(\mathbb{R}^d)} \exp(-s(m - \langle a^+ \rangle)), \quad s \in [0, t]. \quad (103)$$

Indeed, for  $v_s := e^{ms} \|\varrho_s\|_{L^\infty(\mathbb{R}^d)}$  from (100) and (101) we get

$$v_t \leq v_0 + \langle a^+ \rangle \int_0^t v_s ds,$$

which by the Gronwall inequality yields (103). The proof is completed.  $\square$

## 4.2 Properties of the solution

In order to get additional tools for studying the solutions of the problem in (96), from now on we assume that the initial conditions of this problem are taken from the set  $C_b(\mathbb{R}^d)$  of bounded continuous functions  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then the solution  $\varrho_t$  will also belong to  $C_b(\mathbb{R}^d)$  as this set is closed in  $L^\infty(\mathbb{R}^d)$  whereas the map in right-hand side of (101) leaves it invariant. Thus, we can consider (96) in the Banach space obtained by equipping  $C_b(\mathbb{R}^d)$  with the supremum norm. Note that also the map  $\phi \rightarrow (a^\pm * \phi)$  leaves this spaces invariant – by Lebesgue’s dominated convergence theorem this follows from the second equality in (97). By  $\tilde{\Delta}$ ,  $\tilde{\Delta}^+$ , and  $\tilde{\Delta}_b^+$  we denote the intersections of the corresponding sets defined in (99) with  $C_b(\mathbb{R}^d)$ . Our main task is to understand which properties of the model parameters, see (5) and (6), imply that the solution in question is globally bounded. If  $m \geq \langle a^+ \rangle$  then  $\|\varrho_t\|_{L^\infty(\mathbb{R}^d)} \leq \|\varrho_0\|_{L^\infty(\mathbb{R}^d)}$  for all  $t > 0$ , see (103). Thus, it is left to consider the case of  $m < \langle a^+ \rangle$ , in which  $\varrho_t$  has exponential grows in  $t$  if  $a^- \equiv 0$ .

In the sequel, the following alternative situations are studied separately, cf. (50):

- (i)  $\exists \theta > 0 \quad a^+(x) \leq \theta a^-(x) \quad \text{for alm. all } x \in \mathbb{R}^d,$  (104)
- (ii)  $\forall \theta > 0 \quad \exists \Upsilon_\theta \subset \mathbb{R}^d \text{ of positive meas. } \forall x \in \Upsilon_\theta \quad a^+(x) > \theta a^-(x).$

**Theorem 4.4.** *Let (i) in (104) hold. Then, for all  $t > 0$ , the solution of (96) with  $\varrho_0 \in C_b(\mathbb{R}^d)$  lies in  $\tilde{\Delta}_b^+$  for some  $b > 0$ . Furthermore, if  $\varrho_0 \in \tilde{\Delta}_{\theta-\delta}^+$ , for some  $\delta > 0$ , then  $\varrho_t \in \tilde{\Delta}_\theta^+$  for all  $t > 0$ .*

*Remark 4.5.* Theorem 3.4 establishes the existence of solution of (24) on a bounded time interval only. However, by Theorems 4.4 and 4.6 below, as well as by Lemma 4.1, the solution of (24) with  $r_0 = e(\varrho_0, \cdot)$  can be extended to the whole half-axis  $[0, +\infty)$  if the conditions of Theorems 4.4 or 4.6 are satisfied.

