NON-ARITHMETIC HYPERBOLIC REFLECTION GROUPS IN HIGHER DIMENSIONS

E. B. VINBERG

ABSTRACT. We construct examples of non-arithmetic (non-cocompact) cofinite discrete reflection groups in n-dimensional Lobachevsky spaces L^n for $n \leq 18$, $n \neq 13, 15, 16, 17$.

1. INTRODUCTION

In 1988, M. Gromov and I. I. Piatetski-Shapiro [4] proved that in the Lobachevsky space L^n of any dimension n there exist non-arithmetic cofinite discrete groups of motions (both cocompact and non-cocompact). They constructed such groups as a sort of mixture of two different arithmetic groups.

However, non-arithmetic cofinite discrete reflection groups in L^n (both cocompact and non-cocompact) were only known for $n \leq 5$ by that time: see, e.g., [10, Chapter 6, Subsection 3.2]. In 1989, O. Ruzmanov [6] constructed such (non-cocompact) reflection groups for $6 \leq n \leq 10$ as mixtures of arithmetic reflection groups.

In the present paper, a more systematic way of constructing non-arithmetic cofinite discrete reflection groups is suggested and new examples of such groups up to dimension 18, excluding $n = 13, 15, 16, 17$, are constructed.

2. PRELIMINARIES

We use the model of the n-dimensional Lobachevsky (=hyperbolic) space L^n in the $(n+1)$-dimensional Minkowski vector space $E^{n,1}$ as the connected component of the hyperboloid $\{x \in E^{n,1} : (x, x) = -1\}$ lying in the "future cone". The group of motions of L^n in this model is the subgroup $O_{n,1}^+$ of index 2 in the group $O_{n,1}$ of (pseudo)orthogonal transformations of $E^{n,1}$, taking the future cone into itself. The group of proper motions is $SO_{n,1}^+ = O_{n,1}^+ \cap SO_{n,1}$.

The hyperplanes in L^n are the subsets of the form

$$H = H_e = \{x \in L^n : (e, x) = 0\}, \text{ where } (e, e) > 0.$$

Hyperplanes H_e and H_f intersect (are parallel, diverge) if and only if the subspace $\langle e, f \rangle \subset E^{n,1}$ generated by e and f is elliptic (parabolic, hyperbolic). If H_e and H_f intersect, the angle between them is equal to $\pi - \widehat{ef}$, where $\widehat{ef}$ is the angle between the vectors e and f.

The reflection r_H in the hyperplane $H = H_e$ is induced by the linear reflection

$$R_e : x \mapsto x - \frac{2(e, x)}{(e, e)} e.$$

We will consider discrete groups of motions of L^n, i.e. discrete subgroups of the Lie group $\mathrm{O}_{n,1}^+$. A discrete group $\Gamma \subset \mathrm{O}_{n,1}^+$ is called *cofinite (cocompact)*, if $\mathrm{vol}\, \mathrm{L^n}/\Gamma < \infty$ (L^n/Γ is compact).

If $\Gamma \subset \mathrm{O}_{n,1}^+$ ($n \geq 2$) is a cofinite discrete group, then by the Borel density theorem (see, e.g., [5, Theorem 5.5]) the group $\Gamma \cap \mathrm{SO}_{n,1}^+$ is Zariski dense in $\mathrm{SO}_{n,1}^+$ and, hence, absolutely irreducible as a linear group.

In particular, if $L \subset E^{n,1}$ is an (integral) *lattice*, then

$$\mathrm{O}^+(\mathrm{L}) = \{\gamma \in \mathrm{O}_{n,1}^+ : \gamma \mathrm{L} = \mathrm{L}\}$$

is an (arithmetic) cofinite discrete group.

Proposition 1. *Let L_1 and L_2 be two lattices in $E^{n,1}$. The groups $\mathrm{O}^+(\mathrm{L_1})$ and $\mathrm{O}^+(\mathrm{L_2})$ are commensurable if and only if the lattices cL_1 and L_2 are commensurable for some $c \in \mathbb{R}$ with $c^2 \in \mathbb{Q}$.*

Note that the latter condition means that $c\mathbb{Q}L_1 = \mathbb{Q}L_2$.

