Unavoidable sets and Wiener's test for Hunt processes

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Abstract

Let $(X, \mathcal{W})$ be a balayage space, $1 \in \mathcal{W}$, or – equivalently – let $\mathcal{W}$ be the set of excessive functions of a Hunt process on a locally compact space X with countable base such that $\mathcal{W}$ separates points, every function in $\mathcal{W}$ is the supremum of its continuous minorants and there exist strictly positive continuous $u, v \in \mathcal{W}$ such that $u/v \to 0$ at infinity. We suppose that there is a Green function $G > 0$ for X, a metric ρ on X and a decreasing function $g\colon [0, \infty) \to (0, \infty]$ having the doubling property and a mild upper decay at infinity such that $G \approx g \circ \rho$ (which is equivalent to a $3G$-inequality).

Then the corresponding capacity for balls of radius R is bounded by a constant multiple of $1/g(R)$. Assuming that the constant function 1 is harmonic and the capacity of large balls satisfies a reverse estimate or that bounded functions are harmonic if and only if they are constant (Liouville property), it is proven that Wiener's test at infinity shows, if a given set A in X is unavoidable, that is, if the process hits A with probability one, wherever it starts.

An application yields that locally finite unions of pairwise disjoint balls $B(z, r_z)$, $z \in Z$, which have a certain separation property with respect to a suitable measure λ on X are unavoidable if and only if, for some/any point $x_0 \in X$, the series $\sum_{z \in Z} g(\rho(x_0, z))/g(r_z)$ diverges.

The results generalize and, exploiting a zero-one law for hitting probabilities, simplify recent work by S. Gardiner and M. Ghergu, A. Mimica and Z. Vondraček, and the author.

Keywords: unavoidable set; Green function; 3G-property; equilibrium measure; capacity; zero-one law; Wiener's criterion; Hunt process; Lévy process; harmonic space; balayage space.

MSC: 31B15, 31C15, 31D05, 60J25, 60J45, 60J65, 60J75.

1 Preliminaries and main results

Let X be a locally compact space with countable base. Let $\mathcal{C}(X)$ denote the set of all continuous real functions on X and let $\mathcal{B}(X)$ be the set of all Borel measurable numerical functions on X. The set of all (positive) Radon measures on X will be denoted by $\mathcal{M}(X)$.

Moreover, let $\mathcal{W}$ be a convex cone of positive lower semicontinuous numerical functions on X such that $1 \in \mathcal{W}$ and $(X, \mathcal{W})$ is a balayage space (see [2], [6] or [11, Appendix]). In particular, the following holds:

(C) $\mathcal{W}$ separates the points in X,

$$w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) \colon v \le w\}, \qquad \text{for every } w \in \mathcal{W},$$

and there are strictly positive $u, v \in \mathcal{W} \cap \mathcal{C}(X)$ such that $u/v \to 0$ at infinity.

Then there exists a Hunt process $\mathfrak{X}$ on X such that $\mathcal{W}$ is the set $E_{\mathbb{P}}$ of excessive functions for the transition semigroup $\mathbb{P} = (P_t)_{t>0}$ of $\mathfrak{X}$ (see [2, IV.7.6] or [11, Appendix]), that is,

$$\mathcal{W} = \{v \in \mathcal{B}^+(X) \colon \sup_{t>0} P_t v = v\}.$$

We note that, conversely, given any sub-Markov semigroup $\mathbb{P} = (P_t)_{t>0}$ on X such that (C) is satisfied by its convex cone $E_{\mathbb{P}}$ of excessive functions, $(X, E_{\mathbb{P}})$ is a balayage space, and $\mathbb{P}$ is the transition semigroup of a Hunt process (see [6, Corollary 2.3.8] or [11, Corollary A.5]).

For every subset A of X, we have reduced functions R_u^A, $u \in \mathcal{W}$, and reduced measures ε_x^A, $x \in X$, defined by

$$R_u^A := \inf\{v \in \mathcal{W} \colon v \ge u \text{ on } A\} \quad \text{and} \quad \int u \, d\varepsilon_x^A = R_u^A(x).$$

Clearly, $R_u^A \le u$ on X and $R_u^A = u$ on A. If A is open, then

$$R_u^A \in \mathcal{W}. \tag{1.1}$$

For a general subset A, the greatest lower semicontinuous minorant $\hat{R}_1^A$ of R_1^A is contained in $\mathcal{W}$, and $\hat{R}_1^A = R_1^A$ on A^c (see [2, p. 243]).

If A is Borel measurable, then, for every $x \in X$,

$$R_1^A(x) = P^x[T_A < \infty], \tag{1.2}$$

where $T_A(\omega) := \inf\{t \ge 0 \colon X_t(\omega) \in A\}$ (see [2, VI.3.14]) and, for every Borel measurable set B in X,

$$\varepsilon_x^A(B) = P^x[X_{T_A} \in B; T_A < \infty]$$

(see also Remark 4.3).

A set $A \subset X$ is called *unavoidable*, if $R_1^A = 1$ or – equivalently – $\hat{R}_1^A = 1$. Otherwise, it is called *avoidable*, that is, A is avoidable, if there exists $x \in X$ such that $R_1^A(x) < 1$.

The following zero-one law will play an important role (for its proof and the proof of the subsequent corollary see [11, Proposition 2.3]).

PROPOSITION 1.1. *If the function* 1 *is harmonic, then, for every* $A \subset X$,

$$R_1^A = 1 \quad \text{or} \quad \inf_{x \in X} R_1^A(x) = 0. \tag{1.3}$$

COROLLARY 1.2. *Suppose that the function* 1 *is harmonic,* A *is an unavoidable set in* X*, and* $B \subset X$*,* $\gamma > 0$*,* $R_1^B \ge \gamma$ *on* A*. Then* B *is unavoidable.*

By definition, a *potential on* X is a function $p \in \mathcal{W}$ such that, for every relatively compact open set U in X, the function $R_p^{X\setminus U}$ is continuous and real on U and

$$\inf\{R_p^{X\setminus U} : U \text{ relatively compact open in } X\} = 0.$$

By [6, Proposition 4.2.10], a function $p \in \mathcal{W} \cap \mathcal{C}(X)$ is a potential if and only if there exists a strictly positive $q \in \mathcal{W} \cap \mathcal{C}(X)$ such that p/q vanishes at infinity. Let $\mathcal{P}(X)$ denote the set of all continuous real potentials on X.

The following result slightly generalizes [11, Lemma 2.2]), since, for every compact F in X, the function $\hat{R}_1^F$ is $\mathcal{P}(X)$-bounded (we shall also apply it to sets which might perhaps be not compact).

PROPOSITION 1.3. *Suppose that the function* 1 *is harmonic, let* A *be an unavoidable set in* X *and* $B_n \subset X$, $n \in \mathbb{N}$, *such that each function* $R_1^{B_n}$ *is* $\mathcal{P}(X)$-*bounded. Then the following hold.*

(a) *For every* $n \in \mathbb{N}$, *the set* $A \setminus (B_1 \cup B_2 \cup \cdots \cup B_n)$ *is unavoidable.*

(b) *If* $A \subset \bigcup_{n\in\mathbb{N}} B_n$, *then* $\sum_{n\in\mathbb{N}} R_1^{B_n} = \infty$.

Proof. (a) Let $p \in \mathcal{P}(X)$ and $u \in \mathcal{W}$ such that $R_1^{B_1} \le p$ and $u \ge 1$ on $A \setminus B_1$. Then $u + p \in \mathcal{W}$ and $u + p \ge 1$ on A. So $u + p \ge R_1^A = 1$, that is, $u - 1 \ge -p$. Since the function 1 is harmonic, the function $u - 1$ is hyperharmonic and hence, by the minimum principle, $u - 1 \ge 0$ (see [2, III.6.6]). So $u \ge 1$ proving that $R_1^{A\setminus B_1} = 1$. The proof of (a) is completed by induction.