*Proof of Theorem 4.4.* Suppose that the second part of the statement holds true. Then if  $\varrho_0$  is not in  $\tilde{\Delta}_{\theta-\delta}^+$ , we can increase  $\theta$  until this condition is satisfied. For this bigger value of  $\theta$ , (i) in (104) clearly holds. Now let us prove the second part. Since the solution  $\varrho_t(x)$  satisfies (103), we have

$$\|\varrho_s\|_{L^\infty(\mathbb{R}^d)} \leq (\theta - \delta) \exp(-s(m - \langle a^+ \rangle)), \quad s \in [0, t].$$

Hence, for small  $t > 0$ ,  $\varrho_t(x) < \theta$  for all  $x$ . Then either the latter holds for all  $t > 0$ , or there exists  $t_0 > 0$  such that  $\varrho_{t_0}(x_0) = \theta$  for some  $x_0$ , and  $\varrho_{t_0}(x) \leq \theta$  for all other  $x \in \mathbb{R}^d$ . Then

$$\left(\frac{d\varrho_{t_0}}{dt}\right)(x_0) = -m\varrho_{t_0}(x_0) - ((\theta a^- - a^+) * \varrho_{t_0})(x_0) \leq 0. \quad (105)$$

Thus,  $\varrho_t(x_0)$  cannot increase at such a point.  $\square$

Now let us turn to case (ii) in (104). Define

$$f^\pm(\theta) = \int_{\Upsilon_\theta} a^\pm(x) dx, \quad \theta > 0.$$

By (ii) both functions are positive and non-increasing, and

$$g(\theta) := f^+(\theta) - \theta f^-(\theta) > 0,$$

for all  $\theta > 0$ . Hence,  $f^-(\theta) < f^+(\theta)/\theta$ .

**Theorem 4.6.** *Assume that there exists  $\theta > 0$  such that  $g(\theta) \leq m$ . Then the solution of (96) with  $\varrho_0 \in \tilde{\Delta}_{\theta-\delta}^+$  for some  $\delta > 0$ , has the property  $\varrho_t \in \tilde{\Delta}_\theta^+$  for all  $t > 0$ .*

*Proof.* As in the proof of Theorem 4.4, assume that there exists  $t_0 > 0$  such that  $\varrho_{t_0}(x_0) = \theta$  for some  $x_0$ , and  $\varrho_{t_0}(x) \leq \theta$  for all other  $x \in \mathbb{R}^d$ . Then

$$\begin{aligned} \left(\frac{d\varrho_{t_0}}{dt}\right)(x_0) &= -m\theta + \int_{\Upsilon_{\theta, x_0}} [a^+(x_0 - y) - \theta a^-(x_0 - y)] \varrho_{t_0}(y) dy \\ &\quad + \int_{\mathbb{R}^d \setminus \Upsilon_{\theta, x_0}} [a^+(x_0 - y) - \theta a^-(x_0 - y)] \varrho_{t_0}(y) dy \\ &\leq -m\theta + \int_{\Upsilon_{\theta, x_0}} [a^+(x_0 - y) - \theta a^-(x_0 - y)] \varrho(y) dy \\ &\leq \theta[-m + g(\theta)] \leq 0, \end{aligned}$$

which completes the proof. Here  $\Upsilon_{\theta, x} := \{y \in \mathbb{R}^d : x - y \in \Upsilon_\theta\}$ .  $\square$

Let us now make some comments on the result just proven. The condition crucial for the validity of the statement is that  $g(\theta) \leq m$  for  $\theta$  such that  $\varrho_0 \in \tilde{\Delta}_{\theta-\delta}^+$ . If  $a^-$  has finite range, this always holds since

$$\lim_{\theta \rightarrow +\infty} g(\theta) = \int_{\Upsilon_\infty} a^+(x) dx, \quad \Upsilon_\infty := \bigcap_{\theta > 0} \Upsilon_\theta = \{x \in \mathbb{R}^d : a^-(x) = 0\}.$$

Then, the solution  $\varrho_t$  is globally bounded if

$$\int_{\Upsilon_\infty} a^+(x) dx < m, \quad (106)$$

which points to the role of the competition in the considered model – if  $a^- \equiv 0$ , then the left-hand side of (106) is just  $\langle a^+ \rangle$  and the condition in (106) turns into that of the sub-criticality in the contact model. [23] To illustrate this conclusion, let us consider the following example. For  $r > 0$ , set  $B_r = \{x \in \mathbb{R}^d : |x| \leq r\}$ , and let  $\mathbb{1}_r$  and  $|B_r|$  stand for the indicator function and the Euclidean volume of  $B_r$ , respectively. Consider the following choice of the kernels:

$$a^+ = \alpha \mathbb{1}_R, \quad a^- = \alpha \mathbb{1}_r, \quad R > r > 0, \quad \alpha, \beta > 0. \quad (107)$$

