Unavoidable collections of balls for processes with isotropic unimodal Green function

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Abstract

Let us suppose that we have a right continuous Markov semigroup on $\mathbb{R}^d$, $d \geq 1$, such that its potential kernel is given by convolution with a function $G_0 = g(|\cdot|)$, where g is decreasing, has a mild lower decay property at zero, and a very weak decay property at infinity. This captures not only the Brownian semigroup (classical potential theory) and isotropic α-stable semigroups (Riesz potentials), but also more general isotropic Lévy processes, where the characteristic function has a certain lower scaling property, and various geometric stable processes.

There always exists a corresponding Hunt process. A subset A of $\mathbb{R}^d$ is called unavoidable, if the process hits A with probability 1, wherever it starts. It is known that, for any locally finite union of pairwise disjoint balls $B(z, r_z)$, $z \in Z$, which is unavoidable, $\sum_{z\in Z} g(|z|)/g(r_z) = \infty$. The converse is proven assuming, in addition, that, for some $\varepsilon > 0$, $|z - z'| \geq \varepsilon|z|(g(|z|)/g(r_z))^{1/d}$, whenever $z, z' \in Z$, $z \neq z'$. It also holds, if the balls are regularly located, that is, if their centers keep some minimal mutual distance, each ball of a certain size intersects Z, and $r_z = g(\phi(|z|))$, where ϕ is a decreasing function.

The results generalize and, exploiting a zero-one law, simplify recent work by A. Mimica and Z. Vondraček.

1 Introduction and main results

Let $\mathbb{P} = (P_t)_{t>0}$ be a right continuous Markov semigroup on $\mathbb{R}^d$, $d \geq 1$, such that its potential kernel $V_0 := \int_0^\infty P_t\, dt$ is given by convolution with a $\mathbb{P}$-excessive function

$$G_0 = g(|\cdot|),$$

where g is a decreasing function on $[0, \infty)$ such that $0 < g < \infty$ on $(0, \infty)$, $\lim_{r\to 0} g(r) = g(0) = \infty$ and the following holds:

(LD) *Lower decay property*: There are $R_0 \geq 0$ and $C_G \geq 1$ such that

$$d\int_0^r s^{d-1} g(s)\, ds \leq C_G\, r^d g(r), \qquad \text{for all } r > R_0. \tag{1.1}$$

(UD) *Upper decay property at infinity*: There are $R_1 \geq 0$, $\eta \in (0,1)$, and $K > 1$ such that

$$g(Kr) \leq \eta g(r), \qquad \text{for all } r > R_1. \tag{1.2}$$

REMARK 1.1. The inequality (1.1) has a very intuitive meaning: If B is a ball with radius r and center 0, and λ_B denotes normed Lebesgue measure on B, then the potential $G\lambda_B := G_0 * \lambda_B$ of λ_B satisfies $G\lambda_B(0) \leq C_G g(r)$ (and hence $G\lambda_B \leq C_G g(r)$ on $\mathbb{R}^d$), where $g(r)$ is the value of G_0 at the boundary of B.

In Section 2 we shall see the following.

1. For the Brownian semigroup (classical potential theory) and isotropic α-stable semigroups (Riesz potentials) we have $g(r) = r^{\alpha-d}$, $\alpha \in (0,2]$, $\alpha < d$, and our assumptions are satisfied with $R_0 = R_1 = 0$. This holds as well for the more general isotropic unimodal Lévy semigroups considered in [11].

2. If (LD) is satisfied for some $R_0 > 0$, then, for *every* $R > 0$, there exists $C_G \geq 1$ such that (1.1) holds for all $r > R$ (and hence the restriction $r_z > R_0$, for all $z \in Z$, imposed below, reduces to the requirement that $\inf_{z\in Z} r_z > 0$). Analogously for (UD).

3. If $\int_0^1 s^{d-1} g(s)\, ds < \infty$ and $g(r) \approx r^{\beta-d}$, $0 < \beta < d$, as $r \to \infty$, then (LD) and (UD) hold with arbitrary $R_0, R_1 \in (0,\infty)$.

Subordinate Brownian semigroups with subordinators having Laplace exponents of the form

$$\phi(\lambda) = \ln^{\delta}(1 + \lambda^{\alpha/2}), \qquad 0 < \delta \leq 1,\ \alpha \in (0,2],\ \alpha < d,$$

provide examples (symmetric geometric stable processes, if $\delta = 1$), where (LD) does not hold with $R_0 = 0$. Here g satisfies

$$g(r) \approx r^{-d} \ln^{-(1+\delta)}(1/r) \text{ as } r \to 0 \quad \text{ and } \quad g(r) \approx r^{\delta\alpha - d} \text{ as } r \to \infty.$$

Let $\mathfrak{X} = (\Omega, \mathfrak{M}, \mathfrak{M}_t, X_t, \theta_t, P^x)$ be an associated Hunt process on $\mathbb{R}^d$ (for its existence see Remark 3.2,1). A Borel measurable set A in $\mathbb{R}^d$ is called *unavoidable*, if

$$P^x[T_A < \infty] = 1 \qquad \text{for every } x \in \mathbb{R}^d,$$

where $T_A(\omega) := \inf\{t \geq 0 \colon X_t(\omega) \in A\}$. Otherwise, it is called *avoidable*, that is, A is avoidable, if there exists $x \in \mathbb{R}^d$ such that $P^x[T_A < \infty] < 1$.

For all $x \in \mathbb{R}^d$ and $r > 0$, let $B(x,r)$ denote the open ball with center x and radius r, and let $\overline{B}(x,r)$ be its closure. Let us introduce two properties for families of balls which, in the the classical case, have already been considered in [3, 4] (and where it does not make a real difference, if we look at open or closed balls, since a union of open balls is unavoidable if and only if the union of the corresponding closed balls is unavoidable; see Remark 3.2,2).

Let Z be a countable set in $\mathbb{R}^d \setminus \{0\}$ and $r_z > R_0$, $z \in Z$, such that the balls $B(z, r_z)$ are pairwise disjoint. We say that the balls $B(z, r_z)$, $z \in Z$, satisfy the *separation condition*, if Z is locally finite and

$$\inf_{z,z' \in Z,\, z \neq z'} \frac{|z - z'|^d}{|z|^d} \frac{g(r_z)}{g(|z|)} > 0. \tag{1.3}$$

We say that they are *regularly located*, if the following holds:

(a) There exists $\varepsilon > 0$ such that $|z - z'| \geq \varepsilon$, for all $z, z' \in Z$, $z \neq z'$.

