SIMILARITY CLASSES OF INTEGRAL $\mathfrak{p}$-ADIC MATRICES AND REPRESENTATION ZETA FUNCTIONS OF GROUPS OF TYPE A_2

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Abstract. We compute explicitly Dirichlet generating functions enumerating finite-dimensional irreducible complex representations of various p-adic analytic and adèlic profinite groups of type A_2. This has consequences for the representation zeta functions of arithmetic groups $\Gamma \subset \mathbf{H}(k)$, where k is a number field and $\mathbf{H}$ a k-form of SL_3: assuming that Γ possesses the strong Congruence Subgroup Property, we obtain precise, uniform estimates for the representation growth of Γ. Our results are based on explicit, uniform formulae for the representation zeta functions of the p-adic analytic groups $\mathsf{SL}_3(\mathfrak{o})$ and $\mathsf{SU}_3(\mathfrak{o})$, where $\mathfrak{o}$ is a compact discrete valuation ring of characteristic 0. These formulae build on our classification of similarity classes of integral $\mathfrak{p}$-adic 3×3 matrices in $\mathfrak{gl}_3(\mathfrak{o})$ and $\mathfrak{gu}_3(\mathfrak{o})$, where $\mathfrak{o}$ is a compact discrete valuation ring of arbitrary characteristic. Organising the similarity classes by invariants which we call their *shadows* allows us to combine the Kirillov orbit method with Clifford theory to obtain explicit formulae for representation zeta functions. In a different direction we introduce and compute certain similarity class zeta functions.

Our methods also yield formulae for representation zeta functions of various finite subquotients of groups of the form $\mathsf{SL}_3(\mathfrak{o})$, $\mathsf{SU}_3(\mathfrak{o})$, $\mathsf{GL}_3(\mathfrak{o})$, and $\mathsf{GU}_3(\mathfrak{o})$, arising from the respective congruence filtrations; these formulae are valid in case that the characteristic of $\mathfrak{o}$ is either 0 or sufficiently large. Analysis of some of these formulae leads us to observe p-adic analogues of 'Ennola duality'.

Contents

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1. Introduction and discussion of main results

Let G be a group. For $n \in \mathbb{N}$, let $r_n(G)$ denote the number of n-dimensional irreducible complex representations of G up to equivalence. If G is a topological or an algebraic group, the representations are assumed to be continuous or algebraic, respectively. Following [8], we say that G is (*representation*) *rigid* if $r_n(G)$ is finite for all n. In this case one takes interest in the arithmetic function $n \mapsto r_n(G)$. Examples of such rigid groups include most 'semisimple' arithmetic and compact p-adic analytic groups. The *representation zeta function* of a rigid group G is the Dirichlet generating function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s} \quad (s \in \mathbb{C}). \tag{1.1}$$

The group G is said to have *polynomial representation growth* if the growth of $N \mapsto R_N(G) = \sum_{n=1}^{N} r_n(G)$ is bounded by some polynomial in N. In this case the Dirichlet series $\zeta_G(s)$ converges absolutely in a complex half-plane of the form $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$ for some $\alpha \in \mathbb{R}$. The infimum of such numbers α is called the *abscissa of convergence* of $\zeta_G(s)$ and denoted by $\alpha(G)$. If G admits only finitely many irreducible complex representations, then $\alpha(G) = -\infty$ and $\zeta_G(s)$ is holomorphic on the entire complex plane. Otherwise the abscissa of convergence $\alpha(G)$ satisfies

$$\alpha(G) = \limsup_{N \to \infty} \frac{\log\left(R_N(G)\right)}{\log N}$$

and thus gives the degree of polynomial growth. For a range of results on representation growth and zeta functions of arithmetic and profinite groups see, for instance, [41, 2, 30, 38, 51, 1]. The current paper forms part of a series of papers on representation growth; see [3, 4, 5, 6]. For surveys, see [36, 57].

1.1. **Analytic properties of zeta functions of adèlic and arithmetic groups.** The arithmetic groups considered in this paper are of type A_2 and defined in characteristic 0. Let k be a number field with ring of integers $\mathcal{O}$. Let $\mathbf{H}$ be a connected, simply-connected absolutely almost simple algebraic group defined over k, with a fixed embedding into GL_d for some $d \in \mathbb{N}$. For a place v of k, we write k_v for the completion of k at v and, if v is non-archimedean, $\mathcal{O}_v$ for the completion of $\mathcal{O}$ at v. Let S be a finite set of places of k, including all the archimedean ones, and let $\mathcal{O}_S = \{x \in k \mid \forall v \notin S : x \in \mathcal{O}_v\}$ denote the ring of S-integers in k. The arithmetic group $\mathbf{H}(\mathcal{O}_S) = \mathbf{H}(k) \cap \mathsf{GL}_d(\mathcal{O}_S)$ embeds diagonally into the S-adèle group $\mathbf{H}(\mathcal{A}_{k,S}) = \{(g_v) \in \prod_{v \notin S} \mathbf{H}(k_v) \mid g_v \in \mathbf{H}(\mathcal{O}_v) \text{ for almost all } v\}$. By the Strong Approximation Theorem, the congruence completion of $\mathbf{H}(\mathcal{O}_S)$ coincides with the open compact subgroup $\mathbf{H}(\widehat{\mathcal{O}_S}) \simeq \prod_{v \notin S} \mathbf{H}(\mathcal{O}_v)$ of $\mathbf{H}(\mathcal{A}_{k,S})$.

It was shown in [41] that the congruence completion $\mathbf{H}(\widehat{\mathcal{O}_S})$ has polynomial representation growth. In [5, Theorem C] we quantified this result for groups $\mathbf{H}$ of type A_2: in this case, $\alpha(\mathbf{H}(\widehat{\mathcal{O}_S})) = 1$, in other words the representation growth of $\mathbf{H}(\widehat{\mathcal{O}_S})$ is linear. Our first main result establishes finer asymptotic properties of $\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s)$ for groups $\mathbf{H}$ of type A_2.

Theorem A. *Let $\mathbf{H}(\widehat{\mathcal{O}_S})$ be an adèlic profinite group as above, where the algebraic group $\mathbf{H}$ is connected, simply-connected absolutely almost simple of type A_2.*

(1) *The zeta function $\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s)$ can be meromorphically continued to the complex half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 5/6\}$. The only pole of $\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s)$ in this domain is a double pole at $s = 1$.*

(2) *There exists an invariant* $c(\mathbf{H}(\widehat{\mathcal{O}_S})) \in \mathbb{R}_{>0}$ *such that*

$$c(\mathbf{H}(\widehat{\mathcal{O}_S})) = \lim_{N\to\infty} \frac{R_N(\mathbf{H}(\widehat{\mathcal{O}_S}))}{N \log N}.$$

Remark 1.1. In fact, the proof of Theorem A works for a somewhat larger class of profinite groups, including groups of the form $H = \prod_{v \notin S} H_v$, where H_v is commensurable to a compact open subgroup of an absolutely almost simple k_v-algebraic group $\mathbf{H}_v(k_v)$ of type A_2 for each place v and such that H_v is equal to either $\mathsf{SL}_3(\mathcal{O}_v)$ or $\mathsf{SU}_3(\mathcal{O}_v)$ for almost all v; see Section 7 for details. A precise definition of the standard unitary group $\mathsf{SU}_3(\mathfrak{o})$ over a discrete valuation ring $\mathfrak{o}$ is given in Section 3.

The arithmetic group $\mathbf{H}(\mathcal{O}_S)$ has the *weak Congruence Subgroup Property* (wCSP) if the congruence kernel $\ker(\widehat{\mathbf{H}(\mathcal{O}_S)} \to \mathbf{H}(\widehat{\mathcal{O}_S}))$ of the natural projection from the profinite completion onto the congruence completion is finite. We say that $\mathbf{H}(\mathcal{O}_S)$ has the *strong Congruence Subgroup Property* (sCSP) if this congruence kernel is trivial. For instance, the solution of the congruence subgroup problem for the Chevalley group schemes SL_n, $n \geq 3$, by Bass, Milnor, and Serre [9] implies that the group $\mathsf{SL}_n(\mathcal{O})$ always has the wCSP and that it fails to have the sCSP if and only if k is totally imaginary. Theorem A leads to the following corollary.

Corollary B. *Let* $\mathbf{H}(\mathcal{O}_S)$ *be an arithmetic group as above, where the algebraic group* $\mathbf{H}$ *is connected, simply-connected absolutely almost simple of type* A_2*, and suppose that* $\mathbf{H}(\mathcal{O}_S)$ *has the wCSP. Then* $\mathbf{H}(\mathcal{O}_S)$ *contains a finite index subgroup* Γ *such that the following is true.*

(1) *The zeta function* $\zeta_\Gamma(s)$ *can be meromorphically continued to the complex half-plane* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 5/6\}$. *The only pole of* $\zeta_\Gamma(s)$ *in this domain is a double pole at* $s = 1$.
(2) *There exists* $c(\Gamma) \in \mathbb{R}_{>0}$ *such that*

$$c(\Gamma) = \lim_{N\to\infty} \frac{R_N(\Gamma)}{N \log N}.$$

Moreover, if $\mathbf{H}(\mathcal{O}_S)$ *has the sCSP then one may take* $\Gamma = \mathbf{H}(\mathcal{O}_S)$.

Remark 1.2. Within the special class of groups that it covers, Corollary B goes beyond a conjecture of Larsen and Lubotzky on the degrees of polynomial representation growth of arithmetic lattices in higher rank semisimple groups; see [38, Conjecture 1.5]. In the same way, it refines the variant of this conjecture that was proved in [6, Theorem 1.2]. Indeed, Corollary B asserts that, for the relevant arithmetic groups Γ, not only the degree of representation growth but also the order of the pole of the meromorphically continued function at $s = \alpha(\Gamma)$, and thus the exponent of the log-N-term in (2), are invariants of the type A_2. Likewise meromorphic continuation can be achieved uniformly in a strip of width at least $1/6$. The value of the constant $c(\Gamma)$, in contrast, depends subtly on the specific group Γ; see Section 7 for details.

Furthermore, it is not difficult to extend Theorem A and Corollary B to cover adèlic profinite groups arising from semisimple algebraic groups that are not absolutely almost simple, by using the multiplicativity of the representation zeta function, i.e. $\zeta_{H_1 \times H_2}(s) = \zeta_{H_1}(s)\zeta_{H_2}(s)$, for groups whose categories of finite-dimensional complex representations are semisimple.

