

LOCAL WELL-POSEDNESS OF YANG-MILLS EQUATIONS IN LORENZ GAUGE BELOW THE ENERGY NORM

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ABSTRACT. We prove that the Yang-Mills equations in the Lorenz gauge (YM-LG) is locally well-posed for data below the energy norm, in particular, we can take data for the gauge potential A and the associated curvature F in $H^s \times H^{s-1}$ and $H^r \times H^{r-1}$ for $s = (\frac{6}{7}+, -\frac{1}{14}+)$, respectively. This extends a recent result by Selberg and the present author on the local well-posedness of YM-LG for finite energy data (specifically, for $(s, r) = (1-, 0)$). We also prove unconditional uniqueness of the energy class solution, that is, uniqueness in the classical space $C([-T, T]; X_0)$, where X_0 is the energy data space. The key ingredient in the proof is the fact that most bilinear terms in YM-LG contain null structure some of which uncovered in the present paper.

1. INTRODUCTION

The aim of this paper is to prove local well-posedness of YM-LG for data below the energy norm. As a consequence, we show that the energy class solution constructed recently by Selberg and the present author is unconditionally unique.

Let \mathcal{G} be a compact Lie group and \mathfrak{g} its Lie algebra. For simplicity, we shall assume $\mathcal{G} = SO(n, \mathbb{R})$ (the group of orthogonal matrices of determinant one) or $\mathcal{G} = SU(n, \mathbb{C})$ (the group of unitary matrices of determinant one). Then $\mathfrak{g} = so(n, \mathbb{R})$ (the algebra of skew symmetric matrices) or $\mathfrak{g} = su(n, \mathbb{C})$ (the algebra of trace-free skew hermitian matrices).

Given a \mathfrak{g} -valued 1-form A on the Minkowski space-time \mathbb{R}^{1+3} , we denote by $F = F^{(A)}$ the associated curvature $F = dA + [A, A]$. That is, given

$$A_\alpha: \mathbb{R}^{1+3} \rightarrow \mathfrak{g},$$

we define $F_{\alpha\beta} = F_{\alpha\beta}^{(A)}$ by

$$(1.1) \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta],$$

where $\alpha, \beta \in \{0, 1, 2, 3\}$.

In this set up, the Yang-Mills equations (YM) read

$$(1.2) \quad \partial^\alpha F_{\alpha\beta} + [A^\alpha, F_{\alpha\beta}] = 0, \quad \beta \in \{0, 1, 2, 3\},$$

where we follow the convention that repeated upper/lower indices are implicitly summed over their range. Indices are raised and lowered using the Minkowski

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metric $\text{diag}(-1, 1, 1, 1)$ on \mathbb{R}^{1+3} . Roman indices i, j, k, \dots run over $1, 2, 3$ and Greek indices α, β, γ over $0, 1, 2, 3$. Points on \mathbb{R}^{1+3} are written (x^0, x^1, x^2, x^3) with $t = x^0$, and ∂^α denotes the partial derivative with respect to x^α . We write $\partial_t = \partial_0$, $\nabla = (\partial_1, \partial_2, \partial_3)$, and $\partial = (\partial_t, \nabla)$.

The total energy for YM, at time t , is given by

$$\mathcal{E}(t) = \sum_{0 \leq \alpha, \beta \leq 3} \int_{\mathbb{R}^3} |F_{\alpha\beta}(t, x)|^2 dx,$$

and is conserved for a smooth solution decaying sufficiently fast at spatial infinity, i.e.,

$$\mathcal{E}(t) = \mathcal{E}(0).$$

The equation (1.2) is invariant under the *gauge transformation*

$$(1.3) \quad A_\alpha \rightarrow A'_\alpha = U A_\alpha U^{-1} - (\partial_\alpha U) U^{-1}$$

for sufficiently smooth function $U : \mathbb{R}^{1+3} \rightarrow G$. Indeed, if we denote $F' = F^{(A')}$ and $D'_\alpha = D_\alpha^{(A')}$, where $D_\alpha = D_\alpha^{(A)}$ is the *covariant derivative* operator associated to A given by $D_\alpha = \partial_\alpha + [A_\alpha, \cdot]$, then a simple calculation shows that

$$F' = U F U^{-1}, \quad D'_\alpha F' = U [D_\alpha F] U^{-1}.$$

This in turn implies

$$D'^\alpha F'_{\alpha\beta} = \partial^\alpha F'_{\alpha\beta} + [A'^\alpha, F'_{\alpha\beta}] = 0$$

which shows that (1.2) is invariant under the *gauge transformation* (1.3), i.e., if (A, F) satisfies (1.2), so does (A', F') . A solution is therefore a representative of its equivalent class, and hence we may impose an additional gauge condition (on A). The most popular gauges are the temporal gauge: $A_0 = 0$, the Coulomb gauge: $\partial^i A_i = 0$ and the Lorenz gauge: $\partial^\alpha A_\alpha = 0$.

In both temporal and Lorenz gauges, YM can be written as a system of nonlinear wave equations whereas in Coulomb gauge it is expressed as a system of nonlinear wave equations coupled with an elliptic equation. In temporal gauge, Segal [15] proved local and global well-posedness for initial data (for A) in the Sobolev space $^1 H^s \times H^{s-1}$ with $s \geq 3$. This was improved later by Eardley and Moncrief [3, 4] to $s \geq 2$ for the more general Yang-Mills-Higgs equations using the conservation of energy. To prove well-posedness for finite energy data (that is, $s = 1$), however, requires the bilinear terms to be null forms. In Coulomb gauge, Klainerman-Machedon [6] showed that these bilinear terms are in fact null forms and used this fact to prove global well-posedness of YM for finite energy data. This result was later extended for the more general Yang-Mills-Higgs equations by Keel [5]. Also in the temporal gauge YM contains a partial null structure, and Tao [22] used this fact to prove local well-posedness for $s > 3/4$, for data with small norm. Oh [11, 12] developed a new approach based on the *the Yang-Mills heat flow* to recover the finite energy well-posedness result of Klainerman-Machedon [6]. Local and global regularity properties of the YM and Maxwell-Klein-Gordon equations have also been studied in $1 + 4$ dimensions, which is the energy-critical case; see [7, 13, 21, 8].

¹Here $H^s = (I - \Delta)^{-s/2} L^2(\mathbb{R}^3)$.

Recently, Selberg and the present author [18] discovered null structure in most of the bilinear terms in YM-LG, and subsequently proved local well-posedness for finite energy data. This result was later extended for the more general Yang-Mills-Higgs equations by the present author [20]. In the present paper, we uncover additional null structure in YM-LG and prove local well-posedness for data below the energy norm, in particular, we can take data for A and F in $H^s \times H^{s-1}$ and $H^r \times H^{r-1}$ for $(s, r) = (\frac{6}{7}+, -\frac{1}{14}+)$,² respectively. On the other hand, the scaling critical regularity exponents are $(s_c, r_c) = (\frac{1}{2}, -\frac{1}{2})$. Thus, there is still a gap left between the critical regularity and our result, yet this is the first large data well-posedness result for YM below the energy norm. To improve on this one might need to uncover null structure in all of the bilinear terms in YM-LG.

In [18] the authors proved the existence of energy class local solution

$$(A, \partial_t A, F, \partial_t F) \in C([-T, T]; X_0),$$

where X_0 is the energy data space, but uniqueness was known only in the contraction space of $X^{s,b}$ -type, which is strictly smaller than the natural solution space $C([-T, T]; X_0)$. Here we show that uniqueness in fact holds in the latter space, that is, the energy class solution is unconditional unique. To prove this we rely on Strichartz estimates and product estimates in the wave-Sobolev spaces $H^{s,b}$ which is due to D'Ancona, Foschi and Selberg [1]. Using an idea of Zhou [23] we iteratively improve the known regularity of the solution, until we reach a space where uniqueness is known.

Lorenz-gauge null structure was first discovered in [2] for the Maxwell-Dirac equations, and then for the Maxwell-Klein-Gordon equations [16] (see also [17]).

1.1. YM-LG as a system of nonlinear wave equations. Expanding (1.2) in terms of the gauge potentials $\{A_\alpha\}$, we get the following system of second order PDE:

$$(1.4) \quad \square A_\beta = \partial_\beta \partial^\alpha A_\alpha - [\partial^\alpha A_\alpha, A_\beta] - [A^\alpha, \partial^\alpha A_\beta] - [A^\alpha, F_{\alpha\beta}].$$

If we now impose the *Lorenz gauge* condition, then the system (1.4) reduces to the nonlinear wave equation

$$(1.5) \quad \square A_\beta = -[A^\alpha, \partial_\alpha A_\beta] - [A^\alpha, F_{\alpha\beta}].$$

In addition, regardless of the choice of gauge, F satisfies the wave equation

$$(1.6) \quad \begin{aligned} \square F_{\beta\gamma} &= -[A^\alpha, \partial_\alpha F_{\beta\gamma}] - \partial^\alpha [A_\alpha, F_{\beta\gamma}] - [A^\alpha, [A_\alpha, F_{\beta\gamma}]] \\ &\quad - 2[F_{\beta\alpha}^\alpha, F_{\gamma\alpha}]. \end{aligned}$$

Indeed, this will follow if we apply D^α to the Bianchi identity

$$D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0$$

and simplify the resulting expression using the commutation identity

$$D_\alpha D_\beta X - D_\beta D_\alpha X = [F_{\alpha\beta}, X]$$

and (1.2) (see e.g. [18]).

²We use the notation $a_\pm = a \pm \varepsilon$, where ε is assumed to be a sufficiently small positive number.

Expanding the second and fourth terms in (1.6), and also imposing the Lorenz gauge, yields

$$(1.7) \quad \begin{aligned} \square F_{\beta\gamma} = & -2[A^\alpha, \partial_\alpha F_{\beta\gamma}] + 2[\partial_\gamma A^\alpha, \partial_\alpha A_\beta] - 2[\partial_\beta A^\alpha, \partial_\alpha A_\gamma] \\ & + 2[\partial^\alpha A_\beta, \partial_\alpha A_\gamma] + 2[\partial_\beta A^\alpha, \partial_\gamma A_\alpha] - [A^\alpha, [A_\alpha, F_{\beta\gamma}]] \\ & + 2[F_{\alpha\beta}, [A^\alpha, A_\gamma]] - 2[F_{\alpha\gamma}, [A^\alpha, A_\beta]] - 2[[A^\alpha, A_\beta], [A_\alpha, A_\gamma]]. \end{aligned}$$

If we ignore the matrix commutator structure and other special structures, and cubic and quartic terms in the equations (1.5), (1.7), then we would obtain a schematic system of the form

$$\begin{aligned} \square u &= u\partial u + uv, \\ \square v &= u\partial v + \partial u\partial u. \end{aligned}$$

By using Strichartz estimates one can show local well-posedness of this system for data $(u(0), \partial_t u(0)) \in H^{1+\varepsilon} \times H^\varepsilon$ and $(v(0), \partial_t v(0)) \in H^\varepsilon \times H^{-1+\varepsilon}$ (see e.g. [14]). Moreover, in view of the counter examples of Lindblad [9] these results are sharp. However, if the bilinear terms are null forms we can do better and prove local well-posedness for data in the energy class which corresponds to $\varepsilon = 0$. In fact, we can even go below energy and prove local-well posedness for some $\varepsilon < 0$ by using bilinear estimates in $X^{s,b}$ -spaces. The standard null forms are given by

$$(1.8) \quad \begin{cases} Q_0(u, v) = \partial_\alpha u \partial^\alpha v = -\partial_t u \partial_t v + \partial_i u \partial^i v, \\ Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v. \end{cases}$$

In Lorenz gauge, all the bilinear terms in (1.5), (1.7) except $[A^\alpha, F_{\alpha\beta}]$ turn out to be null forms. For this reason, we lose regularity on the solution A starting from finite energy data. However, A is only a potential representing the electromagnetic field F . The most interesting physical quantity here is F , and we do not lose regularity for these quantity starting from finite energy data. In [18] these facts were enough to prove local well-posedness for finite energy data.

