Representation of potentials

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Abstract

Given a balayage space $(X, \mathcal{W})$ and a Borel measurable (Green) function $G : X \times X \to [0, \infty]$ which is locally bounded off the diagonal such that each $G(\cdot, y)$, $y \in Y$, is a potential with superharmonic support $\{y\}$, and each function $G(x, \cdot)$, $x \in X$, is lower semicontinuous on X and continuous on $X \setminus \{x\}$ (even continuous on X if x is finely isolated), the following stability result for potentials $G^\nu := \int G(\cdot, y)\, d\nu(y)$ is obtained: If p is a potential on X such that, for some sequence (μ_n) of (positive Radon) measures on X, the sequence (G^{μ_n}) is bounded by a potential and converges pointwise to p outside a polar set (a semipolar set, if $(X, \mathcal{W})$ is a harmonic space), then there exists a (unique) measure μ on X such that $p = G^\mu$ (and μ is the weak limit of the sequence (μ_n)). An application characterizes the situation, where every potential on X has the form G^μ.

Keywords: Potentials; convergence of potentials; Green function; representation of potentials; balayage spaces; harmonic spaces

MSC: 31D05

1 Introduction

Representation of potentials, as one of the important parts of classical as well as abstract potential theory, has attracted the attention of mathematicians for a long time. In classical potential theory representation of potentials goes back to [22]. Stability of representation with respect to increasing limits of Newtonian and Riesz potentials is established in [16, Theorem 1.26, Theorem 3.9]. Convergence properties of potentials of measures with respect to general kernels on locally compact spaces are investigated in [24]. In abstract potential theory, such results frequently rely on Choquet's theorem (see, for instance, [10, 6, 19, 18, 25, 4]). For representation theorems for excessive functions, see [21, 2, 11, 15].

The question of representability is closely related to the existence of a Green function, the axiom of proportionality, duality theory, and convergence axioms ([10, 14, 8, 5, 23, 1, 12, 20, 7]).

The basic frame for the considerations in this paper is a *balayage space* $(X, \mathcal{W})$, where X is a locally compact space with countable base and $\mathcal{W}$ is the set of all positive hyperharmonic functions on X (see [3]). While the subclass of $\mathcal{P}$-harmonic spaces is designed for a unified discussion of solutions to various types of linear elliptic and parabolic partial differential equations of second order, the notion of a balayage space covers, in addition, Riesz potentials, Markov chains on discrete spaces,

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and integro-differential equations. In contrast to harmonic spaces, where harmonic measures for open sets are concentrated on the boundary, harmonic measures for open sets in balayage spaces may live on the entire complement. Nevertheless, the potential theory on balayage spaces is as rich as that on harmonic spaces.

Moreover, the study of a balayage space $(X, \mathcal{W})$ (with $1 \in \mathcal{W}$, which up to normalization is no restriction) amounts to the study of the set of excessive functions for a sub-Markov resolvent $\mathbb{V} = (V_\lambda)_{\lambda>0}$ (a sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$, a Hunt process $\mathfrak{X}$), where all excessive functions are lower semicontinuous and there are sufficiently many continuous ones.

More precisely, $(X, \mathcal{W})$ is a balayage space, if $\mathcal{W}$ is the set of all excessive functions for a right continuous sub-Markov resolvent $\mathbb{V} = (V_\lambda)_{\lambda>0}$ on X such that its potential kernel $V_0 := \sup_{\lambda>0} V_\lambda$ is proper, every function in $\mathcal{W}$ is the supremum of its continuous minorants in $\mathcal{W}$, and there exist strictly positive, finite, and continuous $u, v \in \mathcal{W}$ such that u/v vanishes at infinity.

Conversely, given any bounded continuous strict potential p on a balayage space $(X, \mathcal{W})$ with $1 \in \mathcal{W}$ (they exist in profusion), there exists a (unique) sub-Markov resolvent $\mathbb{V} = (V_\lambda)_{\lambda>0}$ with $p = V_0 1$, a unique sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ such that $p = \int_0^\infty P_t 1\, dt$, and an associated Hunt process $\mathfrak{X}$ such that $\mathcal{W}$ is the set of all excessive functions with respect to $\mathbb{V}$, $\mathbb{P}$, and $\mathfrak{X}$.