(b) There exists $R > 0$ such that $B(x, R) \cap Z \neq \emptyset$, for every $x \in \mathbb{R}^d$.

(c) There exists a decreasing function $\phi\colon [0, \infty) \to (0, \infty)$ such that $r_z = \phi(|z|)$.

Our main results are the following (where we might bear in our mind that $1/g(r)$ is approximately the capacity of balls having radius r, that is, the total mass of their equilibrium measure; see Proposition 3.5).

THEOREM 1.2. *If the balls $B(z, r_z)$, $z \in Z$, satisfy the separation condition, then their union A is unavoidable provided*

$$\sum\nolimits_{z \in Z} \frac{g(|z|)}{g(r_z)} = \infty.$$

COROLLARY 1.3. *Suppose that the balls $B(z, r_z)$, $z \in Z$, are regularly located. Then their union A is unavoidable if and only if*

$$\int_1^\infty \frac{r^{d-1} g(r)}{g(\phi(r))}\, dr = \infty.$$

The converse in Theorem 1.2 is already known without any restriction on the balls and assuming only $\lim_{r \to \infty} g(r) = 0$ instead of (UD) (see [8, Theorem 6.8]; the inequality $R_1^{\overline{B}(z, r_z)} \leq g(|z|)/g(r_z)$, which is used in its proof, holds trivially, since g is decreasing).

PROPOSITION 1.4. *Let A be an unavoidable union of balls $B(z, r_z)$, $z \in Z$. Then $\sum_{z \in Z} g(|z|)/g(r_z) = \infty$ and $\sum_{z \in Z} 1/g(r_z) = \infty$.*

REMARK 1.5. 1. In the classical case, Theorem 1.2 is [4, Theorem 6] (for unavoidableness under a weaker separation property see [3]) and Corollary 1.3 is [3, Theorem 2].

2. In the more general case of isotropic unimodal Lévy processes, where the characteristic function satisfies a lower scaling condition (and (LD), (UD) hold with $R_0 = R_1 = 0$), both Theorem 1.2, its converse, and Corollary 1.3 are proven in [11]. We shall use the same method of considering finitely many countable unions of concentric shells, but have to overcome additional difficulties caused by having only a rather weak estimate for the exit distribution of balls (compare [11, Lemma 2.2], going back to [5, Corollary 2], and Proposition 3.7). Nevertheless our proof for Theorem 1.2 can be simpler, since starting with an avoidable union A and an arbitrary $\delta > 0$, we may assume without loss of generality that $P^0[T_A < \infty] < \delta$ (using Proposition 3.3 and translation invariance).

3. If the balls $B(z, r_z)$, $z \in Z$, are regularly located, then

$$\sum\nolimits_{z \in Z} \frac{g(|z|)}{g(r_z)} = \infty \quad \text{if and only if} \quad \int_1^\infty \frac{r^{d-1} g(r)}{g(\phi(r))}\, dr = \infty.$$

This is fairly obvious (see [11, Lemma 4.1]) and allows us to reduce Corollary 1.3 to a consequence of Theorem 1.2 by first treating a simple case (see Proposition 5.2).

In view of the second statement in Proposition 1.4 let us mention the following part of [8, Theorem 6.8] (where only $\lim_{r\to\infty} g(r) = 0$ instead of (UD) is needed). See also [7] for the result in classical potential theory.

THEOREM 1.6. *Suppose that* (LD) *holds with* $R_0 = 0$. *Let* $h\colon (0,1) \to (0,1)$ *with* $\lim_{t\to 0} h(t) = 0$, *let* $\varphi \in \mathcal{C}(\mathbb{R}^d)$, $\varphi > 0$, *and* $\delta > 0$. *Then there exist a locally finite set* Z *in* $\mathbb{R}^d$ *and* $0 < r_z \le \varphi(z)$, $z \in Z$, *such that the balls* $\overline{B}(z, r_z)$ *are pairwise disjoint, the union of all* $\overline{B}(z, r_z)$ *is unavoidable, and*

$$\sum\nolimits_{z\in Z} h(r_z)/g(r_z) < \delta.$$

In Section 2, we shall first take a closer look at the properties (LD) and (UD) and then show that our assumptions cover the isotropic unimodal processes considered in [11] and geometric stable processes. In Section 3, we shall discuss some general potential theory of the semigroup $\mathbb{P}$, where, as in [8], at the beginning (LD) and (UD) are replaced by the weaker properties $\int_0^1 r^{d-1} g(r)\, dr < \infty$ and $\lim_{r\to\infty} g(r) = 0$. In Section 4, we prove Theorem 1.2, and the proof of Corollary 1.3 is given in Section 5.

2 Examples

Let us first consider an arbitrary positive decreasing function g on $(0,\infty)$ and write down a few elementary facts justifying, in particular, our statements in Remark 1.1.

Given $R_0 \ge 0$, we say that (LD) holds on (R_0,∞), if there exists $C \ge 1$ such that

$$d\int_0^r s^{d-1} g(s)\, ds \le C r^d g(r), \qquad \text{for every } r > R_0. \tag{2.1}$$

Similarly, given $0 \le R_1 < \infty$, we say that (UD) holds on (R_1,∞), if there exist $K > 1$ and $\eta \in (0,1)$ such that

$$g(Kr) \le \eta g(r), \qquad \text{for every } r > R_1. \tag{2.2}$$

LEMMA 2.1. 1. *If there is a function* $\varphi > 0$ *on* $(0,1)$ *with* $\int_0^1 \gamma^{d-1}\varphi(\gamma)\, d\gamma < \infty$ *and*

$$g(\gamma r) \le \varphi(\gamma) g(r), \qquad \textit{for all } \gamma \in (0,1) \textit{ and } r > 0,$$

then (LD) *holds on* $(0,\infty)$.

2. *Let* $f(r) := r^d g(r)$, $R_0 \ge 0$, $\kappa, C \in (0,\infty)$. *If* $\int_0^1 s^{-1} f(s)\, ds < \infty$, $f \ge \kappa$ *on* (R_0,∞), *and*

$$\int_{R_0}^r s^{-1} f(s)\, ds \le C f(r), \qquad \textit{for every } r > R_0,$$

then (LD) *holds on* (R_0,∞).