Then

$$g(\theta) = \begin{cases} \alpha |B_R| - \theta \beta |B_r|, & \text{if } \theta < \alpha/\beta; \\ \alpha(|B_R| - |B_r|), & \text{otherwise} \end{cases}$$

Hence, the condition of Theorem 4.6 is satisfied if

$$\alpha(|B_R| - |B_r|) \leq m. \quad (108)$$

Case (ii) of (104) contains a subcase where one can get more than the mere global boundedness established in Theorem 4.6. From (101) it clearly follows that the solution as in Theorem 4.3 is independent of  $x$ , i.e., is translation invariant, if so is  $\varrho_0$ . This translation invariant solution can be obtained explicitly. Indeed, let us solve (96) for  $\varrho_t(x) \equiv \psi_t$ . Then it turns into the following

$$\frac{d}{dt} \psi_t = (\langle a^+ \rangle - m) \psi_t - \langle a^+ \rangle \psi_t^2, \quad \psi_t|_{t=0} = \psi_0, \quad (109)$$

which is a Bernoulli equation. For  $m > \langle a_+ \rangle$ , its solution decays to zero exponentially as  $t \rightarrow +\infty$ . For  $m = \langle a_+ \rangle$ , the solution is  $\psi_t = \psi_0 / (1 + \langle a_- \rangle \psi_0 t)$ , and hence also decays to zero as  $t \rightarrow +\infty$ . For  $m < \langle a_+ \rangle$ , we set

$$q = \frac{\langle a_+ \rangle - m}{\langle a_- \rangle}. \quad (110)$$

In this case the solution of (109) has the form

$$\psi_t = \frac{\psi_0 q}{\psi_0 + (q - \psi_0) \exp(-q \langle a_- \rangle t)}, \quad (111)$$

which, in particular, means that  $\psi_t \rightarrow q$  as  $t \rightarrow +\infty$ . Note that  $\psi_t = q$  for all  $t > 0$  whenever  $\psi_0 = q$ .

**Theorem 4.7.** *Suppose that  $q > 0$  and  $a^+(x) \geq qa^-(x)$  for almost all  $x \in \mathbb{R}^d$ . Let also the initial condition  $\varrho_0 \in C_b(\mathbb{R}^d)$  in (96) obeys*

$$0 < \varkappa^- \leq \varrho_0(x) \leq \varkappa^+ < +\infty, \quad (112)$$

*which holds for some  $\varkappa^\pm$  and all  $x \in \mathbb{R}^d$ . Then, for each  $x \in \mathbb{R}^d$  and  $t > 0$ , the solution as in Theorem 4.3 obeys the bounds  $\psi_t^- \leq \varrho_t(x) \leq \psi_t^+$ , where  $\psi_t^\pm$  are given in (111) with  $\psi_0 = \varkappa^\pm$ . Hence  $\varrho_t(x) \rightarrow q$  in  $C_b(\mathbb{R}^d)$  as  $t \rightarrow +\infty$ .*

The result just stated can be interpreted as the asymptotic homogenization of the density function. The condition in Theorem 4.7 can be formulated as  $\Upsilon_\theta = \mathbb{R}^d$  for all  $\theta < q$ . Its another form is

$$\frac{a^+(x)}{\langle a^+ \rangle} \geq \left(1 - \frac{m}{\langle a^+ \rangle}\right) \frac{a^-(x)}{\langle a^- \rangle}, \quad (113)$$

from which we see that the scale of the competition is irrelevant for the result stated in Theorem 4.7 to hold. If  $a^+(x) = \theta a^-(x)$ , for some  $\theta > 0$  and almost all  $x$ , then (113) holds for all  $m \in [0, \langle a^+ \rangle)$ . If the competition has the range shorter than that of dispersal, the mentioned homogenization occurs at nonzero mortality  $m$ . For the example from (107), condition (113) holds if

$$1 - \frac{m}{\langle a^+ \rangle} \leq \left(\frac{r}{R}\right)^d,$$

which is exactly the one given in (108).

