ARITHMETIC GROUPS, BASE CHANGE, AND REPRESENTATION GROWTH

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Abstract. Consider an arithmetic group $\mathbf{G}(O_S)$, where $\mathbf{G}$ is an affine group scheme with connected, simply connected absolutely almost simple generic fiber, defined over the ring of S-integers O_S of a number field K with respect to a finite set of places S. For each $n \in \mathbb{N}$, let $R_n(\mathbf{G}(O_S))$ denote the number of irreducible complex representations of $\mathbf{G}(O_S)$ of dimension at most n. The degree of representation growth $\alpha(\mathbf{G}(O_S)) = \lim_{n\to\infty} \log R_n(\mathbf{G}(O_S))/\log n$ is finite if and only if $\mathbf{G}(O_S)$ has the weak Congruence Subgroup Property.

We establish that for every $\mathbf{G}(O_S)$ with the weak Congruence Subgroup Property the invariant $\alpha(\mathbf{G}(O_S))$ is already determined by the absolute root system of $\mathbf{G}$. To show this we demonstrate that the abscissae of convergence of the representation zeta functions of such groups are invariant under base extensions $K \subset L$. We deduce from our result a variant of a conjecture of Larsen and Lubotzky regarding the representation growth of irreducible lattices in higher rank semi-simple groups. In particular, this reduces Larsen and Lubotzky's conjecture to Serre's conjecture on the weak Congruence Subgroup Property, which it refines.

Contents

1. Introduction and Main Results

1.1. **Background and Motivation.** One of the aims of this paper is to prove a variant of a conjecture of Larsen and Lubotzky on the representation growth of irreducible lattices in higher rank semi-simple groups. We recall that the representation growth of an arbitrary group G is given by the asymptotic behavior of the sequence $R_n(G)$, $n \in \mathbb{N}$, where $R_n(G)$ denotes the number of equivalence classes of irreducible complex representations of G of dimension at most n. Whenever G is a topological (resp. algebraic) group, we restrict the investigation without further comment to continuous (resp. rational) representations. According to Margulis' Arithmeticity Theorem, the lattices in question arise in the following way. Consider an arithmetic group $\mathbf{G}(O_S)$, where O_S is the ring of S-integers of a number field K with respect to a finite set of places S of K and $\mathbf{G}$ is an affine group scheme over O_S whose generic fiber is connected, simply connected absolutely almost simple. Suppose further that the S-rank of $\mathbf{G}$, i.e., $\mathrm{rk}_S\, \mathbf{G} = \sum_{v \in S} \mathrm{rk}_{K_v}\, \mathbf{G}$, is at least 2 and that an infinite place v is included in S if $\mathrm{rk}_{K_v}\, \mathbf{G} \geq 1$. A theorem of Borel and Harish–Chandra shows that the image of $\mathbf{G}(O_S)$ under the diagonal embedding is indeed an irreducible lattice in the higher rank semi-simple group $H = \prod_{v \in S} \mathbf{G}(K_v)$. Moreover, Margulis proved that this construction produces, up to commensurability, essentially all irreducible lattices in higher rank semi-simple groups. Precise notions and a more complete description can be found in [33]. In this paper, we call a group arithmetic if it is commensurable to a group of the form $\mathbf{G}(O_S)$ as above. In particular, all arithmetic groups that we consider are defined in characteristic 0.

The study of representation growth for arithmetic groups was initiated by Lubotzky and Martin in [30]. They showed that, whenever Γ is commensurable to $\mathbf{G}(O_S)$ as above and $\mathrm{rk}_{K_\mathfrak{p}}\, \mathbf{G} \geq 1$ for every finite place $\mathfrak{p} \in S$, then the growth of the sequence $R_n(\Gamma)$, $n \in \mathbb{N}$, is bounded polynomially in n if and only if $\mathbf{G}(O_S)$ has the weak Congruence Subgroup Property. To discuss the latter, let $\widehat{\mathbf{G}(O_S)}$ and $\widehat{O_S}$ denote the profinite completions of the group $\mathbf{G}(O_S)$ and the ring O_S. Furthermore, we write $O_\mathfrak{p}$ for the completion of the ring of integers O of K at a prime $\mathfrak{p}$. The group $\mathbf{G}(O_S)$ has the weak Congruence Subgroup Property (wCSP) if the kernel of the natural map

$$\widehat{\mathbf{G}(O_S)} \longrightarrow \mathbf{G}(\widehat{O_S}) \cong \prod_{\mathfrak{p} \in \mathrm{Spec}(O) \smallsetminus S} \mathbf{G}(O_\mathfrak{p}),$$

is finite. A long-standing conjecture of Serre asserts, in particular, that $\mathbf{G}(O_S)$ has the wCSP whenever $\mathrm{rk}_S \mathbf{G} \geq 2$ and $\mathrm{rk}_{K_\mathfrak{p}} \mathbf{G} \geq 1$ for every finite place $\mathfrak{p} \in S$. This part of Serre's conjecture is known to be true in many cases; e.g., it holds for groups yielding non-uniform irreducible lattices in higher rank semi-simple groups. For more information see [37, Chapter 9.5], [39] or [38], and the references therein.

Next we recall the definition of the representation zeta function of a group G and its abscissa of convergence, which captures the degree of representation growth of G.

Definition. Let G be a group such that $R_n(G)$ is finite for all $n \in \mathbb{N}$. The representation zeta function of G is the Dirichlet generating series

$$\zeta_G(s) = \sum_{\varrho \in \mathrm{Irr}(G)} (\dim \varrho)^{-s},$$

where $\mathrm{Irr}(G)$ is the set of equivalence classes of finite-dimensional irreducible complex representations ϱ of G and $s \in \mathbb{C}$ is a complex variable.

The abscissa of convergence of $\zeta_G(s)$ is the infimum of all $\sigma \in \mathbb{R}$ such that the series $\zeta_G(s)$ converges absolutely for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma$; we denote this invariant by $\alpha(G)$. In particular, $\alpha(G) = \infty$ if $\zeta_G(s)$ diverges for all $s \in \mathbb{C}$.

Whenever a group G, as in the definition above, possesses infinitely many finite-dimensional irreducible complex representations, the abscissa of convergence $\alpha(G)$ is related to the asymptotic behavior of the sequence $R_n(G)$ by the equation

$$\alpha(G) = \limsup_{n \to \infty} \frac{\log R_n(G)}{\log n}. \tag{1.1}$$

In the case of an arithmetic group $\Gamma = \mathbf{G}(O_S)$, as described above, Lubotzky and Martin's result in [30] can therefore be stated as follows: Γ has the wCSP if and only if $\alpha(\Gamma) < \infty$. In this sense the invariant $\alpha(\Gamma)$ provides a means to study the wCSP in a quantitative way. We also remark that if Γ has the wCSP, then $\alpha(\Gamma) = \lim_{n \to \infty} \frac{\log R_n(\Gamma)}{\log n}$ is actually a limit, hence $R_n(\Gamma) = n^{\alpha(\Gamma)+o(1)}$; this is shown implicitly in [2].

1.2. **Discussion of Main Results.** In this paper we establish new quantitative results regarding the representation growth of arithmetic groups with the wCSP. Our first main theorem is the following.

Theorem 1.1. *Let Φ be an irreducible root system. Then there exists a constant α_Φ such that, for every arithmetic group $\mathbf{G}(O_S)$, where O_S is the ring of S-integers of a number field K with respect to a finite set of places S and $\mathbf{G}$ is an affine group scheme over O_S whose generic fiber is connected, simply connected absolutely almost simple with absolute root system Φ, the following holds: if $\mathbf{G}(O_S)$ has the wCSP, then $\alpha(\mathbf{G}(O_S)) = \alpha_\Phi$.*

The theorem highlights two challenging open problems, namely to determine the constants α_Φ and to establish finer asymptotics for the representation growth of arithmetic groups with the wCSP. Even at the conjectural level we are presently very far from solving these problems. The main theorem in [2] shows that $\alpha_\Phi \in \mathbb{Q}$ for all Φ. Furthermore,

$\alpha_\Phi \geq \frac{1}{15}$ (see [28, Theorem 8.1]) and $\alpha_{A_\ell} \leq 22$ for all $\ell \in \mathbb{N}$, with similar bounds on other root systems (see [1]). The only precisely known values are $\alpha_{A_1} = 2$ (see [28, Theorem 10.1]) and $\alpha_{A_2} = 1$ (see [3, Theorem C]). In fact, for arithmetic groups $\Gamma = \mathbf{G}(O_S)$ with the wCSP which arise from affine group schemes $\mathbf{G}$ of type A_1 or A_2, even finer asymptotics of the representation growth of Γ have been established. If $\mathbf{G}$ has absolute root system A_1, then $\zeta_\Gamma(s)$ admits a meromorphic continuation beyond its abscissa of convergence and has a simple pole at $s = 2$ (compare [28] and [4]); consequently, $R_n(\Gamma) = (c_\Gamma + o(1))n^2$ for a constant $c_\Gamma \in \mathbb{R}$. Similarly, if $\mathbf{G}$ has absolute root system A_2, then $\zeta_\Gamma(s)$ has a meromorphic continuation beyond its abscissa of convergence and a double pole at $s = 1$ (see [4]); consequently, $R_n(\Gamma) = (c_\Gamma + o(1))n \log n$ for a constant $c_\Gamma \in \mathbb{R}$. For general Γ, it remains open whether and how far $\zeta_\Gamma(s)$ can be extended meromorphically and, if so, whether the order of the resulting pole at $s = \alpha(\Gamma)$ depends only on the absolute root system Φ.

Besides its intrinsic group theoretic importance, the invariant $\alpha(\mathbf{G}(O_S))$ of an arithmetic group $\mathbf{G}(O_S)$ is also related to the singularities of deformation varieties of surface groups inside $\mathbf{G}(\mathbb{C})$; see [1]. Furthermore it is significant for the volumes of the moduli space of U-local systems on algebraic curves, where U is a compact group. We refer to Witten's paper [43] for the case where U is a compact Lie group and to [1] for the case where U is a maximal compact subgroup in the adelic group $\mathbf{G}(\mathbb{A}_K)$.

Finally, we remark that Theorem 1.1 is similar in spirit to the main result of [31], which pins down the subgroup growth of irreducible lattices in higher rank semi-simple groups (modulo Serre's conjecture and the generalized Riemann Hypothesis). We emphasize that the subgroup growth of such lattices is always faster than polynomial and, in fact, our proofs and methods are completely different from those used in [31].

Let us now focus on the representation growth of irreducible lattices in semi-simple locally compact groups. Let H be a semi-simple group in characteristic 0 of the form $H = \prod_{j=1}^{r} \mathbf{H}_j(F_j)$, where each F_j is a local field of characteristic 0 and each $\mathbf{H}_j$ is a connected, almost simple F_j-group. As indicated at the end of Section 1.1, for every arithmetic irreducible lattice Γ in H we may regard $\alpha(\Gamma)$ as a quantitative measure for the wCSP. Moreover, Serre's conjecture on the Congruence Subgroup Problem asserts that the question whether such a lattice Γ has the wCSP, equivalently whether $\alpha(\Gamma) < \infty$, does not depend on the particular choice of Γ, but is controlled by the ambient group H. Larsen and Lubotzky conjectured that, if H has higher rank, i.e., $\sum_{j=1}^{r} \mathrm{rk}_{F_j} \mathbf{H}_j \geq 2$, then the abscissa of convergence $\alpha(\Gamma)$ is the same for all irreducible lattices Γ in H; see [28, Conjecture 1.5]. We establish the following variant.

Theorem 1.2. *Let H be a semi-simple group in characteristic 0, and let Γ_1, Γ_2 be two arithmetic irreducible lattices in H, both having the wCSP or, equivalently, satisfying $\alpha(\Gamma_1), \alpha(\Gamma_2) < \infty$. Then $\alpha(\Gamma_1) = \alpha(\Gamma_2)$.*

This unconditional result and Margulis' Arithmeticity Theorem immediately reduce Larsen and Lubotzky's conjecture to the original conjecture of Serre.

Theorem 1.3. *Let H be a higher-rank semi-simple group in characteristic* 0*. Assuming Serre's conjecture, for any two irreducible lattices Γ_1, Γ_2 in H we have $\alpha(\Gamma_1) = \alpha(\Gamma_2)$.*

We emphasize that our results are at the same time weaker and stronger than [28, Conjecture 1.5]. They are weaker, since we do not prove Serre's conjecture, and stronger, since Theorem 1.1 shows that the abscissa of convergence depends only on the absolute root system associated to the ambient semi-simple group H.

Central to this paper are new insights into the behavior of the abscissa of convergence under base change. More precisely, we consider the relation between the abscissae of convergence for groups $\mathbf{G}(O_1)$ and $\mathbf{G}(O_2)$, where $O_1 \subset O_2$ is a ring extension. We initiated this study in [3] in the local case, where each O_i is a compact discrete valuation ring of characteristic 0. In the present paper we consider the global case, where each O_i is the ring of S_i-integers in a number field K_i.

It is convenient to organize the results on base change of arithmetic groups into two theorems. Theorem 1.4 links the abscissa of convergence for an arithmetic group $\mathbf{G}(O_S)$ with the wCSP to the abscissa of convergence for the group $\mathbf{G}(\widehat{O})$ over the integral adeles $\widehat{O} = \prod_{\mathfrak{p} \in \mathrm{Spec}(O)} O_{\mathfrak{p}}$. Key to this are results of Larsen and Lubotzky, such as an Euler product factorization for representation zeta functions of arithmetic groups and results on the representation growth of Lie groups, and [3, Theorem B]. The latter is a base change result in the local case; we record a relevant corollary as Theorem 2.2 below.

Theorem 1.4. *Let K be a number field with ring of integers O and let S be a finite set of places of K. Let $\mathbf{G}$ be an affine group scheme defined over O_S whose generic fiber is connected and simply connected semi-simple. Suppose that $\mathbf{G}(O_S)$ has the wCSP. Then $\alpha(\mathbf{G}(O_S)) = \alpha\big(\mathbf{G}(\widehat{O})\big)$.*

Theorem 1.5 incorporates the main thrust of the current paper. It relates the abscissae of convergence for the adelic groups $\mathbf{G}(\widehat{O_K})$ and $\mathbf{G}(\widehat{O_L})$, where $K \subset L$ is an extension of number fields.

Theorem 1.5. *Let $K \subset L$ be number fields with rings of integers $O_K \subset O_L$, and let $\mathbf{G}$ be an affine group scheme defined over O_K whose generic fiber is connected and simply connected semi-simple. Then $\alpha\big(\mathbf{G}(\widehat{O_K})\big) = \alpha\big(\mathbf{G}(\widehat{O_L})\big)$.*

It is noteworthy that the global situation is more rigid than the local one: in the local case, the abscissa of convergence is monotone non-decreasing with respect to base change, but it can be strictly increasing; see Remark 2.3.

At a formal level we can summarize the base change theorems as follows. Theorem 1.4 means that $\alpha(\mathbf{G}(O_1)) = \alpha(\mathbf{G}(O_2))$ when $\mathrm{Spec}(O_2) \to \mathrm{Spec}(O_1)$ is an open embedding. Theorem 1.5 means that $\alpha(\mathbf{G}(O_1)) = \alpha(\mathbf{G}(O_2))$ when $\mathrm{Spec}(O_2) \to \mathrm{Spec}(O_1)$ is finite. Taken together, they mean that $\alpha(\mathbf{G}(O_1)) = \alpha(\mathbf{G}(O_2))$ when $\mathrm{Spec}(O_2) \to \mathrm{Spec}(O_1)$ has finite fibers.

Theorem 1.5 is more difficult to prove than its local counterpart Theorem 2.2. Our approach is based on close approximations of the representation zeta functions associated to groups of the form $\mathbf{G}(O_{\mathfrak{p}})$. A central feature of these approximations is that they are uniform, as O ranges over the set of all finite extensions of a fixed global ring and $\mathfrak{p}$ ranges over a cofinite set of primes of O. We give such approximations in Theorem 2.8. Similar approximations for zeta functions of the finite groups of Lie type $\mathbf{G}(O/\mathfrak{p})$ are derived in Theorem 3.1, using Deligne–Lusztig theory. This allows us to control representations of $\mathbf{G}(\widehat{O})$ that factor through $\prod_{\mathfrak{p}} \mathbf{G}(O/\mathfrak{p})$, but does not account for other representations. In contrast to the situation for finite groups of Lie type, the representation theory of groups over local rings is at present poorly understood. Instead of enumerating representations directly, we follow in the proof of Theorem 2.8 Weil's idea to express local factors of zeta functions as p-adic integrals. We show that in our case the local factors can be approximated by a class of p-adic integrals involving quantifier-free definable functions (Theorem 5.1) and that such integrals admit uniform formulae (Theorem 6.2). In this context we employ tools from the model theory of valued fields, such as partial elimination of quantifiers in the theory of Henselian valued fields of residue characteristic 0. Working with quantifier-free definable functions, we strike a balance between being able to approximate the relevant local factors in the first place and being able to derive uniform formulae for the resulting integrals. At present it is unknown whether one may use more elementary classes of functions, such as polynomials, to carry out an equally effective approximation.

Organization. In Section 2 we prove Theorem 1.4, state Theorem 2.8, and prove Theorems 1.5, 1.1, and 1.2, using Theorem 2.8. The rest of the paper is dedicated to proving Theorem 2.8. In Section 3 we prove Theorem 3.1, which is a variant of Theorem 2.8 for zeta functions of semi-simple algebraic groups over finite fields, and apply it to finite quotients of arithmetic groups. In Section 4 we collect some results about relative representation zeta functions, the Kirillov orbit method, and model theory. In Section 5 we prove that the local factors of representation zeta functions of arithmetic groups are approximated by integrals of quantifier-free definable functions. In Section 6 we prove that integrals of quantifier-free definable functions have a uniform formula, and finish the proof of Theorem 2.8.

Notation. All affine group schemes appearing in this paper are algebraic. For reference purposes we summarize some of the notation used frequently.

- $\mathbf{G}$, $\mathbf{H}$ denote affine group schemes; in this context $\mathfrak{g}$ refers to the Lie algebra of $\mathbf{G}$, but sometimes $\mathfrak{g}$ denotes a more general Lie lattice.
- Φ denotes a root system; $\mathrm{rk}\,\Phi$ its rank and Φ^+ a choice of positive roots; $\mathcal{C}(\mathbf{G}, \mathbf{T}, \mathbb{E})$ and $\mathcal{C}(\Phi)$ are defined in (3.6), (3.7).
- Γ, Δ denote arithmetic groups.
- K, L denote number fields, with rings of integers O_K, O_L.
- $\mathfrak{p}$ denotes a prime of O_K and $\mathfrak{q}$ a prime of O_L.
- $K_{\mathfrak{p}}$ and $O_{K,\mathfrak{p}} = O_{K_{\mathfrak{p}}}$ denote the completions of K and O_K at $\mathfrak{p}$; similarly, $L_{\mathfrak{q}}$ and $O_{L,\mathfrak{q}} = O_{L_{\mathfrak{q}}}$ are the completions of L and O_L at $\mathfrak{q}$.

- $\mathcal{A}$ is a semi-ring with an ideal $\mathcal{A}^+$ and $\xi_{a,q}$ a Dirichlet polynomial; see Definition 2.6.
- $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})$ and $\mathrm{Gr}^{\mathrm{nilp}}_{\mathrm{Lie}}(\mathfrak{g})$ are the Grassmannians of Lie subalgebras and of nilpotent Lie subalgebras of a Lie algebra $\mathfrak{g}$; see Definition 4.11.
- $\mathsf{F}, \mathsf{k}, \Gamma$ are the sorts of the Denef–Pas language of valued fields; see Section 4.4.
- $\mathfrak{T}_{\mathrm{fields}}$, $\mathfrak{T}_{\mathrm{fields},R}$, $\mathfrak{T}_{\mathrm{perf.\text{-}fields},p,R}$, $\mathfrak{T}_{\mathrm{Hen},0}$, and $\mathfrak{T}_{\mathrm{Hen},K,0}$ denote the first-order theories of certain types of fields; see Sections 4.3 and 4.4.
- Π, Ξ and decorations thereof refer to relative orbit method functions; see Section 5.1.
- $\mathscr{X}, \mathscr{Y}$ are quantifier-free definable sets introduced in Definitions 5.2 and 5.12.
- $\mathcal{R}, \mathcal{L}, \mathcal{S}$ and decorations thereof refer to definable functions/families of Lie algebras/groups over $\mathscr{X}$ and $\mathscr{Y}$; see Section 5.1 and also Theorem 5.18.

Additional summaries of more specialised notation can be found in Sections 5.1 and 5.2.

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2. Reduction of the Main Results to Theorem 2.8

In this section we prove Theorems 1.1, 1.2, and 1.5, modulo Theorem 2.8 stated below. For this purpose we fix the following notation.

- K is a number field, $O = O_K$ its ring of integers, and O_S the ring of S-integers for a finite set of places S.
- $\mathbf{G}$ is an affine group scheme defined over O_S whose generic fiber is connected and simply connected semi-simple.
- $\Gamma = \mathbf{G}(O_S)$ has the wCSP.

The groups that we consider in this article have the property that their categories of finite-dimensional complex representations are semi-simple. For example, Γ satisfies this condition since, for every finite-dimensional complex representation ϱ of Γ, the Zariski closure of $\varrho(\Gamma)$ is a reductive algebraic group in characteristic 0.

Our starting point is an Euler product factorization for the representation zeta function of a suitable subgroup $\Delta \subset \Gamma$. The following elementary lemma will be used repeatedly.

Lemma 2.1. *Let G_1 and G_2 be groups such that their categories of finite-dimensional complex representations are semi-simple. If $R_n(G_1)$ and $R_n(G_2)$ are finite for all $n \in \mathbb{N}$, then $\zeta_{G_1 \times G_2} = \zeta_{G_1}\zeta_{G_2}$. In particular, $\alpha(G_1 \times G_2) = \max\{\alpha(G_1), \alpha(G_2)\}$.*

Proof. This follows from the fact that the irreducible complex representations of $G_1 \times G_2$ are the tensor products of irreducible complex representations of G_1 and G_2. $\square$

Considering pro-algebraic completions, one finds that there is a finite-index subgroup $\Delta \subset \Gamma$ such that

$$\zeta_\Delta = \zeta_{\mathbf{G}(\mathbb{C})}^{[K:\mathbb{Q}]} \cdot \prod_{\mathfrak{p} \notin S} \zeta_{\Delta_\mathfrak{p}}, \tag{2.1}$$

where the product ranges over the primes $\mathfrak{p} \in \mathrm{Spec}(O) \smallsetminus S = \mathrm{Spec}(O_S)$ and $\Delta_\mathfrak{p}$ is the closure, in the $\mathfrak{p}$-adic topology, of the image of Δ under the embedding $\Gamma \to \mathbf{G}(O_\mathfrak{p})$; see [28, Theorem 3.3 and Proposition 4.6]. Furthermore, the generating series $\zeta_{\mathbf{G}(\mathbb{C})}$ counts only rational representations, and the generating series $\zeta_{\Delta_\mathfrak{p}}$ count only continuous representations. Since Δ has finite index in Γ, the Strong Approximation Theorem implies that $\Delta_\mathfrak{p}$ is open in $\mathbf{G}(O_\mathfrak{p})$, for every $\mathfrak{p}$, and $\Delta_\mathfrak{p} = \mathbf{G}(O_\mathfrak{p})$, for all but finitely many primes.

It is well-known that the abscissa of convergence for groups is a commensurability invariant (see [30, Lemma 2.2]; we prove a more general version in Lemma 2.11). In particular, $\alpha(\Gamma)$ is equal to $\alpha(\Delta)$. By [28, Theorem 5.1] and [28, Proposition 6.6], we have $\alpha(\mathbf{G}(\mathbb{C})) \leq \alpha(\mathbf{G}(O_\mathfrak{p}))$, for every $\mathfrak{p} \in \mathrm{Spec}(O_S)$. Therefore, (2.1) shows that $\alpha(\Gamma)$ is equal to the abscissa of convergence of the product $\prod_{\mathfrak{p} \notin S} \zeta_{\Delta_\mathfrak{p}}$. By another application of the commensurability invariance, $\alpha(\Gamma)$ is equal to the abscissa of convergence of $\prod_{\mathfrak{p} \notin S} \zeta_{\mathbf{G}(O_\mathfrak{p})}$.

To deduce Theorem 1.4 we need to justify that the abscissa of convergence of the product $\prod_\mathfrak{p} \zeta_{\mathbf{G}(O_\mathfrak{p})}$, ranging over all primes $\mathfrak{p} \in \mathrm{Spec}(O)$, is unchanged by omitting finitely many factors. This is a consequence of the following more general result.

Theorem 2.2. *Let K be a number field with ring of integers O_K, and let $\mathbf{G}$ be an affine group scheme over O_K whose generic fiber is connected and simply connected semi-simple. Then, for every finite extension L of K with ring of integers O_L and every prime $\mathfrak{q}$ of O_L, there are infinitely many primes $\mathfrak{p}$ of O_K such that $\alpha(\mathbf{G}(O_{L,\mathfrak{q}})) \leq \alpha(\mathbf{G}(O_{K,\mathfrak{p}}))$.*

Theorem 2.2 is a corollary of [3, Theorem B], whose proof uses p-adic integrals to analyze the representation zeta functions of $\mathfrak{q}$-adic groups such as $\mathbf{G}(O_{L,\mathfrak{q}})$. The connection to p-adic integrals, and more generally definable integrals in the sense of model theory, is [3, Corollary 3.7] (see also [24, Lemma 4.1]). The notion of a quantifier-free definable function is explained in Section 4. It follows from [3, Corollary 3.7] that there are $d \in \mathbb{N}$ and quantifier-free definable functions φ_1, φ_2 such that, for every finite extension L of K, every prime $\mathfrak{q}$ of O_L, and every sufficiently large integer r, the representation zeta function of the rth principal congruence subgroup $\mathbf{G}^{(r)}(O_{L,\mathfrak{q}}) = \ker(\mathbf{G}(O_{L,\mathfrak{q}}) \to \mathbf{G}(O_L/\mathfrak{q}^r))$ can be expressed as follows:

$$\zeta_{\mathbf{G}^{(r)}(O_{L,\mathfrak{q}})}(s) = |O_L/\mathfrak{q}|^{r\cdot\dim\mathbf{G}} \int_{O_{L,\mathfrak{q}}^d} |\varphi_1(x)|_\mathfrak{q}\ |\varphi_2(x)|_\mathfrak{q}^{-s} d\lambda(x), \tag{2.2}$$

where the absolute value in the integrand is the $\mathfrak{q}$-adic one, and λ is the additive Haar measure on $L_\mathfrak{q}^d$ normalized so that $\lambda(O_{L,\mathfrak{q}}^d) = 1$. Since the abscissae of convergence for $\mathbf{G}(O_{L,\mathfrak{q}})$ and its rth principal congruence subgroup are equal, the claim in Theorem 2.2 may thus be reduced to a similar claim for integrals of the form (2.2). The main point is that the functions φ_1, φ_2 are independent of L and $\mathfrak{q}$, allowing us to compare the integrals for varying fields and primes.

Remark 2.3. Theorem 2.2 can be regarded as a local analog of Theorem 1.5. We explain why a more naive analog is false. Suppose that $K \subset L$ is a finite extension of number fields and that $\mathfrak{q}$ is a prime of O_L lying over a prime $\mathfrak{p}$ of O_K. Then [3, Theorem B] implies that $\alpha(\mathbf{G}(O_{K,\mathfrak{p}})) \leq \alpha(\mathbf{G}(O_{L,\mathfrak{q}}))$. However, in contrast to the global case considered in Theorem 1.5, the inequality can be strict. For example, let D be a central division algebra of degree d over a non-archimedean local field F, and let $\mathbf{G}$ be an affine group scheme over O_F such that $\mathbf{G}(F)$ is the group of norm 1 elements in D. Then the abscissa of convergence for $\mathbf{G}(O_F)$ is $2/d$; see [28, Theorem 7.1]. But if $F \subset E$ is an extension such that D splits over E, then $\alpha(\mathbf{G}(O_E)) \geq 1/15$ by [28, Theorem 8.1], and hence strictly greater than $2/d$ for $d > 30$.

We move on to the proof of Theorem 1.5. The description in (2.2) of representation zeta functions of principal congruence subgroups $\mathbf{G}^{(r)}(O_\mathfrak{p})$ as p-adic integrals is of limited use for determining $\alpha(\mathbf{G}(\widehat{O}))$. This is due to the fact that the infinite product of such congruence subgroups is not of finite index in $\mathbf{G}(\widehat{O}) = \prod_\mathfrak{p} \mathbf{G}(O_\mathfrak{p})$. Facing the challenge to deal with the groups $\mathbf{G}(O_\mathfrak{p})$, and not merely congruence subgroups of sufficiently large index, we approximate their representation zeta functions in the following sense.

Definition 2.4. Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ and $g(s) = \sum_{n=1}^\infty b_n n^{-s}$ be Dirichlet generating series, i.e., Dirichlet series with integer coefficients $a_n, b_n \geq 0$. Let $C \in \mathbb{R}$. Suppose that $\sigma_0 \in \mathbb{R}_{\geq 0}$ is greater than or equal to the abscissae of convergence of f and g. We write

$$f \lesssim_C g \quad \text{for } \sigma > \sigma_0$$

if $f(\sigma) \leq C^{1+\sigma} g(\sigma)$ for every $\sigma \in \mathbb{R}$ with $\sigma > \sigma_0$. We write $f \lesssim_C g$, without specifying the domain, if f and g have the same abscissa of convergence α and if $f \lesssim_C g$ for $\sigma > \max\{0, \alpha\}$. Finally, we write $f \sim_C g$ if $f \lesssim_C g$ and $g \lesssim_C f$.

We routinely use the fact that $f \lesssim_{C_1} g$ and $g \lesssim_{C_2} h$ imply $f \lesssim_{C_1 C_2} h$.

Lemma 2.5. *Let f, g be Dirichlet generating series with abscissae of convergence α_f, α_g. Suppose that $f = \prod_{m=1}^\infty (1 + f_m)$ and $g = \prod_{m=1}^\infty (1 + g_m)$, where f_m, g_m are Dirichlet generating series with vanishing constant terms, and, for every $m \in \mathbb{N}$, let β_m denote the abscissa of convergence of g_m. Suppose further that, for each $\varepsilon > 0$, there is $C(\varepsilon) \in \mathbb{R}_{>0}$ such that, for all m, $f_m \lesssim_{C(\varepsilon)} g_m$ for $\sigma > \beta_m + \varepsilon$. Then $\alpha_f \leq \alpha_g$.*

Proof. The assumptions imply that the abscissa of convergence of f_m is less than or equal to β_m for every m. The abscissa of convergence of a Dirichlet generating series is determined by its behavior on the real axis. Fix $\varepsilon > 0$, and let $\sigma \in \mathbb{R}$ with $\sigma > \max\{0, \alpha_g\} + \varepsilon$, so that $\sigma > \beta_m + \varepsilon$ for all m. Then $g(\sigma) = \prod_m (1 + g_m(\sigma))$ and hence $\sum_m g_m(\sigma)$ converge. As $f_m(\sigma) \leq C(\varepsilon)^{1+\sigma} g_m(\sigma)$ for all m, this implies that $\sum_m f_m(\sigma)$ and hence $f(\sigma) = \prod_m (1 + f_m(\sigma))$ converge. Thus $\sigma > \alpha_f$. Letting ε tend to 0, we deduce that $\alpha_g \geq \alpha_f$. □

We now introduce a semi-ring $\mathcal{A}$ whose elements a are used to index Dirichlet polynomials $\xi_{a,q}$ approximating certain Dirichlet generating series.

Definition 2.6. Let $\mathcal{A}$ be the collection of all finite subsets $a \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0} \cup \{(0,0)\}$. We turn $\mathcal{A}$ into a commutative unital semi-ring by defining the sum of $a, b \in \mathcal{A}$ as $a+b = a \cup b$ and their product as $a \cdot b = \{u+v \mid u \in a, v \in b\}$. Note that the neutral elements in $\mathcal{A}$ are $0 = \varnothing$ and $1 = \{(0,0)\}$. For $a \in \mathcal{A}$ and $n \in \mathbb{N}$ we write $a^{(n)} = \{(nu_1, nu_2) \mid (u_1, u_2) \in a\}$.

The set $\mathcal{A}^+ = \{a \in \mathcal{A} \mid (0,0) \notin a\}$ forms an ideal of the semi-ring $\mathcal{A}$. For $a, b \in \mathcal{A}^+$ we define $a * b \in \mathcal{A}^+$ by the formula $(1+a)(1+b) = 1 + a * b$. For $a \in \mathcal{A}^+$ and $n \in \mathbb{N}$ we write $a^{*n} = a * \cdots * a$ for the n-fold power with respect to $*$.

For $a \in \mathcal{A}$ and $q \in \mathbb{N}_{\geq 2}$ we define the Dirichlet polynomial

$$\xi_{a,q}(s) = \sum_{(m,n) \in a} q^{m-ns}.$$

Remark 2.7. (1) Let $a, b \in \mathcal{A}^+$ and $n \in \mathbb{N}$. The following properties are immediate from the definitions: $a^{(n)} \subset a^{*n}$, $\xi_{a,q^n} = \xi_{a^{(n)},q}$, and there exists $C = C(a,b) \in \mathbb{R}$ such that

$$(1+\xi_{a,q})(1+\xi_{b,q}) - 1 \sim_C \xi_{a*b,q} \quad \text{for all } q \in \mathbb{N}_{\geq 2}.$$

For instance, $C(a,b) = 1 + \min\{|a|, |b|\}$ works. Furthermore, if $a \subset b$, then $\xi_{a,q} \lesssim_1 \xi_{b,q}$ for all $q \in \mathbb{N}_{\geq 2}$.

(2) Let $a, b \in \mathcal{A}$. Let $\mathcal{N}(a)$ denote the "north-west"-Newton polytope associated to a, i.e., the convex hull of $\bigcup\{u + (\mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}) \mid u \in a\}$ in $\mathbb{R}^2$. Then $\mathcal{N}(a) \subset \mathcal{N}(b)$ if and only if there exists $C \in \mathbb{R}$ such that $\xi_{a,q}(s) \lesssim_C \xi_{b,q}(s)$ for all $q \in \mathbb{N}_{\geq 2}$. In fact, if $\mathcal{N}(a) \subset \mathcal{N}(b)$ one can take $C = |a|$.

Recall that the Dedekind zeta function $\zeta_K(s) = \prod_{\mathfrak{p} \in \mathrm{Spec}(O)} \left(1 - |O/\mathfrak{p}|^{-s}\right)^{-1}$ of a number field K has abscissa of convergence 1, and that, for any subset $T \subset \mathrm{Spec}(O)$ with positive analytic density, the abscissa of convergence of $\prod_{\mathfrak{p} \in T} \left(1 - |O/\mathfrak{p}|^{-s}\right)^{-1}$ is equal to 1. Of course, every co-finite subset of $\mathrm{Spec}(O)$ has positive analytic density.

Theorem 2.8. *Let K be a number field with ring of integers O_K, and let $\mathbf{G}$ be an affine group scheme over O_K whose generic fiber is connected, simply connected semi-simple. Then there exist*

- $c(\mathbf{G}) \in \mathcal{A}^+$,
- *for every finite extension L of K with ring of integers O_L, subsets $R(L) \subset T(L) \subset \mathrm{Spec}(O_L)$ with $T(L)$ co-finite and $R(L)$ of positive analytic density in $\mathrm{Spec}(O_L)$,*
- *for every $\varepsilon \in \mathbb{R}_{>0}$, a constant $C(\varepsilon) \in \mathbb{R}_{>0}$*

such that for every finite extension L of K with ring of integers O_L the following hold.

(1) *For every $\mathfrak{q} \in T(L)$, there is a subset $c \subset c(\mathbf{G})$ such that, for every $\varepsilon \in \mathbb{R}_{>0}$,*

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})} - 1 \sim_{C(\varepsilon)} \xi_{c,|O_L/\mathfrak{q}|} \quad \text{for } \sigma > \alpha(\mathbf{G}(O_{L,\mathfrak{q}})) + \varepsilon.$$

(2) *For every* $\mathfrak{q} \in R(L)$ *and every* $\varepsilon \in \mathbb{R}_{>0}$,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})} - 1 \sim_{C(\varepsilon)} \xi_{c(\mathbf{G}),|O_L/\mathfrak{q}|} \quad \textit{for } \sigma > \alpha(\mathbf{G}(O_{L,\mathfrak{q}})) + \varepsilon.$$

We remark that the element $c(\mathbf{G})$ in Theorem 2.8 depends on $\mathbf{G}$, but is not canonically determined by it: in the course of the proof we implicitly make a number of choices, influenced by triangulations of polytopes and resolutions of singularities, and different choices result in different elements $c(\mathbf{G})$. In addition, the proof actually shows that the set $R(L)$ is a Chebotarev set in the sense of Definition 3.14. The proof of Theorem 2.8 occupies a large part of the paper and is completed at the end of Section 6. We now show how Theorem 2.8 implies Theorem 1.5.

Lemma 2.9. *Let* $a \in \mathcal{A}^+ \smallsetminus \{\varnothing\}$, *let* L *be a number field with ring of integers* O_L *and* $R \subset \operatorname{Spec}(O_L)$ *of positive analytic density. Then the abscissa of convergence of the Dirichlet series* $\xi = \prod_{\mathfrak{q}\in R}(1 + \xi_{a,|O_L/\mathfrak{q}|})$ *is equal to* $\max\{(m+1)/n \mid (m,n) \in a\}$.