(b) Let $n \in \mathbb{N}$. Then $A \setminus (B_1 \cup B_2 \cup \cdots \cup B_n) \subset \bigcup_{m>n} B_m =: B$. So B is unavoidable, by (a), and therefore $1 = R_1^B \le \sum_{n\ge m} R_1^{B_n}$. □

For every open set U in X, let $\mathcal{H}^+(U)$ denote the set of all functions in $\mathcal{B}^+(X)$ which are *harmonic on* U (in the sense of [2]), that is, such that $h|_U \in \mathcal{C}(U)$ and

$$\varepsilon_x^{V^c}(h) := \int h \, d\varepsilon_x^{V^c} = h(x), \tag{1.4}$$

for every open V such that $x \in V$ and $\overline{V}$ is a compact in U. It is known that, for every set A in X,

$$R_u^A \in \mathcal{H}^+(X \setminus \overline{A}), \qquad \text{if } u \in \mathcal{W},\ u \le w \in \mathcal{W} \cap \mathcal{C}(X) \tag{1.5}$$

(see [2, VI.2.6]).

ASSUMPTION 1.4. *We have a Borel measurable function* $G\colon X \times X \to (0, \infty]$ *with the following properties:*

(i) *For every* $y \in X$, $G(\cdot, y)$ *is a potential which is harmonic on* $X \setminus \{y\}$.

(ii) *For every potential* p *on* X, *there exists a measure* μ *on* X *such that*

$$p = G\mu := \int G(\cdot, y) \, d\mu(y). \tag{1.6}$$

REMARK 1.5. Having (i), each of the following properties implies (ii).

- G is lower semicontinuous on $X \times X$, continuous outside the diagonal, the potential kernel $V_0 := \int_0^\infty P_t\,dt$ of $\mathfrak{X}$ is proper, and there is a measure μ on X such that $V_0 f := \int G(\cdot, y)\,d\mu(y)$ (see [13] and [2, III.6.6]).
- G is locally bounded off the diagonal, each function $G(x,\cdot)$ is lower semicontinuous on X and continuous on $X \setminus \{x\}$, and there exists a measure ν on X such that $G\nu \in \mathcal{C}(X)$ and $\nu(U) > 0$, for every finely open $U \neq \emptyset$ (the latter holds, for example, if $V_0(x,\cdot) \ll \nu$, $x \in X$). See [10, Theorem 4.1].

The measure in (1.6) is uniquely determined and, given any measure μ on X such that $p := G\mu$ is a potential, the complement of the support of μ is the largest open set, where p is harmonic (see, for example, [10, Proposition 5.2 and Lemma 2.1]).

Suppose that A is a subset of X such that $\hat{R}_1^A$ is a potential. Then there is a unique measure μ_A on X, the *equilibrium measure for* A, such that

$$\hat{R}_1^A = G\mu_A.$$

If A is open, then $\hat{R}_1^A = R_1^A \in \mathcal{H}^+(X \setminus \overline{A})$, and hence μ_A is supported by $\overline{A}$. We observe that, for a general balayage space, this may already fail if A is compact (see [2, V.9.1]).

We define inner capacities for open sets U in X by

$$\operatorname{cap}_* U := \sup\bigl\{\|\mu\| \colon \mu \in \mathcal{M}(X),\ \mu(X \setminus U) = 0,\ G\mu \le 1\bigr\} \tag{1.7}$$

and outer capacities for arbitrary sets A in X by

$$\operatorname{cap}^* A := \inf\bigl\{\operatorname{cap}_* U \colon U \text{ open neighborhood of } A\bigr\}. \tag{1.8}$$

The capacity of open sets U is essentially determined by the total mass of equilibrium measures for open sets which are relatively compact in U:

LEMMA 1.6. *For every open set U in X,*

$$\operatorname{cap} U \ge \sup\{\|\mu_V\| \colon V \textit{ open and } \overline{V} \textit{ compact in } U\} \ge c^{-2} \operatorname{cap} U.$$

Proof. The first inequality is trivial. To prove the second inequality, let $\mu \in \mathcal{M}(X)$ such that $\mu(X \setminus U) = 0$ and $G\mu \le 1$, and let K be a compact in U. We may choose an open neighborhood V of K such that $\overline{V}$ is compact in U. Then

$$\begin{aligned}
\|1_K \mu\| &= \int_K G\mu_V\,d\mu \le \int\int G(x,y)\,d\mu_V(y)\,d\mu(x) \\
&\le c^2 \int\int G(y,x)\,d\mu(x)\,d\mu_V(y) = c^2 \int G\mu(y)\,d\mu_V(y) \le c^2 \|\mu_V\|.
\end{aligned}$$

□

Of course, the function $U \mapsto \operatorname{cap}_* U$, U open in X, is increasing. Hence the function $A \mapsto \operatorname{cap}^* A$, $A \subset X$, is also increasing and $\operatorname{cap}^* A = \operatorname{cap}_* A$, if A is open. If $\operatorname{cap}_* A = \operatorname{cap}^* A$, we may simply write $\operatorname{cap} A$ and speak of the capacity of A.

For a better understanding of our next assumption, we first observe the following equivalences.

PROPOSITION 1.7. *Let $\tilde{G} := G$ outside the diagonal and $\tilde{G} := \infty$ on the diagonal of $X \times X$.*[1] *Then the following properties are equivalent:*

(i) *$\tilde{G}$ has the* triangle property

$$\min\{\tilde{G}(x,z), \tilde{G}(y,z)\} \le C\tilde{G}(x,y), \qquad x, y, z \in X.$$

(ii) *There exists a metric ρ for X and $\gamma > 0$ such that $\tilde{G} \approx \rho^{-\gamma}$.*

(iii) *There exists a metric ρ for X and decreasing continuous $g\colon [0,\infty) \to (0,\infty]$ with*

$$\tilde{G} \approx g \circ \rho,$$

the function g has the doubling property*, that is, for some $c_D > 1$,*

$$g(r) \le c_D g(2r), \qquad r > 0,$$

and there exist $M_0 > 1$ and $\eta_0 \in (0,1)$ such that

$$g(M_0 r) \le \eta_0 g(r), \qquad r > 0.$$

Proof. (i) $\Rightarrow$ (ii): Since $\tilde{G} = \infty$ on the diagonal, the triangle property implies that $\tilde{G}(y,x) \le C\tilde{G}(x,y)$ and $\tilde{\rho}(x,y) := \tilde{G}(x,y)^{-1} + \tilde{G}(y,x)^{-1}$, $x, y \in X$, defines a quasi-metric on X which is equivalent to $\tilde{G}^{-1}$. By [12, Proposition 14.5] (see also [8, pp. 1209–1212] and [5]), there exists a metric ρ for X and $\gamma > 0$ such that $\tilde{\rho} \approx \rho^{\gamma}$, and hence $\tilde{G} \approx \rho^{-\gamma}$.

(ii) $\Rightarrow$ (iii): Trivial defining $g(r) := r^{-\gamma}$.

(iii) $\Rightarrow$ (i): Let $c > 0$ such that $c^{-1} g \circ \rho \le \tilde{G} \le c g \circ \rho$ and $x, y, z \in X$. Since $\rho(x,y) \le \rho(x,z) + \rho(z,y)$, we know that $\rho(x,z) \ge \rho(x,y)/2$ or $\rho(z,y) \ge \rho(x,y)/2$. Therefore

$$\begin{aligned} \min\{\tilde{G}(x,z), \tilde{G}(z,y)\} &\le c \min\{g(\rho(x,z)), g(\rho(z,y))\} \\ &\le c g(\rho(x,y)/2) \le c c_D g(\rho(x,y)) \le c^2 c_D \tilde{G}(x,z). \end{aligned}$$

$\square$

REMARK 1.8. Let us note that the proof of the implication "(iii) $\Rightarrow$ (i)" only uses the property $G \approx g \circ \rho$ and the doubling property of g. Moreover, having these two properties, the continuity of g has the character of "without loss of generality". Indeed, given a decreasing function g, which (perhaps) is not continuous, we may replace it by the function obtained by linear interpolation between the values of g at 2^m, $m \in \mathbb{Z}$.

[1] This definition allows us to cover as well random walks on discrete sets, where G will be finite on the diagonal.

From now on we assume the following.

ASSUMPTION 1.9. *We have a metric ρ for X, a continuous decreasing function $g\colon [0,\infty) \to (0,\infty]$ and $c, c_D, M_0 \in [1,\infty)$, $\eta_0 \in (0,1)$ and $R_0, R_1 \in [0,\infty)$ such that*

$$c^{-1}\, g \circ \rho \le G \le c\, g \circ \rho, \tag{1.9}$$

$$g(r) \le c_D g(2r), \qquad \text{for every } r > R_0, \tag{1.10}$$

$$g(M_0 r) \le \eta_0 g(r), \qquad \text{for every } r > R_1. \tag{1.11}$$

Of course, (1.11) implies that, for any $\eta > 0$, there exists $M > 1$ such that $g(Mr) \le \eta g(r)$, for every $r > R_1$ (it suffices to choose $m \in \mathbb{N}$ such that $\eta_0^m < \delta$ and to take $M := M_0^m$).