Note on the other hand by expanding the last term in the right hand side of (1.5), we could write

$$(1.9) \quad \square A_\beta = -2[A^\alpha, \partial_\alpha A_\beta] + [A^\alpha, \partial_\beta A_\alpha] - [A^\alpha, [A_\alpha, A_\beta]].$$

The cubic term in this equation does not cause a problem, but the new bilinear term $[A^\alpha, \partial_\beta A_\alpha]$ does unless it contains a null structure. In fact it is worse than the term $[A^\alpha, F_{\alpha\beta}]$, since F has better regularity than ∂A , the reason being F solves a nonlinear wave equation with bilinear null form terms. So in [18] it was crucial that equation (1.5) (together with (1.7)) is used instead of the expanded version (1.9) in order to prove finite energy local well-posedness.

In the present paper, we shall nevertheless show the term $[A^\alpha, \partial_\beta A_\alpha]$ in (1.9) does also contain a partial null structure. So expanding $[A^\alpha, F_{\alpha\beta}]$ and writing the equation for A as in (1.9) has in fact an advantage. By using a *div-curl* decomposition of the spatial component of A , we shall write $[A^\alpha, \partial_\beta A_\alpha]$ as a sum of bilinear null form terms, bilinear terms which are smoother, a bilinear term which contains only F and higher order terms in (A, F) which are harmless. The resulting (schematic)

equation for A will look like ³

$$(1.10) \quad \begin{cases} \square A = \Pi(A, \partial A) + \Pi(\langle \nabla \rangle^{-1} A, A) + \Pi(\langle \nabla \rangle^{-2} A, \langle \nabla \rangle A) + \Pi(\langle \nabla \rangle^{-1} F, F) \\ \quad + \Pi(A, A, A) + \Pi(\langle \nabla \rangle^{-1} F, A, A) + \Pi(F, \langle \nabla \rangle^{-1}(AA)) \\ \quad + \Pi(\langle \nabla \rangle^{-1}(AA), A, A), \end{cases}$$

where $\Pi(\dots)$ denotes a multilinear operator in its arguments.

The cubic and quartic terms in (1.10) do not cause problems. We show that the first bilinear term $\Pi(A, \partial A)$ is a sum of null forms, whereas the second and the third bilinear terms are a lot smoother. The only term that causes difficulty is the bilinear term $\Pi(\langle \nabla \rangle^{-1} F, F)$ which as far as we know is not a null form. However, this term has at least better regularity than the term $[A^\alpha, F_{\alpha\beta}]$ in (1.5), since F solves a nonlinear wave equation with null form bilinear terms. Using these facts we are able to prove local well-posedness for large data below the energy norm and consequently show that the energy class solution is unconditionally unique.

1.2. The Cauchy problem and statement of the result. We want to solve the system (1.7)-(1.9) simultaneously for A and F . So to pose the Cauchy problem for this system, we consider initial data for (A, F) at $t = 0$:

$$(1.11) \quad \begin{cases} A(0) = a, & \partial_t A(0) = \dot{a}, \\ F(0) = f, & \partial_t F(0) = \dot{f}. \end{cases}$$

In fact, the initial data for F can be determined from (a, \dot{a}) as follows:

$$(1.12) \quad \begin{cases} f_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j], \\ f_{0i} = \dot{a}_i - \partial_i a_0 + [a_0, a_i], \\ \dot{f}_{ij} = \partial_i \dot{a}_j - \partial_j \dot{a}_i + [\dot{a}_i, a_j] + [a_i, \dot{a}_j], \\ \dot{f}_{0i} = \partial^j f_{ji} + [a^\alpha, f_{\alpha i}] \end{cases}$$

where the first three expressions come from (1.1) whereas the last one comes from (1.2) with $\beta = i$.

Note that the Lorenz gauge condition $\partial^\alpha A_\alpha = 0$ and (1.2) with $\beta = 0$ impose the constraints

$$(1.13) \quad \begin{cases} \dot{a}_0 = \partial^i a_i, \\ \partial^i f_{0i} = [a^i, \dot{a}_i]. \end{cases}$$

It turns out if (A, F) is a solution to (1.7)-(1.9) such that at $t = 0$ (1.12) and (1.13) are satisfied, then the Lorenz gauge condition is satisfied for all times where the solution is sufficiently smooth. That is, the Lorenz gauge condition propagates (see [18] for the details).

We now state our main result.

Theorem 1. *Let $(s, r) = (\frac{6}{7} + \varepsilon, -\frac{1}{14} + \varepsilon)$ or $(1 - \varepsilon, 0)$ for sufficiently small $\varepsilon > 0$.*

³Here we use the notation $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$.

(i) (Local well-posedness.) Given initial data (1.11) with regularity

$$(a, \dot{a}) \in H^s \times H^{s-1}, \quad (f, \dot{f}) \in H^r \times H^{r-1},$$

there exists a time $T > 0$ depending on the initial data norm, and a solution

$$(1.14) \quad \begin{cases} A \in C([-T, T]; H^s) \cap C^1([-T, T]; H^{s-1}), \\ F \in C([-T, T]; H^r) \cap C^1([-T, T]; H^{r-1}), \end{cases}$$

solving the system (1.7)–(1.9) on $S_T = (-T, T) \times \mathbb{R}^3$ in the sense of distributions.

The solution has the regularity

$$\left(A \pm \frac{1}{i\langle \nabla \rangle} \partial_t A \right) \in X_{\pm}^{s, \frac{3}{4}+}(S_T), \quad \left(F \pm \frac{1}{i\langle \nabla \rangle} \partial_t F \right) \in X_{\pm}^{r, \frac{1}{2}+}(S_T),$$

and it is the unique solution with this property (these spaces are defined in Section 3). Moreover, the solution depends continuously on the data and higher regular data persists in time.

(ii) (Unconditional uniqueness of energy class solution.) In the case where $(s, r) = (1 - \varepsilon, 0)$, the solution is in fact unique in the class (1.14).

Remark 1. Local well-posedness for $(s, r) = (1 - \varepsilon, 0)$ is proved in [18]. It will be clear from the estimates that one in fact obtain local well-posedness for (s, r) in some convex region containing the points $(\frac{6}{7} + \varepsilon, -\frac{1}{14} + \varepsilon)$ and $(1 - \varepsilon, 0)$, but we do not pursue this here.

Let us fix some notation. We use \lesssim to mean \leq up to multiplication by a positive constant C which may depend on s, r and T . If A, B are nonnegative quantities, $A \sim B$ means $B \lesssim A \lesssim B$.

The rest of the paper is organized as follows. In the next Section, we reveal the null structure in the key bilinear terms and write the system (1.7)–(1.9) in terms of the null forms. In Section 3, we shall rewrite this new system as a first order system and reduce Theorem 1 to proving nonlinear estimates. In the rest of the Sections we prove the nonlinear estimates.

2. NULL STRUCTURE IN THE BILINEAR TERMS

For \mathfrak{g} -valued u, v , define a commutator version of null forms by

$$(2.1) \quad \begin{cases} Q_0[u, v] = [\partial_\alpha u, \partial^\alpha v] = Q_0(u, v) - Q_0(v, u), \\ Q_{\alpha\beta}[u, v] = [\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v] = Q_{\alpha\beta}(u, v) + Q_{\alpha\beta}(v, u). \end{cases}$$

Note the identity

$$(2.2) \quad [\partial_\alpha u, \partial_\beta u] = \frac{1}{2} ([\partial_\alpha u, \partial_\beta u] - [\partial_\beta u, \partial_\alpha u]) = \frac{1}{2} Q_{\alpha\beta}[u, u].$$

Define

$$(2.3) \quad \Omega[u, v] = -\frac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij} [R^l u^m, v] - Q_{0i} [R^i u_0, v],$$

where ε_{ijk} is the antisymmetric symbol with $\varepsilon_{123} = 1$ and

$$R_i = \langle \nabla \rangle^{-1} \partial_i = (1 - \Delta)^{-1/2} \partial_i$$

are the Riesz transforms.

We split the spatial part $\mathbf{A} = (A_1, A_2, A_3)$ of the potential into divergence-free and curl-free parts and a smoother part:

$$(2.4) \quad \mathbf{A} = \mathbf{A}^{\text{df}} + \mathbf{A}^{\text{cf}} + \langle \nabla \rangle^{-2} \mathbf{A},$$

where

$$\begin{aligned} \mathbf{A}^{\text{df}} &= \langle \nabla \rangle^{-2} \nabla \times \nabla \times \mathbf{A}, \\ \mathbf{A}^{\text{cf}} &= -\langle \nabla \rangle^{-2} \nabla (\nabla \cdot \mathbf{A}). \end{aligned}$$

2.1. Terms of the form $[A^\alpha, \partial_\alpha \phi]$ and $[\partial_t A^\alpha, \partial_\alpha \phi]$. In the Lorenz gauge, terms of the form $[A^\alpha, \partial_\alpha \phi]$, where $A_\alpha, \phi \in \mathcal{S}$ with values in \mathfrak{g} , can be shown to be as a sum of bilinear null forms and a smoother bilinear part whereas the term $[\partial_t A^\alpha, \partial_\alpha \phi]$ is a null form.

Lemma 1. *In the Lorenz gauge, we have the identities*

$$(2.5) \quad [A^\alpha, \partial_\alpha \phi] = \mathfrak{Q} [\langle \nabla \rangle^{-1} A, \phi] + [\langle \nabla \rangle^{-2} A^\alpha, \partial_\alpha \phi],$$

$$(2.6) \quad [\partial_t A^\alpha, \partial_\alpha \phi] = Q_{0i} [A^i, \phi].$$

Proof. To show (2.5) we modify the proof in [18, Lemma 1], whereas (2.6) is proved in the same paper (see identity (2.7) therein).

Using (2.4) we write

$$A^\alpha \partial_\alpha \phi = (-A_0 \partial_t \phi + \mathbf{A}^{\text{cf}} \cdot \nabla \phi) + \mathbf{A}^{\text{df}} \cdot \nabla \phi + \langle \nabla \rangle^{-2} \mathbf{A} \cdot \nabla \phi$$

Let us first consider the first term in the parentheses. We use the Lorenz gauge, $\partial_t A_0 = \nabla \cdot \mathbf{A}$, to write

$$\mathbf{A}^{\text{cf}} \cdot \nabla \phi = -\langle \nabla \rangle^{-2} \partial^i (\partial_t A_0) \partial_i \phi = -\partial_t (\langle \nabla \rangle^{-1} R^i A_0) \partial_i \phi.$$

We can also write

$$\begin{aligned} A_0 \partial_t \phi &= -\langle \nabla \rangle^{-2} \partial_i \partial^i A_0 \partial_t \phi + \langle \nabla \rangle^{-2} A_0 \partial_t \phi \\ &= -\partial_i (\langle \nabla \rangle^{-1} R^i A_0) \partial_t \phi + \langle \nabla \rangle^{-2} A_0 \partial_t \phi. \end{aligned}$$

Combining the above identities, we get

$$-A_0 \partial_t \phi + \mathbf{A}^{\text{cf}} \cdot \nabla \phi = Q_{i0} (\langle \nabla \rangle^{-1} R^i A_0, \phi) - \langle \nabla \rangle^{-2} A_0 \partial_t \phi.$$

Next, we consider the second term. Since

$$(\mathbf{A}^{\text{df}})^i = \varepsilon^{ijk} \varepsilon_{klm} R_j R^l A^m,$$

we have

$$\begin{aligned} \mathbf{A}^{\text{df}} \cdot \nabla \phi &= \varepsilon^{ijk} \varepsilon_{klm} (R_j R^l A^m) \partial_i \phi \\ &= \varepsilon^{ijk} \varepsilon_{klm} \partial_j (\langle \nabla \rangle^{-1} R^l A^m) (\partial_i \phi) \\ &= -\frac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij} (\langle \nabla \rangle^{-1} R^l A_m, \phi). \end{aligned}$$

Thus, we have shown

$$(2.7) \quad \begin{aligned} A^\alpha \partial_\alpha \phi &= -\frac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij} (\langle \nabla \rangle^{-1} R^l A_m, \phi) \\ &\quad + Q_{i0} (\langle \nabla \rangle^{-1} R^i A_0, \phi) + \langle \nabla \rangle^{-2} A^\alpha \partial_\alpha \phi. \end{aligned}$$

Similarly, modifying the above argument one can show

$$(2.8) \quad \begin{aligned} \partial_\alpha \phi A^\alpha &= \frac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij} (\phi, \langle \nabla \rangle^{-1} R^l A_m) \\ &\quad - Q_{i0} (\phi, \langle \nabla \rangle^{-1} R^i A_0) + \partial_\alpha \phi \langle \nabla \rangle^{-2} A^\alpha. \end{aligned}$$

Subtracting (2.7) and (2.8) yields (2.5). \square

2.2. Terms of the form $[A^\alpha, \partial_\beta A_\alpha]$. In the Lorenz gauge, this term can be written as a sum of bilinear null form terms, bilinear terms which are smoother, a bilinear term which contains only F and higher order terms in (A, F) .