In the following let $(X, \mathcal{W})$ be a balayage space. We recall that the ($\mathcal{W}$-)*fine topology* is the coarsest topology, for which all functions in $\mathcal{W}$ are continuous; it is finer (but maybe not strictly finer) than the initial topology on X.

Let us assume that there exists a Borel measurable function $G\colon X \times X \to [0, \infty]$ having the following three properties (for the definition of potentials and superharmonic support see Section 1).

(i) G is locally bounded off the diagonal.

(ii) Each function $G(\cdot, y)$ is a potential with superharmonic support $\{y\}$.

(iii) Each function $G(x, \cdot)$ is lower semicontinuous on X and continuous on $X \setminus \{x\}$. If $x \in X$ is finely isolated (and not isolated), then $G(x, \cdot)$ is even continuous on X.

Let $\mathcal{M}(X)$ be the set of all positive Radon measures on X. For every $\mu \in \mathcal{M}(X)$, we define

$$G^\mu := \int G(\cdot, y)\, d\mu(y). \tag{1.1}$$

The main purpose of this paper is to show that, essentially, the set of all *representable* potentials p, that is, such that $p = G^\mu$ for some $\mu \in \mathcal{M}(X)$, is stable under pointwise limits. More precisely, we have the following stability result.

THEOREM 1.1. *Let p be a potential on X such that, for some sequence (μ_n) in $\mathcal{M}(X)$, the functions G^{μ_n}, $n \in \mathbb{N}$, are bounded by some potential q on X and converge pointwise to p outside a polar set* (*a semipolar set, if* $(X, \mathcal{W})$ *is a harmonic space*).

Then there exists a unique $\mu \in \mathcal{M}(X)$ such that $p = G^\mu$. In fact, μ is the weak limit of the sequence (μ_n) (*that is,* $\lim_{n\to\infty} \mu_n(\varphi) = \mu(\varphi)$ *for every* $\varphi \in \mathcal{K}(X)$)*.*

REMARK 1.2. Let (μ_n) be a sequence in $\mathcal{M}(X)$. Of course, the assumptions of the theorem are satisfied if p is a potential and $G^{\mu_n} \uparrow p$. Moreover, the assumptions are satisfied, if $(X, \mathcal{W})$ is a harmonic space, (G^{μ_n}) is a decreasing sequence of potentials, and p is the lower semicontinuous regularization of $\inf G^{\mu_n}$.

An application of Theorem 1.1 (see Theorem 4.1) yields various characterizations of the possibility to represent all potentials (which, partly under more restrictive assumptions, partly with more complicated proofs, have already been obtained before, see [17, 13]).

2 Preliminaries

Let us first introduce some standard notations. Let $\mathcal{C}(X)$ denote the set of all finite and continuous functions on X. Let $\mathcal{K}(X)$ be the set of all functions $\varphi \in \mathcal{C}(X)$ with compact support $S(\varphi)$, and let $\mathcal{B}_b^+(X)$ be the set of all positive bounded Borel measurable functions on X.

For every $\mu \in \mathcal{M}(X)$ let $S(\mu)$ denote the support of μ. Further, let $\prec$ denote ($\mathcal{W}$-)*specific order* on $\mathcal{M}(X)$, that is $\mu \prec \nu$ if and only if $\mu(u) \le \nu(u)$ for all $u \in \mathcal{W}$. Trivially, $\mu \prec \nu$, if $\mu \le \nu$. We recall that, for every $f\colon X \to [0, \infty]$,

$$R_f := \inf\{v \in \mathcal{W} \colon v \ge f\}$$

and, in particular, for all $A \subset X$ and $u \in \mathcal{W}$,

$$R_u^A := R_{1_A u} = \inf\{v \in \mathcal{W} \colon v \ge u \text{ on } A\}.$$

Moreover, $\hat{R}_u^A(x) := \liminf_{y \to x} R_u^A$, $x \in X$ (so that $\hat{R}_u^A$ is the greatest lower semicontinuous minorant of R_u^A). Let us note that, by the minimum principle (see [3, III.6.6]), for every $y \in X$ and all neighborhoods A of y,

$$R_{G(\cdot,y)}^A = G(\cdot, y). \tag{2.1}$$

For every $x \in X$ and $A \subset X$, let ε_x^A denote the measure obtained *reducing* the Dirac measure ε_x on A, that ist,

