

ZETA FUNCTIONS OF GROUPS AND RINGS – RECENT DEVELOPMENTS

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ABSTRACT. I survey some recent developments in the theory of zeta functions associated to infinite groups and rings, specifically zeta functions enumerating subgroups and subrings of finite index or finite-dimensional complex representations.

1. ABOUT THESE NOTES

Over the last few decades, zeta functions have become important tools in various areas of asymptotic group and ring theory. With the first papers on zeta functions of groups published barely 25 years ago, the subject area is still comparatively young. Recent developments have led to a wealth of results and gave rise to new perspectives on central questions in the field. The aim of these notes is to introduce the nonspecialist reader informally to some of these developments.

I concentrate on two types of zeta functions: firstly, zeta functions in *subgroup and subring growth* of infinite groups and rings, enumerating finite-index subobjects. Secondly, representation zeta functions in *representation growth* of infinite groups, enumerating finite-dimensional irreducible complex representations. I focus on common features of these zeta functions, such as Euler factorizations, local functional equations, and their behaviour under base extension.

Subgroup growth of groups is a relatively mature subject area, and the existing literature reflects this: zeta functions of groups feature in the authoritative 2003 monograph [39] on “Subgroup Growth”, are the subject of the Groups St Andrews 2001 survey [16] and the report [18] to the ICM 2006. The book [21] contains, in particular, a substantial list of explicit examples. Some more recent developments are surveyed in [32, Chapter 3].

On the other hand, few papers on representation zeta functions of infinite groups are older than ten years. Some of the lecture notes in [32] touch on the subject. The recent survey [31] on representation growth of groups complements the current set of notes.

In this text I use, more or less as blackboxes, the theory of p -adic integration and the Kirillov orbit method. The former provides a powerful toolbox for the treatment of a number of group-theoretic counting problems. The latter is a general method to parametrize the irreducible complex representations of certain groups in terms of coadjoint orbits. Rather than to explain in detail how these tools are employed I will refer to specific references at appropriate places in the text. I all but ignore the rich subject of zeta functions enumerating representations or conjugacy classes of finite groups of Lie type; see, for instance, [34].

These notes grew out of a survey talk I gave at the conference Groups St Andrews 2013 in St Andrews. I kept the informal flavour of the talk, preferring instructive examples and sample theorems over the greatest generality of the presented results. As

2010 *Mathematics Subject Classification*. Primary: 22E50, 20E07. Secondary: 22E55, 20F69, 22E40, 20C15, 20G25, 11M41.

a consequence, the text is not a systematic treaty of the subject, but rather the result of a subjective choice.

2. ZETA FUNCTIONS IN ASYMPTOTIC GROUP AND RING THEORY

We consider counting problems of the following general form. Let Γ be a – usually infinite – algebraic object, such as a group or a ring, and assume that, for each $n \in \mathbb{N}$, we are given integers $d_n(\Gamma) \in \mathbb{N}_0$, encoding some algebraic information about Γ . Often this data will have a profinite flavour, in the sense that, for every n , there exists a finite quotient Γ_n of Γ such that $d_n(\Gamma)$ can be computed from Γ_n . In any case, we encode the sequence $(d_n(\Gamma))$ in a generating function.

Definition 2.1. The *zeta function of* $(\Gamma, (d_n(\Gamma)))$ is the Dirichlet generating series

$$(2.1) \quad \zeta_{(d_n(\Gamma))}(s) = \sum_{n=1}^{\infty} d_n(\Gamma) n^{-s},$$

where s is a complex variable. If $(d_n(\Gamma))$ is understood from the context, we simply write $\zeta_{\Gamma}(s)$ for $\zeta_{(d_n(\Gamma))}(s)$.

In the counting problems we consider Dirichlet series often turn out to be preferable over other generating functions, in particular if the arithmetic function $n \mapsto d_n(\Gamma)$ satisfies some of the following properties.

- (A) *Polynomial growth*, i.e. the coefficients $d_n(\Gamma)$ – or, equivalently, their partial sums – have polynomial growth: $D_n(\Gamma) := \sum_{\nu \leq n} d_{\nu}(\Gamma) = O(n^a)$ for some $a \in \mathbb{R}$.
- (B) *Multiplicativity* in the sense of elementary number theory: if $n = \prod_i p_i^{e_i}$ is the prime factorization of n , then $d_n(\Gamma) = \prod_i d_{p_i^{e_i}}(\Gamma)$.

Indeed, polynomial growth implies that $\zeta_{(d_n(\Gamma))}(s)$ converges absolutely on some complex half-plane. If $d_n(\Gamma) \neq 0$ for infinitely many n , then the *abscissa of convergence* of $\zeta_{(d_n(\Gamma))}(s)$ is equal to

$$\alpha((d_n(\Gamma))) := \limsup_{n \rightarrow \infty} \frac{\log \sum_{\nu \leq n} D_{\nu}(\Gamma)}{\log n}.$$

Thus $\alpha((d_n(\Gamma)))$ gives the precise degree of polynomial growth of the partial sums $D_n(\Gamma)$ as n tends to infinity. If the sequence $(d_n(\Gamma))$ is understood from the context, we sometimes write $\alpha(\Gamma)$ for $\alpha((d_n(\Gamma)))$.

Multiplicativity implies that – at least formally – the series (2.1) satisfies an *Euler factorization*, indexed by the prime numbers:

$$(2.2) \quad \zeta_{(d_n(\Gamma))}(s) = \prod_{p \text{ prime}} \zeta_{(d_n(\Gamma)),p}(s),$$

where, for a prime p , the function

$$\zeta_{(d_n(\Gamma)),p}(s) = \zeta_{(d_{p^i}(\Gamma))}(s) = \sum_{i=0}^{\infty} d_{p^i}(\Gamma) p^{-is}$$

is called the *local factor of* $\zeta_{(d_n(\Gamma))}(s)$ *at the prime* p . We will later consider other Euler factorizations, indexed by places of a number field rather than rational prime numbers, which reflect multiplicativity features of the underlying counting problem which are subtler than the multiplicativity of $n \mapsto d_n(\Gamma)$. In any case, there are often *rationality results* which establish that the Euler factors are rational functions, rendering them – at least in principle – amenable to computation. In practice, the study of many (global)

zeta functions of the form (2.1) proceeds via a uniform description of local factors in Euler factorizations like (2.2).

Key questions regarding zeta functions of groups and rings concern the following:

- (1) Analytic properties regarding e.g. the abscissa of convergence, analytic continuation, natural boundaries, location and multiplicities of zeros and poles, residue formulae, special values etc.,
- (2) arithmetic properties of the local factors, e.g. rationality; if so, structure of numerators and denominators, special symmetries (functional equations) etc.,
- (3) the variation of these properties as Γ varies within natural families of groups.

In the sequel we survey some key results and techniques in the study of zeta functions in the context of subgroup and subring growth (Section 3) and of representation growth (Section 4).

3. SUBGROUP AND SUBRING GROWTH

3.1. Subgroup growth of finitely generated nilpotent groups. A finitely generated group Γ has only finitely many subgroups of each finite index n . We set, for $n \in \mathbb{N}$,

$$a_n(\Gamma) := \#\{H \leq \Gamma \mid |\Gamma : H| = n\}.$$

If $s_n(\Gamma) := \sum_{\nu \leq n} a_\nu(\Gamma) = O(n^a)$ for some $a \in \mathbb{R}$, then Γ is said to be of *polynomial subgroup growth* (PSG). Finitely generated, residually finite groups of PSG have been characterized as the virtually solvable groups of finite rank; see [37]. This class of groups includes the torsion-free, finitely generated nilpotent (or \mathcal{T} -)groups. Let Γ be a \mathcal{T} -group. Then the sequence $(a_n(\Gamma))$ is multiplicative. This follows from the facts that every finite index subgroup H of Γ contains a normal such subgroup, and that a finite nilpotent group is isomorphic to the direct product of its Sylow p -subgroups. In [27], Grunewald, Segal, and Smith pioneered the use of zeta functions in the theory of subgroup growth of \mathcal{T} -groups. They studied the *subgroup zeta function*

$$\zeta_\Gamma(s) := \zeta_{(a_n(\Gamma))}(s) = \sum_{n=1}^{\infty} a_n(\Gamma)n^{-s}$$

of Γ via the Euler factorization

$$(3.1) \quad \zeta_\Gamma(s) = \prod_{p \text{ prime}} \zeta_{\Gamma,p}(s),$$

where, for each prime p , the local factor at p is defined via $\zeta_{\Gamma,p}(s) = \sum_{i=0}^{\infty} a_{p^i}(\Gamma)p^{-is}$. One of the main result of [27] is the following fundamental theorem.

Theorem 3.1. [27, Theorem 1] *For all primes p , the function $\zeta_{\Gamma,p}(s)$ is rational in p^{-s} , i.e. there exist polynomials $P_p, Q_p \in \mathbb{Q}[Y]$ such that $\zeta_{\Gamma,p}(s) = P_p(p^{-s})/Q_p(p^{-s})$. The degrees of P_p and Q_p in Y are bounded.*

The following is by now a classical example.

Example 3.2. [27, Proposition 8.1] Let

$$(3.2) \quad \mathbf{H}(\mathbb{Z}) = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix}$$

be the integral Heisenberg group. Then

$$(3.3) \quad \zeta_{\mathbf{H}(\mathbb{Z})}(s) = \zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)\zeta(3s-3)^{-1},$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$ is the Riemann zeta function.