3. *If* $0 < R < R_0$ *and* (LD) *holds on* (R_0,∞), *then* (LD) *holds on* (R,∞).

Proof. 1. For every $r > 0$,

$$\int_0^r s^{d-1} g(s)\, ds = r^d \int_0^1 \gamma^{d-1} g(\gamma r)\, d\gamma \le r^d g(r) \int_0^1 \gamma^{d-1} \varphi(\gamma)\, d\gamma.$$

2. Clearly, $c := \int_0^{R_0} s^{d-1} g(s)\, ds < \infty$. For every $r > R_0$,

$$\int_0^r s^{d-1} g(s)\, ds = c + \int_{R_0}^r s^{-1} f(s)\, ds \le (c\kappa^{-1} + C) f(r) = (c\kappa^{-1} + C) r^d g(r).$$

3. Let $0 < R < R_0 < R_1$ and assume that (2.1) holds. Defining $\tilde{C} := C(R_1/R)^d$ we obtain that, for every $r \in [R, R_0]$,

$$d \int_0^r s^{d-1} g(s)\, ds \le C R_1^d g(R_1) = \tilde{C} R^d g(R_1) \le \tilde{C} r^d g(r).$$

□

LEMMA 2.2. *Let $0 \le R_1 < \infty$.*

1. *If there is a function $\varphi > 0$ on (R, ∞), $R > 0$, with $\lim_{\lambda \to \infty} \varphi(\lambda) = 0$ and $g(\lambda r) \le \varphi(\lambda) g(r)$, for all $\lambda \ge R$ and $r > R_1$, then* (UD) *holds on (R_1, ∞).*
2. *If $0 < R < R_1$ and* (UD) *holds on (R_1, ∞), then* (UD) *holds on (R, ∞).*
3. *If* (UD) *holds on (R_1, ∞), then, for every $\delta > 0$, there exists $K > 1$ such that $g(Kr) \le \delta g(r)$ for every $r > R_1$.*

Proof. 1. We take $K \ge R$ such that $\varphi(K) < \eta$.
2. Let $c := R_1/R$, $r > R$. Then $cr > R_1$ and $g(Kcr) \le \eta g(cr) \le \eta g(r)$.
3. We choose $m \in \mathbb{N}$ such that $\eta^m < \delta$ and replace K by K^m. □

If $0 < \alpha < d$ and $g(r) = r^{\alpha - d}$, then, by Lemmas 2.1 and 2.2, (LD) and (UD) hold on $(0, \infty)$. So our assumptions are satisfied by Brownian motion and isotropic α-stable processes with $0 < \alpha \le 2$, $\alpha < d$.

Let us observe next that, more generally, our assumptions are satisfied by the isotropic unimodal Lévy processes $\mathfrak{X} = (X_t, P^x)$ studied in [11], where the characteristic function ψ for $\mathfrak{X}$ (characterized by $e^{-t\psi(|x|)} = E^0[e^{i\langle x, X_t\rangle}]$, $t > 0$) is supposed to satisfy the following *weak lower scaling condition*: There exist $\alpha > 0$ and $C_L > 0$ such that

$$\psi(\lambda r) \ge C_L \lambda^\alpha \psi(r), \qquad \text{for all } \lambda \ge 1 \text{ and } r > 0$$

(see [11, (1.4)] and the subsequent list of examples in [11]). Then, by [11, Lemma 2.1] (see also [5, Proposition 1, Theorem 3]), there exists a constant $C \ge 1$ such that, for all $r > 0$ and $0 < \gamma \le 1$,

$$\frac{C^{-1}}{r^d \psi(1/r)} \le \quad g(r) \quad \le \frac{C}{r^d \psi(1/r)}, \tag{2.3}$$

$$C^{-1} \gamma^{2-d} g(r) \le \quad g(\gamma r) \quad \le C \gamma^{\alpha - d} g(r). \tag{2.4}$$

By Lemma 2.1,1 and the second inequality of (2.4), (UD) holds on $(0, \infty)$. Replacing, in the first inequality of (2.4), r by λr and γ by $1/\lambda$, we see that $g(\lambda r) \le C \lambda^{2-d} g(r)$,

for all $r > 0$ and $\lambda \geq 1$. Hence (UD) holds on $(0, \infty)$, by Lemma 2.2,1, provided $d \geq 3$. For the case $d \leq 2$, see [11, Section 6].

Further, since the transition kernels P_t are given by convolution with positive functions p_t (see, for example, [9]) satisfying $p_s * p_t = p_{s+t}$, $s, t > 0$, we have

$$G_0 = \int_0^\infty p_t \, dt \in E_{\mathbb{P}}.$$

Moreover, the separation condition (1.6) in [11, Theorem 1.1] is our separation condition (1.3).

Now let us look at a subordinate Brownian semigroup, where $\alpha \in (0, 2]$, $\alpha < d$, $0 < \delta \leq 1$, and the Laplace exponent of the subordinator is

$$\phi(\lambda) = \ln^\delta(1 + \lambda^{\alpha/2}).$$

If $\delta = 1$, then, by [13, Theorem 3.2 and Remark 3.3]),

$$g(r) \approx r^{-d} \ln^{-2}(1/r) \text{ as } r \to 0 \quad \text{and} \quad g(r) \approx r^{\alpha - d} \text{ as } r \to \infty,$$

and (LD) certainly does not hold with $R_0 = 0$, since, for $r > 0$,

$$\int_0^r s^{-1} \ln^{-2}(1/s) \, ds = \ln^{-1}(1/r). \tag{2.5}$$

In the general case[1], we have $\phi'(\lambda)/\phi^2(\lambda) \approx \lambda^{-1} \ln^{-(1+\delta)}(\lambda)$. By [10, Proposition 4.5], we obtain that

$$g(r) \approx r^{-d-2} \phi'(r^{-2})/\phi^2(r^{-2}) \approx r^{-d} \ln^{-(1+\delta)}(1/r) \quad \text{as } r \to 0.$$

Further, by [12, Theorem 3.3], $g(r) \approx r^{\delta\alpha - d}$ as $r \to \infty$. Thus, by Lemmas 2.1 and 2.2, our assumptions in Section 1 are satisfied taking any $R_0, R_1 \in (0, \infty)$.

