On the fluctuations of eigenvalues of multiplicative deformed unitary invariant ensembles

V. Vasilchuk*
B. Verkin Institute for Low Temperatures,
Mathematical Division
47 ave. Lenin 61103 Kharkov, Ukraine
E-mail: vasilchuk@ilt.kharkov.ua

Abstract

We consider the ensemble of $n \times n$ random matrices $W = AU^*BU$, where A and B are non-random, unitary, having the limiting Normalized Counting Measure of eigenvalues (NCM), and U is unitary, uniformly distributed over $U(n)$. We find the leading term of the covariance of traces of resolvent of W and establish the Central Limit Theorem for sufficiently smooth linear eigenvalue statistics of W as $n \to \infty$.

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1 Introduction and main results

Let us consider unitary multiplicative ensemble

$$W = AU^*BU, \tag{1.1}$$

where A and B non-random $n \times n$ unitary matrices, $U \in U(n)$ random, uniformly distributed on $U(n)$. If $\{\lambda_l^A\}_{l=1}^n$ and $\{\lambda_l^B\}_{l=1}^n$ are eigenvalues of A and B and $N_{n,A}$ and $N_{n,B}$ are their Normalized Counting Measures (NCM), defined as

$$N_{n,A}(\Omega) = \#\{\lambda_l^A \in \Omega,\ l = 1, ..., n\}n^{-1},\ N_{n,B}(\Omega) = \#\{\lambda_l^B \in \Omega,\ l = 1, ..., n\}n^{-1} \tag{1.2}$$

for any interval $\Omega \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We assume that the NCMs $N_{n,A}$ and $N_{n,B}$ of the factors A and B converge weakly to probability measures N_A and N_B:

$$\lim_{n\to\infty} N_{n,A} = N_A,\ \lim_{n\to\infty} N_{n,B} = N_B. \tag{1.3}$$

*On visiting SFB-701 "Spectral Structures and Topological Methods in Mathematics", Bielefeld University, Germany

Our goal in this paper is the study of the fluctuations of eigenvalue distribution of W of (1.1). At the first step we need the expression of NCM N_n of of eigenvalues $\{\lambda_l^W\}_{l=1}^n$ of W

$$N_n(\Omega) = \#\{\lambda_l^W \in \Omega,\ l = 1, ..., n\}n^{-1} \qquad (1.4)$$

via NCMs N_A and N_B in the limit $n \to \infty$. It is given via asymptotic analysis of the Stieltjes transform

$$g_n(z) = \int_{\mathbb{T}} \frac{N_n(d\lambda)}{\lambda - z} = n^{-1}\mathrm{Tr}\, G(z),\ G(z) = (W - zI)^{-1},\ |z| \neq 1$$

of the measure N_n. Under assumption (1.3) it was proved in [8] that NCM N_n of the product W converges weakly with probability 1 to N and its Stieltjes transform

$$f(z) = \int_{\mathbb{T}} \frac{N(d\lambda)}{\lambda - z}$$

is the unique solution of the system

$$\begin{cases} \Delta_B(z) & = & f_A\left(\dfrac{z\Delta_A(z)}{1 + zf(z)}\right), \\ \Delta_A(z) & = & f_B\left(\dfrac{z\Delta_B(z)}{1 + zf(z)}\right), \\ f(z)(1 + zf(z)) & = & \Delta_A(z)\Delta_B(z), \end{cases} \qquad (1.5)$$

where

$$f_l(z) = \int_{\mathbb{T}} \frac{N_l(d\lambda)}{\lambda - z},\ l = A, B$$

$f(z)$, $\Delta_A(z)$, $\Delta_B(z)$ are analytic for $|z| < 1$ functions verifying the conditions

$$\Delta_A(0) = \overline{m_B^{(1)}},\ \Delta_B(0) = \overline{m_A^{(1)}},\ m_l^{(1)} = \int_{\mathbb{T}} \lambda N_l(d\lambda),\ l = A, B. \qquad (1.6)$$

It must be noted that such type results for the multiplicative ensemble have been obtained by many people. Originally first was made by Voiculescu [10, 11] in terms of free probability theory under additional restriction

$$m_A^{(1)} m_B^{(1)} \neq 0.$$

Later this result (called in free probability as free multiplicative convolution of measures N_A and N_B) was improved in [1] and [6].

The main result of this paper is the following:

Theorem 1.1 *Assume (1.3) then we have for*

$$g_n(z) = n^{-1}\mathrm{Tr}\, G(z) = \int_{\mathbb{T}} \frac{N_n(d\lambda)}{\lambda - z}$$

and n-independent $|z_{1,2}| < 0$

$$\mathbf{Cov}\{g_n(z_1), g_n(z_2)\} = n^{-2}S(z_1, z_2) + \psi_n(z_1, z_2), \tag{1.7}$$

where

$$S(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \log \frac{1 - z_1 z_2 \dfrac{f(z_1) - f(z_2)}{z_1 - z_2}}{1 - \dfrac{z_1 z_2 I_A(z_1, z_2) I_B(z_1, z_2)}{(1 + z_1 f(z_1))(1 + z_2 f(z_2))}}, \tag{1.8}$$

where

$$\begin{aligned} I_l(z_1, z_2) &= \int_{\mathbb{T}} \frac{N_l(d\lambda)}{(\lambda - \tilde{z}_l(z_1))(\lambda - \tilde{z}_l(z_2))}, \ l = A, B, \\ \tilde{z}_A(z) &= \frac{z\Delta_A(z)}{1 + zf(z)}, \ \tilde{z}_B(z) = \frac{z\Delta_B(z)}{1 + zf(z)}, \end{aligned} \tag{1.9}$$

$\psi_n(z_1, z_2)$ admits the bound $|\psi_n(z_1, z_2)| \le C/n^3$, and C is independent of n and finite if $|z_{1,2}| < 1$.

While such type results are typical for additive deformed hermitian analogs of (1.1) (See e.g. [4], [2], [6], [9] and [5] and links therein) the asymptotic result (1.8) seems to be new. Curiously, its anther form (1.14), applicable in special case $m_A^{(1)} m_B^{(1)} \neq 0$, coincides (but for different functions) to the one obtained for additive deformed ensembles in [4], [2].

Also the following proposition about the "first integral" is valid:

Proposition 1.2 *Suppose that $m_l^{(1)} \neq 0$, $l = A, B$ define new variables*

$$\psi(z) = \frac{zf(z)}{1 + zf(z)} \text{ and } \tilde{z}_A(z), \ \tilde{z}_B(z) \tag{1.10}$$

and transform the system (1.5) to the form

$$\begin{cases} \psi(z) &= \psi_A(\tilde{z}_A(z)), \\ \psi(z) &= \psi_B(\tilde{z}_B(z)), \\ z\psi(z) &= \tilde{z}_A(z)\tilde{z}_B(z), \end{cases} \tag{1.11}$$

where

$$\psi_l(z) = \frac{zf_l(z)}{1 + zf_l(z)}, \ l = A, B$$

then the following identity holds for $|z_{1,2}| < 1$

$$\frac{1 - z_1 z_2 \dfrac{f(z_1) - f(z_2)}{z_1 - z_2}}{1 - \dfrac{z_1 z_2 I_A(z_1, z_2) I_B(z_1, z_2)}{(1 + z_1 f(z_1))(1 + z_2 f(z_2))}} = \frac{(\tilde{z}_A(z_1) - \tilde{z}_A(z_2))\,(\tilde{z}_B(z_1) - \tilde{z}_B(z_2))}{(\psi(z_1) - \psi(z_2))(z_1 - z_2)} \tag{1.12}$$

and

$$\psi'(z) = \frac{-\psi(z)\psi'_A(\tilde{z}_A(z))\psi'_B(\tilde{z}_B(z))}{z\psi'_A(\tilde{z}_A(z))\psi'_B(\tilde{z}_B(z)) - \psi'_A(\tilde{z}_A(z))\tilde{z}_A(z) - \psi'_B(\tilde{z}_B(z))\tilde{z}_B(z)}, \quad (1.13)$$
$$\tilde{z}'_A(z) = \frac{\psi'(z)}{\psi'_A(\tilde{z}_A(z))}, \; l = A, B.$$

Equality (1.12) will be proved in Proposition 4.2 (i). Expressions (1.13) can be obtained by direct calculations.

Corollary 1.3 *It follows from (1.12) and (1.8) that if $m_l^{(1)} \neq 0$, $l = A, B$ then we have in (1.7):*

$$S(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \log \frac{(\tilde{z}_A(z_1) - \tilde{z}_A(z_2))\,(\tilde{z}_B(z_1) - \tilde{z}_B(z_2))}{(\psi(z_1) - \psi(z_2))(z_1 - z_2)}. \quad (1.14)$$

2 Proof of the covariance formula

Our proofs are based on the elementary matrix identities, in particular, on the resolvent identities:

$$\begin{aligned} (M_1 - zI)^{-1} - (M_2 - zI)^{-1} &= -(M_1 - zI)^{-1}(M_1 - M_2)(M_2 - zI)^{-1} \\ &= -(M_2 - zI)^{-1}(M_1 - M_2)(M_1 - zI)^{-1}, \quad (2.1) \\ (M_1 - z_1 I)^{-1} - (M_1 - z_2 I)^{-1} &= (z_1 - z_2)(M_1 - z_1 I)^{-1}(M_1 - z_2 I)^{-1}, \quad (2.2) \\ M_1, M_2 &\in U(n), \; |z| \neq 1 \end{aligned}$$

and on the formula for the derivative of the resolvent $G_1 = (M_1 - zI)^{-1}$ acting on $X \in \mathcal{M}_n$, $\mathcal{M}_n$ – the linear space of $n \times n$ matrices:

$$G'_1 \cdot X = -G_1 X G_1.$$

We denote $\mathbf{E}\{...\}$ the expectation with respect to the normalized to unity Haar measure of $U(n)$. We also use the following two facts on this expectation proved in [3] and [4] correspondently (see also [5]). The first one is the differentiation formula, given by

Proposition 2.1 *Let $\Phi : \mathcal{M}_n \to \mathbb{C}$ be a continuously differentiable function. Then the following relation holds for any element $X \in \mathcal{M}_n$:*

$$\mathbf{E}\{\Phi'(U^*MU) \cdot [X, U^*MU]\} = 0,$$

where

$$[M_1, M_2] = M_1 M_2 - M_1 M_2$$

is the commutator of $M_{1,2} \in \mathcal{M}_n$. Analogously, we have

$$\mathbf{E}\{\Phi'(UMU^*) \cdot [X, UMU^*]\} = 0.$$

The second important technical mean is a "unitary" analog of the Poincare-Nash inequality.

Proposition 2.2 *Let $\Phi : U(n) \to \mathbb{C}$ be a continuously differentiable function. Then*

$$\mathbf{Var}\{\Phi(U_n)\} := \mathbf{E}\{|\Phi(U_n)|^2\} - |\mathbf{E}\{\Phi(U_n)\}|^2 \le \frac{1}{n}\sum_{j,k=1}^{n} \mathbf{E}\{|\Phi'(U_n)\cdot E^{(j,k)}U_n|^2\},$$

where $\{E^{(j,k)}\}_{j,k=1}^n$ is canonical basis in the space $\mathcal{M}_n$ of all $n\times n$ matrices: $E^{(j,k)} = \{E^{(j,k)}_{pq}\}_{p,q=1}^n$, $E^{(j,k)}_{pq} = \delta_{jp}\delta_{kq}$.

Besides, in [8] the proof of uniqueness of the solution of (1.5) implicitly was based on the following proposition which also will be used bellow.