REMARK 1.10. Let us note that Assumption 1.9 is satisfied by rather general isotropic Lévy processes (often with $R_0 = R_1 = 0$; see [4] and [7] for detail).

For $x \in X$ and $0 < r < t$, we define balls $B(x,r)$ and shells $S(x,r,t)$ by

$$B(x,r) := \{y \in X \colon \rho(x,y) < r\}, \quad S(x,r,t) := \{y \in X \colon r \le \rho(x,y) < t\}\,.$$

Let us immediately note some elementary properties of $R_1^{B(x,r)}$ and $\operatorname{cap} B(x,r)$.

PROPOSITION 1.11. *Let $x \in X$, $r > 0$ and $B := B(x,r)$. Then R_1^B is a potential which is $\mathcal{P}(X)$-bounded,*

$$R_1^B \le c\,\frac{G(\cdot,x)}{g(r)} \le c^2\,\frac{g(\rho(\cdot,x))}{g(r)}, \qquad \operatorname{cap} B \le c g(r)^{-1}, \tag{1.12}$$

$$R_1^B \ge c^{-1} \operatorname{cap} B \cdot g(\rho(\cdot,x) + r). \tag{1.13}$$

Proof. We know that $R_1^B \in \mathcal{W}$ (see (1.1)). Moreover, $G(\cdot,x)$ is a potential and $G(\cdot,x) \ge c^{-1} g(r)$ on B. Hence $R_1^B \le \min\{1, cG(\cdot,x)/g(r)\} \in \mathcal{P}(X)$. In particular, R_1^B is a potential.

Moreover, let $\mu \in \mathcal{M}(X)$ such that $\mu(X \setminus B) = 0$ and $\int G(\cdot,z)\,d\mu(z) = G\mu \le 1$. Since $c^{-1}g(r) \le G(x,\cdot)$ on B, we see that $c^{-1}g(r)\|\mu\| \le 1$.

Further, by the minimum principle (see [2, III.6.6]), $R_1^B \ge G\mu$. Let $y \in X$. For all $z \in B$, $\rho(z,y) \le \rho(y,x) + r$, and hence $G(y,z) \ge c^{-1} g(\rho(y,x) + r))$. Thus $R_1^B(y) \ge G\mu(y) \ge c^{-1} g(\rho(y,x) + r)\|\mu\|$. $\square$

COROLLARY 1.12. *Let $B(z, r_z)$, $z \in Z \subset X$, $r_z > 0$, be balls in X such that their union A is unavoidable. Then, for every $x_0 \in X$,*

$$\sum\nolimits_{z \in Z} g(\rho(x_0,z))/g(r_z) = \infty.$$

Proof. Propositions 1.3 and 1.11. $\square$

Our main results are the following (for a discussion of (1.15) see Proposition 5.1 and Remarks 5.3).

THEOREM 1.13 (Wiener's test)**.** *Suppose that the function* 1 *is harmonic, let* $A \subset X$, $x_0 \in X$, $R > 0$, *and* $\gamma > 1$.

1. *If* A *is unavoidable, then*

$$\sum_{n\in\mathbb{N}} g(\gamma^n R)\,\mathrm{cap}^*(A \cap S(x_0, \gamma^n R, \gamma^{n+1} R)) = \infty. \tag{1.14}$$

2 *Suppose that bounded harmonic functions are constant* (Liouville property) *or that there exist* $c_0 \geq 1$ *and* $R_2 \geq 0$ *such that, for all* $x \in X$ *and* $r > R_2$,

$$\mathrm{cap}\, B(x, r) \geq c_0^{-1} g(r)^{-1}. \tag{1.15}$$

Then A *is unavoidable if and only if* (1.14) *holds.*

For the next two corollaries let assume that the function 1 is harmonic, that balls are relatively compact, and that we have $\lambda \in \mathcal{M}(X)$ with $\mathrm{supp}(\lambda) = X$ and such that, for some $c_0 \geq 1$, the normalized restrictions $\lambda_{B(x,r)} := (\lambda(B(x,r)))^{-1} 1_{B(x,r)} \lambda$ of λ on $B(x,r)$, $x \in X$, $r > R_0$, satisfy

$$G\lambda_{B(x,r)} \leq c_0 g(r) \tag{1.16}$$

(so that (1.15) holds with $R_2 = R_0$).

COROLLARY 1.14. *Let* A *be a union of pairwise disjoint balls* $B(z, r_z)$, $z \in Z$, *where* $Z \subset X$ *is locally finite and* $r_z > 4R_0$, *and let* $x_0 \in X \setminus Z$ *such that*

$$\inf_{z,z' \in Z,\, z \neq z'} \frac{\lambda(B(z, \rho(z,z')/4))}{\lambda(B(x_0, 4\rho(x_0, z)))} \cdot \frac{g(r_z)}{g(\rho(x_0,z))} > 0.$$

Then A *is unavoidable if and only if* $\sum_{z\in Z} g(\rho(x_0,z))/g(r_z) = \infty$.

DEFINITION 1.15. *We shall say that pairwise disjoint balls* $B(z, r_z)$, $z \in Z$, $r_z > 4R_0$, *are* regularly located *if the following hold:*

- *There exists* $\varepsilon > 0$ *such that* $\rho(z, z') \geq \varepsilon$, *for all* $z, z' \in Z$, $z \neq z'$.
- *There exists* $R > 0$ *such that every ball of radius* R *contains a point of* Z.
- *There exist a decreasing function* $\phi\colon (0,\infty) \to (0,\infty)$ *such that*

$$r_z \approx \phi(\rho(x_0, z)), \qquad z \in Z. \tag{1.17}$$

Under mild additional assumptions on λ (see Section 7), which are satisfied if $X = \mathbb{R}^d$, ρ is the Euclidean metric and λ is Lebesgue measure, the following holds.

COROLLARY 1.16. *Let* A *be a union of balls* $B(z, r_z)$, $z \in Z$, *in* X *which are regularly located. Then* A *is unavoidable if and only if* $\sum_{z\in Z} g(\rho(x_0,z))/g(r_z) = \infty$.

2 The first part of Wiener's test

The first part of Wiener's test is an easy consequence of Proposition 1.3(b). We only have to use the definition of cap* and note the simple fact that, for every open set V which is contained in an open ball, the reduced function R_1^V is a potential, by Proposition 1.11.

PROPOSITION 2.1. *Suppose that the function* 1 *is harmonic. Let* $A \subset X$ *be an unavoidable set,* $x_0 \in X$*,* $R > 0$*, and* $\gamma > 1$*. Then*

$$\sum\nolimits_{n\in\mathbb{N}} g(\gamma^n R)\,\mathrm{cap}^*(A \cap S(x_0, \gamma^n R, \gamma^{n+1} R)) = \infty.$$

Proof. For $n \in \mathbb{N}$, there are open neighborhoods U_n of $A_n := A \cap S(x_0, \gamma^n R, \gamma^{n+1} R)$ in $S(x_0, \gamma^n R/2, \gamma^{n+1} R)$ such that

$$\mathrm{cap}\, U_n \le \mathrm{cap}^*\, A_n + 2^{-n}. \tag{2.1}$$

Since $A \setminus B(x_0, \gamma R) \subset \bigcup_{n\in\mathbb{N}} U_n$, we know, by Proposition 1.3, that

$$\sum\nolimits_{n\in\mathbb{N}} R_1^{U_n} = \infty. \tag{2.2}$$

By [2, VI.1.7], there exist open sets V_n such that $\overline{V}_n$ is compact in U_n and

$$R_1^{U_n}(x_0) \le R_1^{V_n}(x_0) + 2^{-n}, \qquad n \in \mathbb{N}.$$

Then, by (2.2),

$$\sum\nolimits_{n\in\mathbb{N}} R_1^{V_n}(x_0) = \infty.$$

Let $n_0 \in \mathbb{N}$ such that $\gamma^{n_0} R/2 > R_0$. For $n \ge n_0$, let $\nu_n := \mu_{V_n}$, that is, $G\nu_n = R_1^{V_n}$. Since ν_n is supported by the set $\overline{V}_n$, which does not intersect $B(x_0, \gamma^n R/2)$, and

$$G(x_0, \cdot) \le cg(\rho(x_0, \cdot)) \le cg(\rho(\gamma^n R/2)) \le cc_D g(\gamma^n R) \qquad \text{on } X \setminus B(x_0, \gamma^n R/2),$$

we see that $R_1^{V_n}(x_0) = \int G(x_0, y)\,d\nu_n(y) \le cc_D g(\gamma^n R)\|\nu_n\| \le cc_D \gamma(\gamma^n R)\,\mathrm{cap}\, U_n$. Therefore

$$\infty = \sum\nolimits_{n\ge n_0} R_1^{V_n}(x_0) \le \sum\nolimits_{n\ge n_0} cc_D g(\gamma^n R)\,\mathrm{cap}\, U_n.$$