Proof. For any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of positive real numbers, the product $\prod_{n\in\mathbb{N}}(1 + x_n + y_n)$ converges if and only if $\prod_{n\in\mathbb{N}}(1 + x_n)$ and $\prod_{n\in\mathbb{N}}(1 + y_n)$ converge individually. Thus, if $a = b+c$ in $\mathcal{A}$, then the abscissa of convergence of ξ is the maximum of the abscissae of convergence of $\prod_{\mathfrak{q}\in R}(1+\xi_{b,|O_L/\mathfrak{q}|})$ and $\prod_{\mathfrak{q}\in R}(1+\xi_{c,|O_L/\mathfrak{q}|})$. Thus we may assume that $a = \{(m,n)\}$ is a singleton.

Let $\zeta_{L,\operatorname{Spec}(O_L)\setminus R}(s) = \prod_{\mathfrak{q}\in R}(1 - |O_L/\mathfrak{q}|^{-s})^{-1}$ denote the "restriction" of the Dedekind zeta function of L to the Euler product over factors in R. We observe that

$$\xi(s) = \frac{\zeta_{L,\operatorname{Spec}(O_L)\setminus R}(ns-m)}{\zeta_{L,\operatorname{Spec}(O_L)\setminus R}(2(ns-m))}.$$

Since the abscissa of convergence of $\zeta_{L,\operatorname{Spec}(O_L)\setminus R}(s)$ is equal to 1 and $\zeta_{L,\operatorname{Spec}(O_L)\setminus R}(s) \neq 0$ for $\operatorname{Re}(s) > 1$, we deduce that the abscissa of convergence of ξ is equal to $(m+1)/n$. □

Proof of Theorem 1.5. Let $c(\mathbf{G}) \in \mathcal{A}^+$, $R(L) \subset T(L) \subset \operatorname{Spec}(O_L)$ and $C(\varepsilon)$, for $\varepsilon > 0$, be as in Theorem 2.8. As noted before, Theorem 2.2 implies that $\alpha\big(\mathbf{G}(\widehat{O_L})\big) = \alpha\big(\prod_{\mathfrak{q}\in\operatorname{Spec}(O_L)} \mathbf{G}(O_{L,\mathfrak{q}})\big) = \alpha\big(\prod_{\mathfrak{q}\in T(L)} \mathbf{G}(O_{L,\mathfrak{q}})\big)$. Moreover, $R(L) \subset T(L)$ implies that

$$\alpha\Big(\prod\nolimits_{\mathfrak{q}\in R(L)} \mathbf{G}(O_{L,\mathfrak{q}})\Big) \leq \alpha\Big(\prod\nolimits_{\mathfrak{q}\in T(L)} \mathbf{G}(O_{L,\mathfrak{q}})\Big). \tag{2.3}$$

For $\mathfrak{q} \in R(L)$ and $\varepsilon > 0$, Theorem 2.8 yields that $\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})} - 1 \sim_{C(\varepsilon)} \xi_{c(\mathbf{G}),|O_L/\mathfrak{q}|}$ for $\sigma > \alpha(\mathbf{G}(O_{L,\mathfrak{q}})) + \varepsilon$. Lemma 2.5 implies that the left-hand side of (2.3) is equal to the abscissa of convergence of $\prod_{\mathfrak{q}\in R(L)}(1 + \xi_{c(\mathbf{G}),|O_L/\mathfrak{q}|})$. Similarly, since

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})} - 1 \lesssim_{C(\varepsilon)} \xi_{c,|O_L/\mathfrak{q}|} \lesssim_1 \xi_{c(\mathbf{G}),|O_L/\mathfrak{q}|} \quad \text{for } \sigma > \alpha(\mathbf{G}(O_{L,\mathfrak{q}})) + \varepsilon,$$

for every $\mathfrak{q} \in T(L)$ and suitable $c \subset c(\mathbf{G})$, the right-hand side of (2.3) is less than or equal to the abscissa of convergence of $\prod_{\mathfrak{q}\in T(L)}(1 + \xi_{c(\mathbf{G}),|O_L/\mathfrak{q}|})$. Since $R(L)$ has positive analytic density, the two abscissae are equal. By Lemma 2.9 their common value, and hence $\alpha\big(\mathbf{G}(\widehat{O_L})\big)$, is completely determined by $c(\mathbf{G})$. □

Proof of Theorem 1.1. Let $\mathbf{S}$ be a Chevalley group with root system Φ, i.e., an affine group scheme over $\mathbb{Z}$ whose generic fiber is split, connected, simply connected absolutely almost simple with absolute root system Φ. Consider an arithmetic group $\mathbf{G}(O_S)$ with the wCSP, as in the statement of the theorem. There is a finite extension L of K such that $\mathbf{G}$ and $\mathbf{S}$ are isomorphic over L. Denoting the ring of integers of L by O_L, Theorems 1.4 and 1.5 imply that

$$\alpha(\mathbf{G}(O_S)) = \alpha\big(\mathbf{G}(\widehat{O})\big) = \alpha\big(\mathbf{G}(\widehat{O_L})\big) = \alpha\big(\mathbf{S}(\widehat{O_L})\big) = \alpha\big(\mathbf{S}(\widehat{\mathbb{Z}})\big).$$

Thus $\alpha(\mathbf{G}(O_S))$ depends only on Φ. □

Proof of Theorem 1.2. Let H be a semi-simple group in characteristic 0. Recall that this means that $H = \prod_{j=1}^{r} \mathbf{H}_j(F_j)$, where each F_j is a local field of characteristic 0, and each $\mathbf{H}_j$ is a connected, almost simple group defined over F_j. Let Γ be an arithmetic irreducible lattice in H.

Then there are a number field K with ring of integers O, a finite set S of places of K, an affine group scheme $\mathbf{G}$ defined over O_S whose generic fiber is connected, simply connected absolutely almost simple, and a continuous homomorphism $\psi\colon \prod_{v\in S} \mathbf{G}(K_v) \to H$ whose kernel and cokernel are compact such that $\psi(\mathbf{G}(O_S))$ is commensurable to Γ. Since $\ker(\psi) \cap \mathbf{G}(O_S)$ is finite and $\mathbf{G}(O_S)$ is residually finite, the latter contains a finite-index subgroup that is isomorphic to a finite-index subgroup of Γ. Hence $\alpha(\Gamma) = \alpha(\mathbf{G}(O_S))$.

Fix $j \in \{1, \ldots, r\}$, and denote by $\mathrm{Lie}(\mathbf{G})$ and $\mathrm{Lie}(\mathbf{H}_j)$ the Lie algebras associated to $\mathbf{G}$ and $\mathbf{H}_j$. The homomorphism ψ induces a surjection $\bigoplus_{v\in S} \mathrm{Lie}(\mathbf{G})(K_v) \to \mathrm{Lie}(\mathbf{H}_j)(F_j)$ which is additive and preserves Lie brackets; if F_j is non-archimedean of residue characteristic p, the construction uses the Lie correspondence for p-adic groups. Taking the tensor product with $\mathbb{C}$ – over $\mathbb{R}$ if F_j is archimedean and over $\mathbb{Q}_p$ embedded into $\mathbb{C}$ if F_j is non-archimedean of residue characteristic p – we obtain a Lie algebra epimorphism $\mathrm{Lie}(\mathbf{G})(\mathbb{C})^m \to \mathrm{Lie}(\mathbf{H}_j)(\mathbb{C})^n$, for some $m, n \in \mathbb{N}$. In particular, since $\mathbf{G}(\mathbb{C})$ is almost simple, the group $\mathbf{G}(\mathbb{C})$ and the simple factors of the groups $\mathbf{H}_j(\mathbb{C})$ are all isogenous to one another, and therefore have the same absolute root system Φ. The claim now follows from Theorem 1.1. □

As mentioned above, the abscissa of convergence for groups is a commensurability invariant; see [30, Lemma 2.2]. We conclude the section with a more general lemma about relative zeta functions.

Definition 2.10. Let G be a group with a normal subgroup $N \subset G$ and let ϑ be an irreducible, finite-dimensional complex representation of N. Denote by $\mathrm{Irr}(G|\vartheta)$ the set of (equivalence classes of) finite-dimensional irreducible complex representations ϱ of G such that ϑ is a constituent of the restriction $\mathrm{Res}_N^G \varrho$, which is completely reducible by Clifford's Theorem.

For $n \in \mathbb{N}$ we denote by $R_n(G|\vartheta)$ the number of representations $\varrho \in \mathrm{Irr}(G|\vartheta)$ such that $\dim \varrho \leq n \dim \vartheta$. Suppose that $R_n(G|\vartheta)$ is finite for all $n \in \mathbb{N}$. Then the relative zeta function of G over ϑ is defined as the Dirichlet generating series

$$\zeta_{G|\vartheta}(s) = \sum_{\varrho \in \mathrm{Irr}(G|\vartheta)} \left(\frac{\dim \varrho}{\dim \vartheta} \right)^{-s}.$$

We also set

$$\zeta'_{G|\vartheta}(s) = \sum_{\substack{\varrho \in \mathrm{Irr}(G|\vartheta) \\ \dim \varrho > \dim \vartheta}} \left(\frac{\dim \varrho}{\dim \vartheta} \right)^{-s}.$$

Lemma 2.11. *Let $N \subset H \subset G$ be groups such that $|G : H|$ is finite and N is normal in G. Let $\vartheta \in \mathrm{Irr}(N)$, and suppose that $R_n(G|\vartheta)$, $R_n(H|\vartheta)$ are finite for all $n \in \mathbb{N}$.*

(1) *Let $m \in \mathbb{N}$. Then*
$$|G : H|^{-1} R_{\lfloor m/|G:H| \rfloor}(H|\vartheta) \leq R_m(G|\vartheta) \leq |G : H| R_m(H|\vartheta).$$
(2) *Let $\sigma \in \mathbb{R}_{\geq 0}$. If one of $\zeta_{G|\vartheta}(\sigma)$ or $\zeta_{H|\vartheta}(\sigma)$ converges, then so does the other, and*
$$|G : H|^{-1-\sigma} \zeta_{H|\vartheta}(\sigma) \leq \zeta_{G|\vartheta}(\sigma) \leq |G : H| \zeta_{H|\vartheta}(\sigma).$$
(3) *Suppose that* $\min\{\dim \varrho / \dim \vartheta \mid \varrho \in \mathrm{Irr}(G|\vartheta)$ *with* $\dim \varrho > \dim \vartheta\} > |G : H|$. *Let $\sigma \in \mathbb{R}_{\geq 0}$. If one of $\zeta'_{G|\vartheta}(\sigma)$ or $\zeta'_{H|\vartheta}(\sigma)$ converges, then so does the other, and*
$$|G : H|^{-1-\sigma} \zeta'_{H|\vartheta}(\sigma) \leq \zeta'_{G|\vartheta}(\sigma) \leq |G : H| \zeta'_{H|\vartheta}(\sigma).$$

Proof. Consider the bipartite graph B whose vertex set is $\mathrm{Irr}(G|\vartheta) \sqcup \mathrm{Irr}(H|\vartheta)$ and which has the property that there is an edge between $\varrho_1 \in \mathrm{Irr}(G|\vartheta)$ and $\varrho_2 \in \mathrm{Irr}(H|\vartheta)$ if and only if ϱ_2 is a constituent of $\mathrm{Res}_H^G \varrho_1$. By Nakayama's generalisation of Frobenius reciprocity, the latter condition is equivalent to ϱ_1 being a constituent of $\mathrm{Ind}_H^G \varrho_2$; see [21, Chapter VII, Section 4]. This implies that

- the degree of every vertex of B is positive and bounded by $|G : H|$ and that
- if $\varrho_1 \in \mathrm{Irr}(G|\vartheta)$ and $\varrho_2 \in \mathrm{Irr}(H|\vartheta)$ are connected by an edge, then $\dim \varrho_2 \leq \dim \varrho_1 \leq |G : H| \dim \varrho_2$.

Let $m \in \mathbb{N}$. Write $\mathrm{Irr}(G|\vartheta)_m \subset \mathrm{Irr}(G|\vartheta)$ for the subset consisting of representations of dimension at most $m \dim \vartheta$, and likewise define $\mathrm{Irr}(H|\vartheta)_m$. Then $\mathrm{Irr}(G|\vartheta)_m$ is contained in the set of neighbors of $\mathrm{Irr}(H|\vartheta)_m$, hence

$$R_m(G|\vartheta) = |\mathrm{Irr}(G|\vartheta)_m| \leq |G : H| \, |\mathrm{Irr}(H|\vartheta)_m| = |G : H| R_m(H|\vartheta).$$

Similarly, $\mathrm{Irr}(H|\vartheta)_{\lfloor m/|G:H| \rfloor}$ is contained in the set of neighbors of $\mathrm{Irr}(G|\vartheta)_m$, hence

$$R_{\lfloor m/|G:H| \rfloor}(H|\vartheta) = |\mathrm{Irr}(H|\vartheta)_{\lfloor m/|G:H| \rfloor}| \leq |G : H| \, |\mathrm{Irr}(G|\vartheta)_m| = |G : H| R_m(G|\vartheta).$$

This proves (1). A similar argument yields (2) and (3). For (3) one observes that, if $\min\{\dim \varrho / \dim \vartheta \mid \varrho \in \mathrm{Irr}(G|\vartheta)$ with $\dim \varrho > \dim \vartheta\} > |G : H|$ and if $\varrho_1 \in \mathrm{Irr}(G|\vartheta)$ and $\varrho_2 \in \mathrm{Irr}(H|\vartheta)$ are connected by an edge, then $\dim \varrho_1 > \dim \vartheta$ if and only if $\dim \varrho_2 > \dim \vartheta$. □

3. Base Change for Finite Groups of Lie Type

In this section we prove Theorem 3.1, a variant of Theorem 2.8 for zeta functions of semi-simple algebraic groups over finite fields. In Section 3.2 we apply our results to finite quotients of arithmetic groups.

3.1. Finite Groups of Lie Type.

Given a root system Φ, we denote by $\operatorname{rk}\Phi$ its rank and by Φ^+ a choice of positive roots. By a Lie type $\mathcal{L}$ we mean a pair (Φ, τ), where Φ is a root system with an automorphism τ stabilising Φ^+. We say that a reductive algebraic group $\mathbf{G}$ defined over a finite field $\mathbb{F}_q$ has Lie type $\mathcal{L} = (\Phi, \tau)$ if the absolute root system of $\mathbf{G}$ associated to an $\mathbb{F}_q$-rational and maximally $\mathbb{F}_q$-split maximal torus of $\mathbf{G}$ is Φ and the action of the Frobenius automorphism Frob_q on Φ is given by τ; compare [13, Chapter 3]. For each finite field $\mathbb{F}_q$, the connected semi-simple algebraic groups over $\mathbb{F}_q$ are parametrized up to isogeny by their Lie types. Given a Lie type $\mathcal{L} = (\Phi, \tau)$ we write $\mathcal{L}^{\mathrm{sp}} = (\Phi, \mathrm{Id})$ for the Lie type of a split group with underlying root system Φ. Recall the notation $\mathcal{A}^+$ and $\xi_{a,q}$ from Definition 2.6. In this section we prove the following result.

Theorem 3.1. *Let Φ be a non-trivial root system, and let $\boldsymbol{\mathcal{L}}_\Phi$ denote the collection of Lie types with underlying root system Φ. Let $\mathbf{\Omega}$ denote the set of all prime powers. Then there exist $C \in \mathbb{R}$, $m \in \mathbb{N}$, $a(\Phi) \in \mathcal{A}^+$, and $a(\mathcal{L}, q) \in \mathcal{A}^+$ for $(\mathcal{L}, q) \in \boldsymbol{\mathcal{L}}_\Phi \times \mathbf{\Omega}$ such that the following hold:*

1. *$a(\mathcal{L}, q) \subset a(\Phi)$ for all $(\mathcal{L}, q) \in \boldsymbol{\mathcal{L}}_\Phi \times \mathbf{\Omega}$,*
2. *$a((\Phi, \mathrm{Id}), q) = a(\Phi)$ for all $q \equiv_m 1$,*
3. *for every connected semi-simple algebraic group $\mathbf{G}$ defined over a finite field $\mathbb{F}_q$ which has Lie type $L \in \boldsymbol{\mathcal{L}}_\Phi$,*

$$\zeta_{\mathbf{G}(\mathbb{F}_q)} - |\mathbf{G}(\mathbb{F}_q)/[\mathbf{G}(\mathbb{F}_q), \mathbf{G}(\mathbb{F}_q)]| \sim_C \xi_{a(\mathcal{L},q),q}.$$

Moreover, $(\operatorname{rk}\Phi, |\Phi^+|) \in a(\Phi)$, and every connected semi-simple algebraic group $\mathbf{G}$ defined over a finite field $\mathbb{F}_q$ with absolute root system Φ satisfies

$$\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) \sim_C 1 + q^{\operatorname{rk}\Phi - |\Phi^+| s}. \tag{3.1}$$

Remark 3.2. We remark that if $\mathbf{G}$ is a connected, simply connected semi-simple algebraic group over $\mathbb{F}_q$ and $q > 3$, then $\mathbf{G}(\mathbb{F}_q)$ is perfect and therefore $|\mathbf{G}(\mathbb{F}_q)/[\mathbf{G}(\mathbb{F}_q), \mathbf{G}(\mathbb{F}_q)]| = 1$. Indeed, for simple groups this is a result of Tits [42] (see also [34, Theorem 24.17]), and the semi-simple case follows by taking products.

Example 3.3. The representations of the special linear group $\mathrm{SL}_2(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ are well known; e.g. see [13, Chapter 15]. In particular, we have $\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s) - 1 \sim_2 q^{1-s}$ for sufficiently large q. Consider the following semi-simple algebraic groups defined over $\mathbb{F}_q$ whose absolute root systems are both $A_1 \times A_1$: $\mathbf{G}_1 = \mathrm{SL}_2 \times \mathrm{SL}_2$ and $\mathbf{G}_2 = \mathcal{R}_{\mathbb{F}_{q^2}|\mathbb{F}_q} \mathrm{SL}_2$, the restriction of scalars of SL_2 defined over a quadratic extension. Then $\mathbf{G}_1(\mathbb{F}_q) = \mathrm{SL}_2(\mathbb{F}_q) \times \mathrm{SL}_2(\mathbb{F}_q)$ and $\zeta_{\mathbf{G}_1(\mathbb{F}_q)}(s) - 1 \sim_4 q^{1-s} + q^{2-2s}$, whereas $\mathbf{G}_2(\mathbb{F}_q) = \mathrm{SL}_2(\mathbb{F}_{q^2})$ and $\zeta_{\mathbf{G}_2(\mathbb{F}_q)}(s) - 1 \sim_2 q^{2-2s}$. Since $q^{1-s} + q^{2-2s} \not\sim_C q^{2-2s}$ for any fixed C but unbounded q, we see that the inclusion in part (1) of Theorem 3.1 can be strict.

Whilst Theorem 3.1 is a result on semi-simple groups, it leads to the following corollary on reductive groups.

Corollary 3.4. *Let $\mathbf{G}$ be a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$ with absolute root system Φ. There is a constant $D \in \mathbb{R}$, depending only on Φ and the dimension of $\mathbf{G}$, such that $\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) \sim_D q^{\dim Z(\mathbf{G})}(1 + q^{\mathrm{rk}\,\Phi - |\Phi^+|s})$.*

The proof of Theorem 3.1 and its Corollary 3.4 is given in Section 3.1.4, and prepared in the preceding sections. It is based on Lusztig's classification of irreducible representations of finite groups of Lie type; e.g., see [13, Chapter 13]. We write $\mathbf{G}^*$ for the dual group of a connected reductive group $\mathbf{G}$ and recall that, while $\mathbf{G}$ and $\mathbf{G}^*$ have isomorphic Weyl groups $W \cong W^*$, the absolute root system $\Phi^\vee$ of $\mathbf{G}^*$ is dual to Φ in the sense that the roles of long and short roots are interchanged. We also note that $\mathbf{G}$ is simply connected respectively adjoint if and only if $\mathbf{G}^*$ is adjoint respectively simply connected. Since equivalence classes of irreducible representations of $\mathbf{G}(\mathbb{F}_q)$ are parametrized by the corresponding characters, we use the notation $\mathrm{Irr}(\mathbf{G}(\mathbb{F}_q))$ in a flexible way to denote also the set of irreducible complex characters of $\mathbf{G}(\mathbb{F}_q)$. The set $\mathrm{Irr}(\mathbf{G}(\mathbb{F}_q))$ is the disjoint union of certain Lusztig series $\mathcal{E}(\mathbf{G}(\mathbb{F}_q),(g))$ so that

$$\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) = \sum_{(g)\subset \mathbf{G}^*(\mathbb{F}_q)} \sum_{\chi \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q),(g))} \chi(1)^{-s}, \tag{3.2}$$

where the outer sum ranges over $\mathbf{G}^*$-conjugacy classes of semi-simple elements in $\mathbf{G}^*(\mathbb{F}_q)$.

3.1.1. *Unipotent Zeta Functions.* The elements of $\mathcal{E}(\mathbf{G}(\mathbb{F}_q),(1))$ are known as the unipotent characters of $\mathbf{G}(\mathbb{F}_q)$. We set

$$\zeta^{\mathrm{unip}}_{\mathbf{G}(\mathbb{F}_q)}(s) = \sum_{\chi \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q),(1))} \chi(1)^{-s}.$$

Proposition 3.5. *Let Φ be a non-trivial root system and let $\boldsymbol{\mathcal{L}}_\Phi$ denote the collection of Lie types with underlying root system Φ. Then there are $C \in \mathbb{R}$ and $b^{\mathrm{c}}(\mathcal{L}) \in \mathcal{A}^+$ for $\mathcal{L} \in \boldsymbol{\mathcal{L}}_\Phi$ such that the following hold:*

1. *$b^{\mathrm{c}}(\mathcal{L}) \subset b^{\mathrm{c}}(\mathcal{L}^{\mathrm{sp}})$ for all $L \in \boldsymbol{\mathcal{L}}_\Phi$,*
2. *for every connected reductive algebraic group $\mathbf{G}$ defined over a finite field $\mathbb{F}_q$ which has Lie type $\mathcal{L} \in \boldsymbol{\mathcal{L}}_\Phi$,*

$$\zeta^{\mathrm{unip}}_{\mathbf{G}(\mathbb{F}_q)} - 1 \sim_C \xi_{b^{\mathrm{c}}(\mathcal{L}),q} \qquad \textit{and} \qquad \zeta^{\mathrm{unip}}_{\mathbf{G}(\mathbb{F}_q)} \sim_C 1.$$

Proof. Throughout, all constants – be they real or elements of $\mathcal{A}^+$ – depend only on Φ or a given Lie type $\mathcal{L} \in \boldsymbol{\mathcal{L}}_\Phi$. Let $\mathbf{G}$ be a connected reductive algebraic group of Lie type $\mathcal{L} = (\Phi,\tau) \in \boldsymbol{\mathcal{L}}_\Phi$. Then $\mathbf{G}/Z(\mathbf{G})$ is a semi-simple group over $\mathbb{F}_q$ of adjoint type and, by [13, Proposition 13.20], every unipotent character of $\mathbf{G}(\mathbb{F}_q)$ factors through $(\mathbf{G}/Z(\mathbf{G}))(\mathbb{F}_q)$.

Therefore we may assume that $\mathbf{G}$ is semi-simple of adjoint type and thus a direct product of $\mathbb{F}_q$-simple groups $\mathbf{S}_i$ of Lie type $\mathcal{L}_i = (\Phi_i, \tau_i)$, say, where $i \in \{1,\dots,m\}$. The unipotent characters of $\mathbf{G}(\mathbb{F}_q) = \mathbf{S}_1(\mathbb{F}_q) \times \cdots \times \mathbf{S}_m(\mathbb{F}_q)$ are the irreducible characters that appear in the Deligne–Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(1)$, where $\mathbf{T}$ is a maximal torus defined over $\mathbb{F}_q$. The generalised character $R_{\mathbf{T}}^{\mathbf{G}}(1)$ arises from the action of $\mathbf{G}(\mathbb{F}_q) \times \mathbf{T}(\mathbb{F}_q)$ on the cohomologies of the Deligne–Lusztig variety $\mathbf{X} = \boldsymbol{\ell}^{-1}(\mathbf{U})$, where $\boldsymbol{\ell}$ denotes the Lang map and $\mathbf{U}$ is the unipotent radical of a Borel subgroup containing $\mathbf{T}$; see [13, Section 11]. This construction is compatible with taking products, hence $\mathbf{X}$ can be taken to be the product of the Deligne–Lusztig varieties of the groups $\mathbf{S}_i(\mathbb{F}_q)$. The Künneth formula now implies that

$$\zeta^{\mathrm{unip}}_{\mathbf{G}(\mathbb{F}_q)} = \prod_{i=1}^{m} \zeta^{\mathrm{unip}}_{\mathbf{S}_i(\mathbb{F}_q)}. \tag{3.3}$$

Fix $i \in \{1, \ldots, m\}$. Since $\mathbf{S}_i$ is $\mathbb{F}_q$-simple, τ_i permutes the irreducible components $\Phi_{i,1}$, $\Phi_{i,2}$, $\ldots$, $\Phi_{i,n(i)}$ of Φ_i transitively. Thus $\tau_i^{n(i)}$ restricts to an automorphism of $\Psi_i = \Phi_{i,1}$. Writing $\widetilde{\mathbf{S}}_i$ for the adjoint absolutely simple algebraic group over $\mathbb{F}_{q^{n(i)}}$ of Lie type $\widetilde{\mathcal{L}}_i = (\Psi_i, \tau_i^{n(i)})$, we have $\mathbf{S}_i(\mathbb{F}_q) = \widetilde{\mathbf{S}}_i(\mathbb{F}_{q^{n(i)}})$. By [6, Sections 13.8 and 13.9], the number $k(i)$ of unipotent characters of $\widetilde{\mathbf{S}}_i(\mathbb{F}_{q^{n(i)}})$ depends only on $\widetilde{\mathcal{L}}_i$. This shows that $\zeta^{\mathrm{unip}}_{\mathbf{S}_i(\mathbb{F}_q)} \lesssim_{C_i} 1$ for a suitable constant $C_i \in \mathbb{R}$. Since the trivial character is unipotent, we also get $1 \lesssim_1 \zeta^{\mathrm{unip}}_{\mathbf{S}_i(\mathbb{F}_q)}$. This yields $\zeta^{\mathrm{unip}}_{\mathbf{S}_i(\mathbb{F}_q)} \sim_{C_i} 1$ and thus $\zeta^{\mathrm{unip}}_{\mathbf{G}(\mathbb{F}_q)} \sim_{C_1 \cdots C_m} 1$, using (3.3).

Furthermore, looking through the tables in [6, Sections 13.8 and 13.9], one sees that there are polynomials $f_{i,1}, \ldots, f_{i,k(i)} \in \mathbb{Q}[X]$ depending only on $\widetilde{\mathcal{L}}_i$ such that the degrees of the unipotent characters of $\widetilde{\mathbf{S}}_i(\mathbb{F}_{q^{n(i)}})$ are precisely $f_{i,1}(q^{n(i)}), \ldots, f_{i,k(i)}(q^{n(i)})$. Moreover, only one of these polynomials has degree zero, namely the constant polynomial 1 giving the degree of the trivial character. We may assume that $f_{i,1} = 1$ and define

$$b^{\mathrm{c}}(\widetilde{\mathcal{L}}_i) = \{(0, \deg f_{i,j}) \mid 2 \leq j \leq k(i)\} \in \mathcal{A}^+.$$

We show below that, while the degrees of the $f_{i,j}$ generally depend on $\widetilde{\mathcal{L}}_i$ and not only on Ψ_i, the set of degrees in the split case $\widetilde{\mathcal{L}}_i^{\mathrm{sp}} = (\Psi_i, \mathrm{Id})$ is a superset of the set of degrees in the twisted cases. Granted that this is so, we continue and define $b^{\mathrm{c}}(L_i) = b^{\mathrm{c}}(\widetilde{\mathcal{L}}_i)^{(n(i))}$ so that for a suitable constant $C_i' \in \mathbb{R}$ we obtain

$$\zeta^{\mathrm{unip}}_{\mathbf{S}_i(\mathbb{F}_q)} - 1 = \zeta^{\mathrm{unip}}_{\widetilde{\mathbf{S}}_i(\mathbb{F}_{q^{n(i)}})} - 1 \sim_{C_i'} \xi_{b^{\mathrm{c}}(\widetilde{\mathcal{L}}_i), q^{n(i)}} = \xi_{b^{\mathrm{c}}(L_i), q}. \tag{3.4}$$

We set $b^{\mathrm{c}}(\mathcal{L}) = b^{\mathrm{c}}(L_1) * \cdots * b^{\mathrm{c}}(L_m)$. From our claim regarding the degrees of the polynomials $f_{i,j}$ and Remark 2.7 we deduce that, for each $i \in \{1, \ldots, m\}$,

$$b^{\mathrm{c}}(L_i) = b^{\mathrm{c}}(\widetilde{\mathcal{L}}_i)^{(n(i))} \subset b^{\mathrm{c}}(\Psi_i, \mathrm{Id})^{(n(i))} \subset b^{\mathrm{c}}(\Psi_i, \mathrm{Id})^{*n(i)} = b^{\mathrm{c}}(\mathcal{L}_i^{\mathrm{sp}}). \tag{3.5}$$

In the special case $m = 1$ this establishes assertion (1) of the proposition, and (3.4) directly yields the remaining first assertion in (2). In the general case we conclude the proof based on (3.3) and Remark 2.7.

It remains to justify the claim about the degrees of unipotent characters for split and twisted finite groups of Lie type with the same underlying absolute root system. Simplifying the notation used above, let $\mathbf{S}$ (rather than $\widetilde{\mathbf{S}}_i$) be a connected, adjoint absolutely simple algebraic group defined over $\mathbb{F}_q$ (rather than $\mathbb{F}_{q^{n(i)}}$), and let $\mathbf{S}^{\mathrm{sp}}$ denote a split form of $\mathbf{S}$ over $\mathbb{F}_q$. We consider the possible groups case by case. By the remarks following [13, Theorem 3.17], the relevant twisted Lie types are ${}^2A_n$, ${}^2D_n$, ${}^3D_4$, and ${}^2E_6$, where the left index represents as customary the order of the automorphism of the root system. (In our setup, the Suzuki and Ree groups do not occur. In fact, the analogous statement for groups of Lie type F_4 and ${}^2F_4$ is incorrect; see Remark 3.6.)

Case 1: $\mathbf{S}$ has Lie type ${}^2A_n$ and $\mathbf{S}^{\mathrm{sp}}$ has Lie type A_n. The unipotent characters of each group, $\mathbf{S}(\mathbb{F}_q)$ and $\mathbf{S}^{\mathrm{sp}}(\mathbb{F}_q)$, are parametrized by partitions α of $n+1$. The degree $\chi_\alpha(1)$ of the unipotent character χ_α associated to α in either case is given by a polynomial in q whose degree only depends on α; see [6, p. 465]. This proves that $b^{\mathrm{c}}({}^2A_n) = b^{\mathrm{c}}(A_n)$.

Case 2: $\mathbf{S}$ has Lie type ${}^2D_n$ and $\mathbf{S}^{\mathrm{sp}}$ has Lie type D_n. The unipotent characters of $\mathbf{S}^{\mathrm{sp}}(\mathbb{F}_q)$ are parametrized by so-called Lustzig symbols (or just "symbols" in the terminology of [6]). These are certain pairs (S,T) of finite subsets of $\mathbb{Z}_{\geq 0}$ such that $|S| - |T| \equiv 0 \pmod 4$, where pairs of the form (S,S) correspond to pairs of characters rather than single characters, but this subtlety is irrelevant for our purposes. Similarly, the unipotent characters of $\mathbf{S}(\mathbb{F}_q)$ are parametrized by pairs (S,T) such that $|S| - |T| \equiv 2 \pmod 4$. Let (S,T) be a pair of the second kind and suppose without loss of generality that there exists an element $a \in S \smallsetminus T$. Then $(S',T') = (S \smallsetminus \{a\}, T \cup \{a\})$ is a pair of the first kind. Inspection of the formulae giving the degrees $\chi_{(S,T)}(1)$ and $\chi_{(S',T')}(1)$ of the unipotent characters $\chi_{(S,T)}$ of $\mathbf{S}(\mathbb{F}_q)$ associated with (S,T) and $\chi_{(S',T')}$ of $\mathbf{S}^{\mathrm{sp}}(\mathbb{F}_q)$ associated with (S',T') yields: the polynomials in q giving $\chi_{(S,T)}(1)$ and $\chi_{(S',T')}(1)$ have equal degrees. For details see [6, p. 471 and 475f.]. This proves that $b^{\mathrm{c}}({}^2D_n) \subset b^{\mathrm{c}}(D_n)$.

Case 3: $\mathbf{S}$ has Lie type ${}^3D_4$ and $\mathbf{S}^{\mathrm{sp}}$ has Lie type D_4. The set of degrees of the polynomials expressing the degrees of unipotent characters of $\mathbf{S}(\mathbb{F}_q)$ is $\{0,5,9,11,12\}$ (see [6, p. 478]), whereas the corresponding set for $\mathbf{S}^{\mathrm{sp}}(\mathbb{F}_q)$ is $\{0,5,6,9,10,11,12\}$. This shows that $b^{\mathrm{c}}({}^3D_4) \subset b^{\mathrm{c}}(D_4)$.

Case 4: $\mathbf{S}$ has Lie type ${}^2E_6$ and $\mathbf{S}^{\mathrm{sp}}$ has Lie type E_6. The 30 unipotent characters of these groups are in degree-preserving bijective correspondence; cf. [6, p. 480f.]. This shows that $b^{\mathrm{c}}(E_6) = b^{\mathrm{c}}({}^2E_6)$. □

Remark 3.6. Proposition 3.5 extends only partly to the remaining groups of Lie type, the Suzuki and Ree groups. For the groups $B_2(q)$ and ${}^2B_2(q^2)$ the degrees of the polynomials giving the degrees of the unipotent characters coincide. The same is true for the groups $G_2(q)$ and ${}^2G_2(q^2)$. But the degrees of the polynomials giving the unipotent character degrees of the Ree groups ${}^2F_4(q^2)$ are $\{0,9,14,20,21,22,24\}$ (see [6, p. 489]), whereas the corresponding degrees for the split groups $F_4(q)$ are $\{0,11,15,20,21,22,23,24\}$ (see [6, p. 479]).

Remark 3.7. In fact, the sets $b^{\mathrm{c}}(\mathcal{L})$ in Proposition 3.5 are all contained in $\{0\} \times \mathbb{Z}_{>0}$, so $\xi_{b^{\mathrm{c}}(\mathcal{L}),q}$ could be replaced by $q^{-m(\mathcal{L})s}$, where $m(\mathcal{L}) = \min\{m \in \mathbb{Z}_{>0} \mid (0,m) \in b^{\mathrm{c}}(\mathcal{L})\}$. Moreover, it is conceivable that $m(\mathcal{L})$ actually only depends on the root system Φ. As the proof of the proposition shows, this reduces to a question regarding the degrees of unipotent characters of groups of type D_n. More precisely, it would follow if the degrees of the polynomials giving the degrees of the unipotent characters of the (untwisted) groups of type D_n indexed by special Lusztig symbols of the form (S,S) were already among the corresponding degrees obtained from Lusztig symbols of the form (S,T) with $S \neq T$.

3.1.2. *Connected Centralizers of Semi-Simple Elements.* In preparation for the proof of Theorem 3.1 we record some auxiliary facts about the connected centralisers of semi-simple elements in algebraic groups over finite fields.

Let Φ be a root system. We consider a connected split reductive algebraic group $\mathbf{G}$ over a finite field $\mathbb{F}_q$ whose absolute root system, associated to some $\mathbb{F}_q$-rational split maximal torus $\mathbf{T}$, is isomorphic to and identified with Φ. We denote by $\mathbb{F}_q^{\mathrm{alg}}$ the algebraic closure of $\mathbb{F}_q$. The connected centraliser $C_{\mathbf{G}}(g)^\circ$ of any semi-simple element $g \in \mathbf{G}(\mathbb{F}_q^{\mathrm{alg}})$ is a reductive subgroup of maximal rank in $\mathbf{G}$. Indeed, every semi-simple element of $\mathbf{G}$ is conjugate to an element of $\mathbf{T}$. Furthermore the connected centraliser of $g \in \mathbf{T}(\mathbb{F}_q^{\mathrm{alg}})$ is the reductive group $\langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Psi_g \rangle$ with root system $\Psi_g = \{\alpha \in \Phi \mid \alpha(g) = 1\}$, where each $\mathbf{U}_\alpha$ denotes the root subgroup associated to α; see [13, Proposition 2.3]. For every extension $\mathbb{E}$ of $\mathbb{F}_q$ we put

$$\mathcal{C}(\mathbf{G}, \mathbf{T}, \mathbb{E}) = \{\Psi_g \mid g \in \mathbf{T}(\mathbb{E})\}. \tag{3.6}$$

Furthermore, we set

$$\mathcal{C}(\Phi) = \bigcup_{\mathbf{G},\mathbf{T},\mathbb{F}_q} \mathcal{C}(\mathbf{G}, \mathbf{T}, \mathbb{F}_q), \tag{3.7}$$

where $\mathbf{G}$, $\mathbf{T}$ and $\mathbb{F}_q$ range over all possible choices for the fixed root system Φ. In Proposition 3.9 we show that, in fact, $\mathcal{C}(\mathbf{G}, \mathbf{T}, \mathbb{F}_q) = \mathcal{C}(\Phi)$ as long as $\mathbb{F}_q$ contains a sufficient supply of roots of unity.