For simplicity, consider an arithmetic group of the form $\mathbf{H}(\mathcal{O}_S)$ with the sCSP. A key role in the study of the representation zeta function $\zeta_{\mathbf{H}(\mathcal{O}_S)}(s)$ plays the fact that it admits an Euler product decomposition. Indeed, the triviality of the congruence kernel implies that

$$\zeta_{\mathbf{H}(\mathcal{O}_S)}(s) = \zeta_{\mathbf{H}(\mathbb{C})}(s)^{|k:\mathbb{Q}|} \prod_{v \notin S} \zeta_{\mathbf{H}(\mathcal{O}_v)}(s); \qquad (1.2)$$

see [38, Proposition 1.3]. Here, each archimedean local factor $\zeta_{\mathbf{H}(\mathbb{C})}(s)$, known as the Witten zeta function of the algebraic group $\mathbf{H}(\mathbb{C})$, enumerates the irreducible rational representations of $\mathbf{H}(\mathbb{C})$; see [59]. The non-archimedean local factors $\zeta_{\mathbf{H}(\mathcal{O}_v)}(s)$ are the zeta functions of the rigid compact p-adic analytic groups $\mathbf{H}(\mathcal{O}_v)$. For places v not dividing the prime 2, the representation zeta functions of these groups are given by rational functions; see [30]. If $\mathbf{H}$ is absolutely almost simple of type A_2, then for all but finitely many places v, the groups $\mathbf{H}(\mathcal{O}_v)$ are isomorphic to $\mathsf{SL}_3(\mathcal{O}_v)$ or $\mathsf{SU}_3(\mathcal{O}_v)$; cf. [5, Appendix A]. Our second main result, Theorem C below, describes the rational functions $\zeta_{\mathsf{SL}_3(\mathcal{O}_v)}(s)$ and $\zeta_{\mathsf{SU}_3(\mathcal{O}_v)}(s)$ explicitly, apart from finitely many exceptions; see Corollary D for a concrete, self-contained formula. Our proof of Theorem A is based on an analysis of this formula.

1.2. **Character shadows.** The explicit formulae in Theorem C below, giving the representation zeta functions of various p-adic analytic groups of type A_2 and finite subquotients thereof, are organised in terms of representation-theoretic invariants, which we call shadows, a concept that we now explain.

Let G be a group with a normal subgroup N. Suppose that the category of finite-dimensional complex representations of G, respectively N, is semisimple and that equivalence classes of irreducible finite-dimensional complex representations are parametrised by the corresponding characters. Writing $\mathrm{Irr}(G)$ for the set of irreducible complex characters of G and $\mathrm{CS}(G/N)$ for the set of conjugacy classes of subgroups of G/N, consider the map

$$\mathrm{sh}_{G,N}\colon\ \mathrm{Irr}(G) \to \mathrm{CS}(G/N), \quad \chi \mapsto \{\mathrm{I}_G(\varphi)/N \mid \varphi \text{ an irred. constituent of } \mathrm{Res}^G_N(\chi)\},$$

where $\mathrm{I}_G(\varphi)$ denotes the inertia subgroup of φ in G. We call $\mathrm{sh}_{G,N}(\chi)$ the (*character*) *shadow* of $\chi \in \mathrm{Irr}(G)$ with respect to N. If G is rigid then the map $\mathrm{sh}_{G,N}$ gives rise to a decomposition of the representation zeta function:

$$\zeta_G(s) = \sum_{\sigma \in \mathrm{im}(\mathrm{sh}_{G,N})} \zeta_G^\sigma(s), \qquad \text{where } \zeta_G^\sigma(s) = \sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \mathrm{sh}_{G,N}(\chi)=\sigma}} \chi(1)^{-s}. \tag{1.3}$$

Albeit arguably too general to be of interest for rigid groups at large, this decomposition allows us to give explicit, uniform formulae for the zeta functions of selected classes of groups.

Specifically we consider p-adic analytic groups of the form $G = \mathsf{G}(\mathfrak{o})$, respectively $G = \mathsf{H}(\mathfrak{o})$, where $\mathfrak{o}$ is a compact discrete valuation ring of characteristic 0 with (finite) residue field $\mathbf{k}$ and G is one of the $\mathfrak{o}$-group schemes GL_3 or GU_3, and H is one of SL_3 or SU_3. We set $N = \mathsf{G}^1(\mathfrak{o})$, respectively $N = \mathsf{H}^1(\mathfrak{o})$, the 1st principal congruence subgroup, so that G/N is isomorphic to $\mathsf{G}(\mathbf{k})$ or $\mathsf{H}(\mathbf{k})$. Here, the standard unitary $\mathfrak{o}$-group schemes $\mathsf{GU}_3, \mathsf{SU}_3$ are defined with respect to the standard involution based on the non-trivial Galois automorphism of an unramified quadratic extension of $\mathfrak{o}$; see (3.2) for details. We define

$$\varepsilon = \varepsilon_{\mathsf{G}} = \varepsilon_{\mathsf{H}} = \begin{cases} +1 & \text{if } \mathsf{G} = \mathsf{GL}_3 \text{ and } \mathsf{H} = \mathsf{SL}_3, \\ -1 & \text{if } \mathsf{G} = \mathsf{GU}_3 \text{ and } \mathsf{H} = \mathsf{SU}_3. \end{cases} \tag{1.4}$$

In this setup we describe in an explicit and uniform manner

- the images of the maps $\mathrm{sh}_{G,N}$, i.e. the conjugacy classes of subgroups of the finite groups $\mathsf{G}(\mathbf{k})$ and $\mathsf{H}(\mathbf{k})$ arising as shadows and
- their fibres, i.e. the sets of characters that have a given shadow;

this description is the key to our proof of Theorem C below. More precisely, let

$$\mathbb{T} = \mathbb{T}_{\mathsf{A}_2} = \{\mathcal{G}, \mathcal{L}, \mathcal{J}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{M}, \mathcal{N}, \mathcal{K}_0, \mathcal{K}_\infty\} \tag{1.5}$$

be a set of ten distinct labels, from now referred to as *shadow types*, and set $\mathbb{T}^{(1)} = \mathbb{T}$ and $\mathbb{T}^{(-1)} = \mathbb{T} \smallsetminus \{\mathcal{K}_0, \mathcal{K}_\infty\}$. It turns out that there exist $\mathbf{k}$-forms

$$\mathsf{I}_\varepsilon^{\mathcal{S}} = \mathsf{I}_{\mathsf{A}_2,\varepsilon}^{\mathcal{S}}, \quad \mathcal{S} \in \mathbb{T}^{(\varepsilon)}, \tag{1.6}$$

of algebraic subgroups of GL_3 such that the following hold: if $p := \mathrm{char}(\mathbf{k}) > 3e + 3$, where $e = e(\mathfrak{o}, \mathbb{Z}_p)$ denotes the absolute ramification index of $\mathfrak{o}$, then

- the image of $\mathrm{sh}_{\mathsf{G}(\mathfrak{o}),\mathsf{G}^1(\mathfrak{o})}$ is represented by the groups $\mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k})$, $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$,
- the image of $\mathrm{sh}_{\mathsf{H}(\mathfrak{o}),\mathsf{H}^1(\mathfrak{o})}$ is represented by the groups $\mathsf{H}(\mathbf{k}) \cap \mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k})$, $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$.

In fact, for $\varepsilon = 1$ the groups $\mathsf{I}_\varepsilon^{\mathcal{S}}$ can be defined uniformly over $\mathbb{Z}$, while for $\varepsilon = -1$ the groups $\mathsf{I}_\varepsilon^{\mathcal{S}}$ can be defined uniformly over $\mathbf{k}$ using the Galois automorphism of the quadratic extension $\mathbf{k}_2$ of $\mathbf{k}$. For simplicity, we call the groups $\mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k})$, rather than their respective conjugacy classes, the (*character*) *shadows* of $\mathsf{G}(\mathfrak{o})$. The character shadows $\mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k})$, $\mathcal{S} \in \mathbb{T}^{(\varepsilon)} \smallsetminus \{\mathcal{K}_0, \mathcal{K}_\infty\}$, turn out to be centralisers of elements of $\mathsf{G}(\mathbf{k})$. Table 1.1 lists the isomorphism types of the groups $\mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{K})$, where $\mathbf{K}$ denotes an algebraic closure of $\mathbf{k}$. In the table, Heis stands for the Heisenberg group of upper uni-triangular 3×3 matrices and G_a denotes the additive group. Further details of the character shadows $\mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k})$ and their intersections with $\mathsf{H}(\mathbf{k})$, including their isomorphism types and orders, are compiled in Tables 6.1 and 6.2, where the notation $\mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k}) = \sigma(\mathbf{k})$ and $\mathsf{H}(\mathbf{k}) \cap \mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k}) = \sigma'(\mathbf{k})$ is used; compare Section 1.3. The representation zeta functions of the finite groups $\sigma(\mathbf{k})$ and $\sigma'(\mathbf{k})$ are recorded in Proposition 6.9.

TABLE 1.1. Algebraic groups giving rise to shadows in $\mathsf{G}(\mathbf{k})$ of type $\mathcal{S}$

Type	Isomorphism type of $\mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{K})$ for an algebraic closure $\mathbf{K}$ of $\mathbf{k}$
$\mathcal{G}$	$\mathsf{GL}_3(\mathbf{K})$
$\mathcal{L}$	$\mathsf{GL}_1(\mathbf{K}) \times \mathsf{GL}_2(\mathbf{K})$
$\mathcal{J}$	$\mathsf{Heis}(\mathbf{K}) \rtimes (\mathsf{GL}_1(\mathbf{K}) \times \mathsf{GL}_1(\mathbf{K}))$
$\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$	$\mathsf{GL}_1(\mathbf{K}) \times \mathsf{GL}_1(\mathbf{K}) \times \mathsf{GL}_1(\mathbf{K})$
$\mathcal{M}$	$\mathsf{GL}_1(\mathbf{K}) \times \mathsf{GL}_1(\mathbf{K}) \times \mathsf{G}_\mathrm{a}(\mathbf{K})$
$\mathcal{N}$	$\mathsf{GL}_1(\mathbf{K}) \times \mathsf{G}_\mathrm{a}(\mathbf{K}) \times \mathsf{G}_\mathrm{a}(\mathbf{K})$
$\mathcal{K}_0, \mathcal{K}_\infty$	$\mathsf{GL}_1(\mathbf{K}) \times \mathsf{G}_\mathrm{a}(\mathbf{K}) \times \mathsf{G}_\mathrm{a}(\mathbf{K})$

The same analysis applies, mutatis mutandis, to the character shadows of groups of the form $\mathsf{G}(\mathfrak{o}_\ell)$ and $\mathsf{H}(\mathfrak{o}_\ell)$, $\ell \in \mathbb{N}$, where $\mathfrak{o}_\ell := \mathfrak{o}/\mathfrak{p}^\ell$ with $\mathfrak{p}$ denoting the valuation ideal of $\mathfrak{o}$. Whilst groups of the form $\mathsf{G}(\mathfrak{o})$ are clearly not rigid, the representation zeta functions of their finite quotients $\mathsf{G}(\mathfrak{o}_\ell)$ are of considerable interest. In this situation we may even assume that $\mathfrak{o}$ has positive characteristic, provided that $p = \mathrm{char}(\mathbf{k})$ is large compared to ℓ.