3 Potential theory of $\mathbb{P}$

For the moment, let us assume that the right continuous semigroup $\mathbb{P}$ is only sub-Markov and that, instead of (LD) and (UD),

$$\int_0^1 r^{d-1} g(r) \, dr < \infty \quad \text{and} \quad \lim_{r \to \infty} g(r) = 0. \tag{3.1}$$

Let $\mathcal{B}(\mathbb{R}^d)$ ($\mathcal{C}(\mathbb{R}^d)$, respectively) denote the set of all Borel measurable numerical functions (continuous real functions, respectively) on $\mathbb{R}^d$. We recall that the potential kernel $V_0 = \int_0^\infty P_t \, dt$ is given by

$$V_0 f(x) := G_0 * f(x) = \int G_0(x - y) f(y) \, dy, \qquad f \in \mathcal{B}^+(\mathbb{R}^d), \, x \in \mathbb{R}^d.$$

Let $E_{\mathbb{P}}$ denote the set of all $\mathbb{P}$-excessive functions, that is, $E_{\mathbb{P}}$ is the set of all $v \in \mathcal{B}^+(X)$ such that $\sup_{t>0} P_t v = v$. We note that $V_0(\mathcal{B}^+(\mathbb{R}^d)) \subset E_{\mathbb{P}}$. If $f \in \mathcal{B}^+(\mathbb{R}^d)$ is bounded and has compact support, then $V_0 f \in \mathcal{C}(\mathbb{R}^d)$ and $V_0 f$ vanishes at infinity, by (3.1). This leads to the following results in [8, Section 6] (for the definition of balayage spaces and their connection with sub-Markov semigroups see [2], [6], or [8, Section 8]).

[1]The author is indebted to T. Grzywny for informations in this case.

THEOREM 3.1. *$(\mathbb{R}^d, E_{\mathbb{P}})$ is a balayage space such that every point in $\mathbb{R}^d$ is polar and Borel measurable finely open sets $U \neq \emptyset$ have strictly positive Lebesgue measure.*

REMARK 3.2. 1. There exists a Hunt process $\mathfrak{X} = (\Omega, \mathfrak{M}, \mathfrak{M}_t, X_t, \theta_t, P^x)$ on $\mathbb{R}^d$ with transition semigroup $\mathbb{P}$ (see [2, IV.7.6]).

2. Every open ball $B(x,r)$, $x \in \mathbb{R}^d$, $r > 0$, is finely dense in the closed ball $\overline{B}(x,r)$ (see [8, Proposition 6.4]; the fine topology is the coarsest topology such that every function in $E_{\mathbb{P}}$ is continuous).

For every subset A of $\mathbb{R}^d$, we have a reduced function R_1^A:

$$R_1^A := \inf\{v \in E_{\mathbb{P}} \colon v \geq 1 \text{ on } A\}.$$

Obviously, $R_1^A \leq 1$, since $1 \in E_{\mathbb{P}}$. Hence $R_1^A = 1$ on A. If A is open, then $R_1^A \in E_{\mathbb{P}}$. For a general subset A, the greatest lower semicontinuous minorant $\hat{R}_1^A$ of R_1^A (which is also the greatest finely lower semicontinuous minorant of R_1^A) is contained in $E_{\mathbb{P}}$. It is known that $\hat{R}_1^A = R_1^A$ on A^c. If A is Borel measurable, then, for every $x \in \mathbb{R}^d$,

$$R_1^A(x) = P^x[T_A < \infty] \tag{3.2}$$

(see [2, VI.3.14]). The zero-one law (3.3) will be the key to our proofs of both Theorem 1.2 and Corollary 1.3.

PROPOSITION 3.3. *Suppose that $\mathbb{P}$ is a Markov semigroup. Then the constant function 1 is harmonic and, for each set A in $\mathbb{R}^d$,*

$$R_1^A = 1 \quad \text{or} \quad \inf_{x \in \mathbb{R}^d} R_1^A(x) = 0. \tag{3.3}$$

Proof. Having $P_t 1 = 1$, for every $t > 0$, we know, by [8, Theorem 6.3,4]), that 1 is harmonic. Moreover, by [8, Proposition 2.3,1]), (3.3) holds. □

To illustrate that (3.3) is almost trivial, let us suppose that $R_1^A \in E_{\mathbb{P}}$ (which is true in our applications) and let $\gamma := \inf_{x \in \mathbb{R}^d} R_1^A(x)$. Since $E_{\mathbb{P}}$ is a cone, we trivially have $R_{1-\gamma}^A = (1-\gamma) R_1^A \in E_{\mathbb{P}}$. So $v := \gamma + R_{1-\gamma}^A \in E_{\mathbb{P}}$ and $v = 1$ on A, hence $v \geq R_1^A$. Moreover, $w := R_1^A - \gamma \in E_{\mathbb{P}}$ and $w = 1 - \gamma$ on A, hence $w \geq R_{1-\gamma}^A$. Therefore $R_1^A = v = \gamma + R_{1-\gamma}^A = \gamma + (1-\gamma) R_1^A$. Thus $\gamma R_1^A = \gamma$, that is, $\gamma = 0$ or $R_1^A = 1$.

For all $x, y \in \mathbb{R}^d$, let

$$G_y(x) := G(x,y) := G_0(x-y),$$

and let us recall that, by definition, a potential is a positive superharmonic function with greatest harmonic minorant 0. The next result is essentially [8, Theorem 6.6].

THEOREM 3.4. 1. *The function G is symmetric and continuous.*

2. *For every $y \in \mathbb{R}^d$, G_y is a potential with superharmonic support $\{y\}$.*

3. *If μ is a measure on $\mathbb{R}^d$ with compact support, then $G\mu := \int G_y \, d\mu(y)$ is a potential, and the support of μ is the superharmonic support of $G\mu$.*

4. *For every potential p on $\mathbb{R}^d$, there exists a (unique) measure μ on $\mathbb{R}^d$ such that $p = G\mu$.*

For every ball B let $|B|$ denote the Lebesgue measure of B and let λ_B denote normalized Lebesgue measure on B (the measure on B having density $1/|B|$ with respect to Lebesgue measure).

