# A LOWER BOUND FOR THE NUMBER OF NEGATIVE EIGENVALUES OF SCHRÖDINGER OPERATORS

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ABSTRACT. We prove a lower bound for the number of negative eigenvalues for a Schödinger operator on a Riemannian manifold via the integral of the potential.

## 1. Introduction

Let $(M, g)$ be a compact Riemannian manifold without boundary. Consider the following eigenvalue problem on $M$:

$$-\Delta u - Vu = \lambda u, \tag{1}$$

where $\Delta$ is the Laplace-Beltrami operator on $M$ and $V \in L^\infty(M)$ is a given potential. It is well-known, that the operator $-\Delta - V$ has a discrete spectrum. Denote by $\{\lambda_k(V)\}_{k=1}^\infty$ the sequence of all its eigenvalues arranged in increasing order, where the eigenvalues are counted with multiplicity.

Denote by $\mathcal{N}(V)$ the number of negative eigenvalues of (1), that is,

$$\mathcal{N}(V) = \operatorname{card}\{k : \lambda_k(V) < 0\}.$$

It is well-known that $\mathcal{N}(V)$ is finite. Upper bounds of $\mathcal{N}(V)$ have received enough attention in the literature, and for that we refer the reader to [2], [5], [12], [11], [15] and references therein.

However, a little is known about lower estimates. Our main result is the following theorem. We denote by $\mu$ the Riemannian measure on $M$.

**Theorem 1.1.** *Set $\dim M = n$. For any $V \in L^\infty(M)$ the following inequality is true:*

$$\mathcal{N}(V) \geq \frac{C}{\mu(M)^{n/2-1}} \left( \int_M V d\mu \right)_+^{n/2}, \tag{2}$$

*where $C > 0$ is a constant that in the case $n = 2$ depends only on the genus of $M$ and in the case $n > 2$ depends only on the conformal class of $M$.*

In the case $V \geq 0$ the estimate (2) was proved in [6, Theorems 5.4 and Example 5.12]. Our main contribution is the proof of (2) for signed potentials $V$ (as it was conjectured in [6]), with the same constant $C$ as in [6]. In fact, we reduce the case of a signed $V$ to the case of non-negative $V$ by considering a certain variational problem for $V$ and by showing that the solution of this problem is non-negative. The latter method originates from [14].

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In the case $n = 2$, inequality (2) takes the form

$$\mathcal{N}(V) \geq C \int_M V d\mu. \tag{3}$$

For example, the estimate (3) can be used in the following situation. Let $M$ be a two-dimensional manifold embedded in $\mathbb{R}^3$ and the potential $V$ be of the form $V = \alpha K + \beta H$ where $K$ is the Gauss curvature, $H$ is the mean curvature, and $\alpha, \beta$ are real constants (see [8], [4]). In this case (3) yields

$$\mathcal{N}(V) \geq C\left(K_{total} + H_{total}\right),$$

where $K_{total}$ is the total Gauss curvature and $H_{total}$ is the total mean curvature. We expect in the future many other applications of (2)-(3).

## 2. A variational problem

Fix positive integers $k, N$ and consider the following optimization problem: find $V \in L^\infty(M)$ such that

$$\int_M V d\mu \to \max \ \text{ under restrictions } \lambda_k(V) \geq 0 \text{ and } \|V\|_{L^\infty} \leq N. \tag{4}$$

Clearly, the functional $V \mapsto \int_M V d\mu$ is weakly continuous in $L^\infty(M)$. Since the class of potentials $V$ satisfying the restrictions in (4) is bounded in $L^\infty(M)$, it is weakly precompact in $L^\infty(M)$. In fact, we prove in the next lemma that this class is weakly compact, which will imply the existence of the solution of (4).