Since $g(\gamma^n R) \le g(R) < \infty$, for every $n \in \mathbb{N}$, we finally conclude from (2.1) that $\sum_{n\in\mathbb{N}} g(\gamma^n R)\,\mathrm{cap}^*\, A_n = \infty$. $\square$

3 Wiener's test having the Liouville property

In this section, let us assume that the function 1 is harmonic and that $(X, \mathcal{W})$ has the *Liouville property*, that is, every bounded harmonic function is constant. In this case, a set A in X is avoidable if and only if it is minimally thin at infinity (see [11, Proposition 2.3]). Therefore it suffices to modify the proofs for [2, V.4.15 and V.4.17] (characterizing, in the setting of Riesz potentials, thinness of a set A at a point). In the context of Lévy processes, this has already been noted (see, for example, [14, Proposition 7.3 and Corollary 7.4]). In our situation, the zero-one law will yield a straight forward modification.

PROPOSITION 3.1. *Let A be a subset of X, $x_0 \in X$, $s_n \in (0,\infty)$ and $\delta \in (0,1)$ such that $s_n \le \delta s_{n+1}$, for every $n \in \mathbb{N}$. Then the following hold for the sets $A_n := A \cap S(x_0, s_n, s_{n+1})$:*

(i) *If A is unavoidable, then $\sum_{n\in\mathbb{N}} R_1^{A_n} = \infty$ on X.*

(ii) *If A is avoidable, then $\sum_{n\in\mathbb{N}} R_1^{A_n} < \infty$ on X.*

Proof. (i) Proposition 1.3(b).

(ii) By Proposition 1.1, there exists a point $x_1 \in X$ such that $R_1^A(x_1) < (2c^2 c_D)^{-1}$. By [2, VI.1.2], there exists an open neighborhood V of A such that

$$a := 2c^2 c_D R_1^V(x_1) < 1. \tag{3.1}$$

We define

$$V_n := \{x \in V \colon s_n < \rho(x, x_1) < s_{n+3}\}, \qquad n \in \mathbb{N},$$

and claim that

$$\sum\nolimits_{n\in\mathbb{N}} R_1^{V_n}(x_0) < \infty. \tag{3.2}$$

To that end let $k \in \mathbb{N}$ such that $1 - \delta^{k-3} > 1/2$. We fix $1 \le i \le k$ and define

$$U_n := V_{i+nk}, \qquad n \in \mathbb{N}.$$

To prove (3.2), it clearly suffices to show that $\sum_{n\in\mathbb{N}} R_1^{U_n}(x_0) < \infty$. Let

$$U := \bigcup\nolimits_{n\in\mathbb{N}} U_n, \qquad \mu := \mu_U, \qquad \mu_n := 1_{U_n}\mu,$$

for $n \in \mathbb{N}$ (note that $U \subset V$ and hence, by (3.1) and [11, Proposition 2.3], R_1^U is a potential). Then μ is supported by $\overline{U} = \bigcup_{n\in\mathbb{N}} \overline{U}_n$ and $G\mu = \sum_{n\in\mathbb{N}} G\mu_n$.

For the moment, let us fix $n \in \mathbb{N}$, consider $m \in \mathbb{N}$, $m \ne n$, $x \in U_n$, and $y \in \overline{U}_m$. If $m < n$, then $\rho(x_1, y) \le \delta^{k-3}\rho(x_1, x)$. If $m > n$, then $\rho(x_1, x) \le \delta^{k-3}\rho(x_1, y)$. In both cases,

$$\rho(x, y) \ge (1 - \delta^{k-3})\rho(x_1, y) \ge \rho(x_1, y)/2.$$

Defining $\mu_n' := \mu - \mu_n$ we hence obtain that, for every $x \in U_n$,

$$G\mu_n'(x) \le c \int g(\rho(x,y))\, d\mu_n'(y) \le c c_D \int g(\rho(x_1,y))\, d\mu_n'(y) \le c^2 c_D G\mu_n'(x_1) \le a.$$

Since $G\mu_n + G\mu_n' = G\mu = R_1^U$ and $R_1^U = 1$ on U, we therefore conclude that $1 - a \le G\mu_n$ on U_n, and hence

$$(1-a) R_1^{U_n} \le G\mu_n.$$

Moreover, there exists $n_0 \in \mathbb{N}$, such that, for every $n > n_0$, $\rho(x_0, \cdot) \ge \rho(x_1, \cdot)/2$ on $\overline{U}_n$, and therefore $G\mu_n(x_0) \le c^2 c_D G\mu_n(x_1)$. Thus

$$\begin{aligned}
\sum\nolimits_{n>n_0} R_1^{U_n}(x_0) &\le \frac{1}{1-a} \sum\nolimits_{n>n_0} G\mu_n(x_0) \\
&\le \frac{c^2 c_D}{1-a} \sum\nolimits_{n>n_0} G\mu_n(x_1) \le \frac{c^2 c_D}{1-a} G\mu(x_1) < \infty.
\end{aligned}$$

This proves (3.2). The proof is finished observing that $A_n \subset V_{n-1}$, for every $n \in \mathbb{N}$, which is sufficiently large. $\square$

COROLLARY 3.2. *Let $A \subset X$, $R > 0$ and $\gamma > 1$. Then A is unavoidable if and only if*

$$\sum\nolimits_{n\in\mathbb{N}} g(\gamma^n R)\operatorname{cap}^*(A\cap S(x_0,\gamma^n R,\gamma^{n+1}R)) = \infty. \tag{3.3}$$

Proof. If A is unavoidable, then (3.3) holds, by Proposition 2.1.

So let us assume that A is avoidable. By [2, VI.1.5], there exists an open neighborhood U of A which is avoidable. For every $n \in \mathbb{N}$,

$$U_n := \{x \in U : \gamma^{n-1}R < \rho(x,x_0) < \gamma^{n+1}R\}$$

is an open neighborhood of $A \cap S(x_0,\gamma^n R,\gamma^{n+1}R)$. By Proposition 3.1 (applied to γ^2 in place of γ and both R and $\gamma^{-1}R$),

$$\sum\nolimits_{n\in\mathbb{N}} R_1^{U_n}(x_0) < \infty.$$

By Lemma 1.6, there exist open sets V_n in U_n such that $\overline{V}_n$ is compact in U_n and

$$\operatorname{cap} U_n \le c^2\|\mu_{V_n}\| + 2^{-n}, \qquad n \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ such that $\gamma \le 2^k$, and let $n \in \mathbb{N}$ such that $\gamma^n R > R_0$. Then $g(\gamma^n R) \le c_D^k g(\gamma^{n+1}R) \le c_D^k g(\rho(x_0,\cdot))$ on $\overline{V}_n$, and hence

$$g(\gamma^n R)\|\mu_{V_n}\| \le c_D^k \int g(\rho(x_0,y))\,d\mu_{V_n}(y) \le c c_D^k G\mu_{V_n}(x_0).$$

Since $G\mu_{V_n} = R_1^{V_n} \le R_1^{U_n}$ and $g(\gamma^n R) \le g(R)$, $n \in \mathbb{N}$, we conclude that

$$\sum\nolimits_{n\in\mathbb{N}} g(\gamma^n R)\operatorname{cap} U_n < \infty.$$

Thus $\sum_{n\in\mathbb{N}} g(\gamma^n R)\operatorname{cap}^*(A\cap S(x_0,\gamma^n R,\gamma^{n+1}R)) < \infty$. $\square$