$$\int u \, d\varepsilon_x^A = R_u^A(x) \qquad (u \in \mathcal{W}).$$

Of course, $\varepsilon_x^A \prec \varepsilon_x$, and $\varepsilon_x^A = \varepsilon_x$ if $x \in A$. For every Borel measurable set A in X and every $\mu \in \mathcal{M}(X)$, let

$$\mu^A := \int \varepsilon_x^A \, d\mu(x)$$

(so that $\mu^A \prec \mu$). The harmonic kernels H_U for open sets U in X are then obtained by $H_U(x, \cdot) := \varepsilon_x^{X \setminus U}$, $x \in X$.

Given an open set W in X and a Borel measurable function $v \ge 0$ on X, the function v is *harmonic on* W (*superharmonic on* W, respectively) provided, for every relatively compact open set U such that $\overline{U} \subset W$, the function $H_U v$ is finite and continuous on W and $H_U v = v$ ($H_U v \le v$, respectively). Moreover, in order to avoid pathological situations, we shall tacitly assume that all superharmonic functions we consider are finite at finely isolated points (which other authors did without mentioning it explicitly). The *superharmonic support* $C(v)$ of a superharmonic function $v \geq 0$ on X is the smallest closed set in X such that v is harmonic on its complement.

If $v \geq 0$ is superharmonic on X and (U_n) is a sequence of relatively compact open sets in X such that $U_n \uparrow X$, then the sequence $(H_{U_n})v$ is decreasing and its infimum is the largest harmonic minorant h of v on X. The function v is a *potential on* X if $h = 0$. Let $\mathcal{P}$ denote the set of all finite and continuous potentials on X. It is known that $\mathcal{P}$ is the set of all $p \in \mathcal{W} \cap \mathcal{C}(X)$ such that p/v vanishes at infinity for some strictly positive $v \in \mathcal{W} \cap \mathcal{C}(X)$ (see [3, Section II.5 and III.2.8]).

As in [3, Section III.5], let X_0 denote the set of all points in X such that

$$\lim_{U \downarrow x} H_U(x, \cdot) = \varepsilon_x \tag{2.2}$$

(where U runs over all open neighborhoods of x). Then $X \setminus X_0$ is the set of all finely isolated points,

$$X \setminus X_0 = \{x \in X : \varepsilon_x^{X \setminus \{x\}} \neq \varepsilon_x\},$$

and $X \setminus X_0$ is (at most) countable (see [3, III.7.2]). If X is a discrete space, then, of course, $X_0 = \emptyset$.

If $y \in X \setminus X_0$, then $p_y := R_1^{\{y\}} \in \mathcal{P}$. If y is an isolated point in X, then $G(\cdot, y) \in \mathcal{P}$, and hence $G(\cdot, y) = G(y, y)p_y$ by the minimum principle (see [3, III.6.6]); in particular, $G(y, y) > 0$.

Let $v \geq 0$ be superharmonic on X. Since, for every relatively compact open U in X, the function $H_U v$ is finite and continuous on U, the set $P := \{v = \infty\} \cap X_0$ has no interior points. Moreover, $R_1^P = 0$ on $X \setminus P$ (consider the functions $(1/n)v \in \mathcal{W}$, $n \in \mathbb{N}$) and hence $\hat{R}_1^P = 0$, that is, according to the definition in [3, Section VI.5], P is polar proving that P has no *finely* interior points (see [3, III.5.9]). Having $v < \infty$ on $X \setminus X_0$ (due to our convention above), we see that the set $\{v = \infty\}$ is polar.

For $\mu, \nu \in \mathcal{M}(X)$, let us recall the definition (1.1) and introduce ${}^*G^\nu$ by

$$G^\mu := \int G(\cdot, y)\, d\mu(y) \quad \text{and} \quad {}^*G^\nu := \int G(x, \cdot)\, d\nu(x).$$

By Fatou's lemma, the functions G^μ and ${}^*G^\nu$ are lower semicontinuous. Moreover, by Fubini's theorem, $G^\mu \in \mathcal{W}$ and

$$\int G^\mu\, d\nu = \int {}^*G^\nu d\mu. \tag{2.3}$$

We shall need the following result (which is well known for harmonic spaces). For the convenience of the reader we include its proof.