Lemma 3.8. *Let $\varphi\colon \mathbf{G} \to \mathbf{H}$ be an isogeny, i.e., a surjective morphism with finite central kernel, between algebraic groups defined over a finite field $\mathbb{F}_q$. If $g \in \mathbf{G}(\mathbb{F}_q)$, then φ maps $C_{\mathbf{G}}(g)^\circ$ onto $C_{\mathbf{H}}(\varphi(g))^\circ$.*

Proof. Write $h = \varphi(g)$ and observe that

$$\bigcap_{\tilde{g} \in \mathbf{G}(\mathbb{F}_q^{\mathrm{alg}}),\ \varphi(\tilde{g}) = h} C_{\mathbf{G}}(\tilde{g}) = C_{\mathbf{G}}(g) \subset \varphi^{-1}(C_{\mathbf{H}}(h)). \tag{3.8}$$

Furthermore, $\varphi^{-1}(C_{\mathbf{H}}(h)) / \bigcap_{\varphi(\tilde{g})=h} C_{\mathbf{G}}(\tilde{g})$ acts faithfully by conjugation on the finite set $g\ker(\varphi)$. This shows that the inclusion in (3.8) is of finite index, and hence $C_{\mathbf{G}}(g)^\circ$ is a finite index subgroup of $\varphi^{-1}(C_{\mathbf{H}}(h))$. Since $\varphi(C_{\mathbf{G}}(g)^\circ)$ is connected and of finite index in $C_{\mathbf{H}}(h)$, we conclude that $\varphi(C_{\mathbf{G}}(g)^\circ) = C_{\mathbf{H}}(h)^\circ$. □

Proposition 3.9. *Let Φ be a root system. There exists $m \in \mathbb{N}$ such that for every connected split reductive algebraic group $\mathbf{G}$ over a finite field $\mathbb{F}_q$ whose absolute root system, associated to an $\mathbb{F}_q$-rational split maximal torus $\mathbf{T}$, is isomorphic to and identified with Φ the following holds: if $\mathbb{F}_q$ contains primitive mth roots of unity, equivalently $q \equiv_m 1$, then $\mathcal{C}(\mathbf{G}, \mathbf{T}, \mathbb{F}_q) = \mathcal{C}(\Phi)$.*

In particular, if $\mathbf{G}$ is as above and $q \equiv_m 1$, then for every semi-simple element $g \in \mathbf{G}(\mathbb{F}_q^{\mathrm{alg}})$, there is a semi-simple element $g_0 \in \mathbf{G}(\mathbb{F}_q)$ such that $C_{\mathbf{G}}(g)^\circ$ and $C_{\mathbf{G}}(g_0)^\circ$ are $\mathbf{G}$-conjugate.

Proof. Fix $\Psi \in \mathcal{C}(\Phi)$. Clearly, it suffices to show that there exists $m(\Psi) \in \mathbb{N}$ so that Ψ arises as the absolute root system of a connected centraliser in each instance of $(\mathbf{G}, \mathbf{T}, \mathbb{F}_q)$ with $q \equiv_{m(\Psi)} 1$.

Let $\mathbf{G}$ be a connected split reductive algebraic group over a finite field $\mathbb{F}_q$, with absolute root system Φ. The group $\mathbf{G}$ is the image of an isogeny from the direct product of the connected split semi-simple group $[\mathbf{G}, \mathbf{G}]$ and $Z(\mathbf{G})^\circ$, a torus. Using Lemma 3.8, we may concentrate on the case that $\mathbf{G}$ is semi-simple and hence an isogenous image of a direct product of split simply-connected almost simple algebraic groups. Applying once more Lemma 3.8, we may restrict to the case that $\mathbf{G}$ is simply-connected almost simple, with root datum $(X, \Phi, \mathbb{Z}\Phi^\vee, \Phi^\vee)$, say. The root lattice $\mathbb{Z}\Phi$ is a sublattice of finite index in the free $\mathbb{Z}$-lattice X. We fix free generators $\chi_1, \ldots, \chi_\ell$ of X.

Let $\mathbf{T} \subset \mathbf{G}$ be an $\mathbb{F}_q$-rational split maximal torus. We identify Φ with the root system and X with the character group associated to $\mathbf{T}$. Let $g \in \mathbf{T}(\mathbb{F}_q^{\mathrm{alg}})$. As noted before, the connected centraliser of g is the reductive subgroup

$$C_{\mathbf{G}}(g)^\circ = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Psi_g \rangle$$

with root system $\Psi_g = \{\alpha \in \Phi \mid \alpha(g) = 1\}$. Furthermore, the multiplicative group $(\mathbb{F}_q^{\mathrm{alg}})^*$ is isomorphic to the additive group $\mathbb{Q}_{p'}/\mathbb{Z}$, where p denotes the characteristic of $\mathbb{F}_q$ and $\mathbb{Q}_{p'}$ the ring of all rational numbers whose denominator is not divisible by p. Under this isomorphism $\mathbb{F}_q^*$ corresponds to the additive group $\mathbb{Z}[1/(q-1)]/\mathbb{Z}$.

The conditions $\alpha(g) = 1$ for $\alpha \in \Psi_g$ and $\alpha(g) \neq 1$ for $\alpha \in \Phi \smallsetminus \Psi_g$, which determine $C_{\mathbf{G}}(g)^\circ$, translate into a finite system of linear equations and inequations with integer coefficients in ℓ variables $x_1, \ldots, x_\ell$, corresponding to the generators $\chi_1, \ldots, \chi_\ell$ of X. The element g corresponds to a solution $(a_1/b_1 + \mathbb{Z}, \ldots, a_\ell/b_\ell + \mathbb{Z}) \in (\mathbb{Q}_{p'}/\mathbb{Z})^\ell$ of the system modulo $\mathbb{Z}$. Without loss of generality, we may assume that $\gcd(a_i, b_i) = 1$ for each $i \in \{1, \ldots, \ell\}$, and we set $m(\Psi_g) = \mathrm{lcm}(b_1, \ldots, b_\ell)$. Then $g \in \mathbf{T}(\mathbb{F}_q)$ if and only if $\mathbb{F}_q$ contains primitive roots of unity of degree $m(\Psi_g)$, equivalently $q \equiv_{m(\Psi_g)} 1$.

As $\Psi \in \mathcal{C}(\Phi)$, there is an instance of $(\mathbf{G}, \mathbf{T}, \mathbb{F}_q)$ and $g \in \mathbb{F}_q$ as described above such that $\Psi = \Psi_g$. This single solution shows that $\Psi \in \mathcal{C}(\mathbf{G}, \mathbf{T}, \mathbb{F}_q)$ for all instances $(\mathbf{G}, \mathbf{T}, \mathbb{F}_q)$ with $q \equiv_{m(\Psi)} 1$. ☐

Corollary 3.10. *There exists $c \in \mathbb{N}$, depending only on Φ, such that, for all finite fields $\mathbb{F}_q$ of characteristic $p > c$,*

$$\mathcal{C}(\mathbf{G}, \mathbf{T}, \mathbb{F}_q^{\mathrm{alg}}) = \mathcal{C}(\Phi).$$

Remark 3.11. Proposition 3.9 is related to more detailed investigations into semi-simple conjugacy classes and their centralizers in finite groups of Lie type, which were initiated by Carter and Deriziotis. In particular, their results imply that $\mathcal{C}(\mathbf{G},\mathbf{T},\mathbb{F}_q^{\mathrm{alg}}) = \mathcal{C}(\Phi)$ if the characteristic of $\mathbb{F}_q$ is greater than 5; see [12, Proposition 2.3 and remarks]. For an overview, including also subsequent developments we refer to [20, Chapters 2 and 8] and the references given therein.

Corollary 3.12. *Let Φ be a non-trivial root system, and let $\mathcal{C}(\Phi)$ be as defined in* (3.7). *Let $W(\Phi)$ denote the Weyl group of Φ. Let $\mathbf{G}$ be a connected semi-simple algebraic group defined over a finite field $\mathbb{F}_q$ with absolute root system Φ. Identify Φ with the root system associated to a maximal torus $\mathbf{T}$ of $\mathbf{G}$, and let $\mathcal{C}(\mathbf{G},\mathbf{T},\mathbb{F}_q^{\mathrm{alg}})$ be as defined in* (3.6). *Then there are connected subgroups $\mathbf{H}_\Psi$ of $\mathbf{G}$, not necessarily defined over $\mathbb{F}_q$ and indexed by $\Psi \in \mathcal{C}(\mathbf{G},\mathbf{T},\mathbb{F}_q^{\mathrm{alg}})$, such that the following hold.*

(1) *The groups $\mathbf{H}_\Psi$ form a – typically redundant – set of representatives for the $\mathbf{G}$-conjugacy classes of connected centralisers $C_{\mathbf{G}}(g)^\circ$ of semi-simple elements $g \in \mathbf{G}(\mathbb{F}_q^{\mathrm{alg}})$.*
(2) *The data $(\dim Z(\mathbf{H}_\Psi), \dim \mathbf{H}_\Psi, |\Psi^+|)$ depend only on Φ and Ψ; more precisely, the first two entries satisfy $\dim Z(\mathbf{H}_\Psi) = \mathrm{rk}\,\Phi - \mathrm{rk}\,\Psi$ and $\dim \mathbf{H}_\Psi = \mathrm{rk}\,\Psi + 2|\Psi^+|$.*
(3) *For every Ψ, the number of $\mathbf{G}(\mathbb{F}_q)$-conjugacy classes of connected centralisers $C_{\mathbf{G}}(g)^\circ \subset \mathbf{G}$ of semi-simple elements $g \in \mathbf{G}(\mathbb{F}_q)$ that are $\mathbf{G}$-conjugate to $\mathbf{H}_\Psi$ is at most $|W(\Phi)||\mathcal{C}(\Phi)|$.*

Proof. For each $\Psi \in \mathcal{C}(\mathbf{G},\mathbf{T},\mathbb{F}_q^{\mathrm{alg}})$, choose $\mathbf{H}_\Psi$ as the connected centraliser $C_{\mathbf{G}}(g_\Psi)^\circ$ of a semi-simple element $g_\Psi \in \mathbf{T}(\mathbb{F}_q^{\mathrm{alg}})$ such that the root system associated to $\mathbf{H}_\Psi$ is Ψ.

Clearly, (1) and (2) are satisfied. It remains to justify (3). Fix $\Psi \in \mathcal{C}(\mathbf{G},\mathbf{T},\mathbb{F}_q^{\mathrm{alg}})$. We may assume that $\mathbf{H} = \mathbf{H}_\Psi$ is itself equal to $C_{\mathbf{G}}(g_0)^\circ$ for a semi-simple element $g_0 \in \mathbf{G}(\mathbb{F}_q)$. Let $\mathbf{T}_0$ be an $\mathbb{F}_q$-defined maximal torus of $\mathbf{H}$. We observe that $g_0 \in Z(\mathbf{H}^\circ) \subset \mathbf{T}_0$ (see [6, Proposition 3.5.1]) and that $\mathbf{T}_0$ is also an $\mathbb{F}_q$-defined maximal torus of $\mathbf{G}$. By [6, Proposition 3.3.3], the number of $\mathbf{G}(\mathbb{F}_q)$-conjugacy classes of $\mathbb{F}_q$-defined maximal tori is at most $|W(\Phi)|$. Thus we only need to bound the number of connected centralisers of semi-simple elements g in any fixed $\mathbb{F}_q$-defined maximal torus, but this number is clearly bounded by $|\mathcal{C}(\Phi)|$. □

3.1.3. *Non-Central Semi-Simple Conjugacy Classes.* Let $\mathbf{G}$ be a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$ and let $\mathbf{G}^*$ be the dual group. By [6, Proposition 3.6.8], we have $Z(\mathbf{G}^*(\mathbb{F}_q)) = Z(\mathbf{G}^*)(\mathbb{F}_q)$ so that central geometric conjugacy classes and central rational conjugacy classes in $\mathbf{G}^*(\mathbb{F}_q)$ coincide.

We set

$$\zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}(s) = \sum_{(g)\not\subset Z(\mathbf{G}^*(\mathbb{F}_q))} \; \sum_{\chi \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q),(g))} \chi(1)^{-s}, \tag{3.9}$$

where the outer sum ranges over the non-central semi-simple $\mathbf{G}^*$-conjugacy classes (g) in $\mathbf{G}^*(\mathbb{F}_q)$. For a non-trivial root system Φ, we define

$$b^{\mathrm{nc}}(\Phi) = \big\{(\mathrm{rk}(\Phi) - \mathrm{rk}(\Psi), |\Phi^+| - |\Psi^+|) \mid \Psi \in \mathcal{C}(\Phi) \smallsetminus \{\Phi\}\big\} \in \mathcal{A}^+. \tag{3.10}$$

Lemma 3.13. *Let Φ be a non-trivial root system and let $\mathfrak{G}_\Phi$ denote the collection of pairs $(\mathbf{G}, q)$, where $\mathbf{G}$ is a connected semi-simple algebraic group over a finite field $\mathbb{F}_q$ with absolute root system Φ. Then there are $C \in \mathbb{R}$ and $b^{\mathrm{nc}}(\mathbf{G}, q) \subset b^{\mathrm{nc}}(\Phi)$ for $(\mathbf{G}, q) \in \mathfrak{G}_\Phi$ such that for all $(\mathbf{G}, q), (\mathbf{H}, q) \in \mathfrak{G}_\Phi$ the following hold:*

1. $b^{\mathrm{nc}}(\mathbf{G}, q) \subset b^{\mathrm{nc}}(\mathbf{H}, q)$ *whenever there is an isogeny $\varphi\colon \mathbf{G} \to \mathbf{H}$ defined over $\mathbb{F}_q$,*
2. $b^{\mathrm{nc}}(\mathbf{G}, q) = b^{\mathrm{nc}}(\Phi)$ *whenever $\mathbf{G}$ is split and $q \equiv_m 1$, where m is as in Proposition 3.9,*
3. $\zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)} \sim_C \xi_{b^{\mathrm{nc}}(\mathbf{G},q),q}$*,*
4. $(\mathrm{rk}\,\Phi, |\Phi^+|) \in b^{\mathrm{nc}}(\Phi)$*, and* $1 + \xi_{b^{\mathrm{nc}}(\mathbf{G},q),q}(s) \sim_C 1 + q^{\mathrm{rk}\,\Phi - |\Phi^+| s}$.

Proof. Consider $(\mathbf{G}, q) \in \mathfrak{G}_\Phi$ and let $\mathbf{G}^*$ be the dual group of $\mathbf{G}$. The centralizer $C_{\mathbf{G}^*}(g)$ of a semi-simple element $g \in \mathbf{G}^*(\mathbb{F}_q)$ is a reductive subgroup of maximal rank in $\mathbf{G}^*$. Furthermore, there is a surjection $\psi_g\colon \mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g)) \to \mathcal{E}(C_{\mathbf{G}^*}(g)(\mathbb{F}_q), (1))$ such that the fibers of ψ_g have sizes at most $|Z(\mathbf{G})|$, and the degree of an element $\chi \in \mathcal{E}(\mathbf{G}(\mathbb{F}_q), (g))$ is given by

$$\chi(1) = |\mathbf{G}^*(\mathbb{F}_q) : C_{\mathbf{G}^*}(g)(\mathbb{F}_q)|_{q'} \cdot (\psi_g(\chi))(1),$$

where $n_{q'}$ denotes the prime-to-q part of a number n; see [13, Theorem 13.23 and Remark 13.24] and [6, Proposition 4.4.4] in the case where $\mathbf{G}$ has trivial centre, and [32, Proposition 5.1] for the general case. It follows that

$$\zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}(s) \sim_{|Z(\mathbf{G})|} \sum_{(g) \not\subset Z(\mathbf{G}^*(\mathbb{F}_q))} |\mathbf{G}^*(\mathbb{F}_q) : C_{\mathbf{G}^*}(g)(\mathbb{F}_q)|_{q'}^{-s} \cdot \zeta^{\mathrm{unip}}_{C_{\mathbf{G}^*}(g)(\mathbb{F}_q)}(s).$$

Consider a semi-simple element $g \in \mathbf{G}^*(\mathbb{F}_q)$. The index $|C_{\mathbf{G}^*}(g) : C_{\mathbf{G}^*}(g)^\circ|$ is bounded by a constant $C_1 \in \mathbb{R}$, depending only on Φ; see [13, Remark 2.4]. This gives

$$|\mathbf{G}^*(\mathbb{F}_q) : C_{\mathbf{G}^*}(g)(\mathbb{F}_q)|_{q'}^{-s} \sim_{C_1} |\mathbf{G}^*(\mathbb{F}_q) : C_{\mathbf{G}^*}(g)^\circ(\mathbb{F}_q)|_{q'}^{-s}.$$

Moreover, the unipotent characters of $C_{\mathbf{G}^*}(g)(\mathbb{F}_q)$ are the irreducible characters whose restrictions to $C_{\mathbf{G}^*}(g)^\circ(\mathbb{F}_q)$ are sums of unipotent characters. From Proposition 3.5 we conclude that there is a constant $C_2 \in \mathbb{R}$ such that

$$\zeta^{\mathrm{unip}}_{C_{\mathbf{G}^*}(g)(\mathbb{F}_q)}(s) \sim_{C_2} 1.$$

We get that

$$\zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)} \sim_{|Z(\mathbf{G})| C_1 C_2} \sum_{(g) \not\subset Z(\mathbf{G}^*(\mathbb{F}_q))} \left(\frac{|\mathbf{G}^*(\mathbb{F}_q)|_{q'}}{|C_{\mathbf{G}^*}(g)^\circ(\mathbb{F}_q)|_{q'}} \right)^{-s}.$$

Applying Corollary 3.12 to $\mathbf{G}^*$, we obtain $N \in \mathbb{N}$, bounded by $|\mathcal{C}(\Phi)|$, and algebraic subgroups $\mathbf{H}_1, \ldots, \mathbf{H}_N \subset \mathbf{G}^*$ with the properties described in the corollary. For each $i \in \{1, \ldots, N\}$ denote the absolute root system of $\mathbf{H}_i$ by $\Psi_i \in \mathcal{C}(\Phi)$. Given a non-central semi-simple element $g \in \mathbf{G}^*(\mathbb{F}_q)$, the group $C_{\mathbf{G}^*}(g)^\circ$ is $\mathbf{G}^*$-conjugate to some $\mathbf{H}_i$. There is a constant $C_3 \in \mathbb{R}$ such that $|\mathbf{G}^*(\mathbb{F}_q)|_{q'} \sim_{C_3} q^{\dim \mathbf{G} - |\Phi^+|}$ and $|C_{\mathbf{G}^*}(g)^\circ(\mathbb{F}_q)|_{q'} \sim_{C_3} q^{\dim \mathbf{H}_i - |\Psi_i^+|}$. Therefore

$$\frac{|\mathbf{G}^*(\mathbb{F}_q)|_{q'}}{|C_{\mathbf{G}^*}(g)^\circ(\mathbb{F}_q)|_{q'}} \sim_{C_3^2} q^{\dim \mathbf{G} - |\Phi^+| - \dim \mathbf{H}_i + |\Psi_i^+|} = q^{|\Phi^+| - |\Psi_i^+|}$$

for all $i \in \{1, \ldots, N\}$. Writing $C_4 = |Z(\mathbf{G})| C_1 C_2 C_3^2 N$, we obtain

$$\begin{aligned}(3.11)\quad & \zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}(s) \sim_{C_4} \\ & \sum_{i=1}^{N} |\{(g) \not\subset Z(\mathbf{G}^*(\mathbb{F}_q)) \mid C_{\mathbf{G}^*}(g)^\circ \text{ is } \mathbf{G}^*\text{-conjugate to } \mathbf{H}_i\}| \cdot q^{-(|\Phi^+| - |\Psi_i^+|)s}.\end{aligned}$$

The additional factor N of C_4 is included, because $C_{\mathbf{G}^*}(g)^\circ$ could be conjugate to more than one $\mathbf{H}_i$.

Denote by $I(\mathbf{G}, q)$ the set of all $i \in \{1, \ldots, N\}$ such that there exists a non-central semi-simple element $g \in \mathbf{G}^*(\mathbb{F}_q)$ with $C_{\mathbf{G}^*}(g)^\circ$ being $\mathbf{G}^*$-conjugate to $\mathbf{H}_i$. Fix $i \in I(\mathbf{G}, q)$ and let $\mathbf{K}_{i,1}, \ldots, \mathbf{K}_{i,n(i)}$ be a selection of such centralisers, forming a complete set of representatives up to $\mathbf{G}^*(\mathbb{F}_q)$-conjugacy. By Corollary 3.12, the number $n(i)$ is uniformly bounded in terms of Φ. The collection of non-central semi-simple conjugacy classes $(g) \subset \mathbf{G}^*(\mathbb{F}_q)$ such that $C_{\mathbf{G}^*}(g)^\circ$ is $\mathbf{G}^*$-conjugate to $\mathbf{H}_i$ decomposes as a disjoint union as follows:

$$\begin{aligned}(3.12)\quad & \{(g) \not\subset Z(\mathbf{G}^*(\mathbb{F}_q)) \mid C_{\mathbf{G}^*}(g)^\circ \text{ is } \mathbf{G}^*\text{-conjugate to } \mathbf{H}_i\} = \\ & \bigsqcup_{j=1}^{n(i)} \{(g) \not\subset Z(\mathbf{G}^*(\mathbb{F}_q)) \mid C_{\mathbf{G}^*}(g)^\circ \text{ is } \mathbf{G}^*(\mathbb{F}_q)\text{-conjugate to } \mathbf{K}_{i,j}\}.\end{aligned}$$

Fix also $j \in \{1, \ldots, n(i)\}$. Now [29, Lemma 2.2(ii)] supplies a constant $C_5 \in \mathbb{R}$ such that, for every non-central semi-simple conjugacy class $(g) \subset \mathbf{G}^*(\mathbb{F}_q)$ with $C_{\mathbf{G}^*}(g)^\circ$ being $\mathbf{G}^*(\mathbb{F}_q)$-conjugate to $\mathbf{K}_{i,j}$,

$$1 \leq |\{g^x \in (g) \mid C_{\mathbf{G}^*}(g^x)^\circ = \mathbf{K}_{i,j}\}| \leq C_5.$$

It follows that

$$\begin{aligned}(3.13)\quad & |\{(g) \not\subset Z(\mathbf{G}^*(\mathbb{F}_q)) \mid C_{\mathbf{G}^*}(g)^\circ \text{ is } \mathbf{G}^*(\mathbb{F}_q)\text{-conjugate to } \mathbf{K}_{i,j}\}| \sim_{C_5} \\ & \qquad |\{g \in \mathbf{G}^*(\mathbb{F}_q) \mid C_{\mathbf{G}^*}(g)^\circ = \mathbf{K}_{i,j}\}|.\end{aligned}$$

If $g \in \mathbf{G}^*(\mathbb{F}_q)$ is semi-simple and $C_{\mathbf{G}^*}(g)^\circ = \mathbf{K}_{i,j}$, then $g \in \mathbf{K}_{i,j}(\mathbb{F}_q)$. Choose a maximal torus $\mathbf{T}_{i,j}$ of $\mathbf{K_{i,j}}$. The torus $\mathbf{T}_{i,j}$ is also maximal in $\mathbf{G}^*$. Denote the set of roots of $(\mathbf{G}^*, \mathbf{T}_{i,j})$ by $\Lambda_{i,j} \subset \mathrm{Hom}(\mathbf{T}_{i,j}, \mathbb{G}_{\mathrm{m}})$ and the set of roots of $(\mathbf{K}_{i,j}, \mathbf{T}_{i,j})$ by $\Delta_{i,j} \subset \mathrm{Hom}(\mathbf{T}_{i,j}, \mathbb{G}_{\mathrm{m}})$. Note that $\Lambda_{i,j}$ is isomorphic to Φ and $\Delta_{i,j}$ is isomorphic to Ψ_i. The set of elements $g \in Z(\mathbf{K}_{i,j})^\circ$ with $C_{\mathbf{G}^*}(g)^\circ = \mathbf{K}_{i,j}$ is the complement of the union of the zero loci of the roots in $\Lambda_{i,j} \smallsetminus \Delta_{i,j}$. Each such zero locus is the extension of a proper sub-torus by a finite group. The order of that finite group, i.e., the number of connected components of the zero locus, is bounded by some constant depending only on Φ; see, for example, [15, Corollary 9.7.9]. For every torus $\mathbf{T}$, we have

$$2^{-\dim \mathbf{T}} q^{\dim \mathbf{T}} \leq |\mathbf{T}(\mathbb{F}_q)| \leq 2^{\dim \mathbf{T}} q^{\dim \mathbf{T}}; \tag{3.14}$$

see [35, Lemma 3.5]. Hence there is a constant $D \in \mathbb{R}$ such that, for all sufficiently large q,

$$q^{\dim Z(\mathbf{K}_{i,j})} \lesssim_D |\{g \in \mathbf{G}^*(\mathbb{F}_q) \mid C_{\mathbf{G}^*}(g)^\circ = \mathbf{K}_{i,j}\}|. \tag{3.15}$$

By [15, Corollary 9.7.9], the index $|Z(\mathbf{K}_{i,j}) : Z(\mathbf{K}_{i,j})^\circ|$ for the connected reductive group $\mathbf{K}_{i,j}$ is bounded by some constant depending on Φ and, of course, $\dim Z(\mathbf{K}_{ij}) = \dim Z(\mathbf{H}_i)$. Using (3.12), (3.13), (3.14), and (3.15), it follows that there is a constant $C_6 \in \mathbb{R}$ such that

$$|\{(g) \not\subset Z(\mathbf{G}^*(\mathbb{F}_q)) \mid C_{\mathbf{G}^*}(g)^\circ \text{ is } \mathbf{G}^*\text{-conjugate to } \mathbf{H}_i\}| \sim_{C_6} q^{\dim Z(\mathbf{H}_i)},$$

and from (3.11) we obtain

$$\zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}(s) \sim_{C_4 C_6} \sum_{i \in I(\mathbf{G},q)} q^{\dim Z(\mathbf{H}_i) - (|\Phi^+| - |\Psi_i^+|)s}.$$

Recalling from (3.10) the definition of $b^{\mathrm{nc}}(\Phi)$, we put

$$b^{\mathrm{nc}}(\mathbf{G}, q) = \big\{(\dim Z(\mathbf{H}_i), |\Phi^+| - |\Psi_i^+|) \mid i \in I(\mathbf{G}, q)\big\} \subset b^{\mathrm{nc}}(\Phi)$$

so that

$$\zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)} \sim_{C_4 C_6 N} \xi_{b^{\mathrm{nc}}(\mathbf{G},q),q}.$$

The extra factor N in the index of the $\sim$-symbol accommodates for the fact that $\mathbf{H}_i$ and $\mathbf{H}_j$ may lead to the same data even though $i \neq j$. This completes the proof of assertion (3) of the lemma.

Next we prove (4). For $i \in I(\mathbf{G}, q)$, [29, Lemma 2.5] yields that

$$\frac{\dim Z(\mathbf{H}_i)}{|\Phi^+| - |\Psi_i^+|} \leq \frac{\mathrm{rk}\,\mathbf{G}}{|\Phi^+|} = \frac{\mathrm{rk}\,\Phi}{|\Phi^+|}. \tag{3.16}$$

Moreover, we have $\dim Z(\mathbf{H}_i) \leq \mathrm{rk}\,\mathbf{G}$ for each $i \in I(\mathbf{G}, q)$, and for sufficiently large q (depending only on Φ; see [6, Proposition 3.6.6]) there is at least one $\mathbf{H}_i$, $i \in I(\mathbf{G}, q)$, namely a non-degenerate maximal torus, for which

$$(\dim Z(\mathbf{H}_i), |\Phi^+| - |\Psi_i^+|) = (\mathrm{rk}\,\Phi, |\Phi^+|). \tag{3.17}$$

From this we deduce that, for sufficiently large q,

$$1 + \xi_{b^{\mathrm{nc}}(\mathbf{G},q),q}(s) \sim_{N+1} 1 + \xi_{\{(\mathrm{rk}\,\Phi, |\Phi^+|)\},q}(s) = 1 + q^{\mathrm{rk}\,\Phi - |\Phi^+| s};$$

indeed, the inequality $\gtrsim_{N+1}$ is clear from (3.17), and $\lesssim_{N+1}$ follows from Remark 2.7, noting that $|b^{\mathrm{nc}}(\mathbf{G}, q)| \leq N$. Thus (4) is proved.

Assertion (1) follows from the observation that, if $\varphi\colon \mathbf{G} \to \mathbf{H}$ is an isogeny over $\mathbb{F}_q$, then by Lemma 3.8 we may arrange the labelling so that $I(\mathbf{G}, q) \subset I(\mathbf{H}, q)$. Assertion (2) holds by virtue of Proposition 3.9. $\square$

3.1.4. *Proofs of Theorem 3.1 and Corollary 3.4.*

Proof of Theorem 3.1. Let $\mathbf{G}$ be a connected semi-simple algebraic group over $\mathbb{F}_q$ of Lie type $L \in \boldsymbol{\mathcal{L}}_\Phi$ and let $\mathbf{G}^*$ be the dual group. In analysing the right-hand side of (3.2), we deal separately with the sum over central conjugacy classes and the sum over non-central conjugacy classes. In analogy with the definition (3.9) of $\zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}(s)$ we set

$$\zeta^{\mathrm{c}}_{\mathbf{G}(\mathbb{F}_q)}(s) = \sum_{(g)\subset Z(\mathbf{G}^*(\mathbb{F}_q))} \; \sum_{\chi\in\mathcal{E}(\mathbf{G}(\mathbb{F}_q),(g))} \chi(1)^{-s},$$

where the outer sum ranges over the central $\mathbf{G}^*$-conjugacy classes (g) in $\mathbf{G}^*(\mathbb{F}_q)$ corresponding to semi-simple elements $g \in Z(\mathbf{G}^*(\mathbb{F}_q))$. Thus $\zeta_{\mathbf{G}(\mathbb{F}_q)} = \zeta^{\mathrm{c}}_{\mathbf{G}(\mathbb{F}_q)} + \zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}$; cf. (3.2).

We first consider $\zeta^{\mathrm{c}}_{\mathbf{G}(\mathbb{F}_q)}$. There is a constant $C_0 \in \mathbb{R}$, depending only on Φ, such that

$$|Z(\mathbf{G}^*(\mathbb{F}_q))| \leq C_0. \tag{3.18}$$

Furthermore, if $g \in Z(\mathbf{G}^*(\mathbb{F}_q))$, then the elements of $\mathcal{E}(\mathbf{G}(\mathbb{F}_q),(g))$ are in dimension-preserving bijection with the elements of $\mathcal{E}(\mathbf{G}(\mathbb{F}_q),(1))$; see [32, Theorem 5.1]. Thus,

$$\zeta^{\mathrm{c}}_{\mathbf{G}(\mathbb{F}_q)} = |Z(\mathbf{G}^*(\mathbb{F}_q))|\, \zeta^{\mathrm{unip}}_{\mathbf{G}(\mathbb{F}_q)}, \tag{3.19}$$

and, in combination with (3.18), Proposition 3.5 yields $C_1 \in \mathbb{R}$ and $b^{\mathrm{c}}(\mathcal{L}) \in \mathcal{A}^+$, depending only on Φ respectively L, such that

$$\zeta^{\mathrm{c}}_{\mathbf{G}(\mathbb{F}_q)} - |Z(\mathbf{G}^*(\mathbb{F}_q))| = |Z(\mathbf{G}^*(\mathbb{F}_q))|\,(\zeta^{\mathrm{unip}}_{\mathbf{G}(\mathbb{F}_q)} - 1) \sim_{C_0C_1} \xi_{b^{\mathrm{c}}(\mathcal{L}),q}. \tag{3.20}$$

We set $b^{\mathrm{c}}(\Phi) = b^{\mathrm{c}}(\mathcal{L}^{\mathrm{sp}})$.

Next we account for $\zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}$. We recall the definition (3.10) of the set $b^{\mathrm{nc}}(\Phi)$ and set

$$a(\Phi) = b^{\mathrm{c}}(\Phi) + b^{\mathrm{nc}}(\Phi), \qquad a(\mathbf{G},q) = b^{\mathrm{c}}(\mathcal{L}) + b^{\mathrm{nc}}(\mathbf{G},q), \tag{3.21}$$

where $b^{\mathrm{nc}}(\mathbf{G},q)$ is defined as in Lemma 3.13. Furthermore, we choose m, depending only on Φ, in accordance with Proposition 3.9. Then the following weaker variants of assertions (1) and (2) of the theorem are clearly satisfied:

(1)$'$ $a(\mathbf{G},q) \subset a(\Phi)$,

(2)$'$ $a(\mathbf{G},q) = a(\Phi)$ whenever $\mathbf{G}$ is split and $q \equiv_m 1$.

Moreover, setting C_2 to be the sum of the constant C_0C_1 from (3.20) and the constant that Lemma 3.13 supplies, we obtain

$$\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) - |Z(\mathbf{G}^*(\mathbb{F}_q))| \sim_{C_2} \xi_{a(\mathbf{G},q),q}(s). \tag{3.22}$$

We now prove that, for $q > C_2$,

$$|\mathbf{G}(\mathbb{F}_q)/[\mathbf{G}(\mathbb{F}_q),\mathbf{G}(\mathbb{F}_q)]| = |Z(\mathbf{G}^*(\mathbb{F}_q))|.$$

Indeed, put $\delta(q) = |\mathbf{G}(\mathbb{F}_q)/[\mathbf{G}(\mathbb{F}_q),\mathbf{G}(\mathbb{F}_q)]| - |Z(\mathbf{G}^*(\mathbb{F}_q))|$. Fix $q > C_2$ and let $s = \sigma \in \mathbb{R}$ tend to ∞ in (3.22). The limit of the left-hand side of (3.22) as $s = \sigma \to \infty$ is equal to $\delta(q)$, and $q > C_2$ implies that the limit of $C_2^{1+\sigma}\xi_{a(\mathbf{G},q)}(\sigma)$ as $\sigma \to \infty$ is equal to 0. Thus $\delta(q) \leq 0$. On the other hand, the limit of the right-hand side of (3.22) as $s = \sigma \to \infty$ is equal to 0, and the limit of $C_2^{1+\sigma}(\zeta_{\mathbf{G}(\mathbb{F}_q)}(\sigma) - |Z(\mathbf{G}^*(\mathbb{F}_q))|)$ as $\sigma \to \infty$ is non-negative only if $\delta(q) \geq 0$. Hence $\delta(q) = 0$.

Next we show that in the approximations that we seek it is not necessary to distinguish between different members of the isogeny class of $\mathbf{G}$, as we have done up to this point. We define

$$a(\mathcal{L}, q) = a(\mathbf{G}^{\mathrm{ad}}, q) = b^{\mathrm{c}}(\mathcal{L}) + b^{\mathrm{nc}}(\mathbf{G}^{\mathrm{ad}}, q),$$

where $\mathbf{G}^{\mathrm{ad}}$ denotes the adjoint quotient of $\mathbf{G}$ and claim that assertions (1), (2), (3) in the theorem hold. The statements (1), (2) are clearly special cases of (1)′, (2)′. In order to establish (3) we observe that Lemma 3.13 already gives

$$\xi_{a(\mathbf{G}^{\mathrm{sc}},q),q} \lesssim_1 \xi_{a(\mathbf{G},q),q} \lesssim_1 \xi_{a(\mathbf{G}^{\mathrm{ad}},q),q} = \xi_{a(\mathcal{L},q),q}.$$

where $\mathbf{G}^{\mathrm{sc}}$ denotes the simply connected member in the isogeny class of $\mathbf{G}$.