Lemma 2.3 *Rewrite the system (1.5) in the form*

$$F_j(f(z), \Delta_A(z), \Delta_B(z)) = 0, \ \ j = 1, 2, 3,$$

where

$$\begin{aligned} F_1(w_1,w_2,w_3) &= w_3 - f_A\left(\frac{zw_2}{1+zw_1}\right), \\ F_2(w_1,w_2,w_3) &= w_2 - f_B\left(\frac{zw_3}{1+zw_1}\right), \\ F_3(w_1,w_2,w_3) &= (1+zw_1)w_1 - w_2w_3. \end{aligned}$$

Then the Jacobian

$$J(w_1,w_2,w_3) = \det\left\{\left\{\frac{\partial F_j}{\partial w_k}\right\}_{j,k=1}^3\right\}$$

of the map $F : \mathbb{C}^3 \to \mathbb{C}^3$ has for $|z| \ne 1$ the following expression:

$$\begin{aligned} J(f(z), \Delta_A(z), \Delta_B(z)) &= (1+2zf(z))\left(\frac{z^2 f'_A(\tilde z_A(z)) f'_B(\tilde z_B(z))}{(1+zf(z))^2} - 1\right) \\ &\quad -\tilde z_A(z) f'_A(\tilde z_A(z))\left(\frac{z\tilde z_B(z) f'_B(\tilde z_B(z))}{1+zf(z)} + \tilde z_A(z)\right) \\ &\quad -\tilde z_B(z) f'_B(\tilde z_B(z))\left(\frac{z\tilde z_A(z) f'_A(\tilde z_A(z))}{1+zf(z)} + \tilde z_B(z)\right), \end{aligned} \tag{2.3}$$

and for $|z| \le 1/16$ we have

$$\frac{1}{4} \le |J(f(z), \Delta_A(z), \Delta_B(z))| \le \frac{7}{4}. \tag{2.4}$$

If we suppose in addition that $m_l^{(1)} \neq 0$, $l = A, B$ then the Jacobian J has the following asymptotics at infinity

$$J(f(z), \Delta_A(z), \Delta_B(z)) = \frac{1}{z^2}\left(-\left(m_A^{(1)} m_B^{(1)} \right)^2 + o(1) \right), \ z \to \infty \tag{2.5}$$

Proof. Indeed, we obtain (2.3) by direct calculation. Then, taking into account the following simple bounds

$$\begin{aligned}
|f(z)| &\leq \frac{1}{|1-|z||}, \ |\Delta_l(z)| \leq \frac{1}{|1-|z||}, \ l = A, B, \ |z| \neq 1, \\
|f_l'(z)| &\leq \frac{1}{(1-|z|)^2}, \ l = A, B, \ |z| \neq 1, \\
\frac{1}{|1+zf(z)|} &\geq \frac{1-|z|}{1-2|z|}, \ |\tilde{z}_l(z)| \leq \frac{|z|}{1-2|z|}, \\
|f_l'(\tilde{z}_l(z))| &\leq \left(\frac{1-2|z|}{1-3|z|} \right)^2, \ l = A, B, \ |z| < \frac{1}{3},
\end{aligned} \tag{2.6}$$

we obtain (2.4) for $|z| \leq 1/16$ and, in particular, $J(f(0), \Delta_A(0), \Delta_B(0)) = -1$.

Moreover, introducing new variable

$$p(z) = zf(z)$$

and taking into account the integral representation

$$1 + zf_l(z) = \int_{\mathbb{T}} \frac{\lambda N_l(d\lambda)}{\lambda - z} =: \pi_l(z), \ l = A, B, \tag{2.7}$$

we can rewrite (1.5) as

$$\left\{ \begin{array}{rcl}
p(z) - \pi_A(\tilde{z}_A(z)) & = & -1, \\
p(z) - \pi_B(\tilde{z}_B(z)) & = & -1, \\
p(z)(1+p(z)) - z\Delta_A(z)\Delta_B(z) & = & 0,
\end{array} \right. \tag{2.8}$$

This implies the following linear system for the derivatives $p'(z)$, $\Delta'_{A,B}(z)$:

$$\left\{ \begin{array}{rcl}
\left(1 - \frac{\tilde{z}_A(z)\pi'_A(\tilde{z}_A(z))}{1+p(z)}\right) p'(z) - \frac{z\pi'_A(\tilde{z}_A(z))}{1+p(z)}\Delta'_A(z) & = & \frac{\tilde{z}_A(z)}{z}\pi'_A(\tilde{z}_A(z)), \\
\left(1 - \frac{\tilde{z}_B(z)\pi'_B(\tilde{z}_B(z))}{1+p(z)}\right) p'(z) - \frac{z\pi'_B(\tilde{z}_B(z))}{1+p(z)}\Delta'_B(z) & = & \frac{\tilde{z}_B(z)}{z}\pi'_B(\tilde{z}_B(z)), \\
(1+2p(z))\, p'(z) - z\Delta_B(z)\Delta'_A(z) - z\Delta_A(z)\Delta'_B(z) & = & \Delta_A(z)\Delta_B(z).
\end{array} \right. \tag{2.9}$$

Using direct computations we obtain that the determinant $\hat{J}(z)$ of (2.9) is equal

$$\begin{aligned}\hat{J}(z) &= (1+2p(z))\frac{z^2\pi_A'(\tilde{z}_A(z))\pi_B'(\tilde{z}_B(z))}{(1+p(z))^2} \\ &- \left(1+\frac{\tilde{z}_A(z)\pi_A'(\tilde{z}_A(z))}{1+p(z)}\right) z\tilde{z}_B(z)\pi_B'(\tilde{z}_B(z)) \\ &- \left(1+\frac{\tilde{z}_B(z)\pi_B'(\tilde{z}_B(z))}{1+p(z)}\right) z\tilde{z}_A(z)\pi_A'(\tilde{z}_A(z)) \\ &= \frac{z^2 f(z)}{1+zf(z)} J(f(z),\Delta_A(z),\Delta_B(z)).\end{aligned}$$

Besides, in view of (2.7) we have the asymptotic

$$\begin{aligned}p(z) &= -1+o(1),\ 1+p(z) = \frac{1}{z}\left(-m_A^{(1)}m_B^{(1)}+o(1)\right),\ z\to\infty, \qquad (2.10)\\ \Delta_l(z) &= \frac{1}{z}\left(-m_l^{(1)}+o(1)\right),\ \pi_l'(z) = \frac{1}{z^2}\left(m_l^{(1)}+o(1)\right),\ l=A,B,\ z\to\infty,\\ \tilde{z}_{A,B}(z) &= z\left(\frac{1}{m_{B,A}^{(1)}}+o(1)\right),\ \pi_l'(\tilde{z}_l(z)) = \frac{1}{z^2}\left(m_l^{(1)}+o(1)\right),\ l=A,B,\ z\to\infty.\end{aligned}$$

On other hand, this imply that the determinant $\hat{J}(z)$ has the asymptotics:

$$\hat{J}(z) = -m_A^{(1)}m_B^{(1)}+o(1),\ z\to\infty$$

and hence the determinant

$$J(f(z),\Delta_A(z),\Delta_B(z)) = \frac{1+zf(z)}{z^2 f(z)}\hat{J}(z)$$

has the asymptotics (2.5). ■

Proof of Theorem 1.1. Taking in Proposition 2.1 $\Phi = \left(n^{-1}\mathrm{Tr}G(z_1)\right)^\circ G_{ac}(z_2)$, we obtain

$$\begin{aligned}&\mathbf{E}\{(g(z_1))^\circ\left(G(z_2)A\left[X,U^*BU\right]G(z_2)\right)_{ac}\}\\ &+\mathbf{E}\{\left(n^{-1}\mathrm{Tr}\,G(z_1)A\left[X,U^*BU\right]G(z_1)\right)G_{ac}(z_2)\}=0.\end{aligned}$$

Then taking $X = E^{(a,b)}$, using the resolvent identity (2.1) for the pair $(W,0)$ and applying to the result the operation $n^{-1}\sum\limits_{a=1}^{n}$, we obtain the matrix equality

$$\begin{aligned}\mathbf{E}\{g^\circ(z_1)(1+z_2g_n(z_2))G(z_2)\} &= \mathbf{E}\{g^\circ(z_1)\delta_{n,A}(z_2)U^*BUG(z_2)\} \qquad (2.11)\\ &+ n^{-2}\mathbf{E}\{\left[U^*BU,G^2(z_1)A\right]G(z_2)\},\end{aligned}$$

where

$$\delta_{n,A}(z) = n^{-1}\mathrm{Tr}AG(z). \tag{2.12}$$

Now we rewrite G by using the resolvent identity (2.1) for the pair $(W, 0)$ as

$$G(z) = z^{-1}(AU^*BUG(z) - zI). \tag{2.13}$$

This yields after regrouping terms:

$$\begin{aligned}
& \frac{1 + z_2 f_n(z_2)}{z_2}\,(A - \tilde{z}_{n,A}(z_2)I)\,\mathbf{E}\{g^\circ(z_1)U^*BUG(z_2)\} \qquad (2.14)\\
= \quad & n^{-2}\mathbf{E}\{\left[U^*BU, G^2(z_1)A\right] G(z_2)\}\\
+ \quad & \mathbf{E}\{g^\circ(z_1)\delta^\circ_{n,A}(z_2)U^*BUG(z_2)\}\\
- \quad & z_2\mathbf{E}\{g^\circ(z_1)g^\circ_n(z_2)G(z_2)\},
\end{aligned}$$

where

$$f_n(z) = \mathbf{E}\{g_n(z)\},\ \Delta_{n,A}(z) = \mathbf{E}\{\delta_{n,A}(z)\},\ \tilde{z}_{n,A}(z) = \frac{z\Delta_{n,A}(z)}{1 + zf_n(z)}. \tag{2.15}$$

Introducing centralized values

$$G^\circ = G - \mathbf{E}\{G\},\ (U^*BUG)^\circ = U^*BUG - \mathbf{E}\{U^*BUG\}$$

and regrouping terms in (2.14) we obtain

$$\begin{aligned}
& \frac{1 + z_2 f_n(z_2)}{z_2}\,(A - \tilde{z}_{n,A}(z_2)I)\,\mathbf{E}\{g^\circ(z_1)U^*BUG(z_2)\} \qquad (2.16)\\
= \quad & n^{-2}\mathbf{E}\{\left[U^*BU, G^2(z_1)A\right] G(z_2)\}\\
+ \quad & \mathbf{E}\{g^\circ(z_1)\delta^\circ_{n,A}(z_2)\}\mathbf{E}\{U^*BUG(z_2)\}\\
- \quad & z_2\mathbf{E}\{g^\circ(z_1)g^\circ_n(z_2)\}\mathbf{E}\{G(z_2)\}\\
& +\mathbf{E}\{g^\circ(z_1)\delta^\circ_{n,A}(z_2)\,(U^*BUG(z_2))^\circ\}\\
& -z_2\mathbf{E}\{g^\circ(z_1)g^\circ_n(z_2)G^\circ(z_2)\},
\end{aligned}$$

Besides, we have the bounds

$$\begin{aligned}
||G(z)|| &\leq \frac{1}{|1-|z||},\ |g_n(z)| \leq \frac{1}{|1-|z||},\ |\delta_{n,A}(z)| \leq \frac{1}{|1-|z||}, |z| \neq 1,\\
|\tilde{z}_{n,A}(z)| &\leq \frac{|z|}{1-2|z|},\ |z| < \frac{1}{2},\ |\tilde{z}_{n,A}(z)| \leq \frac{1}{2},\ |z| < \frac{1}{4}. \qquad (2.17)
\end{aligned}$$