It is of great interest to understand how the rational functions giving the local zeta functions in Euler factorizations like (3.1) vary with the prime p . It is known that the denominator polynomials $Q_p(Y)$ can be chosen to be of the form $\prod_{i \in I} (1 - p^{a_i - b_i s})$, for a finite index set I and nonnegative integers a_i, b_i , all depending only on Γ . Computing these integers, or even just a reasonably small set of candidates, however, remains a difficult problem. The numerator polynomials' variation with the prime p is even more mysterious. It follows from fundamental work of du Sautoy and Grunewald that there are finitely many varieties V_1, \dots, V_N defined over \mathbb{Q} , and rational functions $W_1(X, Y), \dots, W_N(X, Y) \in \mathbb{Q}(X, Y)$ such that, for almost all primes p ,

$$(3.4) \quad \zeta_{\Gamma, p}(s) = \sum_{i=1}^N |\overline{V}_i(\mathbb{F}_p)| W_i(p, p^{-s}),$$

where \overline{V}_i denotes the reduction of V_i modulo p ; cf. [17]. One may construct \mathcal{T} -groups where the numbers $|\overline{V}_i(\mathbb{F}_p)|$ are not all polynomials in p ; cf., for instance, [15]. Recent results determine the degree in Y of the rational functions $P_p/Q_p \in \mathbb{Q}(Y)$ in Theorem 3.1 for almost all primes p ; cf. Corollary 3.9.

Variations of the sequence $(a_n(\Gamma))$ include the normal subgroup sequence $(a_n^{\triangleleft}(\Gamma))$, where

$$a_n^{\triangleleft}(\Gamma) := \#\{H \triangleleft \Gamma \mid |\Gamma : H| = n\}.$$

It gives rise to the *normal (subgroup) zeta function*

$$\zeta_{\Gamma}^{\triangleleft}(s) := \zeta_{(a_n^{\triangleleft}(\Gamma))}(s) = \sum_{n=1}^{\infty} a_n^{\triangleleft}(\Gamma) n^{-s}$$

of Γ . It also has an Euler factorization whose factors are rational in p^{-s} and, in principle, given by formulae akin to (3.4). The normal zeta function of the integral Heisenberg group (cf. (3.2)), for example, is equal to

$$\zeta_{\mathbf{H}(\mathbb{Z})}^{\triangleleft}(s) = \zeta(s)\zeta(s-1)\zeta(3s-2) = \prod_{p \text{ prime}} \frac{1}{(1-p^{-s})(1-p^{1-s})(1-p^{2-3s})};$$

cf. [27, Section 8].

It is interesting to ask how subgroup zeta functions of \mathcal{T} -groups, or their variations, vary under base extension. Given a number field K with ring of integers \mathcal{O} one may consider, for instance, the \mathcal{T} -group $\mathbf{H}(\mathcal{O})$ of upper-unitriangular 3×3 -matrices over \mathcal{O} . Then

$$(3.5) \quad \zeta_{\mathbf{H}(\mathcal{O})}^{\triangleleft}(s) = \prod_{p \text{ prime}} \zeta_{\mathbf{H}(\mathcal{O}), p}^{\triangleleft}(s).$$

The following result extends parts of [27, Theorem 2] and makes it more precise.

Theorem 3.3. [46] *For every $r \in \mathbb{N}$ and every finite family $\mathbf{f} = (f_1, \dots, f_r) \in \mathbb{N}^r$, there exist explicitly given rational functions $W_{\mathbf{f}}(X, Y) \in \mathbb{Q}(X, Y)$, such that the following hold.*

- (1) *If p is a prime which is unramified in K and decomposes in K as $p\mathcal{O} = \prod_{i=1}^r \mathfrak{P}_i$ for prime ideals \mathfrak{P}_i of \mathcal{O} with inertia degrees $f_i = \log_p |\mathcal{O} : \mathfrak{P}_i|$ for $i = 1, \dots, r$, then*

$$\zeta_{\mathbf{H}(\mathcal{O}), p}^{\triangleleft}(s) = W_{\mathbf{f}}(p, p^{-s}).$$

(2) *Setting* $d = |K : \mathbb{Q}| = \sum_{i=1}^r f_i$, *we have*

$$(3.6) \quad W_{\mathbf{f}}(X^{-1}, Y^{-1}) = (-1)^n X^{\binom{3d}{2}} Y^{5d} W_{\mathbf{f}}(X, Y).$$

The proof of Theorem 3.3 is essentially combinatorial. In the case that p splits completely, i.e. $\mathbf{f} = (1, \dots, 1)$, it proceeds by organizing the infinite sums defining the local zeta functions as sums indexed by pairs of partitions (λ, μ) , each of at most n parts, where λ dominates μ . We further partition the infinite set of such pairs into $C_n = \frac{1}{n+1} \binom{2n}{n}$ (the n -th Catalan number) parts, indexed by the Dyck words of length $2n$, determined by the “overlap” between λ and μ . This subdivision by Dyck words is suggested by a simple lemma, attributed to Birkhoff, that determines the numbers of subgroups of type μ in a finite abelian p -group of type λ . For each fixed Dyck word, we express the corresponding partial sum of the local zeta function in terms of natural generalizations of combinatorially defined generating functions, first studied by Igusa (cf. [51, Theorem 4]) and Stanley [49]. Remarkably, a functional equation of the form (3.6) is already satisfied by each of the C_n partial sums. If p does not split completely, the strategy above still works after some moderate modification.

The functional equation (3.6) reflects the Gorenstein property of certain face rings. That such a functional equation holds for *almost all* primes p follows from [52, Theorem B]; that it holds in fact for all unramified primes is additional information. Note that $3d = h(\mathbf{H}(\mathcal{O}))$ and $5d = h(\mathbf{H}(\mathcal{O})) + h(\mathbf{H}(\mathcal{O})/Z(\mathbf{H}(\mathcal{O})))$, the sums of the Hirsch lengths of the nontrivial quotients by the terms of the upper central series of $\mathbf{H}(\mathcal{O})$. Here, given a \mathcal{J} -group G , we write $h(G)$ for the Hirsch length of G , i.e. the number of infinite cyclic factors in a decomposition series of G .

Formulae for the Euler factors in (3.5) indexed by primes which are nonsplit (but possibly ramified) in K are given in [47].

3.2. Subring growth of additively finitely generated rings. By a *ring* we shall always mean a finitely generated, torsion-free abelian group, together with a bi-additive multiplication – not necessarily associative, commutative, or unital. Examples of such rings include \mathbb{Z}^d (e.g. with null-multiplication or with componentwise multiplication), the rings of integers in number fields, and Lie rings, that is rings with a multiplication (or “Lie bracket”) which is alternating and satisfies the Jacobi identity. Examples of Lie rings include “semi-simple” matrix rings such as $\mathfrak{sl}_N(\mathbb{Z})$ and the Heisenberg Lie ring

$$\mathfrak{h}(\mathbb{Z}) = \begin{pmatrix} 0 & \mathbb{Z} & \mathbb{Z} \\ & 0 & \mathbb{Z} \\ & & 0 \end{pmatrix},$$

with Lie bracket induced from $\mathfrak{gl}_3(\mathbb{Z})$.

The subring sequence of a ring Λ is $(a_n(\Lambda))$, where

$$a_n(\Lambda) := \#\{H \leq \Gamma \mid |\Gamma : H| = n\}.$$

It is encoded in the *subring zeta function* of Λ , that is the Dirichlet generating series

$$\zeta_{\Lambda}(s) = \zeta_{(a_n(\Lambda))}(s) = \sum_{n=1}^{\infty} a_n(\Lambda) n^{-s}.$$

In contrast to the case of subgroup growth, polynomial growth requires no assumption on the multiplicative structure: indeed, the null-multiplication on \mathbb{Z}^d yields a trivial polynomial upper bound on $s_n(\Lambda) := \sum_{\nu \leq n} a_{\nu}(\Lambda)$. Also, multiplicativity of the subring

growth function $n \mapsto a_n(\Lambda)$ follows from the Chinese Remainder Theorem. Consequently, the subring zeta function of Λ satisfies the following Euler factorization:

$$\zeta_\Lambda(s) = \prod_{p \text{ prime}} \zeta_{\Lambda,p}(s).$$

Many of the structural results for local zeta functions of \mathcal{T} -groups have analogues in the setting of zeta functions of rings. One example is the following.

Theorem 3.4. [27, Theorem 3.5] *For all primes p , the function $\zeta_{\Lambda,p}(s)$ is rational in p^{-s} , i.e. there exist polynomials $P_p, Q_p \in \mathbb{Q}[Y]$ such that $\zeta_{\Lambda,p}(s) = P_p(p^{-s})/Q_p(p^{-s})$. The degrees of P_p and Q_p in Y are bounded.*

As in the context of subgroup growth of \mathcal{T} -groups, one also considers variations such as ideal growth of rings. The *ideal zeta function* of a ring enumerates its ideals of finite additive index. These zeta functions, too, enjoy Euler factorizations indexed by the rational primes. A rationality result analogous to Theorem 3.4 holds for the local factors. From this perspective one recovers, for example, the classical Dedekind zeta function of a number field, enumerating ideals of finite index in the number field's ring of integers.

In fact, the study of subgroup zeta functions of \mathcal{T} -groups as outlined in Section 3.1 may – to a large extent – be reduced to the study of subring zeta functions of nilpotent Lie rings. Indeed, a key tool in the analysis of [27] is a linearization technique: the Mal'cev correspondence associates to each \mathcal{T} -group Γ a nilpotent Lie ring $\Lambda(\Gamma)$, that is a Lie ring whose additive group is isomorphic to \mathbb{Z}^d , where $d = h(\Gamma)$ is the Hirsch length of Γ , which is nilpotent with respect to the Lie bracket; see [27, Section 4] for details on the Mal'cev correspondence and its consequences for zeta functions of \mathcal{T} -groups. One of these consequences is the fact that, for almost all primes p ,

$$(3.7) \quad \zeta_{\Gamma,p}(s) = \zeta_{\Lambda(\Gamma),p}(s);$$

cf. [27, Theorem 4.1]. In nilpotency class at most 2 this equality holds for all primes p . The formula (3.3), for instance, coincides with the subring zeta function of the Heisenberg Lie ring $\mathfrak{h}(\mathbb{Z}) = \Lambda(\mathbf{H}(\mathbb{Z}))$.

Maybe it is due to connections to subgroup growth like the ones just sketched that the study of subring growth has long focussed on *Lie* rings. The following example does not arise in this context.