**Lemma 2.1.** *The class*

$$C_{k,N} = \left\{V \in L^\infty(M) : \lambda_k(V) \geq 0 \text{ and } \|V\|_{L^\infty} \leq N\right\}$$

*is weakly compact in $L^\infty(M)$. Consequently, the problem (4) has a solution $V \in L^\infty(M)$.*

*Proof.* It was already mentioned that the class $C_{k,N}$ is weakly precompact in $L^\infty(M)$. It remains to prove that it is weakly closed, that is, for any sequence $\{V_i\} \subset C_{k,N}$ that converges weakly in $L^\infty$, the limit $V$ is also in $C_{k.N}$. The condition $\|V\|_{L^\infty} \leq N$ is trivially satisfied by the limit potential, so all we need is to prove that $\lambda_k(V) \geq 0$. Let us use the minmax principle in the following form:

$$\lambda_k(V) = \inf_{\substack{E \subset W^{1,2}(M) \\ \dim E = k}} \sup_{u \in E \setminus \{0\}} \frac{\int_M |\nabla u|^2 d\mu - \int_M V u^2 d\mu}{\int_M u^2 d\mu},$$

where $E$ is a subspace of $W^{1,2}(M)$ of dimension $k$. The condition $\lambda_k(V) \geq 0$ is equivalent then to the following:

$$\begin{aligned} &\forall E \subset W^{1,2}(M) \text{ with } \dim E = k \quad \forall \varepsilon > 0 \quad \exists u \in E \setminus \{0\} \\ &\text{such that } \int_M |\nabla u|^2 d\mu - \int_M V u^2 d\mu \geq -\varepsilon \int_M u^2 d\mu. \end{aligned} \tag{5}$$

Fix a subspace $E \subset W^{1,2}(M)$ of dimension $k$ and some $\varepsilon > 0$. Since $\lambda_k(V_i) \geq 0$, we obtain that there exists $u_i \in E \setminus \{0\}$ such that

$$\int_M |\nabla u_i|^2 d\mu - \int_M V_i u_i^2 d\mu \geq -\varepsilon \int_M u_i^2 d\mu. \tag{6}$$

Without loss of generality we can assume that $\|u_i\|_{W^{1,2}(M)} = 1$. Then the sequence $\{u_i\}$ lies on the unit sphere in the finite-dimensional space $E$. Hence, it has a convergent (in $W^{1,2}(M)$-norm) subsequence. We can assume that the whole sequence $\{u_i\}$ converges in $E$ to some $u \in E$ with $\|u\|_{W^{1,2}(M)} = 1$. It remains to verify that $u$ satisfies the inequality (5). By construction we have

$$\int_M |\nabla u_i|^2 \, d\mu \to \int_M |\nabla u|^2 \, d\mu \quad \text{and} \quad \int_M u_i^2 d\mu \to \int_M u^2 d\mu.$$

Next we have

$$\begin{aligned} \left| \int_M V_i u_i^2 d\mu - \int_M V u^2 d\mu \right| &\leq \left| \int_M \left( V_i u_i^2 - V_i u^2 \right) d\mu \right| + \left| \int_M \left( V_i u^2 - V u^2 \right) d\mu \right| \\ &\leq N \|u_i - u\|_{L^2}^2 + \left| \int_M (V_i - V) \, u^2 d\mu \right|. \end{aligned}$$

By construction we have $\|u_i - u\|_{L^2} \to 0$ as $i \to \infty$. Since $u^2 \in L^1(M)$, the $L^\infty$ weak convergence $V_i \rightharpoonup V$ implies that

$$\int_M (V_i - V) \, u^2 d\mu \to 0 \quad \text{as } i \to \infty.$$

Hence, the inequality (5) follows from (6). □

**Lemma 2.2.** *If $N$ is large enough (depending on $k$ and $M$) then any solution $V$ of (4) satisfies $\lambda_k(V) = 0$.*

*Proof.* Assume that $\lambda_k(V) > 0$ and bring this to a contradiction. Consider the family of potentials

$$V_t = (1-t)V + tN \quad \text{where } t \in [0,1].$$

Since $V_t \geq V$, we have by a well-known property of eigenvalues that $\lambda_k(V_t) \leq \lambda_k(V)$. By continuity we have, for small enough $t$, that $\lambda_k(V_t) > 0$. Clearly, we have also $|V_t| \leq N$. Hence, $V_t$ satisfies the restriction of the problem (4), at least for small $t$. If $\mu\{V < N\} > 0$ then we have for all $t > 0$

$$\int_M V_t > \int_M V,$$

which contradicts the maximality of $V$. Hence, we should have $V = N$ a.e.. However, if $N > \lambda_k(-\Delta)$ then $\lambda_k(-\Delta - N) < 0$ and $V \equiv N$ cannot be a solution of (4). This contradiction finishes the proof. □

## 3. Proof of Theorem 1.1

The main part of the proof of Theorem 1.1 is contained in the following lemma.