Thus, the matrix $A - \tilde{z}_{n,A}(z_2)I$ in (2.14) has the inverse

$$\widetilde{G}_A(z) = (A - \tilde{z}_{n,A}(z)I)^{-1} = G_A(\tilde{z}_{n,A}(z))$$

uniformly in n bounded $||\widetilde{G}_A(z)|| \le 2$ for $|z| < 1/4$. .Thus, multiplying (2.16) by

$$\frac{z_2}{1+z_2 f_n(z_2)}\widetilde{G}_A(z_2)$$

from the left and applying then the operation $n^{-1}\mathrm{Tr}\cdot$, we obtain the relation

$$\begin{aligned} &\mathbf{E}\{g^\circ(z_1)\delta_{n,B}(z_2)\} \\ &+\frac{z_2^2}{1+z_2 f_n(z_2)} n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\mathbf{E}\{G(z_2)\}\mathbf{E}\{g^\circ(z_1)g_n(z_2)\} \\ &-\frac{z_2}{1+z_2 f_n(z_2)} n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\mathbf{E}\{U^*BUG(z_2)\}\mathbf{E}\{g^\circ(z_1)\delta_{n,A}(z_2)\} \\ = \ &\frac{1}{n^2}\frac{z_2}{1+z_2 f_n(z_2)} n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\mathbf{E}\left\{\left[U^*BU, G^2(z_1)A\right]G(z_2)\right\} + R_{AB}(z_1,z_2) \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} \delta_{n,B}(z) &= n^{-1}\mathrm{Tr}U^*BUG(z), \\ R_{AB}(z_1,z_2) &= \frac{z_2}{1+z_2 f_n(z_2)}\Big(-z_2\mathbf{E}\left\{g_n^\circ(z_1)g_n^\circ(z_2)n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)G^\circ(z_2)\right\} \\ &\quad +\mathbf{E}\left\{g_n^\circ(z_1)\delta_{n,B}^\circ(z_2)n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\left(U^*BUG(z_2)\right)^\circ\right\}\Big). \end{aligned}$$

It was proved in [8] that

$$\begin{aligned} \mathbf{E}\{U^*BUG(z)\} &= \widetilde{G}_A(z) \\ &+\frac{z}{1+zf_n(z)}\Big(\mathbf{E}\{\delta_{n,A}^\circ(z)\widetilde{G}_A(z)U^*BUG(z)\} \\ &- z\mathbf{E}\{g_n^\circ(z)\widetilde{G}_A(z)G(z)\}\Big). \end{aligned} \tag{2.19}$$

Thus, applying (2.13) we have

$$\begin{aligned} \mathbf{E}\{G(z)\} &= \frac{\tilde{z}_{n,A}(z)}{z}\widetilde{G}_A(z) \\ &+\frac{z}{1+zf_n(z)}\Big(\mathbf{E}\{\delta_{n,A}^\circ(z)\widetilde{G}_A(z)G(z)\} \\ &- \mathbf{E}\{g_n^\circ(z)\widetilde{G}_A(z)AG(z)\}\Big) \end{aligned} \tag{2.20}$$

and hence

$$\begin{aligned} \alpha_A(z) \ &: = \frac{z^2}{1+zf_n(z)}n^{-1}\mathrm{Tr}\widetilde{G}_A(z)\mathbf{E}\{G(z)\} \\ &= \frac{z\tilde{z}_{n,A}(z)}{1+zf_n(z)}n^{-1}\mathrm{Tr}\widetilde{G}_A^2(z) + R_{\alpha A}, \\ \beta_A(z) \ &: = \frac{z}{1+zf_n(z)}n^{-1}\mathrm{Tr}\widetilde{G}_A(z)\mathbf{E}\{U^*BUG(z)\} \\ &= \frac{z}{1+zf_n(z)}n^{-1}\mathrm{Tr}\widetilde{G}_A^2(z) + R_{\beta B}, \end{aligned}$$

where

$$\begin{aligned}
R_{\alpha A} &= \frac{z^3}{(1+zf_n(z))^2}\Big(\mathbf{E}\{\delta^\circ_{n,A}(z)n^{-1}\mathrm{Tr}\widetilde{G}^2_A(z)G(z)\} \\
&\quad - \mathbf{E}\{g^\circ_n(z)n^{-1}\mathrm{Tr}\widetilde{G}^2_A(z)AG(z)\}\Big), \\
R_{\beta A} &= \frac{z^2}{(1+zf_n(z))^2}\Big(\mathbf{E}\{\delta^\circ_{n,A}(z)n^{-1}\mathrm{Tr}\widetilde{G}^2_A(z)U^*BUG(z)\} \\
&\quad - z\mathbf{E}\{g^\circ_n(z)n^{-1}\mathrm{Tr}\widetilde{G}^2_A(z)G(z)\}\Big).
\end{aligned}$$

Thus, sustituting these relations in (2.18) we obtain

$$\begin{aligned}
&\hat{\alpha}_A(z_2)\mathbf{Cov}\{g_n(z_1),g_n(z_2)\} - \hat{\beta}_A(z_2)\mathbf{Cov}\{g_n(z_1),\delta_{n,A}(z_2)\} \\
&+\mathbf{Cov}\{g_n(z_1),\delta_{n,B}(z_2)\} = \frac{1}{n^2}\gamma_{AB}(z_1,z_2) + \hat{R}_{AB}(z_1,z_2),
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
\hat{\alpha}_A(z) &= \frac{z\tilde{z}_{n,A}(z)}{1+zf_n(z)}n^{-1}\mathrm{Tr}\widetilde{G}^2_A(z) = \frac{z\tilde{z}_{n,A}(z)}{1+zf_n(z)}f'_{n,A}(\tilde{z}_{n,A}(z)), \\
\hat{\beta}_A(z) &= \frac{z}{1+zf_n(z)}n^{-1}\mathrm{Tr}\widetilde{G}^2_A(z) = \frac{z}{1+zf_n(z)}f'_{n,A}(\tilde{z}_{n,A}(z)), \\
\gamma_{AB}(z_1,z_2) &= \frac{\partial}{\partial z_1}\frac{z_2}{1+z_2f_n(z_2)}\Big(n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\mathbf{E}\left\{U^*BUG(z_1)AG(z_2)\right\} \\
&\quad -n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\mathbf{E}\left\{AU^*BUG(z_1)G(z_2)\right\}\Big), \qquad (2.22) \\
\hat{R}_{AB}(z_1,z_2) &= R_{AB}(z_1,z_2) - R_{\alpha A}\mathbf{Cov}\{g_n(z_1),g_n(z_2)\} + R_{\beta A}\mathbf{Cov}\{g_n(z_1),\delta_{n,A}(z_2)\}.
\end{aligned}$$

Besides, we have by using Proposition 2.2:

$$\begin{aligned}
\mathbf{Var}\{g_n(z)\} &\le \frac{1}{n^3}\sum_{j,t=1}^n \mathbf{E}\left\{\left|\mathrm{Tr}\,GAU^*\left(E^{(t,j)}B - TE^{(j,t)}\right)UG\right|^2\right\} \\
&= \frac{1}{n^3}\sum_{j,t=1}^n \mathbf{E}\left\{\left|(BUG^2AU^*)_{jt} - (UG^2AU^*B)_{tj}\right|^2\right\} \\
&\le \frac{4}{n^2|1-|z||^4},\ |z|\ne 1
\end{aligned} \tag{2.23}$$

and analogously

$$\mathbf{Var}\left\{\delta_{n,A}(z)\right\} \le \frac{4}{n^2|1-|z||^4},\ |z|\ne 1. \tag{2.24}$$

Moreover, denoting by $\hat{G}$ the resolvent

$$\hat{G} = (UAU^*B - zI)^{-1} = UGU^*, \tag{2.25}$$

we can rewrite

$$\delta_{n,B}(z) = n^{-1}\mathrm{Tr}U^*BUG(z) = n^{-1}\mathrm{Tr}B\hat{G}(z)$$

and hence obtain

$$\mathbf{Var}\left\{\delta_{n,B}(z)\right\} \le \frac{4}{n^2|1-|z||^4},\ |z| \ne 1.$$

Thus, using the above bounds and Schwarz inequality for the expectation $\mathbf{E}\{...\}$ we have uniformly in A and B

$$\begin{aligned} |R_{\alpha A}| \ &\le \ \frac{|z_2|^3}{|1+z_2 f(z_2)|}\frac{\left\|\widetilde{G}_A(z_2)\right\|^2}{|1-|z_2||}\Big(\sqrt{\mathbf{Var}\{g_n(z_2)\}} \\ &+ \sqrt{\mathbf{Var}\left\{\delta_{n,A}(z_2)\right\}}\Big) \le \frac{C}{n},\ |z_2| < \frac{1}{4} \end{aligned}$$

and analogously

$$|R_{\beta A}| \le \frac{C}{n},\ |z_2| < \frac{1}{4}.$$

Besides, we obviously have for $|z_{1,2}| \ne 1$

$$\begin{aligned} \mathbf{Cov}\{g_n(z_1), g_n(z_2)\} \ &\le \ \sqrt{\mathbf{Var}\{g_n(z_1)\}\mathbf{Var}\{g_n(z_2)\}} \le \frac{C}{n^2}, \\ \mathbf{Cov}\{g_n(z_1), \delta_{n,A}(z_2)\} \ &\le \ \sqrt{\mathbf{Var}\{g_n(z_1)\}\mathbf{Var}\{\delta_{n,A}(z_2)\}} \le \frac{C}{n^2} \end{aligned}$$

and according to Proposition 4.1 we have uniformly in A and B for $|z_{1,2}| \le 1/16$

$$|R_{AB}(z_1,z_2)| \le \frac{C}{n^3}.$$

Thus, we conclude that uniformly in A and B for $|z_{1,2}| \le 1/16$ we have

$$|\hat{R}_{AB}(z_1,z_2)| \le \frac{C}{n^3}.$$

Now, taking in Proposition 2.1 $\Phi = \left(n^{-1}\mathrm{Tr}\hat{G}(z_1)\right)^\circ \hat{G}_{ac}(z_2)$ and repeating the procedure above, we obtain the analog of (2.21) with interchanged A and B

$$\begin{aligned} &\hat{\alpha}_B(z_2)\mathbf{Cov}\{g_n(z_1), g_n(z_2)\} - \hat{\beta}_B(z_2)\mathbf{Cov}\{g_n(z_1), \delta_{n,B}(z_2)\} \\ &+\mathbf{Cov}\{g_n(z_1), \delta_{n,A}(z_2)\} = \frac{1}{n^2}\gamma_{BA}(z_1,z_2) + \hat{R}_{BA}(z_1,z_2), \end{aligned} \tag{2.26}$$

where

$$\begin{aligned}
\hat{\alpha}_B(z) &= \frac{z\tilde{z}_{n,B}(z)}{1+zf_n(z)} f'_{n,B}(\tilde{z}_{n,B}(z)),\ \hat{\beta}_B(z) = \frac{z}{1+zf_n(z)} f'_{n,B}(\tilde{z}_{n,B}(z)),\\
\gamma_{BA}(z_1,z_2) &= \frac{\partial}{\partial z_1}\frac{z_2}{1+z_2 f_n(z_2)}\Big(n^{-1}\mathrm{Tr}\mathbf{E}\left\{\hat{G}(z_1)U^*AU\hat{G}(z_2)B\right\}\widetilde{G}_B(z_2)\\
&\quad -n^{-1}\mathrm{Tr}\mathbf{E}\left\{\hat{G}(z_1)\hat{G}(z_2)UAU^*B\right\}\widetilde{G}_B(z_2)\Big),\\
\tilde{z}_{n,B}(z) &= \frac{z\mathbf{E}\{\delta_{n,B}(z)\}}{1+zf_n(z)},\ \widetilde{G}_B(z) = (B-\tilde{z}_{n,B}(z)I)^{-1} = G_B(\tilde{z}_{n,B}(z)),\\
|\hat{R}_{BA}(z_1,z_2)| &\le \frac{C}{n^3},\ |z_{1,2}|\le 1/16.
\end{aligned}$$

On other hand, applying to the (2.11) the operation $n^{-1}\mathrm{Tr}\cdot$ and regrouping terms, we obtain

$$\begin{aligned}
&\hat{\alpha}_0(z_2)\mathbf{Cov}\{g_n(z_1),g_n(z_2)\} - \hat{\beta}_A^{(0)}(z_2)\mathbf{Cov}\{g_n(z_1),\delta_{n,B}(z_2)\}\\
&-\hat{\beta}_B^{(0)}(z_2)\mathbf{Cov}\{g_n(z_1),\delta_{n,A}(z_2)\} = \frac{1}{n^2}\gamma_0(z_1,z_2) + \hat{R}_0(z_1,z_2),
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
\hat{\alpha}_0(z) &= 1+2zf_n(z),\ \hat{\beta}_l^{(0)}(z) = \mathbf{E}\{\delta_{n,l}(z)\}, l=A,B,\\
\gamma_0(z_1,z_2) &= \frac{\partial}{\partial z_1} n^{-1}\mathrm{Tr}\left(\mathbf{E}\{U^*BUG(z_1)AG(z_2)\} - \mathbf{E}\{G(z_1)AU^*BUG(z_2)\}\right),\\
\hat{R}_0(z_1,z_2) &= \mathbf{E}\{g_n^\circ(z_1)\delta_{n,A}^\circ(z_2)\left(n^{-1}\mathrm{Tr}U^*BUG(z_2)\right)^\circ\} - z_2\mathbf{E}\{g_n^\circ(z_1)(g_n^\circ(z_2))^2\}.
\end{aligned}$$