Thus it suffices to show that there is a constant $C_3 \in \mathbb{R}$, depending only on Φ, such that

$$\zeta_{\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q)} - |\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q)/[\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q), \mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q)]| \gtrsim_{C_3} \zeta_{\mathbf{G}^{\mathrm{ad}}(\mathbb{F}_q)} - |\mathbf{G}^{\mathrm{ad}}(\mathbb{F}_q)/[\mathbf{G}^{\mathrm{ad}}(\mathbb{F}_q), \mathbf{G}^{\mathrm{ad}}(\mathbb{F}_q)]|.$$

Write $G = \mathbf{G}^{\mathrm{ad}}(\mathbb{F}_q)$. Let H denote the image of $\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q)$ under the natural homomorphism $\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q) \to \mathbf{G}^{\mathrm{ad}}(\mathbb{F}_q)$. Then $|G| = |\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q)|$, thus $|G : H| = |Z(\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q))|$ is bounded by $C_3 = |Z(\mathbf{G}^{\mathrm{sc}})|$, depending only on Φ. From (3.22) we observe that the smallest degree of a non-linear character of G is given by an increasing function in q. Taking as a normal subgroup the trivial subgroup, we apply part (3) of Lemma 2.11 to deduce that

$$\zeta_{\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q)} - |\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q)/[\mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q), \mathbf{G}^{\mathrm{sc}}(\mathbb{F}_q)]| \gtrsim_1 \zeta_H - |H/[H,H]| \gtrsim_{C_3} \zeta_G - |G/[G,G]|.$$

Finally, it remains to deduce the estimate (3.1). We need to show that for some constant $C_4 \in \mathbb{R}$, depending only on Φ,

$$\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) \sim_{C_4} 1 + q^{\mathrm{rk}\,\Phi - |\Phi^+|s}.$$

By part (4) of Lemma 3.13, there is a constant $C_5 \in \mathbb{R}$ such that

$$1 + \xi_{b^{\mathrm{nc}}(\mathbf{G},q),q}(s) \sim_{C_5} 1 + q^{\mathrm{rk}\,\Phi - |\Phi^+|s}. \tag{3.23}$$

Furthermore, using equation (3.19), part (2) of Proposition 3.5, part (3) of Lemma 3.13 and equations (3.18) and (3.23), we conclude that there is a constant $C_6 \in \mathbb{R}$, depending only on Φ, such that

$$\begin{aligned}\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) &= \zeta^{\mathrm{c}}_{\mathbf{G}(\mathbb{F}_q)}(s) + \zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}(s) = |Z(\mathbf{G}^*(\mathbb{F}_q))|\zeta^{\mathrm{unip}}_{\mathbf{G}(\mathbb{F}_q)}(s) + \zeta^{\mathrm{nc}}_{\mathbf{G}(\mathbb{F}_q)}(s) \\ &\sim_{C_6} |Z(\mathbf{G}^*(\mathbb{F}_q))| - 1 + 1 + \xi_{b^{\mathrm{nc}}(\mathbf{G},q),q}(s) \sim_{C_0+C_5} 1 + q^{\mathrm{rk}\,\Phi - |\Phi^+|s},\end{aligned}$$

proving the claim. This concludes the proof of Theorem 3.1. ☐

Proof of Corollary 3.4. We use the inequality

$$2^{-\dim \mathbf{H}} q^{\dim \mathbf{H}} \leq |\mathbf{H}(\mathbb{F}_q)| \leq 2^{\dim \mathbf{H}} q^{\dim \mathbf{H}}$$

true for every connected algebraic group $\mathbf{H}$ over a finite field $\mathbb{F}_q$; cf. [35, Lemma 3.5]. Let C_1 be the constant from Theorem 3.1. The derived subgroup $\mathbf{G}'$ is connected, semi-simple, and has absolute root system Φ. The quotient $\mathbf{G}/\mathbf{G}'$ is a torus of dimension $\dim Z(\mathbf{G})$. By the Lang–Steinberg theorem [13, Theorem 3.10], there is a short exact sequence $1 \to \mathbf{G}'(\mathbb{F}_q) \to \mathbf{G}(\mathbb{F}_q) \to \mathbf{G}/\mathbf{G}'(\mathbb{F}_q) \to 1$, which implies that

$$\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) \lesssim_1 |\mathbf{G}/\mathbf{G}'(\mathbb{F}_q)|\zeta_{\mathbf{G}'(\mathbb{F}_q)}(s) \lesssim_{2^{\dim Z(\mathbf{G})}} q^{\dim Z(\mathbf{G})}\zeta_{\mathbf{G}'(\mathbb{F}_q)} \lesssim_{C_1} q^{\dim Z(\mathbf{G})}(1+q^{\mathrm{rk}\,\Phi-|\Phi^+|s}).$$

For the opposite inequality, choose a maximal $\mathbb{F}_q$-rational torus $\mathbf{T} \subset \mathbf{G}/Z(\mathbf{G})$ and a Borel subgroup $\mathbf{B} \subset \mathbf{G}/Z(\mathbf{G})$ containing $\mathbf{T}$. Let $\widetilde{\mathbf{T}}, \widetilde{\mathbf{B}} \subset \mathbf{G}$ be the pre-images of $\mathbf{T}, \mathbf{B}$ under the quotient map. By the proof of [6, Lemma 8.4.2], there is $C_2 \in \mathbb{R}$, depending only on Φ, such that the set of characters ϑ of $\widetilde{\mathbf{T}}(\mathbb{F}_q)$ for which the parabolic induction $\mathrm{Ind}_{\widetilde{\mathbf{B}}(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}(\vartheta)$ is irreducible has size at least $C_2|\widetilde{\mathbf{T}}(\mathbb{F}_q)|$. Thus, we get at least $C_2 \cdot 2^{-\dim \widetilde{\mathbf{T}}} q^{\dim \widetilde{\mathbf{T}}} = C_2 \cdot 2^{-\dim \widetilde{\mathbf{T}}} q^{\dim Z(\mathbf{G})+\mathrm{rk}\,\Phi}$ irreducible representations of dimension

$$|\mathbf{G}(\mathbb{F}_q)|/|\widetilde{\mathbf{B}}(\mathbb{F}_q)| \leq 2^{\dim \mathbf{G}+\dim \widetilde{\mathbf{B}}} q^{\dim \mathbf{G}-\dim \widetilde{\mathbf{B}}} = 2^{\dim \mathbf{G}+\dim \widetilde{\mathbf{B}}} q^{|\Phi^+|}.$$

Finally, $\mathbf{G}(\mathbb{F}_q)$ has $|\mathbf{G}/\mathbf{G}'(\mathbb{F}_q)| \geq 2^{-\dim Z(\mathbf{G})} q^{\dim Z(\mathbf{G})}$ one-dimensional representations factoring through $\mathbf{G}/\mathbf{G}'(\mathbb{F}_q)$. Combining these two classes of representations, we get that $\zeta_{\mathbf{G}(\mathbb{F}_q)}(s) \gtrsim_{C_3} q^{\dim Z(\mathbf{G})}(1+q^{\mathrm{rk}\,\Phi-|\Phi^+|s})$ for a suitable $C_3 \in \mathbb{R}$. □

3.2. **Applications to Finite Quotients of Arithmetic Groups.** In order to apply Theorem 3.1 in a global context, we recall the definition of a Chebotarev set.

Definition 3.14. Let K be a number field with ring of integers O, and let $\varrho\colon \mathrm{Gal}_K \to G$ be a continuous homomorphism from the absolute Galois group of K into a finite group G. Let $P(\varrho)$ denote the set of primes $\mathfrak{p} \in \mathrm{Spec}(O)$ such that $\mathfrak{p}$ is unramified in the extension $K \subset \overline{K}^{\ker \varrho}$ and the Frobenius conjugacy class $(\mathrm{Frob}_\mathfrak{p}) \subset \mathrm{Gal}_K$ associated to $\mathfrak{p}$ lies in the kernel of ϱ. A set $P \subset \mathrm{Spec}(O)$ is a Chebotarev set if it almost contains a set of the form $P(\varrho)$, in the sense that $P(\varrho) \smallsetminus P$ is finite.

The intersection of two Chebotarev sets is a Chebotarev set. Furthermore, by Chebotarev's Density Theorem, every Chebotarev set has positive analytic density. We record the following corollary of Theorem 3.1; compare the definition of $a(\Phi) \in \mathcal{A}^+$ in (3.21).

Corollary 3.15. *Let Φ be a non-trivial root system, and let $C \in \mathbb{R}$ and $a(\Phi) \in \mathcal{A}^+$ be as in Theorem 3.1. Let K be a number field with ring of integers O. For every affine group scheme $\mathbf{G}$ over O whose generic fiber is connected semi-simple with absolute root system Φ, the set of primes $\mathfrak{p} \in \mathrm{Spec}(O)$ such that*

$$\zeta_{\mathbf{G}(O/\mathfrak{p})} - |\mathbf{G}(O/\mathfrak{p})/[\mathbf{G}(O/\mathfrak{p}), \mathbf{G}(O/\mathfrak{p})]| \sim_C \xi_{a(\Phi),|O/\mathfrak{p}|}$$

is a Chebotarev set.

Proof. Let $\mathbf{G}$ be as in the corollary, and $m \in \mathbb{N}$ as in Theorem 3.1. The set of primes $\mathfrak{p}$ such that the reduction of $\mathbf{G}$ modulo $\mathfrak{p}$ is connected, semi-simple, and has absolute root system Φ is cofinite. If $M \supset K$ is a splitting field of $\mathbf{G}$ and $\mathfrak{p}$ splits completely in M, then the reduction of $\mathbf{G}$ modulo $\mathfrak{p}$ is split. Hence, there exists a Chebotarev set $P \subset \mathrm{Spec}(O)$ such that, for all $\mathfrak{p} \in P$,

(1) The reduction of $\mathbf{G}$ modulo $\mathfrak{p}$ is connected, split, semi-simple, and has absolute root system Φ.
(2) The field $O/\mathfrak{p}$ contains primitive mth roots of unity, that is $|O/\mathfrak{p}| \equiv_m 1$.

Let $\mathfrak{p} \in P$ and set $q = |O/\mathfrak{p}|$. By property (1), the group $\mathbf{G}(O/\mathfrak{p})$ has Lie type $\mathcal{L} = \mathcal{L}^{\mathrm{sp}} = (\Phi, \mathrm{Id})$. Property (2) together with parts (3) and (2) of Theorem 3.1 yields

$$\zeta_{\mathbf{G}(O/\mathfrak{p})} - |\mathbf{G}(O/\mathfrak{p})/[\mathbf{G}(O/\mathfrak{p}), \mathbf{G}(O/\mathfrak{p})]| \sim_C \xi_{a(\Phi),q}.$$

□

Furthermore, an argument similar to the one used in the proof of Theorem 1.5, now based on Theorem 3.1 and Corollary 3.15 instead of Theorem 2.8, gives the following.

Corollary 3.16. *Let $K \subset L$ be number fields with rings of integers $O_K \subset O_L$, and let $\mathbf{G}$ be an affine group scheme defined over O_K whose generic fiber is connected and simply connected semi-simple. Then $\alpha(\prod_{\mathfrak{p} \in \mathrm{Spec}(O_K)} \mathbf{G}(O_K/\mathfrak{p})) = \alpha(\prod_{\mathfrak{q} \in \mathrm{Spec}(O_L)} \mathbf{G}(O_L/\mathfrak{q}))$.*

We remark that one can use Deligne-Lusztig theory to pin down the precise value of $\alpha(\prod_{\mathfrak{p}} \mathbf{G}(O/\mathfrak{p}))$ as in Corollary 3.16. For instance, if $\mathbf{G}$ is simple of type A_ℓ, then the abscissa of convergence for the product $\prod_{\mathfrak{p}} \mathbf{G}(O/\mathfrak{p})$ of finite groups of Lie type is equal to $2/\ell$ and, in particular, tends to 0 as $\ell \to \infty$. This behavior stands in contrast to the fact the abscissa of convergence $\alpha(\mathbf{G}(O))$ is known to be bounded away from 0; cf. [28, Theorem 8.1]. This underlines that the investigation of $\alpha(\Gamma)$ for arithmetic groups $\Gamma = \mathbf{G}(O_S)$ requires a more careful analysis.

4. Relative Zeta Functions, Kirillov Orbit Method and Model Theoretic Background

4.1. **Relative Zeta Functions and Cohomology.** Throughout this section, let G be a group such that $R_n(G)$ is finite for all $n \in \mathbb{N}$. Let $N \subset G$ be a normal subgroup, and let ϑ be an irreducible finite-dimensional complex representation of N. Recall from Definition 2.10 that $\mathrm{Irr}(G|\vartheta)$ denotes the set of (equivalence classes of) finite-dimensional irreducible complex representations ϱ of G such that ϑ is a constituent of $\mathrm{Res}^G_N \varrho$, the notion of the relative zeta function $\zeta_{G|\vartheta}(s) = \sum_{\varrho \in \mathrm{Irr}(G|\vartheta)} \left(\frac{\dim \varrho}{\dim \vartheta}\right)^{-s}$, and the notation $R_n(G|\vartheta)$ for the number of representations $\varrho \in \mathrm{Irr}(G|\vartheta)$ such that $\dim \varrho \leq n \dim \vartheta$.

Suppose further that, up to equivalence, ϑ is G-invariant. It is typically not true that the relative zeta function $\zeta_{G|\vartheta}$ is equal to the zeta function of the quotient G/N; for instance, if G is non-abelian, step-2 nilpotent, $N = [G, G]$ and ϑ is non-trivial, then $\zeta_{G|\vartheta}$ is not equal to $\zeta_{G/N}$. We describe now a situation in which the two zeta functions are equal. Recall that ϑ defines an element in the second cohomology group $H^2(G/N, \mathbb{C}^\times)$ of G/N with values in $\mathbb{C}^\times$, also known as the Schur multiplier of G/N. The construction is as follows; see [22, Chapter 11]. Suppose $\vartheta\colon N \to \mathrm{GL}_d(\mathbb{C})$. Pick a coset representative $\widetilde{a} \in G$ for every element a of G/N such that $\widetilde{1} = 1$. For every $a \in G/N$, the representations ϑ and $\vartheta^{\widetilde{a}}$ are equivalent, and we choose $T_a \in \mathrm{GL}_d(\mathbb{C})$ such that $T_a \vartheta T_a^{-1} = \vartheta^{\widetilde{a}}$; for T_1

we choose the identity. Then one checks that, for all $a_1, a_2 \in G/N$, the transformation $T_{a_1a_2}^{-1} T_{a_1} T_{a_2} \vartheta(\widetilde{a_1}\widetilde{a_2}(\widetilde{a_1a_2})^{-1})$ commutes with each element of $\vartheta(N)$ and thus defines a scalar $\beta(a_1, a_2) \in \mathbb{C}^\times$. The map $\beta\colon G/N \times G/N \to \mathbb{C}^\times$ is a 2-cocycle representing the cohomology class associated to (G, N, ϑ), which is independent of the choices involved.

Lemma 4.1. *Let G be a profinite group with an open normal subgroup $N \triangleleft G$, and let $\vartheta \in \mathrm{Irr}(N)$ be a complex representation of N which is G-invariant up to equivalence. If the cohomology class in $H^2(G/N, \mathbb{C}^\times)$ associated to (G, N, ϑ) vanishes, then $\zeta_{G|\vartheta} = \zeta_{G/N}$.*

Proof. By [22, Theorem 11.7], the vanishing of the cohomology class implies that ϑ can be extended to a representation $\widetilde{\vartheta}$ of G. By [22, Theorem 6.16], the map $\mathrm{Irr}(G/N) \to \mathrm{Irr}(G|\vartheta)$ given by $\tau \mapsto \tau \otimes \widetilde{\vartheta}$ is a bijection, and the claim of the lemma follows. □

Lemma 4.2. *Let p be a prime number. Let G be a profinite group with an open normal pro-p subgroup $N \triangleleft G$, and let $\vartheta \in \mathrm{Irr}(N)$ be G-invariant up to equivalence. Then the cohomology class in $H^2(G/N, \mathbb{C}^\times)$ associated to (G, N, ϑ) has order a power of p.*

Proof. The dimension of ϑ is a power of p and, for every $h \in N$, the scalar $\det(\vartheta(h))$ is a p^nth root of unity, for some n. For all $a_1, a_2 \in G/N$, taking determinants in the definition of the cocycle β, we get $\beta(a_1, a_2)^{\dim \vartheta} = \det(T_{a_1a_2}^{-1} T_{a_1} T_{a_2} \vartheta(\widetilde{a_1}\widetilde{a_2}(\widetilde{a_1a_2})^{-1}))$. Since we are free to arrange $\det(T_x) = 1$ for $x \in G/N$, we get that $\beta(a, b)$ is a root of unity of order a power of p. □

Lemma 4.3. *Let ℓ be a prime number. Suppose that G is a finite group and that $N \subset G$ is a central subgroup such that* $\gcd(|N|, \ell) = 1$. *Then* $\gcd(|H^2(G, \mathbb{C}^\times)|, \ell) = 1$ *if and only if* $\gcd(|H^2(G/N, \mathbb{C}^\times)|, \ell) = 1$.

Proof. Let $(E_n^{p,q})$ be the Lyndon–Hochschild–Serre spectral sequence associated to the central extension $1 \to N \to G \to G/N \to 1$. Since the order of N is prime to ℓ, so are the orders of $H^1(N, \mathbb{C}^\times)$ and $H^2(N, \mathbb{C}^\times)$. Therefore, the orders of

$$E_2^{0,2} = H^0(G/N, H^2(N, \mathbb{C}^\times)) = H^2(N, \mathbb{C}^\times)$$

and

$$E_2^{1,1} = H^1(G/N, H^1(N, \mathbb{C}^\times)) = \mathrm{Hom}(G/N, H^1(N, \mathbb{C}^\times))$$

are prime to ℓ and, hence, so are the orders of $E_\infty^{0,2}$ and $E_\infty^{1,1}$. The fact that $E_\infty^{p,q}$ converges to $H^*(G, \mathbb{C}^\times)$ implies that $|H^2(G, \mathbb{C}^\times)| = |E_\infty^{0,2}|\,|E_\infty^{1,1}|\,|E_\infty^{2,0}|$, so $|H^2(G, \mathbb{C}^\times)|$ is prime to ℓ if and only if $|E_\infty^{2,0}|$ is prime to ℓ.

Since the order of $E_2^{0,1} = H^0(G/N, H^1(N, \mathbb{C}^\times)) = H^1(N, \mathbb{C}^\times)$ is prime to ℓ, we get that the order of $E_\infty^{2,0} = E_3^{2,0}$ is prime to ℓ if and only if the order of $E_2^{2,0} = H^2(G/N, \mathbb{C}^\times)$ is prime to ℓ, yielding the result. □

Lemma 4.4. *For every root system Φ there is a constant $C \in \mathbb{R}$ such that, for every finite field $\mathbb{F}_q$ of characteristic greater than C and for every connected reductive $\mathbb{F}_q$-algebraic group $\mathbf{G}$ with absolute root system Φ, the size of $H^2(\mathbf{G}(\mathbb{F}_q), \mathbb{C}^\times)$ is prime to q.*

Proof. Let $\mathbb{F}_q$ be a finite field of characteristic p and let $\mathbf{G}$ be a connected reductive $\mathbb{F}_q$-algebraic group with absolute root system Φ.

Assume first that $\mathbf{G}$ is semi-simple. Then there are almost simple groups $\mathbf{G}_1, \dots, \mathbf{G}_n$ such that $\mathbf{G}$ is a quotient of $\mathbf{G}_1 \times \cdots \times \mathbf{G}_n$ by a central subgroup $\mathbf{Z}$, and both n and the ranks of the groups $\mathbf{G}_i$ are bounded in terms of Φ. In particular, the size of $\mathbf{Z}$ is bounded in terms of Φ. From the exact sequence

$$0 \to \mathbf{Z}(\mathbb{F}_q) \to \prod_{i=1}^{n} \mathbf{G}_i(\mathbb{F}_q) \to \mathbf{G}(\mathbb{F}_q) \to H^1(\mathrm{Gal}_{\mathbb{F}_q}, \mathbf{Z})$$

we conclude that both the kernel and the cokernel of the map $\prod_{i=1}^n \mathbf{G}_i(\mathbb{F}_q) \to \mathbf{G}(\mathbb{F}_q)$ have sizes bounded in terms of Φ.

It is known that the sizes of $H^1(\mathbf{G}_i(\mathbb{F}_q), \mathbb{C}^\times)$ and $H^2(\mathbf{G}_i(\mathbb{F}_q), \mathbb{C}^\times)$ are bounded in terms of Φ; see, for example, [11, Table 5]. By the Künneth formula, the sizes of $H^1(\prod_{i=1}^n \mathbf{G}_i(\mathbb{F}_q), \mathbb{C}^\times)$ and $H^2(\prod_{i=1}^n \mathbf{G}_i(\mathbb{F}_q), \mathbb{C}^\times)$ are bounded by some constant $C_1 \in \mathbb{R}$. In particular, if p is greater than C_1, then the size of $H^2(\prod_{i=1}^n \mathbf{G}_i(\mathbb{F}_q), \mathbb{C}^\times)$ is prime to q. By Lemma 4.3, the same is true for the size of $H^2(\mathbf{G}(\mathbb{F}_q), \mathbb{C}^\times)$ if p is larger than the size of the kernel and cokernel of the quotient map $\prod_{i=1}^n \mathbf{G}_i(\mathbb{F}_q) \to \mathbf{G}(\mathbb{F}_q)$.

Now assume that $\mathbf{G}$ is merely reductive. Let $\mathbf{S} = [\mathbf{G}, \mathbf{G}]$ be the derived subgroup of $\mathbf{G}$ and let $\mathbf{T} = Z(\mathbf{G})^\circ$. Then $\mathbf{T}$ is a torus, $\mathbf{S}$ is semi-simple, and $\mathbf{G}(\mathbb{F}_q)$ is a quotient of $\mathbf{T}(\mathbb{F}_q) \times \mathbf{S}(\mathbb{F}_q)$ by a central subgroup, whose size is bounded in terms of Φ. As shown above, if p is sufficiently large, then the size of $H^2(\mathbf{S}(\mathbb{F}_q), \mathbb{C}^\times)$ is prime to q; a similar claim for $H^1(\mathbf{S}(\mathbb{F}_q), \mathbb{C}^\times)$ also holds. Since the size of $\mathbf{T}(\mathbb{F}_q)$ is prime to q, so are the orders of its first and second cohomology groups. By the Künneth formula, the size of $H^2\left(\mathbf{T}(\mathbb{F}_q) \times \mathbf{S}(\mathbb{F}_q), \mathbb{C}^\times\right)$ is prime to q. By Lemma 4.3, if we assume, in addition, that p is larger than the size of the kernel of $\mathbf{T}(\mathbb{F}_q) \times \mathbf{S}(\mathbb{F}_q) \to \mathbf{G}(\mathbb{F}_q)$, then the size of $H^2(\mathbf{G}(\mathbb{F}_q), \mathbb{C}^\times)$ is prime to q. □

4.2. **Kirillov Orbit Method.** All pro-p groups in this section are open subgroups of $\mathbf{G}(O_{L,\mathfrak{q}})$, where $\mathbf{G}$ is an affine group scheme over the ring of integers O_L of a number field L and $O_{L,\mathfrak{q}}$ is the completion of O_L at a prime $\mathfrak{q}$ lying above a rational prime p. We fix an embedding $\mathbf{G} \subset \mathrm{GL}_N$, for a suitable $N \in \mathbb{N}$, and denote by $\mathfrak{g} \subset \mathfrak{gl}_N$ the Lie algebra of $\mathbf{G}$.

Definition 4.5. Let $\mathfrak{q}$ be a prime of O_L. We say that a pro-p subgroup $H \subset \mathbf{G}(O_{L,\mathfrak{q}})$ is good if the following two conditions hold.

(1) The logarithm series $\log(X) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(X-1)^n}{n}$ converges on H, setting up an injective map $\log\colon H \to \log H \subset \mathfrak{g}(O_{L,\mathfrak{q}})$, and the exponential series $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ converges on $\log H$, yielding the inverse map $\exp\colon \log H \to H$.
(2) The image $\log H$ is closed under addition and the Lie bracket, thus forming a $\mathbb{Z}_p$-Lie lattice. It is also closed under the adjoint action of H. For all $A, B \in \log H$, the

Hausdorff formula (e.g., see [23, Chapter V]) holds:

$$\log(\exp(A)\cdot\exp(B)) = \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sum_{r_i+s_i>0}\frac{\left(\sum_{i=1}^{m}(r_i+s_i)\right)^{-1}}{r_1!\cdot s_1!\cdot\ldots\cdot r_m!\cdot s_m!}R_{r_1,s_1,\ldots,r_m,s_m}(A,B),$$

where the Lie polynomials $R_{r_1,s_1,\ldots,r_m,s_m}(A,B)$ are defined by

$$R_{r_1,s_1,\ldots,r_m,s_m}(A,B) = \begin{cases} \mathrm{ad}(A)^{r_1}\,\mathrm{ad}(B)^{s_1}\cdots\mathrm{ad}(A)^{r_m}(B) & \text{if } s_m=1,\\ \mathrm{ad}(A)^{r_1}\,\mathrm{ad}(B)^{s_1}\cdots\mathrm{ad}(B)^{r_{m-1}}(A) & \text{if } r_m=1,\ s_m=0,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 4.6. *Let $\mathbf{G}\subset\mathrm{GL}_N$ be as above and let $\mathfrak{q}$ be a prime of O_L extending a rational prime $p>[L:\mathbb{Q}]N^2$. Then every pro-p subgroup $H\subset\mathbf{G}(O_{L,\mathfrak{q}})$ is good.*

Proof. Pro-p groups which are saturable in the sense of Lazard – for recent characterizations see [26, 17] – are good in the sense of Definition 4.5. The assertion thus follows from [26, Corollary 1.5] which implies that every pro-p subgroup $H\subset\mathrm{GL}_N(O_{L,\mathfrak{q}})$ is saturable. □

A good pro-p group H acts on $\mathfrak{h}=\log H$ via the adjoint action $\mathrm{Ad}\colon H\to\mathrm{Aut}(\mathfrak{h})$. We denote by $\mathrm{Ad}(h)(A)$ the image of $A\in\mathfrak{h}$ under the adjoint action of $h\in H$. The adjoint action induces the co-adjoint action of H on the Pontryagin dual $\mathfrak{h}^\vee=\mathrm{Hom}_{\mathrm{cont}}(\mathfrak{h},\mathbb{C}^\times)$, consisting of all continuous homomorphisms from the abelian pro-p group $\mathfrak{h}$ to $\mathbb{C}^\times$. Concretely, for $h\in H$, $A\in\mathfrak{h}$, and $\vartheta\colon\mathfrak{h}\to\mathbb{C}^\times$, one defines

$$\left(\mathrm{Ad}^*(h)(\vartheta)\right)(A)=\vartheta\left(\mathrm{Ad}(h^{-1})(A)\right).$$

Equivalence classes of irreducible representations of H are parametrized by the corresponding characters. Accordingly, we use the notation $\mathrm{Irr}(H)$ in a flexible way to denote also the set of irreducible complex characters of H. The Kirillov orbit method for p-adic analytic pro-p groups yields the following description of the irreducible characters of H.

Proposition 4.7. *Let $\mathbf{G}\subset\mathrm{GL}_N$ be as above. For almost all primes $\mathfrak{q}$ of O_L, every pro-p subgroup $H\subset\mathbf{G}(O_{L,\mathfrak{q}})$ is good and, setting $\mathfrak{h}=\log H$, the following hold.*

(1) *There is a function $\Omega\colon\mathfrak{h}^\vee\to\mathrm{Irr}(H)$ which is constant on co-adjoint orbits and induces a bijection between the set of co-adjoint orbits in $\mathfrak{h}^\vee$ and the set of irreducible characters of H.*
(2) *For $\vartheta\in\mathfrak{h}^\vee$, the character $\Omega(\vartheta)$ is given by*

$$\Omega(\vartheta)(h)=\frac{1}{|\mathrm{Ad}^*(H)(\vartheta)|^{1/2}}\sum_{\varphi\in\mathrm{Ad}^*(H)(\vartheta)}\varphi(\log(h))\qquad(h\in H).$$

In particular, the degree of $\Omega(\vartheta)$ is $\dim\Omega(\vartheta)=|\mathrm{Ad}^*(H)(\vartheta)|^{1/2}$.
(3) *Every $g\in\mathbf{G}(O_{L,\mathfrak{q}})$ that normalizes the subgroup H also normalizes the Lie lattice $\mathfrak{h}$ and $\Omega(\vartheta)^g=\Omega(\mathrm{Ad}^*(g^{-1})(\vartheta))$ for $\vartheta\in\mathfrak{h}^\vee$.*

Proof. Let S be the finite set of primes $\mathfrak{q}$ of O_L that extend a rational prime p with $p \le [L : \mathbb{Q}]N^2$. The assumptions then imply that $H \subset \mathbf{G}(O_{L,\mathfrak{q}})$ is saturable for $\mathfrak{q} \notin S$ (see [17, Theorem A]), in particular good in the sense of Definition 4.5. As saturable pro-p groups of dimension at most p are potent, the assertions follow from [16, Theorem 5.2]. ☐

We refer to the map Ω in Proposition 4.7 as the orbit method map.

Lemma 4.8. *Let $\mathbf{G} \subset \mathrm{GL}_N$ be as above, and let G be an open subgroup of $\mathbf{G}(O_{L,\mathfrak{q}})$ for some prime $\mathfrak{q}$ of O_L, with open normal subgroups $K \subset H \subset G$. Suppose that H and K are good pro-p groups, with Lie lattices $\mathfrak{h} = \log H$ and $\mathfrak{k} = \log K$, and that the irreducible characters of H and K are described by the orbit method as in Proposition 4.7. Then*

$$\zeta_G(s) = \sum_{\vartheta \in \mathfrak{h}^\vee} \frac{1}{|\mathrm{Ad}^*(G)(\vartheta)|} (\dim \Omega(\vartheta))^{-s} \zeta_{G|\Omega(\vartheta)}(s).$$

Furthermore, if $\tau \in \mathfrak{k}^\vee$, then

$$\zeta_{G|\Omega(\tau)}(s) = \sum_{\vartheta \in \mathfrak{h}^\vee,\ \vartheta|_{\mathfrak{k}} = \tau} \frac{|\mathrm{Ad}^*(H)(\tau)|}{|\mathrm{Ad}^*(G)(\vartheta)|} \left(\frac{\dim \Omega(\vartheta)}{\dim \Omega(\tau)} \right)^{-s} \zeta_{G|\Omega(\vartheta)}(s).$$

Proof. For each $\chi \in \mathrm{Irr}(H)$, the set $\mathrm{Irr}(G|\chi)$ depends only on the G-orbit χ^G. Moreover, the sets $\mathrm{Irr}(G|\chi)$, indexed by the orbits χ^G of irreducible characters of H, form a partition of $\mathrm{Irr}(G)$. Choosing representatives χ_i, $i \in I$, for the G-orbits, we get

$$\zeta_G(s) = \sum_{i \in I} (\dim \chi_i)^{-s} \zeta_{G|\chi_i}(s) = \sum_{\chi \in \mathrm{Irr}(H)} |\chi^G|^{-1} (\dim \chi)^{-s} \zeta_{G|\chi}(s).$$

Consider the orbit method map $\Omega \colon \mathfrak{h}^\vee \to \mathrm{Irr}(H)$. Its fibers are the H-coadjoint orbits. Moreover, Ω is G-equivariant, so the pre-images of the G-orbits in $\mathrm{Irr}(H)$ are the G-orbits in $\mathfrak{h}^\vee$. Hence,

$$\begin{aligned} \zeta_G(s) &= \sum_{\chi \in \mathrm{Irr}(H)} |\chi^G|^{-1} (\dim \chi)^{-s} \zeta_{G|\chi}(s) \\ &= \sum_{\vartheta \in \mathfrak{h}^\vee} \frac{1}{|\mathrm{Ad}^*(H)(\vartheta)| \cdot |\Omega(\vartheta)^G|} (\dim \Omega(\vartheta))^{-s} \zeta_{G|\Omega(\vartheta)}(s) \\ &= \sum_{\vartheta \in \mathfrak{h}^\vee} \frac{1}{|\mathrm{Ad}^*(G)(\vartheta)|} (\dim \Omega(\vartheta))^{-s} \zeta_{G|\Omega(\vartheta)}(s). \end{aligned}$$

The proof of the second statement is similar, using the following consequence of Proposition 4.7: for $\tau \in \mathfrak{k}^\vee$ and $\vartheta \in \mathfrak{h}^\vee$, the character $\Omega(\tau)$ is a constituent of the restriction of $\Omega(\vartheta)$ to K if and only if $\vartheta|_{\mathfrak{k}} = \tau^h$ for a suitable $h \in H$. ☐

4.3. **Quantifier-Free Definable Sets and Functions.** We use several notions from model theory, which we summarize below. For more details we refer to [7]. Fix a first-order language and a theory $\mathfrak{T}$, that is a consistent set of sentences, in that language. Let $(\mathrm{Models}_{\mathfrak{T}})$ be the category whose objects are models of $\mathfrak{T}$ and whose morphisms are elementary embeddings. Let $x = (x_1, \ldots, x_n)$ denote an n-tuple of variables. A formula $\varphi(x)$ gives rise to a functor $\mathcal{X} = \mathcal{X}_\varphi \colon (\mathrm{Models}_{\mathfrak{T}}) \to (\mathrm{Sets})$ that sends a model M of $\mathfrak{T}$ to the set

$$\mathcal{X}(M) = \{(a_1, \ldots, a_n) \in M^n \mid \text{the sentence } \varphi(a_1, \ldots, a_n) \text{ holds in } M\}.$$

We call such a functor $\mathcal{X}$ a *definable functor* or a *definable set* in $\mathfrak{T}$, and denote $\mathcal{X}_\varphi$ also by $\{x \mid \varphi(x)\}$. By Gödel's completeness theorem, the functors associated to formulae φ and ψ are equal if and only if the sentence $\forall x\,(\varphi(x) \leftrightarrow \psi(x))$ can be proved from $\mathfrak{T}$. We say that a definable set is *quantifier-free definable* if it is associated to some quantifier-free formula. For a definable set $\mathcal{X}$ arising from a formula φ, the notation $a \in \mathcal{X}$ is employed in two different contexts: if $a \in M^n$ for some model M of $\mathfrak{T}$ it means $\varphi(a)$ holds in M, whereas if a is a tuple of terms in the underlying language it simply stands for $\varphi(a)$.

The usual pointwise operators $\cap$, $\cup$, and $\times$ on functors to sets take definable sets to definable sets. If $\mathcal{X}$ and $\mathcal{Y}$ are definable sets, we write $\mathcal{X} \subset \mathcal{Y}$ if $\mathcal{X}(M) \subset \mathcal{Y}(M)$ for all models M of $\mathfrak{T}$. If $\mathcal{X}$ and $\mathcal{Y}$ are associated to the formulae φ and ψ, then $\mathcal{X} \subset \mathcal{Y}$ if and only if the sentence $\forall x\,(\varphi(x) \to \psi(x))$ can be proved from $\mathfrak{T}$.

Example 4.9. We consider the first-order language of rings and the theory $\mathfrak{T}_{\text{fields}}$ of fields.

(1) For every $n \in \mathbb{N}$, the formula $0 = 0$ (in variables $x_1, \ldots, x_n$) yields a definable set $\mathcal{X}$ with $\mathcal{X}(F) = F^n$ for every field F. The associated definable set is called *n-dimensional affine space*, and denoted by $\mathbb{A}^n$.
(2) More generally, every affine scheme $\mathbf{X}$ over $\mathbb{Z}$ can be considered as a quantifier-free definable set. This means that there is a quantifier-free definable set $\mathcal{X}$ such that $\mathcal{X}(F) = \mathbf{X}(F)$ for every field F.
(3) The definable set $\mathcal{Y}$ arising from the formula $\exists y\,(y^2 = x)$ is not quantifier-free. Indeed, every quantifier-free definable set in the theory of fields is a Boolean combination of affine varieties. It follows that if $\mathcal{X} \subset \mathbb{A}^1$ is quantifier-free, then there is a constant $C \in \mathbb{R}$ such that $|\mathcal{X}(\mathbb{F}_p)|$ or $|\mathbb{F}_p \smallsetminus \mathcal{X}(\mathbb{F}_p)|$ is bounded by C, uniformly for all primes p. However, $|\mathcal{Y}(\mathbb{F}_p)| = (p+1)/2$ if $p > 2$.

The dimension of a definable set $\mathcal{X} \subset \mathbb{A}^n$ is the dimension of its Zariski closure in $\mathbb{A}^n$; cf. [10, Section 3]. Let $\mathcal{X}$ and $\mathcal{Y}$ be definable sets in a theory $\mathfrak{T}$. A natural transformation $f \colon \mathcal{X} \to \mathcal{Y}$ is called a *definable function* if the functor sending $M \in (\mathrm{Models}_{\mathfrak{T}})$ to the graph of the map $f(M)$ is definable. This means that the graph of f, considered as a functor, is definable. On some occasions, if $f \colon \mathcal{X} \to \mathcal{Y}$ is a definable function, we say that $\mathcal{X}$ is a *definable family* of (definable) sets with base $\mathcal{Y}$. For $y \in \mathcal{Y}$, we denote the fiber $f^{-1}(y)$ by $\mathcal{X}_y$. It is a definable set in the enriched language obtained by adding the coordinates of y as constants.

Throughout the remainder of the section, let R denote a commutative unital ring.

Definition 4.10. Consider the language of rings enriched by constant symbols c_a for all $a \in R$. Let $\mathfrak{T}_{\mathrm{fields},R}$ be the theory consisting of the axioms of fields and the statements $c_a \cdot c_b = c_{ab}$, $c_a + c_b = c_{a+b}$ for all $a, b \in R$.

A model for $\mathfrak{T}_{\mathrm{fields},R}$ is a field F together with a ring homomorphism $R \to F$, which we routinely omit from the notation. As in Example 4.9, every affine scheme over R gives rise to a quantifier-free definable set in $\mathfrak{T}_{\mathrm{fields},R}$.

We explain now how projective spaces arise as definable sets in the theory $\mathfrak{T}_{\mathrm{fields},R}$. Let $\mathfrak{g}$ be a free R-module of finite rank n with an R-basis $e_1, \ldots, e_n$. We view $\mathfrak{g}$ as a definable set in $\mathfrak{T}_{\mathrm{fields},R}$ via $\mathfrak{g}(F) = \mathfrak{g} \otimes_R F$ for $F \in (\mathrm{Models}_{\mathfrak{T}_{\mathrm{fields},R}})$. The chosen basis allows us to identify $\mathfrak{g}$ with $\mathbb{A}^n$. The definable set

$$\mathcal{P} = \{(1)\} \times \mathbb{A}^{n-1} \sqcup \{(0,1)\} \times \mathbb{A}^{n-2} \sqcup \ldots \sqcup \{(0,0,\ldots,1)\} \times \mathbb{A}^0 \subset \mathbb{A}^n$$

plays the role of *projective space* over $\mathfrak{g}$ in the category of definable sets, in the following sense. Consider the definable family $\mathcal{V} \subset \mathcal{P} \times \mathfrak{g}$ given by the condition that $((a_1, \ldots, a_n), (v_1, \ldots, v_n)) \in \mathcal{P} \times \mathfrak{g}$ is in $\mathcal{V}$ if and only if all minors $\begin{vmatrix} a_i & a_j \\ v_i & v_j \end{vmatrix}$, $1 \leq i < j \leq n$, vanish. For every definable set $\mathcal{S}$ and every definable set $\mathcal{X} \subset \mathcal{S} \times \mathfrak{g}$ with the property that, for every $s \in \mathcal{S}$, the fiber $\mathcal{X}_s$ is a line in $\mathfrak{g}$ – that is, a one-dimensional linear subfunctor of $\mathfrak{g}$ – there is a unique definable map $f \colon \mathcal{S} \to \mathcal{P}$ such that the pull-back $f^*\mathcal{V} \subset \mathcal{S} \times \mathfrak{g}$ of $\mathcal{V}$ via f is equal to $\mathcal{X}$. Choosing a different (definable) basis for $\mathfrak{g}$, we obtain a different universal family over $\mathcal{P}$, but there is a quantifier-free definable map from $\mathcal{P}$ to itself that interchanges the two universal families.