LEMMA 2.1. *Let $\mu \in \mathcal{M}(X)$ such that G^μ is a potential. Then the superharmonic support of G^μ is the support of μ.*

Proof. By Fubini's theorem, for every relatively compact open set U such that $\overline{U}$ does not intersect $S(\mu)$, $H_U G^\mu = G^\mu$. So G^μ is harmonic on $X \setminus S(\mu)$. In particular, the superharmonic support C of G^μ is contained in $S(\mu)$.

To prove the reverse inclusion, let K be an arbitrary compact set in $X \setminus C$ and $\nu := 1_K \mu$. Then G^ν is harmonic on $X \setminus K$. Moreover, G^ν is harmonic on $X \setminus C$, since $G^\nu + G^{1_{K^c}\mu} = G^\mu$ and G^μ is harmonic on $X \setminus C$. So, by [3, III.4.5], G^ν is harmonic on X, and hence $G^\nu = 0$, $\nu = 0$. Thus $\mu(X \setminus C) = 0$, that is, $S(\mu) \subset C$. □

Finally, we note the following.

LEMMA 2.2. *Let $\mu \in \mathcal{M}(X)$ such that $G^\mu \in \mathcal{C}(X)$. Then $G^\mu \in \mathcal{P}$.*

Proof. We know that $v := G^\mu \in \mathcal{W} \cap \mathcal{C}(X)$. If U is a relatively compact open set in X, then $H_U v = R_v^{U^c}$ is harmonic on U, by [3, VI.2.6]. So v is superharmonic on X.

Let (U_n) be a sequence of relatively compact open sets in X such that $U_n \uparrow X$. By Fubini's theorem, for every $n \in \mathbb{N}$,

$$H_{U_n} v = \int H_{U_n} G(\cdot, y)\, d\mu(y).$$

For every $y \in X$, the function $G(\cdot, y)$ is a potential, and hence $H_{U_n} G(\cdot, y) \downarrow 0$ as $n \to \infty$. Thus $H_{U_n} v \downarrow 0$ showing that v is a potential. □

3 Limits of representable potentials

LEMMA 3.1. *Suppose that $\nu \in \mathcal{M}(X)$ is supported by a compact set K in X. Then ${}^*G^\nu$ is finite and continuous on $X \setminus K$.*

Proof. Let (y_n) be a sequence in $X \setminus K$ which converges to a point $y_0 \in X \setminus K$. By (i), the function G is bounded on $K \times \{y_n \colon n = 0, 1, 2, \dots\}$. So, by (iii) and Lebesgue's theorem, $\lim_{n\to\infty} {}^*G^\nu(y_n) = {}^*G^\nu(y_0) < \infty$. □

LEMMA 3.2. *Let $\varphi \in \mathcal{K}(X)$ with $0 \le \varphi \le 1$, $x \in X \setminus S(\varphi)$, and*

$$\nu := \int_0^1 \varepsilon_x^{\{\varphi \ge t\}}\, dt.$$

Then ν is supported by $S(\varphi)$, $\nu \prec \varepsilon_x$, and ${}^*G^\nu \in \mathcal{C}(X)$. Moreover, ν does not charge polar sets (semipolar sets, if $(X, \mathcal{W})$ is a harmonic space).

Proof. Each measure $\nu_t := \varepsilon_x^{\{\varphi \ge t\}}$ is supported by $\{\varphi \ge t\}$ and $\nu_t \prec \varepsilon_x$. Therefore ν is supported by $S(\varphi)$ and $\nu \prec \varepsilon_x$. By Lemma 3.1, ${}^*G^\nu$ is finite and continuous at x. So let $y_0, y_1, y_2, \ldots \in X \setminus \{x\}$ with $\lim_{n\to\infty} y_n = y_0$. For all $n \in \mathbb{N}$,

$${}^*G^\nu(y_n) = \int_0^1 {}^*G^{\nu_t}(y_n)\, dt$$

where ${}^*G^{\nu_t}(y_n) \le G(x, y_n)$ and $\sup\{G(x, y_n) \colon n = 0, 1, 2, \dots\} < \infty$ by (i). Let $t \in (0, 1)$. If $\varphi(y_0) < t$, then $\lim_{n\to\infty} {}^*G^{\nu_t}(y_n) = {}^*G^{\nu_t}(y_0)$ by Lemma 3.1. If $\varphi(y_0) > t$, then we know, by (2.1), that ${}^*G^{\nu_t}(y_n) = R_{G(\cdot, y_n)}^{\{\varphi \ge t\}}(x) = G(x, y_n)$ for $n = 0$ and all sufficiently large n, and hence $\lim_{n\to\infty} {}^*G^{\nu_t}(y_n) = {}^*G^{\nu_t}(y_0)$ by (iii). Thus, by Lebesgue's theorem, $\lim_{n\to\infty} {}^*G^\nu(y_n) = {}^*G^\nu(y_0)$.