Example 3.5. Let \mathcal{O} be the ring of integers in a number field K and, for $n \in \mathbb{N}$, let $b_n(\mathcal{O})$ denote the number of subrings of \mathcal{O} of index n , containing $1 \in \mathcal{O}$. The resulting zeta function $\zeta_{(b_n(\mathcal{O}))}(s)$ may be called the *order zeta function* $\eta_K(s)$ of K . The function η_K has an Euler factorization indexed by the rational primes, though – in contrast to the Dedekind zeta function $\zeta_K(s)$ – not generally by the prime ideals of \mathcal{O} . Clearly $\eta_{\mathbb{Q}} = 1$. If $d = |K : \mathbb{Q}| = 2$, then $\eta_K(s) = \zeta(s)$, the Riemann zeta function. For $d = 3$ it is known that

$$\eta_K(s) = \frac{\zeta_K(s)}{\zeta_K(2s)} \zeta(2s) \zeta(3s - 1);$$

see [12]. For $d = 4$, Nakagawa computes in [41] the Euler factors $\eta_{K,p}(s)$, where p ranges over the primes with arbitrary but fixed decomposition behaviour in K . Remarkably, the resulting formulae are rational functions in p and p^{-s} though not, in general, expressible in terms of translates of local Dedekind zeta functions. It is interesting to establish whether this uniformity on sets of primes with equal decomposition behaviour is a general phenomenon. Of particular interest is the case of primes with split totally

in K , i.e. primes p such that $p\mathcal{O} = \mathfrak{P}_1 \cdots \mathfrak{P}_d$, where $\mathfrak{P}_1, \dots, \mathfrak{P}_d$ are pairwise distinct prime ideals of \mathcal{O} with trivial residue field extension. For such primes, it is not hard to see that

$$\eta_{K,p}(s) = \zeta_{\mathbb{Z}^{d-1},p}(s),$$

where we consider \mathbb{Z}^{d-1} as a ring with componentwise multiplication.

Theorem 3.6. [41] and [35, Proposition 6.3]. *Consider \mathbb{Z}^3 as ring with componentwise multiplication. Then $\zeta_{\mathbb{Z}^3}(s) = \prod_{p \text{ prime}} \zeta_{\mathbb{Z}^3,p}(s)$, where, setting $t = p^{-s}$, we have*

$$(3.8) \quad \zeta_{\mathbb{Z}^3,p}(s) = \frac{1 + 4t + 2t^2 + (4p - 3)t^3 + (5p - 1)t^4 + (p^2 - 5p)t^5 + (3p^2 - 4p)t^6 - 2p^2t^7 - 4p^2t^8 - p^2t^9}{(1 - t)^2(1 - p^2t^4)(1 - p^3t^6)}.$$

The evidence available for $d \leq 3$ suggests a positive answer to the following question.

Question 3.7. Do there exist rational functions $W_d(X, Y) \in \mathbb{Q}(X, Y)$, for $d \in \mathbb{N}$, such that, for all primes p ,

$$\zeta_{\mathbb{Z}_p^d,p}(s) = W_d(p, p^{-s})?$$

Local zeta functions such as the ones given in (3.8) exhibit a curious palindromic symmetry under inversion of p . This is no coincidence, as the following result shows.

Theorem 3.8. [52, Theorem A] *Let Λ be a ring with $(\Lambda, +) \cong \mathbb{Z}^d$. Then, for almost all primes p ,*

$$(3.9) \quad \zeta_{\Lambda,p}(s)|_{p \rightarrow p^{-1}} = (-1)^d p^{\binom{d}{2} - ds} \zeta_{\Lambda,p}(s).$$

Corollary 3.9. *For almost all primes p , $\deg_{p^{-s}}(\zeta_{\Lambda,p}(s)) = -d$.*

Via the Mal'cev correspondence, Theorem 3.8 yields an analogous statement for almost all of the local factors $\zeta_{\Gamma,p}(s)$ of the subgroup zeta function $\zeta_{\Gamma}(s)$ of a \mathcal{T} -group Γ ; cf. (3.7). There are analogous results giving functional equations akin to (3.9) for ideal zeta functions of \mathcal{T} -groups – or equivalently, again by the Mal'cev correspondence, nilpotent Lie rings of finite additive rank – of nilpotency class at most 2. There are, however, examples of \mathcal{T} -groups of nilpotency class 3 whose local normal subgroup zeta functions do not satisfy functional equations like (3.9); cf. [21, Theorem 1.1].

Other variants of subgroup zeta functions of \mathcal{T} -groups which have been studied include those encoding the numbers of finite-index subgroups whose profinite completion is isomorphic to the one of the ambient group. These *pro-isomorphic zeta functions* also enjoy Euler product decompositions, indexed by the rational primes, whose factors are rational functions. It is an interesting open problem to characterise the \mathcal{T} -groups for which these local factors satisfy functional equations comparable to (3.9). For positive results in this direction see [20, 10]. An example of a \mathcal{T} -group (of nilpotency class 4 and Hirsch length 25) whose pro-isomorphic zeta function's local factors do not satisfy such functional equations was recently given in [11].

3.3. Taking the limit $p \rightarrow 1$: reduced and topological zeta functions of groups and rings. Numerous mathematical concepts, theorems, and identities allow natural q -analogues. Featuring an additional parameter q – often interpreted as a prime power –, these analogues return the original object upon setting $q = 1$. Examples include the Gaussian q -binomial coefficients, generalizing classical binomial coefficients and Heine's basic hypergeometric series, generalizing ordinary hypergeometric series.

An idea that only recently took hold in the theory of zeta functions of groups and rings is to interpret local such zeta functions as “ p -analogues” of certain limit objects as $p \rightarrow 1$ and to investigate the limit objects with tools from combinatorics or commutative algebra.

3.3.1. Reduced zeta functions. One way to make this idea rigorous leads, for instance, to the concept of the *reduced zeta function* $\zeta_{\Lambda, \text{red}}(t)$ of a ring Λ . Informally, this rational function in a variable t over the rationals is obtained by setting $p = 1$ in the coefficients of the p -adic subring zeta function of Λ , considered as a series in $t = p^{-s}$; formally, it arises by specializing the coefficients of the motivic zeta function associated to Λ via the Euler characteristic; cf. [19, 22]. Under some very restrictive conditions on Λ , the reduced zeta function $\zeta_{\Lambda, \text{red}}(t)$ is known to enumerate the integral points of a rational polyhedral cone. In the language of commutative algebra this means that $\zeta_{\Lambda, \text{red}}(t)$ is the Hilbert series of an affine monoid algebra attached to a Diophantine system of linear inequalities. For general rings a somewhat more multifarious picture seems to emerge, as the following example indicates.

Example 3.10. Consider $\Lambda = \mathbb{Z}^3$ as a ring with componentwise multiplication. Heuristically, setting $p = 1$ in (3.8) we obtain

$$\zeta_{\mathbb{Z}^3, \text{red}}(t) = \frac{1 + 5t + 6t^2 + 3t^3 + 6t^4 + 5t^5 + t^6}{(1-t)(1-t^2)(1-t^6)}.$$

Intriguingly, this rational function is not the generating function of a polyhedral cone, but does exhibit some tell-tale signs of the Hilbert series of a graded Cohen-Macaulay (even Gorenstein) algebra of dimension 3.

3.3.2. Topological zeta functions. Topological zeta functions offer another way to define a limit as $p \rightarrow 1$ of families of p -adic zeta functions. They were first introduced in the realm of Igusa’s p -adic zeta function as singularity invariants of hypersurfaces [13]. Informally, the topological zeta function is the leading term of the expansion of the p -adic zeta function in $p - 1$. Formally, it may be obtained by specialising the motivic zeta function; cf. [14]. Whereas the latter lives in the power series ring over a certain completion of a localization of a Grothendieck ring of algebraic varieties, the topological zeta function is just a rational function in one variable s , say, over the rationals. The topological zeta function $\zeta_{\Lambda, \text{top}}(s)$ of a ring was introduced in [19].

Example 3.11. The topological zeta function of \mathbb{Z}^3 (cf. Example 3.10) is

$$\zeta_{\mathbb{Z}^3, \text{top}}(s) = \frac{9s - 1}{s^2(2s - 1)^2}.$$

In [44] Rossmann develops an effective method for computing topological zeta functions associated to groups, rings, and modules. It is built upon explicit convex-geometric formulae for a class of p -adic integrals under suitable non-degeneracy conditions with respect to associated Newton polytopes. This method yields examples of explicit formulae for topological zeta functions of objects whose p -adic zeta functions are well out of computational reach. For a number of intriguing conjectures about arithmetic properties of topological zeta functions see [44, Section 8]. Rossmann implemented his algorithm in [43] and explained it in detail in [45].

4. REPRESENTATION GROWTH

Let Γ be a group. Consider, for $n \in \mathbb{N}$, the set $\text{Irr}_n(\Gamma)$ of n -dimensional irreducible complex representations of Γ up to isomorphism. If Γ has additional structure, we restrict our attention to representations respecting this structure. For instance, if Γ is a topological group, we only consider continuous representations. The group Γ is called (*representation*) *rigid* if $r_n(\Gamma) := \#\text{Irr}_n(\Gamma)$ is finite for all n . In this case, the Dirichlet generating series

$$\zeta_{\Gamma}^{\text{irr}}(s) := \zeta_{(r_n(\Gamma))}(s) = \sum_{n=1}^{\infty} r_n(\Gamma)n^{-s}$$

is called the *representation zeta function* of Γ . We discuss several classes of groups whose representation zeta functions (or natural variants thereof) have recently attracted attention. These are

- (1) finitely generated nilpotent groups,
- (2) arithmetic groups in characteristic 0,
- (3) algebraic groups,
- (4) compact p -adic analytic groups,
- (5) iterated wreath products and branch groups.