According to Proposition 4.1 we have uniformly in A and B for $|z_{1,2}|\le 1/16$

$$|\hat{R}_0(z_1,z_2)| \le \frac{C}{n^3}.$$

Thus, the relations (2.21), (2.26) and (2.27) for the triple of the covariances

$$C_0 := \mathbf{Cov}\{g_n(z_1),g_n(z_2)\},\ C_A := \mathbf{Cov}\{g_n(z_1)\},\delta_{n,A}(z_2),\ C_B := \mathbf{Cov}\{g_n(z_1),\delta_{n,B}(z_2)\}$$

lead to the following linear system

$$\left\{\begin{aligned}
\hat{\alpha}_A(z_2)C_0 - \hat{\beta}_A(z_2)C_A + C_B &= \gamma_{AB}(z_1,z_2)/n^2 + O(n^{-3}),\\
\hat{\alpha}_B(z_2)C_0 + C_A - \hat{\beta}_B(z_2)C_B &= \gamma_{BA}(z_1,z_2)/n^2 + O(n^{-3}),\\
\hat{\alpha}_0(z_2)C_0 - \hat{\beta}_B^{(0)}(z_2)C_A - \hat{\beta}_A^{(0)}C_B &= \gamma_0(z_1,z_2)/n^2 + O(n^{-3}).
\end{aligned}\right. \tag{2.28}$$

The determinant of this system for $|z_{1,2}|\le 1/16$ is equal

$$\begin{aligned}
D(z_2) &= \hat{\alpha}_0(z_2)(\hat{\beta}_A(z_2)\hat{\beta}_B(z_2)-1) - \hat{\alpha}_{Br}(z_2)(\hat{\beta}_A(z_2)\hat{\beta}_A^{(0)}(z_2) + \hat{\beta}_B^{(0)}(z_2))\\
&\quad -\hat{\alpha}_A(z_2)(\hat{\beta}_B(z_2)\hat{\beta}_B^{(0)}(z_2) + \hat{\beta}_A^{(0)}(z_2)),
\end{aligned}$$

where

$$\begin{aligned}\hat{\alpha}_0(z) &= 1+2zf(z)+o(1),\ \hat{\alpha}_l(z)=\frac{z\tilde{z}_l(z)}{1+zf(z)}f_l'(\tilde{z}_l(z))+o(1),\ l=A,B,\ n\to\infty,\\ \hat{\beta}_A^{(0)}(z) &= \Delta_A(z)+o(1),\ \hat{\beta}_B^{(0)}(z)=\Delta_B(z)+o(1),\ n\to\infty,\\ \hat{\beta}_l(z) &= \frac{z}{1+zf(z)}f_l'(\tilde{z}_l(z))+o(1),\ l=A,B,\ n\to\infty.\end{aligned}$$

Thus, it coincides asymptotically with the Jacobian J (2.3)

$$D(z_2)=J(f(z_2),\Delta_A(z_2),\Delta_B(z_2))+o(1),\ n\to\infty.$$

Hence, according to Lemma 2.3 for sufficiently large n the determinant $D(z_2)$ will be non-zero uniformly in A and B for $|z_{1,2}|\le 1/16$. Thus, for sufficiently large n the system (2.28) is uniquely soluble and its solution is

$$\begin{aligned}\mathbf{Cov}\{g_n(z_1),g_n(z_2)\} &= \frac{1}{J(f(z_2),\Delta_A(z_2),\Delta_B(z_2))}\\ &\times\Bigg(\frac{\gamma_0(z_1,z_2)}{n^2}\left(\frac{z_2^2f_A'(\tilde{z}_A(z_2))f_B'(\tilde{z}_B(z_2))}{(1+z_2f(z_2))^2}-1\right)\\ &-\frac{\gamma_{AB}(z_1,z_2)}{n^2}(\tilde{z}_B(z_2)f_B'(\tilde{z}_B(z_2))+\Delta_A(z_2))\\ &-\frac{\gamma_{BA}(z_1,z_2)}{n^2}(\tilde{z}_A(z_2)f_A'(\tilde{z}_A(z_2))+\Delta_B(z_2))\Bigg)+o(n^{-2}).\end{aligned}\tag{2.29}$$

Now, to apply Proposition 4.2 (ii) and (iii) and calculate the asymptotics of $\gamma_0(z_1,z_2)$, $\gamma_{AB}(z_1,z_2)$ and $\gamma_{BA}(z_1,z_2)$ we suppose for a while that $m_A^{(1)}m_B^{(1)}\neq 0$. Passing to the variables (1.10) in the coefficients of (2.29) and regrouping terms according to (1.11), we have

$$\begin{aligned}\frac{z_2^2f_A'(\tilde{z}_A(z_2))f_B'(\tilde{z}_B(z_2))}{(1+z_2f(z_2))^2}-1 &= \frac{z_2\psi_A'(\tilde{z}_A(z_2))\psi_B'(\tilde{z}_B(z_2))}{\psi(z_2)(1-\psi(z_2))^2}\\ &-\frac{z_2\psi_A'(\tilde{z}_A(z_2))}{(1-\psi(z_2))\tilde{z}_B(z_2)}-\frac{z_2\psi_B'(\tilde{z}_B(z_2))}{(1-\psi(z_2))\tilde{z}_A(z_2)},\\ \tilde{z}_B(z_2)f_B'(\tilde{z}_B(z_2))+\Delta_A(z_2) &= \frac{\psi_B'(\tilde{z}_B(z_2))}{(1-\psi(z_2))^2},\\ \tilde{z}_A(z_2)f_A'(\tilde{z}_A(z_2))+\Delta_B(z_2) &= \frac{\psi_A'(\tilde{z}_A(z_2))}{(1-\psi(z_2))^2}.\end{aligned}$$

Applying the same procedure to the expressions (4.2), (4.3) and (4.4), we have

$$\begin{aligned}
\gamma_0(z_1,z_2) &= \frac{\partial}{\partial z_1}\frac{1}{z_2}\left(\frac{\delta\tilde{z}_A\delta\tilde{z}_B}{\delta z\delta\psi}-1\right)+o(1)\\
&= \frac{\partial}{\partial z_1}\frac{z_1\psi(z_2)+z_2\psi(z_1)-\tilde{z}_A(z_1)\tilde{z}_B(z_2)-\tilde{z}_A(z_2)\tilde{z}_B(z_1)}{z_2\delta z\delta\psi}+o(1),\\
\gamma_{AB}(z_1,z_2) &= \frac{\partial}{\partial z_1}z_1z_2\frac{\delta(\tilde{z}_A/z)}{\delta z\delta\psi}\left(\frac{\delta\Delta_B}{\delta\tilde{z}_A}(1-\psi(z_1))(1-\psi(z_2))\right.\\
&\left.-\frac{(1-\psi(z_2))^2}{\tilde{z}_A(z_2)}\left(\frac{\psi_A'(\tilde{z}_A(z_2))}{(1-\psi(z_2))^2}-\Delta_B(z_2)\right)\right)+o(1),\\
\gamma_{BA}(z_1,z_2) &= \frac{\partial}{\partial z_1}z_1z_2\frac{\delta(\tilde{z}_B/z)}{\delta z\delta\psi}\left(\frac{\delta\Delta_A}{\delta\tilde{z}_B}(1-\psi(z_1))(1-\psi(z_2))\right.\\
&\left.-\frac{(1-\psi(z_2))^2}{\tilde{z}_B(z_2)}\left(\frac{\psi_B'(\tilde{z}_B(z_2))}{(1-\psi(z_2))^2}-\Delta_A(z_2)\right)\right)+o(1).
\end{aligned}$$

Substituting this expressions in (2.29) and regrouping terms, we obtain

$$\begin{aligned}
\mathbf{Cov}\{g_n(z_1),g_n(z_2)\} &= \frac{1}{n^2}\frac{\partial}{\partial z_1}\frac{1}{J(f(z_2),\Delta_A(z_2),\Delta_B(z_2))}\frac{1}{\delta z\delta\psi}\\
&\times(\psi_A'(\tilde{z}_A(z_2))\psi_B'(\tilde{z}_B(z_2))K_0\\
&-\psi_B'(\tilde{z}_B(z_2))K_A-\psi_A'(\tilde{z}_A(z_2))K_B)+o(n^{-2}),
\end{aligned}$$

where

$$\begin{aligned}
K_0 &= \frac{1}{\psi(z_2)(1-\psi(z_2))^2}(-z_1(\tilde{z}_B(z_2)\delta(\tilde{z}_A/z)+\tilde{z}_A(z_2)\delta(\tilde{z}_B/z))\\
&+z_1\psi(z_2)+z_2\psi(z_1)-\tilde{z}_A(z_1)\tilde{z}_B(z_2)-\tilde{z}_A(z_2)\tilde{z}_B(z_1))\\
&= \frac{z_2\delta\psi-\psi(z_2)\delta z}{\psi(z_2)(1-\psi(z_2))^2}
\end{aligned}$$

and

$$\begin{aligned}
K_A &= z_1z_2\left(\frac{\delta\Delta_B}{\delta\tilde{z}_A}\frac{1-\psi(z_1)}{1-\psi(z_2)}+\frac{\Delta_B(z_2)}{\tilde{z}_A(z_2)}\right)\delta\frac{\tilde{z}_A}{z}+\frac{1}{(1-\psi(z_2))\,\tilde{z}_A(z_2)}\\
&\times(z_1\psi(z_2)+z_2\psi(z_1)-\tilde{z}_A(z_1)\tilde{z}_B(z_2)-\tilde{z}_A(z_2)\tilde{z}_B(z_1))\\
&= \frac{z_2\psi(z_1)-\tilde{z}_A(z_2)\tilde{z}_B(z_1)}{(1-\psi(z_2))\,\tilde{z}_A(z_2)\delta\tilde{z}_A}\tilde{z}_A(z_1)-\frac{1-\psi(z_1)}{1-\psi(z_2)}\frac{z_2\tilde{z}_A(z_1)-z_1\tilde{z}_A(z_2)}{\delta\tilde{z}_A}\Delta_B(z_2)\\
&= \frac{z_2\tilde{z}_A(z_1)-z_1\tilde{z}_A(z_2)}{(1-\psi(z_2))\,\tilde{z}_A(z_2)\delta\tilde{z}_A}\left(\psi(z_1)-\frac{1-\psi(z_1)}{1-\psi(z_2)}\psi(z_2)\right)\\
&= \frac{\delta\psi(z_2\delta\tilde{z}_A-\tilde{z}_A(z_2)\delta z)}{(1-\psi(z_2))^2\,\tilde{z}_A(z_2)\delta\tilde{z}_A},\\
K_B &= \frac{\delta\psi(z_2\delta\tilde{z}_B-\tilde{z}_B(z_2)\delta z)}{(1-\psi(z_2))^2\,\tilde{z}_B(z_2)\delta\tilde{z}_B}.
\end{aligned}$$