**Lemma 3.1.** *Let $V_{\max}$ be a maximizer of the variational problem (4). Then $V_{\max}$ satisfies the inequality*

$$V_{\max} \geq 0 \text{ a.e. on } M$$

3.1. **Proof of Theorem 1.1 assuming Lemma 3.1.** Choose $N$ large enough, say

$$N > \sup_M |V|.$$

Set $k = \mathcal{N}(V) + 1$ so that $\lambda_k(V) \geq 0$. For the maximizer $V_{\max}$ of (4) we have

$$\int_M V\, d\mu \leq \int_M V_{\max}\, d\mu.$$

On the other hand, since $V_{\max} \geq 0$, we have by [6]

$$\mathcal{N}(V_{\max}) \geq \frac{C}{\mu(M)^{n/2-1}} \left( \int_M V_{\max}\, d\mu \right)^{n/2}.$$

Also , we have

$$\lambda_k(V_{\max}) \geq 0,$$

which implies

$$\mathcal{N}(V_{\max}) \leq k - 1 = \mathcal{N}(V).$$

Hence, we obtain

$$\mathcal{N}(V) \geq \mathcal{N}(V_{\max}) \geq \frac{C}{\mu(M)^{n/2-1}} \left( \int_M V_{\max}\, d\mu \right)^{n/2} \geq \frac{C}{\mu(M)^{n/2-1}} \left( \int_M V\, d\mu \right)_+^{n/2},$$

which was to be proved.

3.2. **Some auxiliary results.** Before we can prove Lemma 3.1, we need some auxiliary lemmas. The following lemma can be found in [9].

**Lemma 3.2.** *Let $V(t,x)$ be a function on $\mathbb{R} \times M$ such that, for any $t \in \mathbb{R}$, $V(t,\cdot) \in L^\infty(M)$ and $\partial_t V(t,\cdot) \in L^\infty(M)$. For any $t \in \mathbb{R}$, consider the Schrödinger operator $L_t = -\Delta - V(t,\cdot)$ on $M$ and denote by $\{\lambda_k(t)\}_{k=1}^\infty$ the sequence of the eigenvalues of $L_t$ counted with multiplicities and arranged in increasing order. Let $\lambda$ be an eigenvalue of $L_0$ with multiplicity $m$; moreover, let*

$$\lambda = \lambda_{k+1}(0) = \ldots = \lambda_{k+m}(0).$$

*Let $U_\lambda$ be the eigenspace of $L_0$ that corresponds to the eigenvalue $\lambda$ and $\{u_1, \ldots, u_m\}$ be an orthonormal basis in $U_\lambda$. Set for all $i, j = 1, \ldots, m$*

$$Q_{ij} = \int_M \left. \frac{\partial V}{\partial t} \right|_{t=0} u_i u_j d\mu.$$

*and denote by $\{\alpha_i\}_{i=1}^m$ the sequence of the eigenvalues of the matrix $\{Q\}_{i,j=1}^m$ counted with multiplicities and arranged in increasing order. Then we have the following asymptotic, for any $i = 1, \ldots, m$,*

$$\lambda_{k+i}(t) = \lambda_{k+i}(0) - t\alpha_i + o(t) \text{ as } t \to 0.$$

The following lemma is multi-dimensional extension of [14, Lemmas 3.4,3.6]. Given a connected open subset $\Omega$ of $M$ with smooth boundary, the Dirichlet problem

$$\begin{cases} \Delta u = 0 \;\; mboxin\ \Omega \\ u|_{\partial\Omega} = f \end{cases}$$

has for any $f \in C(\partial\Omega)$ a unique solution that can be represented in the form