Moreover, the measures ν_t, $0 < t \le 1$, do not charge (Borel) polar sets (see [3, VI.5.6]). So ν does not charge polar sets.

Suppose now that $(X, \mathcal{W})$ is a harmonic space (see [3, p. 129] or [6]) and let A be a semipolar Borel set in X. Then each measure ν_t, $0 < t \le 1$, is supported by the boundary of the closed set $\{\varphi \ge t\}$ (see [3, VI.2.12,2] or [6, Proposition 7.1.3]), which is a subset of $\{\varphi = t\}$. Since the level sets $\{\varphi = t\}$ are pairwise disjoint, we see, by [9, Corollary 1.6], that all intersections $A \cap \{\varphi = t\}$ except at most countably many are polar. Thus $\nu(A) = 0$. $\square$

COROLLARY 3.3. *Let $x \in X_0$ and $u, v \in \mathcal{W}$ such that $u(x) < v(x)$. Then there exists $\nu \in \mathcal{M}(X)$ which has compact support, does not charge (Borel) polar sets (semipolar sets if $(X, \mathcal{W})$ is a harmonic space) and satisfies $\nu \prec \varepsilon_x$, ${}^*G^\nu \in \mathcal{C}(X)$, and $\nu(u) < \nu(v)$.*

Proof (cf. [17]). Let (K_n) be a sequence of compact sets in X such that $K_n \uparrow X \setminus \{x\}$. Then $R_v^{K_n} \uparrow R_v^{X \setminus \{x\}} = v$, since $X \setminus \{x\}$ is finely dense in X. So there exists $n \in \mathbb{N}$, such that $u(x) < R_v^{K_n}(x)$. We may choose $\varphi \in \mathcal{K}(X)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on K_n and $\varphi = 0$ in a neighborhood of x. Then $K_n \subset \{\varphi \ge t\}$ for every $t \in (0, 1)$. Defining ν as in Lemma 3.2 we hence obtain that $\nu(u) \le u(x) < R_v^{K_n}(x) \le \nu(v)$. $\square$

LEMMA 3.4. *Let τ be a measure on a compact set K in X and let L be a compact neighborhood of K. Then ${}^*G^{\tau^{X \setminus L}} = {}^*G^\tau$ on $X \setminus L$.*

Proof. Let $y \in V := X \setminus L$. By (2.1), $R^V_{G(\cdot,y)} = G(\cdot, y)$, that is, $\varepsilon_x^V(G(\cdot, y)) = G(x, y)$ for every $x \in X$. In particular,

$$ {}^*G^{\tau^V}(y) = \tau^V(G(\cdot, y)) = \int \varepsilon_x^V(G(\cdot, y))\, d\tau(x) = \int G(x, y)\, d\tau(x) = {}^*G^\tau(y). $$

$\square$

Proof of Theorem 1.1. If $\mu, \nu \in \mathcal{M}(X)$ such that $p = G^\mu = G^\nu$, then $\mu = \nu$ (see the Appendix, where this is shown under more general assumptions on G).

Let us first assume that (G^{μ_n}) converges to p outside a polar set. Then

$$ D := \{x \in X : q(x) < \infty, \lim_{n\to\infty} G^{\mu_n}(x) = p(x)\}. $$

is polar. For the moment, let us fix $y \in X$. If y is isolated, then

$$ \limsup_{n\to\infty} G(y, y)\mu_n(\{y\}) \le \limsup_{n\to\infty} G^{\mu_n}(y) \le q(y) < \infty, $$

where we know that $G(y, y) > 0$. Let us suppose now that y is not isolated. Then, by (ii) and the density of D, there exists $x_y \in D$ such that $x_y \ne y$ and $G(x_y, y) > 0$. By (iii), there exists an open neighborhood V of y and $\gamma > 0$ such that $G(x_y, \cdot) \ge \gamma$ on V. Then, for every $n \in \mathbb{N}$,

$$ \gamma\mu_n(V) \le \int G(x_y, y')\, d\mu_n(y') \le q(x_y) < \infty. $$

Hence every subsequence of (μ_n) has a subsequence which is weakly converging. Let (ν_n) be a subsequence of (μ_n) which converges weakly to $\mu \in \mathcal{M}(X)$.