Throughout, let K be a number field with ring of integers $\mathcal{O} = \mathcal{O}_K$. We write

$$\zeta_K(s) = \sum_{I \triangleleft \mathcal{O}} |\mathcal{O} : I|^{-s} = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}} (1 - |\mathcal{O}/\mathfrak{p}|^{-s})^{-1}$$

for the Dedekind zeta function of K . Note that $\zeta_{\mathbb{Q}}(s) = \zeta(s)$. By representations we will always mean complex representations.

4.1. Finitely generated nilpotent groups. Let Γ be a \mathcal{J} -group. Unless Γ is trivial, the sets $\text{Irr}_n(\Gamma)$ are not all finite. Indeed, a nontrivial \mathcal{J} -group surjects onto the infinite cyclic group and thus has infinitely many one-dimensional representations. We therefore consider finite-dimensional representations up to twists by one-dimensional representations. More precisely, two representations $\rho_1, \rho_2 \in \text{Irr}_n(\Gamma)$ are said to be *twist-equivalent* if there exists $\chi \in \text{Irr}_1(\Gamma)$ such that ρ_1 is isomorphic to $\rho_2 \otimes \chi$. The numbers $\tilde{r}_n(\Gamma)$ of isomorphism classes of irreducible, complex n -dimensional representations of Γ are all finite; cf. [36, Theorem 6.6]. We define the *representation zeta function* of Γ to be the Dirichlet generating series

$$\zeta_{\Gamma}^{\tilde{\text{irr}}}(s) := \zeta_{(\tilde{r}_n(\Gamma))}(s) = \sum_{n=1}^{\infty} \tilde{r}_n(\Gamma)n^{-s}.$$

The coefficients $\tilde{r}_n(\Gamma)$ grow polynomially, so $\zeta_{\Gamma}^{\tilde{\text{irr}}}(s)$ converges on some complex right-half plane. The precise abscissa of convergence of $\zeta_{\Gamma}^{\tilde{\text{irr}}}(s)$ is an interesting invariant of Γ .

The function $n \mapsto \tilde{r}_n(\Gamma)$ is multiplicative, which yields the Euler factorization

$$(4.1) \quad \zeta_{\Gamma}^{\tilde{\text{irr}}}(s) = \prod_{p \text{ prime}} \zeta_{\Gamma,p}^{\tilde{\text{irr}}}(s),$$

where, for a prime p , the local factor $\zeta_{\Gamma,p}^{\tilde{\text{irr}}}(s) = \sum_{i=0}^{\infty} \tilde{r}_{p^i}(\Gamma)(p^{-s})^i$ enumerates twist-isoclasses of representations of Γ of p -power dimension.

Theorem 4.1. [28, Theorem 8.5] *For all primes p , the function $\zeta_{\Gamma,p}^{\tilde{\text{irr}}}(s)$ is rational in p^{-s} , i.e. there exist polynomials $P_p, Q_p \in \mathbb{Q}[Y]$ such that $\zeta_{\Gamma,p}^{\tilde{\text{irr}}}(s) = P_p(p^{-s})/Q_p(p^{-s})$. The degrees of P_p and Q_p in Y are bounded.*

The proof uses model-theoretic results on definable equivalence classes. We illustrate this important rationality result with a simple but instructive example.

Example 4.2. Consider the integral Heisenberg group $\mathbf{H}(\mathbb{Z})$; cf. (3.2). Then

$$(4.2) \quad \zeta_{\mathbf{H}(\mathbb{Z})}^{\widetilde{\text{irr}}}(s) = \sum_{n=1}^{\infty} \varphi(n)n^{-s} = \frac{\zeta(s-1)}{\zeta(s)} = \prod_{p \text{ prime}} \frac{1-p^{-s}}{1-p^{1-s}},$$

where φ is the Euler totient function; cf. [42].

It turns out that the formula in (4.2) behaves uniformly under some base extensions, as we shall now explain. Consider, for example, the Heisenberg group $\mathbf{H}(\mathcal{O})$ over \mathcal{O} , i.e. the group of upper-unitriangular 3×3 -matrices over \mathcal{O} . Then

$$(4.3) \quad \zeta_{\mathbf{H}(\mathcal{O})}^{\widetilde{\text{irr}}}(s) = \frac{\zeta_K(s-1)}{\zeta_K(s)} = \prod_{\mathfrak{p} \in \text{Spec } \mathcal{O}} \frac{1-|\mathcal{O}/\mathfrak{p}|^{-s}}{1-|\mathcal{O}/\mathfrak{p}|^{1-s}}.$$

For quadratic number fields this was proven by Ezzat in [24]. The general case follows from [50, Theorem B].

Each factor $\zeta_{\mathbf{H}(\mathcal{O}),\mathfrak{p}}^{\widetilde{\text{irr}}}(s) := \frac{1-|\mathcal{O}/\mathfrak{p}|^{-s}}{1-|\mathcal{O}/\mathfrak{p}|^{1-s}}$ of the Euler factorization (4.3) is interpretable as a representation zeta function associated to a pro- p group. Indeed, for $\mathfrak{p} \in \text{Spec } \mathcal{O}$, we denote by $\mathcal{O}_{\mathfrak{p}}$ the completion of \mathcal{O} at \mathfrak{p} . Then $\zeta_{\mathbf{H}(\mathcal{O}),\mathfrak{p}}^{\widetilde{\text{irr}}}(s)$ is equal to the zeta function $\zeta_{\mathbf{H}(\mathcal{O}_{\mathfrak{p}})}^{\widetilde{\text{irr}}}(s)$ of the pro- p group

$$\mathbf{H}(\mathcal{O}_{\mathfrak{p}}) = \begin{pmatrix} 1 & \mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \\ & 1 & \mathcal{O}_{\mathfrak{p}} \\ & & 1 \end{pmatrix},$$

enumerating *continuous* irreducible representations of $\mathbf{H}(\mathcal{O}_{\mathfrak{p}})$ up to twists by *continuous* one-dimensional representations. We note the following features of $\zeta_{\mathbf{H}(\mathcal{O})}^{\widetilde{\text{irr}}}(s)$.

- (1) Whilst the Euler factorization (4.2) illustrates the general factorization (4.1), the factorization (4.3) is finer than (4.1). In fact, for each rational prime p ,

$$\zeta_{\mathbf{H}(\mathcal{O}),p}^{\widetilde{\text{irr}}}(s) = \prod_{\mathfrak{p}|p\mathcal{O}} \zeta_{\mathbf{H}(\mathcal{O}_{\mathfrak{p}})}^{\widetilde{\text{irr}}}(s).$$

- (2) The factors of the “fine” Euler factorization (4.3) are indexed by the nonzero prime ideals \mathfrak{p} of \mathcal{O} , and are each given by a rational functions in q^{-s} , where $q = |\mathcal{O}/\mathfrak{p}|$ denotes the residue field cardinality.
- (3) Each factor of the Euler factorization (4.3) satisfies the functional equation

$$\zeta_{\mathbf{H}(\mathcal{O}),\mathfrak{p}}^{\widetilde{\text{irr}}}(s) \Big|_{q \rightarrow q^{-1}} = \frac{1-q^{-s}}{1-q^{1-s}} \Big|_{q \rightarrow q^{-1}} = \frac{1-q^s}{1-q^{-1+s}} = q \zeta_{\mathbf{H}(\mathcal{O}),\mathfrak{p}}^{\widetilde{\text{irr}}}(s).$$

As we shall see, all of these points are special cases of general phenomena.

We consider in the sequel families of groups obtained from Lie lattices. Let, more precisely, Λ be an \mathcal{O} -Lie lattice, i.e. a free and finitely generated \mathcal{O} -module, together with an antisymmetric, bi-additive form $[\cdot, \cdot] : \Lambda \times \Lambda \rightarrow \Lambda$, called ‘Lie bracket’, which satisfies the Jacobi identity. Assume further that Λ is nilpotent with respect to $[\cdot, \cdot]$ of class c , and let Λ' denote the derived Lie lattice $[\Lambda, \Lambda]$. If Λ satisfies $\Lambda' \subseteq c!\Lambda$, then it gives rise to a unipotent group scheme \mathbf{G}_{Λ} over \mathcal{O} , via the Hausdorff series as we shall now explain. The Hausdorff series $F(X, Y)$ is a formal power series in two noncommuting variables

X and Y , with rational coefficients. The Hausdorff formula gives an expression for this series in terms of Lie terms:

$$(4.4) \quad F(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([[[X, Y], Y] - [[X, Y], X]]) + \dots,$$

where $[A, B] := AB - BA$. See, e.g., [32, Chapter I, Section 7.4] for further details on the Hausdorff series.

For an \mathcal{O} -algebra R , let $\Lambda(R) := \Lambda \otimes_{\mathcal{O}} R$. The assumption that $\Lambda' \subseteq c!\Lambda$ allows one to define on the set $\Lambda(R)$ a group structure $*$ by setting, for $x, y \in \Lambda(R)$,

$$x * y := F(x, y), \quad x^{-1} = -x.$$

Note that, by the nilpotency of Λ , the Hausdorff formula (4.4) yields an expression for $x * y$ as a linear combination of Lie terms in x and y . In this way one obtains a unipotent group scheme \mathbf{G}_{Λ} over \mathcal{O} , representing the functor $R \mapsto (\Lambda(R), *)$. In nilpotency class $c = 2$, one may define the group scheme \mathbf{G}_{Λ} directly and avoiding the condition $\Lambda' \subseteq c!\Lambda$; cf. [50, Section 2.4].

By taking rational points of \mathbf{G}_{Λ} we obtain a multitude of groups, all originating from the same global Lie lattice Λ . The group $\mathbf{G}_{\Lambda}(\mathcal{O})$ of \mathcal{O} -rational points, for instance, is a \mathcal{T} -group of Hirsch length $\text{rk}_{\mathbb{Z}}(\mathcal{O}) \text{rk}_{\mathcal{O}}(\Lambda)$. By considering the $\mathcal{O}_{\mathfrak{p}}$ -rational points of \mathbf{G}_{Λ} for a nonzero prime ideal \mathfrak{p} of \mathcal{O} , we obtain the nilpotent pro- p group $\mathbf{G}_{\Lambda}(\mathcal{O}_{\mathfrak{p}})$. It is remarkable that many features of the representation growth of groups of the form $\mathbf{G}_{\Lambda}(\mathcal{O}_{\mathfrak{p}})$ only depend on the lattice Λ , and not on the local ring $\mathcal{O}_{\mathfrak{p}}$.