Let us now fix $R_0 \geq 0$ and assume that (1.1) holds, that is, for $r > R_0$,

$$G\lambda_{B(0,r)}(0) = \frac{1}{|B(0,r)|}\int_{B(0,r)} G_y(0)\,dy \leq C_G\, g(r). \tag{3.4}$$

Then, in fact (see [8, (6.9)]),

$$G\lambda_{B(0,r)} \leq C_G g(r), \quad \text{for every } r > R_0. \tag{3.5}$$

Moreover, since $g(r/2) \leq g$ on $(0, r/2)$ and $d\int_0^{r/2} s^{d-1}\,ds = (r/2)^d$, we see that there exists $1 \leq C_D \leq 2^d C_G$ such that, for all $r > R_0$,

$$g(r/2) \leq C_D g(r) \qquad \text{(doubling property)}. \tag{3.6}$$

To simplify our estimates, let us define, once and for all,

$$c := \max\{C_D, C_G\}.$$

If B is an open ball, then $R_1^B = R_1^{\overline{B}}$ is a continuous potential and, by Theorem 3.4, there exists a unique measure μ on $\overline{B}$, the *equilibrium measure* for B, such that $R_1^B = G\mu$. For measures ν on $\mathbb{R}^d$, let $\|\nu\|$ denote their total mass. The following holds (cf. [8, Proposition 6.7] in the case $r_0 = \infty$; assuming $r > R_0$ its simple proof carries over word by word).

PROPOSITION 3.5. *Let $r > R_0$, $B := B(0,r)$, and let μ be the equilibrium measure for B. Then*

$$\frac{g(|\cdot|)}{g(r)} \geq R_1^B \geq c^{-1}\,\frac{g(|\cdot|+r)}{g(r)} \quad \text{and} \quad c^{-1}\,\frac{1}{g(r)} \leq \|\mu\| \leq c\,\frac{1}{g(r)}.$$

The following well known fact will be used in the proofs of Proposition 3.7 and Lemma 4.1. By (3.2), it is an immediate consequence of the strong Markov property. For the convenience of the reader we write down its short proof (a corresponding argument based on iterated balayage can be given using [2, VI.2.9]).

LEMMA 3.6. *Let A be a Borel measurable set in an open set $U \subset \mathbb{R}^d$ and $\gamma > 0$ such that $R_1^A \leq \gamma$ on U^c. Then $P^x[T_A < T_{U^c}] \geq R_1^A(x) - \gamma$, for every $x \in U$.*

Proof. Let $\tau := T_{U^c}$ and $x \in U$. We obviously have the identity

$$[T_A < \infty] \setminus [T_A < \tau] = [\tau \leq T_A < \infty] = [\tau \leq T_A] \cap \theta_\tau^{-1}([T_A < \infty]).$$

Since $X_\tau \in U^c$ on $[\tau < \infty]$, the strong Markov property yields that

$$P^x([\tau < T_A] \cap \theta_\tau^{-1}[T_A < \infty]) = \int_{[\tau < T_A]} P^{X_\tau}[T_A < \infty]\,dP^x \leq \gamma,$$

and hence $P^x[T_A < \infty] - P^x[T_A < \tau] \leq \gamma$. □

For every $r > 0$, we introduce the (closed) shell

$$S(r) := \overline{B}(0, 3r) \setminus B(0, r).$$

The following estimate of the probability for hitting a shell $S(r)$ before leaving a ball $B(0, Mr)$, M large, will be sufficient for us (see [11, Lemma 2.2], going back to [5, Corollary 2], for a much stronger estimate which is used [11]).

PROPOSITION 3.7. *Let $r > R_0$, $\eta := c^{-3}/2$, $M > 3$, and $g((M-2)r) \le \eta g(r)$. Then*

$$P^0[T_{S(r)} < T_{B(0,Mr)^c}] \ge \eta. \tag{3.7}$$

Proof. We choose $z \in \partial B(0, 2r)$ and take $B := B(z, r)$. Then B is contained in $S(r)$. By Proposition 3.5,

$$R_1^B(0) \ge c^{-1} \frac{g(|z| + r)}{g(r)} = c^{-1} \frac{g(3r)}{g(r)} \ge c^{-3} = 2\eta, \tag{3.8}$$

whereas, for every $y \in B(0, Mr)^c$,

$$R_1^B(y) \le \frac{g(|y - z|)}{g(r)} \le \frac{g((M-2)r)}{g(r)} \le \eta. \tag{3.9}$$

The proof is finished by Lemma 3.6. □

The next simple result on comparison of potentials will be sufficient for us (see the proof of [8, Theorem 5.3] for a much more delicate version; cf. also the proof of [1, Theorem 3]).

LEMMA 3.8. *Let $Z \subset \mathbb{R}^d$ be finite and $r_z > R_0$, $z \in Z$, such that, for $z \ne z'$, $B(z, r_z) \cap B(z', 3r_{z'}) = \emptyset$. Let $w \in E_{\mathbb{P}}$ and, for every $z \in Z$, let μ_z, ν_z be measures on $\overline{B}(z, r_z)$ such that $G\mu_z \in \mathcal{C}(\mathbb{R}^d)$, $G\mu_z \le w$, and $\|\mu_z\| \le \|\nu_z\|$. Then $\mu := \sum_{z \in Z} \mu_z$ and $\nu := \sum_{z \in Z} \nu_z$ satisfy*

$$G\mu \le w + cG\nu. \tag{3.10}$$

Proof. Let $z, z' \in Z$, $z' \ne z$, and $x \in \overline{B}(z, r_z)$. For all $y, y' \in \overline{B}(z', r_{z'})$, $|y - y'| \le 2r_{z'} \le |x - y'|$, hence $R_0 < r_{z'} < |x - y| \le 2|x - y'|$ and $g(|x - y'|) \le cg(|x - y|)$. By integration, $G\mu_{z'}(x) \le cG\nu_{z'}(x)$. Therefore

$$G\mu(x) = G\mu_z(x) + \sum\nolimits_{z' \in Z, z' \ne z} G\mu_{z'}(x) \le w(x) + cG\nu(x). \tag{3.11}$$

Thus $G\mu \le w + cG\nu$ on the union of the balls $\overline{B}(z, r_z)$, $z \in Z$. By the minimum principle [2, III.6.6], the proof is finished. □

REMARK 3.9. If each $G\mu_z$ is only bounded by some potential in $\mathcal{C}(\mathbb{R}^d)$, but there exists $\gamma > 1$ such that $B(z, \gamma r_z) \cap B(z', 3r_{z'}) = \emptyset$, whenever $z \ne z'$, then (3.11) holds for all $x \in B(z, \gamma r_z)$, $z \in Z$, and (3.10) follows as well.