In Proposition 6.1 we introduce, for $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$ and $\ell \in \mathbb{N}_0$, certain Dirichlet polynomials $\Xi_{\varepsilon,q,\ell}^{\mathcal{S}}(s) = \Xi_{\mathsf{A}_2,\varepsilon,q,\ell}^{\mathcal{S}}(s) \in \mathbb{Z}[\frac{1}{6}][q, q^{-s}]$. Their formal limits $\Xi_{\varepsilon,q}^{\mathcal{S}}(s)$ as $\ell \to \infty$ are given in Corollary 6.2. Deferring precise definitions and motivations of these functions for the moment, we now state our second main result.

Theorem C. *Let $\mathfrak{o}$ be a compact discrete valuation ring of residue characteristic $p = \mathrm{char}(\mathbf{k})$. Let G, H be either $\mathsf{GL}_3, \mathsf{SL}_3$ or $\mathsf{GU}_3, \mathsf{SU}_3$ as above and $\ell \in \mathbb{N}$. Assume that $p \geq \min\{3\ell, 3e+3\}$ if*

$\mathrm{char}(\mathfrak{o}) = 0$, *and* $p \geq 3\ell$ *if* $\mathrm{char}(\mathfrak{o}) = p$. *Then the following hold:*

$$\zeta_{\mathsf{G}(\mathfrak{o}_\ell)}(s) = q^{\ell-1} \sum_{\mathcal{S} \in \mathbb{T}^{(\varepsilon)}} [\mathsf{G}(\mathbf{k}) : \mathsf{I}^{\mathcal{S}}_{\varepsilon}(\mathbf{k})]^{-1-s}\, \zeta_{\mathsf{I}^{\mathcal{S}}_{\varepsilon}(\mathbf{k})}(s)\, \Xi^{\mathcal{S}}_{\varepsilon,q,\ell-1}(s), \tag{1.7}$$

$$\zeta_{\mathsf{H}(\mathfrak{o}_\ell)}(s) = \sum_{\mathcal{S} \in \mathbb{T}^{(\varepsilon)}} [\mathsf{H}(\mathbf{k}) : (\mathsf{H}(\mathbf{k}) \cap \mathsf{I}^{\mathcal{S}}_{\varepsilon}(\mathbf{k}))]^{-1-s}\, \zeta_{\mathsf{H}(\mathbf{k}) \cap \mathsf{I}^{\mathcal{S}}_{\varepsilon}(\mathbf{k})}(s)\, \Xi^{\mathcal{S}}_{\varepsilon,q,\ell-1}(s). \tag{1.8}$$

Moreover, if $\mathrm{char}(\mathfrak{o}) = 0$ *and* $p > 3e + 3$, *then*

$$\zeta_{\mathsf{H}(\mathfrak{o})}(s) = \sum_{\mathcal{S} \in \mathbb{T}^{(\varepsilon)}} [\mathsf{H}(\mathbf{k}) : (\mathsf{H}(\mathbf{k}) \cap \mathsf{I}^{\mathcal{S}}_{\varepsilon}(\mathbf{k}))]^{-1-s}\, \zeta_{\mathsf{H}(\mathbf{k}) \cap \mathsf{I}^{\mathcal{S}}_{\varepsilon}(\mathbf{k})}(s)\, \Xi^{\mathcal{S}}_{\varepsilon,q}(s). \tag{1.9}$$

For $\mathrm{char}(\mathfrak{o}) = 0$, we record a self-contained formula for the zeta functions $\zeta_{\mathsf{H}(\mathfrak{o})}(s)$ that can be read off from the structural formulation in (1.9). For this purpose we define

$$\iota(\mathsf{H}, \mathbf{k}) = \iota(\varepsilon, q) = \gcd(q - \varepsilon, 3), \qquad \text{where } q = |\mathbf{k}|. \tag{1.10}$$

Corollary D. *Let* $\mathfrak{o}$, q, H, *and* ε *be as above. Suppose that* $\mathrm{char}(\mathfrak{o}) = 0$ *and* $p > 3e + 3$. *Then*

$$\zeta_{\mathsf{H}(\mathfrak{o})}(s) = \zeta_{\mathsf{H}(\mathbf{k})}(s) + \psi_{\varepsilon,q}(s),$$

where $\zeta_{\mathsf{H}(\mathbf{k})}(s)$ *is the zeta function of the finite group of Lie type* $\mathsf{SL}_3(\mathbf{k})$ *for* $\varepsilon = 1$, *respectively* $\mathsf{SU}_3(\mathbf{k})$ *for* $\varepsilon = -1$, *given by the uniform formula*

$$\begin{aligned}
\zeta_{\mathsf{H}(\mathbf{k})}(s) = {} & 1 + (q^2 + \varepsilon q)^{-s} + (q - 1 - \varepsilon)(q^2 + \varepsilon q + 1)^{-s} \\
& + \tfrac{1}{2}(q^2 - q - 1 + \varepsilon)(q^3 - \varepsilon)^{-s} + q^{-3s} + (q - 1 - \varepsilon)(q^3 + \varepsilon q^2 + q)^{-s} \\
& + \tfrac{1}{3}(q^2 + \varepsilon q - 2)\left((q + \varepsilon)(q - \varepsilon)^2\right)^{-s} + \tfrac{2}{3}\iota(\varepsilon, q)^2 \left((q + \varepsilon)(q - \varepsilon)^2/\iota(\varepsilon, q)\right)^{-s} \\
& + \tfrac{1}{6}(q - \varepsilon)(q - 3 - \varepsilon)\left((q^2 + \varepsilon q + 1)(q + \varepsilon)\right)^{-s} \\
& + \tfrac{1}{3}\iota(\varepsilon, q)^2 \left((q^2 + \varepsilon q + 1)(q + \varepsilon)/\iota(\varepsilon, q)\right)^{-s},
\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}
\psi_{\varepsilon,q}(s) = {} & \tfrac{1}{2}\frac{(q-1)(q-\varepsilon)\left(2 + 2q^{-s} + (q-2)(q+1)^{-s} + q(q-1)^{-s}\right)}{1 - q^{1-2s}}\left(q^2(q^2 + \varepsilon q + 1)\right)^{-s} \\
& + \frac{\left((q-\varepsilon) + (q+\varepsilon)\iota(\varepsilon,q)^2\left((q-\varepsilon)/\iota(\varepsilon,q)\right)^{-s} + (q-1)(q-\varepsilon)q^{-s}\right)}{1 - q^{1-2s}}\left((q^3 - \varepsilon)(q + \varepsilon)\right)^{-s} \\
& + \tfrac{1}{6}\frac{(q-1)(q-\varepsilon)^2\left(q - 2 + 2q^{2-2s} - q^{1-2s}\right)}{(1 - q^{1-2s})(1 - q^{2-3s})}\left(q^3(q^2 + \varepsilon q + 1)(q + \varepsilon)\right)^{-s} \\
& + \tfrac{1}{2}\frac{(q-1)(q^2-1)q(1 - q^{-2s})}{(1 - q^{1-2s})(1 - q^{2-3s})}\left(q^3(q^3 - \varepsilon)\right)^{-s} \\
& + \tfrac{1}{3}\frac{(q^2-1)(q^2 + \varepsilon q + 1)}{1 - q^{2-3s}}\left(q^3(q^2 - 1)(q - \varepsilon)\right)\!)^{-s} \\
& + \frac{(q-1)(q-\varepsilon)q(1 + q^{1-2s})}{(1 - q^{1-2s})(1 - q^{2-3s})}\left(q^2(q^3 - \varepsilon)(q + \varepsilon)\right)^{-s} \\
& + \frac{(\iota(\varepsilon,q)q)^2(1 - q^{-2s})}{(1 - q^{1-2s})(1 - q^{2-3s})}\left(q(q^3 - \varepsilon)(q^2 - 1)/\iota(\varepsilon,q)\right)^{-s} \\
& + (\varepsilon + 1)\frac{(\iota(\varepsilon,q)q^{1-s})^2}{(1 - q^{1-2s})(1 - q^{2-3s})}\left((q^3 - 1)(q^2 - 1)q/\iota(\varepsilon,q)\right)^{-s}.
\end{aligned}$$

Here the order of summation follows the ordering of the shadow types $\mathcal{L}$, $\mathcal{J}$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{M}$, $\mathcal{N}$, *and* $\mathcal{K}_0, \mathcal{K}_\infty$ *in Table 1.1.*

Remark 1.3. Assume that $\mathrm{char}(\mathfrak{o}) = 0$. For general reasons, the zeta functions $\zeta_{\mathsf{H}(\mathfrak{o})}(s)$ vanish at $s = -2$ for $p > 2$; cf. [24, Corollary 2]. Assume further that p is as in Corollary D. Computations with the explicit formulae in Corollary D suggest that then $\zeta_{\mathsf{SL}_3(\mathfrak{o})}(s)$ has no further integral zeros. In contrast, $\zeta_{\mathsf{SU}_3(\mathfrak{o})}(s)$ also vanishes at $s = 0$. In addition, it vanishes at $s = -1$ if and only if $\iota(-1, q) = \gcd(q + 1, 3) = 1$.

We further remark that the special values of the zeta functions of the finite groups $\zeta_{\mathsf{GU}_3(\mathfrak{o}_\ell)}(s)$ – as far as they are given by (1.7) – at $s = -1$, i.e. the sum of the character degrees of these finite groups, yield the number of invertible symmetric matrices in $\mathsf{GU}_3(\mathfrak{o}_\ell)$, viz.

$$\zeta_{\mathsf{GU}_3(\mathfrak{o}_\ell)}(-1) = (1 + q^{-1})(1 + q^{-3})q^{6\ell}.$$

The corresponding assertion for groups of the form $\mathsf{GL}_3(\mathfrak{o}_\ell)$ seems to hold only for $\ell = 1$, i.e. for $\mathsf{GL}_3(\mathbf{k})$. In this case, the phenomenon is a special case of [56, Corollary 5.2], concerning the sums of character degrees of unitary groups of the form $\mathsf{GU}_d(\mathbf{k})$. This result, in turn, is a unitary analogue of results of Gow and Klyachko for groups of the form $\mathsf{GL}_d(\mathbf{k})$; see [56, Section 5.2.2].