Using these relations, (1.13) and the following equality

$$\begin{aligned} & J(f(z_2), \Delta_A(z_2), \Delta_B(z_2)) \\ = \; & \frac{z\psi_A'(\tilde z_A(z_2))\psi_B'(\tilde z_B(z_2) - \tilde z_A(z_2)\psi_A'(\tilde z_A(z_2)) - \tilde z_B(z_2)\psi_B'(\tilde z_B(z_2)}{\psi(z_2)(1-\psi(z_2))^2}, \end{aligned}$$

we conclude that

$$\mathbf{Cov}\{g_n(z_1), g_n(z_2)\} = \frac{1}{n^2}\frac{\partial}{\partial z_1}\left(\frac{\tilde z_A'(z_2)}{\delta \tilde z_A} + \frac{\tilde z_B'(z_2)}{\delta \tilde z_B} - \frac{\psi'(z_2)}{\delta\psi} - \frac{1}{\delta z}\right) + o(n^{-2}).$$

This leads to (1.14) and hence to (1.8). Now we note that asymptotics (1.8) can be used not only in the case $m_A^{(1)}m_B^{(1)} \neq 0$, but for any A and B and $|z_{1,2}| \le 1/16$, since it has no singularity. The remainders $\hat R_0$, $\hat R_{AB}$ and $\hat R_{BA}$ are also of the order $o(n^{-2})$ for any A and B and $|z_{1,2}| \le 1/16$. Thus, we can omit the restriction $m_A^{(1)}m_B^{(1)} \neq 0$. ■

Remark 2.4 Due to the non-zero asymptotics (2.5) (Lemma 2.3) in the case $m_A^{(1)}m_B^{(1)} \neq 0$ the explicit expressions for asymptotics $S(z_1, z_2)$ (1.14) and (1.8) are valid not only in the interior of the unit disk $|z| < 1$, but in the exterior $|z| > 1$ too, at least for sufficiently large z. Indeed, as it was shown in Lemma 2.3 and due to the relation

$$1 + zf(z) = -\frac{1}{z}\left(m_A^{(1)}m_B^{(1)} - \int_{\mathbb{T}} \frac{\lambda^2}{\lambda - z} N(d\lambda)\right) \tag{2.30}$$

for sufficiently large but finite z (e.g. for $|z| > 1 + |m_A^{(1)}m_B^{(1)}|^{-1}$) we Jacobian J will be non-zero. Hence, the linear system (2.28) for the covariances is uniquely soluble for sufficiently large n. The correspondent bounds for variances and higher centralized moments in the remainders are valid outside unit circle. Thus, the explicit formulas for the asymptotics of coefficients and of terms in r.h.s. of (2.28) are valid too.

3 Central Limit Theorem for linear statistics

In this section we prove the Central Limit Theorem for linear eigenvalue statistics

$$\mathcal{N}_n[\varphi] := \operatorname{Tr}\varphi(H_n) = \sum_{l=1}^{n} \varphi(\lambda_l^W) = n\int_{\mathbb{T}} \varphi(\lambda) N_n(d\lambda)$$

of some sufficiently smooth function $\varphi : \mathbb{T} \to \mathbb{R}$. We apply the approach used in [5] and [7] to study the additive analogs of the ensemble (1.1). It based on the two propositions: the a priory bound on the variance of linear statistic and on the following proposition proved in [5]:

Proposition 3.1 *If $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi \in \mathcal{L}$-vector space with the norm $\|\cdot\|$ and we have:*

• *the variance $V_n[\varphi] = \mathbf{Var}\{\mathcal{N}_n[\varphi]\}$ admits uniform in n bound $V_n[\varphi] \le C||\varphi||^2$ for all $\varphi \in \mathcal{L}$;*

• *there exists a dense linear manifold $\mathcal{L}_1 \subset \mathcal{L}$ such that Central Limit Theorem is valid for*

$$\mathcal{N}_n^\circ[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}$$

with $\varphi \in \mathcal{L}_1$i.e., if $Z_n[x\varphi] = \mathbf{E}\left\{e^{ix\mathcal{N}_n^\circ[\varphi]}\right\}$ then there exists a continuous quadratic functional $V_{kk} : \mathcal{L}_1 \to \mathbb{R}_+$ such that uniformly in x on any compact

$$\lim_{n\to\infty} Z_n[x\varphi] = e^{-x^2 V[\varphi]/2}, \ \forall \varphi \in \mathcal{L}_1$$

then V admits a continuous extension to $\mathcal{L}$ and Central Limit Theorem is valid for all $\mathcal{N}_n^\circ[\varphi]$ with $\varphi \in \mathcal{L}$.

We will consider as the space $\mathcal{L}$ the Hilbert space $\mathcal{H}_s(\mathbb{T})$, $s \ge 0$ of functions $\varphi : \mathbb{T} \to \mathbb{C}$ with the norm:

$$||\varphi||_s^2 = \sum_{k=-\infty}^{+\infty} (1 + 2|k|)^{2s} |\hat{\varphi}_k|^2, \ \hat{\varphi}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{i\theta}) e^{-ik\theta} d\theta.$$

For hermitian ensemble in [7] the a priory bound for the variance $\mathbf{Var}\{\mathcal{N}_n[\varphi]\}$ needed in Proposition 3.1 have been obtained via some inequality. Here we consider its unitary analog:

Proposition 3.2 *For any $\varphi \in \mathcal{H}_s(\mathbb{T})$, $s > 0$ and any random unitary W we have the bound*

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \le C_s ||\varphi||_s^2 \int_0^{+\infty} dt e^{-t} t^{2s-1} \int_{-\pi}^{\pi} \mathbf{Var}\{\mathrm{Tr}\mathrm{Re} Q(e^{-t} e^{i\theta})\} d\theta, \tag{3.1}$$

where $Q(z) = (W + zI)(W - zI)^{-1}$, $|z| < 1$.

Proof. We just follow the proof of the analogous proposition for hermitian case in [7]. Consider the operator $D_s : \mathcal{H}_s(\mathbb{T}) \to L^2(\mathbb{T})$ defined as follows for any $\psi \in \mathcal{H}_s(\mathbb{T})$:

$$\widehat{(D_s\psi)}_k = (1 + 2|k|)^s \hat{\psi}_k.$$

For fixed n $\mathbf{Var}\{\mathcal{N}_n[\varphi]\}$ is a bounded quadratic form in the Hilbert space $\mathcal{H}_s(\mathbb{T})$ with the inner product $(u, v)_s = (D_s u, D_s v)$, where $(\cdot, \cdot)$ denotes inner product in $L^2(\mathbb{T})$. Then there exist positive self adjoint operator $\mathcal{V}_{jj}$ which defines this quadratic form

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = (\mathcal{V}\varphi, \varphi) = \mathrm{Tr}(\Pi_\varphi \mathcal{V} \Pi_\varphi),$$

where Π_φ is the modified projection operator: $\Pi_\varphi u = \varphi(u,\varphi)/||\varphi||_{L^2}$. Besides, we have

$$\mathrm{Tr}(\Pi_\varphi \mathcal{V}\Pi_\varphi) = \mathrm{Tr}(\Pi_\varphi D_s D_s^{-1}\mathcal{V}D_s^{-1}D_s\Pi_\varphi) \le ||D_s\Pi_\varphi||^2\mathrm{Tr}(D_s^{-1}\mathcal{V}D_s^{-1}),$$

where $||D_s\Pi_\varphi|| \le ||D_s\varphi||_{L^2} = ||\varphi||_s$. On the other hand, for any $u, v \in L^2(\mathbb{T})$ we have

$$\begin{aligned}
\Gamma(2s)(D_s^{-2}u,v) &= \Gamma(2s)\sum_{k=-\infty}^{+\infty}(1+2|k|)^{-2s}\hat{u}_k\hat{v}_k \\
&= \int_0^{+\infty} dte^{-t}t^{2s-1}\sum_{k=-\infty}^{+\infty}\left(e^{-t}\right)^{2|k|}\hat{u}_k\hat{v}_k \\
&= \int_0^{+\infty} dte^{-t}t^{2s-1}\left(P_{e^{-t}} * u, P_{e^{-t}} * v\right) \\
&= \int_0^{+\infty} dte^{-t}t^{2s-1}\int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_{e^{-t}}(\theta-\lambda)P_{e^{-t}}(\theta-\mu)u(\lambda)\overline{v(\mu)}d\lambda d\mu,
\end{aligned}$$

where Γ denotes Γ-function, $\cdot * \cdot$ means the convolution of functions and $P_r(\theta)$ is the Poisson kernel with the parameter $0 \le r < 1$:

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = \mathrm{Re}\frac{1+re^{i\theta}}{1-re^{i\theta}}.$$

This implies the explicit form of the integral kernel of $\Gamma(2s)(D_s^{-2}u,v)$

$$\Gamma(2s)D_s^{-2}(\lambda,\mu) = \int_0^{+\infty} dte^{-t}t^{2s-1}\int_{-\pi}^{\pi} d\theta P_{e^{-t}}(\theta-\lambda)P_{e^{-t}}(\theta-\mu)$$

and

$$\begin{aligned}
\Gamma(2s)\mathrm{Tr}(D_s^{-1}\mathcal{V}D_s^{-1}) &= \int_0^{+\infty} dte^{-t}t^{2s-1}\int_{-\pi}^{\pi} d\theta\left(\mathcal{V}P_{e^{-t}}(\theta-.), P_{e^{-t}}(\theta-.)\right) \\
&= \int_0^{+\infty} dte^{-t}t^{2s-1}\int_{-\pi}^{\pi} d\theta\mathbf{Var}\{\mathcal{N}_n[P_{e^{-t}}(\theta-.)]\} \\
&= \int_0^{+\infty} dte^{-t}t^{2s-1}\int_{-\pi}^{\pi}\mathbf{Var}\{\mathrm{Tr}\mathrm{Re}Q(e^{-t}e^{i\theta})\}d\theta,
\end{aligned}$$

which completes the proof. ■

The main result of this section is follows:

Theorem 3.3 *If S and T are non-random, satisfying conditions (1.3) and $m_A^{(1)} m_B^{(1)} \neq 0$, and the statistic function $\varphi : \mathbb{T} \to \mathbb{R}$, $\varphi \in \mathcal{H}_{2+\varepsilon}(\mathbb{T})$, $\varepsilon > 0$ then*

$$\mathcal{N}_n^\circ[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}$$

converges in distribution to the Gaussian random variable with zero mean and the variance

$$V[\varphi] = \lim_{r_1, r_2 \uparrow 1} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \varphi(e^{i\theta_1})\varphi(e^{i\theta_2}) \left(\mathbf{R}_{1,2} \cdot T\right)(r_1 e^{i\theta_1}, r_2 e^{i\theta_1}) d\theta_1 d\theta_2,$$

where

$$\begin{aligned} T(z_1, z_2) \ &: = z_1 z_2 S(z_1, z_2), \\ \left(\mathbf{R}_{1,2} \cdot T\right)(z_1, z_2) \ &: = T(z_1, z_2) - T\left(z_1, \frac{1}{\overline{z_2}}\right) - T\left(\frac{1}{\overline{z_1}}, z_2\right) + T\left(\frac{1}{\overline{z_1}}, \frac{1}{\overline{z_2}}\right) \end{aligned}$$

and $S(z_1, z_2)$ is defined in (1.14) and (1.8).