Lemma 2. *In the Lorenz gauge, we have the identity*

$$[A^\alpha, \partial_\beta A_\alpha] = \sum_{i=1}^4 \Gamma_\beta^i(A, \partial A, F, \partial F),$$

where

$$(2.9) \quad \left\{ \begin{aligned} \Gamma_\beta^1(A, \partial A, F, \partial F) &= -[A_0, \partial_\beta A_0] + [\langle \nabla \rangle^{-1} R_j (\partial_t A_0), \langle \nabla \rangle^{-1} R^j \partial_t (\partial_\beta A_0)], \\ \Gamma_\beta^2(A, \partial A, F, \partial F) &= -\frac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} \left\{ Q_{ij} [\langle \nabla \rangle^{-1} R^n \mathbf{A}_n, \langle \nabla \rangle^{-1} R^l \partial_\beta \mathbf{A}^m] \right. \\ &\quad \left. + Q_{ij} [\langle \nabla \rangle^{-1} R^n \partial_\beta \mathbf{A}_n, \langle \nabla \rangle^{-1} R^l \mathbf{A}^m] \right\}, \\ \Gamma_\beta^3(A, \partial A, F, \partial F) &= [\langle \nabla \rangle^{-2} \nabla \times \mathbf{F}, \langle \nabla \rangle^{-2} \nabla \times \partial_\beta \mathbf{F}] \\ &\quad - [\langle \nabla \rangle^{-2} \nabla \times \mathbf{F}, \langle \nabla \rangle^{-2} \partial_\beta \nabla \times (\mathbf{A} \times \mathbf{A})] \\ &\quad - [\langle \nabla \rangle^{-2} \nabla \times (\mathbf{A} \times \mathbf{A}), \langle \nabla \rangle^{-2} \nabla \times \partial_\beta \mathbf{F}] \\ &\quad + [\langle \nabla \rangle^{-2} \nabla \times (\mathbf{A} \times \mathbf{A}), \langle \nabla \rangle^{-2} \partial_\beta \nabla \times (\mathbf{A} \times \mathbf{A})], \\ \Gamma_\beta^4(A, \partial A, F, \partial F) &= [\mathbf{A}^{\text{cf}} + \mathbf{A}^{\text{df}}, \langle \nabla \rangle^{-2} \partial_\beta \mathbf{A}] + [\langle \nabla \rangle^{-2} \mathbf{A}, \partial_\beta \mathbf{A}]. \end{aligned} \right.$$

Here $\mathbf{F} = (F_{23}, F_{31}, F_{12})$.

Thus, Γ_β^2 is a combination of the commutator version Q -type null forms. The term Γ_β^1 is also a null form (of non Q -type) as can be shown as follows.

We write

$$\begin{aligned} \Gamma_\beta^1 &= (-A_0 \partial_\beta A_0 + \langle \nabla \rangle^{-1} R_j (\partial_t A_0) \langle \nabla \rangle^{-1} R^j \partial_t (\partial_\beta A_0)) \\ &\quad + (\partial_\beta A_0 A_0 - \langle \nabla \rangle^{-1} R_j \partial_t (\partial_\beta A_0) \langle \nabla \rangle^{-1} R_j (\partial_t A_0)) \\ &=: R_1 + R_2. \end{aligned}$$

Now if we denote the space-time Fourier variables of the first and second A_0 in the product in R_1 by (τ, ξ) and (λ, η) , respectively, where $\tau, \lambda \in \mathbb{R}$ are temporal frequencies and $\xi, \eta \in \mathbb{R}^3$ spatial frequencies, then R_1 has symbol

$$(2.10) \quad i \left(-1 + \frac{\tau \lambda}{\langle \xi \rangle^2 \langle \eta \rangle^2} \xi \cdot \eta \right) \eta_\beta = -i \langle \xi \rangle^{-2} \langle \eta \rangle^{-2} (\langle \xi \rangle^2 \langle \eta \rangle^2 - (\tau \lambda) (\xi \cdot \eta)) \eta_\beta.$$

If we write

$$\langle \xi \rangle^2 \langle \eta \rangle^2 - (\tau \lambda)(\xi \cdot \eta) = (\langle \xi \rangle^2 \langle \eta \rangle^2 - |\xi|^2 |\eta|^2) + (|\xi|^2 |\eta|^2 - \tau \lambda \xi \cdot \eta),$$

then the first term in parenthesis satisfies the estimate

$$\langle \xi \rangle^2 \langle \eta \rangle^2 - |\xi|^2 |\eta|^2 = 1 + |\xi|^2 + |\eta|^2 \leq \langle \xi \rangle^2 + \langle \eta \rangle^2.$$

Combined with (2.10), this will imply a gain in two derivatives. On the other hand, the second term $(|\xi|^2 |\eta|^2 - \tau \lambda)(\xi \cdot \eta)$ vanishes if (τ, ξ) and (λ, η) are parallel null vectors, i.e., if $\tau = \pm |\xi|$ and $(\lambda, \eta) = c(\tau, \xi)$ for some $c \in \mathbb{R}$. Thus, R_1 is a null form up to a smooth bilinear term. A similar argument shows R_2 is also a null form up to a smooth bilinear term.

Proof of Lemma 2. Using (2.4), we write

$$\begin{aligned} [A^\alpha, \partial_\beta A_\alpha] &= -[A_0, \partial_\beta A_0] + [\mathbf{A}, \partial_\beta \mathbf{A}] \\ &=: \sum_{i=1}^4 \Gamma_\beta^i(A, \partial A, F, \partial F), \end{aligned}$$

where

$$\begin{aligned} \Gamma_\beta^1(A, \partial A, F, \partial F) &= -[A_0, \partial_\beta A_0] + [\mathbf{A}^{\text{cf}}, \partial_\beta \mathbf{A}^{\text{cf}}], \\ \Gamma_\beta^2(A, \partial A, F, \partial F) &= [\mathbf{A}^{\text{cf}}, \partial_\beta \mathbf{A}^{\text{df}}] + [\mathbf{A}^{\text{df}}, \partial_\beta \mathbf{A}^{\text{cf}}], \\ \Gamma_\beta^3(A, \partial A, F, \partial F) &= [\mathbf{A}^{\text{df}}, \partial_\beta \mathbf{A}^{\text{df}}], \\ \Gamma_\beta^4(A, \partial A, F, \partial F) &= [\mathbf{A}^{\text{cf}} + \mathbf{A}^{\text{df}}, \langle \nabla \rangle^{-2} \partial_\beta \mathbf{A}] + [\langle \nabla \rangle^{-2} \mathbf{A}, \partial_\beta \mathbf{A}]. \end{aligned}$$

So Γ_β^4 is exactly as in (2.9). It remains to show that Γ_β^i ($i = 1, 2, 3$) can also be rewritten as in (2.9).

First consider Γ_β^1 . We have

$$\Gamma_\beta^1 = (-A_0 \partial_\beta A_0 + \mathbf{A}^{\text{cf}} \cdot \partial_\beta \mathbf{A}^{\text{cf}}) + (\partial_\beta A_0 A_0 - \partial_\beta \mathbf{A}^{\text{cf}} \cdot \mathbf{A}^{\text{cf}})$$

Using the Lorenz gauge, we write

$$\begin{aligned} -A_0 \partial_\beta A_0 + \mathbf{A}^{\text{cf}} \cdot \partial_\beta \mathbf{A}^{\text{cf}} &= -A_0 \partial_\beta A_0 + \langle \nabla \rangle^{-2} \nabla(\nabla \cdot \mathbf{A}) \cdot \partial_\beta \langle \nabla \rangle^{-2} \nabla(\nabla \cdot \mathbf{A}) \\ &= -A_0 \partial_\beta A_0 + \langle \nabla \rangle^{-2} \nabla(\partial_t A_0) \cdot \langle \nabla \rangle^{-2} \nabla \partial_t(\partial_\beta A_0) \\ &= -A_0(\partial_\beta A_0) + \langle \nabla \rangle^{-1} R_j(\partial_t A_0) \langle \nabla \rangle^{-1} R^j \partial_t(\partial_\beta A_0). \end{aligned}$$

Similarly, one can write

$$\partial_\beta A_0 A_0 - \partial_\beta \mathbf{A}^{\text{cf}} \cdot \mathbf{A}^{\text{cf}} = (\partial_\beta A_0) A_0 - \langle \nabla \rangle^{-1} R^j \partial_t(\partial_\beta A_0) \langle \nabla \rangle^{-1} R_j(\partial_t A_0).$$

Summing up, we obtain the desired identity for Γ_β^1 .

Next, we consider Γ_β^2 . Let

$$B = -\langle \nabla \rangle^{-2}(\nabla \cdot \mathbf{A}) = -\langle \nabla \rangle^{-1} R^n \mathbf{A}_n, \quad \mathbf{C} = \langle \nabla \rangle^{-2} \nabla \times \mathbf{A}.$$

Then

$$\begin{aligned}
\mathbf{A}^{\text{cf}} \cdot \partial_\beta \mathbf{A}^{\text{df}} &= \nabla B \cdot (\nabla \times \partial_\beta \mathbf{C}) = (\nabla B \times \nabla \partial_\beta \mathbf{C}_k)^k \\
&= \frac{1}{2} \varepsilon^{ijk} Q_{ij}(B, \partial_\beta \mathbf{C}_k) \\
&= -\frac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij}(\langle \nabla \rangle^{-1} R^n \mathbf{A}_n, \langle \nabla \rangle^{-1} R^l \partial_\beta \mathbf{A}^m),
\end{aligned}$$

where we used $\mathbf{C}_k = \varepsilon_{klm} \langle \nabla \rangle^{-1} R^l \mathbf{A}^m$. Similarly,

$$\begin{aligned}
\mathbf{A}^{\text{df}} \cdot \partial_\beta \mathbf{A}^{\text{cf}} &= \frac{1}{2} \varepsilon^{ijk} Q_{ij}(\partial_\beta \mathbf{C}_k, B) \\
&= -\frac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij}(\langle \nabla \rangle^{-1} R^l \partial_\beta \mathbf{A}^m, \langle \nabla \rangle^{-1} R^n \mathbf{A}_n).
\end{aligned}$$

The terms $\partial_\beta \mathbf{A}^{\text{cf}} \cdot \mathbf{A}^{\text{df}}$ and $\partial_\beta \mathbf{A}^{\text{df}} \cdot \mathbf{A}^{\text{cf}}$ in Γ_β^2 can also be written as in $\mathbf{A}^{\text{cf}} \cdot \partial_\beta \mathbf{A}^{\text{df}}$ and $\mathbf{A}^{\text{df}} \cdot \partial_\beta \mathbf{A}^{\text{cf}}$, respectively, except that now ∂_β falls on \mathbf{A}_n instead of \mathbf{A}_m . Combining these facts gives the desired identity for Γ_β^2 .