Formulae for the representation zeta functions of principal congruence subgroups of the groups considered in Theorem C are provided in Theorem J below.

A key tool in the analysis of zeta functions of groups is the Kirillov orbit method, describing the irreducible characters of suitable pro-p subgroups of p-adic analytic groups such as $\mathsf{G}(\mathfrak{o})$ in terms of co-adjoint orbits in the duals of the corresponding $\mathbb{Z}_p$-Lie lattices; see Section 5 for details. This approach leads naturally to the study of similarity classes of $\mathfrak{p}$-adic matrices, where invariants called *similarity class shadows* – very much analogous to the character shadows of the matrix groups considered in the present section – play an important role, as we explain next.

1.3. **Similarity classes and their shadows.** Let $\mathfrak{o}$ be a compact discrete valuation ring with valuation ideal $\mathfrak{p}$ and residue field $\mathbf{k}$ of cardinality q. We impose no restriction on the characteristic of $\mathfrak{o}$. Recall that $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$ for $\ell \in \mathbb{N}$. The problem of classifying and enumerating *similarity classes* in $\mathsf{Mat}_n(\mathfrak{o}_\ell)$, or equivalently orbits of the adjoint action of $\mathsf{GL}_n(\mathfrak{o})$ on $\mathfrak{gl}_n(\mathfrak{o}_\ell)$, has attracted much attention over the years. In the field case, i.e. for $\ell = 1$, a classification is achieved, for instance, by the Frobenius normal form. For the case $\ell = 2$ see, for example, [31, 49]. In Theorem 2.11 we give a complete and irredundant list of representatives of the similarity classes in $\mathfrak{gl}_3(\mathfrak{o}_\ell)$, for any $\ell \in \mathbb{N}$, building on and refining results from [7]. We also study the analogous problem of classifying and enumerating similarity classes of anti-hermitian integral $\mathfrak{p}$-adic matrices, i.e. orbits of the adjoint action of the unitary group $\mathsf{GU}_n(\mathfrak{o})$ on the unitary Lie lattices $\mathfrak{gu}_n(\mathfrak{o}_\ell)$. Here the residue characteristic of $\mathfrak{o}$ is assumed to be odd, and the relevant objects are defined by means of the non-trivial Galois automorphism of an unramified quadratic extension of $\mathfrak{o}$; see (3.2) for details. In Theorem 3.14 we provide an explicit list of matrices parametrising $\mathsf{GU}_3(\mathfrak{o})$-similarity classes in $\mathfrak{gu}_3(\mathfrak{o}_\ell)$, for any $\ell \in \mathbb{N}$.

1.3.1. *Similarity class shadows.* A fundamental idea of the current paper is to organise similarity classes by invariants called shadows, which we now explain. Given $\ell \in \mathbb{N}$ and $A \in \mathfrak{gl}_n(\mathfrak{o}_\ell)$, the *group centraliser shadow* $\mathrm{sh}_{\mathsf{GL}}(A)$ of A is the image $\overline{\mathrm{C}_{\mathsf{GL}_n(\mathfrak{o})}(A)} \leq \mathsf{GL}_n(\mathbf{k})$ of the centraliser $\mathrm{C}_{\mathsf{GL}_n(\mathfrak{o})}(A)$ under reduction modulo $\mathfrak{p}$. Evidently, similar matrices have conjugate group centraliser shadows. Roughly speaking, the *shadow* $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ *of a similarity class* $\mathcal{C} \subset \mathfrak{gl}_n(\mathfrak{o}_\ell)$ is the conjugacy class of $\mathrm{sh}_{\mathsf{GL}}(A)$ in $\mathsf{GL}_n(\mathbf{k})$, for any $A \in \mathcal{C}$. (More precisely, we also keep track of Lie objects associated to the shadows; see Definition 2.2 and the discussion following it.) We write $\mathfrak{Sh}_{\mathsf{GL}_n(\mathfrak{o})}$ for the set of shadows arising. Similar to the definitions for $\mathfrak{gl}_n(\mathfrak{o})$, we define shadows of similarity classes of anti-hermitian integral $\mathfrak{p}$-adic matrices. Broadly speaking, these may be thought of as conjugacy classes of subgroups in $\mathsf{GU}_n(\mathbf{k})$; see Definition 3.7. We write $\mathfrak{Sh}_{\mathsf{GU}_n(\mathfrak{o})}$ for the set of shadows in the unitary setting.

In order to discuss the general linear and unitary scenarios for type A_{n-1} in parallel, let G be one of the $\mathfrak{o}$-group schemes $\mathsf{GL}_n, \mathsf{GU}_n$ and, accordingly, let g be one of the $\mathfrak{o}$-Lie lattice schemes $\mathsf{gl}_n, \mathsf{gu}_n$. As above, GU_n and gu_n are defined over $\mathfrak{o}$ using the non-trivial Galois automorphism of an unramified quadratic extension of $\mathfrak{o}$. We continue to use the parameter $\varepsilon = \varepsilon_{\mathsf{G}} \in \{1, -1\}$ defined in (1.4). The following questions naturally present themselves:

(1) Describe the set $\mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$ of shadows. How does it vary with the ring $\mathfrak{o}$?
(2) Let $\ell \in \mathbb{N}$ and let $\mathcal{C}$ be a similarity class in $\mathsf{g}(\mathfrak{o}_\ell)$. Which shadows arise among the similarity classes $\widetilde{\mathcal{C}}$ in $\mathsf{g}(\mathfrak{o}_{\ell+1})$ lifting $\mathcal{C}$, and with what multiplicities? What can be said about the cardinalities $|\mathcal{C}|$ and $|\widetilde{\mathcal{C}}|$?

For $n = 3$, i.e. groups and Lie lattices of type A_2, we answer these questions completely. Let us restrict to this setting. Theorems 2.8 and 3.12, two of the paper's main technical results, yield:

(1) The elements of $\mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$ are represented by the groups $\mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k})$, $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$; cf. (1.6).
(2) Let $\sigma, \tau \in \mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$. Given a similarity class $\mathcal{C}$ in $\mathsf{g}(\mathfrak{o}_\ell)$ of shadow σ, the number of similarity classes $\widetilde{\mathcal{C}}$ in $\mathsf{g}(\mathfrak{o}_{\ell+1})$ of shadow τ lifting $\mathcal{C}$ is given by a rational polynomial in q, depending only on the types of σ and τ, but not on ℓ or ε. The quotients $|\widetilde{\mathcal{C}}|/|\mathcal{C}|$ are given by integral polynomials in q, depending only on the types of σ and τ and mildly on ε, but not on ℓ.

Theorem 2.8 and 3.12 deliver these groups and polynomials explicitly; cf. Tables 2.1 and 3.1 for the shadows' isomorphism types and Table 2.2 for the polynomial data.

Our results on shadows unveil a remarkable recursive structure on the collection $\mathcal{Q}_{\mathfrak{o}}^{\mathsf{g}} := \coprod_{\ell \in \mathbb{N}_0} \mathrm{Ad}(\mathsf{G}(\mathfrak{o}))\backslash \mathsf{g}(\mathfrak{o}_\ell)$ of similarity classes over all $\ell \in \mathbb{N}_0$. Indeed, informally speaking we may view $(\mathrm{Ad}(\mathsf{G}(\mathfrak{o}))\backslash \mathsf{g}(\mathfrak{o}_\ell))_{\ell \in \mathbb{N}_0}$ as a memory-less stochastic process with finite state space $\mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$, indexed by $\ell \in \mathbb{N}_0$: in order to enumerate, for instance, similarity classes in $\mathsf{g}(\mathfrak{o}_{\ell+1})$ it suffices to enumerate similarity classes in $\mathsf{g}(\mathfrak{o}_\ell)$, sorted by their shadows, and process the 'transition data' provided by Table 2.2. Formally, we define on $\mathcal{Q}_{\mathfrak{o}}^{\mathsf{g}}$ the structure of an infinite rooted *similarity class tree*; see Definitions 2.1 and 3.6. Remarkably, the tree's structure is completely determined by local branching rules, given by the data provided by Theorems 2.8 and 3.12. This data may also be organised in a finite *shadow graph* $\Gamma^{(\varepsilon)}$ with vertex set $\mathbb{T}^{(\varepsilon)}$; cf. Figure 4.1.

1.3.2. *Enumerating similarity classes.* Our first application of the concept of similarity class shadows is to the enumeration of similarity classes of integral $\mathfrak{p}$-adic 3×3 matrices. As above, let G be one of the $\mathfrak{o}$-group schemes $\mathsf{GL}_3, \mathsf{GU}_3$ and accordingly g one of the Lie lattice schemes $\mathsf{gl}_3, \mathsf{gu}_3$; let $\varepsilon = \varepsilon_{\mathsf{G}} \in \{1, -1\}$ as in (1.4). We write $\mathfrak{Sh}$ for the shadow set $\mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$. In Proposition 4.7 we give explicit formulae for the *partial similarity class zeta functions*

$$\gamma_\ell^\sigma(s) = \sum_{\substack{\mathcal{C} \in \mathrm{Ad}(\mathsf{G}(\mathfrak{o}))\backslash \mathsf{g}(\mathfrak{o}_\ell) \\ \mathrm{sh}(\mathcal{C}) = \sigma}} |\mathcal{C}|^{-s} \quad \text{for } \sigma \in \mathfrak{Sh} \text{ and } \ell \in \mathbb{N}_0,$$

enumerating similarity classes in $\mathsf{g}(\mathfrak{o}_\ell)$ of shadow σ; cf. Definition 4.1.

These formulae and variants thereof appear throughout the paper. Indeed, for our applications to representation zeta functions it is useful to consider the related Dirichlet polynomials

$$\xi_\ell^\sigma(s) = [\mathsf{G}(\mathbf{k}) : \mathsf{I}_\varepsilon^{\mathcal{S}}(\mathbf{k})]^{1+s/2}\, q^{-\ell}\, \gamma_\ell^\sigma(s/2) \quad \text{for } \sigma \in \mathfrak{Sh} \text{ of type } \mathcal{S} \text{ and } \ell \in \mathbb{N}_0;$$

cf. Definition 5.14. In Proposition 6.1 we establish that the Dirichlet polynomials $\xi_\ell^\sigma(s)$ are, in fact, equal to the functions $\Xi_{\varepsilon,q,\ell}^{\mathcal{S}}(s)$ featuring in Theorem C; the proposition provides explicit formulae for these functions.