$$u(y) = \int_{\partial\Omega} Q(x, y) f(x) d\sigma(x)$$

for any $y \in \Omega$, where $Q(x, y)$ is the Poisson kernel of this problem and $\sigma$ is the surface measure on $\partial\Omega$. For any $y \in \Omega$, the function $q(x) = Q(x, y)$ on $\partial\Omega$ will be called the Poisson kernel at the source $y$. Note that $q(x)$ is continuous, positive and

$$\int_{\partial\Omega} q d\sigma = 1.$$

**Lemma 3.3.** *Let $\Omega$ be a connected open subset of $M$ with smooth boundary and $x_0$ be a point in $\Omega$. Then, for any constant $N \geq 1$ there exists $\varepsilon = \varepsilon(\Omega, N, x_0) > 0$ such that for any measurable set $E \subset \Omega$ with*

$$\mu(E) \leq \varepsilon$$

*and for any positive solution $v \in C^2(\Omega)$ of the inequality*

$$\Delta v + Wv \geq 0 \text{ in } \Omega, \tag{7}$$

*where*

$$W = \begin{cases} N & \text{in } E, \\ -\frac{1}{N} & \text{in } \Omega \setminus E, \end{cases} \tag{8}$$

*the following inequality holds*

$$v(x_0) < \int_{\partial\Omega} v\, q d\sigma, \tag{9}$$

*where $q$ is the Poisson kernel of the Laplace operator at the source $x_0$.*

*Proof.* For any $\delta > 0$ denote by $A_\delta$ the set of points in $\Omega$ at the distance $\leq \delta$ from $\partial\Omega$ (see Fig. 1) and consider the potential $V_\delta$ in $\Omega$ defined by

$$V_\delta = \begin{cases} N & \text{in } A_\delta, \\ -\frac{1}{N} & \text{in } \Omega \setminus A_\delta. \end{cases} \tag{10}$$

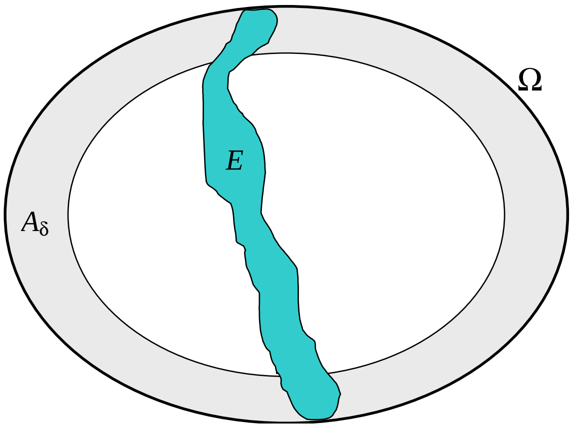


FIGURE 1.

Since $\left\|V_\delta^+\right\|_{L^p(\Omega)}$ can be made sufficiently small by the choice of $\delta > 0$, the following boundary value problem has a unique positive solution:

$$\begin{cases} \Delta w + V_\delta w = 0 \text{ in } \Omega \\ w = f \text{ on } \partial\Omega, \end{cases} \tag{11}$$

for any positive continuous function $f$ on $\partial\Omega$. Denote by $q_\delta(x)$, $x \in \partial\Omega$, the Poisson kernel of (11) at the source $x_0$. Letting $\delta \to 0$, we obtain that the solution of (11) converges to that of

$$\begin{cases} \Delta w - \frac{1}{N} w = 0 \text{ in } \Omega \\ w = f \text{ on } \partial\Omega. \end{cases} \tag{12}$$

Denoting by $q_0$ the Poisson kernel of (12) at the source $x_0$, we obtain that $q_\delta \searrow q_0$ on $\partial\Omega$ as $\delta \searrow 0$ and, moreover, the convergence is uniform.