Finally, we consider Γ_β^3 . Using the definition for F_{ij} , we have $\nabla \times \mathbf{A} = \mathbf{F} - \mathbf{A} \times \mathbf{A}$, where $\mathbf{F} = (F_{23}, F_{31}, F_{12})$. This in turn implies

$$\mathbf{A}^{\text{df}} = \langle \nabla \rangle^{-2} \{ \nabla \times \mathbf{F} - \nabla \times (\mathbf{A} \times \mathbf{A}) \}.$$

Inserting this in place of \mathbf{A}^{df} gives the desired identity for Γ_β^3 .

□

2.3. The system (1.7)–(1.9) in terms of the null forms. Let us now look at the nonlinear terms in (1.7)–(1.9). In view of Lemma 1 the first, second and third bilinear terms in (1.7) are null forms up to some smoother bilinear terms. By the identity (2.2), the fourth and fifth terms are identical to $2Q_0[A_\beta, A_\gamma]$ and $Q_{\beta\gamma}[A^\alpha, A_\alpha]$, respectively.

By Lemma 1, the first term in (1.9) is a null form up to some smoother bilinear terms. By Lemma 2 the second term in (1.9) is a sum of bilinear null form terms, bilinear terms which are smoother, a bilinear term which contains only F and higher order terms in (A, F) .

In conclusion, the system (1.7)–(1.9) can be written as

$$(2.11) \quad \begin{cases} \square A_\beta = \mathfrak{M}_\beta(A, \partial_t A, F, \partial_t F), \\ \square F_{\beta\gamma} = \mathfrak{N}_{\beta\gamma}(A, \partial_t A, F, \partial_t F), \end{cases}$$

where

$$\begin{cases} \mathfrak{M}_\beta(A, \partial_t A, F, \partial_t F) = -2\Omega[\langle \nabla \rangle^{-1} A, A_\beta] + \sum_{i=1}^4 \Gamma_\beta^i(A, \partial A, F, \partial F) - 2[\langle \nabla \rangle^{-2} A^\alpha, \partial_\alpha A_\beta] \\ \quad - [A^\alpha, [A_\alpha, A_\beta]], \\ \mathfrak{N}_{ij}(A, \partial_t A, F, \partial_t F) = -2\Omega[\langle \nabla \rangle^{-1} A, F_{ij}] + 2\Omega[\langle \nabla \rangle^{-1} \partial_j A, A_i] - 2\Omega[\langle \nabla \rangle^{-1} \partial_i A, A_j] \\ \quad + 2Q_0[A_i, A_j] + Q_{ij}[A^\alpha, A_\alpha] - 2[\langle \nabla \rangle^{-2} A^\alpha, \partial_\alpha F_{ij}] \\ \quad + 2[\langle \nabla \rangle^{-2} \partial_j A^\alpha, \partial_\alpha A_i] - 2[\langle \nabla \rangle^{-2} \partial_i A^\alpha, \partial_\alpha A_j] \\ \quad - [A^\alpha, [A_\alpha, F_{ij}]] + 2[F_{\alpha i}, [A^\alpha, A_j]] - 2[F_{\alpha j}, [A^\alpha, A_i]] \\ \quad - 2[[A^\alpha, A_i], [A_\alpha, A_j]], \\ \mathfrak{N}_{0i}(A, \partial_t A, F, \partial_t F) = -2\Omega[\langle \nabla \rangle^{-1} A, F_{0i}] + 2\Omega[\langle \nabla \rangle^{-1} \partial_i A, A_0] - 2Q_{0j}[A^j, A_i] + 2Q_0[A_0, A_i] \\ \quad + Q_{0i}[A^\alpha, A_\alpha] - 2[\langle \nabla \rangle^{-2} A^\alpha, \partial_\alpha F_{0i}] + 2[\langle \nabla \rangle^{-2} \partial_i A^\alpha, \partial_\alpha A_0] \\ \quad - [A^\alpha, [A_\alpha, F_{0i}]] + 2[F_{\alpha 0}, [A^\alpha, A_i]] - 2[F_{\alpha i}, [A^\alpha, A_0]] \\ \quad - 2[[A^\alpha, A_0], [A_\alpha, A_i]] \end{cases}$$

and $\Gamma_\beta^i(A, \partial A, F, \partial F)$ for $i = 1, \dots, 4$ are as in (2.9).

3. REDUCTION OF THEOREM 1 TO NONLINEAR ESTIMATES

3.1. Rewriting (2.11) as a first order system. We rewrite (2.11) as a first order system and reduce Theorem 1 to proving nonlinear estimates in $X^{s,b}$ -spaces. We subtract A and F to each side of the equations in (2.11) and hence replace the wave operator \square by the Klein-Gordon operator $\square - 1$ at the expense of adding linear term on the right hand side. This is done to avoid the singular operator $|\nabla|^{-1}$, and instead obtain the non-singular operator $\langle \nabla \rangle^{-1}$ in the change of variables

$$(A, \partial_t A, F, \partial_t F) \rightarrow (A_+, A_-, F_+, F_-),$$

where

$$A_\pm = \frac{1}{2} \left(A \pm \frac{1}{i\langle \nabla \rangle} \partial_t A \right), \quad F_\pm = \frac{1}{2} \left(F \pm \frac{1}{i\langle \nabla \rangle} \partial_t F \right).$$

Equivalently,

$$(3.1) \quad \begin{cases} (A, \partial_t A) = (A_+ + A_-, i\langle \nabla \rangle(A_+ - A_-)) \\ (F, \partial_t F) = (F_+ + F_-, i\langle \nabla \rangle(F_+ - F_-)). \end{cases}$$

Then the system (2.11) transforms to

$$(3.2) \quad \begin{cases} (i\partial_t \pm \langle \nabla \rangle)A_\pm = \mp \frac{1}{2\langle \nabla \rangle} \mathfrak{M}'(A_+, A_-, F_+, F_-), \\ (i\partial_t \pm \langle \nabla \rangle)F_\pm = \mp \frac{1}{2\langle \nabla \rangle} \mathfrak{N}'(A_+, A_-, F_+, F_-), \end{cases}$$

where

$$(3.3) \quad \begin{cases} \mathfrak{M}'(A_+, A_-, F_+, F_-) = -A + \mathfrak{M}_\beta(A, \partial_t A, F, \partial_t F), \\ \mathfrak{N}'(A_+, A_-, F_+, F_-) = -F + \mathfrak{N}_{\beta\gamma}(A, \partial_t A, F, \partial_t F), \end{cases}$$

In the right-hand side of (3.3) it is understood that we use the substitution (3.1) on (A, F) and the arguments of \mathfrak{M} and \mathfrak{N} .

The initial data transforms to

$$(3.4) \quad \begin{cases} A_{\pm}(0) = a_{\pm} := \frac{1}{2} \left(a \pm \frac{1}{i\langle \nabla \rangle} \dot{a} \right) \in H^s, \\ F_{\pm}(0) = f_{\pm} := \frac{1}{2} \left(f \pm \frac{1}{i\langle \nabla \rangle} \dot{f} \right) \in H^r. \end{cases}$$

3.2. Spaces used: $X^{s,b}$ -Spaces and their properties. We prove local well-posedness of (3.2)–(3.4) by iterating in the $X^{s,b}$ -spaces adapted to the dispersive operators $i\partial_t \pm \langle \nabla \rangle$. These spaces are defined to be the completion of $\mathcal{S}(\mathbb{R}^{1+3})$ with respect to the norm

$$\|u\|_{X_{\pm}^{s,b}} = \left\| \langle \xi \rangle^s \langle -\tau \pm \langle \xi \rangle \rangle^b \tilde{u}(\tau, \xi) \right\|_{L_{\tau, \xi}^2},$$

where $\tilde{u}(\tau, \xi) = \mathcal{F}_{t,x} u(\tau, \xi)$ is the space-time Fourier transform of $u(t, x)$.

Let $X_{\pm}^{s,b}(S_T)$ denote the restriction space to a time slab $S_T = (-T, T) \times \mathbb{R}^3$ for $T > 0$. We recall the fact that

$$(3.5) \quad X_{\pm}^{s,b}(S_T) \hookrightarrow C([-T, T]; H^s) \quad \text{for } b > \frac{1}{2}.$$

Moreover, it is well known that the linear initial value problem

$$(i\partial_t \pm \langle \nabla \rangle)u = G \in X_{\pm}^{s,b-1+\varepsilon}(S_T), \quad u(0) = u_0 \in H^s,$$

for any $s \in \mathbb{R}$, $b > \frac{1}{2}$, $0 < \varepsilon \ll 1$, has a unique solution satisfying

$$(3.6) \quad \|u\|_{X_{\pm}^{s,b}(S_T)} \leq C \left(\|u_0\|_{H^s} + T^{\varepsilon} \|G\|_{X_{\pm}^{s,b-1+\varepsilon}(S_T)} \right)$$

for $0 < T < 1$.

In addition to $X_{\pm}^{s,b}$, we shall also need the wave-Sobolev spaces $H^{s,b}$, defined to be the completion of $\mathcal{S}(\mathbb{R}^{1+3})$ with respect to the norm

$$\|u\|_{H^{s,b}} = \left\| \langle \xi \rangle^s \langle |\tau| - \langle \xi \rangle \rangle^b \tilde{u}(\tau, \xi) \right\|_{L_{\tau, \xi}^2}.$$

We shall make a frequent use of the relations

$$(3.7) \quad \begin{cases} \|u\|_{H^{s,b}} \leq \|u\|_{X_{\pm}^{s,b}} & \text{if } b \geq 0, \\ \|u\|_{X_{\pm}^{s,b}} \leq \|u\|_{H^{s,b}} & \text{if } b \leq 0. \end{cases}$$

In particular, (3.7) allows us to pass from estimates in $X_{\pm}^{s,b}$ to corresponding estimates in $H^{s,b}$.

3.3. Reduction to nonlinear estimates using iteration. Using (3.6) and a standard iteration argument, local well-posedness can be deduced from the following nonlinear estimates.

Lemma 3. *Let $0 < T < 1$, $(a, b) = (\frac{3}{4} + \delta, \frac{1}{2} + \delta)$, $(a', b') = (a + \delta, b + \delta)$ and $(s, r) = (1 - \varepsilon, 0)$ or $(\frac{6}{7} + \varepsilon, -\frac{1}{14} + \varepsilon)$ where ε is sufficiently small and $0 < \delta \ll \varepsilon \ll 1$. Then we have the estimates*

$$(3.8) \quad \begin{cases} \|\mathfrak{M}'(A_+, A_-, F_+, F_-)\|_{X_{\pm}^{s-1, a'-1}(S_T)} \lesssim N(1 + N^3), \\ \|\mathfrak{M}'(A_+, A_-, F_+, F_-)\|_{X_{\pm}^{r-1, b'-1}(S_T)} \lesssim N(1 + N^3), \end{cases}$$

where

$$N = \sum_{\pm} \left(\|A_{\pm}\|_{X_{\pm}^{s, a}(S_T)} + \|F_{\pm}\|_{X_{\pm}^{r, b}(S_T)} \right).$$

Moreover, we have the difference estimates

$$(3.9) \quad \begin{cases} \|\mathfrak{M}'(A_+, A_-, F_+, F_-) - \mathfrak{M}'(A'_+, A'_-, F'_+, F'_-)\|_{X_{\pm}^{s-1, a'-1}(S_T)} \lesssim \sigma(1 + N^3), \\ \|\mathfrak{M}'(A_+, A_-, F_+, F_-) - \mathfrak{M}'(A'_+, A'_-, F'_+, F'_-)\|_{X_{\pm}^{r-1, b'-1}(S_T)} \lesssim \sigma(1 + N^3), \end{cases}$$

where

$$\sigma = \sum_{\pm} \left(\|A_{\pm} - A'_{\pm}\|_{X_{\pm}^{s, a}(S_T)} + \|F_{\pm} - F'_{\pm}\|_{X_{\pm}^{r, b}(S_T)} \right).$$

Then by iteration we obtain a solution (A_+, A_-, F_+, F_-) of the transformed system (3.2)–(3.4) on S_T for $T > 0$. The solution has the regularity

$$(3.10) \quad A_{\pm} \in X_{\pm}^{s, a}(S_T), \quad F_{\pm} \in X_{\pm}^{r, b}(S_T),$$

and is unique in this space. By (3.5)

$$(3.11) \quad A_{\pm} \in C([-T, T]; H^s), \quad F_{\pm} \in C([-T, T]; H^r).$$

Continuous dependence of solution on the data and persistence of higher regularity data also follow from standard arguments which we omit here. Once we have obtained the solution (A_+, A_-, F_+, F_-) of the system (3.2), we can then define $A = A_+ + A_-$ and $F = F_+ + F_-$, and show that (A, F) solves the original system (2.11) (see [18] for details).