Remark 4.3. We comment on connections between the above construction and the Mal'cev correspondence between \mathcal{T} -groups and nilpotent Lie rings. Starting from a \mathcal{T} -group Γ , there exists a \mathbb{Q} -Lie algebra $\mathcal{L}_{\Gamma}(\mathbb{Q})$ and an injective mapping $\log : \Gamma \rightarrow \mathcal{L}_{\Gamma}(\mathbb{Q})$, such that $\log(\Gamma)$ spans $\mathcal{L}_{\Gamma}(\mathbb{Q})$ over \mathbb{Q} . Whilst $\log(\Gamma)$ needs not, in general, be a Lie lattice inside $\mathcal{L}_{\Gamma}(\mathbb{Q})$, there always exists a subgroup H of Γ of finite index with this property, satisfying $\log(H)' \subseteq c!\log(H)$, where c is the nilpotency class of Γ . Setting $\Lambda = \log(H)$, we recover H as the group of \mathbb{Z} -rational points of \mathbf{G}_{Λ} .

Let now Λ be again a nilpotent \mathcal{O} -Lie lattice of class c , and suppose that $\Lambda' \subseteq c!\Lambda$. Denote by \mathbf{G}_{Λ} the associated unipotent group scheme. For every finite extension L of K , with ring of integers \mathcal{O}_L , we obtain a \mathcal{T} -group $\mathbf{G}_{\Lambda}(\mathcal{O}_L)$ and, for every nonzero prime ideal $\mathfrak{P} \in \text{Spec } \mathcal{O}_L$, a pro- p group $\mathbf{G}_{\Lambda}(\mathcal{O}_{L, \mathfrak{P}})$.

Theorem 4.4. [50] *For every finite extension L of K , with ring of integers \mathcal{O}_L ,*

$$(4.5) \quad \zeta_{\mathbf{G}_{\Lambda}(\mathcal{O}_L)}^{\text{irr}}(s) = \prod_{\mathfrak{P} \in \text{Spec } \mathcal{O}_L} \zeta_{\mathbf{G}_{\Lambda}(\mathcal{O}_{L, \mathfrak{P}})}^{\text{irr}}(s),$$

where, for each prime ideal $\mathfrak{P} \in \text{Spec } \mathcal{O}_L$, the factor $\zeta_{\mathbf{G}_{\Lambda}(\mathcal{O}_{L, \mathfrak{P}})}^{\text{irr}}(s)$ enumerates the continuous finite-dimensional irreducible representations of $\mathbf{G}_{\Lambda}(\mathcal{O}_{L, \mathfrak{P}})$ up to twisting by continuous one-dimensional representations. Moreover, the following hold.

(1) *For each rational prime p ,*

$$\zeta_{\mathbf{G}_{\Lambda}(\mathcal{O}_L, p)}^{\text{irr}}(s) = \prod_{\mathfrak{P} | p\mathcal{O}} \zeta_{\mathbf{G}_{\Lambda}(\mathcal{O}_{L, \mathfrak{P}})}^{\text{irr}}(s).$$

(2) *There exists a finite subset $S \subset \text{Spec } \mathcal{O}$, an integer $t \in \mathbb{N}$, and a rational function $R(X_1, \dots, X_t, Y) \in \mathbb{Q}(X_1, \dots, X_t, Y)$ such that, for every prime ideal $\mathfrak{p} \notin S$, the*

following holds. There exist algebraic integers $\lambda_1, \dots, \lambda_t$, depending on \mathfrak{p} , such that, for all finite extensions \mathfrak{D} of $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$,

$$(4.6) \quad \zeta_{\mathbf{G}_{\Lambda}(\mathfrak{D})}^{\widetilde{\text{irr}}}(s) = R(\lambda_1^f, \dots, \lambda_t^f, q^{-fs}),$$

where $q = |\mathcal{O} : \mathfrak{p}|$ and $|\mathcal{O}_L : \mathfrak{P}| = q^f$.

(3) Setting $d = \dim_K(\Lambda' \otimes_{\mathcal{O}} K)$, the following functional equation holds:

$$(4.7) \quad \zeta_{\mathbf{G}_{\Lambda}(\mathfrak{D})}^{\widetilde{\text{irr}}}(s) \Big|_{\substack{q \rightarrow q^{-1} \\ \lambda_i \rightarrow \lambda_i^{-1}}} = q^{fd} \zeta_{\mathbf{G}_{\Lambda}(\mathfrak{D})}^{\widetilde{\text{irr}}}(s).$$

As a corollary, we obtain that $\zeta_{\mathbf{G}_{\Lambda}(\mathfrak{D})}^{\widetilde{\text{irr}}}(s)$ is rational in q^{-fs} . In particular, the dimensions of the continuous representations of the pro- p group $\mathbf{G}_{\Lambda}(\mathfrak{D})$ are all powers of q^f .

Example 4.2 illustrates Theorem 4.4. Indeed, the Heisenberg group scheme \mathbf{H} is defined over $K = \mathbb{Q}$. We have $d = 1$ and, in (2), we may take $S = \emptyset$, $t = 1$, $R(X, Y) = \frac{1-Y}{1-XY}$ and $\lambda_1 = p$.

We say a few words about the proof of Theorem 4.4, referring to [50] for all details. The Euler factorization (4.5) and the statement (1) follow easily from strong approximation for unipotent groups. The key tool to enumerate the representation zeta functions of pro- p groups like $\mathbf{G}_{\Lambda}(\mathfrak{D})$ is the Kirillov orbit method. Wherever this method is applicable, it parametrizes the irreducible representations of a group in terms of the co-adjoint orbits in the Pontryagin dual of a corresponding Lie algebra. In the case at hand, it reduces the problem of enumerating twist-isoclasses of continuous finite-dimensional irreducible representations of groups of the form $\mathbf{G}_{\Lambda}(\mathfrak{D})$ to that of enumerating certain orbits in the duals of the derived \mathfrak{D} -Lie lattices $\Lambda(\mathfrak{D})' = (\Lambda \otimes_{\mathcal{O}} \mathfrak{D})'$. By translating the latter into the problem of evaluating p -adic integrals, one reduces the problem further to the problem of enumerating p -adic points on certain algebraic varieties, which only depend on Λ . In this way, one can show that there exist finitely many smooth projective varieties V_i defined over \mathcal{O} , and rational functions $W_i(X, Y) \in \mathbb{Q}(X, Y)$, $i = 1, \dots, N$, such that, if \mathfrak{p} avoids a finite set $S \subset \text{Spec } \mathcal{O}$,

$$\zeta_{\mathbf{G}_{\Lambda}(\mathfrak{D})}^{\widetilde{\text{irr}}}(s) = \sum_{i=1}^N |\overline{V}_i(\mathbb{F}_{q^f})| W_i(q^f, q^{-fs}),$$

where \overline{V}_i denotes reduction modulo \mathfrak{p} . By the Weil conjectures there exist, for each $i \in \{1, \dots, N\}$, algebraic integers λ_{ij} , $j = \{0, \dots, 2 \dim V_i\}$, such that

$$|\overline{V}_i(\mathbb{F}_{q^f})| = \sum_{j=0}^{2 \dim V_i} (-1)^j \lambda_{ij}^f$$

and

$$\sum_{j=0}^{2 \dim V_i} (-1)^j \lambda_{ij}^{-f} = q^{f \dim V_i} \sum_{j=0}^{2 \dim V_i} (-1)^j \lambda_{ij}^f.$$

This remarkable symmetry is behind the functional equations for the Hasse-Weil zeta functions of the varieties \overline{V}_i and also functional equations such as (3.9). The rational functions W_i come from the enumeration of rational points of rational polyhedral cones.

Question 4.5. Given \mathbf{G}_{Λ} and \mathcal{O}_L as above. Is the abscissa $\alpha^{\text{irr}}(\mathbf{G}_{\Lambda}(\mathcal{O}_L))$ of $\zeta_{\mathbf{G}_{\Lambda}(\mathcal{O}_L)}^{\widetilde{\text{irr}}}(s)$ always a rational number? Is it independent of L ?

In general, the algebraic varieties V_i are obtained from resolutions of singularities of certain – in general highly singular – varieties, and are difficult to compute explicitly. We give some of the relatively few explicit examples of representation zeta functions of \mathcal{T} -groups we have at the moment.

Example 4.6. Let $d \in \mathbb{N}_{>1}$ and $\mathfrak{f}_{d,2}$ the free nilpotent Lie ring on d generators of nilpotency class 2, of additive rank $d + \binom{d}{2} = \binom{d+1}{2}$. We write $\mathbf{F}_{d,2}$ for the unipotent group scheme $\mathbf{G}_{\mathfrak{f}_{d,2}}$ associated to this \mathbb{Z} -Lie lattice. For $d = 2$ we obtain $\mathbf{F}_{2,2} = \mathbf{H}$, the Heisenberg group scheme. We also recover the free class-2-nilpotent group on d generators as $\mathbf{F}_{d,2}(\mathbb{Z})$. We write $d = 2\lfloor d/2 \rfloor + \varepsilon$ for $\varepsilon \in \{0, 1\}$. The following generalizes (4.3).

Theorem 4.7. [50, Theorem B] *Let \mathcal{O} be the ring of integers of a number field K . Then*

$$\zeta_{\mathbf{F}_{d,2}(\mathcal{O})}^{\text{irr}}(s) = \prod_{i=0}^{\lfloor d/2 \rfloor} \frac{\zeta_K(s - 2(\lfloor d/2 \rfloor + i + \varepsilon) + 1)}{\zeta_K(s - 2i)}.$$

E. Avraham has computed the local factors of the representation zeta function of the groups $\mathbf{F}_{2,3}(\mathcal{O}[\frac{1}{6}])$; see [7]. For further explicit examples of representation zeta functions of \mathcal{T} -groups see [23, 48].