Let $q$ be the Poisson kernel of the Laplace operator $\Delta$ in $\Omega$, as in the statement of the theorem. Since any solution of (12) is strictly subharmonic in $\Omega$, we obtain that $q_0 < q$ on $\partial\Omega$. In particular, there is a constant $\eta > 0$ depending only on $\Omega, N, x_0$ such that

$$q_0 < (1-\eta)\, q \text{ on } \partial\Omega.$$

Since the convergence $q_\delta \to q$ is uniform on $\partial\Omega$, we obtain that, for small enough $\delta$ (depending on $\Omega, N, x_0$),

$$q_\delta < (1-\eta/2)\, q \text{ on } \partial\Omega.$$

Fix such $\delta$. Consequently, we obtain for the solution $w$ of (11) that

$$w(x_0) < (1-\eta/2) \int_{\partial\Omega} fq d\sigma. \tag{13}$$

Note that the function $W$ from (8) can be increased without violating (7). Define a new potential $W_\delta$ by

$$W_\delta = \begin{cases} N & \text{in } A_\delta \cup E, \\ -\frac{1}{N} & \text{in } \Omega \setminus A_\delta \setminus E. \end{cases} \tag{14}$$

Observe that, for any $p > 1$

$$\left\|W_\delta^+\right\|_{L^p(\Omega)}^p \leq N^p \left(\mu(A_\delta) + \varepsilon\right),$$

so that by the choice of $\varepsilon$ and further reducing $\delta$ this norm can be made arbitrarily small. By a well-known fact (see [13]), if $\left\|W_\delta^+\right\|_{L^p(\Omega)}$ is sufficiently small, then the operator $-\Delta - W_\delta$ in $\Omega$ with the Dirichlet boundary condition on $\partial\Omega$ is positive definite, provided $p = n/2$ for $n > 2$ and $p > 1$ for $n = 2$.

So, we can assume that the operator $-\Delta - W_\delta$ is positive definite. In particular, the following boundary value problem

$$\begin{cases} \Delta u + W_\delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = v \end{cases} \tag{15}$$

has a unique positive solution $u$. Comparing this with (7) and using the maximum principle for the operator $\Delta + W_\delta$, we obtain $u \geq v$ in $\Omega$. Since $u = v$ on $\partial\Omega$, the required inequality (9) will follow if we prove that

$$u(x_0) < \int_{\partial\Omega} uq d\sigma. \tag{16}$$

Set $\Omega_\delta = \Omega \setminus A_\delta$ and prove that

$$\sup_{\Omega_\delta} u \leq C \int_{\partial\Omega} u d\sigma, \tag{17}$$

for some constant $C$ that depends on $\Omega, N, \delta, n$. By choosing $\varepsilon$ and $\delta$ sufficiently small, the norm $\|W_\delta\|_{L^p}$ can be made arbitrarily small for any $p$. Hence, function $u$ satisfies the Harnack inequality

$$\sup_{\Omega_\delta} u \leq C \int_{\Omega_\delta} u d\mu \tag{18}$$

where $C$ depends on $\Omega, N, \delta$ (see [1], [7]). Let $h$ be the solution of the following boundary value problem

$$\begin{cases} -\Delta h - W_\delta h = 1_{\Omega_\delta} \text{ in } \Omega \\ h = 0 \text{ on } \partial\Omega. \end{cases}$$

where $\Omega_\delta = \Omega \setminus A_\delta$. Since $\|W_\delta\|_{L^q}$ is bounded for any $q$, we obtain by the known a priori estimates, that

$$\|h\|_{W^{2,p}(\Omega)} \leq C \|1_{\Omega_\delta}\|_{L^p(\Omega)},$$

where $p > 1$ is arbitrary and $C$ depends on $\Omega, N, \delta, p$ (see [10]). Choose $p > n$ so that by the Sobolev embedding

$$\|h\|_{C^1(\Omega)} \leq C \|h\|_{W^{2,p}(\Omega)}.$$

Since $\|1_{\Omega_\delta}\|_{L^p(\Omega)}$ is uniformly bounded, we obtain by combining the above estimates that

$$\|h\|_{C^1(\Omega)} \leq C,$$

with a constant $C$ depending on $\Omega, N, \delta, n$.

Multiplying the equation $-\Delta h - W_\delta h = 1_{\Omega_\delta}$ by $u$ and integrating over $\Omega$, we obtain

$$\int_{\Omega_\delta} u d\mu = \int_{\partial\Omega} \frac{\partial h}{\partial\nu} u \, d\sigma \leq C \int_{\partial\Omega} u d\sigma$$

which together with (18) implies (17).