4 Hitting of sets before leaving large balls

Let us first recall a simple statement on the probability for hitting a set A before leaving an open neighborhood U (see, for example, [7]). For the convenience of the reader we include the short proof. As usual, we define, for every open set U in X,

$$\tau_U := T_{U^c}.$$

LEMMA 4.1. *Let A be a Borel measurable set in an open set $U \subset X$ and $\eta > 0$. If $R_1^A \le \eta$ on U^c, then*

$$P^x[T_A < \tau_U] \ge R_1^A(x) - \eta, \qquad \textit{for every } x \in U.$$

Proof. Let $\tau := T_{U^c}$. Obviously,

$$[T_A < \infty] \setminus [T_A < \tau] = [\tau \le T_A < \infty] \subset [\tau < \infty] \cap \theta_\tau^{-1}([T_A < \infty]).$$

Let $x \in U$. By the strong Markov property,

$$P^x([\tau < \infty] \cap \theta_\tau^{-1}[T_A < \infty]) = \int_{[\tau<\infty]} P^{X_\tau}[T_A < \infty]\,dP^x \le \gamma,$$

since $X_\tau \in U^c$ on $[\tau < \infty]$. Therefore $P^x[T_A < \infty] - P^x[T_A < \tau] \le \gamma$. $\square$

We now easily obtain the following lower estimate for the probability of hitting a subset of a ball before leaving a much larger ball (cf. [7]).

PROPOSITION 4.2. *Let $\gamma > 1$. There exists $m \in \mathbb{N}$ and $\eta_1 \in (0,1)$ such that, for all $x_0 \in X$, $r > R_0 \vee R_1$, $x \in B(x_0, \gamma r)$, and Borel measurable sets A in $B(x_0, \gamma r)$,*

$$P^x[T_A < \tau_{B(x_0,\gamma^m r)}] \ge \eta_1 g(r) \operatorname{cap}^*(A). \tag{4.1}$$

Proof. We choose $k \in \{0, 1, 2, \dots\}$ such that $1 \le \gamma \le 2^k$ and define

$$\eta_1 := (2c^3 c_D^{k+1})^{-1}.$$

By (1.11), there exists $m \in \mathbb{N}$ such that

$$g((\gamma^m - \gamma)r) \le c\eta_1 g(r), \qquad \text{for all } r > R_1.$$

Now let $x_0 \in X$, $r > R_0 \vee R_1$, $x \in B(x_0, \gamma r)$, and $A \subset B(x_0, \gamma r)$ Borel measurable. To prove (4.1) we may assume without loss of generality that A is open. Let V be an open set such that $\overline{V}$ is compact in A. Since $\rho(x, \cdot) \le 2\gamma r$ on $\overline{V}$, we have

$$R_1^V(x) = \int G(x,z)\, d\mu_V(z) \ge c^{-1} g(2\gamma r) \|\mu_V\| \ge 2\eta_1 c^2 g(r) \|\mu_V\|.$$

If $y \in X \setminus B(x_0, \gamma^m r)$, then $\rho(y, \cdot) \ge (\gamma^m - \gamma)r$ on $\overline{V}$, and therefore

$$R_1^V(y) = \int G(y,z)\, d\mu_V(z) \le cg((\gamma^m - \gamma)r) \|\mu_V\| \le \eta_1 c^2 g(r) \|\mu_V\|.$$

So, using Lemma 4.1,

$$P^x[T_A < \tau_{B(x_0,\gamma^m r)}] \ge P^x[T_V < \tau_{B(x_0,\gamma^m r)}] \ge \eta_1 g(r) c^2 \|\mu_V\|.$$

An application of Lemma 1.6 completes the proof. $\square$

REMARK 4.3. Our probabilistic statements and proofs can be replaced by purely analytic ones using that, for all Borel measurable sets A, B in an open set U,

$$P^x[X_{T_A} \in B; T_A < \tau_U] = \varepsilon_x^{A \cup U^c}(B) \tag{4.2}$$

(see [2, VI.2.9]) and, for all Borel measurable sets B in X and $B \subset A \subset X$,

$$\varepsilon_x^B = \varepsilon_x^A|_B + \left(\varepsilon_x^A|_{B^c}\right)^B. \tag{4.3}$$

(If $x \in B$, then (4.3) holds trivially. If $x \notin B$ and $p \in \mathcal{P}(X)$, then, by [2, VI.9.1],

$$\hat{R}_p^B(x) = R_p^B(x) = \int R_p^B\, d\varepsilon_x^A = \int_B p\, d\varepsilon_x^A + \int_{B^c} \hat{R}_p^B\, d\varepsilon_x^A.)$$

5 Wiener's test having a lower bound for the capacity of large balls

To deal with the case, where we do not have the Liouville property or do not know, if it holds, we note first that there is a close relation between estimates which are reverse to the ones in (1.12).

PROPOSITION 5.1. *Let $x \in X$, $r > R_0$, $B := B(x,r)$, and $C \geq 1$. Then the following hold.*

(a) *If* $\operatorname{cap} B \geq C^{-1} g(r)^{-1}$, *then* $R_1^B \geq (c^2 c_D C)^{-1} G(\cdot, x)/g(r)$ *on* $X \setminus B$.

(b) *If* $R_1^B \geq C^{-1} G(\cdot, x)/g(r)$ *on* $X \setminus B$ *and if there exists a point* z *such that* $2r \leq \rho(z,x) \leq 4r$, *then* $\operatorname{cap} B \geq (Cc^2 c_D^2)^{-1} g(r)^{-1}$.

Proof. (a) Immediate consequence of (1.13): it suffices to note that, for every $y \in B^c$, $\rho(y,x) + r \leq 2\rho(y,x)$ and hence $g(\rho(y,x)+r) \geq c_D^{-1} g(\rho(y,x)) \geq (cc_D)^{-1} G(y,x)$.

(b) Since $G(z,x) \geq c^{-1} g(\rho(z,x)) \geq c^{-1} g(4r) \geq (cc_D^2)^{-1} g(r)$, we obtain that

$$R_1^B(z) \geq (Ccc_D^2)^{-1} := a.$$

For every $y \in B$, $\rho(z,y) \geq r$ and hence $G(z,y) \leq cg(\rho(z,y)) \leq cg(r)$. Given $\varepsilon > 0$, there exists $0 < r' < r$ such that $V := B(x,r')$ satisfies $R_1^V(z) > a - \varepsilon$, and hence

$$a - \varepsilon < R_1^V(z) = \int G(z,y)\, d\mu_V(y) \leq cg(r)\|\mu_V\| \leq cg(r) \operatorname{cap} B.$$

Thus $\operatorname{cap} B \geq c^{-1} a g(r)^{-1}$. □

In Proposition 5.1(b), we may, of course, replace the upper bound $\rho(z,x) \leq 4r$ by any bound $\rho(z,x) \leq 2^{k+1} r$, $k \in \mathbb{N}$, at the expense of obtaining $\operatorname{cap}(B) \geq (Cc^2 c_D^{k+1} g(r))^{-1}$.

In this section, we assume from now on that, in addition to the Assumptions (1.4) and (1.9), the following holds.

ASSUMPTION 5.2. *The function* 1 *is harmonic and there exist* $c_0 \geq 1$ *and* $R_2 \geq 0$ *such that, for all* $x \in X$ *and* $r > R_2$,

$$\operatorname{cap} B(x,r) \geq c_0^{-1} g(r)^{-1}. \tag{5.1}$$

Then, by Proposition 1.11, for all $x \in X$ and $r > R_2$,

$$R_1^{B(x,r)} \geq (cc_0)^{-1} \frac{g(\rho(\cdot,x)+r)}{g(r)}. \tag{5.2}$$

EXAMPLES 5.3. 1. Assume that $(X, \mathcal{W})$ is a harmonic space, that is, $\mathfrak{X}$ is a diffusion. Moreover, suppose that X is non-compact, but balls are relatively compact. Then Assumption 5.2 is satisfied with $R_2 = R_0$.