4.2. Arithmetic lattices in semisimple groups. Let S be a finite set of places of a number field K , including all archimedean ones, and let \mathcal{O}_S denote the S -integers of K . Let further \mathbf{G} be an affine group scheme over \mathcal{O}_S whose generic fibre is connected, simply-connected semi-simple algebraic group defined over K , together with a fixed embedding $\mathbf{G} \hookrightarrow \text{GL}_N$ for some $N \in \mathbb{N}$. Let $\Gamma = \mathbf{G}(\mathcal{O}_S)$. Then Γ has polynomial representation growth if and only if Γ has the weak Congruence Subgroup Property, i.e. the congruence kernel, that is the kernel of the natural surjection

$$(4.8) \quad \widehat{\mathbf{G}(\mathcal{O}_S)} \rightarrow \mathbf{G}(\widehat{\mathcal{O}_S}) \cong \prod_{\mathfrak{p} \in (\text{Spec } \mathcal{O}) \setminus S} \mathbf{G}(\mathcal{O}_{\mathfrak{p}}),$$

is finite; cf. [38]. Here $\mathbf{G}(\widehat{\mathcal{O}_S})$ denotes the congruence completion of $\mathbf{G}(\mathcal{O}_S)$. For simplicity we assume in the sequel that Γ actually has the strong Congruence Subgroup Property, i.e. that the congruence kernel is trivial, so that the surjection (4.8) is an isomorphism. A prototypical example of such a group is the group $\text{SL}_N(\mathbb{Z})$ for $N \geq 3$.

On the level of representation zeta functions, the triviality of the congruence kernel is reflected by an Euler factorization, similar to but different from those previously discussed, be it in the context of subgroup and subring growth or of representation growth of \mathcal{T} -groups. The Euler factorization features two types of factors: the *archimedean factors* are equal to $\zeta_{\mathbf{G}(\mathbb{C})}^{\text{irr}}(s)$, the so-called *Witten zeta function*, that is the Dirichlet generating series enumerating the rational finite-dimensional irreducible complex representations of the algebraic group $\mathbf{G}(\mathbb{C})$. The *non-archimedean factors*, on the other hand, are the representation zeta functions $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}^{\text{irr}}(s)$, where $\mathfrak{p} \notin S$. These Dirichlet generating series enumerate the continuous finite-dimensional irreducible complex representations of the p -adic analytic groups $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$.

Proposition 4.8. [33, Proposition 4.6] *The following Euler factorization holds:*

$$(4.9) \quad \zeta_{\mathbf{G}(\mathcal{O}_S)}^{\text{irr}}(s) = \zeta_{\mathbf{G}(\mathbb{C})}^{\text{irr}}(s)^{|K:\mathbb{Q}|} \prod_{\mathfrak{p} \in (\text{Spec } \mathcal{O}) \setminus S} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}^{\text{irr}}(s).$$

It is a problem of central importance to compute the abscissa of convergence $\alpha(\mathbf{G}(\mathcal{O}_S))$ of the representation zeta function $\zeta_{\mathbf{G}(\mathcal{O}_S)}^{\text{irr}}(s)$. It is known that $\alpha(\mathbf{G}(\mathcal{O}_S))$ is always a rational number; see [2, Theorem 1.2] and compare Question 4.5.

The two types of factors of $\zeta_{\mathbf{G}(\mathcal{O})}^{\text{irr}}(s)$ in (4.9) turn out to have quite distinct flavours. We discuss the archimedean local factors in Section 4.2.1, the non-archimedean local factors in 4.2.2, and return to global zeta functions of arithmetic groups in Section 4.2.3.

4.2.1. Witten zeta functions. In this section let $\Gamma = \mathbf{G}(\mathbb{C})$. For $n \in \mathbb{N}$ we denote by $r_n(\Gamma)$ the number of n -dimensional rational, irreducible complex representations of Γ . Let Φ be the root system of \mathbf{G} of rank $r = \text{rk}(\Phi)$, let Φ^+ a choice of positive roots of Φ and set $\rho = \sum_{\alpha \in \Phi^+} \alpha$. We write w_1, \dots, w_r for the fundamental weights. The rational irreducible representations of Γ are all of the form W_λ , where $\lambda = \sum_{i=1}^r a_i w_i$ for $a_i \in \mathbb{N}_0$. The Weyl dimension formula asserts that

$$\dim W_\lambda = \prod_{\alpha \in \Phi^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

Note that the numerator is a product of $\kappa := |\Phi^+|$ affine linear functions f_1, \dots, f_κ in the integer coordinates of λ , whilst the denominator $C = \prod_{\alpha \in \Phi^+} \langle \rho, \alpha \rangle$ is a constant depending only on Φ . Thus

$$(4.10) \quad \zeta_\Gamma^{\text{irr}}(s) = \sum_{\lambda} (\dim W_\lambda)^{-s} = C^s \sum_{a \in \mathbb{N}_0^r} \prod_{i=1}^{\kappa} f_i(a)^{-s}.$$

Example 4.9. Assume that \mathbf{G} is of type G_2 . Then $C = 120$, $r = 6$ and we may take

$$\begin{aligned} f_1 = f_2 &= X_1 + 1, & f_3 &= X_1 + X_2 + 2, & f_4 &= X_1 + 2X_2 + 3, \\ f_5 &= X_1 + 3X_2 + 4, & f_6 &= 2X_1 + 3X_2 + 5 \quad . \end{aligned}$$

Theorem 4.10. [33, Theorem 5.1] *The abscissa of convergence of $\zeta_{\mathbf{G}(\mathbb{C})}^{\text{irr}}(s)$ is r/κ .*

Multivariable generalisations of zeta functions like (4.10) have been considered by Matsumoto ([40]), among others. Functions of the form

$$\zeta(s_1, \dots, s_r; \mathbf{G}) = \sum_{a \in \mathbb{N}_0^r} \prod_{i=1}^r f_i(a)^{-s_i},$$

where s_1, \dots, s_r are complex variables, are, in particular, known to have meromorphic continuation to the whole complex plane; cf. [40, Theorem 3].

Special values of Witten zeta functions are interpretable as volumes of moduli spaces of certain vector bundles; cf. [54, Section 7] and [53]. From (4.10), Zagier deduces

Theorem 4.11. [54] *If $s \in 2\mathbb{N}$, then $\zeta_\Gamma^{\text{irr}}(s) \in \mathbb{Q}\pi^{\kappa s}$.*

4.2.2. Representation zeta functions of compact p -adic analytic groups. Let Γ be a profinite group. For $n \in \mathbb{N}$ we denote by $r_n(\Gamma)$ the number of continuous finite-dimensional irreducible complex representations of Γ . If Γ is finitely generated, then $r_n(\Gamma)$ is finite for all $n \in \mathbb{N}$ if and only if Γ is FAb, i.e. has the property that every open subgroup of Γ has finite abelianization.

Theorem 4.12. [29, Theorem 1] *Let p be an odd prime and Γ a FAb compact p -adic analytic group. Then there are natural numbers n_1, \dots, n_k and rational functions $W_1(Y), \dots, W_k(Y) \in \mathbb{Q}(Y)$ such that*

$$(4.11) \quad \zeta_\Gamma^{\text{irr}}(s) = \sum_{i=1}^k n_i^{-s} W_i(p^{-s}).$$

Example 4.13. Let R be a compact discrete valuation ring whose (finite) residue field \mathbb{F}_q has odd characteristic. The representation zeta function of the group $\mathrm{SL}_2(R)$ was computed in [29, Section 7]:

$$(4.12) \quad \zeta_{\mathrm{SL}_2(R)}^{\mathrm{irr}}(s) = \zeta_{\mathrm{SL}_2(\mathbb{F}_q)}^{\mathrm{irr}}(s) + \frac{4q \left(\frac{q^2-1}{2}\right)^{-s} + \frac{q^2-1}{2}(q^2-q)^{-s} + \frac{(q-1)^2}{2}(q^2+q)^{-s}}{1-q^{1-s}},$$

where

$$\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}^{\mathrm{irr}}(s) = 1 + q^{-s} + \frac{q-3}{2}(q+1)^{-s} + \frac{q-1}{2}(q-1)^{-s} + 2 \left(\frac{q+1}{2}\right)^{-s} + 2 \left(\frac{q-1}{2}\right)^{-s}$$

is the representation zeta function of the finite group of Lie type $\mathrm{SL}_2(\mathbb{F}_q)$.

If R is a finite extension of \mathbb{Z}_p , the ring of p -adic integers, then (4.12) illustrates (4.11). It is remarkable that the same formula applies in the characteristic p case, that is if $R = \mathbb{F}_q[[X]]$, the ring of formal power series over \mathbb{F}_q .

The proof of Theorem 4.12 utilizes the fact that a FAb compact p -adic analytic group Γ is virtually pro- p : it has an open normal subgroup N which one may assume to be uniformly powerful. The Kirillov orbit method for uniformly powerful groups and methods from model theory and the theory of definable p -adic integration may be used to describe the distribution of the representations of N . Clifford theory is then applied to extend the analysis for N to an analysis for Γ . The integers n_1, \dots, n_k are closely related to the dimensions of the irreducible representations of the finite group Γ/N .

Computing zeta functions of FAb compact p -adic analytic groups – such as the groups $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$ in (4.8) – explicitly is in general very difficult. The situation is more tractable for pro- p groups. Theorem 4.12 states that if Γ is a FAb compact p -adic analytic pro- p group, then $\zeta_{\Gamma}^{\mathrm{irr}}(s)$ is rational in p^{-s} . That this generating function is a power series in p^{-s} is obvious. Indeed, the irreducible *continuous* representations of a pro- p group Γ all have p -power dimensions, as they factorize over finite quotients of Γ , which are all finite p -groups.