Our first application, however, of the *similarity class zeta functions*

$$\gamma_\ell(s) := \sum_{\sigma \in \mathfrak{Sh}} \gamma_\ell^\sigma(s)$$

is based on the observation that for, all $\ell \in \mathbb{N}_0$,

$$s_\ell(\mathsf{g}(\mathfrak{o})) := \gamma_\ell(0) = |\mathrm{Ad}(\mathsf{G}(\mathfrak{o}))\backslash \mathsf{g}(\mathfrak{o}_\ell)|$$

is just the total number of similarity classes in $\mathsf{g}(\mathfrak{o}_\ell)$.

Theorem E. *Let $\mathfrak{o}$, G, g, and $\varepsilon = \varepsilon_\mathsf{G}$ be as above; if $\varepsilon = -1$ suppose that $\mathfrak{o}$ has odd residue characteristic. Then*

$$\zeta^{\mathrm{sc}}_{\mathsf{g}(\mathfrak{o})}(s) := \sum_{\ell=0}^{\infty} s_\ell(\mathsf{g}(\mathfrak{o}))q^{-\ell s} = \frac{1+\varepsilon q^{2-2s}}{(1-q^{1-s})(1-q^{2-s})(1-q^{3-s})}. \tag{1.12}$$

For $\varepsilon = 1$, this confirms the relevant part of [7, Theorem 5.2]; for $\varepsilon = -1$ the formula is new. In any case, the local results may be put in an adèlic context as follows. Let k be a number field with ring of integers $\mathcal{O}$. Let $\mathbf{G}$ be one of the k-algebraic groups GL_3 or $\mathsf{GU}_3(K,f)$, where the unitary group $\mathsf{GU}_3(K,f)$ is defined with respect to the standard hermitian form f associated to the non-trivial Galois automorphism of a quadratic extension K of k. Accordingly, let $\mathfrak{g}$ be one of the Lie algebra schemes $\mathfrak{gl}_3$ or $\mathfrak{gu}_3(K,f)$. Put $\varepsilon_\mathbf{G} = 1$ if $\mathbf{G} = \mathsf{GL}_3$, and $\varepsilon_\mathbf{G} = -1$ if $\mathbf{G} = \mathsf{GU}_3(K,f)$. For the ring of S-integers $\mathcal{O}_S$, where S is a finite set of places of k including all the archimedean ones, $\mathcal{V}_k^\infty \subset S$, we consider the Dirichlet series

$$\zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_S)}(s) = \sum_{n=1}^{\infty} s_n(\mathfrak{g}(\mathcal{O}_S))n^{-s} := \sum_{I \lhd \mathcal{O}_S} |\mathrm{Ad}(\mathbf{G}(\widehat{\mathcal{O}_S}))\backslash \mathfrak{g}(\mathcal{O}_S/I)|\ [\mathcal{O}_S : I]^{-s}, \tag{1.13}$$

where – in absence of the strong approximation property for $\mathbf{G}$ – we count adjoint orbits of the congruence completion $\mathbf{G}(\widehat{\mathcal{O}_S}) = \varprojlim_{I \lhd \mathcal{O}_S} \mathbf{G}(\mathcal{O}_S/I)$ rather than $\mathbf{G}(\mathcal{O}_S)$. As $\mathcal{O}_S$ is a Dedekind domain, this Dirichlet series admits the Euler product

$$\zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_S)}(s) = \prod_{v \notin S} \zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_v)}(s). \tag{1.14}$$

Writing $\zeta_k(s) = \prod_{v \notin \mathcal{V}_k^\infty}(1-\mathrm{N}(\mathfrak{p}_v)^{-s})^{-1}$ for the Dedekind zeta function of the number field k, and $\zeta_{k,S}(s) = \prod_{v \notin S}(1-\mathrm{N}(\mathfrak{p}_v)^{-s})^{-1}$ for the same product with the factors indexed by non-archimedean places in S omitted, we obtain the following corollary.

Corollary F. *Let $\mathcal{O}_S \subset k$ and $\mathbf{G}$, $\mathfrak{g}$, $\varepsilon_\mathbf{G}$ be as above; if $\varepsilon_\mathbf{G} = -1$ suppose that S includes all dyadic places of k as well as those places which ramify in the quadratic extension K of k defining $\mathbf{G} = \mathsf{GU}_3(K,f)$. Then*

$$\zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_S)}(s) = \begin{cases} \zeta_{k,S}(2s-2)\ \zeta_{k,S}(4s-4)^{-1} \prod_{i=1}^{3} \zeta_{k,S}(s-i) & \text{if } \varepsilon_\mathbf{G} = 1, \\ \zeta_{k,S}(2s-2)^{-1}\zeta_{K,S}(2s-2)\ \zeta_{k,S}(4s-4)^{-1} \prod_{i=1}^{3} \zeta_{k,S}(s-i) & \text{if } \varepsilon_\mathbf{G} = -1. \end{cases}$$

In particular, there exists an invariant $\delta(\varepsilon_\mathbf{G}, \mathcal{O}_S) \in \mathbb{R}_{>0}$ such that

$$\delta(\varepsilon_\mathbf{G}, \mathcal{O}_S) = \lim_{N \to \infty} \frac{\sum_{n=1}^{N} s_n(\mathfrak{g}(\mathcal{O}_S))}{N^4}.$$

For instance, if $\varepsilon_\mathbf{G} = 1$ and $S = \mathcal{V}_k^\infty$ comprises just the archimedean places of k, then

$$\delta(1, \mathcal{O}) = \frac{\zeta_k(6)\zeta_k(3)\zeta_k(2)}{4\zeta_k(12)}.$$

We briefly return to the local setting. Evaluating $\gamma_\ell(s)$ in $s = 0$ as above means, of course, to disregard most of the information encoded in the similarity class zeta functions. In our second application, we retain this information and define suitable limits (as $\ell \to \infty$) and Euler products, which we now explain. In Proposition 4.2 we verify that the normalised polynomials $q^{-\ell}\gamma_\ell^\sigma(s)$ converge coefficientwise. Apart from the exceptional case that $p = \mathrm{char}(\mathbf{k})$ divides n, the presence of scalar matrices implies that the coefficients of the Dirichlet polynomials $\gamma_\ell^\sigma(s)$ are actually integers divisible by q^ℓ, whence the normalised polynomials $q^{-\ell}\gamma_\ell^\sigma(s)$ have integral coefficients. The limit functions $\lim_{\ell\to\infty} q^{-\ell}\gamma_\ell^\sigma(s)$ are recorded in Corollary 4.8. They, too, may be put in an adèlic context, as follows.

Let k be a number field with ring of integers $\mathcal{O}$. As above, let $\mathbf{G}$ be one of the k-algebraic groups GL_3 or $\mathsf{GU}_3(K, f)$ and, accordingly, let $\mathfrak{g}$ be one of the Lie algebra schemes $\mathfrak{gl}_3$ or $\mathfrak{gu}_3(K, f)$. Let S be a finite set of places of k including all the archimedean ones, $\mathcal{V}_k^\infty \subset S$. For any non-zero ideal $I \lhd \mathcal{O}_S$, consider the normalised Dirichlet generating polynomial

$$Z_{\mathfrak{g}(\mathcal{O}_S/I)}(s) := [\mathcal{O}_S/I]^{-1} \sum_{\mathcal{C}\in \mathrm{Ad}(\mathsf{G}(\widehat{\mathcal{O}_S}))\backslash \mathfrak{g}(\mathcal{O}_S/I)} |\mathcal{C}|^{-s}$$

enumerating similarity classes in $\mathfrak{g}(\mathcal{O}_S/I)$ by their cardinality. We consider the Dirichlet series

$$Z_{\mathfrak{g}(\mathcal{O}_S)}(s) = \sum_{n=1}^{\infty} \mathrm{sim}_n(\mathfrak{g}(\mathcal{O}_S))n^{-s} := \lim_{I\lhd \mathcal{O}_S} Z_{\mathfrak{g}(\mathcal{O}_S/I)}(s). \tag{1.15}$$

Informally speaking, $\mathrm{sim}_n(\mathfrak{g}(\mathcal{O}_S))$ is the number of similarity classes of cardinality n in $\mathfrak{g}(\mathcal{O}_S/I)$ modulo scalars, for ideals I such that the index $[\mathcal{O}_S : I]$ is divisible by a 'relatively large' power of n. By construction, the Dirichlet generating function $Z_{\mathfrak{g}(\mathcal{O}_S)}(s)$ satisfies an Euler product decomposition of the form

$$Z_{\mathfrak{g}(\mathcal{O}_S)}(s) = \prod_{v\notin S} \lim_{\ell\to\infty} q^{-\ell}\gamma_\ell(s). \tag{1.16}$$

Theorem G. *Let $\mathcal{O}_S \subset k$ and $\mathbf{G}$, $\mathfrak{g}$, $\varepsilon_{\mathbf{G}}$ be as above; if $\varepsilon_{\mathbf{G}} = -1$ suppose that S includes all dyadic places of k as well as those places which ramify in the quadratic extension K of k defining $\mathbf{G} = \mathsf{GU}_3(K, f)$. Then the following hold:*

1. *The abscissa of convergence of $Z_{\mathfrak{g}(\mathcal{O}_S)}(s)$ is equal to $1/2$.*
2. *The zeta function $Z_{\mathfrak{g}(\mathcal{O}_S)}(s)$ has meromorphic continuation to the complex half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 2/5\}$. The only pole of $Z_{\mathfrak{g}(\mathcal{O}_S)}(s)$ in this domain is a double pole at $s = 1/2$.*
3. *There exists an invariant $\delta'(\varepsilon_{\mathbf{G}}, \mathcal{O}_S) \in \mathbb{R}_{>0}$ such that*
$$\delta'(\varepsilon_{\mathbf{G}}, \mathcal{O}_S) = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} \mathrm{sim}_n(\mathfrak{g}(\mathcal{O}_S))}{N^{1/2}\log N}.$$

To put Theorem G into perspective, we remark that $Z_{\mathfrak{g}(\mathcal{O}_S)}(s/2)$ can be regarded as an 'approximation' of the non-achimedean part $\prod_{v\notin S} \zeta_{\mathbf{H}(\mathcal{O}_v)}(s)$ of the representation zeta function $\zeta_{\mathbf{H}(\mathcal{O}_S)}(s)$ in (1.2); cf. (1.9). The zeta functions $Z_{\mathfrak{g}(\mathcal{O}_S)}(s)$ may well turn out to be more tractable than representation zeta functions and thus serve as a tool for studying the latter.

1.4. **Character degrees: 'Ennola duality' and estimates.** Our results – or sometimes rather their proofs – have a number of consequences regarding the finer asymptotic and arithmetic properties of character degrees of the groups under consideration.