Indeed, let $x \in X$, $r > 0$, and $B := B(x,r)$. Then $p := G(\cdot,x) \wedge (cg(r)) \in \mathcal{P}(X)$, $p = G(\cdot,x)$ on $X \setminus B$, and hence p is harmonic on $X \setminus \overline{B}$. By the minimum principle (see [2, III.6.6]), $R_1^B \geq (cg(r))^{-1}p$. Finally, let $2r < s < 4r$ and $y \in X \setminus \overline{B}(x,s)$. Since we have the minimum principle for $B(x,s)$ and $G(\cdot,y) \in \mathcal{H}^+(X\setminus\{y\})$ is strictly positive, we see that $\emptyset \neq \partial B(x,s) \subset B(x,4r) \setminus B(x,2r)$. So the claim follows by Proposition 5.1(b).

2. Suppose that $X = \mathbb{R}^d$ and $\rho(x,y) = |x-y|$. Let $x \in \mathbb{R}^d$ and $r > 0$. Then (5.1) holds provided

$$d\int_0^r s^{d-1}g(s)\,ds \leq c_0 r^d g(r),$$

since then the normalized Lebesgue measure $\lambda_{B(x,r)}$ on $B(x,r)$ satisfies $G\lambda_{B(x,r)} \leq G\lambda_{B(x,r)}(x) \leq c_0 g(r)$ (see [7]).

So rather general isotropic unimodular Lévy processes satisfy Assumption 5.2 (for further details see [7]).

We shall need a lower estimate for the outer capacity of shells:

LEMMA 5.4. *Given $\gamma > 1$, there exists $\eta_2 \in (0,1)$, such that, for all $x_0 \in X$ and $r > R := \max\{1, (\gamma-1)^{-1}\} \cdot \max\{R_0, R_2\}$,*

$$g(r)\,\mathrm{cap}^*\,S(x_0,r,\gamma^3 r) \geq \eta_2 \qquad \textit{provided } S(x_0,\gamma r,\gamma^2 r) \neq \emptyset.$$

Proof. We choose $k \in \{0,1,2,\dots\}$ with $2^{-k} \leq \gamma - 1$ (if $\gamma \geq 2$ we may take $k=0$). Let $x_0 \in X$, $r > R$, and $x \in S(x_0,\gamma r,\gamma^2 r)$. Then $V := B(x,(\gamma-1)r) \subset S(x_0,r,\gamma^3 r)$ and hence $\mathrm{cap}^*\,S(x_0,r,\gamma^3 r) \geq \mathrm{cap}\,V \geq c_0^{-1} g((\gamma-1)r)^{-1} \geq (c_0 c_D^k)^{-1} g(r)^{-1}$. □

Now we are ready to prove the following which, together with Proposition 2.1, completes the proof of Theorem 1.13 (see also [7]) for the main idea).

THEOREM 5.5. *Let $A \subset X$ be avoidable, $x_0 \in X$, $R > 0$, and $\gamma > 1$. Then*

$$\sum\nolimits_{n\in\mathbb{N}} g(\gamma^n R)\,\mathrm{cap}^*(A \cap S(x_0,\gamma^n R,\gamma^{n+1}R)) < \infty. \tag{5.3}$$

Proof. We may assume that A is open. Indeed, given $x \in X$ such that $R_1^A(x) < 1$, there exists an open neighborhood U of A such that $R_1^U(x) < 1$ (see [2, VI.1.5]).

Let us fix $m \in \mathbb{N}$ and $\eta_1, \eta_2 \in (0,1)$ according to Proposition 4.2 and Lemma 5.4, and let

$$\delta := \eta_1\eta_2.$$

By Proposition 1.1, there exists $x_1 \in X$ such that

$$P^{x_1}[T_A < \infty] < \delta/2.$$

Replacing R by some $\gamma^{n_0}R$ we may assume without loss of generality that

$$R > 2\rho(x_0,x_1) + R_1 + \max\{1,(\gamma-1)^{-1}\}(R_0+R_2). \tag{5.4}$$

For $n = 0, 1, 2, \dots$, let

$$a_n := \operatorname{cap}^*(A \cap S(x_0, \gamma^{n+1}R, \gamma^{n+2}R)).$$

To show that $\sum_{n\geq 0} g(\gamma^{n+1}R)a_n < \infty$ it suffices to show that

$$\sum_{n\geq 0} g(\gamma^{mn+1}R)a_{mn} < \infty, \tag{5.5}$$

since, having (5.5), we may replace R by $\gamma^j R$, $1 \leq j < m$, and thus obtain (5.3). Let $n \in \mathbb{N}$,

$$r := \gamma^{mn}R, \quad a := a_{mn}, \quad S := S(x_0, r, \gamma^3 r), \quad \tau := \tau_{B(0,r)}, \quad \tau' := \tau_{B(0,\gamma^m r)}.$$

For the moment, let us assume that $S(x_0, \gamma r, \gamma^2 r) \neq \emptyset$. Then $g(r)\operatorname{cap}^*(S) \geq \eta_2$, by Lemma 5.4. Therefore, by Proposition 4.2,

$$P^{x_1}[T_S < \tau'] \geq \eta_1 g(r)\operatorname{cap}^*(S) \geq \delta,$$

$$P^{x_1}[T_S < T_A \wedge \tau'] \geq P^{x_1}[T_S < \tau'] - P^{x_1}[T_A < \infty] > \delta/2. \tag{5.6}$$

By Proposition 4.2, for every $x \in \overline{B}(x_0, \gamma r)$,

$$P^x[T_A < \tau'] \geq P^x[T_{A\cap S} < \tau'] \geq \eta_1 g(r)\operatorname{cap}^*(A \cap S) \geq \eta_1 g(r)a. \tag{5.7}$$

Clearly, $T_S + T_A \circ \theta_{T_S} = T_A$ and $T_S + \tau' \circ \theta_{T_S} = \tau'$ on $[T_S < T_A \wedge \tau']$. Hence

$$[T_S < T_A < \tau'] = [T_S < T_A \wedge \tau', T_A < \tau'] = [T_S < T_A \wedge \tau'] \cap \theta_{T_S}^{-1}([T_A < \tau']).$$

Since $X_{T_S} \in \overline{B}(x_0, \gamma r)$ on $[T_S < \infty]$, the strong Markov property, (5.7) and (5.6) imply that

$$P^{x_1}[T_S < T_A < \tau'] = \int_{[T_S < T_A \wedge \tau']} P^{X_{T_S}}[T_A < \tau']\, dP^{x_1} \geq \frac{\delta\eta_1}{2}\, g(r)a.$$

Of course, $\tau \leq T_S$. Hence the sets $[T_S < T_A < \tau']$, obtained for different n, are pairwise disjoint subsets of $[T_A < \infty]$. Of course, $a = 0$, if $S(x_0, \gamma r, \gamma^2 r) = \emptyset$. Therefore

$$\sum_{n\geq 0} g(\gamma^{mn+1}R)a_{mn} \leq \frac{2}{\delta\eta_1}\, P^{x_1}[T_A < \infty] \leq \frac{1}{\eta_1}$$

finishing the proof. $\square$

We note that the preceding proof could also be given in a purely analytic way using iterated reducing of measures (see Remark 4.3).

6 Application to collections of balls having the separation property

The next simple result on comparison of potentials (cf. [7]) will be sufficient for us (see the proof of [11, Theorem 5.3] for a much more delicate version; cf. also the proof of [1, Theorem 3]).