More generally, for $0 \leq d \leq n$, there is a quantifier-free definable set that functions as the Grassmannian of d-dimensional subspaces in $\mathfrak{g} \otimes_R F$ for every model F of $\mathfrak{T}_{\mathrm{fields},R}$. The union of these, over all dimensions d, is the *Grassmannian* $\mathrm{Gr}(\mathfrak{g})$ of $\mathfrak{g}$.

Definition 4.11. Let $\mathfrak{g} \subset \mathfrak{gl}_N(R)$ be a free R-module which is closed under Lie brackets. In the theory $\mathfrak{T}_{\mathrm{fields},R}$, let $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})$ be the subfunctor of $\mathrm{Gr}(\mathfrak{g})$ given by

$$F \mapsto \{\text{Lie subalgebras of } \mathfrak{g} \otimes_R F\},$$

and let $\mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g})$ be the subfunctor of $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})$ given by

$$F \mapsto \{\text{Lie subalgebras of } \mathfrak{g} \otimes_R F \text{ consisting of nilpotent matrices}\}.$$

Proposition 4.12. *Let R be a commutative unital ring, and let $\mathfrak{g} \subset \mathfrak{gl}_N(R)$ be a free R-module which is closed under Lie brackets. View $\mathfrak{g}$ as a definable set in $\mathfrak{T}_{\mathrm{fields},R}$ via $\mathfrak{g}(F) = \mathfrak{g} \otimes_R F$ for $F \in (\mathrm{Models}_{\mathfrak{T}_{\mathrm{fields},R}})$. Then the following hold.*

(1) $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})$ *and* $\mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g})$ *are quantifier-free definable subfunctors of* $\mathrm{Gr}(\mathfrak{g})$.

(2) *The natural transformation* $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}) \to \mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g})$ *induced by taking a Lie algebra $h \subset \mathfrak{g}(F)$ to the subalgebra of nilpotent matrices in the solvable radical* $\mathrm{Rad}(h)$ *of h is quantifier-free definable.*

(3) *The natural transformation* $\mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g}) \to \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})$ *induced by taking a Lie algebra* $h \subset \mathfrak{g}(F)$ *consisting of nilpotent matrices to its normalizer* $N_{\mathfrak{g}(F)}(h)$ *in* $\mathfrak{g}(F)$ *is quantifier-free definable.*
(4) *There are a constant* $p_0 \in \mathbb{N}$ *and a quantifier-free definable function from* $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})$ *to the set of all root systems of rank at most* N *such that the following is true: for every Lie algebra* $h \subset \mathfrak{g}(F)$ *with* F *satisfying* $\mathrm{char}(F) = 0$ *or* $\mathrm{char}(F) \geq p_0$*, the Lie algebra* $h/\mathrm{Rad}(h)$ *is semi-simple of classical type and the value of the function at* h *is the absolute root system of* $h/\mathrm{Rad}(h)$*.*
(5) *Suppose that* $\mathfrak{g}$ *is the Lie algebra of an affine group scheme* $\mathbf{G}$ *over* R*. Then there is a quantifier-free definable subset of* $\mathbf{G} \times \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})$ *whose fiber over a Lie algebra* $h \subset \mathfrak{g}(F)$ *is the normalizer* $N_{\mathbf{G}(F)}(h)$ *of* h *in* $\mathbf{G}(F)$*.*

We use the following well-known characterization of quantifier-free definable sets; e.g., see [25, Theorem 8.11].

Lemma 4.13. *Let* R *be a commutative unital ring, and let* $\mathcal{X} \subset \mathbb{A}^n$ *be a definable set in the theory* $\mathfrak{T}_{\mathrm{fields},R}$*. Then the following are equivalent.*
(1) *For every two models* $F \subset E$ *of* $\mathfrak{T}_{\mathrm{fields},R}$*, we have*
$$\mathcal{X}(F) = \mathcal{X}(E) \cap F^n.$$
(2) *For every model* F *of* $\mathfrak{T}_{\mathrm{fields},R}$*, we have*
$$\mathcal{X}(F) = \mathcal{X}(F^{\mathrm{alg}}) \cap F^n.$$
(3) $\mathcal{X}$ *is quantifier-free definable.*

Proof. The implications (1) ⇒ (2) and (3) ⇒ (1) are clear. We prove (2) ⇒ (3). By elimination of quantifiers over algebraically closed fields with coefficients in R, there is a quantifier-free definable set $\mathcal{Y}$ such that $\mathcal{X}(E) = \mathcal{Y}(E)$ for every algebraically closed model E of $\mathfrak{T}_{\mathrm{fields},R}$. Let F be a model of $\mathfrak{T}_{\mathrm{fields},R}$. By (2) and the implication (3) ⇒ (1), applied to $\mathcal{Y}$, we obtain $\mathcal{X}(F) = \mathcal{X}(F^{\mathrm{alg}}) \cap F^n = \mathcal{Y}(F^{\mathrm{alg}}) \cap F^n = \mathcal{Y}(F)$, so $\mathcal{X} = \mathcal{Y}$. □

Remark 4.14. We also use variants of Lemma 4.13 which characterize quantifier-free definable sets in (i) the theory $\mathfrak{T}_{\text{perf.-fields},p,R}$ of perfect fields of characteristic p together with a homomorphism from R and (ii) the theory $\mathfrak{T}_{\mathrm{Hen},K,0}$ of Henselian valued fields of residue characteristic 0 together with a homomorphism from field K; cf. Definition 4.17. The proof proceeds in the same way.

Proof of Proposition 4.12. We show in each case that the functor or the graph of the natural transformation in question is definable. In some cases, the formulae that we supply are polynomial, and hence quantifier-free. In the remaining cases, we explain how to apply Lemma 4.13 in order to obtain that the functor or graph in question is actually quantifier-free definable.

Put $n = \dim \mathfrak{g}$ and fix an R-basis $e_1, \ldots, e_n$ of $\mathfrak{g}$. There are a finite Zariski open affine cover $(\mathcal{U}_\iota)_{\iota \in I}$ of $\mathrm{Gr}(\mathfrak{g})$, non-negative integers $(d_\iota)_{\iota \in I}$, and regular functions $x_{\iota,1}, \ldots, x_{\iota,n} \colon \mathcal{U}_\iota \to \mathfrak{g}$, for $\iota \in I$, such that for each model F of $\mathfrak{T}_{\mathrm{fields},R}$,

- the dimension of every $h \in \mathcal{U}_\iota(F)$ is equal to d_ι,
- for every $h \in \mathcal{U}_\iota(F)$, the elements $x_{\iota,1}(h), \ldots, x_{\iota,d_\iota}(h)$ yield a linear basis for h and the elements $x_{\iota,1}(h), \ldots, x_{\iota,n}(h)$ yield a linear basis for $\mathfrak{g}(F)$.

In order to prove that a subfunctor $\mathcal{X} \subset \mathrm{Gr}(\mathfrak{g})$ is definable, it is enough to show that $\mathcal{X} \cap \mathcal{U}_\iota$ is definable for every $\iota \in I$. A similar claim holds for natural transformations. Fix therefore $\iota, \iota' \in I$ and write

$$\mathcal{U} = \mathcal{U}_\iota,\ \mathcal{U}' = \mathcal{U}_{\iota'}, \quad d = d_\iota,\ d' = d_{\iota'}, \quad \text{and} \quad x_i = x_{\iota,i},\ x'_i = x_{\iota',i} \text{ for } i \in \{1, \ldots, n\}.$$

(1) Consider the $\left(\binom{d+1}{2} \times n\right)$-matrix A whose first d rows record the coordinates of the functions $x_1, \ldots, x_d$ and whose last $\binom{d}{2}$ rows record the coordinates of the Lie brackets $[x_i, x_j]$, $1 \leq i < j \leq a$, all with respect to the basis $e_1, \ldots, e_n$. The functor $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}) \cap \mathcal{U}$ is definable by the polynomial condition $\mathrm{rk}\, A = d$.

The functor $\mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g}) \cap \mathcal{U}$ is definable by the conjunction of the previous condition and the polynomial condition that all products of the x_i of length N vanish, i.e., that $\prod_{j=1}^{N} x_{i_j} = 0$ for all $(i_1, \ldots, i_N) \in \{1, \ldots, n\}^N$.

(2) Let F be a model of $\mathfrak{T}_{\mathrm{fields},R}$ and $h \in (\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}) \cap \mathcal{U})(F)$. Recall that an element $X \in h$ is in the solvable radical $\mathrm{Rad}(h)$ of h if and only if the Lie ideal $[X, h]$ generated by X in h is solvable. As an F-vector space, $[X, h]$ is spanned by the set $\mathcal{S}$ consisting of all elements $[X, x_{i_1}(h), \ldots, x_{i_{d-1}}(h)]$, where $i_1, \ldots, i_{d-1} \in \{1, \ldots, d\}$. The Lie words w_i, $i \in \mathbb{N}$, defining the terms of the derived series are $w_1(z_1, z_2) = [z_1, z_2]$ and $w_i(z_1, \ldots, z_{2^i}) = [w_{i-1}(z_1, \ldots, z_{2^{i-1}}), w_{i-1}(z_{2^{i-1}+1}, \ldots, z_{2^i})]$ for $i \geq 2$. By linearity, $[X, h]$ is solvable if and only if $w_n(Z_1, \ldots, Z_{2^n}) = 0$ for all $Z_1, \ldots, Z_{2^n} \in \mathcal{S}$.

Using (1), we deduce that the functor

$$F \mapsto \{(h, X) \in (\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}) \cap \mathcal{U})(F) \times \mathfrak{g}(F) \mid X \in \mathrm{Rad}(h)\} \tag{4.1}$$

is quantifier-free definable. Clearly, we can express nilpotency of an element $X \in h$ by a polynomial formula. Consequently, also the functor

$$F \mapsto \{(h, X) \in (\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}) \cap \mathcal{U})(F) \times \mathfrak{g}(F) \mid X \in \mathrm{Rad}(h) \text{ is nilpotent}\}$$

is quantifier-free definable. Using quantifiers, we deduce that the functor

$$F \mapsto \left\{ \begin{array}{c} (h, k) \in (\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}) \cap \mathcal{U})(F) \times (\mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g}) \cap \mathcal{U}')(F) \mid \\ k \text{ is the collection of nilpotent elements in } \mathrm{Rad}(h) \end{array} \right\} \tag{4.2}$$

is definable and thus the graph of a natural transformation, namely the one we are interested in.

It remains to prove that the functor (4.2) is *quantifier-free* definable. By Lemma 4.13, it suffices to consider the following. Let $F \subset E$ be models of $\mathfrak{T}_{\mathrm{fields},R}$, and for subalgebras $h, k \subset \mathfrak{g}(F)$ write $h_E = h \otimes_F E$ and $k_E = k \otimes_F E$. We claim: if k_E consists of the nilpotent elements of $\mathrm{Rad}(h_E)$, then k consists of the nilpotent elements of $\mathrm{Rad}(h)$.

Clearly, $X \in h$ is nilpotent as an element of h_E if and only if it is nilpotent as an element of h. Moreover, because the functor (4.1) is quantifier-free definable, we have $\mathrm{Rad}(h_E) \cap \mathfrak{g}(F) = \mathrm{Rad}(h)$. This proves the claim.

(3) We first show that the functor $\mathcal{N}$ given by

$$F \mapsto \left\{(h,k) \in \mathrm{Gr}(\mathfrak{g})(F)^2 \mid \forall Y \in \mathfrak{g}(F)\colon \; (Y \in k \;\leftrightarrow\; (\forall X \in h\colon [X,Y] \in h))\right\}$$

is definable. As before, it is enough to prove this Zariski locally around fixed elements $h \in \mathcal{U}(F)$ and $k \in \mathcal{U}'(F)$, of dimensions d and d'. The intersection $\mathcal{N} \cap (\mathcal{U} \times \mathcal{U}')$ is given by the formula

$$\text{(4.3)} \quad \forall a_1, \dots, a_n \\ \left(\left(\bigwedge\nolimits_{i=1}^{d} \exists b_1, \dots, b_d \left(\left[x_i, \sum\nolimits_{j=1}^{n} a_j x'_j \right] = \sum\nolimits_{m=1}^{d} b_m x_m \right) \right) \longleftrightarrow \; a_{d'+1} = \ldots = a_n = 0 \right).$$

We now show that this formula is equivalent to a quantifier-free formula. To this end, consider first the $(d(d'+1) \times n)$-matrix A whose first d rows record the coordinates of $x_1, \dots, x_d$ and whose last dd' rows record the coordinates of the Lie brackets $[x_i, x'_j]$, where $1 \le i \le d$ and $1 \le j \le d'$, all with respect to the basis $e_1, \dots, e_n$. The polynomial condition $\mathrm{rk}(A) = d$ ensures that $\langle x'_1, \dots, x'_{d'} \rangle$ is contained in the Lie normalizer of $\langle x_1 \dots, x_d \rangle$. Consider now, for $1 \le i \le d$ and coordinates $a_{d'+1}, \dots, a_n$, the $((d+1) \times n)$-matrix $B_i(a_{d'+1}, \dots, a_n)$ whose first d rows record the coordinates of $x_1, \dots, x_d$ and whose last row records the coordinates of $[x_i, \sum_{j=d'+1}^{n} a_j x'_j]$. Formula (4.3) is equivalent to

$$(\mathrm{rk}(A) = d) \wedge \\ \forall a_{d'+1}, \dots, a_n \left(\left(\bigwedge\nolimits_{i=1}^{d} \mathrm{rk}(B_i(a_{d'+1}, \dots, a_n)) = d \right) \longleftrightarrow \; a_{d'+1} = \cdots = a_n = 0 \right).$$

We claim that this is equivalent to a polynomial condition in the coordinate functions $x_1, \dots, x_d$, $x'_1, \dots, x'_{d'}$. Indeed, for every $i \in \{1, \dots, d\}$, the condition $\mathrm{rk}(B_i(a_{d'+1}, \dots, a_n)) = d$ is linear in $a_{d'+1}, \dots, a_n$ and polynomial in $x_1, \dots, x_d, x'_1, \dots, x'_{d'}$. The condition that the resulting system of linear equations for $a_{d'+1}, \dots, a_n$ only has the trivial solution is polynomial in $x_1, \dots, x_d, x'_1, \dots, x'_{d'}$.

(4) The theory of modular Lie algebras, and in particular the classification of semi-simple Lie algebras, is rather more involved than in characteristic 0. However, there exists $p_0 \in \mathbb{N}$ such that the classification of semi-simple Lie algebras of dimension at most n over every algebraically closed field of characteristic at least p_0 is completely analogous to the well-known classification in characteristic 0: the Lie algebras are of classical type and parametrized by suitable root systems. One can deduce this, for instance, from Robinson's Principle, according to which a first order statement in the language of fields is true in algebraically closed fields of characteristic 0 if and only if it is true in algebraically closed fields of sufficiently large characteristic. We may thus restrict attention to fields F with $\mathrm{char}(F) = 0$ or $\mathrm{char}(F) \ge p_0$.

Since the absolute root system of a Lie algebra does not change under extension of scalars, it suffices, by Lemma 4.13, to produce a *definable* function with the required properties.

For every root system Φ, let $\mathfrak{s}_\Phi$ be the split Lie algebra, given in terms of a corresponding Chevalley basis, with root system Φ; see [23, Chapter IV]. For every subalgebra h of $\mathfrak{gl}_N(F)$, the algebra $h/\operatorname{Rad}(h)$ is semi-simple and has rank at most N. By adjoining the characteristic roots of basis elements of a Cartan subalgebra of h in the adjoint action, one obtains a splitting field E for h. Clearly, the degree of such a field E over F is bounded in terms of N. Therefore, it suffices to show that, for any given $m \in \mathbb{N}$, the functor

$$F \mapsto \left\{ \begin{array}{l} h \in \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})(F) \mid \\ \quad \exists \text{ field ext. } F \subset E \text{ of degree } m \text{ such that } (h/\operatorname{Rad}(h)) \otimes_F E \cong \mathfrak{s}_\Phi(E) \end{array} \right\}$$

is definable. Using Boolean combinations of formulae, it is enough to show that the functor

$$F \mapsto \left\{ \begin{array}{l} h \in \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})(F) \mid \\ \quad \exists \text{ field ext. } F \subset E \text{ of degree } m \text{ and an embedding } \mathfrak{s}_\Phi(E) \to h \otimes_F E \end{array} \right\} \tag{4.4}$$

is definable.

It is enough to check definability on an open neighborhood $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}) \cap \mathcal{U}$ of an element $h \in \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})(F)$. Field extensions $F \subset E$ of degree m can be modelled on the vector space F^m via a set of structure constants $\underline{c} = (c_{ij}^k)_{i,j,k=1}^m$ in F. The latter supply a binary operation

$$F^m \times F^m \to F^m, \quad (a_1, \ldots, a_m) * (b_1, \ldots, b_m) = \left(\sum\nolimits_{i,j=1}^m a_i b_j c_{ij}^k \right)_{k=1}^m.$$

The condition that the multiplication $*$, together with vector addition, defines a field extension of F is a first-order condition on $\underline{c}$. Furthermore, for every $\underline{c}$ giving rise to an extension $F \subset E$, an E-linear map $T \colon \mathfrak{s}_\Phi(E) \to h \otimes_F E$ can be described, locally, by a $(\dim \mathfrak{s}_\Phi \times d)$-matrix over F^m, with respect to the Chevalley basis of $\mathfrak{s}_\Phi$ and the basis $x_1(h), \ldots, x_d(h)$ of h. The condition that the map T is an embedding and preserves Lie brackets is polynomial in the entries of the corresponding $(\dim \mathfrak{s}_\Phi \times d \times m)$-array $\underline{t}$ over F. Thus the functor (4.4) is indeed definable.

(5) Let $\mathcal{X}$ be the functor

$$F \mapsto \{(g,h) \in \mathbf{G}(F) \times \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g})(F) \mid \operatorname{Ad}(g)h = h\}.$$

The functor $\mathcal{X} \cap (\mathbf{G} \times (\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}) \cap \mathcal{U}))$ is definable by the formula

$$(\forall a_1, \ldots, a_d)(\exists b_1, \ldots, b_d) \left(\operatorname{Ad}(g) \left(\sum\nolimits_{j=1}^d a_j x_j(h) \right) = \sum\nolimits_{k=1}^d b_k x_k(h) \right), \tag{4.5}$$

where the operation of $\operatorname{Ad}(g)$ is given by polynomial expressions involving the entries of the matrix g. To see that the functor is quantifier-free definable, consider the $(2d \times n)$-matrix $A = A(g,h)$, whose first d rows record the coordinates of $x_1, \ldots, x_d$ and whose last d rows record the coordinates of $\operatorname{Ad}(g)x_1, \ldots, \operatorname{Ad}(g)x_d$, all with respect to the basis $e_1, \ldots, e_n$. The polynomial condition $\operatorname{rk}(A(g,h)) = d$ is equivalent to (4.5). ∎

The following proposition can be found, for example, in [8, Théorème 6.4] or [9, Main Theorem].

Proposition 4.15. *Suppose that $\varphi(x,y)$ is a first-order formula in the language of rings, where $x=(x_1,\dots,x_m)$ and $y=(y_1,\dots,y_n)$. There is a constant $C\in\mathbb{R}$ such that, for every finite field $\mathbb{F}_q$ and every $a\in\mathbb{F}_q^n$, there is a natural number d such that the size of the set $\left\{x\in\mathbb{F}_q^m \mid \varphi(x,a) \text{ holds in } \mathbb{F}_q\right\}$ is either 0, or between $C^{-1}q^d$ and Cq^d.*

Lemma 4.16. *Let $\mathbf{G}$ be an affine algebraic group over a finite field $\mathbb{F}_q$ with at most C connected components, and let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$.*

1. *Writing $D_1 = C\,2^{\dim\mathbf{G}}$, the estimates $D_1^{-1}|\mathfrak{g}(\mathbb{F}_{q^n})| < |\mathbf{G}(\mathbb{F}_{q^n})| < D_1|\mathfrak{g}(\mathbb{F}_{q^n})|$ hold for every finite extension $\mathbb{F}_q\subset\mathbb{F}_{q^n}$.*
2. *Suppose that $\mathbf{G}$ acts on a variety $\mathbf{X}$ in such a way that the stabilizer $\mathbf{H}$ of a point $x\in\mathbf{X}(\mathbb{F}_{q^n})$ in $\mathbf{G}$ has less than C connected components. Writing $\mathfrak{h}$ for Lie algebra of $\mathbf{H}$ and $D_2 = C^2 2^{\dim\mathbf{G}}$, the following estimates hold:*
$$D_2^{-1}\frac{|\mathfrak{g}(\mathbb{F}_{q^n})|}{|\mathfrak{h}(\mathbb{F}_{q^n})|} \le |\mathbf{G}(\mathbb{F}_{q^n})x| \le D_2\frac{|\mathfrak{g}(\mathbb{F}_{q^n})|}{|\mathfrak{h}(\mathbb{F}_{q^n})|}.$$

Proof. We use the inequality
$$2^{-\dim\mathbf{G}}q^{n\dim\mathbf{G}} \le |\mathbf{G}^\circ(\mathbb{F}_{q^n})| \le 2^{\dim\mathbf{G}}q^{n\dim\mathbf{G}}$$
for the connected $\mathbb{F}_q$-algebraic group $\mathbf{G}^\circ$; cf. [35, Lemma 3.5]. Observing that $|\mathbf{G}^\circ(\mathbb{F}_{q^n})| \le |\mathbf{G}(\mathbb{F}_{q^n})| \le |\mathbf{G}/\mathbf{G}^\circ|\cdot|\mathbf{G}^\circ(\mathbb{F}_{q^n})|$ and $|\mathfrak{g}(\mathbb{F}_{q^n})| = q^{n\dim\mathbf{G}}$, we deduce (1). Claim (2) follows from (1), by applying the orbit stabilizer theorem. □

4.4. **Valued Fields.** We use the Denef–Pas language of valued fields; see, for example [10, Section 2]. It is a three-sorted, first-order language. The three sorts are the valued field sort F, the residue field sort k, and the value group sort Γ. The function symbols are

$+_{\mathrm{val}}, \times_{\mathrm{val}}$	from pairs of valued field sort variables to one valued field sort variable,
$+_{\mathrm{res}}, \times_{\mathrm{res}}$	from pairs of residue field sort variables to one residue field sort variable,
$+_{\mathrm{gr}}$	from pairs of value group sort variables to one value group sort variable,
val	from one valued field sort to one value group sort,
ac	from one valued field sort to one residue field sort.

In addition there is one binary relation symbol, $<$, between two value group sort variables.

For us, the important structures for the language of valued fields come from discrete valuation fields. Given a discrete valuation field F with a uniformizer ϖ, we interpret the valued field sort F as F, the residue field sort k as the residue field of F, and the value group sort Γ as the value group of F which we identify with $\mathbb{Z}$. The functions $+_{\mathrm{val}}, \times_{\mathrm{val}}, +_{\mathrm{res}}, \times_{\mathrm{res}}, +_{\mathrm{gr}}$, and the relation $<$ are interpreted as the usual operations and order relation. Finally, the function symbol val is interpreted as the valuation map, and the function symbol ac is interpreted as the angular component map
$$\mathrm{ac}(x) \equiv x\varpi^{-\mathrm{val}(x)} \pmod{\varpi} \qquad \text{for } x\in F\smallsetminus\{0\}.$$

The values of val(0) and ac(0) are chosen to be ∞ and 0.

We will only use theories for which F is a valued field with residue field k and value group Γ. Definable sets and functions are introduced similarly as for languages with only one variable sort. The definable set $\{x \in \mathsf{F} \mid \mathrm{val}(x) \geq 0\}$ is denoted by $\mathcal{O}$. We let $\mathrm{red} : \mathcal{O} \to \mathsf{k}$ denote the reduction modulo the maximal ideal, i.e. the definable map

$$\mathrm{red}(x) = \begin{cases} \mathrm{ac}(x), & \text{if } \mathrm{val}(x) = 0, \\ 0, & \text{if } \mathrm{val}(x) > 0. \end{cases}$$

When several valuation rings are involved, we sometimes use subscripts to distinguish between the various realizations of the reduction map. We use the function symbol $\mathrm{red}(\cdot)$ also to denote the componentwise reduction of a matrix or a tuple. In the latter case we also write $\mathrm{red}^{\times n}$ to highlight the n-arity. Likewise, we write $\mathrm{ac}^{\times n}(x)$ and $\mathrm{val}^{\times n}(x)$ when we apply the map ac or val coordinatewise to a tuple $x \in \mathsf{F}^n$.

For every discrete valuation field F, the set $\mathcal{O}(F)$ is the valuation ring O_F. Every O_F-scheme $\mathbf{X}$ gives rise to three definable sets in the Denef–Pas language augmented by constants from O_F. Indeed, let $x = (x_1, \ldots, x_n)$ denote an n-tuple of variables and suppose that $\mathbf{X}$ is given as the vanishing set of polynomials $f_1(x), \ldots, f_m(x) \in O_F[x]$. The first definable set is the set $\mathbf{X}_{\mathsf{F}}$ of all zeros of $f_i(x)$, $1 \leq i \leq m$, in F^n. The second is the set $\mathbf{X}_{\mathsf{k}}$ of zeros of the reductions of $f_i(x)$, $1 \leq i \leq m$, in k^n. The third is $\mathbf{X}_{\mathcal{O}} = \mathbf{X}_{\mathsf{F}} \cap \mathcal{O}^n$. For instance, if $\mathbf{G}$ is an affine group scheme defined over O_F, and $F \subset E$ is a finite extension of discrete valuation fields, with ring of integers O_E and residue field $\mathbb{F}_q$, then

$$\mathbf{G}_{\mathsf{F}}(E) = \mathbf{G}(E), \qquad \mathbf{G}_{\mathsf{k}}(E) = \mathbf{G}(\mathbb{F}_q), \qquad \mathbf{G}_{\mathcal{O}}(E) = \mathbf{G}(O_E).$$

The constructions of Section 4.3 can be applied to definable sets of sorts F and k. For example, let F be a valued field and let $\mathfrak{g}$ be a Lie algebra scheme over O_F. Applying Proposition 4.12 to the quantifier-free definable set $\mathfrak{g}_{\mathsf{k}}$, we get a quantifier-free definable set $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ such that, for every extension $F \subset E$, the set $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})(E)$ is the collection of all Lie subalgebras of $\mathfrak{g}_{\mathsf{k}}(E) = \mathfrak{g} \otimes_{O_F} \mathsf{k}(E)$. The assertions of Proposition 4.12 carry over to analogous statements for definable sets of sorts F and k.

Definition 4.17. Let $\mathcal{T}_{\mathrm{Hen},0}$ be the theory of Henselian valued fields of residue characteristic 0, that is the theory generated by the axioms stating that a discrete valuation field is Henselian (i.e., the valuation ring satisfies the conclusions of Hensel's Lemma), and that its residue characteristic is not equal to p, for every rational prime p. Furthermore, given a field K of characteristic 0, we consider also the Denef–Pas language enriched by constant symbols c_a of valued field sort for all $a \in K$ and $\mathcal{T}_{\mathrm{Hen},K,0}$ denotes the theory of Henselian valued fields of residue characteristic 0 together with the statements $c_a \cdot c_b = c_{ab}$, $c_a + c_b = c_{a+b}$ for all $a, b \in K$; cf. Definition 4.10.

The theory $\mathcal{T}_{\mathrm{Hen},0}$ admits partial elimination of quantifiers; see [36, Theorem 4.1] or [19]. By a standard argument, the same holds for the theory $\mathcal{T}_{\mathrm{Hen},K,0}$ in every extended language, as discussed above. We record this fact as follows.

Theorem 4.18. *Every Denef–Pas formula φ is $\mathfrak{T}_{\mathrm{Hen},0}$-equivalent to a formula ψ without valued field quantifiers. For every field K of characteristic 0, the analogous statement holds true for $\mathfrak{T}_{\mathrm{Hen},K,0}$.*

Consider a number field L. None of the local fields $L_{\mathfrak{q}}$, where $\mathfrak{q}$ ranges over the primes of O_L, is a model for $\mathfrak{T}_{\mathrm{Hen},0}$. Nevertheless, we will use theorems proved in $\mathfrak{T}_{\mathrm{Hen},K,0}$ in the following way. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two definable sets, and that $\mathcal{X} = \mathcal{Y}$ holds in $\mathfrak{T}_{\mathrm{Hen},K,0}$. Then the equivalence of $\mathcal{X}$ and $\mathcal{Y}$ can be proved by only finitely many axioms of $\mathfrak{T}_{\mathrm{Hen},K,0}$. Hence, $\mathcal{X}(E) = \mathcal{Y}(E)$ is true is true for $K \subset E$, assuming only that the valued field E is Henselian and that the characteristic of the residue field of E is greater than some constant (depending on $\mathcal{X}$ and $\mathcal{Y}$). In particular, we deduce that $\mathcal{X}(L_{\mathfrak{q}}) = \mathcal{Y}(L_{\mathfrak{q}})$ for almost all primes $\mathfrak{q}$ of O_L.

5. Parametrizing Representations

In this section, we consider an affine group scheme $\mathbf{G} \subset \mathrm{GL}_N$ over the ring of integers O_K of a number field K whose generic fiber is semi-simple. Let $\mathfrak{g} \subset \mathfrak{gl}_N$ be the Lie algebra of $\mathbf{G}$. We consider $\mathbf{G}$ and $\mathfrak{g}$ as quantifier-free definable sets over the first-order language of valued fields enriched by adding constant symbols for the elements of K, and work in the theory $\mathfrak{T}_{\mathrm{Hen},K,0}$ of Henselian fields over K with residue characteristic 0. Our aim is to prove the following result.

Theorem 5.1. *Let K be a number field with ring of integers O_K, and let $\mathbf{G} \subset \mathrm{GL}_N$ be an affine group scheme over O_K whose generic fiber is semi-simple. There are a $(\dim \mathbf{G} + 1)$-dimensional quantifier-free definable set $\mathscr{Z} \subset \mathcal{O}^{\dim \mathbf{G}+1}$, quantifier-free definable functions $f_1, f_2 \colon \mathscr{Z} \to \Gamma$, and a constant $C \in \mathbb{R}$ such that, for every finite field extension $K \subset L$ and almost all primes $\mathfrak{q}$ of O_L,*

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) \sim_C \int_{\mathscr{Z}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s} d\lambda(z),$$

where λ is the additive Haar measure on $L_{\mathfrak{q}}^{\dim \mathbf{G}+1}$ normalized so that $\lambda(O_{L,\mathfrak{q}}^{\dim \mathbf{G}+1}) = 1$.

Throughout the proof of Theorem 5.1, there are places where we omit finitely many primes $\mathfrak{q}$ of O_L. Observation 5.3 collects most of the restrictions that we impose. In addition, we exclude in the proof of Proposition 5.9 and all consequences thereof finitely many primes that are not specified explicitly; this is due to partial elimination of quantifiers. In the actual proof of Theorem 5.1 in Section 5.4, an application of Proposition 4.12 also requires us to disregard finitely many primes. Throughout we collectively write "for almost all primes" to refer to these restrictions. The choice of primes we omit may depend on L. However, we emphasize that the definable set $\mathscr{Z}$ and the definable functions f_1, f_2 in Theorem 5.1 do not depend on the choice of omitted primes.

5.1. **Relative Orbit Method.** We continue to use the notation set up to formulate Theorem 5.1.

Definition 5.2. Let $\mathscr{X}$ be the quantifier-free definable set $\mathfrak{g}_{\mathcal{O}} \times (\mathcal{O} \smallsetminus \{0\})$.

Throughout we fix a non-degenerate, invariant and $\mathrm{Ad}(\mathbf{G})$-invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, e.g., the Killing form, and we consider finite extensions $K \subset L$. For $\mathfrak{q} \in \mathrm{Spec}(O_L)$ lying above a rational prime p, we denote by $\mathfrak{g}(O_{L,\mathfrak{q}})^\vee$ the Pontryagin dual of the abelian pro-p group $\mathfrak{g}(O_{L,\mathfrak{q}})$.

Observation 5.3. By omitting finitely many primes of O_L, we may concentrate in the proof of Theorem 5.1 on $\mathfrak{q} \in \mathrm{Spec}(O_L)$ such that the following conditions hold.

- $p\mathbb{Z} = \mathfrak{q} \cap \mathbb{Z}$ is unramified in $\mathbb{Q} \subset L$, i.e., the valuation of the integer p in $L_\mathfrak{q}$ is 1; furthermore, $p > N+1$, and $p \nmid |H^2(\mathbf{H}(\mathbb{F}_q), \mathbb{C}^\times)|$ for every finite field $\mathbb{F}_q$ of characteristic p and every connected reductive $\mathbb{F}_q$-algebraic group $\mathbf{H}$ with $\dim \mathbf{H} \leq \dim \mathbf{G}$; see Lemma 4.4.
- The form $\langle \cdot, \cdot \rangle$ is non-degenerate on $\mathfrak{g}(O_{L,\mathfrak{q}})$.
- There is a surjective map $\Pi_\mathfrak{q} \colon \mathscr{X}(L_\mathfrak{q}) \to \mathfrak{g}(O_{L,\mathfrak{q}})^\vee$, taking the pair $x = (A_x, z_x)$ to the homomorphism of abelian groups
$$\Pi_\mathfrak{q}(x) \colon \mathfrak{g}(O_{L,\mathfrak{q}}) \to \mathbb{C}^\times, \quad B \mapsto \exp\left(2\pi i \cdot \mathrm{Tr}_{L_\mathfrak{q}|\mathbb{Q}_p}\left(\frac{\langle A_x, B\rangle}{z_x}\right)\right).$$
- Every pro-nilpotent Lie subring of $\mathfrak{g}(O_{L,\mathfrak{q}})$ containing the 1st principal congruence Lie sublattice $\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})$ yields a pro-p subgroup of $\mathbf{G}(O_{L,\mathfrak{q}})$ via the exponential map, and the Kirillov orbit method applies to pro-p subgroups of $\mathbf{G}(O_{L,\mathfrak{q}})$ as described in Proposition 4.7. (For instance, $p > [L : \mathbb{Q}]N^2$ suffices.)

In particular, by restricting one of the homomorphisms $\Pi_\mathfrak{q}(x)$ to $\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}}) = \mathfrak{q} \cdot \mathfrak{g}(O_{L,\mathfrak{q}})$ and applying the orbit method map Ω, we get an irreducible character $\Xi_\mathfrak{q}(x) = \Omega(\Pi_\mathfrak{q}(x)|_{\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})})$ of the 1st principal congruence subgroup $\mathbf{G}^{(1)}(O_{L,\mathfrak{q}})$. When the prime $\mathfrak{q}$ is clear from the context, it may be dropped from the notation.

For every finite extension $K \subset L$ and every $\mathfrak{q} \in \mathrm{Spec}(O_L)$, the set $\mathscr{X}(L_\mathfrak{q})$ is an open subset of $L_\mathfrak{q}^{\dim \mathbf{G}+1}$. We normalize the additive Haar measure on $L_\mathfrak{q}$ so that the ring of integers $O_{L,\mathfrak{q}}$ has measure 1, and denote by λ the product measure on $L_\mathfrak{q}^{\dim \mathbf{G}+1}$. In [24, Lemma 4.1 and Corollary 4.6], Jaikin–Zapirain proved the following result.

Theorem 5.4. *There exist quantifier-free definable functions $\varphi_1, \varphi_2 \colon \mathscr{X} \to \Gamma$ such that, for every finite extension $K \subset L$, almost all $\mathfrak{q} \in \mathrm{Spec}(O_L)$, and every $x \in \mathscr{X}(L_\mathfrak{q})$,*

(1) $\lambda(\Pi_\mathfrak{q}^{-1}(\Pi_\mathfrak{q}(x))) = |O_L/\mathfrak{q}|^{\varphi_1(x)}$,
(2) $\dim \Xi_\mathfrak{q}(x) = |\mathrm{Ad}^*(\mathbf{G}^{(1)}(O_{L,\mathfrak{q}}))(\Pi_\mathfrak{q}(x))|^{1/2} = |O_L/\mathfrak{q}|^{\varphi_2(x)}$.

Remark 5.5. More explicitly, one can take $\varphi_1(x) = \dim \mathbf{G}\, \mathrm{val}(z_x)$, and $\varphi_2(x) = \frac{1}{2}\mathrm{val}(\alpha(x))$, where α is essentially the function appearing in [24, Corollary 4.6]. According to its definition in [24], the function α, and hence φ_2, is quantifier-free.

We need a generalization of the construction leading to Theorem 5.4. Employing the notation introduced in Sections 4.3 and 4.4 (compare, in particular, Definition 4.11), the definable set $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ is the Grassmannian of Lie subalgebras of $\mathfrak{g}_{\mathsf{k}}$, and the definable set $\mathrm{Gr}^{\mathrm{nilp}}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ is the subset of $\mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ parametrizing Lie subalgebras that consist of nilpotent matrices. By applying Proposition 4.12, part (1), we see that both sets are, in fact, quantifier-free definable with respect to the theory $\mathbf{\mathcal{T}}_{\mathrm{fields},O_K}$.