Representation zeta functions of pro- p groups for which a version of the Kirillov orbit method is available may be computed in terms of p -adic integrals associated to polynomial mappings; see [4, Part 1] for details. These integrals are of a much simpler type than the general definable integrals used in the proof of Theorem 4.12. In the following we discuss some cases where this approach allows for an explicit computation of representation zeta functions.

We concentrate on groups of the form $\mathbf{G}(\mathfrak{o})$, where \mathfrak{o} is a finite extension of $\mathcal{O}_{\mathfrak{p}}$ for some $\mathfrak{p} \in (\mathrm{Spec} \mathcal{O}) \setminus S$. Then \mathfrak{o} is a compact discrete valuation ring of characteristic 0, with maximal ideal \mathfrak{m} , say, and finite residue field of characteristic p , where $\mathfrak{p}|p\mathcal{O}$. For $m \in \mathbb{N}$ we consider the m -th principal congruence subgroup $\mathbf{G}^m(\mathfrak{o})$, that is the kernel of the natural surjection

$$\mathbf{G}(\mathfrak{o}) \rightarrow \mathbf{G}(\mathfrak{o}/\mathfrak{m}^m).$$

The groups $\mathbf{G}^m(\mathfrak{o})$ are FAb p -adic analytic pro- p groups and, for sufficiently large $m \in \mathbb{N}$, the Kirillov orbit method is applicable. This follows from the fact that the groups $\mathbf{G}^m(\mathfrak{o})$ are saturable and potent for $m \gg 0$; cf. [4, Proposition 2.3] and [25]. (In fact, if \mathfrak{o} is an unramified extension of \mathbb{Z}_p , then $m = 1$ suffices.) One would like to understand the representation zeta functions $\zeta_{\mathbf{G}^m(\mathfrak{o})}^{\mathrm{irr}}(s)$, and their variation with

- the prime ideal $\mathfrak{p} \in (\mathrm{Spec} \mathcal{O}) \setminus S$,
- the ring extension \mathfrak{o} , and
- the congruence level $m \in \mathbb{N}$.

The following result achieves much of this for the special linear groups $\mathrm{SL}_3(\mathfrak{o})$ and the special unitary groups $\mathrm{SU}_3(\mathfrak{o})$, assuming that $p \neq 3$. Here, the special unitary groups $\mathrm{SU}_3(\mathfrak{o})$ are defined in terms of the nontrivial Galois automorphism of the unramified quadratic extension of the field of fractions of \mathfrak{o} ; see [4, Section 6] for details.

Theorem 4.14. [4, Theorem E] *Let \mathfrak{o} be a compact discrete valuation ring of characteristic 0 whose residue field has cardinality q and characteristic different from 3. Let $\mathbf{G}(\mathfrak{o})$ be either $\mathrm{SL}_3(\mathfrak{o})$ or $\mathrm{SU}_3(\mathfrak{o})$. Then, for all sufficiently large $m \in \mathbb{N}$,*

$$(4.13) \quad \zeta_{\mathbf{G}^m(\mathfrak{o})}^{\mathrm{irr}}(s) = q^{8m} \frac{1 + u(q)q^{-3-2s} + u(q^{-1})q^{-2-3s} + q^{-5-5s}}{(1 - q^{1-2s})(1 - q^{2-3s})},$$

where

$$u(X) = \begin{cases} X^3 + X^2 - X - 1 - X^{-1} & \text{if } \mathbf{G}(\mathfrak{o}) = \mathrm{SL}_3(\mathfrak{o}), \\ -X^3 + X^2 - X + 1 - X^{-1} & \text{if } \mathbf{G}(\mathfrak{o}) = \mathrm{SU}_3(\mathfrak{o}). \end{cases}$$

Furthermore, the following functional equation holds:

$$\zeta_{\mathbf{G}^m(\mathfrak{o})}^{\mathrm{irr}}(s) \Big|_{q \rightarrow q^{-1}} = q^{8(1-2m)} \zeta_{\mathbf{G}^m(\mathfrak{o})}^{\mathrm{irr}}(s).$$

Remark 4.15. We note that $\zeta_{\mathbf{G}^m(\mathfrak{o})}^{\mathrm{irr}}(s)$ is a rational function in q^{-s} whose coefficients are given by polynomials in q , that 8 is the dimension of the algebraic group SL_3 , and that $\zeta_{\mathbf{G}^m(\mathfrak{o})}^{\mathrm{irr}}(s)/q^{8m}$ is independent of the congruence level m . Only a few signs in the numerators reflect the difference between special linear and unitary groups.

In general, one can give formulae for the representation zeta functions of groups of the form $\mathbf{G}^m(\mathfrak{o})$ — valid for all sufficiently large m and virtually independent of m — which are uniform both under variation of \mathfrak{p} and \mathfrak{o} , and all but independent of m . More precisely, [4, Theorem A] implies the following result, which in turn generalizes Theorem 4.14.

Theorem 4.16. [4, Theorem A] *There exist a finite subset $T \subset (\mathrm{Spec} \mathcal{O}) \setminus S$, an integer $t \in \mathbb{N}$, and a rational function $R(X_1, \dots, X_t, Y) \in \mathbb{Q}(X_1, \dots, X_t, Y)$ such that, for every prime ideal $\mathfrak{p} \notin S \cup T$, the following holds.*

There exist algebraic integers $\lambda_1, \dots, \lambda_t$, depending on \mathfrak{p} , such that, for all finite extensions \mathfrak{D} of $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$, and all sufficiently large $m \in \mathbb{N}$,

$$(4.14) \quad \zeta_{\mathbf{G}^m(\mathfrak{D})}^{\mathrm{irr}}(s) = q^{fdm} R(\lambda_1^f, \dots, \lambda_t^f, q^{-fs}),$$

where $q = |\mathcal{O} : \mathfrak{p}|$, $|\mathfrak{D} : \mathfrak{P}| = q^f$, and $d = \dim \mathbf{G}$.

Furthermore, the following functional equation holds:

$$(4.15) \quad \zeta_{\mathbf{G}^m(\mathfrak{D})}^{\mathrm{irr}}(s) \Big|_{\substack{q \rightarrow q^{-1} \\ \lambda_i \rightarrow \lambda_i^{-1}}} = q^{fd(1-2m)} \zeta_{\mathbf{G}^m(\mathfrak{D})}^{\mathrm{irr}}(s).$$

We note the close analogy between this result and Theorem 4.4, which it precedes. Generalizing points made in Remark 4.15, we further note that Theorem 4.16 implies that $\zeta_{\mathbf{G}^m(\mathfrak{D})}^{\mathrm{irr}}(s)$ is rational in q^{-fs} and $\zeta_{\mathbf{G}^m(\mathfrak{D})}^{\mathrm{irr}}(s)/q^{fdm}$ is independent of m . In general we do not expect that the coefficients of $\zeta_{\mathbf{G}^m(\mathfrak{D})}^{\mathrm{irr}}(s)$ are given by polynomials in q^f . In fact, as in Theorem 4.4, the algebraic integers λ_i arise from formulae for the numbers of rational points of certain algebraic varieties over finite fields. One may ask, however, whether these numbers are given by polynomials for interesting classes of pro- p groups arising from classical groups, such as groups of the form $\mathrm{SL}_N^m(\mathfrak{o})$.

Question 4.17. Let $N, m \in \mathbb{N}$ and \mathfrak{o} be a compact discrete valuation ring of characteristic 0 whose residue field has cardinality q and characteristic not dividing N . Does there exist a rational function $W_N(X, Y) \in \mathbb{Q}(X, Y)$ such that, for sufficiently large m ,

$$\zeta_{\mathrm{SL}_N^m(\mathfrak{o})}^{\mathrm{irr}}(s) = q^{(N^2-1)m} W_N(q, q^{-s})?$$

The answer is “yes” in case $N = 2$ (cf. [3, Theorem 1.2]) and $N = 3$ (cf. Theorem 4.14).

The striking similarity between the formulae for the representation zeta functions of groups of the form $\mathrm{SL}_3^m(\mathfrak{o})$ and $\mathrm{SU}_3(\mathfrak{o})$ is reminiscent of Ennola duality for the characters of the finite groups $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{GU}_n(\mathbb{F}_q)$; cf. [30]. I am not aware of such a duality in the realm of compact p -adic analytic groups, but read (4.13) as a strong indication for a connection like this.

Computing the representation zeta functions of the “full” p -adic analytic groups $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$ is significantly harder than those of their principal congruence subgroups. In principle, Clifford theory allows one to describe the representations of the former groups in terms of the representations of their open normal subgroups $\mathbf{G}^m(\mathcal{O}_{\mathfrak{p}})$. However, how to tie in explicit Clifford theory with the theory that leads to results like Theorem 4.16 in a way that is uniform in \mathfrak{p} and \mathfrak{o} is not clear in general.

The paper [6] contains formulae for the representation zeta functions of special linear groups of the form $\mathrm{SL}_3(\mathfrak{o})$ and special unitary groups of the form $\mathrm{SU}_3(\mathfrak{o})$, where \mathfrak{o} is an unramified extension of \mathbb{Z}_p and $p \neq 3$. The resulting formulae of the form (4.11) are significantly more complicated than the formulae (4.13) for the principal congruence subgroups, and are omitted here. We just record the fact that

$$(1 - q^{1-2s})(1 - q^{2-3s})$$

is a common denominator for the rational functions involved, just as in (4.13).

It is of great interest if these formulae also apply in characteristic p , i.e. for groups like $\mathrm{SL}_3(\mathbb{F}_q[[X]])$. In contrast to the hands-on computations in [29], the computations in [6] do rely on the Kirillov orbit method for uniformly powerful subgroups of the relevant p -adic analytic groups, which is only available in characteristic 0.