LEMMA 6.1. *Let $Z \subset \mathbb{R}^d$ be finite and $r_z > R_0$, $z \in Z$, such that, for $z \neq z'$, $B(z, r_z) \cap B(z', 3r_{z'}) = \emptyset$. Let $w \in \mathcal{W} \cap \mathcal{C}(X)$ and, for every $z \in Z$, let μ_z, ν_z be measures on $B(z, r_z)$ such that $G\mu_z \leq w$, and $\|\mu_z\| \leq \|\nu_z\|$. Then $\mu := \sum_{z \in Z} \mu_z$ and $\nu := \sum_{z \in Z} \nu_z$ satisfy*

$$G\mu \leq w + c^2 c_D G\nu. \tag{6.1}$$

Proof. Let $z, z' \in Z$, $z' \neq z$, and $x \in B(z, r_z)$. For all $y, y' \in B(z', r_{z'})$, $\rho(y, y') \leq 2r_{z'} \leq \rho(x, y')$, hence $R_0 < r_{z'} < \rho(x, y) \leq 2\rho(x, y')$, $g(\rho(x, y') \leq c_D g(\rho(x, y))$, and $G(x, y') \leq c^2 c_D G(x, y)$. By integration, $G\mu_{z'}(x) \leq c^2 c_D G\nu_{z'}(x)$. Therefore

$$G\mu(x) = G\mu_z(x) + \sum_{z' \in Z, z' \neq z} G\mu_{z'}(x) \leq w(x) + c^2 c_D G\nu(x).$$

Thus $G\mu \leq w + c^2 c_D G\nu$ on the union A of the balls $B(z, r_z)$, $z \in Z$. By the minimum principle (see [2, III.6.6]), the proof is finished. □

LEMMA 6.2. *Let $x_0 \in X$, $R > 2R_0$ and $B := B(x_0, R)$. Suppose that there exist $C \geq 1$ and a probability measure λ on B such that $G\lambda \leq Cg(r)$. Let Z be a finite subset of $B(x_0, R/2)$ and $R_0 < r_z \leq R/2$, $z \in Z$, such that the balls $B(z, 3r_z)$ are pairwise disjoint and, for some $\varepsilon \in (0, 1)$,*

$$g(r_z)\lambda(B(z, \rho(z, z')/4)) \geq \varepsilon g(R), \qquad \textit{whenever } z \neq z'. \tag{6.2}$$

Then the union A of the balls $B(z, r_z)$, $z \in Z$, satisfies

$$\operatorname{cap} A \geq \varepsilon (2c^3 c_D C)^{-1} \sum_{z \in Z} \operatorname{cap} B(z, r_z).$$

Proof. It clearly suffices to consider the case, where Z contains more than one point. Then, for $z \in Z$,

$$\tilde{r}_z := \max\{r_z, \operatorname{dist}(z, Z \setminus \{z\})/4\} \leq R/2, \tag{6.3}$$

hence $B(z, \tilde{r}_z) \subset B$ and $\lambda(B(z, \tilde{r}_z)) > 0$, by (6.2). Further, $B(z, \tilde{r}_z) \cap B(z', 3\tilde{r}_{z'}) = \emptyset$, whenever $z \neq z'$,

For $z \in Z$, let $\mu_z \in \mathcal{M}(X)$ with $\mu_z(X \setminus B(z, r_z)) = 0$ and $G\mu_z \leq 1$, and let

$$\alpha_z := \|\mu_z\| / \lambda(B(z, \tilde{r}_z)), \qquad \nu_z := \alpha_z 1_{B(z, \tilde{r}_z)} \lambda.$$

Then $\|\nu_z\| = \|\mu_z\|$ and, by Proposition 1.11, (6.3), and (6.2),

$$\alpha_z \leq c g(r_z)^{-1} / \lambda(B(z, \tilde{r}_z)) \leq c(\varepsilon g(R))^{-1}.$$

Since the balls $B(z,\tilde r_z)$, $z\in Z$, are pairwise disjoint subsets of B, the measure $\nu := \sum_{z\in Z}\nu_z$ satisfies

$$G\nu \le c(\varepsilon g(R))^{-1}G\lambda \le cC\varepsilon^{-1}.$$

By Lemma 6.1, $\mu := \sum_{z\in Z}\mu_z$ satisfies $G\mu \le 1 + c^2c_DG\nu$. Thus $G\mu \le 2c^3c_DC\varepsilon^{-1}$. Since $\mu(X\setminus A)=0$, we see that $\operatorname{cap} A \ge \varepsilon(2c^3c_DC)^{-1}\sum_{z\in Z}\|\mu_z\|$ completing the proof. □

In addition to the assumptions 1.4 and 1.9, we assume from now on the following.

ASSUMPTION 6.3. *The constant function* 1 *is harmonic and balls are relatively compact. Moreover, we have a measure $\lambda\in\mathcal{M}(X)$ with $\operatorname{supp}(\lambda)=X$ and constants $c_0\ge 1$ and $R_2\ge 0$ such that, for all $x\in X$ and $r>R_2$,*

$$G\lambda_{B(x,r)} \le c_0 g(r) \tag{6.4}$$

(*where, as before, $\lambda_B := \lambda(B)^{-1}1_B\lambda$ for every ball B*).

We note that then Assumption 5.2 is satisfied.

Let us say that a family of pairwise disjoint balls $B(z,r_z)$, $z\in Z\subset X$, $r_z>4R_0$, has the *separation property with respect to* λ, if Z is locally finite and, for some $x_0\in X$,

$$\inf\nolimits_{z,z'\in Z, z\ne z'} \frac{\lambda(B(z,\rho(z,z')/4))}{\lambda(B(x_0,4\rho(x_0,z)))}\cdot\frac{g(r_z)}{g(\rho(x_0,z))} \;>\; 0. \tag{6.5}$$

REMARK 6.4. If, for example, λ is Lebesgue measure on $X=\mathbb{R}^d$, $x_0=0$, $\rho(x,y)=|x-y|$, and $g(r)=r^{\alpha-d}$, then (6.5) means that

$$\inf\nolimits_{z,z'\in Z, z\ne z'} \frac{|z-z'|^d}{|z|^\alpha r_z^{d-\alpha}} \;>\; 0,$$

which, in the classical case $\alpha=2$, is the separation property in [3, Theorem 6].

THEOREM 6.5. *Let A be an avoidable union of pairwise disjoint balls $B(z,r_z)$, $z\in Z\subset X$, having the separation property with respect to λ. Then*

$$\sum\nolimits_{z\in Z} g(\rho(x_0,z))\operatorname{cap} B(z,r_z) < \infty.$$

Proof. We may assume without loss of generality that

$$r_z \le \rho(x_0,z)/2, \qquad \text{for every } z\in Z. \tag{6.6}$$

Indeed, replacing r_z by $r_z' := \min\{r_z,\rho(x_0,z)/2\}$ our assumptions are preserved as well. Suppose we have shown that $\sum_{z\in Z} g(\rho(x_0,z))/g(r_z')<\infty$. Since $g(r)/g(r/2)\ge c_D^{-1}$, $r>2R_0$, the set Z' of all points $z\in Z$ such that $r_z'=\rho(x_0,z)/2$ is finite, and hence $\sum_{z\in Z'} g(\rho(x_0,z))/g(r_z)<\infty$. So we may assume without loss of generality that $r_z'=r_z$, for all $z\in Z$, that is, (6.6) holds.

Moreover, we may suppose that $\rho(x_0,z)>4R_0$, for every $z\in Z$ (we simply omit finitely many points from Z). Further, we may assume that the balls $B(z,4r_z)$ are pairwise disjoint. Indeed, since $g(r) \le g(r/4) \le c^2 g(r)$, $r > 4R_0$, a replacement of r_z by $r_z/4$ does neither affect (6.5) nor the convergence of $\sum_{z\in Z} g(\rho(x_0,z))/g(r_z)$, and the new, smaller union is, of course, avoidable.

By (6.5), there exists $\varepsilon \in (0,1)$ such that, for $z, z' \in Z$, $z \neq z'$,

$$g(r_z)\lambda(B(z,\rho(z,z')/4)) \ge \varepsilon g(\rho(x_0,z))\lambda(B(x_0,4\rho(x_0,z))). \tag{6.7}$$

Let $R > 4R_0$. For $n \in \mathbb{N}$, let

$$Z_n := Z \cap S(x_0, 2\cdot 8^n R, 4\cdot 8^n R) \quad \text{and} \quad A_n := \bigcup\nolimits_{z\in Z_n} B(z,r_z).$$

Then $A_n \subset A \cap S(x_0, 8^n R, 8^{n+1}R)$. Moreover, for every $z \in Z_n$, $\rho(x_0,z) \ge 8^n R$, and hence $g(\rho(x_0,z)) \le g(8^n R)$. Therefore, by Lemma 6.2 and Theorem 1.13,

$$\begin{aligned}\sum\nolimits_{n\in\mathbb{N}} \sum\nolimits_{z\in Z_n} & g(\rho(x_0,z)) \operatorname{cap} B(z,r_z) \\ &\le 2c^3 c_D c_0 \varepsilon^{-1} \sum\nolimits_{n\in\mathbb{N}} g(8^n R) \operatorname{cap}^*(A \cap S(x_0, 8^n R, 8^{n+1}R)) < \infty.\end{aligned}$$

Applying this estimate as well to $2R$ and $4R$ in place of R we obtain that

$$\sum\nolimits_{z\in Z} g(\rho(x_0,z)) \operatorname{cap} B(z,r_z) < \infty.$$

□

COROLLARY 6.6. *Suppose that* (6.4) *holds for all* $R > R_0$. *Let* A *be a union of pairwise disjoint balls* $B(z,r_z)$, $z \in Z \subset X$, *having the separation property with respect to* λ. *Then the following statements are equivalent.*

(1) *The set* A *is unavoidable.*

(2) $\sum_{z\in Z} g(\rho(x_0,z)) \operatorname{cap} B(z,r_z) = \infty$.