Given a group G, we denote the collection of its *irreducible character degrees* by

$$\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\},$$

and, for any prime p, we write $\mathrm{cd}(G)_{p'} = \{\chi(1)_{p'} \mid \chi \in \mathrm{Irr}(G)\}$ for the prime-to-p parts of the irreducible character degrees of G. Let $n \in \mathbb{N}$. In the 1960s Ennola observed an intriguing duality between the character tables of the finite groups $\mathsf{GL}_n(\mathbb{F}_q)$ and $\mathsf{GU}_n(\mathbb{F}_q)$. In particular, he noted that there exist a finite index set $I = I(n)$ and polynomials $g_i \in \mathbb{Z}[t]$, $i \in I$, such that

$$\mathrm{cd}(\mathsf{GL}_n(\mathbb{F}_q)) = \{g_i(q) \mid i \in I\} \qquad \text{and} \qquad \mathrm{cd}(\mathsf{GU}_n(\mathbb{F}_q)) = \{(-1)^{\deg g_i} g_i(-q) \mid i \in I\};$$

cf. [25, 20] and [42, Chapter IV, Section 6]. This phenomenon, known as 'Ennola duality', was explained only later, culminating in work by Kawanaka; cf. [34] and also [56]. While we cannot offer an analogous theory for the character degrees of compact p-adic Lie groups $\mathsf{GL}_n(\mathfrak{o})$ and $\mathsf{GU}_n(\mathfrak{o})$, our approach allows us to generalise Ennola's observation as follows.

Let $\mathfrak{o}$ be a compact discrete valuation ring of residue characteristic p and residue cardinality q. If $\mathrm{char}(\mathfrak{o}) = 0$, let $e = e(\mathfrak{o}, \mathbb{Z}_p)$ denote the absolute ramification index of $\mathfrak{o}$.

Theorem H. *Let $\mathfrak{o}$ be as above. Let G be one of the $\mathfrak{o}$-group schemes $\mathsf{GL}_3, \mathsf{GU}_3$, and let $\varepsilon = \varepsilon_{\mathsf{G}} \in \{1, -1\}$ as in (1.4). Let $\ell \in \mathbb{N}$. Suppose that $p \geq \min\{3\ell, 3e+3\}$ if $\mathrm{char}(\mathfrak{o}) = 0$, and $p \geq 3\ell$ if $\mathrm{char}(\mathfrak{o}) = p$. Then the prime-to-$p$ parts of the character degrees of $\mathsf{G}(\mathfrak{o}_\ell)$ are as follows.*

$$\mathrm{cd}(\mathsf{G}(\mathfrak{o}_\ell))_{p'} = \begin{cases} \{1, q+\varepsilon, q^2+\varepsilon q+1, (q+\varepsilon)(q^2+\varepsilon q+1), q^3-\varepsilon, (q-\varepsilon)^2(q+\varepsilon)\} & \text{for } \ell = 1, \\ \mathrm{cd}(\mathsf{G}(\mathfrak{o}_1))_{p'} \cup \{(q^3-\varepsilon)(q^2-1), (q^3-\varepsilon)(q+\varepsilon)\} & \text{for } \ell \geq 2. \end{cases}$$

Furthermore, for all $g \in \mathbb{Z}[t]$ and $\ell \in \mathbb{N}$,

$$g(q) \in \mathrm{cd}(\mathsf{GL}(\mathfrak{o}_\ell)) \quad \text{if and only if} \quad (-1)^{\deg g} g(-q) \in \mathrm{cd}(\mathsf{GU}(\mathfrak{o}_\ell)). \tag{1.17}$$

It is of great interest to determine the precise scope of this phenomenon, in the first place for the groups GL_n and GU_n for $n > 3$; see Section 1.6.5. We remark that whilst the theorem addresses the character degrees' prime-to-p parts, the explicit formulae underpinning its proof would also allow for a uniform, albeit somewhat technical description of the powers of q entering into the character degrees.

Our next main result concerns the character degrees of groups of the form $\mathsf{SL}_3(\mathfrak{o})$ and $\mathsf{SU}_3(\mathfrak{o})$. Let H denote one of the $\mathfrak{o}$-group schemes $\mathsf{SL}_3, \mathsf{SU}_3$. By convention, the *level* of an irreducible character $\chi \in \mathrm{Irr}(\mathsf{H}(\mathfrak{o}))$ is equal to $\ell - 1$, where $\ell \in \mathbb{N}$ is minimal such that χ is trivial on the ℓ-th principal congruence subgroup $\mathsf{H}^\ell(\mathfrak{o})$. The following theorem relates the degree of an irreducible character in $\mathrm{Irr}(\mathsf{H}(\mathfrak{o}))$ to its level.

Theorem I. *There exist absolute constants $C_1, C_2 \in \mathbb{R}_{>0}$ such that the following holds. Let $\mathfrak{o}$ be as above. Let H be one of the $\mathfrak{o}$-group schemes $\mathsf{SL}_3, \mathsf{SU}_3$, and let $\varepsilon = \varepsilon_{\mathsf{H}} \in \{1, -1\}$ as in (1.4). Let $\ell \in \mathbb{N}$. Suppose that $p \geq \min\{3\ell, 3e+3\}$ if $\mathrm{char}(\mathfrak{o}) = 0$, and $p \geq 3\ell$ if $\mathrm{char}(\mathfrak{o}) = p$. For every non-trivial $\chi \in \mathrm{Irr}(\mathsf{H}(\mathfrak{o}))$ of level $\ell - 1$ the degree of χ is bounded by the inequalities:*

$$C_1 q^{2\ell} < \chi(1) < C_2 q^{3\ell}.$$

In fact, our proof of Theorem I yields slightly more precise estimates. The constants C_1 and C_2, for instance, may be taken arbitrarily close to 1 at the cost of excluding finitely many values of q. Note that the groups $\mathsf{GL}_3(\mathfrak{o})$ and $\mathsf{GU}_3(\mathfrak{o})$ have 1-dimensional representations of arbitrary level, namely those factoring through the determinant map. Therefore there is no non-trivial lower bound for the irreducible character degrees of these groups in relation to the level. However, similar considerations as in the proof of Theorem I apply to these groups so that the upper bound holds for them as well. Bounds as in Theorem I are of interest, for instance, in the study of the 'Gelfand-Kirillov dimensions' of admissible smooth complex representations of the locally compact group $\mathsf{H}(\mathfrak{f})$, where $\mathfrak{f}$ denotes the fraction field of $\mathfrak{o}$; cf. [14, Remark 1.19].

1.5. **Principal congruence subgroups.** Finally we record applications to principal congruence subgroups and subquotients defined in terms of the congruence filtration. As above, let $\mathfrak{o}$ denote a compact discrete valuation ring of residue characteristic p and residue cardinality q. If $\mathrm{char}(\mathfrak{o}) = 0$, let $e = e(\mathfrak{o}, \mathbb{Z}_p)$ denote the absolute ramification index of $\mathfrak{o}$. Let G be one of the $\mathfrak{o}$-group schemes $\mathsf{GL}_3, \mathsf{GU}_3$ and, accordingly, let H be one of the $\mathfrak{o}$-group schemes $\mathsf{SL}_3, \mathsf{SU}_3$, the choice being reflected in the value of the parameter $\varepsilon = \varepsilon_{\mathsf{G}} = \varepsilon_{\mathsf{H}} \in \{1, -1\}$; see (1.4). For $m \in \mathbb{N}$, let $\mathsf{G}^m(\mathfrak{o})$ and $\mathsf{H}^m(\mathfrak{o})$ denote the mth principal congruence subgroups of $\mathsf{G}(\mathfrak{o})$ and $\mathsf{H}(\mathfrak{o})$. We put

$$u_\varepsilon(t) = \varepsilon t^3 + t^2 - t - \varepsilon - t^{-1} \in \mathbb{Z}[t, t^{-1}].$$

Our last main result generalises and yields an different approach to [5, Theorem E] which, for $\mathrm{char}(\mathfrak{o}) = 0$ and $p > 3$, implies that for all $m \in \mathbb{N}$ with $m \geq e/(p-2)$,

$$\zeta_{\mathsf{H}^m(\mathfrak{o})}(s) = q^{8m} \frac{1 + u_\varepsilon(q) q^{-3-2s} + u_\varepsilon(q^{-1}) q^{-2-3s} + q^{-5-5s}}{(1 - q^{1-2s})(1 - q^{2-3s})}. \tag{1.18}$$

Recall the notation introduced in Section 1.2, in particular the Dirichlet polynomials $\Xi^{\mathcal{S}}_{\varepsilon,q,\ell}(s)$ and their limits $\Xi^{\mathcal{S}}_{\varepsilon,q}(s)$, first mentioned just before Theorem C.

Theorem J. *Let $\mathfrak{o}$ and G, H, $\varepsilon = \varepsilon_{\mathsf{G}} = \varepsilon_{\mathsf{H}}$ be as above. Let $\ell, m \in \mathbb{N}$ with $\ell \geq m$. Suppose that $p > 3$; suppose further that $m \geq \min\{\ell/p, e/(p-2)\}$ if $\mathrm{char}(\mathfrak{o}) = 0$, and $m \geq \ell/p$ if $\mathrm{char}(\mathfrak{o}) = p$. Then*

$$\zeta_{\mathsf{H}^m(\mathfrak{o})/\mathsf{H}^\ell(\mathfrak{o})}(s) = \begin{cases} q^{8(\ell-m)} & \text{if } \ell \leq 2m, \\ q^{8(m-1)} \sum_{\mathcal{S} \in \mathbb{T}^{(\varepsilon)}} \Xi^{\mathcal{S}}_{\varepsilon,q,\ell-2m+1}(s) & \text{if } \ell > 2m, \end{cases} \tag{1.19}$$

$$\zeta_{\mathsf{G}^m(\mathfrak{o})/\mathsf{G}^\ell(\mathfrak{o})}(s) = q^{\ell-m} \zeta_{\mathsf{H}^m(\mathfrak{o})/\mathsf{H}^\ell(\mathfrak{o})}(s). \tag{1.20}$$

Moreover, if $\mathrm{char}(\mathfrak{o}) = 0$ and $m \geq e/(p-2)$ then

$$\zeta_{\mathsf{H}^m(\mathfrak{o})}(s) = q^{8(m-1)} \sum_{\mathcal{S} \in \mathbb{T}^{(\varepsilon)}} \Xi^{\mathcal{S}}_{\varepsilon,q}(s). \tag{1.21}$$

Remark 1.4. The zeta functions $\zeta_{\mathsf{H}^m(\mathfrak{o})}(s)$ vanish at $s = -2$ for $p > 2$; cf. [24, Corollary 2]. Inspection of the right hand side of (1.18) shows that it vanishes, in addition, at $s = -1$ if $\varepsilon = 1$ and at $s = 0$ if $\varepsilon = -1$, but not vice versa.