We claim that $G^\mu = p$. Due to the lower semicontinuity of the functions $G(x, \cdot)$,

$$G^\mu = \int G(\cdot, y)\, d\mu(y) \le \lim_{n\to\infty} \int G(\cdot, y)\, d\nu_n(y) = p \le q < \infty \qquad \text{on } D.$$

To prove the reverse inequality, let us suppose that there exists $x \in D$ such that $G^\mu(x) < p(x)$. By Corollary 3.3, there exists a measure $\nu \in \mathcal{M}(X)$ which is supported by a compact set K in X, does not charge $X \setminus D$ and satisfies $\nu \prec \varepsilon_x$, ${}^*G^\nu \in \mathcal{C}(X)$, and $\nu(G^\mu) < \nu(p)$ (take $\nu = \varepsilon_x$ if $x \in X \setminus X_0$). Let $0 < \delta < \nu(p) - \nu(G^\mu)$. By [3, III.6.1], there exists a compact neighborhood L of K in X such that $R_p^{X\setminus L} < \delta/\nu(K)$ on K. Let

$$\sigma := \nu^{X\setminus L}. \tag{3.1}$$

Then $\sigma(p) = \int R_p^{X\setminus L}\, d\nu < \delta$. Moreover, $\sigma(X \setminus D) = 0$ by [3, VI.5.6]. The function ${}^*G^\sigma$ is lower semicontinuous and, by Lemma 3.4, ${}^*G^\sigma = {}^*G^\nu$ on $X \setminus L$. Thus the function ${}^*G^\nu - {}^*G^\sigma$ is upper semicontinuous and vanishes on $X \setminus L$. Therefore

$$\lim_{n\to\infty} \nu_n({}^*G^\nu - {}^*G^\sigma) \le \mu({}^*G^\nu - {}^*G^\sigma). \tag{3.2}$$

Since $\sigma(X \setminus D) = \nu(X \setminus D) = 0$, we know that

$$\lim_{n\to\infty} \nu(G^{\nu_n}) = \nu(p) < \infty, \quad \lim_{n\to\infty} \sigma(G^{\nu_n}) = \sigma(p) < \infty, \quad \sigma(G^\mu) \le \nu(G^\mu) < \infty.$$

So, by (3.2) and (2.3), $\nu(p) - \sigma(p) = \lim_{n\to\infty}(\nu - \sigma)(G^{\nu_n}) \le (\nu - \sigma)(G^\mu) \le \nu(G^\mu)$, hence $\nu(p) - \nu(G^\mu) \le \sigma(p) < \delta$ contradicting our choice of δ.

Therefore $p = G^\mu$ on D, and hence $G^\mu = p$. Because of the uniqueness of a representing measure for p this implies that the sequence (μ_n) itself converges to μ.

Finally, let us assume that $(X, \mathcal{W})$ is a harmonic space and that the set, where (G^{μ_n}) does not converge to p may be semipolar. Then we only know that $X \setminus D$ is semipolar so that $\nu^{X\setminus L}$ might charge $X \setminus D$. Hence we replace (3.1) by

$$\sigma := \int_0^1 \nu^{\{\varphi \ge t\}}\, dt,$$

where $\varphi \in \mathcal{C}(X)$ satisfying $0 \le \varphi \le 1$, $\varphi = 0$ on L and $\varphi = 1$ outside a compact neighborhood L' of L. We recall that $S(\nu) \subset K$ and note that, for $t \in (0, 1)$, $X \setminus L' \subset \{\varphi \ge t\} \subset X \setminus L$. Arguing as at the end of the proof of Lemma 3.2, we obtain that σ does not charge semipolar sets, and therefore $\sigma(X \setminus D) = 0$. It is easily verified that $\sigma(p) < \delta$ as before and now ${}^*G^\sigma = {}^*G^\nu$ on $X \setminus L'$. Hence the proof can be finished as before. □