Suppose that $\mathcal{R}\colon \mathscr{X} \to \mathrm{Gr}^{\mathrm{nilp}}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ is a definable function. We denote by $\widetilde{\mathcal{R}} \subset \mathscr{X} \times \mathfrak{g}_{\mathcal{O}}$ the definable set of tuples (x, X) such that the reduction of X to $\mathfrak{g}_{\mathsf{k}}$ is in $\mathcal{R}(x)$. Recall that $\mathbf{G} \subset \mathrm{GL}_N$. By Observation 5.3, we may assume that the residue field characteristic p satisfies $p > N$ so that, over the residue field, one can evaluate without problems the logarithm series up to its Nth term. We define $\exp \mathcal{R} \subset \mathscr{X} \times \mathbf{G}_{\mathsf{k}}$ to be the definable set of pairs (x, g) such that g is unipotent and $\log g \in \mathcal{R}(x)$. By Observation 5.3, we may assume that the residue field characteristic p is unramified in L and satisfies $p - 1 > N$. In this case, a result of Lazard implies that $\log(g)$ converges for elements g of any pro-p subgroup of $\mathbf{G}(O_{L,\mathfrak{q}})$, cf. [26, Lemma B.1]. Denoting by $\widetilde{\mathcal{R}}_x$ the fiber of $\widetilde{\mathcal{R}}$ at x, we define $\exp \widetilde{\mathcal{R}} \subset \mathscr{X} \times \mathbf{G}_{\mathcal{O}}$ to consist of all pairs (x, g) such that the reduction of g to $\mathbf{G}_{\mathsf{k}}$ is unipotent and $\log(g) \in \widetilde{\mathcal{R}}_x$. More generally, if $\mathcal{S} \subset \mathscr{X} \times \mathbf{G}_{\mathsf{k}}$ is a definable family over $\mathscr{X}$, let $\widetilde{\mathcal{S}} \subset \mathscr{X} \times \mathbf{G}_{\mathcal{O}}$ be the definable set of all pairs (x, g) such that the reduction of g to $\mathbf{G}_{\mathsf{k}}$ lies in $\mathcal{S}_x$. We observe that $\exp \widetilde{\mathcal{R}} = \widetilde{\exp \mathcal{R}}$ and will use the lighter notation.

By Observation 5.3, for every finite extension $K \subset L$, for almost all primes $\mathfrak{q}$ of O_L, and for every $x \in \mathscr{X}(L_{\mathfrak{q}})$, the additive group $\widetilde{\mathcal{R}}_x(L_{\mathfrak{q}})$ is closed under Lie commutators and it is the Lie ring associated to the pro-p group $\exp \widetilde{\mathcal{R}}_x(L_{\mathfrak{q}})$.

Definition 5.6. Denote by $\Pi_{\mathcal{R},\mathfrak{q}}(x)$ the restriction of $\Pi_{\mathfrak{q}}(x)$ to $\widetilde{\mathcal{R}}_x(L_{\mathfrak{q}})$. For almost all primes $\mathfrak{q}$, the orbit method map applied to $\Pi_{\mathcal{R},\mathfrak{q}}(x)$ yields an irreducible character $\Xi_{\mathcal{R},\mathfrak{q}}(x)$ of the group $\exp \widetilde{\mathcal{R}}_x(L_{\mathfrak{q}})$. When the prime $\mathfrak{q}$ is clear from the context, it may be dropped from the notation.

Note that, if $\mathcal{R}\colon \mathscr{X} \to \mathrm{Gr}^{\mathrm{nilp}}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ is the constant function with common value $\{0\}$, then the irreducible character $\Xi_{\{0\},\mathfrak{q}}(x)$ coincides with the previously defined $\Xi_{\mathfrak{q}}(x)$. For general $\mathcal{R}$, we are interested in possible extensions of the character $\Xi_{\mathcal{R},\mathfrak{q}}(x)$ to its stabilizer in the normalizer $N_{\mathbf{G}(O_{L,\mathfrak{q}})}(\exp \widetilde{\mathcal{R}}_x(L_{\mathfrak{q}}))$. This stabilizer is known as the inertia group of $\Xi_{\mathcal{R},\mathfrak{q}}(x)$.

We summarize the described set-up for $x \in \mathscr{X}(L_\mathfrak{q})$ in the following diagram.

characters	groups			Lie lattices	functionals
	$\mathbf{G}(O_{L,\mathfrak{q}})$	•	•	$\mathfrak{g}(O_{L,\mathfrak{q}})$	$\Pi_\mathfrak{q}(x)$
$\Xi_{\mathcal{R},\mathfrak{q}}(x)$	$\exp \widetilde{\mathcal{R}}_x(L_\mathfrak{q})$	• ⟵ exp-log correspondence ⟶	•	$\widetilde{\mathcal{R}}_x(L_\mathfrak{q})$	$\Pi_{\mathcal{R},\mathfrak{q}}(x)$
$\Xi_\mathfrak{q}(x)$	$\mathbf{G}^{(1)}(O_{L,\mathfrak{q}})$	•	•	$\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})$	$\Pi_\mathfrak{q}(x)\vert_{\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})}$

5.2. **The Stabilizer of $\Xi_\mathcal{R}$.** Suppose now that $\mathcal{R}\colon \mathscr{X} \to \mathrm{Gr}^{\mathrm{nilp}}_{\mathrm{Lie}}(\mathfrak{g}_\mathsf{k})$ is a *quantifier-free* definable function with respect to the theory $\boldsymbol{\mathcal{T}}_{\mathrm{Hen},K,0}$. This means that $\mathcal{R}$ can be described by a quantifier-free definable formula for all models $L_\mathfrak{q}$ with sufficiently large residue field characteristic. We remark that many of the results in this section do not yet depend essentially on $\mathcal{R}$ being quantifier-free, but it is convenient to focus on this situation in preparation of Section 5.3, where the extra condition plays a crucial role. By Proposition 4.12, parts (3) and (5), there are quantifier-free definable sets $\mathcal{N}_{\mathcal{R},\mathrm{Lie}} \subset \mathscr{X} \times \mathfrak{g}_\mathsf{k}$ and $\mathcal{N}_\mathcal{R} \subset \mathscr{X} \times \mathbf{G}_\mathsf{k}$ whose fibers over a point $x \in \mathscr{X}(L_\mathfrak{q})$ are the stabilizers of $\mathcal{R}(x)$ under the adjoint actions in the Lie algebra and in the group respectively.

By Proposition 4.7, part (3), the stabilizer of $\Xi_{\mathcal{R},\mathfrak{q}}(x)$ in $\mathbf{G}(O_{L,\mathfrak{q}})$ is equal to the stabilizer in $\mathbf{G}(O_{L,\mathfrak{q}})$ of the $\exp \widetilde{\mathcal{R}}_x(L_\mathfrak{q})$-orbit of $\Pi_{\mathcal{R},\mathfrak{q}}(x)$. Hence, this stabilizer is the product of $\exp \widetilde{\mathcal{R}}_x(L_\mathfrak{q})$ and the stabilizer of $\Pi_{\mathcal{R},\mathfrak{q}}(x)$ in $\mathbf{G}(O_{L,\mathfrak{q}})$. Writing

$$N_{\mathbf{G}(O_{L,\mathfrak{q}})}(\widetilde{\mathcal{R}}_x(L_\mathfrak{q})) = \{g \in \mathbf{G}(O_{L,\mathfrak{q}}) \mid \mathrm{Ad}(g)(\widetilde{\mathcal{R}}_x(L_\mathfrak{q})) = \widetilde{\mathcal{R}}_x(L_\mathfrak{q})\},$$

we have

$$\mathrm{Stab}_{\mathbf{G}(O_{L,\mathfrak{q}})}(\Pi_{\mathcal{R},\mathfrak{q}}(x)) = \\ \left\{ g \in N_{\mathbf{G}(O_{L,\mathfrak{q}})}(\widetilde{\mathcal{R}}_x(L_\mathfrak{q})) \mid \forall Y \in \widetilde{\mathcal{R}}_x(L_\mathfrak{q})\colon \Pi_{\mathcal{R},\mathfrak{q}}(x)(Y) = \Pi_{\mathcal{R},\mathfrak{q}}(x)(\mathrm{Ad}(g)Y) \right\}.$$

In the following we consider a prime $\mathfrak{q}$ of O_L, lying above a prime $\mathfrak{p}$ of O_K and different from the previously-omitted primes; see Observation 5.3. In addition we fix an element $x = (A_x, z_x) \in \mathscr{X}(L_\mathfrak{q})$. Recall that $\mathrm{red}_{\mathcal{O}|\mathsf{k}} : \mathcal{O} \to \mathsf{k}$ denotes the reduction map (applied component-wise to entries of a matrix or vector).

Let $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}_{\mathcal{R},\mathfrak{q},x} \subset \mathbf{G}_\mathcal{O}$ be the definable group, in $\boldsymbol{\mathcal{T}}_{\mathrm{Hen},K,0}$, given by the formula

$$\text{(5.1)} \qquad \varphi(x,g) =_{\mathrm{def}} \mathrm{red}_{\mathcal{O}|\mathsf{k}}(g) \in (\mathcal{N}_\mathcal{R})_x \wedge \Big(\forall Z \in \widetilde{\mathcal{R}}_x \big(\mathrm{val}\left(\langle A_x^g - A_x, Z\rangle\right) > \mathrm{val}(z_x) \big) \Big),$$

where we think of x as a parameter and g as a "free variable".

We denote by $\mathcal{S} = \mathcal{S}_{\mathcal{R},\mathfrak{q},x}$ the reduction of $\widehat{\mathcal{S}}$ modulo the maximal ideal, i.e., the definable subgroup of $\mathbf{G}_{\mathsf{k}}$, in $\boldsymbol{\mathcal{T}}_{\mathrm{Hen},K,0}$, given by the formula

$$\psi(x,h) =_{\mathrm{def}} \exists g \in \mathbf{G}_{\mathcal{O}} \left(\mathrm{red}_{\mathcal{O}|\mathsf{k}}(g) = h \wedge \varphi(x,g)\right), \tag{5.2}$$

where again we think of x as a parameter and h as a "free variable".

Denote the characteristic of the residue field $O_K/\mathfrak{p}$ by p. The functor of p-Witt vectors

$$\mathbb{F} \mapsto \mathrm{Witt}(\mathbb{F}) = \varprojlim \mathrm{Witt}_n(\mathbb{F})$$

associates to every perfect field $\mathbb{F}$ of characteristic p canonically a strict p-ring $\mathrm{Witt}(\mathbb{F})$ with residue field $\mathbb{F}$; see [40, Chapter II]. The integral domain $\mathrm{Witt}(\mathbb{F})$ is complete and Hausdorff with respect to the p-adic topology and $\mathrm{Witt}_n(\mathbb{F}) \cong \mathrm{Witt}(\mathbb{F})/p^n \,\mathrm{Witt}(\mathbb{F})$ is called the ring of truncated Witt vectors of length n. For short we denote by $\mathrm{FWitt}(\mathbb{F})$ the field of fractions of $\mathrm{Witt}(\mathbb{F})$.

The functor Witt is pro-representable; the underlying set is represented by $\prod_{i=1}^{\infty} \mathbb{A}^1$; cf. [18]. Using this fact, one sees that there are a pro-algebraic group scheme $\widehat{\mathbf{S}} = \widehat{\mathbf{S}}_{\mathcal{R},\mathfrak{q},x}$ and a definable group $\overline{\mathcal{S}} = \overline{\mathcal{S}}_{\mathcal{R},\mathfrak{q},x}$, in $\boldsymbol{\mathcal{T}}_{\text{perf.-fields},p,O_L/\mathfrak{q}}$, such that, for every (possibly infinite) perfect extension $\mathbb{F}$ of $O_L/\mathfrak{q}$,

$$\begin{aligned} \widehat{\mathbf{S}}(\mathbb{F}) &= \widehat{\mathcal{S}}(\mathrm{FWitt}(\mathbb{F})) \subset \mathbf{G}_{\mathcal{O}}(\mathrm{FWitt}(\mathbb{F})), \\ \overline{\mathcal{S}}(\mathbb{F}) &= \mathcal{S}(\mathrm{FWitt}(\mathbb{F})) \subset \mathbf{G}_{\mathsf{k}}(\mathrm{FWitt}(\mathbb{F})) \cong \mathbf{G}(\mathbb{F}). \end{aligned} \tag{5.3}$$

Our next goal, Proposition 5.7, is to show that there exists an algebraic group $\mathbf{S}$ over $O_L/\mathfrak{q}$ such that, for every perfect extension $\mathbb{F}$ of $O_L/\mathfrak{q}$, one has $\mathbf{S}(\mathbb{F}) = \overline{\mathcal{S}}(\mathbb{F})$. The following table summarizes some of the notation that will feature in this discussion.

$\boldsymbol{\mathcal{T}}_{\mathrm{Hen},K,0}$-def.	$\boldsymbol{\mathcal{T}}_{\text{perf.-fields},p,O_L/\mathfrak{q}}$-def.	$\mathrm{Witt}_n(\mathbb{F})$-algebraic	$\mathbb{F}$-(pro-)algebraic
$\widehat{\mathcal{S}} \subset \mathbf{G}_{\mathcal{O}}$		$\mathbf{S}_n$	$\widehat{\mathbf{S}} = \varprojlim \mathcal{F}_n(\mathbf{S}_n)$
$\mathcal{S} \subset \mathbf{G}_{\mathsf{k}}$	$\overline{\mathcal{S}}$		$\mathbf{S}$

We briefly recall further details regarding the functor Witt; compare [18], or [5, p. 276] for a summary. As above, let $\mathbb{F}$ be a perfect field of characteristic $p > 0$ and fix $n \in \mathbb{N}$. For an $\mathbb{F}$-scheme $\mathbf{X}$, let $\mathcal{G}_n(\mathbf{X})$ be the locally ringed space whose underlying topological space is the same as the topological space of $\mathbf{X}$ and whose sheaf of rings is the sheaf of germs of morphisms $\mathbf{X} \to \mathrm{Witt}_n$. For example,

$$\mathcal{G}_n(\mathrm{Spec}(\mathbb{F})) = \mathrm{Spec}(\mathrm{Witt}_n(\mathbb{F})) \quad \text{and} \quad \mathcal{G}_n(\mathrm{Spec}(\mathbb{F}[e]/(e^2))) = \mathrm{Spec}(\mathrm{Witt}_n(\mathbb{F})[e]/(e^2)),$$

which we will shortly put to good use. By the main theorem of [18, §4], there is a functor

$$\mathcal{F}_n \colon (\mathrm{Witt}_n(\mathbb{F})\text{-schemes of finite type}) \to (\mathbb{F}\text{-schemes of finite type}),$$

the Greenberg functor of degree n, such that, for every $\mathbb{F}$-scheme $\mathbf{X}$ of finite type and every $\mathrm{Witt}_n(\mathbb{F})$-scheme $\mathbf{Y}$ of finite type, there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Spec}(\mathbb{F})}(\mathbf{X}, \mathcal{F}_n(\mathbf{Y})) \cong \mathrm{Hom}_{\mathrm{Spec}(\mathrm{Witt}_n(\mathbb{F}))}(\mathcal{G}_n(\mathbf{X}), \mathbf{Y}). \tag{5.4}$$

Returning to the situation at hand, by Observation 5.3, we may suppose that $p\mathbb{Z} = \mathfrak{q} \cap \mathbb{Z}$ is unramified in $\mathbb{Q} \subset L$. For $n \in \mathbb{N}$, there is an affine group scheme $\mathbf{S}_n$ over $\mathrm{Witt}_n(O_L/\mathfrak{q})$ such that, for every perfect extension $\mathbb{F}$ of $O_L/\mathfrak{q}$,

$$\begin{aligned}\mathbf{S}_n(\mathrm{Witt}_n(\mathbb{F})) = \big\{ g \in \mathbf{G}(\mathrm{Witt}_n(\mathbb{F})) \mid \mathrm{red}_{\mathrm{Witt}_n(\mathbb{F})|\mathbb{F}}(g) \in (\mathcal{N}_{\mathcal{R}})_x(\mathbb{F}) \wedge \\ \forall Z \in \widetilde{\mathcal{R}}_x(\mathrm{FWitt}(\mathbb{F}))\big(\mathrm{val}(\langle A_x^g - A_x, Z\rangle) > \min\{n, \mathrm{val}(z_x)\}\big) \big\}.\end{aligned}$$

Here $\mathrm{red}_{\mathrm{Witt}_n(\mathbb{F})|\mathbb{F}} \colon \mathrm{Witt}_n(\mathbb{F}) \to \mathrm{Witt}_1(\mathbb{F}) \cong \mathbb{F}$ denotes the natural reduction map. The pro-algebraic group scheme $\widehat{\mathbf{S}} = \widehat{\mathbf{S}}_{\mathcal{R},\mathfrak{q},x}$ in (5.3) is the inverse limit of the $O_L/\mathfrak{q}$-group schemes $\mathcal{F}_n(\mathbf{S}_n)$, $n \in \mathbb{N}$.

Next we discuss the pro-Lie algebra $\widehat{\mathbf{T}}$ of $\widehat{\mathbf{S}}$. We use the following consequence of (5.4). If $\mathbf{V} = \mathrm{Spec}(\mathrm{Witt}_n(\mathbb{F})[x_1, \ldots, x_d]/(f_1, \ldots, f_m))$ is an affine $\mathrm{Witt}_n(\mathbb{F})$-scheme and $v \colon \mathrm{Spec}(\mathrm{Witt}_n(\mathbb{F})) \to \mathbf{V}$ a $\mathrm{Witt}_n(\mathbb{F})$-point, then the tangent space of $\mathcal{F}_n(\mathbf{V})$ at $\mathcal{F}_n(v)$ is the affine subspace of $\mathcal{F}_n(\mathbb{A}^d_{\mathrm{Witt}_n(\mathbb{F})}) \cong \mathbb{A}^{nd}_{\mathbb{F}}$ defined by the "polynomials" $\mathcal{F}_n(\mathrm{d}f_i(v))$, $i \in \{1, \ldots, m\}$. This can be seen from applying (5.4) to $\mathbf{X} = \mathrm{Spec}(\mathbb{F}[e]/(e^2))$ and $\mathbf{Y} = \mathbf{V}$. In particular, the Lie algebra of $\mathcal{F}_n(\mathbf{S}_n)$ is isomorphic to $\mathcal{F}_n(\mathbf{T}_n)$, where $\mathbf{T}_n$ denotes the $\mathrm{Witt}_n(O_L/\mathfrak{q})$-scheme satisfying, for every perfect extension $\mathbb{F}$ of $O_L/\mathfrak{q}$,

$$\begin{aligned}\mathbf{T}_n(\mathrm{Witt}_n(\mathbb{F})) = \big\{ X \in \mathfrak{g}(\mathrm{Witt}_n(\mathbb{F})) \mid \mathrm{red}_{\mathrm{Witt}_n(\mathbb{F})|\mathbb{F}}(X) \in (\mathcal{N}_{\mathcal{R},\mathrm{Lie}})_x(\mathbb{F}) \wedge \\ \forall Z \in \widetilde{\mathcal{R}}_x(\mathrm{FWitt}(\mathbb{F}))\big(\mathrm{val}(\langle [A_x, X], Z\rangle) > \min\{n, \mathrm{val}(z_x)\}\big) \big\}.\end{aligned}$$

The pro-Lie algebra $\widehat{\mathbf{T}}$ is the inverse limit of the $O_L/\mathfrak{q}$-Lie algebra schemes $\mathcal{F}_n(\mathbf{T}_n)$, $n \in \mathbb{N}$.

Next we show that the definable group $\overline{\mathcal{S}}$ in (5.3) is actually an algebraic group.

Proposition 5.7. *Let $\mathfrak{q} \in \mathrm{Spec}(O_L)$, $x \in \mathscr{X}(L_{\mathfrak{q}})$ and $\overline{\mathcal{S}} = \overline{\mathcal{S}}_{\mathcal{R},\mathfrak{q},x}$ be as above. In particular, suppose that $\mathfrak{q}$ satisfies the conditions listed in Observation* 5.3*. Then $\overline{\mathcal{S}}$ is (equivalent to) an algebraic group over $O_L/\mathfrak{q}$.*

Proof. Recall that $\mathbf{G} \subset \mathrm{GL}_N$. We show that $\overline{\mathcal{S}}$ is quantifier-free in $\boldsymbol{\mathcal{T}}_{\text{perf.-fields},p,O_L/\mathfrak{q}}$, using Remark 4.14. Since constructable subgroups are Zariski-closed, this implies that $\overline{\mathcal{S}}$ is (equivalent to) a Zariski-closed subgroup of the algebraic group GL_N over $O_L/\mathfrak{q}$.

We need to check that

$$\overline{\mathcal{S}}(\mathbb{F}) = \overline{\mathcal{S}}(\mathbb{F}^{\mathrm{alg}}) \cap \mathbf{G}(\mathbb{F}) \tag{5.5}$$

for every (possibly infinite) perfect extension $\mathbb{F}$ of $O_L/\mathfrak{q}$. Fix such an extension $\mathbb{F}$, set $\mathbb{O} = \mathrm{Witt}(\mathbb{F})$ with residue field $\mathbb{O}/p\mathbb{O} \cong \mathbb{F}$, and write $\mathbb{L} = \mathrm{FWitt}(\mathbb{F})$. Let $\mathbb{O}^{\mathrm{unr}} = \mathrm{Witt}(\mathbb{F}^{\mathrm{alg}})$ and $\mathbb{L}^{\mathrm{unr}} = \mathrm{FWitt}(\mathbb{F}^{\mathrm{alg}})$ denote the maximal unramified extensions. Since $L_{\mathfrak{q}}$ is unramified over $\mathbb{Q}_p$, we have $L_{\mathfrak{q}} \subset \mathbb{L}$.

The inclusion $\subset$ in (5.5) is clear. To prove the other inclusion we consider $A \in \mathfrak{g}(\mathbb{O})$, $\overline{g} \in \mathbf{G}(\mathbb{F})$, and $z \in \mathbb{O} \smallsetminus \{0\}$. Writing $x = (A, z) \in \mathscr{X}(\mathbb{L})$ and $\gamma = \mathrm{val}(z)$, we suppose that there exists $\widetilde{g} \in \mathbf{G}(\mathbb{O}^{\mathrm{unr}})$ such that $\mathrm{red}_{\mathbb{O}|\mathsf{k}}(\widetilde{g}) = \overline{g}$ and $\mathrm{val}(\langle A^{\widetilde{g}} - A, X\rangle) \geq \gamma + 1$ for all $X \in \widetilde{\mathcal{R}}_x(\mathbb{L}^{\mathrm{unr}})$. The task is to produce $g \in \mathbf{G}(\mathbb{O})$ with the same properties as $\widetilde{g}$. Clearly, there exists $g_1 \in \mathbf{G}(\mathbb{O})$ such that $\mathrm{red}_{\mathbb{O}|\mathsf{k}}(g_1) = \overline{g}$. Put $B = A^{g_1} \in \mathfrak{g}(\mathbb{O})$ and consider the definable set

$$\mathcal{Y} = \mathcal{Y}_{A,B,\mathcal{R}(x),\gamma} = \left\{ g \in \mathbf{G}_{\mathcal{O}}^{(1)} \mid \forall Z \in \widetilde{\mathcal{R}}_x \left(\mathrm{val}(\langle B^g - A, Z\rangle) \geq \gamma + 1\right) \right\},$$

where the labeling is permissible, because $\mathcal{R}(x)$ determines $\widetilde{\mathcal{R}}_x$. Clearly, $g_1^{-1}\widetilde{g} \in \mathcal{Y}(\mathbb{O}^{\mathrm{unr}})$, and it suffices to show that $\mathcal{Y}(\mathbb{O})$ is not empty. Furthermore, by forming the quotient of $\mathcal{Y}$ by the $(\gamma+1)$st principal congruence subgroup $\mathbf{G}_{\mathcal{O}}^{(\gamma+1)}$ of $\mathbf{G}_{\mathcal{O}}$, we obtain a $\mathrm{Witt}_{\gamma+1}(O_L/\mathfrak{q})$-scheme. Using the Greenberg functor, this quotient can be identified with an algebraic variety $\mathbf{Y} = \mathbf{Y}_{A,B,\mathcal{R}(x),\gamma}$ over $O_L/\mathfrak{q}$. Our aim $\mathcal{Y}(\mathbb{O}) \neq \varnothing$ is equivalent to $\mathbf{Y}(\mathbb{F}) \neq \varnothing$.

It is convenient to treat first the special case $\mathcal{R}(x) = \{0\}$. This means that $\widetilde{\mathcal{R}}_x$ gives the 1st principal congruence Lie sublattice $\mathfrak{g}^{(1)}$. Since the form $\langle \cdot, \cdot \rangle$ is non-degenerate, the defining condition of $\mathcal{Y}$ is equivalent to $B^g \equiv A \pmod{p^\gamma}$. By induction on γ, we may further assume that $B \equiv A \pmod{p^{\gamma-1}}$, that is $B = A + p^{\gamma-1}D$ for some $D \in \mathfrak{g}(\mathbb{O})$. By Observation 5.3, the logarithm map is a well-defined polynomial map on $\mathbf{G}^{(1)}(\mathbb{O}^{\mathrm{unr}})$. For $g \in \mathbf{G}^{(1)}(\mathbb{O}^{\mathrm{unr}})$, the formula

$$B^g = B + \sum_{i=1}^{\infty} \frac{1}{i!}[B, \underbrace{\log(g), \ldots, \log(g)}_{i}] \equiv B + [B, \log(g)] \pmod{p^{\delta+1}}, \tag{5.6}$$

where $\delta = \sup\{d \in \mathbb{N} \mid [B, \log(g)] \equiv 0 \pmod{p^d}\}$, shows that the defining condition of $\mathcal{Y}$ can be replaced by

$$[A, \log(g)] + p^{\gamma-1}D \equiv 0 \pmod{p^\gamma}.$$

Passing to $\mathbf{Y}$, this translates into a system of linear equations over $\mathbb{F}$. Since $\mathbf{Y}(\mathbb{F}^{\mathrm{alg}})$ is non-empty we deduce that $\mathbf{Y}(\mathbb{F})$ is non-empty.

Now we return to the general case. Based on the trivial inclusion $\{0\} \subset \mathcal{R}(x)$, the special case yields $g_2 \in \mathbf{G}^{(1)}(\mathbb{O})$ such that $B^{g_2} \equiv A \pmod{p^\gamma}$. Thus we may assume that B itself is already of the form $B = A + p^\gamma E$ for some $E \in \mathfrak{g}(\mathbb{O})$. By choosing a basis for the Lie lattice $\widetilde{\mathcal{R}}_x(\mathbb{L})$, the defining condition of $\mathcal{Y}$ can be phrased as $\ell(B^g - A) \equiv 0 \pmod{p^{\gamma+1}}$ for a linear operator $\ell = \ell_{\mathcal{R}(x)} \colon \mathfrak{g} \to \mathfrak{g}$. The elementary divisors of ℓ are 1 and p, with multiplicities $\dim \mathcal{R}(x)$ and $\dim \mathfrak{g}_{\mathsf{k}} - \dim \mathcal{R}(x)$. Thus (5.6) implies that the defining condition of $\mathcal{Y}$ can be replaced by

$$\ell([A, \log(g)] + p^\gamma E) \equiv 0 \pmod{p^{\gamma+1}},$$

because necessarily $[A, \log(g)] \equiv 0 \pmod{p^\gamma}$ and all higher terms vanish modulo $p^{\gamma+1}$. Passing to $\mathbf{Y}$, this translates once more into a system of linear equations over $\mathbb{F}$. Since $\mathbf{Y}(\mathbb{F}^{\mathrm{alg}})$ is non-empty we deduce that $\mathbf{Y}(\mathbb{F})$ is non-empty. $\square$

Remark 5.8. The proof of Proposition 5.7 admits the following short interpretation. In the special case $\mathcal{R}(0) = \{0\}$, one can regard $\mathbf{Y} = \mathbf{Y}_{A,B,\mathcal{R}(x),\gamma}$ as a torsor of the connected unipotent group

$$\mathcal{U}_{A,\gamma} = \left\{ g\mathbf{G}_{\mathcal{O}}^{(\gamma+1)} \in \mathbf{G}_{\mathcal{O}}^{(1)}/\mathbf{G}_{\mathcal{O}}^{(\gamma+1)} \mid A^g - A \equiv 0 \pmod{p^\gamma} \right\},$$

and the logarithm map sets up a bijection between $\mathbf{Y}$ and the affine space

$$\{Z + \mathfrak{g}_{\mathcal{O}}^{(\gamma+1)} \in \mathfrak{g}_{\mathcal{O}}^{(1)}/\mathfrak{g}_{\mathcal{O}}^{(\gamma+1)} \mid [A, Z] \equiv 0 \pmod{p^\gamma}\}.$$

It is known that connected unipotent groups have trivial first Galois cohomology groups, and thus every torsor over such a group has a rational point.

We denote the algebraic group equivalent to $\overline{\mathcal{S}} = \overline{\mathcal{S}}_{\mathcal{R},\mathfrak{q},x}$ by $\mathbf{S} = \mathbf{S}_{\mathcal{R},\mathfrak{q},x}$ and refer to it as the stabilizer of $\Xi_{\mathcal{R},\mathfrak{q}}(x)$ modulo the 1st principal congruence subgroup.

Proposition 5.9. *There is a constant $C \in \mathbb{R}$, depending only on K and $\mathbf{G}$, such that, for every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of O_L, every quantifier-free definable function $\mathcal{R}\colon \mathscr{X} \to \mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g}_{\mathsf{k}})$, and every $x = (A_x, z_x) \in \mathscr{X}(L_{\mathfrak{q}})$, the number of connected components of $\mathbf{S} = \mathbf{S}_{\mathcal{R},\mathfrak{q},x}$ is less than C.*

Proof. Recall that the notation $\mathrm{ac}^{\times n}$, respectively $\mathrm{val}^{\times n}$, indicates that the angular component map, respectively valuation map, is to be applied coordinatewise to vectors of length n. We apply partial elimination of quantifiers in the theory $\mathfrak{T}_{\mathrm{Hen},K,0}$ (cf. Theorem 4.18) by treating the entries of x, which involve elements of $L_{\mathfrak{q}}$, as parameters: after omitting finitely many primes, the formula (5.2) defining $\mathcal{S} = \mathcal{S}_{\mathcal{R},\mathfrak{q},x}$ is equivalent to a formula of the form

$$\eta(x, h) =_{\mathrm{def}} \bigvee_{i=1}^{M} \big(H_i(A_x) = 0 \,\wedge\, \varphi_i(\mathrm{ac}^{\times m(i)}(H_i'(A_x)), \mathcal{R}(x), h) \,\wedge\, \psi_i(\mathrm{val}^{\times n(i)}(H_i''(A_x)), \mathrm{val}(z_x))\big),$$

where the H_i, $H_i' = (H_{i,1}', \ldots, H_{i,m(i)}')$ and $H_i'' = (H_{i,1}'', \ldots, H_{i,n(i)}'')$ are polynomial functions over K (of sort F), the φ_i are formulae in the language of rings (of sort k), and the ψ_i are formulae in the language of ordered groups (of sort Γ). This can be proved by induction on the length of the formula.

It follows that $\mathcal{S}$ is a finite union of some of the definable sets $\big\{h \mid \varphi_i(\mathrm{ac}^{\times m(i)}(H_i'(A_x)), \mathcal{R}(x), h)\big\}$. Write $\overline{\mathcal{S}} = \overline{\mathcal{S}}_{\mathcal{R},\mathfrak{q},x}$ and $\mathbb{F}_q = O_L/\mathfrak{q}$. Since $\overline{\mathcal{S}}(\mathbb{F}_q)$ always contains the identity, it is non-empty. Proposition 4.15 implies that there is a constant $C \in \mathbb{R}$ such that, for every unramified finite extension $L_{\mathfrak{q}} \subset M_{\mathfrak{r}}$ with residue field $\mathbb{F}_{q^r}$, there exists $d \in \mathbb{N}_0$ such that $C^{-1}q^{rd} \leq |\overline{\mathcal{S}}(M_{\mathfrak{r}})| = |\overline{\mathcal{S}}(\mathbb{F}_{q^r})| \leq Cq^{rd}$.

Using Proposition 5.7, we regard $\overline{\mathcal{S}}$ as an algebraic variety $\mathbf{S}$ over $\mathbb{F}_q$ and apply the Lang–Weil bound [27]. There are infinitely many $r \in \mathbb{N}$ such that all absolutely irreducible components of $\mathbf{S}$ are defined over $\mathbb{F}_{q^r}$, and $|\mathbf{S}(\mathbb{F}_{q^r})| = (|\mathbf{S} : \mathbf{S}^\circ| + O(q^{-r/2}))q^{r \dim \mathbf{S}}$ for such r. Comparing the two estimates for $|\overline{\mathcal{S}}(\mathbb{F}_{q^r})| = |\mathbf{S}(\mathbb{F}_{q^r})|$ as r tends to infinity, we obtain $|\mathbf{S} : \mathbf{S}^\circ| \le C$. □

Next we identify the Lie algebra of $\mathbf{S} = \mathbf{S}_{\mathcal{R},\mathfrak{q},x}$, where $\mathcal{R}$ and $x = (A_x, z_x) \in \mathscr{X}(L_\mathfrak{q})$ are as above. In analogy to (5.1), let $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}_{\mathcal{R},\mathfrak{q},x} \subset \mathfrak{g}_\mathcal{O}$ be the definable set given by

$$\xi(x, X) =_{\text{def}} \text{red}_{\mathcal{O}|\mathsf{k}}(X) \in (\mathcal{N}_{\mathcal{R},\text{Lie}})_x \wedge \left(\forall Z \in \widetilde{\mathcal{R}}_x \, (\text{val}(\langle [A_x, X], Z\rangle) > \text{val}(z_x))\right),$$

where we think of x as a parameter and X as a free variable. Furthermore, let $\mathcal{T} = \mathcal{T}_{\mathcal{R},\mathfrak{q},x}$ denote the reduction of $\widehat{\mathcal{T}}$ modulo the maximal ideal, a definable subset of $\mathfrak{g}_\mathsf{k}$; compare (5.2). Arguing similarly as in the proof of Proposition 5.7, we see that for every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of O_L, and every $x \in \mathscr{X}(L_\mathfrak{q})$, the set $\mathcal{T}(L_\mathfrak{q})$ is a linear space over the residue field of $L_\mathfrak{q}$. In fact, we give an independent proof of this fact in Corollary 5.15.

Proposition 5.10. *For every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of O_L, and every $x \in \mathscr{X}(L_\mathfrak{q})$, the definable set $\mathcal{T}_{\mathcal{R},\mathfrak{q},x}$ is (equivalent to) the Lie algebra of $\mathbf{S}_{\mathcal{R},\mathfrak{q},x}$.*

Proof. Write $x = (A_x, z_x)$ and $\gamma = \text{val}(z_x)$. We drop all subscripts $\mathcal{R}, \mathfrak{q}, x$. Recall the construction of the pro-algebraic group $\widehat{\mathbf{S}} = \varprojlim \mathcal{F}_n(\mathbf{S}_n)$ and its pro-Lie algebra $\widehat{\mathbf{T}} = \varprojlim \mathcal{F}_n(\mathbf{T}_n)$ via Witt vectors. The reduction map $\text{red}_{\mathcal{O}|\mathsf{k}} \colon \widehat{\mathcal{S}} \to \mathcal{S}$ translates into $\widehat{\mathbf{S}} \to \mathbf{S}$ which factors through the homomorphism $f_\gamma \colon \mathcal{F}_\gamma(\mathbf{S}_\gamma) \to \mathbf{S}$ of $O_L/\mathfrak{q}$-algebraic groups. By definition, this homomorphism is onto and therefore flat; see [14, Proposition 6.1.5]. As indicated in Remark 5.8, the fibers of f_γ, each isomorphic to the kernel of f_γ, are affine spaces, and hence smooth. By [15, Theorem 17.5.1], the map f_γ is smooth, and so its differential at 1 is surjective. It follows that $\mathcal{F}_\gamma(\mathbf{T}_\gamma)$ maps onto the Lie algebra $\mathbf{T}$ of $\mathbf{S}$. In this way we can identify $\mathcal{T}$ with $\mathbf{T}$. □

Using Proposition 5.9 and Lemma 4.16, we obtain the following consequence.

Corollary 5.11. *There is a constant $C \in \mathbb{R}$ such that, for every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of O_L, and every $x \in \mathscr{X}(L_\mathfrak{q})$,*

$$C^{-1}|\mathcal{T}_{\mathcal{R},\mathfrak{q},x}(L_\mathfrak{q})| \le |\mathcal{S}_{\mathcal{R},\mathfrak{q},x}(L_\mathfrak{q})| \le C|\mathcal{T}_{\mathcal{R},\mathfrak{q},x}(L_\mathfrak{q})|.$$

5.3. **The Lie Algebra Associated to the Stabilizer of $\Xi_\mathcal{R}$.** We continue to work in the set-up introduced in Sections 5.1 and 5.2. In particular, we consistently omit finitely many primes, as specified by Observation 5.3 and in the proof of Proposition 5.9. It is easier, and more transparent, to handle the Lie algebra associated to the stabilizer $\mathbf{S}_{\mathcal{R},\mathfrak{q},x}$ of $\Xi_{\mathcal{R},\mathfrak{q}}(x)$ modulo the 1st principal congruence subgroup, rather than the stabilizer itself. For this purpose, we introduce the following cover of $\mathscr{X}$.

Definition 5.12. Let $\mathscr{Y} \subset \mathscr{X} \times (\Gamma \cup \{\infty\})^{\dim \mathfrak{g}} \times (\text{Aut}(\mathfrak{g})_\mathcal{O})^2$ be the quantifier-free definable set consisting of tuples $((A, z), (\gamma_1, \ldots, \gamma_{\dim \mathfrak{g}}), (U_1, U_2))$ such that, in the chosen standard basis of $\mathfrak{g}$, the linear operator $T = U_1 \, (\text{ad}\, A) \, U_2$ is diagonal and the valuations of the diagonal elements are given by $(\gamma_1, \ldots, \gamma_{\dim \mathfrak{g}})$.