4 Proof of Theorem 1.2

From now on let us suppose that the assumptions introduced at the beginning of Section 1 are satisfied. We recall that in many cases (LD) and (UD) hold with $R_0 = 0$ and $R_1 = 0$. If not, we may assume without loss of generality that R_0 and R_1, respectively, while being strictly positive, are as small as we want.

We prepare the proof of Theorem 1.2 by a first application of Lemma 3.8.

LEMMA 4.1. *Let $\rho > \max\{R_0, R_1\}$, $0 < \varepsilon \le 1/4$. Let Z be a finite subset of $S(\rho)$ and $R_0 < r_z \le |z|/4$, $z \in Z$, such that the balls $B(z, 4r_z)$ are pairwise disjoint and*

$$|z - z'| \ge 4\varepsilon|z|\big(g(|z|)/g(r_z)\big)^{1/d}, \qquad \text{whenever } z \ne z'.$$

Let $C := 1 + (4/\varepsilon)^d c^3$, $\delta := (2Cc^4)^{-1}$, $M > 4$, and suppose $g((M-3)\rho) \le \delta g(\rho)$.

Then the union A of the balls $B(z, r_z)$, $z \in Z$, satisfies

$$P^x[T_A < T_{B(0,M\rho)^c}] \ge \delta \sum\nolimits_{z\in Z} g(|z|)/g(r_z), \qquad \text{for every } x \in B(0, 3\rho).$$

Proof. Let $B := B(0, 4\rho)$. By (3.5),

$$G\lambda_B \le cg(4\rho). \tag{4.1}$$

For $z \in Z$, let

$$\tilde r_z := \max\big\{r_z, \varepsilon|z|\big(g(|z|)/g(r_z)\big)^{1/d}\big\}$$

so that $B(z, \tilde r_z) \cap B(z', 3\tilde r_{z'}) = \emptyset$, whenever $z \ne z'$.

For the moment, fix $z \in Z$. Since $\max\{r_z, \varepsilon|z|\} \le |z|/4 < \rho$ and $g(|z|)/g(r_z) \le 1$, we know that $B(z, \tilde r_z) \subset B$. Moreover,

$$\tilde r_z^{-d} \le \varepsilon^{-d}\,\frac{g(r_z)}{|z|^d g(|z|)} \le \varepsilon^{-d}\,\frac{g(r_z)}{\rho^d g(4\rho)}. \tag{4.2}$$

Let μ_z be the equilibrium measure for $B(z, r_z)$, that is, $G\mu_z = R_1^{B(z,r_z)}$. Then $\|\mu_z\| \le cg(r_z)^{-1}$, by Proposition 3.5. We define

$$\nu_z := \|\mu_z\|\lambda_{B(z,\tilde r_z)} = \beta_z 1_{B(z,\tilde r_z)}\lambda_B, \tag{4.3}$$

where, by (4.2),

$$\beta_z = \|\mu_z\|\,\frac{|B|}{|B(z,\tilde r_z)|} \le cg(r_z)^{-1}(4\rho/\tilde r_z)^d \le (4/\varepsilon)^d cg(4\rho)^{-1} =: \beta. \tag{4.4}$$

Let $\nu := \sum_{z\in Z}\nu_z$. Since the balls $B(z, \tilde r_z)$, $z \in Z$, are pairwise disjoint subsets of B, we conclude, by (4.3), (4.4), and (4.1), that

$$G\nu \le \beta G\lambda_B \le (4/\varepsilon)^d c^2. \tag{4.5}$$

Next let $\mu := \sum_{z\in Z}\mu_z$ so that

$$p := \sum\nolimits_{z\in Z} R_1^{B(z,r_z)} = G\mu.$$

By Lemma 3.8, $G\mu \le 1 + cG\nu$, and hence $p \le C$, by (4.5) and our definition of C. Therefore, by the minimum principle [2, III.6.6], we obtain that $C^{-1}p \le R_1^{\overline{A}} = R_1^A$. Trivially, $R_1^A \le p$. Thus

$$C^{-1}p \le R_1^A \le p. \tag{4.6}$$

Let $U := B(0, M\rho)$ and $z \in Z$. By Proposition 3.5, for $y \in U^c$,

$$g(r_z)R_1^{B(z,r_z)}(y) \le g(|y - z|) \le g((M-3)\rho) \le \delta g(\rho),$$

whereas, for every $x \in B(0, 3\rho)$,

$$g(r_z)R_1^{B(z,r_z)}(x) \ge c^{-1}g(|x - z| + \rho) \ge c^{-1}g(7\rho) \ge c^{-4}g(\rho).$$

Defining $\gamma := \sum_{z\in Z} g(\rho)/g(r_z)$ we hence see, by (4.6), that

$$R_1^A \le \delta\gamma \quad \text{on } U^c, \qquad R_1^A \ge 2\delta\gamma \quad \text{on } B(0, 3\rho).$$

By Lemma 3.6, for every $x \in B(0, 3\rho)$,

$$P^x[T_A < T_{B(0,M\rho)^c}] \ge \delta\gamma.$$

Observing that $g(\rho) \ge g$ on $S(\rho)$ the proof is finished. □

Now let us fix a locally finite subset Z of $\mathbb{R}^d \setminus \{0\}$ and $r_z > 4R_0$, $z \in Z$, such that the balls $B(z, r_z)$ are pairwise disjoint and satisfy the separation condition (1.3). Let A denote the union of these balls. For a proof of Theorem 1.2 we show the following.