4 Application

THEOREM 4.1. *The following statements are equivalent.*

1. *For every $\varphi \in \mathcal{K}^+(X)$, there exists $\nu \in \mathcal{M}(X)$ such that $R_\varphi = G^\nu$.*

2. *For every finite continuous potential p on X having compact superharmonic support, there exists $\mu \in \mathcal{M}(X)$ such that $p = G^\mu$.*

3. *For every potential p on X, there exists $\mu \in \mathcal{M}(X)$ such that $p = G^\mu$.*

4. *There exists $\nu \in \mathcal{M}(X)$ such that G^ν is a finite continuous strict potential.*

5. *There exists $\nu \in \mathcal{M}(X)$, charging every finely open $V \neq \emptyset$, with $G^\nu \in \mathcal{C}(X)$.*

Proof. If $\varphi \in \mathcal{K}(X)$, $\varphi \geq 0$, then $R_\varphi \in \mathcal{P}$ and $C(R_\varphi) \subset S(\varphi)$ (see [3, III.5.6 and III.6.8]). Moreover, there are finite continuous potentials which are strict (see [3, I.1.5]). So the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) hold trivially.

By Theorem 1.1, (1) implies (3). Indeed, if p is any potential on X, there is a sequence (φ_n) in $\mathcal{K}^+(X)$ such that $\varphi_n \uparrow p$, and hence $R_{\varphi_n} \uparrow p$.

To prove the equivalence of (4) and (5) let us consider $\nu \in \mathcal{M}(X)$ such that $G^\nu \in \mathcal{C}(X)$. By Lemma 2.2, $G^\nu \in \mathcal{P}$. It is obvious that the potential kernel associated with G^ν is given by $f \mapsto G^{f\nu}$, $f \in \mathcal{B}_b^+(X)$ (see Lemma 5.1 and [3, II.6.17]). Moreover, ν charges every finely open non-empty set if and only if G^ν is strict (see [3, VI.8.2]).

It remains to show that (4) implies (3). So let p be a potential on X and let $\nu \in \mathcal{M}(X)$ such that $q := G^\nu$ is a finite continuous strict potential. Since, of course, $q > 0$, we may consider the balayage space $(X, \tilde{\mathcal{W}})$, where $\tilde{\mathcal{W}} := (1/q)\mathcal{W}$. Then p/q is a potential and $1 = q/q$ is a strict potential with respect to $(X, \tilde{\mathcal{W}})$. We define

$$\tilde{G}(x,y) := (1/q(x))G(x,y) \qquad (x, y \in X).$$

Then $\tilde{G}$ satisfies our assumptions and, for every $\mu \in \mathcal{M}(X)$, $\tilde{G}^\mu = (1/q)G^\mu$. In particular, $\tilde{G}^\nu = 1$. There exists a sub-Markov resolvent $\mathbb{V} = (V_\lambda)_{\lambda>0}$ on X such that $\tilde{\mathcal{W}}$ is the set of all $\mathbb{V}$-excessive functions and the potential kernel of $\mathbb{V}$ is the potential kernel associated with 1 (see [3, II.7.8]). By [3, II.3.11], there exists a sequence (f_n) in $\mathcal{B}_b^+(X)$ such that $\tilde{G}^{f_n\nu} \uparrow p/q$. Clearly, $\tilde{G}^{f_n\nu} \in \mathcal{C}(X)$, $n \in \mathbb{N}$. Thus an application of Theorem 1.1 yields a measure $\mu \in \mathcal{M}(X)$ such that $p/q = \tilde{G}^\mu$, that is, $p = G^\mu$. □

5 Appendix: Uniqueness

In this section we shall only need that $G\colon X \times X \to [0, \infty]$ is Borel measurable and has the following property:

(ii′) For every $y \in X$, $G(\cdot, y) \in \mathcal{W} \setminus \{0\}$ and $G(\cdot, y)$ is harmonic on $X \setminus \{y\}$.

Let us observe that under this weak hypothesis it is not excluded that $G(\cdot, y)$ is harmonic on X (which, of course, means that such a $G(\cdot, y)$ is useless for the representation of potentials).