Proof. The "if" part is obvious, since $\mathrm{O}^+(\mathrm{cL}) = \mathrm{O}^+(\mathrm{L})$ for any lattice L, and any subgroup of finite index in L is invariant under some subgroup of finite index in $\mathrm{O}^+(\mathrm{L})$.

Conversely, if the groups $\mathrm{O}^+(\mathrm{L_1})$ and $\mathrm{O}^+(\mathrm{L_2})$ are commensurable, then their intersection Γ is a cofinite discrete group leaving invariant both L_1 and L_2. Writing elements of Γ in some bases of the lattices L_1 and L_2 yields some matrix representations of Γ over $\mathbb{Q}$, which are isomorphic over $\mathbb{R}$ and, hence, also over $\mathbb{Q}$. This means that there is a linear isomorphism $C : \mathbb{Q}L_1 \to \mathbb{Q}L_2$ commuting with the action of Γ. Since Γ is absolutely irreducible, $C = c\mathrm{Id}$ for some $c \in \mathbb{R}$. Since $c\mathbb{Q}L_1 = \mathbb{Q}L_2$ is a quadratic vector space over $\mathbb{Q}$, one must have $c^2 \in \mathbb{Q}$. $\qquad \square$

Let $\Gamma \subset \mathrm{O}_{n,1}^+$ be a discrete group containing a reflection $r = r_H$ such that

(*) for any $\gamma \in \Gamma$ either $\gamma H = H$, or $\gamma H \cap H = \emptyset$.

Note that this condition automatically holds if Γ does not contain non-trivial proper motions of finite order. If the group Γ is finitely generated (in particular, if it is cofinite), the latter can be always achieved by passing to a suitable subgroup of finite index.

The hyperplanes γH, $\gamma \in \Gamma$, decompose the space into some (closed) pieces transitively permuted by Γ. Each of these pieces is a fundamental domain for the subgroup $N \subset \Gamma$. generated by the reflections in these hyperplanes (the normal closure of r in Γ). Let D be one of these pieces, and let

$$\Delta = \{\gamma \in \Gamma : \gamma D = D\}.$$

Then

$$\Gamma = N \rtimes \Delta.$$

3. Mixtures of discrete groups

Let now $\Gamma_1, \Gamma_2 \subset O_{n,1}^+$ be two cofinite discrete groups containing a reflection $r = r_H$ and satisfying condition (*). Assume that Γ_1 and Γ_2 induce the same group Γ_0 on $H = L^{n-1}$. This means that

$$Z_{\Gamma_1}(r) = Z_{\Gamma_2}(r) = \langle r \rangle \times \Gamma_0,$$

where $\Gamma_0 \subset O_{n-1,1}^+$ is embedded into $O_{n,1}^+$ so that it leaves invariant each of the two half-spaces bounded by H.

Let N_1 (resp. N_2) be the normal closure of r in G_1 (resp. G_2). Choose fundamental domains D_1 and D_2 for N_1 and N_2 containing the hyperplane H and lying on different sides of it. Set

$$\Delta_1 = \{\gamma \in \Gamma_1 : \gamma D = D\}, \quad \Delta_2 = \{\gamma \in \Gamma_2 : \gamma D = D\}.$$

Then

$$\Gamma_0 = \Delta_1 \cap \Delta_2.$$

As above, we have

$$\Gamma_1 = N_1 \rtimes \Delta_1, \quad \Gamma_2 = N_2 \rtimes \Delta_2.$$

Theorem 1. *(M. Gromov, I. I. Piatetski-Shapiro [4]) The group*

$$\Gamma = \langle \Delta_1, \Delta_2 \rangle \subset O_{n,1}^+,$$

generated by Δ_1 and Δ_2, is a cofinite discrete group. If Γ_1 and Γ_2 are non-commensurable arithmetic groups and $n \geq 3$, then Γ is not arithmetic.