Let $w$ be the solution (11) with the boundary condition $f = u$, that is,

$$\begin{cases} \Delta w + V_\delta w = 0 \text{ in } \Omega \\ w = u \text{ on } \partial\Omega. \end{cases}$$

Let us consider the difference

$$\varphi = u - w.$$

Clearly, we have in $\Omega$

$$\Delta\varphi + V_\delta\varphi = (\Delta u + V_\delta u) - (\Delta w + V_\delta w) = (V_\delta - W_\delta) u$$

and $\varphi = 0$ on $\partial\Omega$. Denoting by $G_{V_\delta}$ the Green function of the operator $-\Delta - V_\delta$ in $\Omega$ with the Dirichlet boundary condition, we obtain

$$\varphi(x_0) = \int_\Omega G_{V_\delta}(x_0, y)\,(W_\delta - V_\delta)\, u(y)\, d\mu(y).$$

Since we are looking for an upper bound for $\varphi(x_0)$, we can restrict the integration to the domain $\{V_\delta \le W_\delta\}$. By (14) and (10) we have

$$\{V_\delta \le W_\delta\} = (\Omega \setminus A_\delta) \cap (A_\delta \cup E) = E \setminus A_\delta =: E'$$

and, moreover, on $E'$ we have

$$W_\delta - V_\delta = N + \frac{1}{N} < 2N,$$

whence it follows that

$$\varphi(x_0) \le 2N \int_{E'} G_{V_\delta}(x_0, y)\, u(y)\, d\mu(y).$$

Using (17) to estimate here $u(y)$, we obtain

$$\varphi(x_0) \le 2NC \left( \int_{E'} G_{V_\delta}(x_0, y)\, d\mu(y) \right) \int_{\partial\Omega} u d\sigma$$

Since $\mu(E') \le \varepsilon$ and the Green function $G_{V_\delta}(x_0, \cdot)$ is integrable, we see that $\int_{E'} G_{V_\delta}(x_0, \cdot)\, d\mu$ can be made arbitrarily small by choosing $\varepsilon > 0$ small enough. Choose $\varepsilon$ so small that

$$2NC \int_{E'} G_{V_\delta}(x_0, y)\, d\mu(y) < \eta/2 \inf_{\partial\Omega} q,$$

which implies that

$$\varphi(x_0) < \eta/2 \int_{\partial\Omega} uq d\sigma.$$

Since by (13)

$$w(x_0) < (1 - \eta/2) \int_{\partial\Omega} uq d\sigma,$$

we obtain

$$u(x_0) = \varphi(x_0) + w(x_0) < \int_{\partial\Omega} uq d\sigma,$$

which was to be proved. □

Let $V_{\max}$ be a solution of the problem (4). Denote by $U$ the eigenspace of $-\Delta - V_{\max}$ associated with the eigenvalue $\lambda_k(V_{\max}) = 0$ assuming that $N$ is sufficiently large.

**Lemma 3.4.** *Fix some $c > 0$ and consider the set*

$$F = \{V_{max} \le -c\}.$$

*Then, for any Lebesgue point $x \in F$, then there exists a non-negative function $q \in L^\infty(M)$ such that*

(1) $\int_M q\, d\mu = 1$;
(2) *for any $u \in U \setminus \{0\}$ we have*

$$u^2(x) < \int_M u^2 q\, d\mu. \tag{19}$$

*Proof.* Set $V = V_{\max}$. Any function $u \in U$ satisfies $\Delta u + Vu = 0$, which implies by a simple calculation that the function $v = u^2$ satisfies

$$\Delta v + 2Vv \geq 0.$$

Next, we apply Lemma 3.3 with $J = \max(2N, \frac{1}{2c})$. Choose $r$ so small that the density of the set $F$ in $B(x,r)$ is sufficiently close to 1, namely,

$$\mu\left(F \cap B\left(x,r\right)\right) > \left(1-\varepsilon\right)\mu\left(B\left(x,r\right)\right),$$

where $\varepsilon = \varepsilon\left(J\right)$ is given in Lemma 3.3. Since $h \leq 2N \leq J$ in $B\left(x,r\right)$ and