Let $\mathcal{R}\colon \mathscr{Y} \to \mathrm{Gr}^{\mathrm{nilp}}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ be a definable function. The following definitions are analogous to those in Section 5.1. Let $\widetilde{\mathcal{R}} \subset \mathscr{Y} \times \mathfrak{g}_{\mathcal{O}}$ be the definable set of tuples (y, X) such that the reduction of X to $\mathfrak{g}_{\mathsf{k}}$ is in $\mathcal{R}(y)$. Recall that $\mathbf{G} \subset \mathrm{GL}_N$ and that, by virtue of Observation 5.3, the residue field characteristic p satisfies $p > N$. We define $\exp \mathcal{R} \subset \mathscr{Y} \times \mathbf{G}_{\mathsf{k}}$ to be the set of pairs (y, g) such that g is unipotent and $\log g \in \mathcal{R}(y)$. Finally, again based on Observation 5.3, we define $\exp \widetilde{\mathcal{R}} \subset \mathscr{Y} \times \mathbf{G}_{\mathcal{O}}$ to consist of all pairs (y, g) such that the reduction of g to $\mathbf{G}_{\mathsf{k}}$ is unipotent and $\log g \in \widetilde{\mathcal{R}}_y$.

We denote by $\mathrm{pr}_{\mathscr{Y}\downarrow\mathscr{X}}\colon \mathscr{Y} \to \mathscr{X}$ the natural projection. For every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of O_L, and for every $y \in \mathscr{Y}(L_{\mathfrak{q}})$, the additive group $\widetilde{\mathcal{R}}_y(L_{\mathfrak{q}})$ is closed under Lie commutators and it is the Lie ring of the pro-p group $\exp \widetilde{\mathcal{R}}_y(L_{\mathfrak{q}})$. Given $y \in \mathscr{Y}(L_{\mathfrak{q}})$, we write $\Pi_{\mathcal{R},\mathfrak{q}}(y)$ for the restriction of $\Pi_{\mathfrak{q}}(\mathrm{pr}_{\mathscr{Y}\downarrow\mathscr{X}}(y))$ to $\widetilde{\mathcal{R}}_y(L_{\mathfrak{q}})$ and we denote by $\Xi_{\mathcal{R},\mathfrak{q}}(y)$ the resulting irreducible character of $\exp \widetilde{\mathcal{R}}_y(L_{\mathfrak{q}})$. As in the analogous Definition 5.6, the subscript $\mathfrak{q}$ is sometimes omitted. Similarly, we write $\mathbf{S}_{\mathcal{R},\mathfrak{q},y} = \mathbf{S}_{\mathcal{R},\mathfrak{q},x}$, where $x = \mathrm{pr}_{\mathscr{Y}\downarrow\mathscr{X}}(y)$.

The following lemma is evident.

Lemma 5.13. *Let $\mathfrak{O}$ be a complete, discrete valuation ring with a uniformizer ϖ. Let $M = \mathfrak{O}^n$ be a free $\mathfrak{O}$-module of rank $n \in \mathbb{N}$, and let $\overline{N} \subset M/\varpi M$ be a linear subspace with pre-image N in M. Assume that T is an endomorphism of M which, in the standard basis, is given by a diagonal matrix with diagonal entries $\varpi^{\gamma_1}, \ldots, \varpi^{\gamma_n}$, where $\gamma_1, \ldots, \gamma_n \in \mathbb{Z}$ satisfy $0 \leq \gamma_1 \leq \ldots \leq \gamma_n$. For $l \in \mathbb{Z}$, let $i(l) = \max(\{0\} \cup \{i \mid \gamma_i \leq l\})$ and $j(l) = \min(\{n+1\} \cup \{j \mid \gamma_j \geq l\})$. Then the following hold.*

(1) *The pre-image $T^{-1}(\varpi^l N)$ is equal to*

$$\Big\{(a_1, \ldots, a_n) \in M \mid \forall i \in \{1, \ldots, i(l)\} : \mathrm{val}(a_i) \geq l - \gamma_i \\ \text{and } (\overline{\varpi^{\gamma_1 - l} a_1}, \ldots, \overline{\varpi^{\gamma_{i(l)} - l} a_{i(l)}}, 0, \ldots, 0) \in \overline{N}\Big\}.$$

(2) *The reduction of $T^{-1}(\varpi^l N)$ modulo ϖ is the set of all $(\overline{a_1}, \ldots, \overline{a}_n) \in (\mathfrak{O}/\varpi\mathfrak{O})^n$ such that $\overline{a_1} = \ldots = \overline{a_{j(l)-1}} = 0$ and $(0, \ldots, 0, \overline{a_{j(l)}}, \ldots, \overline{a_{i(l)}}, 0, \ldots, 0) \in \overline{N}$.*

Proposition 5.14. *Let $\mathcal{R}\colon \mathscr{Y} \to \mathrm{Gr}^{\mathrm{nilp}}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ be a quantifier-free definable function. Then there is a quantifier-free definable function $\mathcal{L}\colon \mathscr{Y} \to \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ such that, for every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of O_L, and every $y \in \mathscr{Y}(L_{\mathfrak{q}})$, the Lie algebra of the stabilizer $\mathbf{S}_{\mathcal{R},\mathfrak{q},y}$ is given by $\mathcal{L}(y)$.*

Proof. For every $y \in \mathscr{Y}(L_{\mathfrak{q}})$, the Lie algebra of the stabilizer $\mathbf{S}_{\mathcal{R},\mathfrak{q},y}$ is the sum of $\mathcal{R}(y)$ and the Lie algebra of the stabilizer of $\Pi_{\mathcal{R},\mathfrak{q}}(y)$ modulo the maximal ideal. Thus it suffices to prove that there is a quantifier-free definable function $\mathcal{L}\colon \mathscr{Y} \to \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}})$ such that the image $\mathcal{L}(y)$ of $y \in \mathscr{Y}(L_{\mathfrak{q}})$ is the Lie algebra of the stabilizer of $\Pi_{\mathcal{R},\mathfrak{q}}(y)$ modulo the maximal ideal. For $y = ((A_y, z_y), (\gamma_{y,1}, \ldots, \gamma_{y,\dim \mathfrak{g}}), (U_{y,1}, U_{y,2})) \in \mathscr{Y}(L_{\mathfrak{q}})$, this is the intersection of the normalizer of $\mathcal{R}(y)$, which is given by a quantifier-free function, and the reduction modulo the maximal ideal of

$$\begin{aligned}\mathcal{V}_y(L_\mathfrak{q}) &= \Big\{ Y \in \mathfrak{g}_\mathcal{O}(L_\mathfrak{q}) \mid \forall X \in (\widetilde{\mathcal{R}})_y(L_\mathfrak{q}) \, \big(\Pi_{\mathcal{R},\mathfrak{q}}(y)([X,Y]) = 1 \big) \Big\} \\ &= \Big\{ Y \in \mathfrak{g}_\mathcal{O}(L_\mathfrak{q}) \mid \forall X \in (\widetilde{\mathcal{R}})_y(L_\mathfrak{q}) \, \big(\mathrm{val}(\langle A_y, [X,Y] \rangle) \geq \mathrm{val}(z_y) \big) \Big\} \\ &= \Big\{ Y \in \mathfrak{g}_\mathcal{O}(L_\mathfrak{q}) \mid \forall X \in (\widetilde{\mathcal{R}})_y(L_\mathfrak{q}) \, \big(\mathrm{val}(\langle X, [A_y, Y] \rangle) \geq \mathrm{val}(z_y) \big) \Big\}.\end{aligned}$$

The set $\mathcal{V}_y$ can be interpreted as the fiber of a definable set $\mathcal{V} \subset \mathscr{Y} \times \mathfrak{g}_\mathcal{O}$, and it remains to prove that the reduction of $\mathcal{V}_y$ is quantifier-free. Let $\mathcal{R}^\perp(y)$ be the orthogonal subspace to $\mathcal{R}(y)$ in $\mathfrak{g}_\mathsf{k}$ with respect to the form $\langle \cdot, \cdot \rangle$. Recall that by omitting finitely many $\mathfrak{q}$, we arranged that $\langle \cdot, \cdot \rangle$ is non-degenerate on $\mathfrak{g}_\mathsf{k}(L_\mathfrak{q})$. We observe that $\mathcal{V}_y(L_\mathfrak{q})$ is the pre-image of $z_y \big((\widetilde{\mathcal{R}^\perp})_y(L_\mathfrak{q}) \big)$ under the map $\mathrm{ad}(A_y)$. According to the definition of $\mathscr{Y}$, we have $\mathrm{ad}(A_y) = U_{y,1}^{-1} T_y U_{y,2}^{-1}$, where $U_{y,1}, U_{y,2} \in \mathrm{Aut}(\mathfrak{g})_\mathcal{O}(L_\mathfrak{q})$ and T_y is diagonal with respect to the standard basis. Thus $U_{y,2}^{-1}(V(L_\mathfrak{q}))$ is the pre-image of $z_y \, U_{y,1}\big((\widetilde{\mathcal{R}^\perp})_y(L_\mathfrak{q}) \big)$ under T_y. By Lemma 5.13, statement (2), its reduction modulo the maximal ideal is quantifier-free. Hence, the reduction of $\mathcal{V}_y$ is quantifier-free. □

Corollary 5.15. *For every quantifier-free definable function $\mathcal{R}\colon \mathscr{X} \to \mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g}_\mathsf{k})$, there is a quantifier-free definable function $\mathcal{L}\colon \mathscr{X} \to \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_\mathsf{k})$ such that for every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of O_L, and every $x \in \mathscr{X}(L_\mathfrak{q})$, the Lie algebra of the stabilizer $\mathbf{S}_{\mathcal{R},\mathfrak{q},x}$ is given by $\mathcal{L}(x)$.*

Proof. Pre-composing $\mathcal{R}$ with the projection $\mathrm{pr}_{\mathscr{Y}\downarrow\mathscr{X}}\colon \mathscr{Y} \to \mathscr{X}$ we get a quantifier-free definable function $\mathcal{R}'\colon \mathscr{Y} \to \mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g}_\mathsf{k})$. Applying Proposition 5.14, we obtain a quantifier-free definable function $\mathcal{L}'\colon \mathscr{Y} \to \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_\mathsf{k})$ such that, for all $y \in \mathscr{Y}(L_\mathfrak{q})$, the vector space $\mathcal{L}'(y)$ is the Lie algebra of the stabilizer $\mathbf{S}_{\mathcal{R},\mathfrak{q},y}$. Since $\mathcal{L}'(y)$ depends only on $\mathrm{pr}_{\mathscr{Y}\downarrow\mathscr{X}}(y)$, we get a definable function $\mathcal{L}\colon \mathscr{X} \to \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_\mathsf{k})$ such that $\mathcal{L}(x)$ is the Lie algebra of the stabilizer $\mathbf{S}_{\mathcal{R},\mathfrak{q},x}$ for all $x \in \mathscr{X}(L_\mathfrak{q})$.

It suffices to show that $\mathcal{L}$ is quantifier-free definable. By an analogue of Lemma 4.13 for valued fields (cf. Remark 4.14), it is enough to show that if $F \subset E$ is an extension of valued fields, $x \in \mathscr{X}(F)$, and $v \in \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_\mathsf{k})(F)$, then $\mathcal{L}(x) = v$ holds in F if and only if $\mathcal{L}(x) = v$ holds in E. Since the map $\mathscr{Y} \to \mathscr{X}$ is onto, we can choose $y \in \mathscr{Y}(F) \subset \mathscr{Y}(E)$ such that $\mathrm{pr}_{\mathscr{Y}\downarrow\mathscr{X}}(y) = x$. Then the following assertions are pairwise equivalent: $\mathcal{L}(x) = v$ holds in F; $\mathcal{L}'(y) = v$ holds in F; $\mathcal{L}'(y) = v$ holds in E (because $\mathcal{L}'$ is quantifier-free); $\mathcal{L}(x) = v$ holds in E. □

Proposition 5.16. *Let $\mathcal{R}_1, \mathcal{R}_2\colon \mathscr{X} \to \mathrm{Gr}_{\mathrm{Lie}}^{\mathrm{nilp}}(\mathfrak{g}_\mathsf{k})$ be quantifier-free definable maps and assume that, for every $x \in \mathscr{X}$, the Lie algebra $\mathcal{R}_1(x)$ is normalized by $\mathcal{R}_2(x)$. Then there is a quantifier-free definable function $\varphi\colon \mathscr{X} \to \mathsf{\Gamma}$ such that, for every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of L, and every $x \in \mathscr{X}(L_\mathfrak{q})$,*

$$\Big| \mathrm{Ad}^*(\exp \widetilde{\mathcal{R}_2}(x))(\Pi_{\mathcal{R}_1,\mathfrak{q}}(x)) \Big| = |O_L/\mathfrak{q}|^{\varphi(x)}.$$

Proof. Similarly to the proof of Proposition 5.14, the first claim of Lemma 5.13 gives a quantifier-free function $\varphi_1 \colon \mathscr{Y} \to \Gamma$ such that, for every $y \in \mathscr{Y}(L_{\mathfrak{q}})$,

$$\left|\mathrm{Ad}^*(\exp \widetilde{\mathcal{R}_2}(\mathrm{pr}(y)))(\Pi_{\mathcal{R}_1 \,\mathrm{pr},\mathfrak{q}}(y))\right| = |O_L/\mathfrak{q}|^{\varphi_1(y)},$$

where $\mathrm{pr} = \mathrm{pr}_{\mathscr{Y}\downarrow\mathscr{X}} \colon \mathscr{Y} \to \mathscr{X}$ is the projection. Since $\varphi_1(y)$ depends only on the image of y in $\mathscr{X}$, we get a definable function $\varphi_2 \colon \mathscr{X} \to \Gamma$ such that

$$\left|\mathrm{Ad}^*(\exp \widetilde{\mathcal{R}_2}(x))(\Pi_{\mathcal{R}_1,\mathfrak{q}}(x))\right| = |O_L/\mathfrak{q}|^{\varphi_2(x)}.$$

An argument similar to the one in Corollary 5.15 shows that φ_2 is a quantifier-free definable function, so we can take $\varphi = \varphi_2$. □

Remark 5.17. We can now indicate a proof of part (1) of Theorem 5.4.

We write $x = (A_x, z_x)$. Then $\Pi_{\{0\}}^{-1}(\Pi_{\{0\}}(x))$ consists of the pairs (B, w) such that, for all $X \in \mathfrak{g}^1(O_{L,\mathfrak{q}})$,

$$\mathrm{val}\left(\left\langle \frac{A_x}{z_x} - \frac{B}{w}, X \right\rangle\right) > 0,$$

or, equivalently, such that $\mathrm{val}(A_x w - B z_x) \geq \mathrm{val}(z_x w)$. The claim follows.

5.4. **Proof of Theorem 5.1.** We continue to work in the same set-up as in the previous sections and recall that, if $\mathcal{S} \subset \mathscr{X} \times \mathbf{G}_{\mathsf{k}}$ is a definable family over $\mathscr{X}$, then $\widetilde{\mathcal{S}} \subset \mathscr{X} \times \mathbf{G}_{\mathcal{O}}$ denotes the definable set of all pairs (x, g) such that the reduction of g to $\mathbf{G}_{\mathsf{k}}$ lies in $\mathcal{S}_x$.

Theorem 5.18. *There are, for $n \in \mathbb{N}_0$, quantifier-free definable functions*

$$g_n, h_n \colon \mathscr{X} \to \Gamma, \qquad \mathcal{R}_n \colon \mathscr{X} \to \mathrm{Gr}^{\mathrm{nilp}}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}}), \qquad \mathcal{L}_n \colon \mathscr{X} \to \mathrm{Gr}_{\mathrm{Lie}}(\mathfrak{g}_{\mathsf{k}}),$$

constants $C_n \in \mathbb{R}$, and definable families $\mathcal{S}_n \subset \mathscr{X} \times \mathbf{G}_{\mathsf{k}}$ of subgroups of $\mathbf{G}_{\mathsf{k}}$ such that the following hold.

1. *$\mathcal{R}_0$ is the constant function $\{0\}$, $\mathcal{L}_0$ is the constant function $\mathfrak{g}_{\mathsf{k}}$, and $\mathcal{S}_0 = \mathscr{X} \times \mathbf{G}_{\mathsf{k}}$.*
2. *For $n \geq 1$, every finite extension $K \subset L$, almost all primes $\mathfrak{q}$ of O_L, and every $x \in \mathscr{X}(L_{\mathfrak{q}})$, the following hold:*
   - *$(\widetilde{\mathcal{S}_n})_x$ is the stabilizer of $\Xi_{\mathcal{R}_{n-1}}(x)$ in $(\widetilde{\mathcal{S}_{n-1}})_x$,*
   - *$\mathcal{L}_n(x)$ is the Lie algebra of $(\mathcal{S}_n)_x$, and*
   - *$\mathcal{R}_n(x)$ is the Lie subalgebra of nilpotent matrices in the solvable radical of $\mathcal{L}_n(x)$.*
3. *There exists $n_0 \in \mathbb{N}$ such that the sequences $g_n, h_n, \mathcal{R}_n, \mathcal{L}_n, \mathcal{S}_n$ stabilize for $n \geq n_0$.*
4. *For every n, every finite extension $K \subset L$, and almost all primes $\mathfrak{q}$ of O_L,*

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) \sim_{C_n} \int_{\mathscr{X}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{g_n(x)-h_n(x)s} \zeta_{(\widetilde{\mathcal{S}_n})_x(L_{\mathfrak{q}})|\Xi_{\mathcal{R}_n}(x)}(s)\, d\lambda(x).$$

Proof. We first construct $\mathcal{R}_n, \mathcal{L}_n, \mathcal{S}_n$ using recursion on $n \in \mathbb{N}_0$. Suitable functions g_n, h_n will be obtained in a second step. The functions $\mathcal{R}_0, \mathcal{L}_0$ and the family $\mathcal{S}_0$ are prescribed by (1). Suppose $\mathcal{R}_n, \mathcal{L}_n, \mathcal{S}_n$ have been constructed. The discussion at the beginning of Section 5.2 implies that there is a definable family of subgroups of $\mathbf{G}_{\mathcal{O}}$ over $\mathscr{X}$ whose fiber at any $x \in \mathscr{X}$ is the stabilizer of $\Xi_{\mathcal{R}_n}(x)$ in $(\widetilde{\mathcal{S}_n})_x$. Take $\mathcal{S}_{n+1}$ to be the reduction of this family modulo the maximal ideal. Similarly, we get $\mathcal{L}_{n+1}$, using Corollary 5.15, and $\mathcal{R}_{n+1}$, using Proposition 4.12. Thus (2) is taken care of.

Next, we show that the sequences $\mathcal{R}_n, \mathcal{L}_n$, and $\mathcal{S}_n$, $n \in \mathbb{N}_0$, stabilize as required by (3). Note that the sequence $\dim \mathcal{R}_n$, $n \in \mathbb{N}_0$, is (pointwise) non-decreasing and the sequence $\dim \mathcal{L}_n$, $n \in \mathbb{N}_0$, is non-increasing. Proposition 5.9 implies that, for every $n \in \mathbb{N}_0$, there is an upper bound $D(n)$ for the number of connected components of each of the groups $(\mathcal{S}_n)_x$.

We claim that, if $n_1 \in \mathbb{N}_0$ is such that $\dim \mathcal{R}_{n_1}(x) = \dim \mathcal{R}_{n_1+D(n_1)}(x)$ and $\dim \mathcal{L}_{n_1}(x) = \dim \mathcal{L}_{n_1+D(n_1)}(x)$, then the sequences $\mathcal{R}_n(x), \mathcal{L}_n(x)$, and $\mathcal{S}_n(x)$, $n \in \mathbb{N}_0$, stabilize for $n > n_1 + D(n_1)$. Indeed, suppose that $n \in \mathbb{N}_0$ with $n_1 \le n < n_1 + D(n_1)$. If $\dim \mathcal{L}_n(x) = \dim \mathcal{L}_{n+1}(x)$, then $\mathcal{L}_n(x) = \mathcal{L}_{n+1}(x)$ and similarly for $\mathcal{R}_n(x)$. Since $(\mathcal{S}_{n+1})_x$ is a subgroup of $(\mathcal{S}_n)_x$ and they have the same Lie algebra, either $\mathcal{S}_n(x) = \mathcal{S}_{n+1}(x)$ or $\mathcal{S}_{n+1}(x)$ has fewer connected components than $\mathcal{S}_n(x)$. It follows that there is $n \in \mathbb{N}_0$ with $n_1 \le n < n_1 + D(n_1)$ such that $\mathcal{S}_n(x) = \mathcal{S}_{n+1}(x)$. It now follows that the sequences of functions $\mathcal{R}_n, \mathcal{L}_n$, and $\mathcal{S}_n$ stabilize for sufficiently large $n \in \mathbb{N}_0$. Once $\mathcal{R}_n$ and $\mathcal{S}_n$ stabilize, we can keep also the functions g_n and h_n unchanged. Thus (3) is satisfied.

It remains to construct g_n and h_n for $n \in \mathbb{N}_0$ so that (4) holds. We start with $n = 0$. Fix a finite extension $K \subset L$, and consider primes $\mathfrak{q}$ of O_L that satisfy, in particular, the conditions of Lemma 4.6. For short we write $X = \mathscr{X}(L_\mathfrak{q})$, and we put

$$X' = \bigsqcup \left\{ X_\vartheta \mid \vartheta \in \mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})^\vee \smallsetminus \{1\} \right\},$$

where $X_\vartheta = \{x \in X \mid \Pi_{\{0\},\mathfrak{q}}(x) = \vartheta\}$ for every homomorphism ϑ from the additive group of the 1st principal congruence Lie sublattice $\mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})$ to $\mathbb{C}^\times$. Summing over $\vartheta \in \mathfrak{g}^{(1)}(O_{L,\mathfrak{q}})^\vee \smallsetminus \{1\}$, applying the orbit method map Ω, and using Lemma 4.8, we obtain

$$\begin{aligned}
&\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) \\
&= \sum_{\vartheta \ne 1} \frac{1}{|\mathrm{Ad}^*(\mathbf{G}(O_{L,\mathfrak{q}}))(\vartheta)|} (\dim \Omega(\vartheta))^{-s} \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})|\Omega(\vartheta)}(s) \\
&= \sum_{\vartheta \ne 1} \int_{X_\vartheta} \frac{1}{\lambda(X_\vartheta)} \frac{1}{|\mathrm{Ad}^*(\mathbf{G}(O_{L,\mathfrak{q}}))(\Pi_{\{0\}}(x))|} \dim(\Xi_{\{0\}}(x))^{-s} \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})|\Xi_{\{0\}}(x)}(s)\, d\lambda(x) \\
&= \int_{X'} \frac{1}{\lambda(\Pi_{\{0\}}^{-1}(\Pi_{\{0\}}(x)))} \frac{1}{|\mathrm{Ad}^*(\mathbf{G}(O_{L,\mathfrak{q}}))(\Pi_{\{0\}}(x))|} \dim(\Xi_{\{0\}}(x))^{-s} \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})|\Xi_{\{0\}}(x)}(s)\, d\lambda(x).
\end{aligned}$$

By Theorem 5.4 and Corollary 5.15, there are quantifier-free definable functions $\varphi_1, \varphi_2, \varphi_3 \colon \mathscr{X} \to \Gamma$ such that, for almost all primes $\mathfrak{q}$ of O_L,

(1) $\lambda(\Pi_{\{0\}}^{-1}(\Pi_{\{0\}}(x))) = |O_L/\mathfrak{q}|^{\varphi_1(x)}$,
(2) $\dim \Xi_{\{0\}}(x) = |\mathrm{Ad}^*(\mathbf{G}^{(1)}(O_{L,\mathfrak{q}}))(\Pi_{\{0\}}(x))|^{1/2} = |O_L/\mathfrak{q}|^{\varphi_2(x)}$,
(3) $\dim \mathcal{L}_1(x) = |O_L/\mathfrak{q}|^{\varphi_3(x)}$.

By Lemma 4.16, there is a constant $C \in \mathbb{R}$ such that, for every $x \in X$,

$$|\mathrm{Ad}^*(\mathbf{G}(O_{L,\mathfrak{q}}))(\Pi_{\{0\}}(x))| = \\ |\mathrm{Ad}^*(\mathbf{G}^{(1)}(O_{L,\mathfrak{q}}))(\Pi_{\{0\}}(x))| \cdot |\mathbf{G}(O_L/\mathfrak{q})/(\mathcal{S}_1)_x(L_\mathfrak{q})| \sim_C |O_L/\mathfrak{q}|^{2\varphi_2(x)+\dim\mathfrak{g}-\varphi_3(x)}.$$

Therefore,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) \sim_C \\ \int_{X'} |O_L/\mathfrak{q}|^{(-\varphi_1(x)-2\varphi_2(x)+\varphi_3(x)-\dim\mathfrak{g})-\varphi_2(x)s}\, \zeta_{\mathbf{G}(O_{L,\mathfrak{q}})|\Xi_{\{0\}}(x)}(s)\, d\lambda(x),$$

and we set $g_0(x) = -\varphi_1(x) - 2\varphi_2(x) + \varphi_3(x) - \dim\mathfrak{g}$, $h_0(x) = \varphi_2(x)$, and $C_0 = C$.

Finally, we suppose that g_n, h_n are given for some $n \in \mathbb{N}_0$ and we explain how to construct g_{n+1}, h_{n+1}. Again, fix a finite extension $K \subset L$, and consider primes $\mathfrak{q}$ of O_L that are different from any of the finitely many primes omitted.

Let $\mathfrak{a} \subset \mathfrak{g}(O_L/\mathfrak{q})$ be a nilpotent Lie algebra, and recall that $\widetilde{\mathfrak{a}}$ denotes the pre-image of $\mathfrak{a}$ under the map $\mathfrak{g}(O_{L,\mathfrak{q}}) \to \mathfrak{g}(O_L/\mathfrak{q})$; it is a Lie ring, and, in particular, an additive group. Let τ be a homomorphism from $\widetilde{\mathfrak{a}}$ to $\mathbb{C}^\times$, and consider the set

$$X_{\mathfrak{a},\tau} = \Pi_{\mathcal{R}_n}^{-1}(\tau) = \{x \in \mathscr{X}(L_\mathfrak{q}) \mid \mathcal{R}_n(x) = \mathfrak{a}, \Pi_{\mathcal{R}_n}(x) = \tau\}.$$

Inductively, we see that on $X_{\mathfrak{a},\tau}$, the values of $\mathcal{R}_0(x), \mathcal{L}_0(x), \ldots, \mathcal{R}_n(x), \mathcal{L}_n(x)$ and thus the values of $(\mathcal{S}_n)_x(L_\mathfrak{q})$, $(\mathcal{S}_{n+1})_x(L_\mathfrak{q})$, and $\mathcal{R}_{n+1}(x)$ are constant. Denote the latter by S_n, S_{n+1}, and $\mathfrak{b}$ respectively. Writing $m = |\mathfrak{b} : \mathfrak{a}|$, let $\vartheta_1, \ldots, \vartheta_m$ denote the homomorphisms from $\widetilde{\mathfrak{b}}$ to $\mathbb{C}^\times$ that extend τ. We define, for $i \in \{1, \ldots, m\}$,

$$X_{\mathfrak{a},\tau}^i = \Pi_{\mathcal{R}_{n+1}}^{-1}(\vartheta_i) = \{x \in X_{\mathfrak{a},\tau} \mid \Pi_{\mathcal{R}_{n+1}}(x) = \vartheta_i\}.$$

These sets form a partition of $X_{\mathfrak{a},\tau}$ into m parts of equal measure. Using Lemma 4.8 and Clifford theory, we deduce that, for every $x \in X_{\mathfrak{a},\tau}$,

$$\zeta_{\widetilde{(\mathcal{S}_n)}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)}(s) = \zeta_{\widetilde{S_n}|\Omega(\tau)}(s) = |\widetilde{S_n} : \widetilde{S_{n+1}}|^{-s} \zeta_{\widetilde{S_{n+1}}|\Omega(\tau)}(s) \\ = |\widetilde{S_n} : \widetilde{S_{n+1}}|^{-s} \sum_{i=1}^m \frac{|\mathrm{Ad}^*(\exp\widetilde{\mathcal{R}_{n+1}})(\tau)|}{|\mathrm{Ad}^*(\widetilde{S_{n+1}})(\vartheta_i)|} \left(\frac{\dim\Omega(\vartheta_i)}{\dim\Omega(\tau)}\right)^{-s} \zeta_{\widetilde{S_{n+1}}|\Omega(\vartheta_i)}(s).$$

Therefore,

$$\begin{aligned}
&\int_{X_{\mathfrak{a},\tau}} \zeta_{\widetilde{(\mathcal{S}_n)}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)}(s)\,d\lambda(x)\\
&= |\widetilde{S_n} : \widetilde{S_{n+1}}|^{-s} \sum_{i=1}^{m} \int_{X^i_{\mathfrak{a},\tau}} \underbrace{\frac{\lambda(X_{\mathfrak{a},\tau})}{\lambda(X^i_{\mathfrak{a},\tau})}}_{=m} \frac{|\mathrm{Ad}^*(\exp\widetilde{\mathcal{R}_{n+1}})(\tau)|}{|\mathrm{Ad}^*(\widetilde{S_{n+1}})(\vartheta_i)|} \left(\frac{\dim\Omega(\vartheta_i)}{\dim\Omega(\tau)}\right)^{-s} \zeta_{\widetilde{S_{n+1}}|\Omega(\vartheta_i)}(s)\,d\lambda(x)\\
&= \sum_{i=1}^{m} \int_{X^i_{\mathfrak{a},\tau}} |(\mathcal{S}_n)_x(L_\mathfrak{q}) : (\mathcal{S}_{n+1})_x(L_\mathfrak{q})|^{-s}\, \underbrace{|\mathcal{R}_{n+1}(x) : \mathcal{R}_n(x)|}_{=m}\cdot\\
&\qquad \frac{|\mathrm{Ad}^*(\exp\widetilde{\mathcal{R}_{n+1}})(\Pi_{\mathcal{R}_n}(x))|}{|\mathrm{Ad}^*(\widetilde{(\mathcal{S}_{n+1})}_x(L_\mathfrak{q}))(\Pi_{\mathcal{R}_{n+1}}(x))|} \left(\frac{\dim\Xi_{\mathcal{R}_{n+1}}(x)}{\dim\Xi_{\mathcal{R}_n}(x)}\right)^{-s} \zeta_{\widetilde{(\mathcal{S}_{n+1})}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_{n+1}}(x)}(s)\,d\lambda(x).
\end{aligned}$$

Now there are quantifier-free definable functions $\psi_1, \psi_2, \psi_3, \psi_4 \colon \mathscr{X} \to \Gamma$ and a constant $C \in \mathbb{R}$ – all independent of L and $\mathfrak{q}$ – such that, for almost all primes $\mathfrak{q}$ of O_L, and all $x \in \mathscr{X}(L_\mathfrak{q})$, the following hold:

(1) $|(\mathcal{S}_n)_x(L_\mathfrak{q}) : (\mathcal{S}_{n+1})_x(L_\mathfrak{q})| \sim_C |O_L/\mathfrak{q}|^{\psi_1(x)}$, by Corollary 5.15 and Lemma 4.16 (1), because the number of connected components of $(\mathcal{S}_n)_x$ is bounded by $D(n)$,
(2) $|\mathcal{R}_{n+1}(x) : \mathcal{R}_n(x)| = |O_L/\mathfrak{q}|^{\psi_2(x)}$, as $\mathcal{R}_n$ and $\mathcal{R}_{n+1}$ are quantifier-free definable,
(3) $|\mathrm{Ad}^*(\exp\widetilde{\mathcal{R}_{n+1}})(\Pi_{\mathcal{R}_n}(x))|/|\mathrm{Ad}^*(\widetilde{(\mathcal{S}_{n+1})}_x(L_\mathfrak{q}))(\Pi_{\mathcal{R}_{n+1}}(x))| \sim_C |O_L/\mathfrak{q}|^{\psi_3(x)}$, as

$$\begin{aligned}
&|\mathrm{Ad}^*(\widetilde{(\mathcal{S}_{n+1})}_x(L_\mathfrak{q}))(\Pi_{\mathcal{R}_{n+1}}(x))| =\\
&\qquad |\mathrm{Ad}^*(\exp\widetilde{\mathcal{R}_{n+1}}(x))(\Pi_{\mathcal{R}_{n+1}}(x))| \cdot |(\mathcal{S}_n)_x(L_\mathfrak{q}) : (\mathcal{S}_{n+1})_x(L_\mathfrak{q})|,
\end{aligned}$$

by part (1), and by Proposition 5.16,
(4) $\dim\Xi_{\mathcal{R}_{n+1}}(x)/\dim\Xi_{\mathcal{R}_n}(x) \sim_C |O_L/\mathfrak{q}|^{\psi_4(x)}$ because, for example, $\dim\Xi_{\mathcal{R}_n}(x) = |\mathrm{Ad}^*(\exp\widetilde{\mathcal{R}_n}(x))(\Pi_{\mathcal{R}_n}(x))|^{1/2}$ and by Proposition 5.16.

Writing $\alpha_n = \psi_2 + \psi_3$ and $\beta_n = \psi_1 + \psi_4$, we obtain

$$\int_{X_{\mathfrak{a},\tau}} \zeta_{\widetilde{(S_n)}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)}(s)\,d\lambda(x) \sim_{C^3} \int_{X_{\mathfrak{a},\tau}} |O_L/\mathfrak{q}|^{\alpha_n(x)-\beta_n(x)s}\, \zeta_{\widetilde{(\mathcal{S}_{n+1})}_x(\mathfrak{q})|\Xi_{\mathcal{R}_{n+1}}(x)}(s)\,d\lambda(x).$$

Defining $g_{n+1} = g_n + \alpha_n$ and $h_{n+1} = h_n + \beta_n$ we obtain, from the corresponding properties of g_n, h_n,

$$\begin{aligned}
\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) &\sim_{C_n} \int_{\mathscr{X}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{g_n(x)-h_n(x)s}\, \zeta_{\widetilde{(\mathcal{S}_n)}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)}(s)\, d\lambda(x) \\
&= \sum_{\mathfrak{a},\tau} \int_{X_{\mathfrak{a},\tau}} |O_L/\mathfrak{q}|^{g_n(x)-h_n(x)s}\, \zeta_{\widetilde{(\mathcal{S}_n)}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)}(s)\, d\lambda(x) \\
&\sim_{C^3} \sum_{\mathfrak{a},\tau} \int_{X_{\mathfrak{a},\tau}} |O_L/\mathfrak{q}|^{g_{n+1}(x)-h_{n+1}(x)s}\, \zeta_{\widetilde{(\mathcal{S}_{n+1})}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_{n+1}}(x)}(s)\, d\lambda(x) \\
&= \int_{\mathscr{X}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{g_{n+1}(x)-h_{n+1}(x)s}\, \zeta_{\widetilde{(\mathcal{S}_{n+1})}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_{n+1}}(x)}(s)\, d\lambda(x).
\end{aligned}$$

Here $\mathfrak{a}$ ranges over the finite set of nilpotent Lie subalgebras of $\mathfrak{g}(O_L/\mathfrak{q})$ and τ ranges over the countable set of characters of $\widetilde{\mathfrak{a}}$. The required properties of g_{n+1}, h_{n+1} hold for $C_{n+1} = C_n C^3$. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. We continue to use the notation set up in this section. In particular, let $g_n, h_n, \mathcal{R}_n, \mathcal{L}_n, \mathcal{S}_n, C_n$ be the sequences constructed in Theorem 5.18. Suppose $n_0 \in \mathbb{N}$ is sufficiently large so that $g_n, h_n, \mathcal{R}_n, \mathcal{L}_n, \mathcal{S}_n$ are stable for $n \geq n_0$. Fix $n \geq n_0$. By Theorem 5.1, for all finite extensions $K \subset L$ and almost all $\mathfrak{q}$,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) \sim_{C_n} \int_{\mathscr{X}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{g_n(x)-h_n(x)s} \cdot \zeta_{\widetilde{(\mathcal{S}_n)}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)}(s) d\lambda(x). \tag{5.7}$$

Recall that $(\mathcal{S}_n)_x \subset \mathbf{G}_\mathsf{k}$ is an algebraic group. Let $(\mathcal{S}_n)_x^\circ$ denote the connected component of the identity, and let $\widetilde{(\mathcal{S}_n)_x^\circ} \subset \mathbf{G}_\mathcal{O}$ be the pre-image of $(\mathcal{S}_n)_x^\circ$ under the reduction map modulo the maximal ideal. It was shown in Proposition 5.9 that there is a constant $D_1 \in \mathbb{R}$ such that, for almost all $\mathfrak{q}$ and all $x \in \mathscr{X}(L_\mathfrak{q})$,

$$|\widetilde{(\mathcal{S}_n)}_x(L_\mathfrak{q}) : \widetilde{(\mathcal{S}_n)_x^\circ}(L_\mathfrak{q})| \leq |(\mathcal{S}_n)_x : (\mathcal{S}_n)_x^\circ| \leq D_1.$$

By Lemma 2.11 (2), we get for almost all $\mathfrak{q}$ and all $x \in \mathscr{X}(L_\mathfrak{q})$,

$$\zeta_{\widetilde{(\mathcal{S}_n)}_x(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)} \sim_{D_1} \zeta_{\widetilde{(\mathcal{S}_n)_x^\circ}(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)}.$$

For every prime $\mathfrak{q}$ and every $x \in \mathscr{X}(L_\mathfrak{q})$, the group $Q_x = (\mathcal{S}_n)_x^\circ / \exp \mathcal{R}_n(x)$ is a reductive group over the field $O_L/\mathfrak{q}$ of dimension at most $\dim \mathbf{G}$. Omitting finitely many primes $\mathfrak{q}$ of O_L as specified in Observation 5.3, none of the Schur multipliers of the groups $\widetilde{(\mathcal{S}_n)_x^\circ}(L_\mathfrak{q}) / \exp \widetilde{\mathcal{R}_n(x)}(L_\mathfrak{q}) = Q_x(O_L/\mathfrak{q})$, for $x \in \mathscr{X}(L_\mathfrak{q})$, contains elements of order $\mathrm{char}(O_L/\mathfrak{q})$. Using Lemmas 4.2 and 4.1, we thus obtain

$$\zeta_{\widetilde{(\mathcal{S}_n)_x^\circ}(L_\mathfrak{q})|\Xi_{\mathcal{R}_n}(x)} = \zeta_{Q_x(O_L/\mathfrak{q})}.$$

The Lie algebra of Q_x is $\mathcal{L}_n(x)/\mathcal{R}_n(x)$. By Proposition 4.12, there is a finite, quantifier-free partition of $\mathscr{X}$ such that, on each part, the absolute root system Φ_x of $Q_x/Z(Q_x)$ and the dimension of the center of Q_x – which can be read off from the Lie algebra of Q_x, cf. Proposition 4.12 – are constant. By Corollary 3.4, there is a constant $D_2 \in \mathbb{R}$ such that

$$\zeta_{Q_x(O_L/\mathfrak{q})}(s) \sim_{D_2} |O_L/\mathfrak{q}|^{\dim Z(Q_x)}(1 + |O_L/\mathfrak{q}|^{\operatorname{rk}\Phi_x - |\Phi_x^+|s}).$$

Setting $\mathscr{Z} = \mathscr{X} \times \{0,1\}$,

$$f_1(z) = \begin{cases} g_n(x) + \dim Z(Q_x) & \text{for } z = (x,0) \in \mathscr{Z}, \\ g_n(x) + \dim Z(Q_x) + \operatorname{rk}\Phi_x & \text{for } z = (x,1) \in \mathscr{Z}, \end{cases}$$

and

$$f_2(z) = \begin{cases} -h_n(x) & \text{for } z = (x,0) \in \mathscr{Z}, \\ -h_n(x) + |\Phi_x^+| & \text{for } z = (x,1) \in \mathscr{Z}, \end{cases}$$

we obtain, with $C = C_n D_1 D_2$,

$$\begin{aligned} &\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) \\ &\quad \sim_C \int_{\mathscr{X}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{g_n(x)+\dim Z(Q_x)+h_n(x)s}(1 + |O_L/\mathfrak{q}|^{\operatorname{rk}\Phi_x - |\Phi_x^+|s})d\lambda(x) \\ &\quad = \int_{\mathscr{Z}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s} d\lambda(z). \end{aligned}$$

This completes the proof of Theorem 5.1. □

6. Quantifier-Free Integrals

In this section we complete the proof of Theorem 2.8. The section's main result is Theorem 6.2, expressing the dependence of an integral such as the one in Theorem 5.1 on the local field the integral is interpreted at. We state the precise result in Section 6.1 and prove it in Section 6.2.