Moreover,

$$\mathrm{vol}\, L^n/\Gamma = \mathrm{vol}\, L^n/\Gamma_1 + \mathrm{vol}\, L^n/\Gamma_2,$$

and Γ is cocompact if and only if both Γ_1 and Γ_2 are cocompact. As an abstract group, $\Gamma = \Delta_1 \underset{\Gamma_0}{*} \Delta_2$ (the free product of Δ_1 and Δ_2 amalgamated along Γ_0).

We call Γ a **(Gromov–Piatetski-Shapiro) mixture** of Γ_1 and Γ_2.

For (discrete) reflection groups, there is a quite transparent version of a Gromov–Piatetski-Shapiro mixture.

Let $\Gamma_1, \Gamma_2 \subset O_{n,1}$ be two cofinite reflection groups. Assume that their fundamental polyhedra P_1, P_2 have congruent hyperfaces, which are perpendicular to all adjacent hyperfaces. Gluing P_1 and P_2 together along these hyperfaces, we obtain a new Coxeter polyhedron P. The corresponding reflection group Γ is a Gromov–Piatetski-Shapiro mixture of Γ_1 and Γ_2.

More precisely, let H be the common wall of P_1 and P_2, i.e. the hyperplane containing their common hyperface. Then the group Δ_1 (resp. Δ_2) is generated by the reflections in all the walls of P_1 (resp. P_2) but H, the group Γ_0 is generated by the reflections in all the common walls of P_1 and P_2 but H, and the group Γ is generated by the reflections in all the walls of P_1 and P_2 but H.

4. Roots of lattices

Let $L \subset E^{n,1}$ be an (integral) lattice. If the reflection R_α defined by a vector $\alpha \in E^{n,1}$ with $(\alpha, \alpha) > 0$ belongs to $O^+(L)$, then α is proportional to a vector of L, so one may assume without loss of generality that α is a primitive vector of L. Under this condition, $R_\alpha \in O^+(L)$ if and only if

$$(\alpha, \alpha) | 2(\alpha, x) \text{ for all } x \in L.$$

(This holds automatically, if $(\alpha, \alpha) = 1$ or 2.) Such vectors α are called *roots* of L. More precisely, if $(\alpha, \alpha) = k$, then α is called a *k-root*.

Definition 1. *The subgroup $W(L) \subset \mathrm{O}^+(\mathrm{L})$ generated by all reflections contained in $\mathrm{O}^+(\mathrm{L})$ is called the* **Weyl group** *of L.*

Definition 2. *The lattice L is called* **reflective** *if $[\mathrm{O}^+(L) : W(L)] < \infty$ or, equivalently, if the group $W(L)$ is cofinite.*

Definition 3. *Having fixed a fundamental polyhedron P of the group $W(L)$, the normal roots of its hyperfaces are called the* **simple roots** *of L.*

There is an algorithm for determining, whether a lattice is reflective or not, and for finding its simple roots in the case, when it is reflective: see [7].

Definition 4. *A root α of a lattice L is called* **isolated** *if it is orthogonal to all roots $\beta \neq \pm\alpha$ such that the subspace $\langle \alpha, \beta \rangle \subset E^{n,1}$ is elliptic (i.e., the hyperplanes H_α and H_β intersect).*

If a simple root of a lattice L is isolated, then the corresponding hyperface of the fundamental polyhedron of the group $W(L)$ is perpendicular to all adjacent hyperfaces.

We will consider lattices $L \subset E^{n,1}$ of the form

$$L = L_0 \oplus \mathbb{Z}\epsilon$$

(an orthogonal direct sum), where L_0 is a lattice in $E^{n-1,1}$ and $(\epsilon, \epsilon) > 0$. Obviously, ϵ is a root of L and the roots of L orthogonal to it are exactly the roots of L_0.