$$\begin{aligned}\mu\left(\left\{h > -\frac{1}{J}\right\} \cap B\left(x,r\right)\right) &\leq \mu\left(\left\{h > -2c\right\} \cap B\left(x,r\right)\right) \\ &= \mu\left(\left\{V > -c\right\} \cap B\left(x,r\right)\right) \\ &< \varepsilon\mu\left(B\left(x,r\right)\right),\end{aligned}$$

all the hypotheses of Lemma 3.3 are satisfied. Let $q$ be the function that exists by Lemma 3.3 in some small ball $B\left(x,r\right)$. Extending $q$ by setting $q = 0$ outside $B\left(x,r\right)$ we obtain a desirable function. □

3.3. **Proof of main Lemma 3.1.** We can now prove Lemma 3.1, that is, that $V_{\max} \geq 0$. Consider again the set

$$F = \left\{V_{max} \leq -c\right\},$$

where $c > 0$. We want to show that, for any $c > 0$,

$$\mu(F) = 0,$$

which will imply the claim. Assume the contrary, that is $\mu(F) > 0$ for some $c > 0$. Denote by $F_L$ the set of Lebesgue points of $F$. For any $x \in F_L$ denote by $q_x$ the function $q$ that is given by Lemma 3.4. For $x \notin F_L$ set $q_x = \delta_x$. Then $x \mapsto q_x$ is a Markov kernel and, for all $x \in M$ and $u \in U$

$$u^2\left(x\right) \leq \int_M u^2 q_x d\mu. \tag{20}$$

Denote by $\mathcal{M}$ the set of all probability measures on $M$. Define on $\mathcal{M}$ a partial order: $\nu_1 \preceq \nu_2$ if and only if

$$\int_M u^2 d\nu_1 \leq \int_M u^2 d\nu_2 \text{ for all } u \in U \setminus \left\{0\right\}. \tag{21}$$

Define $\nu_0 \in \mathcal{M}$ by

$$d\nu_0 = \frac{1}{\mu\left(F_L\right)} \mathbf{1}_{F_L} d\mu$$

and measure $\nu_1 \in \mathcal{M}$ by

$$\nu_1 = \int_M q_x d\nu_0\left(x\right).$$

Since $\nu_0(F_L) > 0$, we obtain for any $u \in U \setminus \{0\}$ that

$$\begin{aligned}\int_M u^2 d\nu_1 &= \int_M \left( \int_M u^2 q_x d\mu \right) d\nu_0(x) \\ &\geq \int_{F_L} \left( \int_M u^2 q_x d\mu \right) d\nu_0(x) + \int_{M\setminus F_L} \left( \int_M u^2 q_x d\mu \right) d\nu_0(x) \\ &> \int_{F_L} u^2(x)\, d\nu_0(x) + \int_{M\setminus F_L} u^2(x)\, d\nu_0(x) \\ &= \int_M u^2 d\nu_0. \end{aligned} \tag{22}$$

In particular, we have $\nu_0 \preceq \nu_1$. Consider the following subset of $\mathcal{M}$:

$$\mathcal{M}_1 = \{\nu \in \mathcal{M} : \nu \succeq \nu_1\}.$$

Let us prove that $\mathcal{M}_1$ has a maximal element. By Zorn's Lemma, it suffices to show that any chain (=totally ordered subset) $\mathcal{C}$ of $\mathcal{M}_1$ has an upper bound in $\mathcal{M}_1$. It follows from $\dim U < \infty$ that there exists an increasing sequence $\{\nu_i\}_{i=1}^\infty$ of elements of $\mathcal{C}$ such that, for all $u \in U$,

$$\lim_{i\to\infty} \int_M u^2 d\nu_i \to \sup_{\{\nu\in\mathcal{C}\}} \int_M u^2 d\nu.$$

The sequence $\{\nu_i\}_{i=1}^\infty$ of probability measures is $w^*$-compact. Without loss of generality we can assume that this sequence is $w^*$-convergent. It follows that the measure

$$\nu_{\mathcal{C}} = w^*\text{-}\lim \nu_i \in \mathcal{M}_1$$

is an upper bound for $\mathcal{C}$.