1.6. **Outlook and conjectures.** The results discussed above raise many interesting questions. We highlight and discuss some of these.

1.6.1. *Analytic properties of zeta functions of arithmetic groups.* It is of interest to investigate whether the assertions in Corollary B for Γ hold more generally also for arithmetic groups of type A_2 satisfying just the wCSP. Let $\mathbf{H}(\mathcal{O}_S)$ be such a group and $\Gamma \leq \mathbf{H}(\mathcal{O}_S)$ as in the corollary. That the abscissae of convergence of $\zeta_\Gamma(s)$ and $\zeta_{\mathbf{H}(\mathcal{O}_S)}(s)$ coincide is well known (see, for instance, [38, Corollary 4.5]), but we do not know whether they also share the finer analytic properties described in Corollary B, such as meromorphic continuation, pole order et cetera. Note that subgroups of arithmetic groups satisfying the sCSP also satisfy this property.

As we mentioned in Remark 1.2, Corollary B transcends – for the groups it covers – general results for arithmetic groups under base extension. It is interesting to decide whether such uniformity also governs the analytic behaviour of representation zeta functions of arithmetic groups of other types.

1.6.2. *Similarity classes of matrices.* Our results exhibit similarity class shadows and associated combinatorial structures as an effective tool to analyse and enumerate similarity classes of integral $\mathfrak{p}$-adic 3×3 matrices, uniformly in the linear and unitary setting. Of particular relevance is the shadows' capacity to uniformly describe the lifting behaviour of similarity classes in Lie lattices such as $\mathfrak{gl}_3(\mathfrak{o}_\ell)$. It is of great interest to investigate whether shadows of similarity classes in more general Lie lattices, say of type A_{n-1} or the other classical types, also share this feature. The (simpler) case of type A_1 is treated in Appendix A.

Another remarkable fact in type A_2 is that the shadows are represented by a finite number of algebraic subgroups of GL_3; in particular, their number is uniformly bounded independently of the residue cardinality q. We do not know whether this is a general phenomenon, even in type A_{n-1}; for type A_1 see Appendix A. It is worth exploring potential connections between shadows in 'semisimple' Lie lattices and decomposition classes; cf. [12, 13].

The adèlic results Corollary F and Theorem G are phrased in such a way that the relevant Euler products (1.14) and (1.16) extend over places for which our results give precise formulae for the involved Euler factors. It seems reasonable to expect that global features such as the adèlic zeta functions' abscissae of convergence, meromorphic continuation, pole order et cetera remain unchanged in the general case, in which the Euler products are enlarged by finitely many 'exceptional' factors. In particular, it would be of interest to set up a universal $\mathfrak{p}$-adic integration formalism that covers these factors, too; cf. [5, Theorem B] and [10].

1.6.3. *Positive characteristic.* All our local results assume that the discrete valuation ring $\mathfrak{o}$ has characteristic 0 or – in the case of finite groups over rings of the form $\mathfrak{o}_\ell$ – residue characteristic large in comparison to ℓ. This restriction is owed to the limitations of the linearisation techniques we use, which allow us to employ the Kirillov orbit method. It is natural to ask for results in the remaining cases, i.e. in 'small' positive characteristic. There are some indications that the formulae we obtain could – to a large extent – be characteristic-independent, just depending on the residue field.

This is, for instance, the case for the zeta functions of groups of the form $\mathsf{SL}_2(\mathfrak{o})$, where $\mathfrak{o}$ is an arbitrary compact discrete valuation ring of odd residue characteristic. In [30, Section 7], Jaikin-Zapirain computed a uniform formula for the zeta functions of such groups, which only depends on the residue field of $\mathfrak{o}$; see [4, Section 3.4] for a discussion of the case of even residue characteristic. In light of this, it would be interesting to compute, for instance, the zeta functions of groups of the form $\mathsf{SL}_3(\mathbf{k}[\![x]\!])$ and $\mathsf{SU}_3(\mathbf{k}[\![x]\!])$, where $\mathbf{k}$ is a finite field with $\mathrm{char}(\mathbf{k}) \neq 3$, as well as their principal congruence subquotients. We expect that the resulting formulae coincide with those given in Theorem C.

The results in [10, Theorem C] on 'conjugacy class zeta functions' – enumerating the total numbers of irreducible characters of principal congruence quotients, such as $\mathsf{H}(\mathfrak{o}_\ell)$, as opposed to enumerating them by their degrees – also point towards a very general 'characteristic independence' of representation zeta functions associated to suitable group schemes.

1.6.4. *Uniformity.* All the explicit formulae of zeta functions for p-adic analytic groups provided in this paper – notably in Theorems C and J – display a high degree of uniformity in the residue field of the underlying compact discrete valuation ring: the character degrees and their multiplicities for the groups in question are given by (quasi-)polynomials in q, the residue field's cardinality, whose coefficients only depend on the residue class of q modulo some small, well-understood modulus and, possibly, the splitting behavior of the place determined by the local ring in some quadratic extension. We speculate that these features are not specific to type A_2.

Let k be a number field with ring of integers $\mathcal{O}$, and $\mathbf{H}$ a connected, simply-connected semisimple algebraic group defined over k, with a fixed embedding into GL_d for some $d \in \mathbb{N}$. It is natural to ask under which conditions on $\mathbf{H}$ the following uniformity property holds.

Property. *There exist $N \in \mathbb{N}$, finite index sets I and J, polynomials $f_{\tau,i}, g_{\tau,i} \in \mathbb{Q}[t]$ for $(\tau, i) \in \{1, \ldots, N\} \times I$, non-negative integers A_j, B_j for $j \in J$, and a finite set S of places of k, containing all archimedean ones, all depending on $\mathbf{H}$, such that the following holds.*

If v is a place of k not in S and the residue cardinality q_v satisfies $q_v \equiv_N \tau$, then

$$\zeta_{\mathbf{H}(\mathcal{O}_v)}(s) = \frac{\sum_{i \in I} f_{\tau,i}(q_v)\ g_{\tau,i}(q_v)^{-s}}{\prod_{j \in J}(1 - q_v^{A_j - B_j s})}. \tag{1.22}$$

Theorem C establishes that $\mathbf{H} = \mathsf{SL}_3$ has this property. Finite groups of Lie type – giving rise to representations of 'level 0' of $\mathbf{H}(\mathcal{O}_v)$ – satisfy an analogous property; see [40, Theorem 1.7].

In the following we formulate a more specific conjecture on the shape of almost all local factors of an arithmetic group of type A_{n-1}, generalizing Theorem C. Suppose that the group $\mathbf{H}$ is absolutely almost simple of type A_{n-1}. Then $\mathbf{H}$ is either an inner form, i.e. of type ${}^1\mathsf{A}_{n-1}$, arising from a matrix algebra over a central division algebra over k, or an outer form, i.e. of type ${}^2\mathsf{A}_{n-1}$, arising from a matrix algebra over a central division algebra over a quadratic extension K of k, equipped with an involution and with reference to a suitable hermitian form; see [48, Propositions 2.17 and 2.18]. For almost all non-archimedean places v of k, the completed group $\mathbf{H}(\mathcal{O}_v)$ is of the form $\mathsf{SL}_n(\mathcal{O}_v)$ or $\mathsf{SU}_n(\mathcal{O}_v)$; compare [5, Appendix A]. The latter case distinction – which occurs infinitely many often if and only if $\mathbf{H}$ is an outer form – is, for all but finitely many places v of k, described by the *Artin symbol* $\varepsilon(v) = \left(\frac{K \mid k}{v}\right) \in \{1, -1\}$, which dictates whether or not v is decomposed in $K \mid k$. For each non-archimedean place v of k, we set $\left(\frac{k \mid k}{v}\right) = 1$ and we write $\iota(v) := \gcd(q_v - \varepsilon(v), n)$. For $\varepsilon(v) = 1$ the latter gives the number of nth roots of unity in the residue field $\mathbf{k}_v$ of $\mathcal{O}_v$; for $\varepsilon(v) = -1$ it gives the number of norm-1 elements in the residue field extension $\mathbf{K}_w \mid \mathbf{k}_v$, associated to the induced quadratic extension $K_w \mid k_v$, whose order divides n. We write $\mathrm{Div}(n) = \{m \in \mathbb{N} \mid m \mid n\}$ for the set of divisors of n.

Conjecture 1.5. *Let $n \in \mathbb{N}_{\geq 2}$. There exist finite index sets I and J, polynomials $f_{\iota,\varepsilon,i}, g_{\iota,\varepsilon,i} \in \mathbb{Q}[t]$ for $(\iota, \varepsilon, i) \in \mathrm{Div}(n) \times \{1, -1\} \times I$ and non-negative integers A_j, B_j for $j \in J$, such that the following holds.*

Let k be a number field with ring of integers $\mathcal{O}$, and $\mathbf{H}$ a connected, simply-connected absolutely almost simple k-algebraic group of type A_{n-1}. If $\mathbf{H}$ is an outer form, let K denote the quadratic extension of k appearing in the definition of $\mathbf{H}$; if $\mathbf{H}$ is an inner form, put $K = k$. Then there exists a finite set of places S of k, containing all archimedean ones and depending on $\mathbf{H}$, such that for every place v of k not in S,

$$\zeta_{\mathbf{H}(\mathcal{O}_v)}(s) = \frac{\sum_{i \in I} f_{\iota(v),\varepsilon(v),i}(q_v)\ g_{\iota(v),\varepsilon(v),i}(q_v)^{-s}}{\prod_{j \in J}(1 - q_v^{A_j - B_j s})}, \tag{1.23}$$

where q_v denotes the residue cardinality of $\mathcal{O}_v$.

1.6.5. *Ennola duality.* Let $n \in \mathbb{N}$, and let $\mathfrak{o}$ be a compact discrete valuation ring with residue cardinality q. Examples suggest that a dependence of the representation zeta function on the residue class of q modulo N as in (1.22) does not occur for general linear groups. Although the groups $\mathsf{GL}_n(\mathfrak{o})$ do not have convergent zeta functions one may consider the zeta functions of the finite principal congruence quotients $\mathsf{GL}_n(\mathfrak{o}_\ell)$, $\ell \in \mathbb{N}$. In all known cases these zeta functions are uniform in q, i.e. both the occurring character degrees and their multiplicities are given by polynomials in q with constant coefficients. In the case $(n \in \mathbb{N}, \ell = 1)$ this follows from [25], the case ($n \in \mathbb{N}, \ell = 2$) appears in [53], and the case ($n = 2, \ell \in \mathbb{N}$) in [47]. Theorem C confirms that this is the case also for ($n = 3, \ell \in \mathbb{N}$), at least for sufficiently large p; we expect that these restrictions on the primes covered are limitations of the methods we use rather than genuine exceptions. We phrase the following general conjecture.