Lemma 1. *If the lattice L_0 cannot be decomposed as $L_0 = L_0' \oplus \mathbb{Z}\epsilon'$ with $(\epsilon', \epsilon') = (\epsilon, \epsilon)$, then ϵ is an isolated root of L.*

Proof. Let $\eta \neq \pm\epsilon$ be a root such that the subspace $\langle \epsilon, \eta \rangle$ is elliptic but $(\epsilon, \eta) > 0$. Since the projection of η to $\mathbb{R}\epsilon$ must be in $\mathbb{Z}\epsilon$, we have $\frac{(\epsilon, \eta)}{(\epsilon, \epsilon)} \in \mathbb{Z}$. But also $\frac{2(\epsilon, \eta)}{(\eta, \eta)} \in \mathbb{Z}$ (since η is a root). It follows that

$$\frac{2(\epsilon, \eta)^2}{(\epsilon, \epsilon)(\eta, \eta)} = 2\cos^2 \widehat{\epsilon\eta} \in \mathbb{Z},$$

whence $\widehat{\epsilon\eta} = \frac{\pi}{4}$ and

$$\eta = \epsilon + \epsilon', \quad \text{where} \quad \epsilon' \in L_0 \quad \text{and} \quad (\epsilon', \epsilon') = (\epsilon, \epsilon).$$

Further, since $\epsilon' = -R_\eta \epsilon$, we have $L = R_\eta L_0 \oplus \mathbb{Z}\epsilon'$, whence $L_0 = (L_0 \cap R_\eta L_0) \oplus \mathbb{Z}\epsilon'$, a contradiction. $\square$

Corollary 1. *If L_0 is an even lattice with odd discriminant $d(L_0)$, then ϵ is an isolated root of L.*

Proof. Suppose that $L_0 = L_0' \oplus \mathbb{Z}\epsilon'$ with $(\epsilon', \epsilon') = (\epsilon, \epsilon) = k$. Since the lattice L_0 is even, k must be even; but then $d(L_0) = kd(L_0')$ is also even, a contradiction. $\square$

5. Main construction

Remind that an odd unimodular quadratic lattice of signature (k, l) exists for any k, l, and it is unique for $k, l > 0$. We denote it by $I_{k,l}$. It is known that the lattice $I_{n,1}$ is reflective if and only if $n \leq 19$ [7], [9], [3].

An even unimodular quadratic lattice of signature (k, l) exists if and only if $k \equiv l \pmod{8}$, and it is unique for $k, l > 0$. We denote it by $J_{k,l}$. The lattice $J_{8m+1,1}$ $(m > 0)$ is reflective if and only if $m = 1, 2$ [8].

For $n \equiv 2 \pmod 8$, take

$$L_0 = J_{n-1,1} \subset E^{n-1,1},$$
$$L_1 = L_0 \oplus \mathbb{Z}\epsilon_1 \subset E^{n,1} \quad \text{with} \quad (\epsilon_1, \epsilon_1) = 1,$$
$$L_2 = L_0 \oplus \mathbb{Z}\epsilon_2 \subset E^{n,1} \quad \text{with} \quad (\epsilon_2, \epsilon_2) = 2.$$

Then ϵ_1 (resp. ϵ_2) is an isolated root of L_1 (resp. L_2). Note that the lattice L_1 is unimodular and, hence, isomorphic to $I_{n,1}$. By Proposition 1 the groups $O^+(L_1)$ and $O^+(L_2)$ are not commensurable.

Assume that the lattices L_0, L_1, L_2 are reflective, and set

$$\Gamma_0 = W(L_0), \quad \Gamma_1 = W(L_1), \quad \Gamma_2 = W(L_2).$$

Let $H = L^n \cap E^{n-1,1}$, and let P_0 be a fundamental polyhedron for Γ_0 in H. It is a common face of some fundamental polyhedra P_1 and P_2 of Γ_1 and Γ_2 lying on different sides of H. The union $P = P_1 \cup P_2$ is a fundamental polyhedron of a non-arithmetic cofinite discrete group Γ, a mixture of Γ_1 and Γ_2.

To investigate reflectivity of the lattice L_2, the following lemma is useful.