Note that uniqueness of solution is only known only in the iteration space (3.10) and not in (3.11). But we can show that a solution in the space (3.11) with $(s, r) = (1 - \varepsilon, 0)$ also belongs to (3.10) with $(s, r) = (\frac{6}{7} + \varepsilon, -\frac{1}{14} + \varepsilon)$ in which the solution is known to be unique. So unconditional uniqueness of the energy class solution follows from the following Lemma.

Lemma 4. *Let*

$$A_{\pm} \in C([-T, T]; H^{1-\varepsilon}), \quad F_{\pm} \in C([-T, T]; L^2)$$

be the solution to the system (3.2) with initial data $(a_{\pm}, f_{\pm}) \in H^{1-\varepsilon} \times L^2$. Then

$$A_{\pm} \in X_{\pm}^{\frac{6}{7}+\varepsilon, \frac{3}{4}+\delta}(S_T), \quad F_{\pm} \in X_{\pm}^{-\frac{1}{14}+\varepsilon, \frac{1}{2}+\delta}(S_T)$$

where ε and δ are as in Lemma 3.

The proof of Lemma 4 is given in the last Section.

3.4. Further reduction of Lemma 3 to null form and multilinear estimates. For the proof of Lemma 3, it suffices to show only (3.8) since the proof for (3.9) is similar.

The estimates for the linear terms in \mathfrak{M}' and \mathfrak{N}' are trivial, and so we ignore them. Thus, we remain to prove the estimates for \mathfrak{M} and \mathfrak{N} . So we reduce to

$$(3.12) \quad \begin{cases} \|\mathfrak{M}(A_+, A_-, F_+, F_-)\|_{X_{\pm}^{s-1, a'-1}} \lesssim N(1 + N^3), \\ \|\mathfrak{N}(A_+, A_-, F_+, F_-)\|_{X_{\pm}^{r-1, b'-1}} \lesssim N(1 + N^3), \end{cases}$$

where

$$N = \sum_{\pm} \left(\|A_{\pm}\|_{X_{\pm}^{s, a}} + \|F_{\pm}\|_{X_{\pm}^{r, b}} \right).$$

To this end, to simplify notation, we write

$$\|u\|_{X^{s, b}} = \|u_+\|_{X_+^{s, b}} + \|u_-\|_{X_-^{s, b}}$$

for $u = u_+ + u_-$.

Now, looking at the terms in \mathfrak{M} and \mathfrak{N} and noting the fact that the Riesz transforms R_i are bounded in the spaces involved, the estimates in (3.12) reduce to proving

(i) the corresponding estimates for the null forms $Q = Q_0, Q_{0i}, Q_{ij}$:

$$(3.13) \quad \|Q[\langle \nabla \rangle^{-1} A, A]\|_{H^{s-1, a'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

$$(3.14) \quad \|Q_{ij}[\langle \nabla \rangle^{-1} A, \langle \nabla \rangle^{-1} \partial A]\|_{H^{s-1, a'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

$$(3.15) \quad \|Q[\langle \nabla \rangle^{-1} A, F]\|_{H^{r-1, b'-1}} \lesssim \|A\|_{X^{s, a}} \|F\|_{X^{r, b}},$$

$$(3.16) \quad \|Q[A, A]\|_{H^{r-1, b'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

(ii) the following estimates for Γ^1 (which is a non Q -type null form) and bilinear terms:

$$(3.17) \quad \|\Gamma^1(A, \partial A)\|_{H^{s-1, a'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

$$(3.18) \quad \|\Pi(A, \langle \nabla \rangle^{-2} \partial A)\|_{H^{s-1, a'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

$$(3.19) \quad \|\Pi(\langle \nabla \rangle^{-2} A, \partial A)\|_{H^{s-1, a'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

$$(3.20) \quad \|\Pi(\langle \nabla \rangle^{-1} F, \langle \nabla \rangle^{-1} \partial F)\|_{H^{s-1, a'-1}} \lesssim \|F\|_{X^{r, b}} \|F\|_{X^{r, b}},$$

$$(3.21) \quad \|\Pi(\langle \nabla \rangle^{-2} A, \partial F)\|_{H^{r-1, b'-1}} \lesssim \|A\|_{X^{s, a}} \|F\|_{X^{r, b}},$$

$$(3.22) \quad \|\Pi(\langle \nabla \rangle^{-1} A, \partial A)\|_{H^{r-1, b'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}}$$

and

(iii) the following trilinear and quadrilinear estimates:

$$(3.23) \quad \|\Pi(\langle \nabla \rangle^{-1} F, \langle \nabla \rangle^{-1} \partial(AA))\|_{H^{s-1, a'-1}} \lesssim \|F\|_{X^{r, b}} \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

$$(3.24) \quad \|\Pi(\langle \nabla \rangle^{-1} \partial F, \langle \nabla \rangle^{-1} (AA))\|_{H^{s-1, a'-1}} \lesssim \|F\|_{X^{r, b}} \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

(3.25)

$$\|\Pi(\langle \nabla \rangle^{-1} (AA), \langle \nabla \rangle^{-1} \partial(AA))\|_{H^{s-1, a'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}} \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

$$(3.26) \quad \|\Pi(A, A, A)\|_{H^{s-1, a'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

$$(3.27) \quad \|\Pi(A, A, F)\|_{H^{r-1, b'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}} \|F\|_{X^{r, b}},$$

$$(3.28) \quad \|\Pi(A, A, A, A)\|_{H^{r-1, b'-1}} \lesssim \|A\|_{X^{s, a}} \|A\|_{X^{s, a}} \|A\|_{X^{s, a}} \|A\|_{X^{s, a}},$$

where $\Pi(\dots)$ denotes a multilinear operator in its arguments.

Remark 2. There is room for improvement in all of the estimates except for (3.16) and (3.20) which seem to be sharp. In fact, it will be clear from the estimates below that if not for one of these (or both) terms, we could have improved our result. In particular, (3.20) enforces the condition $2r > s - 1$ and gives rise to the restriction $r > -\frac{1}{14}$ if $s > \frac{6}{7}$.

4. PROOF OF (3.13)–(3.16)

The matrix commutator null forms are linear combinations of the ordinary ones, in view of (2.1). Since the matrix structure plays no role in the estimates under consideration, we reduce (3.13)–(3.16) to estimates to the ordinary null forms for \mathbb{C} -valued functions u and v (as in (1.8)).

Substituting

$$\begin{aligned} u &= u_+ + u_-, & \partial_t u &= i\langle \nabla \rangle (u_+ - u_-), \\ v &= v_+ + v_-, & \partial_t v &= i\langle \nabla \rangle (v_+ - v_-), \end{aligned}$$

one obtains

$$\begin{aligned} Q_0(u, v) &= \sum_{\pm, \pm'} (\pm 1)(\pm' 1) [\langle D \rangle u_{\pm} \langle D \rangle v_{\pm'} - (\pm D^i) u_{\pm} (\pm' D_i) v_{\pm'}], \\ Q_{0i}(u, v) &= \sum_{\pm, \pm'} (\pm 1)(\pm' 1) [-\langle D \rangle u_{\pm} (\pm' D_i) v_{\pm'} + (\pm D_i) u_{\pm} \langle D \rangle v_{\pm'}], \\ Q_{ij}(u, v) &= \sum_{\pm, \pm'} (\pm 1)(\pm' 1) [-(\pm D_i) u_{\pm} (\pm' D_j) v_{\pm'} + (\pm D_j) u_{\pm} (\pm' D_i) v_{\pm'}], \\ Q_{ij}(u, \partial_t v) &= \sum_{\pm, \pm'} (\pm 1) [-(\pm D_i) u_{\pm} \langle D \rangle (\pm' D_j) v_{\pm'} + (\pm D_j) u_{\pm} \langle D \rangle (\pm' D_i) v_{\pm'}], \end{aligned}$$

where

$$D = (D_1, D_2, D_3) = \frac{\nabla}{i}$$

has Fourier symbol ξ . In terms of the Fourier symbols

$$\begin{aligned} q_0(\xi, \eta) &= \langle \xi \rangle \langle \eta \rangle - \xi \cdot \eta, \\ q_{0i}(\xi, \eta) &= -\langle \xi \rangle \eta_i + \xi_i \langle \eta \rangle, \\ q_{ij}(\xi, \eta) &= -\xi_i \eta_j + \xi_j \eta_i, \\ q_{ij}^t(\xi, \eta) &= \langle \eta \rangle q_{ij}(\xi, \eta), \end{aligned}$$

we have

$$\begin{aligned} Q_0(u, v) &= \sum_{\pm, \pm'} (\pm 1)(\pm' 1) B_{q_0(\pm\xi, \pm'\eta)}(u_{\pm}, v_{\pm'}), \\ Q_{0i}(u, v) &= \sum_{\pm, \pm'} (\pm 1)(\pm' 1) B_{q_{0i}(\pm\xi, \pm'\eta)}(u_{\pm}, v_{\pm'}), \\ Q_{ij}(u, v) &= \sum_{\pm, \pm'} (\pm 1)(\pm' 1) B_{q_{ij}(\pm\xi, \pm'\eta)}(u_{\pm}, v_{\pm'}), \\ Q_{ij}(u, \partial_t v) &= \sum_{\pm, \pm'} (\pm 1) B_{\langle \eta \rangle \cdot q_{ij}(\pm\xi, \pm'\eta)}(u_{\pm}, v_{\pm'}), \end{aligned}$$

where for a given symbol $\sigma(\xi, \eta)$ we denote by $B_{\sigma(\xi, \eta)}(\cdot, \cdot)$ the operator defined by

$$\mathcal{F}_{t,x} \{ B_{\sigma(\xi, \eta)}(u, v) \}(\tau, \xi) = \int \sigma(\xi - \eta, \eta) \tilde{u}(\tau - \lambda, \xi - \eta) \tilde{v}(\lambda, \eta) d\lambda d\eta.$$

The symbols appearing above satisfy the following estimates.

Lemma 5. [18] *For all nonzero $\xi, \eta \in \mathbb{R}^3$,*

$$\begin{aligned} |q_0(\xi, \eta)| &\lesssim |\xi| |\eta| \theta(\xi, \eta)^2 + \frac{1}{\min(\langle \xi \rangle, \langle \eta \rangle)}, \\ |q_{0j}(\xi, \eta)| &\lesssim |\xi| |\eta| \theta(\xi, \eta) + \frac{|\xi|}{\langle \eta \rangle} + \frac{|\eta|}{\langle \xi \rangle}, \\ |q_{ij}(\xi, \eta)| &\leq |\xi| |\eta| \theta(\xi, \eta), \end{aligned}$$

where $\theta(\xi, \eta) = \arccos\left(\frac{\xi \cdot \eta}{|\xi| |\eta|}\right) \in [0, \pi]$ is the angle between ξ and η . It is this angle which quantifies the null structure in the bilinear terms. In particular, this angle satisfies the following estimate that allows us to trade in hyperbolic regularity and gain a corresponding amount of elliptic regularity.