Proof. According to Propositions 3.2 and the bound (2.23) we have for any $\varphi \in \mathcal{H}_{2+\varepsilon}(\mathbb{T})$

$$\begin{aligned} \mathbf{Var}\{\mathcal{N}_n[\varphi]\} &\leq C_s ||\varphi||_s^2 \int_0^{+\infty} dt e^{-t} t^{2s-1} \int_{-\pi}^{\pi} \mathbf{Var}\{\mathrm{Tr}\mathrm{Re} Q(e^{-t} e^{i\theta})\} d\theta \\ &\leq C_s ||\varphi||_s^2 \int_0^{+\infty} dt e^{-t} t^{2s-1} \int_{-\pi}^{\pi} n^2 \mathbf{Var}\{\mathrm{Re} g_n(e^{-t} e^{i\theta})\} d\theta \leq \tilde{C} ||\varphi||_s^2. \end{aligned}$$

Thus, according to Proposition 3.1 it suffices to prove the theorem for some dense set in $\mathcal{H}_{2+\varepsilon}(\mathbb{T})$ (say, trigonometric polynomials) and then extend it on whole $\mathcal{H}_{2+\varepsilon}(\mathbb{T})$. The procedure is the same as used in [9] for additive hermitian analog of (1.1) with the difference in the representation of analytic in $\mathbb{C}\backslash\{0\}$ statistic function φ via contour integral by the Cauchy formula

$$\begin{aligned} \mathcal{N}_n^\circ[\varphi] &= \sum_{l=1}^{n} \mathrm{Tr}\left(\varphi(W) - \mathbf{E}\left\{\varphi(W)\right\}\right) \\ &= \frac{n}{2\pi i} \int_\Gamma \varphi(z) \left(-g_n(z) + \mathbf{E}\left\{g_n(z)\right\}\right) dz \\ &= -\frac{n}{2\pi i} \int_\Gamma \varphi(z) g_n^\circ(z) dz, \end{aligned}$$

We start with the contour $\Gamma = \Gamma_r \cup \Gamma_\beta^+ \cup \Gamma_{1/r} \cup \Gamma_r^-$ (see Figure 1.(a)) encircling the spectrum of the ensemble (1.1) for any realization. Of course this contour have realization-dependent part (we chose the angle β in such way to have contour Γ outside the eigenvalues for each realization), but we can cancel the integrals over Γ_β^+ and Γ_r^- since they are the same contour integrals in opposite directions. Thus we obtain the

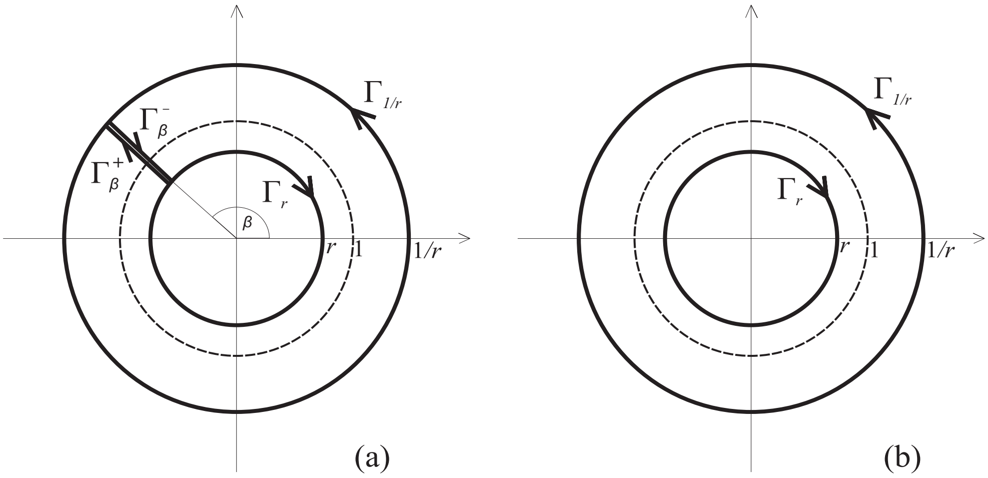

Figure 1: Contours (a) and (b)

integral over the realization-independent contour $\hat{\Gamma}_r = \Gamma_r \cup \Gamma_{1/r}$ (see Figure 1.(b))

$$\begin{aligned} \mathcal{N}_n^\circ[\varphi] &= -\frac{n}{2\pi i}\int\limits_{\hat{\Gamma}_r} \varphi(z) g^\circ(z) dz \\ &= \frac{n}{2\pi}\int\limits_{-\pi}^{\pi} \left(\varphi\left(re^{i\theta}\right) re^{i\theta} g^\circ\left(re^{i\theta}\right) - \varphi\left(\frac{e^{i\theta}}{r}\right)\frac{e^{i\theta}}{r} g^\circ\left(\frac{e^{i\theta}}{r}\right)\right) d\theta. \end{aligned}$$

Define the characteristic function

$$Z_n(x) = \mathbf{E}\left\{e_n(x)\right\}, \ x \in \mathbb{R},$$

where

$$e_n(x) = e^{ix\mathcal{N}_n^\circ[\varphi]} = \exp\left\{-\frac{nx}{2\pi}\int\limits_{\hat{\Gamma}_r} \varphi(z) g_n^\circ(z) dz\right\}.$$

Since $Z_n(0) = 1$ and

$$e_n(x) = 1 + \int\limits_0^x e_n'(y) dy, \ Z_n(x) = 1 + \int\limits_0^x Z_n'(y) dy, \tag{3.2}$$

is suffices to prove that there exist subsequences $\{Z_{n_j}(x)\}$ and $\{Z'_{n_j}(x)\}$ that converge uniformly on any finite interval and

$$\lim_{n_j\to\infty} Z_{n_j}(x) = Z(x), \quad \lim_{n_j\to\infty} Z'_{n_j}(x) = -xV[\varphi]Z(x).$$

Besides, due to the Cauchy theorem

$$\begin{aligned} \frac{d}{dx}e_n(x) &= -\frac{n}{2\pi}e_n(x)\int\limits_{\hat\Gamma_{r_1}}\varphi(z)g_n^\circ(z)dz \\ &= -\frac{n}{2\pi}\int\limits_{\hat\Gamma_{r_1}}\varphi(z_1)e_n(x)g_n^\circ(z_1)dz_1, \\ Z'_n(x) &= -\frac{1}{2\pi}\int\limits_{\hat\Gamma_{r_1}}\varphi(z_1)\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\}dz_1, \end{aligned} \tag{3.3}$$

where we choose the contour $\hat\Gamma_{r_1}$ in the domain

$$D = \{z\in\mathbb{C} : |z| < |m_A^{(1)}m_B^{(1)}|/8 \le 1/8\} \cup \{z\in\mathbb{C} : |z| > 1 + |m_A^{(1)}m_B^{(1)}|^{-1}\}.$$

To find $\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\}$, we apply the same procedure as in the previous section and obtain for the triple

$$(\hat C_0 := \mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\}, \hat C_A := \mathbf{E}\left\{ne_n^\circ(x)\delta_{n,A}(z_1)\right\}, \hat C_B := \mathbf{E}\left\{ne_n^\circ(x)\delta_{n,B}(z_1)\right\})$$

the uniquely soluble system

$$\left\{\begin{array}{rcl} \hat\alpha_A(z_1)\hat C_0 - \hat\beta_A(z_1)\hat C_A + \hat C_B &=& C_{AB}, \\ \hat\alpha_B(z_1)\hat C_0 + \hat C_A - \hat\beta_B(z_1)\hat C_B &=& C_{BA}, \\ \hat\alpha_0(z_1)\hat C_0 - \hat\beta_B^{(0)}(z_1)\hat C_A - \hat\beta_A^{(0)}\hat C_B &=& C_0. \end{array}\right. \tag{3.4}$$

where

$$\begin{aligned} C_{AB} &= -\frac{xZ_n(x)}{2\pi}\int\limits_{\hat\Gamma_{r_2}}\varphi(z_2)\gamma_{AB}(z_2,z_1)dz_2 + n\hat R_{AB}(z_2,z_1) - \tau_{AB}(z_1,z_2), \\ \tau_{AB}(z_1,z_2) &= \frac{z_1\int_{\hat\Gamma_{r_2}} x\varphi(z_2)\mathbf{Cov}\{e_n(x), n^{-1}\mathrm{Tr}\widetilde G_A(z_1)[U^*BU, G^2(z_2)]AG(z_1)\}dz_2}{1+z_1f_n(z_1)}, \\ C_0 &= -\frac{xZ_n(x)}{2\pi}\int\limits_{\hat\Gamma_{r_2}}\varphi(z_2)\gamma_0(z_2,z_1)dz_2 + n\hat R_0(z_2,z_1) - \tau_0(z_1,z_2), \\ \tau_0(z_1,z_2) &= \int\limits_{\hat\Gamma_{r_2}} x\varphi(z_2)\mathbf{Cov}\{e_n(x), n^{-1}\mathrm{Tr}[U^*BU, G^2(z_2)]AG(z_1)\}dz_2 \end{aligned}$$

contour $\hat\Gamma_{r_2}\subset D$ and C_{BA} is defined analogously to the C_{AB} with interchanged A and B. Using Proposition 4.1 and Schwarz inequality, we obtain uniformly in x on any finite interval and in $z_{1,2}\in K$, K-compact, $K\subset D$

$$n\hat R_l(z_2,z_1)=O(n^{-1}),\ \tau_l(z_1,z_2)=O(n^{-1}),\ l=0,AB,BA.$$

Then, solving (3.4), we obtain uniformly in x on any finite interval

$$\begin{aligned}\mathbf{E}\left\{ne_n^\circ(x)g_n(z_1)\right\} &= \frac{xZ_n(x)}{2\pi}\int\limits_{\hat\Gamma_{r_2}} dz_2\frac{\varphi(z_2)}{D(z_1)}\Big(\gamma_0(z_2,z_1)(\hat\beta_A(z_1)\hat\beta_B(z_1)-1)\\ &\quad -\gamma_{AB}(z_2,z_1)(\hat\beta_B(z_1)\hat\beta_B^{(0)}(z_1)+\hat\beta_B^{(0)}(z_1))\\ &\quad -\gamma_{BA}(z_2,z_1)(\hat\beta_A(z_1)\hat\beta_A^{(0)}(z_1)+\hat\beta_B^{(0)}(z_1)))+O(n^{-1})\\ &= \frac{xZ_n(x)}{2\pi}\int\limits_{\hat\Gamma_{r_2}}\varphi(z_2)S(z_2,z_1)dz_2+o(1),\ n\to\infty.\end{aligned}$$

Substituting this in (3.3), we obtain uniformly in x on any finite interval in view of finiteness of the contours $\hat\Gamma_{r_1}$ and $\hat\Gamma_{r_2}$

$$Z_n'(x)=-\frac{xZ_n(x)}{4\pi}\int\limits_{\hat\Gamma_{r_1}}\int\limits_{\hat\Gamma_{r_2}}\varphi(z_1)\varphi(z_2)S(z_1,z_2)dz_1dz_2+o(1),\ n\to\infty,$$

which completes the proof, due to the analyticity of $\varphi(z_1)\varphi(z_2)S_n(z_1,z_2)$ in $z_{1,2}$ for $|z_{1,2}|\neq 1$. ■

4 Appendix

Proposition 4.1 *For the ensemble (1.1) we have*
(i) for $|z_{1,2}|\neq 1$

$$\hat R_0(z_1,z_2)=\mathbf{E}\{g_n^\circ(z_1)\delta_{n,A}^\circ(z_2)\delta_{n,B}^\circ(z_2)\}-z_2\mathbf{E}\{g_n^\circ(z_1)(g_n^\circ(z_2))^2\}=O(n^{-3}),$$

(ii) for $|z_{1,2}|<1$

$$\begin{aligned}R_{AB}(z_1,z_2) &= \frac{z_2}{1+z_2f_n(z_2)}\Big(-z_2\mathbf{E}\left\{g_n^\circ(z_1)g_n^\circ(z_2)n^{-1}\mathrm{Tr}\widetilde G_A(z_2)G^\circ(z_2)\right\}\\ &\quad +\mathbf{E}\left\{g_n^\circ(z_1)\delta_{n,B}^\circ(z_2)n^{-1}\mathrm{Tr}\widetilde G_A(z_2)\left(U^*BUG(z_2)\right)^\circ\right\}\Big)=O(n^{-3}).\end{aligned}$$

(iii)

$$\mathbf{Var}\left\{n^{-1}\mathrm{Tr}\widetilde G_A(z_1)[U^*BU,G^2(z_2)]AG(z_1)\right\}\le O(n^{-2})$$

and if we suppose $m_A^{(1)}m_B^{(1)}\neq 0$ *then this relation is valid also for* $|z|>1$ *and* z *sufficiently large.*

Proof. (i) Note, that for

$$|\mathbf{E}\{g_n^\circ(z_1)(g_n^\circ(z_2))^2\}| \le \mathbf{Var}^{1/2}\{g_n(z_1)\}\mathbf{Var}^{1/2}\{(g_n^\circ(z_2))^2\} \le \frac{C}{n}\mathbf{Var}^{1/2}\{(g_n^\circ(z_2))^2\}.$$

Then, using Proposition 2.2 we obtain

$$\begin{aligned}\mathbf{Var}\{(g_n^\circ(z))^2\} &\le \frac{4}{n^3}\sum_{j,t=1}^{n}\mathbf{E}\left\{|g_n^\circ(z)\mathrm{Tr}\, GAU^*\left(E^{(t,j)}B - TE^{(j,t)}\right)UG|^2\right\}\\ &\le \frac{16}{n^2|1-|z||^4}\mathbf{Var}\{g_n(z)\} \le \frac{64}{n^4|1-|z||^8}\end{aligned}$$

and analogous bound for $\mathbf{E}\{g_n^\circ(z_1)\delta_{n,A}^\circ(z_2)\delta_{n,B}^\circ(z_2)\}$ and hence we obtain (i).