Lemma 2. *The lattice L_2 has only 2-roots.*

Proof. Let $\alpha \neq \pm\epsilon_2$ be a k-root of L_2. Since the lattice L_2 is even, k is even. One has $\alpha = l\alpha_0 + m\epsilon_2$, where α_0 is a primitive vector of the lattice L_0 and l, m are coprime non-negative integers. Since the lattice L_0 is unimodular, there is a vector $x \in L_0$ such that $(\alpha_0, x) = 1$. It follows that $k | 2(\alpha, x) = 2l$. On the other hand, $k | 2(\alpha, \epsilon_2) = 4m$, This means that $k = 2$ or 4. The latter case is only possible if l is even; but then m is odd and

$$k = l^2(\alpha_0, \alpha_0) + 2m^2 \equiv 2 \pmod 8,$$

a contradiction. $\square$

6. Case $n = 10$

For $n = 10$, both $L_0 \cong J_{9,1}$ and $L_1 \cong I_{10,1}$ are reflective. The Coxeter diagrams of their Weyl groups are presented on Fig. 1, where the black vertex of the second diagram denotes the isolated root ϵ_1, and the crossed vertex denotes the simple root, which is not orthogonal to ϵ_1.

Turn now to the lattice L_2. There is an orthogonal basis $\{e_0, e_1, \ldots, e_9, f = \epsilon_2\}$ of $E^{10,1}$ such that

$$(e_0, e_0) = -1, (e_1, e_1) = \cdots = (e_9, e_9) = 1, (f, f) = 2$$

and
$$L_2 = \{x = x_0 e_0 + x_1 e_1 + \cdots + x_9 e_9 + yf : x_0, x_1, \ldots, x_9 \in \frac{1}{2}\mathbb{Z},$$
$$x_i - x_j \in \mathbb{Z} \text{ for all } i, j, \quad \sum_{i=0}^{9} x_i \in 2\mathbb{Z}, \quad y \in \mathbb{Z}\}.$$

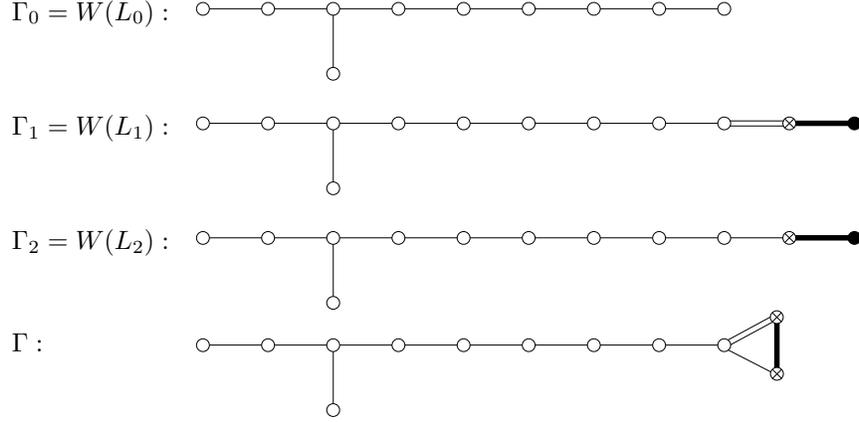By Lemma 2, the lattice L_2 has only 2-roots.

FIGURE 1. $\Gamma_0 = W(L_0)$, $\Gamma_1 = W(L_1)$, $\Gamma_2 = W(L_2)$ and Γ for $n = 10$

Apply the algorithm of [7] to the lattice L_2 taking e_0 for the base point. The stabilizer of e_0 in $\mathrm{O}^+(L_2)$is a finite reflection group of type $D_9 + A_1$ permuting the vectors $e_1, \ldots, e_9$ with multiplying an even number of them by -1 and multiplying f by ± 1. A fundamental polyhedron P_{e_0} of it is given by the inequalities