LEMMA 5.1. *Let $\mu \in \mathcal{M}(X)$. Then $G^\mu \in \mathcal{W}$. Moreover, if G^μ is superharmonic on $X \setminus S(\mu)$, then G^μ is harmonic on $X \setminus S(\mu)$.*

Proof. By Fatou's lemma, G^μ is lower semicontinuous. Let U be a relatively compact open set in X. We know by (ii′) that $H_U G(\cdot, y) \le G(\cdot, y)$ for every $y \in X$. Hence, by Fubini's theorem, $H_U G^\mu \le G^\mu$. Therefore $G^\mu \in \mathcal{W}$.

Next we suppose that G^μ is superharmonic on $X \setminus S(\mu)$ and $\overline{U} \cap S(\mu) = \emptyset$. Then $H_U G^\mu$ is finite and continuous on U. Moreover, by Fubini's theorem, $H_U G^\mu = G^\mu$, since, by (ii′), $H_U G(\cdot, y) = G(\cdot, y)$ for every $y \in X \setminus S(\mu)$. □

The proof of the uniqueness of representations is a slight modification of the proof given in [1, p. 44] for Brelot spaces (cf. also [8, Théorème 7], [12, Proposition 6.1] and [20, Lemma 2.5] for harmonic spaces, and [13, Proposition 1.3] for balayage spaces).

PROPOSITION 5.2. *For every potential p on X, there exists at most one measure $\mu \in \mathcal{M}(X)$ such that $p = G^\mu$.*

Proof. Let us assume that $\mu, \nu \in \mathcal{M}(X)$ with $p = G^\mu = G^\nu$, and let $D := \{p < \infty\}$.

Let $\tilde{\mu} := (\mu - \nu)^+$ and $\tilde{\nu} := (\nu - \mu)^+$ so that $\tilde{\mu} \wedge \tilde{\nu} = 0$. Since $\mu = \tilde{\mu} + \mu \wedge \nu$ and $\nu = \tilde{\nu} + \mu \wedge \nu$ we see that $G^{\tilde{\mu}} = G^{\tilde{\nu}}$ on D, and hence on X. By Lemma 5.1, $G^{\tilde{\mu}} \in \mathcal{W}$ so that $G^{\tilde{\mu}}$ is a potential. In other words, to prove that $\mu = \nu$ we may assume without loss of generality that $\mu \wedge \nu = 0$.

Then there exist $\mu_n, \nu_n \in \mathcal{M}(X)$ such that $\mu_n \uparrow \mu$, $\nu_n \uparrow \nu$, and $S(\mu_n) \cap S(\nu_n) = \emptyset$ for every $n \in \mathbb{N}$. For the moment we fix $n \in \mathbb{N}$ and define

$$f_n := 1_{\{G^{\nu - \nu_n} < \infty\}} (G^{\mu_n} - G^{\nu - \nu_n})^+, \qquad g_n := 1_{\{G^{\mu - \mu_n} < \infty\}} (G^{\nu_n} - G^{\mu - \mu_n})^+.$$

Then $R_{f_n}, R_{g_n} \in \mathcal{W}$ and there exist functions $u_n, v_n \in \mathcal{W}$ such that

$$R_{f_n} + u_n = G^{\mu_n} \quad \text{and} \quad R_{g_n} + v_n = G^{\nu_n} \tag{5.1}$$

(see [3, bottom of p. 43 and II.4.4]). In particular, $p_n := R_{f_n}$ is a potential which is harmonic on $X \setminus S(\mu_n)$, and $q_n := R_{g_n}$ is a potential which is harmonic on $X \setminus S(\nu_n)$. Since p_n is finely continuous, g_n is finely lower semicontinuous, and $p_n \ge f_n = g_n$ on D, we see that $p_n \ge g_n$ on X, and hence $p_n \ge q_n$. Similarly, $q_n \ge p_n$. Thus $p_n = q_n$, which implies that p_n is harmonic on $(X \setminus S(\mu_n)) \cup (X \setminus S(\nu_n)) = X$ (see [3, III.4.5]). So $p_n = 0$. Consequently, by the definition of p_n, $G^{\mu_n} \le G^{\nu - \nu_n}$ on D.

Letting n tend to infinity, we conclude that $G^\mu \le 0$ on D. Hence $G^\nu = G^\mu = 0$ on X. Thus $\mu = 0$ and $\nu = 0$. □

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