(ii) and (iii) These assertions can be proved analogously, with the appearance of the new factor

$$\frac{|z_2|}{|1+z_2f_n(z_2)|}||\widetilde{G}_A(z_2)||^2$$

in the numerator. But this factor is bounded uniformly in A and B for small z and according to (2.30) it will be also bounded in the case $m_A^{(1)}m_B^{(1)} \ne 0$ for sufficiently large z (e.g. for $|z| > 1 + |m_A^{(1)}m_B^{(1)}|^{-1}$) due to the relation (2.10). ■

Proposition 4.2 *For the ensemble (1.1) we have for* $m_A^{(1)}m_B^{(1)} \ne 0$ *and* $|z| \le |m_A^{(1)}m_B^{(1)}|/8 \le 1/8$:

(i)

$$\frac{1 - z_1z_2\dfrac{\delta f}{\delta z}}{1 - \dfrac{z_1z_2I_A(z_1,z_2)I_B(z_1,z_2)}{(1+z_1f(z_1))(1+z_2f(z_2))}} = \frac{\delta\tilde{z}_A\delta\tilde{z}_B}{\delta(zf)\delta z}(1+z_1f(z_1))(1+z_2f(z_2)); \tag{4.1}$$

where

$$\delta\tilde{z}_l = \tilde{z}_l(z_1) - \tilde{z}_l(z_2),\ l = A, B,\ \delta(zf) = z_1f(z_1) - z_2f(z_2);$$

(ii)

$$\begin{aligned}\gamma_0(z_1,z_2) &= \frac{\partial}{\partial z_1}\left(n^{-1}\mathrm{Tr}\mathbf{E}\{U^*BUG(z_1)AG(z_2)\} - \frac{\delta(zf_n)}{\delta z}\right)\\ &= \frac{\partial}{\partial z_1}\frac{1}{z_2}\left(\frac{\delta\tilde{z}_A\delta\tilde{z}_B}{\delta z\delta(zf)}(1+z_1f(z_1))(1+z_2f(z_2)) - 1\right) + o(1),\end{aligned} \tag{4.2}$$

where

$$\delta(zf_n) = z_1f_n(z_1) - z_2f_n(z_2);$$

(iii)

$$\begin{aligned}
\gamma_{AB}(z_1,z_2) &= \frac{\partial}{\partial z_1}\frac{z_2}{1+z_2f_n(z_2)}\Big(n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\mathbf{E}\{U^*BUG(z_1)AG(z_2)\}\\
&\quad -\frac{1}{\delta z}n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\delta\left(z\mathbf{E}\{G\}\right)\Big)\\
&= \frac{\partial}{\partial z_1}\frac{z_1z_2}{1+z_2f_n(z_2)}\frac{\delta(\tilde{z}_A/z)}{\delta z\delta(zf)} \qquad (4.3)\\
&\quad\times\left((1+z_2f(z_2))\frac{\delta\Delta_B}{\delta\tilde{z}_A}-(1+z_1f(z_1))f_A'(\tilde{z}_A(z_2))\right)+o(1),\\
\gamma_{BA}(z_1,z_2) &= \frac{\partial}{\partial z_1}\frac{z_1z_2}{1+z_2f_n(z_2)}\frac{\delta(\tilde{z}_B/z)}{\delta z\delta(zf)} \qquad (4.4)\\
&\quad\times\left((1+z_2f(z_2))\frac{\delta\Delta_A}{\delta\tilde{z}_B}-(1+z_1f(z_1))f_B'(\tilde{z}_B(z_2))\right)+o(1),
\end{aligned}$$

where

$$\begin{aligned}
\delta\left(z\mathbf{E}\{G\}\right) &= z_1\mathbf{E}\{G(z_1)\}-z_2\mathbf{E}\{G(z_2)\},\\
\delta\frac{\tilde{z}_l}{z} &= \frac{\tilde{z}_l(z_1)}{z_1}-\frac{\tilde{z}_l(z_2)}{z_2},\ \delta\Delta_l=\Delta_l(z_1)-\Delta_l(z_2),\ l=A,B.
\end{aligned}$$

Proof. (i) Taking in Proposition 2.1 $\Phi=(G(z_1)AG(z_2))_{ac}$ we obtain

$$\begin{aligned}
&\mathbf{E}\{(G(z_1)A\left[X,U^*BU\right]G(z_1)AG(z_2))_{ac}\}\\
+\ &\mathbf{E}\{(G(z_1)AG(z_2)A\left[X,U^*BU\right]G(z_2))_{ac}\}=0.
\end{aligned}$$

Then take $X=E^{(a,b)}$ and apply to the result the operation $n^{-1}\sum_{a=1}^{n}$. This yields the matrix equality

$$\begin{aligned}
&\mathbf{E}\{\delta_{n,A}(z_1)U^*BUG(z_1)AG(z_2)\}-\mathbf{E}\{(1+z_1g_n(z_1))G(z_1)AG(z_2)\}\\
+\ &\mathbf{E}\left\{\left(\frac{1}{n}\mathrm{Tr}\,G(z_1)AG(z_2)A\right)U^*BUG(z_2)\right\}\\
-\ &\mathbf{E}\left\{\left(\frac{1}{n}\mathrm{Tr}\,G(z_1)AG(z_2)AU^*BU\right)G(z_2)\right\}=0. \qquad (4.5)
\end{aligned}$$

Regrouping terms using the resolvent identity and the centralized values we obtain

$$\begin{aligned}
-\ &\frac{1+z_1f_n(z_1)}{z_1}\left(A-\tilde{z}_{n,A}(z_1)I\right)\mathbf{E}\{U^*BUG(z_1)AG(z_2)\}\\
+\ &\mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\,G(z_1)AG(z_2)A\right\}\mathbf{E}\{U^*BUG(z_2)\}\\
=\ &-\frac{1+z_1f_n(z_1)}{z_1}A\mathbf{E}\{G(z_2)\}+\frac{\delta(z\Delta_A)}{\delta z}\mathbf{E}\{G(z_2)\}+O(n^{-1}).
\end{aligned}$$

Then taking the inverse $\widetilde{G}_A(z)$ and regrouping terms, we have

$$\begin{aligned}
& \mathbf{E}\{U^*BUG(z_1)AG(z_2)\} \\
= \; & \mathbf{E}\{G(z_2)\} + \frac{z_1}{1+z_1 f_n(z_1)}\left(\mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, G(z_1)AG(z_2)A\right\}\right. \\
- \; & \left.\frac{\delta\Delta_{n,A}}{\delta z}\tilde{z}_{n,A}(z_2)\right)\widetilde{G}_A(z_1)\widetilde{G}_A(z_2) + O(n^{-1}). \qquad (4.6)
\end{aligned}$$

After taking the operation $\frac{1}{n}\mathrm{Tr}\,\cdot$ over (4.6), we obtain the following relation

$$\begin{aligned}
& \mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, U^*BUG(z_1)AG(z_2)\right\} \qquad (4.7) \\
- \; & \frac{z_1}{1+z_1 f_n(z_1)}\frac{1}{n}\mathrm{Tr}\,\widetilde{G}_A(z_1)\widetilde{G}_A(z_2)\mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, G(z_1)AG(z_2)A\right\} \\
= \; & f_n(z_2) - \frac{z_1\tilde{z}_{n,A}(z_2)}{1+z_1 f_n(z_1)}\frac{\delta\Delta_{n,A}}{\delta z}\frac{1}{n}\mathrm{Tr}\,\widetilde{G}_A(z_1)\widetilde{G}_A(z_2) + O(n^{-2}).
\end{aligned}$$

On other hand, let us consider the modified resolvent $\hat{G}$ (2.25). Due to the trace property we have the following identities

$$\begin{aligned}
\frac{1}{n}\mathrm{Tr}\, B\hat{G}(z_1)B^{-1}\hat{G}(z_2) &= \frac{1}{z_1}\left(\frac{1}{n}\mathrm{Tr}\, U^*BUG(z_1)AG(z_2) - g_n(z_2)\right) \\
\mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, UAU^*\hat{G}(z_1)B^{-1}\hat{G}(z_2)\right\} &= \frac{1}{z_1}\left(\mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, G(z_1)AG(z_2)A\right\}\right. \\
&\quad \left. -\frac{1}{n}\mathrm{Tr}\, B^{-1}\widetilde{G}_B(z_2)\right) + O(n^{-1}).
\end{aligned}$$

Then using the identities above and Proposition 2.1 with $\Phi = (\hat{G}(z_1)B^{-1}\hat{G}(z_2))_{ac}$ and the procedure similar to used above we finally obtain

$$\begin{aligned}
- \; & \frac{z_2}{1+z_2 f_n(z_2)}\frac{1}{n}\mathrm{Tr}\,\widetilde{G}_B(z_1)\widetilde{G}_B(z_2)\mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, U^*BUG(z_1)AG(z_2)\right\} \qquad (4.8) \\
+ \; & \mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, G(z_1)AG(z_2)A\right\} \\
= \; & \frac{1}{1+z_2 f_n(z_2)}\left(1 - \tilde{z}_{n,B}(z_1)z_2\frac{\delta\Delta_{n,A}}{\delta z}\right)\frac{1}{n}\mathrm{Tr}\,\widetilde{G}_B(z_1)\widetilde{G}_B(z_2) + O(n^{-1}).
\end{aligned}$$

Thus, from relations (4.7) and (4.8) we obtain for the pair

$$\left(m_{BA} := \mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, U^*BUG(z_1)AG(z_2)\right\}, m_{AA} := \mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\, G(z_1)AG(z_2)A\right\}\right)$$

the following linear system

$$\begin{aligned} m_{BA} - \frac{z_1 I_{n,A}(z_1,z_2)}{1+z_1 f_n(z_1)} m_{AA} &= b_{BA} + O(n^{-1}) \\ -\frac{z_2 I_{n,B}(z_1,z_2)}{1+z_2 f_n(z_2)} m_{BA} + m_{AA} &= b_{AA} + O(n^{-1}), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} I_{n,l}(z_1,z_2) &= \int_{\mathbb{T}} \frac{N_{n,l}(d\lambda)}{(\lambda-\tilde{z}_{n,l}(z_1))(\lambda-\tilde{z}_{n,l}(z_2))}, \ l=A,B, \\ b_{BA} &= f_n(z_2) - \frac{z_1 \tilde{z}_{n,A}(z_2)}{1+z_1 f_n(z_1)} \frac{\delta\Delta_{n,A}}{\delta z} I_{n,A}(z_1,z_2), \\ b_{AA} &= \frac{I_{n,B}(z_1,z_2)}{1+z_2 f_n(z_2)} \left(1 - \tilde{z}_{n,B}(z_1) z_2 \frac{\delta\Delta_{n,A}}{\delta z}\right). \end{aligned}$$