Lemma 6. *Let $\alpha, \beta, \gamma \in [0, 1/2]$. Then for all pairs of signs (\pm, \pm') , all $\tau, \lambda \in \mathbb{R}$ and all nonzero $\xi, \eta \in \mathbb{R}^3$,*

$$\theta(\pm\xi, \pm'\eta) \lesssim \left(\frac{\langle |\tau + \lambda| - |\xi + \eta| \rangle}{\min(\langle \xi \rangle, \langle \eta \rangle)} \right)^\alpha + \left(\frac{\langle -\tau \pm |\xi| \rangle}{\min(\langle \xi \rangle, \langle \eta \rangle)} \right)^\beta + \left(\frac{\langle -\lambda \pm' |\eta| \rangle}{\min(\langle \xi \rangle, \langle \eta \rangle)} \right)^\gamma.$$

For a proof, see for example [19, Lemma 2.1].

In view of Lemma 5, and since the norms we use only depend on the absolute value of the space-time Fourier transform, we can reduce any estimate for $Q(u, v)$ to a corresponding estimate for the three expressions:

$$B_{\theta(\pm\xi, \pm'\eta)}(|\nabla|u, |\nabla|v), \quad \langle \nabla \rangle u \langle \nabla \rangle^{-1} v \quad \text{and} \quad \langle \nabla \rangle^{-1} u \langle \nabla \rangle v.$$

Consequently, we can reduce (3.13)–(3.16) to the following:

$$(4.1) \quad \left\{ \begin{array}{l} \|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{s-1, a'-1}} \lesssim \|u\|_{X_{\pm}^{s, a}} \|v\|_{X_{\pm'}^{s-1, a}}, \\ \|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{r-1, b'-1}} \lesssim \|u\|_{X_{\pm}^{s, a}} \|v\|_{X_{\pm'}^{r-1, b}}, \\ \|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{r-1, b'-1}} \lesssim \|u\|_{X_{\pm}^{s-1, a}} \|v\|_{X_{\pm'}^{s-1, a}}, \\ \|uv\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{s, a}} \|v\|_{H^{s+1, a}}, \\ \|uv\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{s+2, a}} \|v\|_{H^{s-1, a}}, \\ \|uv\|_{H^{r-1, b'-1}} \lesssim \|u\|_{H^{s, a}} \|v\|_{H^{r+1, b}}, \\ \|uv\|_{H^{r-1, b'-1}} \lesssim \|u\|_{H^{s+1, a}} \|v\|_{H^{r-1, b}}, \\ \|uv\|_{H^{r-1, b'-1}} \lesssim \|u\|_{H^{s-1, a}} \|v\|_{H^{s+1, a}}, \\ \|uv\|_{H^{r-1, b'-1}} \lesssim \|u\|_{H^{s+1, a}} \|v\|_{H^{s-1, a}}, \end{array} \right.$$

where the first, fourth and fifth estimates are reductions of (3.13)–(3.14) and the rest comes from (3.15)–(3.16).

The estimates in (4.1) follow from the following atlas of product estimates in $H^{s, b}$ -spaces which is due to D'Ancona, Foschi and Selberg [1] and null form estimates which is proved in [18]. In fact, the null form estimates follow from the estimate for the angles in Lemma 6 and the product estimates in Theorem 2 below.

Theorem 2 (Product estimates, [1]). *Let $s_0, s_1, s_2 \in \mathbb{R}$ and $b_0, b_1, b_2 \geq 0$. Assume that*

$$\left\{ \begin{array}{l} \sum b_i > \frac{1}{2}, \\ \sum s_i > 2 - \sum b_i, \\ \sum s_i > \frac{3}{2} - \min_{i \neq j} (b_i + b_j), \\ \sum s_i > \frac{3}{2} - \min(b_0 + s_1 + s_2, s_0 + b_1 + s_2, s_0 + s_1 + b_2), \\ \sum s_i \geq 1, \\ \min_{i \neq j} (s_i + s_j) \geq 0, \end{array} \right.$$

and that the last two inequalities are **not both equalities**. Then

$$\|uv\|_{H^{-s_0, b_0}} \lesssim \|u\|_{H^{s_1, b_1}} \|v\|_{H^{s_2, b_2}}$$

holds for all $u, v \in \mathcal{S}(\mathbb{R}^{1+3})$.

Theorem 3 (Null form estimates, [18]). *Let $\sigma_0, \sigma_1, \sigma_2, \beta_0, \beta_1, \beta_2 \in \mathbb{R}$. Assume that*

$$\left\{ \begin{array}{l} 0 \leq \beta_0 < \frac{1}{2} < \beta_1, \beta_2 < 1, \\ \sum \sigma_i + \beta_0 > \frac{3}{2} - (\beta_0 + \sigma_1 + \sigma_2), \\ \sum \sigma_i > \frac{3}{2} - (\sigma_0 + \beta_1 + \sigma_2), \\ \sum \sigma_i > \frac{3}{2} - (\sigma_0 + \sigma_1 + \beta_2), \\ \sum \sigma_i + \beta_0 \geq 1, \\ \min(\sigma_0 + \sigma_1, \sigma_0 + \sigma_2, \beta_0 + \sigma_1 + \sigma_2) \geq 0, \end{array} \right.$$

and that the last two inequalities are **not both equalities**. Then we have the null form estimate

$$\|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{-\sigma_0, -\beta_0}} \lesssim \|u\|_{X_{\pm}^{\sigma_1, \beta_1}} \|v\|_{X_{\pm}^{\sigma_2, \beta_2}}.$$

5. PROOF OF (3.17)–(3.22)

5.1. Proof of (3.17). We write $\Gamma^1 = (\Gamma_0^1, \Gamma_1^1, \Gamma_2^1, \Gamma_3^1)$. Using the substitution (3.1), we write

$$\begin{aligned} \Gamma_0^1(A, \partial A) &= -A_0 \partial^i \mathbf{A}_i + \langle \nabla \rangle^{-1} R_j(\partial_t A_0) \langle \nabla \rangle^{-1} R^j \partial^i(\partial_t \mathbf{A}_i) \\ &\quad + \partial^i \mathbf{A}_i A_0 - \langle \nabla \rangle^{-1} R^j \partial^i(\partial_t \mathbf{A}_i) \langle \nabla \rangle^{-1} R_j(\partial_t A_0) \\ &= - \sum_{\pm, \pm'} \left\{ A_{0, \pm} \partial^i \mathbf{A}_{i, \pm'} + R_j(\pm A_{0, \pm}) R^j(\pm' \partial^i \mathbf{A}_{i, \pm'}) \right. \\ &\quad \left. - \partial^i \mathbf{A}_{i, \pm} A_{0, \pm'} - R^j(\pm \partial^i \mathbf{A}_{i, \pm}) R_j(\pm' A_{0, \pm'}) \right\}, \end{aligned}$$

where in the first two lines we used the Lorenz gauge condition $\partial_t A_0 = \partial^i \mathbf{A}_i$ to replace $\partial_t A_0$ by $\partial^i \mathbf{A}_i$. We do this in order to avoid too much time derivative in the nonlinearity.

Similarly,

$$\begin{aligned} \Gamma_i^1(A, \partial A) &= -A_0 \partial_i A_0 + \langle \nabla \rangle^{-1} R_j(\partial_t A_0) \langle \nabla \rangle^{-1} R^j \partial_t(\partial_i A_0) \\ &\quad + \partial_i A_0 A_0 - \langle \nabla \rangle^{-1} R^j \partial_t(\partial_i A_0) \langle \nabla \rangle^{-1} R_j(\partial_t A_0) \\ &= - \sum_{\pm, \pm'} \left\{ A_{0, \pm} \partial_i A_{0, \pm'} + R_j(\pm A_{0, \pm}) R^j(\pm' \partial_i A_{0, \pm'}) \right. \\ &\quad \left. - \partial_i A_{0, \pm} A_{0, \pm'} - R^j(\pm \partial_i A_{0, \pm}) R_j(\pm' A_{0, \pm'}) \right\}. \end{aligned}$$

We can write

$$(5.1) \quad \Gamma_0^1(A, \partial A) = - \sum_{\pm, \pm'} \{ \mathcal{P}_{\pm, \pm'}(A_{0, \pm}, \partial^i \mathbf{A}_{i, \pm'}) - \mathcal{P}_{\pm, \pm'}(\partial^i \mathbf{A}_{i, \pm}, A_{0, \pm'}) \},$$

$$(5.2) \quad \Gamma_i^1(A, \partial A) = - \sum_{\pm, \pm'} \{ \mathcal{P}_{\pm, \pm'}(A_{0, \pm}, \partial_i A_{0, \pm'}) - \mathcal{P}_{\pm, \pm'}(\partial_i A_{0, \pm}, A_{0, \pm'}) \},$$

where

$$(5.3) \quad \mathcal{P}_{\pm, \pm'}(u, v) = uv + R_j(\pm u)R^j(\pm' v)$$

which has symbol

$$(5.4) \quad p_{\pm, \pm'}(\eta, \zeta) = 1 - \frac{(\pm\eta) \cdot (\pm'\zeta)}{\langle \eta \rangle \langle \zeta \rangle}.$$

This symbol satisfies the estimate

$$(5.5) \quad |p_{\pm, \pm'}(\eta, \zeta)| \lesssim \theta^2(\pm\eta, \pm'\zeta) + \frac{1}{\langle \eta \rangle \langle \zeta \rangle}.$$

Indeed, if $\pm = +$ and $\pm' = \pm$, then

$$p_{+, \pm}(\eta, \zeta) = 1 - \frac{\eta \cdot (\pm\zeta)}{\langle \eta \rangle \langle \zeta \rangle} = \frac{(|\eta||\zeta| - \eta \cdot (\pm\zeta)) + (\langle \eta \rangle \langle \zeta \rangle - |\eta||\zeta|)}{\langle \eta \rangle \langle \zeta \rangle}.$$

But

$$|\eta||\zeta| - \eta \cdot (\pm\zeta) \leq |\eta||\zeta|(1 - \cos(\theta(\eta, \pm\zeta))) \lesssim |\eta||\zeta|\theta^2(\eta, \pm\zeta)$$

and

$$\langle \eta \rangle \langle \zeta \rangle - |\eta||\zeta| = \frac{1 + |\eta|^2 + |\zeta|^2}{\langle \eta \rangle \langle \zeta \rangle + |\eta||\zeta|} \leq 1.$$

Then the estimate (5.5) follows.

So in view of (5.1)–(5.5) and by symmetry, the estimate (3.17) can be reduced to the following:

$$\begin{cases} \|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{s-1, a'-1}} \lesssim \|u\|_{X_{\pm}^{s, a}} \|v\|_{X_{\pm'}^{s-1, a}}, \\ \|uv\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{s+1, a}} \|v\|_{H^{s, a}}, \end{cases}$$

where the first estimate reduces to the first estimate in (4.1) while the second estimate holds by Theorem 2.

5.2. Proof of (3.18)–(3.22). Using (3.1), if necessary, the estimates (3.18)–(3.22) reduce to

$$\begin{cases} \|uv\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{s, a}} \|v\|_{H^{s+1, a}} \\ \|uv\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{s+2, a}} \|v\|_{H^{s-1, a}} \\ \|uv\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{r+1, b}} \|v\|_{H^{r, b}} \\ \|uv\|_{H^{r-1, b'-1}} \lesssim \|u\|_{H^{s+2, a}} \|v\|_{H^{r-1, b}} \\ \|uv\|_{H^{r-1, b'-1}} \lesssim \|u\|_{H^{s+1, a}} \|v\|_{H^{s-1, a}}, \end{cases}$$

all of which hold by Theorem 2.