(1) $$x_1 \geq \cdots \geq x_8 \geq |x_9|, \, y \geq 0.$$

There is a unique fundamental polyhedron P of $W(L)$ containing e_0 and contained in P_{e_0}. The first simple roots of L with respect to P corresponding to the walls of P_{e_0} are

$$\alpha_1 = -e_1 + e_2, \ldots, \alpha_8 = -e_8 + e_9, \alpha_9 = -e_8 - e_9, \alpha_{10} = -f(= -\epsilon_2).$$

The subsequent simple roots α should satisfy inequalities (1) and the inequality $x_0 \geq 0$. They are ordered by the coordinate x_0, which increases together with the distance from the base point e_0 to the hyperplane H_α. The scalar products of each subsequent simple root with all the preceding ones should be non-positive.

Proceeding in this way, we find two extra simple roots

$$\alpha_{11} = \frac{1}{2}(e_0 + e_1 + \cdots + e_9), \quad \alpha_{12} = e_0 + e_1 + f.$$

The Coxeter diagram of the roots $\alpha_1, \ldots, \alpha_{12}$ is shown on Fig. 1. The corresponding Coxeter polyhedron turns to be of finite volume, a pyramid with ideal apex over a 9-dimensional symplicial prism (cf. [7, Subsection 2.5, Example 2]). It follows that the lattice L_2 is reflective, and $\alpha_1, \ldots, \alpha_{12}$ are all of its simple roots.

The last diagram of Fig. 1 is the Coxeter diagram of the mixture Γ of the groups $\Gamma_1 = W(L_1)$ and $\Gamma_2 = W(L_2)$ defined by the isolated roots ϵ_1 and ϵ_2 of them.

In all the diagrams of Fig. 1, the black vertices denote the isolated roots. and the crossed vertices denote the simple roots, which are not orthogonal to the isolated root.

The group Γ is a non-arithmetic cofinite discrete reflection group in L^{10}. It coincides with Ruzmanov's example [6] for this dimension.

7. CASE $n = 18$

For $n = 18$, both $L_0 \cong J_{17,1}$ and $L_1 \cong I_{18,1}$ are also reflective lattices.
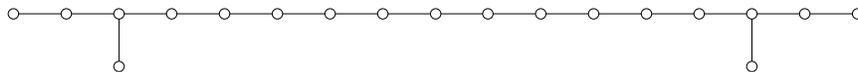The Coxeter diagram of the group $W(L_0)$ is shown on Fig. 2.

FIGURE 2. $\Gamma_0 = W(L_0)$ for $n = 18$

The lattice L_1 has 37 simple roots: 22 "long" 2-roots, and 15 "short" 1-roots [9]. The Coxeter diagram of the long simple roots is shown on Fig. 3a. It has a symmetry group of type S_4, which extends to the entire Coxeter diagram. The short simple roots decompose into two orbits of this group, consisting of 3 and 12 roots, which we call the short roots of the first and second kind, respectively. They join with the long roots as shown on Fig. 3b,c. All the short roots of the first (resp. second) kind are joined with one another by thick (resp. dashed) lines. The short roots of the first kind are joined with the short roots of the second kind by thick and dashed lines according to some rule described in [?]. One can check that the simple roots orthogonal to the isolated root (the black vertex of the diagram) constitute a subdiagram isomorphic to the Coxeter diagram of $W(L_0)$ (as was expected).
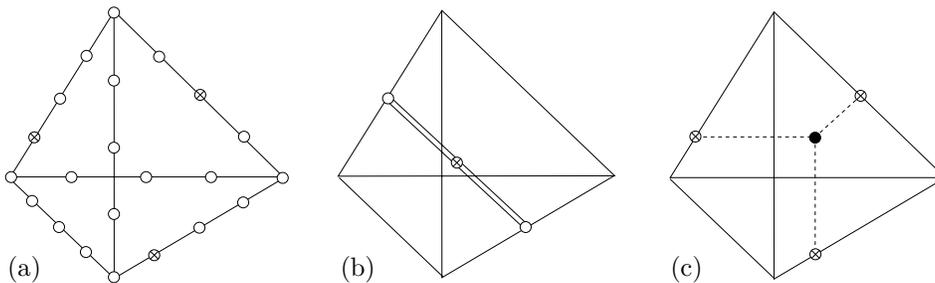