By Zorn's Lemma, there exists a maximal element $\nu$ in $\mathcal{M}_1$. Note that the measure $\nu$ can be alternatively constructed by using a standard balayage procedure (see e.g. [3, Proposition 2.1, p. 250]). Consider first the measure $\nu'$ defined by $\nu' = \int_M q_x d\nu(x)$. It follows from (20) that for any $u \in U$

$$\begin{aligned}\int_M u^2 d\nu' &= \int_M \left( \int_M u^2 q_x d\mu \right) d\nu \\ &\geq \int_M u^2 d\nu,\end{aligned}$$

that is, $\nu' \succeq \nu$, in particular, $\nu' \in \mathcal{M}_1$. Since $\nu$ is a maximal element in $\mathcal{M}_1$, it follows that $\nu' = \nu$, which implies the identity

$$\int_M u^2 d\nu = \int_M \left( \int_M u^2 q_x d\mu \right) d\nu. \tag{23}$$

Now we can prove that $\nu(F_L) = 0$. Assuming from the contrary that $\nu(F_L) > 0$, we obtain, for any $u \in U \setminus \{0\}$.

$$\begin{aligned}\int_M u^2 d\nu &= \int_M \left(\int_M u^2 q_x d\mu\right) d\nu(x) \\ &\geq \int_{F_L} \left(\int_M u^2 q_x d\mu\right) d\nu(x) + \int_{M\setminus F_L} \left(\int_M u^2 q_x d\mu\right) d\nu(x) \\ &> \int_{F_L} u^2(x)\, d\nu(x) + \int_{M\setminus F_L} u^2(x)\, d\nu(x) \\ &= \int_M u^2 d\nu, \end{aligned} \tag{24}$$

which is a contradiction. Finally, it follows from (22) and $\nu \in \mathcal{M}_1$ that, for any $u \in U \setminus \{0\}$,

$$\int_M u^2 d\nu_0 < \int_M u^2 d\nu.$$

Measure $\nu$ can be approximated in $w^*$-sense by measures with bounded densities sitting in $M \setminus F_L$. Therefore, there exists a non-negative function $\varphi \in L^\infty(M)$ that vanishes on $F_L$ and such that

$$\int_M \varphi d\mu = 1$$

and, for any $u \in U \setminus \{0\}$,

$$\int_M u^2 \varphi_0 d\mu < \int_M u^2 \varphi d\mu \tag{25}$$

where $\varphi_0 = \frac{1}{\mu(F_L)} \mathbf{1}_{F_L}$. Consider now the potential

$$V_t = V_{max} + t\varphi_0 - t\varphi.$$

We have for all $t$

$$\int_M V_t d\mu = \int_M V_{\max} d\mu$$

and for $t \to 0$

$$\lambda_k(V_t) = \lambda_k(V_{\max}) - t\alpha + o(t),$$

where $\alpha$ is the minimal eigenvalue of the quadratic form

$$Q(u,u) = \int_M u^2 (\varphi_0 - \varphi)\, d\mu,$$

which by (25) is negative definite. Therefore, $\alpha < 0$, which together with $\lambda_k(V_{\max}) = 0$ implies that, for all small enough $t > 0$

$$\lambda_k(V_t) > 0.$$

Finally, let us show that $|V_t| \leq N$ a.e. Indeed, on $F$ we have

$$V_t \leq -c + t\varphi_0 < N$$

for small enough $t > 0$, and on $M \setminus F_L$ we have

$$V_t \leq V_{\max} - t\varphi \leq V_{\max} \leq N.$$

Therefore, $V \le N$ a.e. for small enough $t > 0$. Similarly, we have on $F_L$

$$V_t \ge V_{\max} + t\varphi_0 \ge V_{\max} \ge -N$$

and on $M \setminus F$

$$V_t \ge -c - t\varphi \ge -N$$

for small enough $t > 0$, which implies that $|V_t| \le N$ a.e. for small enough $t > 0$.

Hence, we obtain that $V_t$ is a solution to our optimization problem (4), but it satisfies $\lambda_k(V_t) > 0$, which contradicts the optimality of $V_t$ by Lemma 2.2.

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