5.3. Proof of (3.23)–(3.28). The estimates (3.23)–(3.28) reduce to the following:

$$(5.6) \quad \|u\langle \nabla \rangle^{-1}(vw)\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{r+1, b}} \|v\|_{H^{s, a}} \|w\|_{H^{s-1, a}},$$

$$(5.7) \quad \|u\langle \nabla \rangle^{-1}(vw)\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{r, b}} \|v\|_{H^{s, a}} \|w\|_{H^{s, a}},$$

$$(5.8) \quad \|\langle \nabla \rangle^{-1}(uv)\langle \nabla \rangle^{-1}(wz)\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{s, a}} \|v\|_{H^{s, a}} \|w\|_{H^{s, a}} \|z\|_{H^{s-1, a}},$$

$$(5.9) \quad \|uvw\|_{H^{s-1, a'-1}} \lesssim \|u\|_{H^{s, a}} \|v\|_{H^{s, a}} \|w\|_{H^{s, a}},$$

$$(5.10) \quad \|uvw\|_{H^{r-1, b'-1}} \lesssim \|u\|_{H^{s, a}} \|v\|_{H^{s, a}} \|w\|_{H^{r, b}},$$

$$(5.11) \quad \|uvwz\|_{H^{r-1, b'-1}} \lesssim \|u\|_{H^{s, a}} \|v\|_{H^{s, a}} \|w\|_{H^{s, a}} \|z\|_{H^{s, a}}.$$

The trilinear estimates (5.6), (5.7), (5.9) and (5.10) follows by applying Theorem 2 twice as follows:

$$\begin{aligned}
\|u\langle\nabla\rangle^{-1}(vw)\|_{H^{s-1,a'-1}} &\lesssim \|u\|_{H^{r+1,b}} \|\langle\nabla\rangle^{-1}(vw)\|_{H^{s-\frac{1}{2},0}} \\
&= \|u\|_{H^{r+1,b}} \|vw\|_{H^{s-\frac{3}{2},0}} \\
&\lesssim \|u\|_{H^{r+1,b}} \|v\|_{H^{s,a}} \|w\|_{H^{s-1,a}}, \\
\|u\langle\nabla\rangle^{-1}(vw)\|_{H^{s-1,a'-1}} &\lesssim \|u\|_{H^{r,b}} \|\langle\nabla\rangle^{-1}(vw)\|_{H^{s+\frac{1}{2},0}} \\
&= \|u\|_{H^{r,b}} \|vw\|_{H^{s-\frac{1}{2},0}} \\
&\lesssim \|u\|_{H^{r,b}} \|v\|_{H^{s,a}} \|w\|_{H^{s,a}}, \\
\|uvw\|_{H^{s-1,a'-1}} &\lesssim \|u\|_{H^{s,a}} \|vw\|_{H^{s-\frac{1}{2},0}} \\
&\lesssim \|u\|_{H^{s,a}} \|v\|_{H^{s,a}} \|w\|_{H^{s,a}}, \\
\|uvw\|_{H^{r-1,b'-1}} &\lesssim \|uv\|_{H^{s-\frac{1}{2},0}} \|w\|_{H^{r,b}} \\
&\lesssim \|u\|_{H^{s,a}} \|v\|_{H^{s,a}} \|w\|_{H^{r,b}}.
\end{aligned}$$

The quadrilinear (5.8) and (5.11) follow by applying Theorem 2 and [1, Theorem 8.1] follows:

$$\begin{aligned}
\|\langle\nabla\rangle^{-1}(uv)\langle\nabla\rangle^{-1}(wz)\|_{H^{s-1,a'-1}} &\lesssim \|\langle\nabla\rangle^{-1}(uv)\|_{H^{s+\frac{2}{7},\frac{1}{2}}} \|\langle\nabla\rangle^{-1}(wz)\|_{H^{0,0}} \\
&= \|uv\|_{H^{s-\frac{5}{7},\frac{1}{2}}} \|wz\|_{H^{-1,0}} \\
&\lesssim \|u\|_{H^{s,a}} \|v\|_{H^{s,a}} \|w\|_{H^{s,a}} \|z\|_{H^{s-1,a}}, \\
\|uvwz\|_{H^{r-1,b'-1}} &\lesssim \|uv\|_{H^{\frac{2}{7},\frac{1}{4}}} \|vwz\|_{H^{\frac{2}{7},\frac{1}{4}}} \\
&\lesssim \|u\|_{H^{s,a}} \|v\|_{H^{s,a}} \|w\|_{H^{s,a}} \|z\|_{H^{s,a}}.
\end{aligned}$$

6. PROOF OF LEMMA 4

By assumption the solution (A_{\pm}, F_{\pm}) to the system (3.2) lies in the regularity class

$$(6.1) \quad A_{\pm} \in C([-T, T]; H^{1-\varepsilon}), \quad F_{\pm} \in C([-T, T]; L^2).$$

For sufficiently small σ such that $1 \gg \sigma \gg \varepsilon$, we claim

$$(6.2) \quad A_{\pm} \in X_{\pm}^{1-\sigma, 1-\sigma}(S_T),$$

$$(6.3) \quad F_{\pm} \in X_{\pm}^{-\sigma, \frac{1}{2}+\sigma}(S_T),$$

which are in fact far better than the desired regularities in Lemma 4.

To prove (6.2) we start with the assumed regularity for A_{\pm} in (6.1) and use the equations for A_{\pm} to successively improve the regularity by applying Strichartz, Hölder and Sobolev inequalities until we reach (6.2). The regularity of F_{\pm} in (6.3) is determined from (6.2) and the relation (1.1) by applying the product estimates in Theorem 2. We do not need the null structures in the nonlinear terms, and so we write the equations for A_{\pm} in the original form.

Consider the solutions A_{\pm} to the equations

$$(6.4) \quad (i\partial_t \pm \langle\nabla\rangle)A_{\pm} = \mp \frac{1}{2\langle\nabla\rangle} M'(A_+, A_-),$$

where

$$M'(A_+, A_-) = -A - 2[A^\alpha, \partial_\alpha A] + [A^\alpha, \partial A_\alpha] - [A^\alpha, [A_\alpha, A]].$$

We have thus replaced \mathfrak{M}' in (3.2) which was written in terms of the null forms by M' which is the same but in its original form. Now write

$$A_\pm = A_\pm^{(0)} + A_\pm^{(1)} + A_\pm^{(2)},$$

where $A_\pm^{(0)}$ is the homogeneous part while $A_\pm^{(1)}$ and $A_\pm^{(2)}$ are the inhomogeneous parts corresponding to the linear and nonlinear terms, respectively. First note that by (3.6) and assumption (6.1)

$$A_\pm^{(0)}, A_\pm^{(1)} \in X_\pm^{1-\varepsilon, 1}(S_T)$$

which imply (also using (6.1))

$$A_\pm^{(2)} = A_\pm - A_\pm^{(0)} - A_\pm^{(1)} \in X_\pm^{1-\varepsilon, 0}(S_T).$$

It remains to prove

$$(6.5) \quad A_\pm^{(2)} \in X_\pm^{1-\sigma, 1-\sigma}(S_T).$$

To this end, we start with (6.1) and use the following Strichartz estimates to successively improve the regularity.

Lemma 7. [16, Lemma 7.1] *Suppose $2 < q \leq \infty$ and $2 \leq r < \infty$ satisfy $\frac{1}{2} \leq \frac{1}{q} + \frac{1}{r} \leq 1$. Then*

$$\|u\|_{L_t^q L_x^r} \lesssim \| |\nabla|^{1-\frac{2}{r}} u \|_{H^{0, 1-(\frac{1}{q}+\frac{1}{r})+\gamma}}$$

holds for any $\gamma > 0$.

In what follows we always assume γ to be sufficiently small.

We also need the following simple inequality:

$$(6.6) \quad \|f|\nabla|g\|_{L^2} \lesssim \| |\nabla|^\varepsilon f |\nabla|^{1-\varepsilon} g \|_{L^2} + \|f|\nabla|^{1-\varepsilon} g\|_{H^\varepsilon}.$$

Indeed, let the Fourier variables of f and g be η and $\xi - \eta$, respectively, so that ξ is the output frequency for $f|\nabla|g$. Now, if $|\xi| \lesssim |\eta| \sim |\xi - \eta|$ then the left hand side of (6.6) is bounded by the first term on the right hand side, while if $|\eta| \lesssim |\xi - \eta| \sim |\xi|$ or $|\xi - \eta| \lesssim |\eta| \sim |\xi|$ it is bounded by the second term on the right hand side.

6.1. First estimate for $A_\pm^{(2)}$. We claim that

$$(6.7) \quad A_\pm^{(2)} \in X_\pm^{\frac{2}{3}-2\varepsilon, \frac{5}{6}-2\varepsilon}(S_T).$$

By (3.6) and using (3.1), it suffices to show

$$A|\nabla|A, A^3 \in X_\pm^{-\frac{1}{3}-2\varepsilon, -\frac{1}{6}-2\varepsilon}(S_T).$$

In view of (6.6), we reduce to proving

$$A|\nabla|^{1-\varepsilon}A \in X_\pm^{-\frac{1}{3}-\varepsilon, -\frac{1}{6}-2\varepsilon}(S_T), \quad |\nabla|^\varepsilon A|\nabla|^{1-\varepsilon}, A^3 \in X_\pm^{-\frac{1}{3}-2\varepsilon, -\frac{1}{6}-2\varepsilon}(S_T).$$

Using Lemma 7 and duality, we have for some $q' < 2$

$$\begin{aligned} \|A|\nabla|^{1-\varepsilon}A\|_{X_{\pm}^{-\frac{1}{3}-\varepsilon, -\frac{1}{6}-2\varepsilon}(S_T)} &\lesssim \|A|\nabla|^{1-\varepsilon}A\|_{L_t^{q'}L_x^{\frac{6}{4+3\varepsilon}}(S_T)} \\ &\lesssim \|A\|_{L_t^\infty L_x^{\frac{6}{1+3\varepsilon}}(S_T)} \| |\nabla|^{1-\varepsilon}A \|_{L_t^\infty L_x^2(S_T)} \\ &\lesssim \|A\|_{L_t^\infty H^{1-\varepsilon}(S_T)}^2, \end{aligned}$$

where we used Sobolev embedding in the last inequality. Similarly,

$$\begin{aligned} \| |\nabla|^\varepsilon A |\nabla|^{1-\varepsilon} A \|_{X_{\pm}^{-\frac{1}{3}-2\varepsilon, -\frac{1}{6}-2\varepsilon}} &\lesssim \| |\nabla|^\varepsilon A |\nabla|^{1-\varepsilon} A \|_{L_t^{q'}L_x^{\frac{3}{2+3\varepsilon}}(S_T)} \\ &\lesssim \| |\nabla|^\varepsilon A \|_{L_t^\infty L_x^{\frac{6}{1+6\varepsilon}}(S_T)} \| |\nabla|^{1-\varepsilon} A \|_{L_t^\infty L_x^2(S_T)} \\ &\lesssim \|A\|_{L_t^\infty H^{1-\varepsilon}(S_T)}^2 \end{aligned}$$

and

$$\|A^3\|_{X_{\pm}^{-\frac{1}{3}-2\varepsilon, -\frac{1}{3}-2\varepsilon}(S_T)} \lesssim \|A^3\|_{L_t^{q'}L_x^{\frac{3}{2+3\varepsilon}}(S_T)} \lesssim \|A\|_{L_t^\infty L_x^{\frac{9}{2+3\varepsilon}}(S_T)}^3 \lesssim \|A\|_{L_t^\infty H^{1-\varepsilon}(S_T)}^3.$$

6.2. Inductive estimates. We claim that, for $m = 1, 2, \dots$,

$$(6.8) \quad A_{\pm}^{(2)} \in X_{\pm}^{s_m - \varepsilon_m, b_m - \varepsilon_m}(S_T),$$

where

$$s_m = 1 - \frac{1}{2m+1}, \quad b_m = 1 - \frac{1}{2m+1+2}, \quad \varepsilon_m = (m+1)\varepsilon.$$

Granted this claim we can then choose m sufficiently large to obtain (6.5).

Note that the claim (6.8) is true for $m = 1$ by (6.7). We shall prove that if (6.8) holds for some $m \geq 1$, then it holds for $m+1$ also.