Throughout this section, we fix a number field K and work within the first order language of valued fields together with a constant, of the value field sort, for every element of K. We will use the theory $\mathfrak{T}_{\mathrm{Hen},K,0}$ of Henselian valued fields over K of characteristic 0; cf. Definition 4.17.

6.1. **Uniform Formulae for Quantifier-Free Integrals.**

We make no notational distinction between an algebraic variety and the corresponding functor of points.

Definition 6.1. Let X be a smooth algebraic variety of dimension n over K and let ω be a regular differential n-form on X. For any local field F containing K, the set $X(F)$ has the structure of a p-adic analytic manifold; cf. [41, Part II, Chapter III]. We define a measure $|\omega|_F$ on $X(F)$ as follows: given a compact open set $U \subset X(F)$, an open compact subset $W \subset F^n$, and an analytic diffeomorphism $f\colon U \to W$, we write

$$f^*\omega = g dx_1 \wedge \cdots \wedge dx_n,$$

for some function $g\colon W \to F$, and define

$$|\omega|_F(U) = \int_W |g(x)|_F d\lambda(x),$$

where $|\cdot|_F$ is the normalized absolute value of F and λ is the Haar measure on F^n normalized so that $\lambda(O_F^n) = 1$. The assignment $U \mapsto |\omega|_F(U)$ extends uniquely to a non-negative (possibly infinite) Radon measure on $X(F)$, which we also denote by $|\omega|_F$. See [1, Section 3.1] for further details.

We now state the section's main theorem.

Theorem 6.2. *Let K be a number field with ring of integers O_K, and let $X \subset \mathbb{A}^M$ be a smooth affine K-variety, and ω a regular differential top form on X. Suppose that $\mathscr{X} \subset X \cap \mathcal{O}^M$ is a quantifier-free definable set and that $f_1, f_2\colon \mathscr{X} \to \Gamma$ are quantifier-free definable functions. There exist $N \in \mathbb{N}$, quasi-affine O_K-schemes $\mathbf{W}_i$ and integers $\alpha_i, \beta_i \in \mathbb{N}_0$ and $n_i \in \mathbb{N}$, for $i \in \{1, \ldots, N\}$, and $A_{ij}, B_{ij} \in \mathbb{Z}$, for $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n_i\}$, such that the following holds: for every finite extension $K \subset L$ and almost all primes $\mathfrak{q}$ of O_L,*

$$\int_{\mathscr{X}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s} |\omega|_{L_\mathfrak{q}} = \sum_{i=1}^{N} |O_L/\mathfrak{q}|^{\alpha_i - \beta_i s} |\mathbf{W}_i(O_L/\mathfrak{q})| \cdot \prod_{j=1}^{n_i} \frac{|O_L/\mathfrak{q}|^{A_{ij}-B_{ij}s}}{1 - |O_L/\mathfrak{q}|^{A_{ij}-B_{ij}s}}, \tag{6.1}$$

for every $s \in \mathbb{C}$ for which the integral on the left converges.

Theorem 6.2 may possibly be deduced from [19, Proposition 4.5 and Proposition 10.10] in a similar way that [19, Theorem 1.3] is. In the next section we give a direct proof, in terms of resolutions of singularities.

6.2. **Proof of Theorem 6.2.** In this section we decorate the reduction, angular component and valuation maps with a superscript to indicate the arity of their domain and write, e.g., $\mathrm{red}^{\times n}$, $\mathrm{ac}^{\times n}$ and $\mathrm{val}^{\times n}$ (cf. Section 4.4).

Definition 6.3. A rational polyhedral cone in $\mathbb{Q}^n$ is the intersection of finitely many open or closed linear half-spaces, i.e. subsets of the form $\{x \in \mathbb{Q}^n \mid \langle x, v\rangle > 0\}$ or $\{x \in \mathbb{Q}^n \mid \langle x, v\rangle \geq 0\}$, where $v \in \mathbb{Q}^n$ and $\langle\, ,\rangle$ denotes the usual inner product on $\mathbb{R}^n$.

We will reduce Theorem 6.2 to the following special case:

Special Case. There exist an O_K-scheme $\mathbf{X} \subset \mathbb{A}^M_{O_K}$ such that the structure map $\mathbf{X} \to \mathrm{Spec}(O_K)$ is smooth and its non-empty fibers are irreducible of dimension n, an étale map $\underline{\xi} = (\underline{\xi}_1, \ldots, \underline{\xi}_n)\colon \mathbf{X} \to \mathbb{A}^n_{O_K}$, a rational polyhedral cone $\mathcal{D} \subset \mathbb{Q}^n_{\geq 0}$, an O_K-scheme $\mathbf{M} \subset (\mathbb{G}_\mathrm{m}^n)_{O_K} \times \mathbf{X}$, and integers $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that

(1) $X = \mathbf{X} \times \mathrm{Spec}(K)$, i.e. $\mathbf{X}$ is an O_K-model of X,
(2) $\omega = \underline{\xi}^*(dx_1 \wedge \cdots \wedge dx_n)$, where $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n_{O_K}$,
(3) $f_1(z) = \sum_{i=1}^n a_i \,\mathrm{val}(\underline{\xi}_i(z))$,
(4) $f_2(z) = \sum_{i=1}^n b_i \,\mathrm{val}(\underline{\xi}_i(z))$,

(5) the quantifier-free definable set $\mathscr{Z}$ is defined by the formula

$$z \in X \cap \mathcal{O}^M \ \wedge \ \mathrm{val}^{\times n}(\underline{\xi}(z)) \in \mathcal{D} \ \wedge \ \left(\mathrm{ac}^{\times n}(\underline{\xi}(z)), \mathrm{red}^{\times M}(z)\right) \in \mathbf{M}_{\mathsf{k}}.$$

6.2.1. *Proof of Theorem 6.2 in the Special Case.* Let $\mathbf{Z} \subset \mathbf{X}$ be the image of $\mathbf{M}$ under the projection to $\mathbf{X}$. After reordering the coordinates of the map $\underline{\xi}$ and passing, if necessary, to one of the parts of a finite, quantifier-free partition of $\mathbf{M}$, we can assume that, for some $0 \leq t \leq n$, the functions $\underline{\xi}_1, \dots, \underline{\xi}_t$ vanish on $\mathbf{Z} \times \mathrm{Spec}(K)$ and $\underline{\xi}_{t+1}, \dots, \underline{\xi}_n$ are invertible on $\mathbf{Z} \times \mathrm{Spec}(K)$. This implies that, for every finite extension $K \subset L$ and almost all primes $\mathfrak{q}$ of O_L, the functions $\underline{\xi}_1, \dots, \underline{\xi}_t$ vanish on $\mathbf{Z} \times \mathrm{Spec}(O_L/\mathfrak{q})$ and $\underline{\xi}_{t+1}, \dots, \underline{\xi}_n$ are invertible on $\mathbf{Z} \times \mathrm{Spec}(O_L/\mathfrak{q})$. We may thus assume, without loss of generality, that $\mathcal{D} \subset \mathbb{Q}^t_{>0} \times \{0\}^{n-t}$.

For each $\mathrm{p} \in \mathbf{X}(O_L/\mathfrak{q})$, the map $\underline{\xi}$ induces a measure-preserving diffeomorphism between $\left(\left(\mathrm{red}^{\times M}\right)^{-1}(\mathrm{p}) \cap \mathbf{X}(L_\mathfrak{q}), |\omega|_{L_\mathfrak{q}}\right)$ and $\left(\left(\mathrm{red}^{\times n}\right)^{-1}(\underline{\xi}(\mathrm{p})), |dx_1 \wedge \cdots \wedge dx_n|_{L_\mathfrak{q}}\right)$. The image of $\left(\mathrm{red}^{\times M}\right)^{-1}(\mathrm{p}) \cap \mathscr{Z}(L_\mathfrak{q})$ under $\underline{\xi}$ is the set of all $x = (x_1, \dots, x_n) \in O^n_{L,\mathfrak{q}}$ that satisfy

(1) $\mathrm{val}^{\times n}(x) \in \mathcal{D} \cap \mathbb{Z}^n$,
(2) $\mathrm{red}^{\times n}(x) = \underline{\xi}(\mathrm{p})$, and
(3) $(\mathrm{ac}^{\times n}(x), \mathrm{p}) \in \mathbf{M}(O_L/\mathfrak{q})$,

or, equivalently,

(1)' $\mathrm{val}^{\times n}(x) \in \mathcal{D} \cap \mathbb{Z}^n$,
(2)' $(\mathrm{ac}(x_{t+1}), \dots, \mathrm{ac}(x_n)) = \left(\underline{\xi}_{t+1}(\mathrm{p}), \dots, \underline{\xi}_n(\mathrm{p})\right)$,
(3)' $(\mathrm{ac}(x_1), \dots, \mathrm{ac}(x_t)) \in \mathbf{M}^{(\mathrm{p})}(O_L/\mathfrak{q})$,

where

$$\mathbf{M}^{(\mathrm{p})}(O_L/\mathfrak{q}) = \left\{(y_1, \dots, y_t) \in ((O_L/\mathfrak{q})^\times)^t \mid (y_1, \dots, y_t, \underline{\xi}_{t+1}(\mathrm{p}), \dots, \underline{\xi}_n(\mathrm{p}), \mathrm{p}) \in \mathbf{M}(O_L/\mathfrak{q})\right\}.$$

Setting $\mathcal{W}(\mathrm{p}) \subset O^n_{L,\mathfrak{q}}$ to be the set defined by the conjunction of these three conditions, we get

$$\begin{aligned} &\int_{\left(\mathrm{red}^{\times M}\right)^{-1}(\mathrm{p})} 1_{\mathscr{Z}(L_\mathfrak{q})}(z) |O_L/\mathfrak{q}|^{f_1(z) - f_2(z)s} |\omega|_{L_\mathfrak{q}} \\ &= \int_{\left(\mathrm{red}^{\times n}\right)^{-1}(\underline{\xi}(\mathrm{p}))} 1_{\mathcal{W}(\mathrm{p})}(x) |O_L/\mathfrak{q}|^{\sum_{i=1}^n (a_i - b_i s)\,\mathrm{val}(x_i)} |dx_1 \wedge \cdots \wedge dx_n|_{L_\mathfrak{q}} \\ &= \left|\mathbf{M}^{(\mathrm{p})}(O_L/\mathfrak{q})\right| \cdot \sum_{\gamma \in \mathcal{D} \cap \mathbb{Z}^n} |O_L/\mathfrak{q}|^{\sum_{i=1}^n (a_i - b_i s)\gamma_i}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
(6.2) \qquad & \int_{\mathscr{X}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s}|\omega|_{L_{\mathfrak{q}}} \\
&= \int_{\mathbf{X}(O_{L,\mathfrak{q}})} 1_{\mathscr{X}(L_{\mathfrak{q}})}(z)|O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s}|\omega|_{L_{\mathfrak{q}}} \\
&= \sum_{\mathrm{p}\in\mathbf{X}(O_L/\mathfrak{q})} \int_{\left(\mathrm{red}^{\times M}\right)^{-1}(\mathrm{p})} 1_{\mathscr{X}(L_{\mathfrak{q}})}(z)|O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s}|\omega|_{L_{\mathfrak{q}}} \\
&= \sum_{\mathrm{p}\in\mathbf{X}(O_L/\mathfrak{q})} \left|\mathbf{M}^{(\mathrm{p})}(O_L/\mathfrak{q})\right| \cdot \sum_{\gamma\in\mathcal{D}\cap\mathbb{Z}^n} |O_L/\mathfrak{q}|^{\sum_{i=1}^n (a_i-b_is)\gamma_i}.
\end{aligned}
$$

Let $\mathbf{W}$ be the fiber product $\mathbf{M} \times_{(\mathbb{G}_{\mathrm{m}}^{n-t})_{O_K}\times\mathbf{X}} \mathbf{X}$, where the map $\mathbf{M} \to (\mathbb{G}_{\mathrm{m}}^{n-t})_{O_K} \times \mathbf{X}$ is the projection onto the last $n-t+M$ coordinates, and the map $\mathbf{X} \to (\mathbb{G}_{\mathrm{m}}^{n-t})_{O_K} \times \mathbf{X}$ is $(\underline{\xi}_{t+1}, \dots, \underline{\xi}_n) \times \mathrm{Id}_{\mathbf{X}}$. As

$$|\mathbf{W}(O_L/\mathfrak{q})| = \sum_{\mathrm{p}\in\mathbf{X}(O_L/\mathfrak{q})} |\mathbf{M}^{(\mathrm{p})}(O_L/\mathfrak{q})|,$$

(6.2) implies that

$$(6.3) \qquad \int_{\mathscr{X}(L_{\mathfrak{q}})} |O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s}|\omega|_{L_{\mathfrak{q}}} = |\mathbf{W}(O_L/\mathfrak{q})| \cdot \sum_{\gamma\in\mathcal{D}\cap\mathbb{Z}^n} |O_L/\mathfrak{q}|^{\sum_{i=1}^n (a_i-b_is)\gamma_i}.$$

The set $\mathcal{D}\cap\mathbb{Z}^n$ may be decomposed into a finite, disjoint union of cosets of free monoids. We may thus replace $\mathcal{D}\cap\mathbb{Z}^n$ by such a coset. Sums as the ones in (6.3) over free monoids are just finite products of geometric progressions of the form $\frac{|O_L/\mathfrak{q}|^{A-Bs}}{1-|O_L/\mathfrak{q}|^{A-BS}}$, for suitable numerical data $A, B \in \mathbb{Z}$. Translating the monoid amounts to multiplying the relevant product by a factor of the form $|O_L/\mathfrak{q}|^{\alpha-\beta s}$, for suitable $\alpha, \beta \in \mathbb{Z}$. This proves Theorem 6.2 in the Special Case.

6.2.2. *Reduction to the Special Case.* Let $X, \omega, \mathscr{X}, f_1, f_2$ be as in Theorem 6.2. As X is smooth, the integral (6.1) in the theorem is the sum of the respective integrals over the irreducible components of X. Hence, we can assume that X is irreducible.

Lemma 6.4. *If $h\colon \mathcal{O}^M \to \Gamma$ is a quantifier-free definable function, then there exist $N \in \mathbb{N}$, a finite quantifier-free definable partition*

$$\mathcal{O}^M = \Omega_1 \sqcup \cdots \sqcup \Omega_N,$$

and, for each $i \in \{1, \dots, N\}$, finitely many polynomials $P_{i1}, \dots, P_{in_i}$ over K and rational numbers $r_{i1}, \dots, r_{in_i}$ such that $h|_{\Omega_i} = \sum_{j=1}^{n_i} r_{ij} \operatorname{val} \circ P_{ij}$.

Proof. By assumption, the graph of h is a Boolean combination of quantifier-free formulae in $M+1$ variables $x_1, \dots, x_M, \gamma$, where x_i are valued field sort and γ is value group sort.

The quantifier-free formulae, in turn, are Boolean combinations of formulae of the form

$$P(x)=0,\ Q(\mathrm{ac}(R_1(x)),\dots,\mathrm{ac}(R_{N'}(x)))=0,\ \text{and } n_0\gamma+\sum_{j\in J} n_j\,\mathrm{val}(P_j(x))=0,$$

where $x=(x_1,\dots,x_M)$, J is a finite index set, P, $R_1,\dots,R_{N'}$, and P_j, $j\in J$, are polynomials over K, Q is a polynomial over $\mathrm{ac}(K)$, and n_0 and n_j, $j\in J$, are integers. This implies the claim. □

By Lemma 6.4, we can assume that the functions f_1 and f_2 have the form

$$f_1=\sum_{k\in J} r_k\,\mathrm{val}\circ F_k,\quad f_2=\sum_{k\in J} r'_k\,\mathrm{val}\circ F_k,$$

where J is a finite indexing set, $r_k,r'_k\in\mathbb{Q}$, and F_k are regular functions on X. By definition, the definable set $\mathscr{X}$ is a disjoint union of sets that are defined using formulas of the form

$$z\in X\cap\mathcal{O}^M\ \wedge\ \varphi(\mathrm{ac}^{\times M'}(H(z)))\ \wedge\ \psi(\mathrm{val}^{\times M'}(H(z)))\ \wedge\ H'(z)=0, \tag{6.4}$$

where $H,H'\colon X\to\mathbb{A}_K^{M'}$ are regular maps defined over K, φ is a quantifier-free formula in the language of fields (in variables of sort k), and ψ is a quantifier-free formula in the language of ordered groups (in variables of sort $\mathsf{\Gamma}$). Hence, it is enough to prove the theorem assuming that $\mathscr{X}$ is defined by a formula of the form (6.4).

If $H'\neq 0$, then the integral in (6.1) is zero. Hence, we can assume that $H'=0$. The definable set defined by φ is equivalent to the disjoint union of finitely many quasi-affine varieties. Partitioning $\mathscr{X}$ according to these varieties, we can assume that φ defines a single quasi-affine K-variety V. Since we are only interested in evaluating the integral (6.1) over local fields with large residue field characteristic, we may replace V by one of its O_K-models, which we denote by $\mathbf{V}$. We apply a similar argument for ψ: after passing to one of the parts of a finite, quantifier-free partition of $\mathscr{X}$, we can assume that ψ defines a translation of a rational polyhedral cone in $\mathbb{Q}_{\geq 0}^{M'}$ by a vector of the form $\mathrm{val}^{\times M'}(e)$, where $e\in K^{M'}$. For every finite extension $K\subset L$ and almost all $\mathfrak{q}\in\mathrm{Spec}(O_L)$, we have $\mathrm{val}^{\times M'}(e)=0$. We may thus assume that ψ defines a rational polyhedral cone $\mathcal{C}\subset\mathbb{Q}_{\geq 0}^{M'}$. In conclusion, we can assume that $\mathscr{X}$ is defined by the formula

$$z\in X\cap\mathcal{O}^M\ \wedge\ \mathrm{ac}^{\times M'}(H(z))\in\mathbf{V}\ \wedge\ \mathrm{val}^{\times M'}(H(z))\in\mathcal{C}. \tag{6.5}$$

6.2.3. *Resolution of Singularities.* We write $H=(H_1,\dots,H_{M'})$, let $P=\prod_{i=1}^{M'}H_i\cdot\prod_{k\in J}F_k$, and consider the divisor $D=\mathrm{div}(P\cdot\omega)$, i.e. the union of the vanishing loci of ω and P. By Hironaka's theorem on strict resolution of singularities (cf. [1, Definition B.5.1]), applied to the divisor D, there exist $m\in\mathbb{N}$ and a smooth variety $Y\subset X\times\mathbb{P}^m$ defined over K such that the projection $\pi\colon Y\to X$ is birational, is an isomorphism above the complement of D, and the pullback of D under π is a divisor with normal crossings.

Denote the dimension of X by n. By the definition of divisor with normal crossings, there is an open cover $Y=\bigcup_{i\in I}U_i$ by affine K-varieties U_i, for some finite index set I, and, for each $i\in I$, there is an étale map $\xi_i\colon U_i\to\mathbb{A}^n$ such that $\pi^{-1}(\mathrm{supp}(D))\cap U_i$ is contained in the pre-image under ξ_i of the coordinate hyperplanes in $\mathbb{A}^n$. The divisor of $\pi^*\omega$ is supported on $\pi^{-1}(D)$, and $\xi_i^*(dx_1 \wedge \cdots \wedge dx_n)$ is an invertible top differential form. Hence, the function $\frac{\pi^*\omega}{\xi_i^*(dx_1\wedge\cdots\wedge dx_n)}$ is regular and its divisor is supported on $\pi^{-1}(D)$. Therefore, there is a regular function $\vartheta_i : U_i \to \mathbb{G}_\mathrm{m}$ and $a_{ij} \in \mathbb{Z}$, $j \in \{1, \ldots, n\}$, such that, for $y \in U_i$,

$$\frac{\pi^*\omega}{\xi_i^*(dx_1 \wedge \cdots \wedge dx_n)}(y) = \vartheta_i(y) \cdot \prod_{j=1}^{n} (\xi_i(y))_j^{a_{ij}},$$

For each $i \in I$, fix an embedding $U_i \subset \mathbb{A}^N$ for some N. For every finite extension $K \subset L$ and almost all primes $\mathfrak{q}$ of O_L, the function ϑ_i is the restriction of a polynomial in N variables with coefficients in $O_{L,\mathfrak{q}}$. In particular, its restriction to $O_{L,\mathfrak{q}}^N$ has non-negative valuation. The same is true for $1/\vartheta_i$ and so, for almost all $\mathfrak{q}$, the restriction of $\mathrm{val}(\vartheta_i)$ to $U_i(L_\mathfrak{q}) \cap O_{L,\mathfrak{q}}^N$ is 0. Consequently, we obtain, for $y \in U_i(L_\mathfrak{q}) \cap O_{L,\mathfrak{q}}^N$,

$$\mathrm{val}\left(\frac{\pi^*\omega}{\xi_i^*(dx_1 \wedge \cdots \wedge dx_n)}(y)\right) = \sum_{j=1}^{n} a_{ij}\,\mathrm{val}(\xi_i(y)_j).$$

Similarly there are, for $j \in \{1, \ldots, n\}$ and $t \in \{1, \ldots, M'\}$, integers b_{ij}, c_{ij}, d_{itj} and functions $\eta_{it} : U_i \to \mathbb{G}_\mathrm{m}$, such that, for $y \in U_i(L_\mathfrak{q}) \cap O_{L,\mathfrak{q}}^N$,

$$f \circ \pi(y) = \sum_{j=1}^{n} b_{ij}\,\mathrm{val}(\xi_i(y)_j), \quad g \circ \pi(y) = \sum_{j=1}^{n} c_{ij}\,\mathrm{val}(\xi_i(y)_j),$$

and, for $t \in \{1, \ldots, M'\}$,

$$\mathrm{val}(H_t \circ \pi(y)) = \sum_{j=1}^{n} d_{itj}\,\mathrm{val}(\xi_i(y)_j), \quad \mathrm{ac}(H_t \circ \pi(y)) = \mathrm{ac}(\eta_{it}(y)) \cdot \prod_{j=1}^{n} \mathrm{ac}(\xi_i(y)_j)^{d_{itj}}.$$

6.2.4. *Reduction Modulo* $\mathfrak{q}$. Fix a total ordering $<$ on the finite index set I. For $i \in I$ set $Z_i = U_i \smallsetminus \bigcup_{j<i} U_j \hookrightarrow \mathbb{A}^N$, so that $Y = \bigcup_{i \in I} U_i = \bigsqcup_{i \in I} Z_i$, the latter union being disjoint. Further choose O_K-models $\mathbf{X} \subset \mathbb{A}^M_{O_K}$, $\mathbf{Y} \subset \mathbf{X} \times \mathbb{P}^m_{O_K}$, and $\mathbf{Z}_i \subset \mathbb{A}^N_{O_K}$ for the varieties X, Y, and Z_i, for $i \in I$. We also fix O_K-models for the maps η_{it}, for $t \in \{1, \ldots, M'\}$, and ξ_i which – by slight abuse of notation – we continue to denote by these letters.

Lemma 6.5. *For every finite extension $K \subset L$ and almost all primes $\mathfrak{q}$ of O_L,*

$$\pi^{-1}(\mathbf{X}(O_{L,\mathfrak{q}})) = \bigsqcup_{i \in I} \mathrm{red}^{-1}(\mathbf{Z}_i(O_L/\mathfrak{q})).$$

Proof. We first contend that $\pi^{-1}(\mathbf{X}(O_{L,\mathfrak{q}})) = \mathbf{Y}(O_{L,\mathfrak{q}})$ for almost all $\mathfrak{q}$. One containment follows from the projectivity of π, the other follows from the fact that, for almost all $\mathfrak{q}$, the map π is defined over $O_{L,\mathfrak{q}}$. Next, we claim that, for almost all $\mathfrak{q}$, the sets $\mathbf{Z}_i(O_L/\mathfrak{q})$ form a partition of $\mathbf{Y}(O_L/\mathfrak{q})$. Indeed, we know that $\mathbf{Z}_i(\mathbb{C})$ form a partition of $\mathbf{Y}(\mathbb{C})$, so the $\mathbf{Z}_i((O_L/\mathfrak{q})^{\mathrm{alg}})$, $i \in I$, form a partition of $\mathbf{Y}((O_L/\mathfrak{q})^{\mathrm{alg}})$ for almost all $\mathfrak{q}$, using Robinson's

Principle. Since $\mathbf{Z}_i$ and $\mathbf{Y}_i$ are quantifier-free, we get that $\mathbf{Z}_i(O_L/\mathfrak{q})$ form a partition of $\mathbf{Y}(O_L/\mathfrak{q})$ for these primes $\mathfrak{q}$. Altogether this yields, for almost all $\mathfrak{q}$,

$$\mathbf{Y}(O_{L,\mathfrak{q}}) = \bigsqcup_{\mathrm{p}\in\mathbf{Y}(O_L/\mathfrak{q})} \mathrm{red}^{-1}(\mathrm{p}) = \bigsqcup_{i\in I}\bigsqcup_{\mathrm{p}\in\mathbf{Z}_i(O_L/\mathfrak{q})} \mathrm{red}^{-1}(\mathrm{p}) = \bigsqcup_{i\in I}\mathrm{red}^{-1}(\mathbf{Z}_i(O_L/\mathfrak{q})).$$

□

We deduce from Lemma 6.5 that, for every finite extension $K \subset L$ and almost all primes $\mathfrak{q}$ of O_L,

$$\begin{aligned}\int_{\mathscr{X}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s}|\omega|_{L_\mathfrak{q}} &= \int_{\mathbf{X}(O_{L,\mathfrak{q}})} 1_{\mathscr{X}(L_\mathfrak{q})}(x)|O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s}|\omega|_{L_\mathfrak{q}} \\ &= \int_{\pi^{-1}(\mathbf{X}(O_{L,\mathfrak{q}}))} 1_{\mathscr{X}(L_\mathfrak{q})}(\pi(y))|O_L/\mathfrak{q}|^{f_1(\pi(y))-f_2(\pi(y))s}|\pi^*\omega|_{L_\mathfrak{q}} \\ &= \sum_{i\in I}\int_{\mathrm{red}_\mathfrak{q}^{-1}(\mathbf{Z}_i(O_L/\mathfrak{q}))} 1_{\pi^{-1}(\mathscr{X})(L_\mathfrak{q})}(y)|O_L/\mathfrak{q}|^{f_1(\pi(y))-f_2(\pi(y))s}|\pi^*\omega|_{L_\mathfrak{q}}.\end{aligned}$$

Recall from Section 6.2.2 the rational polyhedral cone $\mathcal{C} \subset \mathbb{Q}_{\geq 0}^{M'}$ and the O_K-model $\mathbf{V}$ of the quasi-affine K-variety V, featuring in the definition (6.5) of the quantifier-free definable set $\mathscr{X}$, as well as the various data defined in Section 6.2.3. For each $i \in I$, let $\mathcal{D}_i \subset \mathbb{Q}_{\geq 0}^n$ be the rational polyhedral cone defined by

$$(\gamma_1, \ldots, \gamma_n) \in \mathcal{D}_i \iff \left(\sum_{j=1}^n d_{itj}\gamma_j\right)_{t=1}^{M'} \in \mathcal{C},$$

let $\mathbf{M}_i \subset (\mathbb{G}_\mathrm{m}^n)_{O_K} \times \mathbf{Y}$ be the O_K-scheme defined by

$$(x_1, \ldots, x_n, y) \in \mathbf{M}_i \iff \left(\eta_{it}(y)\prod_{j=1}^n x_j^{d_{itj}}\right)_{t=1}^{M'} \in \mathbf{V},$$

and let $\mathscr{X}_i$ be the definable set defined by

$$y \in U_i \cap \mathcal{O}^{M+m} \wedge \mathrm{val}^{\times n}(\xi_i(y)) \in \mathcal{D}_i \wedge \left(\mathrm{ac}^{\times n}(\xi_i(y)), \mathrm{red}^{\times(M+m)}(y)\right) \in (\mathbf{M}_i)_\mathsf{k}.$$

Then

$$\int_{\mathscr{X}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s}|\omega|_{L_\mathfrak{q}} = \sum_{i\in I}\int_{\mathscr{X}_i(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{\sum_{j=1}^n (a_{ij}+b_{ij}-c_{ij}s)\,\mathrm{val}(\xi_i(y)_j)}|\xi_i^*(dx_1 \wedge \cdots \wedge dx_n)|_{L_\mathfrak{q}},$$

and each summand on the right hand side is of the form covered by the Special Case. This concludes the proof of Theorem 6.2.

6.3. **Proof of Theorem 2.8.** Let Φ be the absolute root system of $\mathbf{G}$. We show that the assertions of the theorem hold for

$$c(\mathbf{G}) = a(\Phi) \cup b(\mathbf{G}),$$

where $a(\Phi) \in \mathcal{A}^+$ is the element constructed in Theorem 3.1 and $b(\mathbf{G})$ is obtained as follows. By Theorem 5.1, there are a quantifier-free definable set $\mathscr{X}$, quantifier-free definable functions $f_1, f_2 \colon \mathscr{X} \to \Gamma$, and a constant $C_1 \in \mathbb{R}$ such that, for every finite extension $K \subset L$ and almost all primes $\mathfrak{q}$ of O_L,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) \sim_{C_1} \int_{\mathscr{X}(L_\mathfrak{q})} |O_L/\mathfrak{q}|^{f_1(z)-f_2(z)s} d\lambda(z).$$

Furthermore, Theorem 6.2 gives that, for almost all primes $\mathfrak{q}$,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}(s) - \zeta_{\mathbf{G}(O_L/\mathfrak{q})}(s) \sim_{C_1} \sum_{i=1}^{N} |O_L/\mathfrak{q}|^{\alpha_i-\beta_i s}|\mathbf{W}_i(O_L/\mathfrak{q})| \cdot \prod_{j=1}^{n_i} \frac{|O_L/\mathfrak{q}|^{A_{ij}-B_{ij}s}}{1-|O_L/\mathfrak{q}|^{A_{ij}-B_{ij}s}}, \tag{6.6}$$

where the $\mathbf{W}_1, \dots, \mathbf{W}_N$ are quasi-affine O_K-schemes and n_i, $A_{ij}, B_{ij}, \alpha_i, \beta_i$ are integers specified in Theorem 6.2. We can assume that the generic fiber of each $\mathbf{W}_i$ is non-empty and irreducible, and we set

$$b(\mathbf{G}) = \left\{ \left(\alpha_i + \dim \mathbf{W}_i + \sum\nolimits_{j=1}^{n_i} A_{ij}, -\beta_i - \sum\nolimits_{j=1}^{n_i} B_{ij} \right) \mid 1 \le i \le N \right\} \in \mathcal{A}^+.$$

Let $Q \subset \operatorname{Spec}(O_L)$ denote the set of all primes $\mathfrak{q}$ such that (6.6) holds. By the Lang–Weil estimates [27], there is a constant $C_2 \in \mathbb{R}$ such that, for each $i \in \{1, \dots, N\}$ and for almost all $\mathfrak{q} \in Q$, either $\mathbf{W}_i(O_L/\mathfrak{q}) = \varnothing$ or

$$\frac{1}{2} \le \frac{|\mathbf{W}_i(O_L/\mathfrak{q})|}{|O_L/\mathfrak{q}|^{\dim \mathbf{W}_i}} \le C_2.$$

For each $\mathfrak{q} \in Q$, there exists $i \in \{1, \dots, N\}$ such that $\mathbf{W}_i(O_L/\mathfrak{q}) \neq \varnothing$, and we define

$$\beta_\mathfrak{q} = \max\{A_{ij}/B_{ij} \mid 1 \le i \le N,\ 1 \le j \le n_i,\ \mathbf{W}_i(O_L/\mathfrak{q}) \neq \varnothing,\ B_{ij} \neq 0\} \in \mathbb{Q}_{>0},$$
$$b_\mathfrak{q} = \left\{ \left(\alpha_i + \dim \mathbf{W}_i + \sum\nolimits_{j=1}^{n_i} A_{ij}, -\beta_i - \sum\nolimits_{j=1}^{n_i} B_{ij} \right) \mid 1 \le i \le N,\ \mathbf{W}_i(O_L/\mathfrak{q}) \neq \varnothing \right\} \in \mathcal{A}^+.$$

We observe that, for each prime $\mathfrak{q} \in Q$, the abscissa of convergence of $\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})}$ is equal to $\beta_\mathfrak{q}$ by (6.6) and $b_\mathfrak{q} \subset b(\mathbf{G})$.

Let $\varepsilon \in \mathbb{R}_{>0}$. Since $|O_L/\mathfrak{q}| \ge 2$ for all $\mathfrak{q} \in \operatorname{Spec}(O_L)$, there is a constant $\delta(\varepsilon) \in \mathbb{R}_{>0}$ such that, for each $i \in \{1, \dots, N\}$, each $\mathfrak{q} \in Q$, and all $\sigma \in \mathbb{R}$ with $\sigma > \beta_\mathfrak{q} + \varepsilon$,

$$\delta(\varepsilon) < \prod\nolimits_j \left(1 - |O_L/\mathfrak{q}|^{A_{ij}-B_{ij}\sigma}\right) \le 1.$$

From (6.6) it follows that there is a constant $C_3(\varepsilon) \in \mathbb{R}$ such that, for every $q \in Q$,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})} - \zeta_{\mathbf{G}(O_L/\mathfrak{q})} \sim_{C_3(\varepsilon)} \xi_{b_\mathfrak{q}, |O_L/\mathfrak{q}|} \quad \text{for } \sigma > \beta_\mathfrak{q} + \varepsilon. \tag{6.7}$$

By Theorem 3.1 and Remark 3.2, there is a constant $C_4 \in \mathbb{R}$ such that, for almost all primes $\mathfrak{q} \in \operatorname{Spec}(O_L)$, there is $a_\mathfrak{q} \subset a(\Phi)$ such that

$$\zeta_{\mathbf{G}(O_L/\mathfrak{q})} - 1 \sim_{C_4} \xi_{a_\mathfrak{q}, |O_L/\mathfrak{q}|}. \tag{6.8}$$

Combining (6.7) and (6.8), we get that, for almost all primes $\mathfrak{q} \in Q$, the following holds: for every $\varepsilon \in \mathbb{R}_{>0}$,

$$\zeta_{\mathbf{G}(O_{L,\mathfrak{q}})} - 1 \sim_{2\max\{C_3(\varepsilon), C_4\}} \xi_{a_\mathfrak{q} \cup b_\mathfrak{q}, |O_L/\mathfrak{q}|} \quad \text{for } \sigma > \beta_\mathfrak{q} + \varepsilon.$$

These primes form a co-finite subset $T(L) \subset \operatorname{Spec}(O_L)$, and this proves the assertion (1) of the theorem.

Assertion (2) of the theorem is derived from the argument above as follows. The set $R_1(L) = \{\mathfrak{q} \in \operatorname{Spec}(O_L) \mid a_\mathfrak{q} = a(\Phi)\}$ is a Chebotarev set by Corollary 3.15. Moreover, the Chebotarev Density Theorem implies that $\{\mathfrak{q} \in Q \mid \forall i \in \{1, \ldots, N\} : \mathbf{W}_i(O_L/\mathfrak{q}) \neq \varnothing\}$ is a Chebotarev set. Hence $R_2(L) = \{\mathfrak{q} \in Q \mid b_\mathfrak{q} = b(\mathbf{G})\}$ is a Chebotarev set. It follows that $R(L) = R_1(L) \cap R_2(L)$ is a Chebotarev set, in particular of positive analytic density. This concludes the proof of Theorem 2.8.

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