PROPOSITION 4.2. *If A is avoidable, then $\sum_{z\in Z} g(|z|)/g(r_z) < \infty$.*

Proof. So let us suppose that A is avoidable. To prove that $\sum_{z\in Z} g(|z|)/g(r_z) < \infty$ we may assume that $|z| > 8R_0$, for every $z \in Z$ (we simply omit finitely many points from Z). Further, we may assume that the balls $B(z, 4r_z)$ are pairwise disjoint. Indeed, since $g(r) \le g(r/4) \le c^2 g(r)$, $r > R_0$, a replacement of r_z by $r_z/4$ does neither affect (1.3) nor the convergence of $\sum_{z\in Z} g(|z|)/g(r_z)$, and the new, smaller union is, of course, avoidable. Moreover, similarly as at the beginning of the proof of [11, Theorem 1.1]), we may assume without loss of generality that

$$r_z \le |z|/8, \qquad \text{for every } z \in Z. \tag{4.7}$$

Indeed, replacing r_z by $r'_z := \min\{r_z, |z|/8\}$ our assumptions are preserved as well. Suppose we have shown that $\sum_{z\in Z} g(|z|)/g(r'_z) < \infty$. Since $g(|z|)/g(|z|/8) \ge c^{-3}$, we see that the set Z' of all $z \in Z$ such that $r'_z = |z|/8$ is finite, and hence certainly $\sum_{z\in Z'} g(|z|)/g(r_z) < \infty$. So we may assume without loss of generality that $r'_z = r_z$, for all $z \in Z$, that is, (4.7) holds.

By (1.3), we may choose $0 < \varepsilon < 1/4$ such that, for $z, z' \in Z$, $z \ne z'$,

$$|z - z'| \ge 8c^{1/d}\varepsilon|z|\big(g(|z|)/g(r_z)\big)^{1/d}. \tag{4.8}$$

As in Lemma 4.1, we define

$$C := 1 + (4/\varepsilon)^d c^3, \qquad \delta := (2Cc^4)^{-1}.$$

By Lemma 2.2, there exists $M := 3^m$, $m \in \mathbb{N}$, such that

$$g((M-3)\rho) \le \delta g(\rho), \qquad \text{for every } \rho > R_1.$$

Moreover, let us define

$$R := 1 + \max\{R_0, R_1\}.$$

By Proposition 3.3, there is a point x_0 in $\mathbb{R}^d$ such that

$$P^{x_0}[T_A < \infty] = R_1^A(x_0) < \delta/2. \tag{4.9}$$

Deleting finitely many points from Z, we obtain $Z \cap B(0, 2|x_0| + R) = \emptyset$. Then, for every $z \in Z$,

$$|z|/2 \le |z - x_0| \le 2|z|, \qquad c^{-1} g(|z|) \le g(|z - x_0|) \le c g(|z|). \tag{4.10}$$

Hence, by (4.7) and (4.8), $r_z < |z - x_0|/4$ and, for $z, z' \in Z$, $z \ne z'$,

$$|z - z'| \ge 4\varepsilon |z - x_0| \big(g(|z - x_0|)/g(r_z)\big)^d. \tag{4.11}$$

By translation invariance, we may therefore assume without loss of generality that $x_0 = 0$, $Z \cap B(0, R) = \emptyset$, and (4.11) holds instead of (4.8).

For every $0 \le j < m$, let

$$Z_j := \bigcup\nolimits_{n=0}^{\infty} Z \cap S(3^{nm+j} R).$$

Then Z is the union of $Z_0, Z_1, \dots, Z_{m-1}$. Therefore it suffices to show that

$$\sum\nolimits_{z \in Z_j} g(|z|)/g(r_z) < \infty, \qquad \text{for every } 0 \le j < m. \tag{4.12}$$

So let us fix $0 \le j < m$. For the moment, we also fix $n \in \{0, 1, 2, \dots\}$ and define $\rho := 3^{nm+j} R$,

$$S := T_{S(\rho)}, \quad \tau := T_{B(0,\rho)^c}, \quad \tau' := T_{B(0,M\rho)^c}, \quad T := \min\{T_A, \tau'\}.$$

By Lemma 4.1,

$$P^y[T_A < \tau'] \ge \delta \sum\nolimits_{z \in Z \cap S(\rho)} g(|z|)/g(r_z), \quad \text{for every } y \in S(\rho).$$

By Proposition 3.7, $P^0[S < \tau'] \ge \delta$, and hence, by (4.9),

$$P^0[S < T] \ge P^0[S < \tau'] - P^0[T_A < \infty] > \delta/2.$$

Clearly, $S + T_A \circ \theta_S = T_A$ and $S + \tau' \circ \theta_S = \tau'$ on $[S < T]$. Hence

$$[S < T_A < \tau'] = [S < T, T_A < \tau'] = [S < T] \cap \theta_S^{-1}([T_A < \tau']).$$

Since $X_S \in S(\rho)$ on $[S < \infty]$, the strong Markov property yields that

$$P^0[S < T_A < \tau'] = \int_{[S<T]} P^{X_S}[T_A < \tau']\, dP^0 \ge (\delta^2/2) \sum\nolimits_{z \in Z \cap S(\rho)} \frac{g(|z|)}{g(r_z)}.$$

Of course, $\tau \le S$. Hence the sets $[S < T_A < \tau']$, obtained for different n, are pairwise disjoint subsets of $[T_A < \infty]$ (recall that $M = 3^m$). Thus, by (4.9),

$$\sum\nolimits_{z \in Z_j} g(|z|)/g(r_z) \le (2/\delta^2) P^0[T_A < \infty] \le 1/\delta.$$

□

Let us note that the preceding proof could also be presented in a purely analytic way using iterated balayage of measures.

5 Proof of Corollary 1.3

Again we suppose that the assumptions from the beginning of Section 1 are satisfied. Let Z be a countable set in $\mathbb{R}^d$ and $r_z > 0$, $z \in Z$, such that the balls $B(z, r_z)$ are pairwise disjoint and regularly located. So there exist $\varepsilon, R \in (0, \infty)$ such that the points in Z have a mutual distance which is at least ε and every open ball of radius R contains some point of Z. Moreover, $r_z = \phi(|z|)$, where the function ϕ is decreasing. If (LD) does not hold with $R_0 = 0$, we assume that $\kappa := \inf_{x \in \mathbb{R}^d} \phi(x) > 0$. By Lemma 2.1, we then know that (LD) holds, if we define $R_0 := \kappa/8$. By Lemma 2.2, (UD) holds with, say, $R_1 := R_0 + 1$.