FIGURE 3. $\Gamma_1 = W(L_1)$ for $n = 18$

Let us now prove that the lattice L_2 is also reflective.
Similarly to the case $n = 10$, there is an orthogonal basis $\{e_0, e_1, \ldots, e_{17}, f = \epsilon_2\}$ of $E^{18,1}$ such that

$$(e_0, e_0) = -1, \ (e_1, e_1) = \cdots = (e_{17}, e_{17}) = 1, \ (f, f) = 2$$

and

$$L_2 = \{x = x_0 e_0 + x_1 e_1 + \cdots + x_{17} e_{17} + yf : x_0, x_1, \ldots, x_{17} \in \frac{1}{2}\mathbb{Z},$$

$$x_i - x_j \in \mathbb{Z} \text{ for all } i, j, \quad \sum_{i=0}^{17} x_i \in 2\mathbb{Z}, \quad y \in \mathbb{Z}\}.$$

By Lemma 2, the lattice L_2 has only 2-roots.

Applying to L_2 the algorithm of [7] with the base point e_0, we find simple roots

$$\alpha_1 = -e_1 + e_2, \ldots, \alpha_{16} = -e_{16} + e_{17}, \alpha_{17} = -e_{16} - e_{17}, \alpha_{18} = -f(= -\epsilon_2),$$

$$\alpha_{19} = e_0 + e_1 + e_2 + e_3,$$

$$\alpha_{20} = e_0 + e_1 + f,$$

$$\alpha_{21} = \frac{3}{2}e_0 + \frac{1}{2}(e_1 + \cdots + e_{17}),$$

$$\alpha_{22} = 3e_0 + e_1 + \cdots + e_9 + f,$$

$$\alpha_{23} = 8e_0 + 2(e_1 + \cdots + e_{12}),$$

$$\alpha_{24} = 9e_0 + 3(e_1 + \cdots + e_6) + e_7 + \cdots + e_{16} - e_{17} + 3f.$$

Their Coxeter diagram is shown on Fig. 4. Making use of the method of [7], one can prove that this diagram defines a Coxeter polyhedron of finite volume. Thus, the lattice L_2 is reflective, and $\alpha_1, \ldots, \alpha_{24}$ are all of its simple roots. One can check that the simple roots orthogonal to the isolated root constitute a subdiagram isomorphic to the Coxeter diagram of $W(L_0)$.
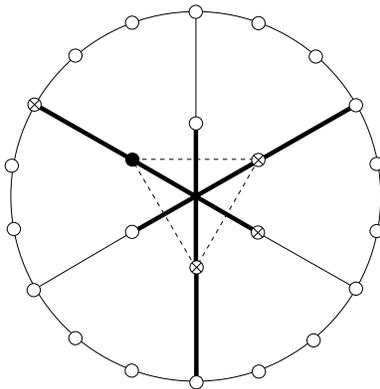

FIGURE 4. $\Gamma_2 = W(L_2)$ for $n = 18$

Thus, the mixture Γ of the groups $\Gamma_1 = W(L_1)$ and $\Gamma_2 = W(L_2)$ is a non-arithmetic cofinite discrete reflection group in L^{18}.

8. FACES OF COXETER POLYHEDRA AND FURTHER EXAMPLES

Any $(n - r)$-dimensional face of a Coxeter polyhedron $P \subset L^n$, if it is not an ideal vertex, belongs exactly to r hyperfaces, and the corresponding subdiagram of the Coxeter diagram of P is elliptic. Conversely, every elliptic subdiagram of rank r

of the Coxeter diagram defines an $(n-r)$-dimensional face of P as the intersection of the corresponding hyperfaces.

A face of a Coxeter polyhedron does not need to be a Coxeter polyhedron itself. However, sometimes this happens. The following useful observation belongs to Borcherds.