Due to the bounds (2.17) and

$$|I_{n,l}(z_1,z_2)| \le \frac{1}{|(1-|\tilde{z}_{n,l}(z_1)|)(1-|\tilde{z}_{n,l}(z_2)|)|}, \ l=A,B$$

the determinant D_m of (4.9)

$$D_m = 1 - \frac{z_1 z_2 I_{n,A}(z_1,z_2) I_{n,B}(z_1,z_2)}{(1+z_1 f_n(z_1))(1+z_2 f_n(z_2))}$$

for $|z_{1,2}| \le 1/4$ satisfies the inequalities

$$7/16 \le |D_m| \le 25/16.$$

Thus the system (4.9) is uniquely soluble and its solution for $|z_{1,2}| \le 1/4$ is equal

$$\begin{aligned} m_{BA} &= \frac{1}{D_m}\left(b_{BA} + b_{AA}\frac{z_1 I_{n,A}(z_1,z_2)}{1+z_1 f_n(z_1)}\right) + O(n^{-1}) \\ &= \frac{1}{D_m}\left(f_n(z_2) + \frac{z_1 I_{n,A}(z_1,z_2)}{1+z_1 f_n(z_1)}\left(\frac{I_{n,B}(z_1,z_2)}{1+z_2 f_n(z_2)}\right.\right. \\ &\quad \left.\left. - \frac{z_2 \delta\Delta_{n,A}}{(1+z_2 f_n(z_2))\delta z}\left(\Delta_{n,A}(z_2) + \tilde{z}_{n,B}(z_1) I_{n,B}(z_1,z_2)\right)\right)\right) + O(n^{-1}). \end{aligned} \tag{4.10}$$

Besides, from the resolvent identity, (2.19) and its analogs we have

$$\begin{aligned} 1 + z f_n(z) &= \int_{\mathbb{T}} \frac{\lambda N_{n,A}(d\lambda)}{\lambda - \tilde{z}_{n,B}(z)} + O(n^{-1}), \\ \Delta_{n,A}(z) &= \int_{\mathbb{T}} \frac{N_{n,B}(d\lambda)}{\lambda - \tilde{z}_{n,B}(z)} + O(n^{-1}), \\ \delta\Delta_{n,A} &= I_{n,B}(z_1,z_2)\delta\tilde{z}_{n,B} + O(n^{-1}). \end{aligned} \tag{4.11}$$

Using these relations we can rewrite

$$\begin{aligned}
&\delta\Delta_{n,A}\left(\Delta_{n,A}(z_2)+\tilde{z}_{n,B}(z_1)I_{n,B}(z_1,z_2)\right)\\
=&\ \delta\Delta_{n,A}\int_{\mathbb{T}}\frac{\lambda N_{n,A}(d\lambda)}{(\lambda-\tilde{z}_{n,B}(z_1))(\lambda-\tilde{z}_{n,B}(z_2))}\\
=&\ I_{n,B}(z_1,z_2)\int_{\mathbb{T}}\frac{\delta\tilde{z}_{n,B}\lambda N_{n,A}(d\lambda)}{(\lambda-\tilde{z}_{n,B}(z_1))(\lambda-\tilde{z}_{n,B}(z_2))}+O(n^{-1})\\
=&\ I_{n,B}(z_1,z_2)\left(\int_{\mathbb{T}}\frac{\lambda N_{n,A}(d\lambda)}{\lambda-\tilde{z}_{n,B}(z_1)}-\int_{\mathbb{T}}\frac{\lambda N_{n,A}(d\lambda)}{\lambda-\tilde{z}_{n,B}(z_2)}\right)+O(n^{-1})\\
=&\ I_{n,B}(z_1,z_2)\left(z_1f_n(z_1)-z_2f_n(z_2)\right)+O(n^{-1}).
\end{aligned}$$

Substituting this in (4.10) and regrouping the terms, we obtain

$$m_{BA}=f_n(z_2)+z_1\frac{\left(1-z_1z_2\dfrac{\delta f_n}{\delta z}\right)\dfrac{I_{n,A}(z_1,z_2)I_{n,B}(z_1,z_2)}{(1+z_1f_n(z_1))\,(1+z_2f_n(z_2))}}{1-\dfrac{z_1z_2I_{n,A}(z_1,z_2)I_{n,B}(z_1,z_2)}{(1+z_1f_n(z_1))\,(1+z_2f_n(z_2))}}+O(n^{-1}) \tag{4.12}$$

On other hand, multiplying (4.5) by A from the left and taking $n^{-1}\mathrm{Tr}\cdot$, we obtain

$$\delta(zf_n)\mathbf{E}\left\{\frac{1}{n}\mathrm{Tr}\,G(z_1)AG(z_2)A\right\}=\frac{\delta\Delta_{n,A}\delta(z\Delta_{n,A})}{\delta z}+R_{AA},$$

and due to (2.23) and (2.24) we have

$$\begin{aligned}
R_{AA}\ =&\ \mathbf{E}\left\{\left(\frac{1}{n}\mathrm{Tr}\,G(z_1)AG(z_2)A\right)(z_2g_n^\circ(z_2)-z_1g_n^\circ(z_1))\right\}\\
&+\mathbf{E}\left\{\delta_{n,A}^\circ(z_1)\frac{1}{n}\mathrm{Tr}AU^*BUG(z_1)AG(z_2)\right\}\\
&-\mathbf{E}\left\{\delta_{n,A}^\circ(z_2)\frac{1}{n}\mathrm{Tr}AU^*BUG(z_1)AG(z_2)\right\}\\
=&\ O(n^{-1}),\ \ |z|\neq 1.
\end{aligned}$$

Now, suppose that $m_A^{(1)}m_B^{(1)}\neq 0$ then for $|z_{1,2}|\leq|m_A^{(1)}m_B^{(1)}|/8\leq 1/8$ we have

$$\begin{aligned}
zf_n(z)\ =&\ z\int_{\mathbb{T}}\left(\frac{1}{\lambda}+\frac{z}{\lambda\,(\lambda-z)}\right)N_n(d\lambda)\\
=&\ z\left(\overline{m_{n,A}^{(1)}m_{n,B}^{(1)}}+z\int_{\mathbb{T}}\frac{N_n(d\lambda)}{\lambda\,(\lambda-z)}\right),\ |zf_n(z)|\leq\frac{8}{7}|z|, \qquad (4.13)\\
\delta(zf_n)\ =&\ \delta z\left(\overline{m_{n,A}^{(1)}m_{n,B}^{(1)}}+\int_{\mathbb{T}}\frac{\lambda(z_1+z_2)-z_1z_2}{\lambda\,(\lambda-z_1)\,(\lambda-z_2)}N_n(d\lambda)\right),
\end{aligned}$$

where for sufficiently large n

$$\left|\overline{m_{n,A}^{(1)}m_{n,B}^{(1)}}+\int_{\mathbb{T}}\frac{\lambda(z_1+z_2)-z_1z_2}{\lambda\left(\lambda-z_1\right)\left(\lambda-z_2\right)}N_n(d\lambda)\right|\geq\left|m_A^{(1)}m_B^{(1)}\right|/2.$$

Thus, we can divide by $\delta(zf_n)$ and

$$\mathbf{E}\left\{\frac{1}{n}\operatorname{Tr}G(z_1)AG(z_2)A\right\}=\frac{\delta\Delta_{n,A}\delta(z\Delta_{n,A})}{\delta z\delta(zf_n)}+O(n^{-1}).$$

Substituting this in (4.6) we have

$$\begin{aligned}
& \mathbf{E}\left\{U^*BUG(z_1)AG(z_2)\right\}=\mathbf{E}\left\{G(z_2)\right\}+\frac{z_1}{1+z_1f_n(z_1)} \\
\times\quad & \frac{\delta\Delta_{n,A}}{\delta z}\left(\frac{\delta(z\Delta_{n,A})}{\delta(zf_n)}-\tilde{z}_{n,A}(z_2)\right)\widetilde{G}_A(z_1)\widetilde{G}_A(z_2)+O(n^{-1}) \\
=\quad & \mathbf{E}\left\{G(z_2)\right\}+z_1\frac{\delta\Delta_{n,A}\delta\tilde{z}_{n,A}}{\delta z\delta(zf_n)}\widetilde{G}_A(z_1)\widetilde{G}_A(z_2)+O(n^{-1}).
\end{aligned}\tag{4.14}$$

Then, taking $n^{-1}\mathrm{Tr}\cdot$ over (4.14) we obtain

$$m_{BA}=f_n(z_2)+z_1\frac{\delta\Delta_{n,A}\delta\Delta_{n,B}}{\delta z\delta(zf_n)}+O(n^{-1}).$$

Now, comparing this relation with (4.12) we obtain

$$\frac{\delta\Delta_{n,A}\delta\Delta_{n,B}}{\delta z\delta(zf_n)}=\frac{\left(1-z_1z_2\dfrac{\delta f_n}{\delta z}\right)\dfrac{I_{n,A}(z_1,z_2)I_{n,B}(z_1,z_2)}{\left(1+z_1f_n(z_1)\right)\left(1+z_2f_n(z_2)\right)}}{1-\dfrac{z_1z_2I_{n,A}(z_1,z_2)I_{n,B}(z_1,z_2)}{\left(1+z_1f_n(z_1)\right)\left(1+z_2f_n(z_2)\right)}}+O(n^{-1}).$$

This relation, (4.11), (4.13) and their analogs lead to (4.1).

(ii) Using (4.12) and (4.1), we obtain for $|z_{1,2}|\leq|m_A^{(1)}m_B^{(1)}|/8\leq 1/8$ and $n\to\infty$

$$\begin{aligned}
\gamma_0(z_1,z_2) &= \frac{\partial}{\partial z_1}\left(m_{BA}-\frac{\delta(zf_n)}{\delta z}\right) \\
&= \frac{\partial}{\partial z_1}\frac{1}{z_2}\left(\frac{1-z_1z_2\dfrac{\delta f_n}{\delta z}}{1-\dfrac{z_1z_2I_{n,A}(z_1,z_2)I_{n,B}(z_1,z_2)}{\left(1+z_1f_n(z_1)\right)\left(1+z_2f_n(z_2)\right)}}-1\right)+O(n^{-1}) \\
&= \frac{\partial}{\partial z_1}\frac{1}{z_2}\left(\frac{\delta\tilde{z}_A\delta\tilde{z}_B}{\delta z\delta(zf)}(1+z_1f(z_1))(1+z_2f(z_2))-1\right)+o(1).
\end{aligned}$$

(iii) It is sufficient to prove (4.3). Using (4.14),)(2.20) and the resolvent identity, we obtain

$$\begin{aligned}
\gamma_{AB}(z_1,z_2) &= \frac{\partial}{\partial z_1}\frac{z_2}{1+z_2 f_n(z_2)}\Big(n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\mathbf{E}\left\{U^*BUG(z_1)AG(z_2)\right\} \\
&\quad -\frac{1}{\delta z}n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\delta\left(z\mathbf{E}\left\{G\right\}\right)\Big) \\
&= \frac{\partial}{\partial z_1}\frac{z_1 z_2}{1+z_2 f_n(z_2)}n^{-1}\mathrm{Tr}\widetilde{G}_A(z_2)\left(\frac{\delta\Delta_{n,A}}{\delta z\delta(zf_n)}\delta\widetilde{G}_A-\frac{1}{\delta z}\delta\mathbf{E}\left\{G\right\}\right)+O(n^{-1}) \\
&= \frac{\partial}{\partial z_1}\frac{z_1 z_2}{1+z_2 f(z_2)}\frac{1}{\delta z}\left(\frac{\delta\Delta_B}{\delta\tilde{z}_A}\left(\frac{\delta\Delta_A}{\delta(zf)}-\frac{\tilde{z}_A(z_1)}{z_1}\right)\right. \\
&\quad \left.-f_A'(\tilde{z}_A(z_2))\left(\frac{\delta\Delta_A}{\delta(zf)}-\frac{\tilde{z}_A(z_2)}{z_2}\right)\right)+o(1)
\end{aligned}$$

where

$$\delta\widetilde{G}_A=\widetilde{G}_A(z_1)-\widetilde{G}_A(z_2),\ \delta\mathbf{E}\left\{G\right\}=\mathbf{E}\left\{G(z_1)\right\}-\mathbf{E}\left\{G(z_2)\right\}.$$

Then, canceling terms we arrive to (4.3). ■

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