Interpolating (6.8) with $A_{\pm}^{(2)} \in X_{\pm}^{1-\varepsilon, 0}(S_T)$, we get

$$A_{\pm}^{(2)} \in X_{\pm}^{\theta(1-\varepsilon)+(1-\theta)(s_m-\varepsilon_m), (1-\theta)(b_m-\varepsilon_m)}(S_T)$$

for all $0 \leq \theta \leq 1$ and $\varepsilon > 0$. Take $\theta = \frac{2^m}{2^{m+1}+1}$ to obtain

$$(6.9) \quad A_{\pm}^{(2)} \in X_{\pm}^{s_{m+1}-\varepsilon_m, \frac{1}{2}-\varepsilon_m}(S_T).$$

Then by Lemma 7

$$(6.10) \quad A_{\pm}^{(2)} \in L_t^q L_x^r(S_T), \quad |\nabla|^\varepsilon A_{\pm}^{(2)} \in L_t^{\frac{2q}{2-q\varepsilon}} L_x^{\frac{2r}{2+r\varepsilon}}(S_T),$$

where

$$\frac{1}{q} = \frac{2m+1}{2m+2} + \frac{\varepsilon_m}{2} + \gamma, \quad \frac{1}{r} = \frac{1}{2m+2} + \frac{\varepsilon_m}{2}.$$

Recall also that $A_{\pm}^{(0)}$ and $A_{\pm}^{(1)}$ has better regularity than (6.9). This in turn implies they also satisfy the property (6.10), and we can therefore conclude

$$(6.11) \quad A_{\pm} \in L_t^q L_x^r(S_T), \quad |\nabla|^\varepsilon A_{\pm} \in L_t^{\frac{2q}{2-q\varepsilon}} L_x^{\frac{2r}{2+r\varepsilon}}(S_T).$$

Using (3.6), the induction claim

$$A_{\pm}^{(2)} \in X_{\pm}^{s_{m+1}-\varepsilon_{m+1}, b_{m+1}-\varepsilon_{m+1}}(S_T)$$

reduces to

$$A|\nabla|A, \quad A^3 \in X_{\pm}^{-1+s_{m+1}-\varepsilon_{m+1}, -1+b_{m+1}-\varepsilon_{m+1}}(S_T).$$

For the quadratic term $A|\nabla|A$, in view of (6.6), it suffices to show

$$\begin{aligned} A|\nabla|^{1-\varepsilon}A &\in X_{\pm}^{-1+s_{m+1}-\varepsilon_m, -1+b_{m+1}-\varepsilon_{m+1}}(S_T), \\ |\nabla|^{\varepsilon}A|\nabla|^{1-\varepsilon}A &\in X_{\pm}^{-1+s_{m+1}-\varepsilon_{m+1}, -1+b_{m+1}-\varepsilon_{m+1}}(S_T). \end{aligned}$$

By Lemma 7 and duality, we thus reduce to proving for some $l \leq 2$

$$A|\nabla|^{1-\varepsilon}A \in L_t^l L_x^{\frac{2r}{2+r}}(S_T), \quad |\nabla|^{\varepsilon}A|\nabla|^{1-\varepsilon}A, \quad A^3 \in L_t^l L_x^{\frac{2r}{2+r(1+\varepsilon)}}(S_T).$$

But by Hölder and Sobolev inequality

$$\begin{aligned} \|A|\nabla|^{1-\varepsilon}A\|_{L_t^l L_x^{\frac{2r}{2+r}}(S_T)} &\lesssim \|A\|_{L_t^q L_x^r(S_T)} \|A\|_{L_t^{\infty} H^{1-\varepsilon}(S_T)}, \\ \| |\nabla|^{\varepsilon}A|\nabla|^{1-\varepsilon}A \|_{L_t^l L_x^{\frac{2r}{2+r(1+\varepsilon)}}(S_T)} &\lesssim \| |\nabla|^{\varepsilon}A \|_{L_t^{\frac{2q}{2-q\varepsilon}} L_x^{\frac{2r}{2+r\varepsilon}}(S_T)} \|A\|_{L_t^{\infty} H^{1-\varepsilon}(S_T)} \end{aligned}$$

and

$$\|A^3\|_{L_t^l L_x^{\frac{2r}{2+r(1+\varepsilon)}}(S_T)} \lesssim \|A\|_{L_t^{\infty} L_x^{\frac{6r}{2+r(1+\varepsilon)}}(S_T)}^3 \lesssim \|A\|_{L_t^{\infty} H^{1-\varepsilon}(S_T)}^3,$$

where in the first two inequalities we used (6.11).

6.3. Proof of (6.3). Using (1.1) and (3.1), we can write

$$(6.12) \quad \begin{cases} F_{ij,\pm} = \partial_i A_{j,\pm} - \partial_j A_{i,\pm} + \frac{1}{2} \left([A_i, A_j] \pm \frac{1}{i\langle \nabla \rangle} \partial_t [A_i, A_j] \right), \\ F_{0i,\pm} = \partial_t A_{i,\pm} - \partial_i A_{0,\pm} + \frac{1}{2} \left([A_0, A_i] \pm \frac{1}{i\langle \nabla \rangle} \partial_t [A_0, A_i] \right), \end{cases}$$

where $\partial_t A$ in the parentheses can also be written in terms of A_{\pm} .

In view of (6.12), the claim (6.3) reduces to

$$(6.13) \quad A^2, \quad \langle \nabla \rangle^{-1}(A\partial A) \in X_{\pm}^{-\sigma, \frac{1}{2}+\sigma}(S_T),$$

$$(6.14) \quad \partial A_{\pm} \in X_{\pm}^{-\sigma, \frac{1}{2}+\sigma}(S_T).$$

Let us first prove (6.13). Using (6.2) and the substitution (3.1), this reduces to

$$(6.15) \quad \|uv\|_{X_{\pm}^{-\sigma, \frac{1}{2}+\sigma}} \lesssim \|u\|_{X_{\pm 1}^{1-\sigma, 1-\sigma}} \|v\|_{X_{\pm 2}^{1-\sigma, 1-\sigma}},$$

$$(6.16) \quad \|uv\|_{X_{\pm}^{-1-\sigma, \frac{1}{2}+\sigma}} \lesssim \|u\|_{X_{\pm 1}^{1-\sigma, 1-\sigma}} \|v\|_{X_{\pm 2}^{-\sigma, 1-\sigma}}.$$

We would like to reduce (6.15)–(6.16) to the corresponding estimates where all the X norms are replaced by H norms so that we can apply the product estimates in Theorem 2 to prove them. In view of (3.7), we can do this to the right hand side, but not to the left hand side since the hyperbolic exponents to the X norms are positive, i.e., $\frac{1}{2} + \sigma$. We can nevertheless do the following trick to fix the problem. Let (λ, η) and $(\tau - \lambda, \xi - \eta)$ be the Fourier variables for u and v respectively, such that (τ, ξ) is the Fourier variable for the product uv . Then we have the identity

$$-\tau \pm |\xi| = (-\lambda \pm_1 |\eta|) + (-\tau - \lambda) \pm_2 |\xi - \eta| + (\pm |\xi| \mp_1 |\eta| \mp_2 |\xi - \eta|),$$

which implies

$$(6.17) \quad |-\tau \pm |\xi|| \lesssim |-\lambda \pm_1 |\eta|| + |-\tau - \lambda) \pm_2 |\xi - \eta|| + \max(|\eta|, |\xi - \eta|).$$

Now, using this estimate we can reduce (6.15) (here we also use symmetry)

$$\begin{cases} \|uv\|_{H^{-\sigma,0}} \lesssim \|u\|_{H^{1-\sigma,\frac{1}{2}-2\sigma}} \|v\|_{H^{1-\sigma,1-\sigma}}, \\ \|uv\|_{H^{-\sigma,0}} \lesssim \|u\|_{H^{\frac{1}{2}-2\sigma,1-\sigma}} \|v\|_{H^{1-\sigma,1-\sigma}} \end{cases}$$

both of which hold by Theorem 2. Similarly, (6.16) reduces to

$$\begin{cases} \|uv\|_{H^{-1-\sigma,0}} \lesssim \|u\|_{H^{1-\sigma,\frac{1}{2}-2\sigma}} \|v\|_{H^{-\sigma,1-\sigma}} \\ \|uv\|_{H^{-1-\sigma,0}} \lesssim \|u\|_{H^{1-\sigma,1-\sigma}} \|v\|_{H^{-\sigma,\frac{1}{2}-2\sigma}} \\ \|uv\|_{H^{-1-\sigma,0}} \lesssim \|u\|_{H^{\frac{1}{2}-2\sigma,1-\sigma}} \|v\|_{H^{-\sigma,1-\sigma}} \\ \|uv\|_{H^{-1-\sigma,0}} \lesssim \|u\|_{H^{1-\sigma,1-\sigma}} \|v\|_{H^{-\frac{1}{2}-2\sigma,1-\sigma}}, \end{cases}$$

all of which hold by Theorem 2.

Finally, we prove (6.14). Clearly, $\partial_j A_{\pm} \in X_{\pm}^{-\sigma, \frac{1}{2}+\sigma}(S_T)$ by (6.2). We remain to prove

$$\partial_t A_{\pm} \in X_{\pm}^{-\sigma, \frac{1}{2}+\sigma}(S_T).$$

To do this, we first use the equation (6.4) to write

$$\partial_t A_{\pm} = \pm i \langle \nabla \rangle A_{\pm} \pm i (2 \langle \nabla \rangle)^{-1} M'(A_+, A_-).$$

The term $\langle \nabla \rangle A_{\pm}$ is clearly in $X_{\pm}^{-\sigma, \frac{1}{2}+\sigma}(S_T)$. Looking at the terms in M' , we then reduce to proving

$$A, A \partial A, A^3 \in X_{\pm}^{-1-\sigma, \frac{1}{2}+\sigma}(S_T).$$

For the linear term $A = A_+ + A_-$, it suffices to show $A_{\mp} \in X_{\pm}^{-1-\sigma, \frac{1}{2}+\sigma}(S_T)$. But using the simple inequality $\langle -\tau \pm |\xi| \rangle \lesssim \langle -\tau \mp |\xi| \rangle \langle \xi \rangle$, this reduces to proving $A_{\mp} \in X_{\mp}^{-\frac{1}{2}, \frac{1}{2}+\sigma}(S_T)$, which holds by (6.2). By (6.13) the bilinear term $A \partial A$ is also in $X_{\pm}^{-1-\sigma, \frac{1}{2}+\sigma}(S_T)$.

It remains to prove that the cubic term $A^3 \in X_{\pm}^{-1-\sigma, \frac{1}{2}+\sigma}(S_T)$. Since $A^2 \in X_{\pm}^{-\sigma, \frac{1}{2}+\sigma}(S_T)$ by (6.13), we reduce to

$$\|uv\|_{X_{\pm}^{-1-\sigma, \frac{1}{2}+\sigma}} \lesssim \|u\|_{X_{\pm 1}^{1-\sigma, 1-\sigma}} \|v\|_{X_{\pm 2}^{-\sigma, \frac{1}{2}+\sigma}}.$$

Again using (6.17), this reduces to

$$\begin{cases} \|uv\|_{H^{-1-\sigma,0}} \lesssim \|u\|_{H^{1-\sigma,\frac{1}{2}-2\sigma}} \|v\|_{H^{-\sigma,\frac{1}{2}+\sigma}}, \\ \|uv\|_{H^{-1-\sigma,0}} \lesssim \|u\|_{H^{1-\sigma,1-\sigma}} \|v\|_{H^{-\sigma,0}}, \\ \|uv\|_{H^{-1-\sigma,0}} \lesssim \|u\|_{H^{\frac{1}{2}-2\sigma,1-\sigma}} \|v\|_{H^{-\sigma,\frac{1}{2}+\sigma}}, \\ \|uv\|_{H^{-1-\sigma,0}} \lesssim \|u\|_{H^{1-\sigma,1-\sigma}} \|v\|_{H^{-\frac{1}{2}-2\sigma,\frac{1}{2}+\sigma}} \end{cases}$$

all of which hold by Theorem 2.

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