Definition 5. *An elliptic Coxeter diagram is called* **admissible** *if it has no connected components of types A_n and D_5.*

Theorem 2. *(*Borcherds[2, Example 5.6]*.) Any face of a Coxeter polyhedron P corresponding to an admissible elliptic subdiagram of the Coxeter diagram of P is a Coxeter polyhedron.*

This can be applied to the fundamental polyhedron P of the non-arithmetic reflection group $\Gamma \subset O_{18,1}^+$ obtained as above as a mixture of arithmetic reflection groups Γ_1 and Γ_2. In the previous notation, take any admissible elliptic subdiagram of rank $r < n - 1$ of the Coxeter diagram of $P_0 = P_1 \cap P_2$. It defines faces F_0, F_1, F_2, F of the polyhedra P_0, P_1, P_2, P so that $F_0 = F_1 \cap F_2$ and $F = F_1 \cup F_2$. These faces are Coxeter polyhedra in an $(18-r)$-dimensional plane perpendicular to H, where we obtain a picture similar to that we had in the whole space. It follows that the reflection group associated with F is a non-arithmetic cofinite discrete reflection group in L^{18-r}.

Taking, for example, subdiagrams D_r, $r = 4, 6, 7, \ldots, 16$ in the Coxeter diagram of P_0, we obtain examples of non-arithmetic cofinite discrete reflection groups in L^n, $n = 14, 12, 11, \ldots, 2$. Note that all these groups are non-cocompact. Examples of non-arithmetic cocompact discrete reflection groups in L^n are still known only for $n \leq 5$.

References

[1] D.V. Alekseevskij, E. B. Vinberg and A. S. Solodovnikov. Geometry of spaces of constant curvature.(Russian) Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya **29**(1988), 5–146. [English translation: Geometry II. Spaces of constant curvature. Encyclopaedia of Mathematical Sciences **29**, Springer-Verlag, 1993, 1–138.]

[2] R. E. Borcherds. Coxeter groups, Lorentzian lattices, and $K3$ surfaces.*Internat. Math. Res. Notices* **19**(1998), 1011–1031.

[3] F. Esselmann. Uber die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe. *J. Number Theory* **61**(1996), 103–144.

[4] M. Gromov and I. Piatetski-Shapiro. Nonarithmetic groups in Lobachevsky spaces. *Inst. Hautes Études Sci. Publ. Math.* **66** (1988), 93–103.

[5] M. S. Raghunathan. Discrete subgroups of Lie groups. Springer, 1972.

[6] O. P. Ruzmanov. Examples of nonarithmetic crystallographic Coxeter groups in n-dimensional Lobachevsky space for $6 \leq n \leq 10$. (Russian) *Problems in group theory and homological algebra*, Yaroslav. Gos. Univ., Yaroslavl, 1989, 138–142.

[7] E. B. Vinberg. On the unit groups of some quadratic forms. (Russian) *Mat. Sbornik*, N. ser. **87** (1972), 18–36. [English tranls.:*Math. USSR, Sbornik* **16**(1972), 17–35.]

[8] E. B. Vinberg. On unimodular integral quadratic forms. (Russian) *Funkts. Anal. Prilozh.* **6** (1972), 24–31. [English transl.: *Funct. Anal. Appl.* **6** (1972), 105–111 .]

[9] E. B. Vinberg and I. M. Kaplinskaya. On the groups $O_{18,1}(\mathbb{Z})$ and $O_{19,1}(\mathbb{Z})$. (Russian) *Dokl. Akad. Nauk SSSR* **238** (1978), 1273-1275. [English translation: *Soviet Math., Dokl.* **19** (1978), 194–197.]

[10] E. B. Vinberg and O. V. Shvartsman. Discrete groups of motions of spaces of constant curvature. (Russian) Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya **29** (1988), 147–259. [English translation: Geometry II. Spaces of Constant Curvature. Encyclopaedia of Mathematical Sciences **29**, Springer-Verlag, 1993, 139–248.]

Prof. E. B. Vinberg, Department of Mechanics and Mathematics, Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow, 119991, Russian Federation