*Proof of Theorem 4.7.* In general, we assume in (112) that  $\varkappa^- < q < \varkappa^+$ , and hence  $\psi_t^- < q < \psi_t^+$  for all  $t \geq 0$ , cf. (111). First we prove the upper bound  $\varrho_t(x) \leq \psi_t^+$ . Similarly as in the proof of Theorem 4.4, let  $t_0$  be the first moment at which  $\varrho_{t_0}(x_0) = \psi_{t_0}^+$  for some  $x_0$  and  $\varrho_{t_0}(x) \leq \psi_{t_0}^+$  for other  $x \in \mathbb{R}^d$ . By (110), the condition in (113) means that  $a^+(x) \geq qa^-(x)$  for almost all  $x$ . Then, cf. (105) and (110),

$$\begin{aligned} \left(\frac{d\varrho_{t_0}}{dt}\right)(x_0) &= -m\psi_{t_0}^+ + ((a^+ - qa^-) * \varrho_{t_0})(x_0) - (\psi_{t_0}^+ - q)(a^- * \varrho_{t_0})(x_0) \\ &\leq -m\psi_{t_0}^+ + ((a^+ - qa^-) * \psi_{t_0}^+)(x_0) - (\psi_{t_0}^+ - q)(a^- * \varrho_{t_0})(x_0) \\ &= -(\psi_{t_0}^+ - q)(a^- * \varrho_{t_0})(x_0) < 0. \end{aligned}$$

Thus,  $\varrho_t$  cannot exceed  $\psi_t^+$  for all  $t \geq 0$ . Now let  $t_0$  be the first moment at which  $\varrho_{t_0}(x_0) = \psi_{t_0}^-$  for some  $x_0$  and  $\varrho_{t_0}(x) \geq \psi_{t_0}^-$  for other  $x \in \mathbb{R}^d$ . Then

$$\begin{aligned} \left(\frac{d\varrho_{t_0}}{dt}\right)(x_0) &= -m\psi_{t_0}^- + ((a^+ - \psi_{t_0}^- a^-) * \varrho_{t_0})(x_0) \\ &\geq -m\psi_{t_0}^- + ((a^+ - \psi_{t_0}^- a^-) * \psi_{t_0}^-)(x_0) \\ &= \frac{\varkappa^- q^2 (q - \varkappa^-) \langle a^- \rangle \exp(-q \langle a^- \rangle t)}{[\varkappa^- + (q - \varkappa^-) \exp(-q \langle a^- \rangle t)]^2} > 0. \end{aligned}$$

For the same reasons as above,  $\varrho_t$  cannot get smaller than  $\psi_t^-$ .  $\square$

### 4.3 Conclusion remarks

Let us now make some comments on the results of this section. An analog of (96) was non-rigorously deduced in [12] from a microscopic model on  $\mathbb{Z}^d$ . Then this

equation with  $a^+ = a^-$  was studied in [31]. Note that the case of equal kernels is covered by both Theorems 4.4 and 4.7. According to Theorem 4.3, with no assumption on the parameters of the model we have the existence of the global evolution of  $\varrho_t$ , which is in accord with Theorem 3.4. This can be interpreted as that the mesoscopic description based on the scaling applied here is insensitive to the relationship between  $a^+$  and  $a^-$ . This relationship is, however, important if one wants to get more detailed information, which is contained in Theorems 4.4 and 4.7.

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