Of course, we may assume that $R \geq 1 + \phi(1)$. We already know (see Remark 1.5,3 and Proposition 1.4) that it suffices to show that the union A of all $B(z, r_z)$, $z \in Z$, is unavoidable provided

$$\sum\nolimits_{z \in Z} g(|z|)/g(r_z) = \infty. \tag{5.1}$$

So let us suppose that (5.1) holds. Moreover, let us assume for the moment that

$$\limsup\nolimits_{\rho \to \infty} \rho^d g(\rho)/g(\phi(\rho)) < \infty. \tag{5.2}$$

Then $\beta := \inf_{z \in Z} g(r_z)(|z|^d g(|z|))^{-1} > 0$. Since $|z - z'| \geq \varepsilon > 0$, whenever $z \neq z'$, this implies that

$$\inf\nolimits_{z, z' \in Z, z \neq z'} \frac{|z - z'|^d}{|z|^d} \, \frac{g(r_z)}{g(|z|)} \geq \varepsilon^d \beta.$$

Hence the balls $B(z, r_z)$, $z \in Z$, satisfy the separation condition (1.3), and A is unavoidable, by Theorem 1.2. Thus already the following lemma would finish the proof of Corollary 1.3.

LEMMA 5.1. *If* $\limsup_{\rho \to \infty} \rho^d g(\rho)/g(\phi(\rho)) = \infty$, *the set* A *is unavoidable.*

For Lévy processes considered in [11], this is [11, Lemma 4.2]. However, its proof (by contradiction) is almost as involved as the proof of [11, Theorem 1.1].

By Proposition 3.3, we only have to show that $\inf_{x \in \mathbb{R}^d} R_1^A(x) > 0$. Hence a second application of Lemma 3.8, which is yet another variation of the arguments for quasi-additivity of capacities in [1], will allow us even to prove the following.

PROPOSITION 5.2. *Suppose that* $\limsup_{\rho \to \infty} \rho^d g(\rho)/g(\phi(\rho)) > \eta > 0$. *Then the union* A *of all* $B(z, r_z)$, $z \in Z$, *is unavoidable.*

Proof. We define

$$a := (2c)^{-1}(18R)^{-d}, \qquad b := c^2 R^{-d},$$

and fix $x \in \mathbb{R}^d$. There exists $\rho > 9R + 2|x| + 4R_1$ such that

$$\gamma := \rho^d g(\rho)/g(\phi(\rho)) > \eta. \tag{5.3}$$

Let

$$r := \phi(\rho), \qquad B := B(0, \rho) \quad \text{and} \quad S := \overline{B}(0, \rho/2) \setminus B(\rho/4).$$

There exist finitely many points $y_1, \dots, y_m \in S$ such that $B(y_1, 3R), \dots, B(y_m, 3R)$ are pairwise disjoint and S is covered by the balls $B(y_1, 9R), \dots, B(y_m, 9R)$. Obviously, $m \geq (1/2)(\rho/18R)^d$. There exist points $z_j \in Z \cap B(y_j, R)$, $1 \leq j \leq m$. Then, for all $i, j \in \{1, \dots, m\}$ with $i \neq j$, $|z_i - z_j| \geq |y_i - y_j| - 2R \geq 4R$, and hence

$$B(z_i, R) \cap B(z_j, 3R) = \emptyset. \tag{5.4}$$

Let $1 \leq j \leq m$. Clearly, $\rho \geq \rho/2 + R \geq |z_j| \geq \rho/4 - R \geq R \geq 1$, and hence

$$r = \phi(\rho) \leq \phi(|z_j|) = r_{z_j} \leq \phi(1) \leq R. \tag{5.5}$$

Moreover, $r + |x - z_j| \leq R + |x| + \rho/2 + R \leq \rho$, and hence $g(|x - z_j| + r) \geq g(\rho)$. So, by translation invariance and Proposition 3.5,

$$R_1^{B(z_j,r)}(x) \geq c^{-1} g(|x - z_j| + r)/g(r) \geq c^{-1} g(\rho)/g(r). \tag{5.6}$$

Let

$$A_x := B(z_1, r) \cup \cdots \cup B(z_m, r) \quad \text{and} \quad p := \sum\nolimits_{j=1}^{m} R_1^{B(z_j,r)}.$$

Then $A_x \subset A$, by (5.5). So, by (5.6) and our definitions of γ, a, and r,

$$R_1^{A_x} \leq R_1^A \quad \text{and} \quad p(x) \geq mc^{-1} g(\rho)/g(r) \geq a\gamma. \tag{5.7}$$

Now let μ_0 denote the equilibrium measure for $B(0, r)$, $G\mu_0 = R_1^{B(0,r)}$. By Proposition 3.5, $\|\mu_0\| \leq cg(r)^{-1}$. We define

$$\nu_j := \|\mu_0\| \lambda_{B(z_j,R)} = \|\mu_0\| (\rho/R)^d 1_{B(z_j,R)} \lambda_B, \qquad 1 \leq j \leq m,$$

and $\nu := \sum_{j=1}^m \nu_j$. Since $B(z_1, R), \dots, B(z_j, R)$ are pairwise disjoint subsets of B and $G\lambda_B \leq cg(\rho)$, we see that

$$G\nu \leq \|\mu_0\| (\rho/R)^d G\lambda_B \leq c^2 R^{-d} g(r)^{-1} \rho^d g(\rho) = b\gamma.$$

For every $1 \leq j \leq m$, $R_1^{B(z_j,r)} = G\mu_j$, where μ_j is obtained from μ_0 translating by z_j. Let $\mu := \sum_{j=1}^m \mu_j$. By (5.4), (5.5), and Lemma 3.8,

$$p = G\mu \leq 1 + cG\nu \leq 1 + cb\gamma.$$

Since μ is supported by $\overline{A}_x$ and p is continuous, we get that

$$R_1^{A_x} = R_1^{\overline{A}_x} \geq (1 + cb\gamma)^{-1} p,$$

by the minimum principle [2, III.6.6]. In particular,

$$R_1^A(x) \geq R_1^{A_x}(x) \geq \frac{a\gamma}{1 + cb\gamma} = \frac{a}{\gamma^{-1} + cb} > \frac{a}{\eta^{-1} + cb}$$

by (5.7) and (5.3). Thus A is unavoidable, by Proposition 3.3. □

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