(3) $\sum_{z\in Z} g(\rho(x_0,z))/g(r_z) = \infty$.

Proof. Since (6.4) implies that $\operatorname{cap} B(z,r_z) \ge g(r_z)^{-1}$ for every $z \in Z$, the equivalences follow immediately from Corollary 1.12 and Theorem 6.5. □

7 Application to regularly located balls

In this section we suppose as before that the Assumptions 1.4, 1.9, and 6.3 are satisfied. Moreover, let us assume that we have a distinguished point $x_0 \in X$, and that there exists $R_2 \ge 0$ such that the measure λ has the following additional properties:

(i) For all $x, y \in X$ and $r > R_2$,

$$\lambda(B(y,r)) \le c_0 \lambda(B(x,r)).$$

(ii) There exist $C_D \in (1,\infty)$, $\kappa \in (0,1)$ such that, for all $r > R_2$,

$$\lambda(B(x_0,2r)) \le C_D \lambda(B(x_0,r)) \quad \text{and} \quad \lambda(B(x_0,r)) \le C_D \lambda(S(x_0,\kappa r,r)).$$

We first prove the following proposition.

PROPOSITION 7.1. *Let $\phi\colon (0,\infty) \to (0,\infty)$ be decreasing, $C > 1$, and let $B(z,r_z)$, $z \in Z$, $r_z > R_0$, be balls in X such that the following hold.*

- *There exists $R > 0$ such that every ball of radius R contains a point of Z.*
- *There exists $C > 1$ such that, for every $z \in Z$,*

$$\phi(\rho(x_0,z)) < r_z < C\phi(\rho(x_0,z)). \tag{7.1}$$

- $\limsup_{r\to\infty} \lambda(B(x_0,r))g(r)/g(\phi(r)) > 0$.

Then the union A of all $B(z,r_z)$, $z \in Z$, is unavoidable.

Proof. We may assume without loss of generality that $R > \max\{1,\phi(1),R_0,R_2\}$. We define

$$a := (cc_0^2 C_D^5 \lambda(B(x_0,R))^{-1} \quad \text{and} \quad b := cc_0^2 \lambda(B(x_0,R))^{-1}.$$

Let $0 < \beta < \limsup_{r\to\infty} \lambda(B(x_0,r))g(r)/g(\phi(r))$.

We now fix $x \in X$ and choose $r > \kappa^{-1}(4R + 2\rho(x_0,x))$ such that $r > \phi(r)$ and

$$\gamma := \lambda(B(x_0,r))g(r)/g(\phi(r)) > \beta. \tag{7.2}$$

Let

$$S := S(x_0,\kappa r/2, r/2), \qquad B := B(x_0,r) \quad \text{and} \quad r_0 := \phi(r)$$

so that

$$\gamma = \lambda(B)g(r)/g(r_0).$$

There are finitely many points $y_1,\dots,y_m \in \overline{S}$ such that $B(y_1,3R),\dots,B(y_m,3R)$ are pairwise disjoint and $\overline{S}$ is covered by $B(y_1,9R),\dots,B(y_m,9R)$. We may choose $z_j \in Z \cap B(y_j,R)$, $1 \le j \le m$. Then $\rho(z_i,z_j) \ge \rho(y_i,y_j) - 2R \ge 4R$, and hence

$$B(z_i,R) \cap B(z_j,3R) = \emptyset, \tag{7.3}$$

for all $i,j \in \{1,\dots,m\}$ with $i \ne j$. Moreover,

$$\lambda(B) \le C_D^2 \lambda(S) \le C_D^2 \sum_{j=1}^m \lambda(B(y_j,9R)) \le mC_D^5 c_0 \lambda(B(x_0,R)). \tag{7.4}$$

Let $1 \le j \le m$. Clearly,

$$r > r - R > r/2 + R \ge \rho(x_0,z_j) \ge \kappa r/2 - R \ge R \ge 1.$$

Therefore $B(z_j,R)\subset B$ and $r_0=\phi(r)\le\phi(\rho(x_0,z_j))<r_{z_j}$, hence $\overline{B}(z_j,r_0)\subset A$. Moreover, $r_0\le\phi(1)\le R$ and $r_0+\rho(x,z_j)\le R+\rho(x_0,x)+r/2+R\le r$. So $g(\rho(x,z_j)+r_0)\ge g(r)$ and, by (5.2),

$$R_1^{B(z_j,r_0)}(x)\ge(cc_0)^{-1}g(\rho(x,z_j)+r_0)/g(r_0)\ge(cc_0)^{-1}g(r)/g(r_0). \tag{7.5}$$

Let μ_j be the equilibrium measure for $B(z_j,r_0)$, $1\le j\le m$. We define

$$p:=\sum\nolimits_{j=1}^m R_1^{B(z_j,r_0)}=\sum\nolimits_{j=1}^m G\mu_j.$$

Then, by (7.5) and (7.4),

$$p(x)\ge m(cc_0)^{-1}g(r)/g(r_0)\ge a\gamma. \tag{7.6}$$

Finally, let $\nu:=\sum_{j=1}^m\nu_j$, where

$$\nu_j:=cg(r_0)^{-1}\lambda_{B(z_j,R)}\le cc_0g(r_0)^{-1}\frac{\lambda(B)}{\lambda(B(x_0,R))}\,1_{B(z_j,R)}\lambda_B.$$

Since $B(z_1,R),\dots,B(z_j,R)$ are pairwise disjoint subsets of B and $G\lambda_B\le c_0g(r)$,

$$G\nu\le cc_0g(r_0)^{-1}\frac{\lambda(B)}{\lambda(B(x_0,R))}\,G\lambda_B\le cc_0^2\frac{\lambda(B)}{\lambda(B(x_0,R))}\,g(r)/g(r_0)=b\gamma.$$

By Proposition 1.11, $\|\mu_j\|\le cg(r_0)^{-1}$, $1\le j\le m$. Thus, by (7.3) and Lemma 6.1,

$$p\le 1+c^2c_DG\nu\le 1+c^2c_Db\gamma.$$

Since μ is supported by the compact $\overline{B}(z_1,r_0)\cup\dots\cup\overline{B}(z_m,r_0)$ in A, this implies that

$$R_1^A\ge(1+c^2c_Db\gamma)^{-1}p,$$

by the minimum principle (see [2, III.6.6]). In particular,

$$R_1^A(x)\ge\frac{a\gamma}{1+c^2c_Db\gamma}=\frac{a}{\gamma^{-1}+c^2c_Db}>\frac{a}{\beta^{-1}+c^2c_Db}\,,$$

by (7.6) and (7.2). Thus A is unavoidable, by Proposition 1.1. □

Proof of Corollary 1.16. Let us assume that (6.4) holds for all $r>R_0$ and that A is the union of balls $B(z,r_z)$, $z\in Z$, which are regularly located (see Definition 1.15).

If A is unavoidable, then $\sum_{z\in Z}g(\rho(x_0,z))/g(r_z)=\infty$, by Corollary 1.12.

To prove the converse, suppose that $\sum_{z\in Z}g(\rho(x_0,z))/g(r_z)=\infty$. By Proposition 7.1, it suffices to consider the case

$$\limsup\nolimits_{r\to\infty}\lambda(B(x_0,r))g(r)/g(\phi(r))=0. \tag{7.7}$$

Then $\inf_{z\in Z}g(r_z)(\lambda(B(x_0,\rho(x_0,z)))g(\rho(x_0,z)))^{-1}>0$ and

$$\lambda(B(z,\rho(z,z')/4))\ge\inf\nolimits_{x\in X}\lambda(B(x,\varepsilon/4))>0,$$

whenever, $z,z'\in Z$, $z\ne z'$. So the balls $B(z,r_z)$, $z\in Z$, have the separation property, and A is unavoidable, by Corollary 6.6. □

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