In [6] we also compute the representation zeta functions of finite quotients of groups of the form

$$\begin{aligned} & \mathrm{SL}_3(\mathfrak{o}), \mathrm{SU}_3(\mathfrak{o}), \mathrm{GL}_3(\mathfrak{o}), \mathrm{GU}_3(\mathfrak{o}), \\ & \mathrm{SL}_3^m(\mathfrak{o}), \mathrm{SU}_3^m(\mathfrak{o}), \mathrm{GL}_3^m(\mathfrak{o}), \mathrm{GU}_3^m(\mathfrak{o}) \end{aligned}$$

by principal congruence subgroups, subject to some restrictions on the residue field characteristic p . Some further examples of representation zeta functions of p -adic analytic groups are contained in [3]. Recent results of Aizenbud and Avni bound the abscissa of convergence of zeta functions of groups of the form $\mathrm{SL}_N(\mathfrak{o})$, where \mathfrak{o} is a compact discrete valuation ring of characteristic 0; cf. [1, Theorem A].

We close this section by mentioning a vanishing theorem for representation zeta functions.

Theorem 4.18. [26] *Let p be an odd prime and Γ an infinite FAb compact p -adic analytic group. Then $\zeta_{\Gamma}^{\mathrm{irr}}(-2) = 0$.*

The proof of this result uses the fact that, while the series $\zeta_{\Gamma}^{\mathrm{irr}}(s)$ does not converge in the usual topology for $s \in \mathbb{R}_{<0}$, the expressions $\zeta_{\Gamma}^{\mathrm{irr}}(e)$ do converge in the p -adic topology for all negative integers e .

4.2.3. *Representation zeta functions of arithmetic lattices.* We now return to the global representation zeta function of $\mathbf{G}(\mathcal{O}_S)$.

For the purpose of analyzing $\zeta_{\mathbf{G}(\mathcal{O}_S)}^{\text{irr}}(s)$ via the Euler factorization (4.9), uniform formulae for zeta functions of the form $\zeta_{\mathbf{G}^m(\mathcal{O}_{\mathfrak{p}})}^{\text{irr}}(s)$ — as provided, e.g., by Theorem 4.16 — are of limited value. Indeed, whilst the index of $\mathbf{G}^m(\mathcal{O}_{\mathfrak{p}})$ in $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$ is finite for each \mathfrak{p} and all m , the representation zeta function of every finite index subgroup of $\mathbf{G}(\mathcal{O})$ will share all but finitely many of its non-archimedean factors with those of $\zeta_{\mathbf{G}(\mathcal{O}_S)}^{\text{irr}}(s)$.

Essentially only for groups of type A_2 do we know how to use Clifford effectively to deduce explicit uniform formulae for the representation zeta functions $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}^{\text{irr}}(s)$; cf. [6]. This allows for precise asymptotic results about the representation growth of arithmetic groups of type A_2 .

Theorem 4.19. [6] *Let \mathbf{G} be a connected, simply-connected absolutely simple algebraic group defined over K of type A_2 , and assume that $\Gamma = \mathbf{G}(\mathcal{O}_S)$ has the strong Congruence Subgroup Property. Then $\alpha(\Gamma) = 1$. Moreover, $\zeta_{\Gamma}^{\text{irr}}(s)$ admits meromorphic continuation to $\{s \in \mathbb{C} \mid \Re(s) > 5/6\}$. The continued function is analytic on this half-plane, except for a double pole at $s = 1$. Consequently, there exists a constant $c(\Gamma) \in \mathbb{R}_{>0}$, such that*

$$\sum_{i=1}^n r_i(\Gamma) \sim c(\Gamma) \cdot n \log n.$$

We comment briefly on the proof of Theorem 4.19. Let Γ be as in the theorem. It is a key fact that all but finitely many of the Euler factors of $\zeta_{\Gamma}^{\text{irr}}(s)$ are of the form $\text{SL}_3(\mathcal{O}_{\mathfrak{p}})$ or $\text{SU}_3(\mathcal{O}_{\mathfrak{p}})$, where \mathfrak{p} is a prime ideal of \mathcal{O} . To see that $\alpha(\Gamma) = 1$, it suffices to prove that the abscissa of convergence of the product over these factors is equal to 1. Indeed, [4, Theorem B] implies that the abscissa of convergence of the Euler factorization (4.9) remains unchanged by removing finitely many non-archimedean factors. The archimedean factors' abscissa of convergence is $2/3$; cf. Theorem 4.10. To compute the abscissa of convergence of the Euler factorization of the factors of the form $\text{SL}_3(\mathcal{O}_{\mathfrak{p}})$ or $\text{SU}_3(\mathcal{O}_{\mathfrak{p}})$, one may either inspect the explicit formulae given in [6], or argue with “approximative Clifford theory” as in [4].

The existence of meromorphic continuation is evident from inspection of the explicit formulae for $\zeta_{\text{SL}_3(\mathcal{O}_{\mathfrak{p}})}^{\text{irr}}(s)$ and $\zeta_{\text{SU}_3(\mathcal{O}_{\mathfrak{p}})}^{\text{irr}}(s)$. The key here is that the relevant Euler factorization can be approximated by the following product of translates of (partial) Dedekind zeta functions:

$$(4.16) \quad \zeta_{K,S}(2s-1)\zeta_{K,S}(3s-2) = \prod_{\mathfrak{p} \in (\text{Spec } \mathcal{O}) \setminus S} \frac{1}{(1 - |\mathcal{O} : \mathfrak{p}|^{1-2s})(1 - |\mathcal{O} : \mathfrak{p}|^{2-3s})}.$$

Roughly speaking, dividing $\zeta_{\text{SL}_3(\mathcal{O}_{\mathfrak{p}})}^{\text{irr}}(s)$ or $\zeta_{\text{SU}_3(\mathcal{O}_{\mathfrak{p}})}^{\text{irr}}(s)$ by the appropriate local factor of (4.16) clears their common denominator $(1 - q^{1-2s})(1 - q^{2-3s})$, and the Euler factorization of the remaining numerators converges strictly better than the original Euler factorization.

Theorem 4.19 states, in particular, that the abscissa of convergence of the representation zeta function of an arithmetic group of type A_2 is always equal to 1: the degree of representation growth of very different groups — such as, for example, $\text{SL}_3(\mathcal{O})$ and $\text{SU}_3(\mathcal{O})$, for various number rings \mathcal{O} — only depends on the root system of the underlying algebraic group. This remarkable fact is vastly generalized by the following result.

Theorem 4.20. [5, Theorem 1.1] *Let Φ be an irreducible root system. Then there exists a constant $\alpha_{\Phi} \in \mathbb{Q}$ such that, for every arithmetic group $\mathbf{G}(\mathcal{O}_S)$, where \mathcal{O}_S is the ring*

of S -integers of a number field K with respect to a finite set of places S and \mathbf{G} is a connected absolutely almost simple algebraic group over K with absolute root system Φ , the following holds: if $\mathbf{G}(\mathcal{O}_S)$ has the CSP, then $\alpha(\mathbf{G}(\mathcal{O}_S)) = \alpha_\Phi$.

Theorem 4.20 reduces a conjecture of Larsen and Lubotzky on the invariance of representation growth of lattices in higher rank semisimple locally compact groups to a conjecture of Serre on the CSP; see [5, Theorem 1.3]. A key idea of its proof is to approximate the local factors of the representation zeta function $\zeta_{\mathbf{G}(\mathcal{O}_S)}^{\text{irr}}(s)$ uniformly by certain definable integrals, in a way that leaves the abscissa of convergence unchanged. The proof uses deep, nonconstructive techniques from model theory, which hold little promise to yield an explicit description of the function $\Phi \mapsto \alpha_\Phi$. So far, the only explicitly known values of this function are $\alpha_{A_1} = 2$ and $\alpha_{A_2} = 1$.

Question 4.21. What is the value of α_Φ in Theorem 4.20, for various root systems Φ ?

4.3. Iterated wreath products and branch groups. Let Q be a finite group, acting on a finite set X of cardinality $|X| = d \geq 2$. We define iterated permutational wreath products as follows. Set $W(Q, 0) := \{1\}$ and, for $k \in \mathbb{N}$, set $W(Q, k+1) = W(Q, k) \wr_X Q$. Passing to the inverse limit yields the profinite group $W(Q) := \varprojlim_k W(Q, k)$. Recall that, for a profinite group G , we denote by $r_n(G)$ the number of continuous n -dimensional irreducible complex representations of G up to isomorphism and that G is called rigid if $r_n(G) < \infty$ for all $n \in \mathbb{N}$.

Theorem 4.22. [9] *$W(Q)$ is rigid if and only if the group Q is perfect, i.e. $G = [G, G]$. In this case, the following hold.*

- (1) *The abscissa of convergence $\alpha := \alpha(\zeta_{W(Q)}^{\text{irr}}(s))$ is positive and finite, i.e. $\alpha \in \mathbb{R}_{>0}$.*
- (2) *Locally around α , the function $\zeta_{W(Q)}^{\text{irr}}(s)$ allows for a Puiseux expansion of the form*

$$\sum_{n=0}^{\infty} c_n (s - \alpha)^{n/e}$$

for suitable $c_n \in \mathbb{C}$, $n \in \mathbb{N}$, and $e \in \{2, 3, \dots, d\}$.

- (3) *Let p_1, \dots, p_ℓ denote the primes dividing $|Q|$. There exists a nontrivial polynomial $\Psi \in \mathbb{Q}[X_1, \dots, X_d, Y_1, \dots, Y_\ell]$ such that*

$$(4.17) \quad \Psi(\zeta_{W(Q)}^{\text{irr}}(s), \zeta_{W(Q)}^{\text{irr}}(2s), \dots, \zeta_{W(Q)}^{\text{irr}}(ds), p_1^{-s}, \dots, p_\ell^{-s}) = 0.$$

For examples illustrating in particular the functional equations (4.17), see [9]. For generalizations of these results to self-similar profinite branched groups, see [8].

Acknowledgements. Over several years, my work has been generously supported by numerous funding bodies, including the EPSRC, the DFG, and the Nuffield Foundation. I am also grateful to the organizers of Groups St Andrews 2013 in St Andrews.

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