Conjecture 1.6. *Let $n \in \mathbb{N}$. There exist*

- *a finite index set I,*
- *polynomials $f_i^{(\varepsilon)} \in \mathbb{Z}[\frac{1}{n!}][t]$ and $g_i^{(\varepsilon)} \in \mathbb{Z}[t]$ for $i \in I$ and $\varepsilon \in \{-1, 1\}$,*
- *ascending chains of finite sets $\mathcal{B}_{i,1} \subset \mathcal{B}_{i,2} \subset \ldots \subset \mathbb{N}$ for $i \in I$,*
- *non-negative integers $A_{ij}^{(\varepsilon)}, B_{ij}$ for $(i, j) \in I \times \mathcal{B}_{i,\ell}$ and $\varepsilon \in \{-1, 1\}$*

such that the following hold.

(1) *Let $\mathfrak{o}$ be a compact discrete valuation ring with residue cardinality q, let G be one of the $\mathfrak{o}$-group schemes $\mathsf{GL}_n, \mathsf{GU}_n$, and $\varepsilon = \varepsilon_{\mathsf{G}} \in \{-1, 1\}$ accordingly. For every $\ell \in \mathbb{N}$, the character degrees of the finite group $\mathsf{G}(\mathfrak{o}_\ell)$ are given by*

$$\mathrm{cd}(\mathsf{G}(\mathfrak{o}_\ell)) = \{g_i^{(\varepsilon)}(q)\, q^{B_{ij}} \mid i \in I, j \in \mathcal{B}_{i,\ell}\},$$

and its representation zeta function is given by

$$\zeta_{\mathsf{G}(\mathfrak{o}_\ell)}(s) = \sum_{i \in I} \sum_{j \in \mathcal{B}_{i,\ell}} f_i^{(\varepsilon)}(q)\, q^{A_{ij}^{(\varepsilon)}} \left(g_i^{(\varepsilon)}(q)\, q^{B_{ij}} \right)^{-s}.$$

(2) *Ennola duality holds for character degrees: for all $i \in I$,*

$$g_i^{(-1)}(t) = (-1)^{\deg(g_i^{(1)})} g_i^{(1)}(-t).$$

1.7. **Notation and organisation.** Throughout, $\mathfrak{o}$ denotes a compact discrete valuation ring, with valuation ideal $\mathfrak{p}$ and residue field $\mathbf{k}$. We write $q = |\mathbf{k}|$ and $p = \mathrm{char}(\mathbf{k})$. Often, but not always, we assume $\mathrm{char}(\mathfrak{o}) = 0$. In this case we write $e = e(\mathfrak{o}, \mathbb{Z}_p)$ for the absolute ramification index of $\mathfrak{o}$. Superscripts usually denote cartesian powers. In other contexts they form part of our notation for principal congruence subgroups, powers of maximal ideals, and subgroups generated by powers. Table 1.2 summarises some further frequently used notation.

The paper's broad organisation is as follows. Part 1 relates to questions about the classification and enumeration of similarity classes of integral $\mathfrak{p}$-adic $n \times n$ matrices, with a focus on $n = 3$. The main results are Theorems 2.8 and 3.12, classifying shadows of similarity classes in $\mathfrak{gl}_3(\mathfrak{o}_\ell)$ and $\mathfrak{gu}_3(\mathfrak{o}_\ell)$, respectively. In Section 4 these two results are applied to compute similarity class zeta functions in type A_2.

In Part 2 we apply our results on similarity classes of 3×3 matrices to representation zeta functions of groups of type A_2. To this end, methodology from p-adic Lie theory, the Kirillov orbit method, and Clifford theory are prepared in Section 5. Section 6 contains explicit computations of (local) representation zeta functions of various groups of type A_2. Results on global zeta functions, obtained as Euler products of local ones, are proved in Section 7.

Complementing the paper's main ideas, we collect, in Appendix A, a number of results in type A_1 that are analogous to and easier to derive than those obtained in type A_2.

Table 1.3 collects the locations of the proofs of the main results stated in the introduction.

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Table 1.2. Some frequently used notation.

Notation	Description
G	group scheme GL_n or GU_n, depending on $\varepsilon \in \{1,-1\}$
H	group scheme SL_n or SU_n, depending on $\varepsilon \in \{1,-1\}$
G_ℓ, H_ℓ	$\mathfrak{o}_\ell$-rational points $\mathsf{G}(\mathfrak{o}_\ell)$, $\mathsf{H}(\mathfrak{o}_\ell)$
G^m, H^m	principal congruence subgroups $\mathsf{G}^m(\mathfrak{o})$, $\mathsf{H}^m(\mathfrak{o})$
G^m_ℓ, H^m_ℓ	principal congruence subquotients $\mathsf{G}^m(\mathfrak{o})/\mathsf{G}^\ell(\mathfrak{o})$, $\mathsf{H}^m(\mathfrak{o})/\mathsf{H}^\ell(\mathfrak{o})$
g	Lie lattice scheme gl_n or gu_n, depending on $\varepsilon \in \{1,-1\}$
$\mathfrak{g} = \mathsf{g}(\mathfrak{o}_\ell)$	$\mathfrak{o}_\ell$-rational points $\mathsf{gl}_n(\mathfrak{o}_\ell)$, $\mathsf{gu}_n(\mathfrak{o}_\ell)$
$\mathfrak{g}^m = \mathsf{g}^m(\mathfrak{o})$	principal congruence sublattices $\mathfrak{p}^m\mathsf{gl}_n(\mathfrak{o})$, $\mathfrak{p}^m\mathsf{gu}_n(\mathfrak{o})$
$\mathfrak{g}^m_\ell$	principal congruence subquotient $\mathfrak{g}^m/\mathfrak{g}^\ell$
$\mathfrak{Sh}$	set of shadows $\mathfrak{Sh}_{\mathsf{GL}_3(\mathfrak{o})}$ or $\mathfrak{Sh}_{\mathsf{GU}_3(\mathfrak{o})}$
$\mathbb{T}^{(\varepsilon)}$	set of shadow types for $\varepsilon \in \{1,-1\}$
$\sigma(\mathbf{k}) = \mathsf{l}^{\mathcal{S}}_\varepsilon(\mathbf{k})$	$\mathbf{k}$-rational points of shadow σ of type $\mathcal{S}$ for $\varepsilon \in \{1,-1\}$
$\mathrm{sh}_{\mathsf{GL}}(A)$, $\mathrm{sh}_{\mathsf{GU}}(A)$	group centraliser shadows
$\mathrm{sh}_{\mathsf{gl}}(A)$, $\mathrm{sh}_{\mathsf{gu}}(A)$	Lie centraliser shadows
$\Gamma^{(\varepsilon)}$	shadow graph for $\varepsilon \in \{1,-1\}$
$\mathcal{Q}^{\mathsf{gl}_n}_\mathfrak{o}$, $\mathcal{Q}^{\mathsf{gu}_n}_\mathfrak{o}$	similarity class trees

Table 1.3. Location of proofs.

Result	Proved in Section
Theorem A, Corollary B	7.1
Theorem C	6.2
Corollary D	6.3
Theorem E, Corollary F	4.2
Theorem G	7.2
Theorem H, Theorem I	6.4
Theorem J	6.1

Part 1. Similarity classes of $\mathfrak{p}$-adic matrices

2. Similarity classes of integral $\mathfrak{p}$-adic matrices

Let $\mathfrak{o}$ be a compact discrete valuation ring, with valuation ideal $\mathfrak{p}$ and finite residue field $\mathbf{k}$. Put $p = \mathrm{char}(\mathbf{k})$ and $q = |\mathbf{k}|$. Let π be a fixed uniformiser of $\mathfrak{o}$ so that $\mathfrak{p} = \pi\mathfrak{o}$, and let $v\colon \mathfrak{o} \to \mathbb{Z} \cup \{\infty\}$ denote the valuation map on $\mathfrak{o}$. In this section there is no restriction on either $\mathrm{char}(\mathfrak{o})$ or $\mathrm{char}(\mathbf{k})$. In the simplest cases, $\mathfrak{o}$ is the ring $\mathrm{Witt}(\mathbf{k})$ of Witt vectors over $\mathbf{k}$, i.e. the unique unramified extension of the p-adic integers $\mathbb{Z}_p$ with residue field $\mathbf{k}$, or the ring $\mathbf{k}[\![x]\!]$ of formal power series over $\mathbf{k}$.

For $\ell \in \mathbb{N}_0$ let $\mathfrak{o}_\ell$ denote the finite quotient ring $\mathfrak{o}/\mathfrak{p}^\ell$. Let $n \in \mathbb{N}$, and let $\mathsf{gl}_n(\mathfrak{o}_\ell)$ denote the collection of $n \times n$ matrices over $\mathfrak{o}_\ell$, with the standard structure as an $\mathfrak{o}_\ell$-Lie ring.

Definition 2.1. Let $\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},\ell} = \mathrm{Ad}(\mathsf{GL}_n(\mathfrak{o}))\backslash\mathfrak{gl}_n(\mathfrak{o}_\ell)$ denote the set of similarity classes in $\mathfrak{gl}_n(\mathfrak{o}_\ell)$, that is, orbits in $\mathfrak{gl}_n(\mathfrak{o}_\ell)$ under the adjoint action of $\mathsf{GL}_n(\mathfrak{o})$. We endow

$$\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o}} = \coprod_{\ell=0}^{\infty} \mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},\ell}$$

with the structure of a directed graph, induced by reduction modulo powers of $\mathfrak{p}$, as follows. Vertices $\mathcal{C} \in \mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},\ell}$ and $\widetilde{\mathcal{C}} \in \mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},\ell+1}$ are connected by a directed edge $(\mathcal{C}, \widetilde{\mathcal{C}})$ if the reduction of $\widetilde{\mathcal{C}}$ modulo $\mathfrak{p}^\ell$ is equal to $\mathcal{C}$, and we say that $\widetilde{\mathcal{C}}$ lies above $\mathcal{C}$. In this way $\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o}}$ becomes an infinite rooted tree, its root being the single element $\{0\}$ of $\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},0}$. We refer to $\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o}}$ as the *similarity class tree* in degree n over $\mathfrak{o}$.

The aim in this section is to provide a framework for analysing the structure of $\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o}}$ and to apply this method in the concrete case $n = 3$. In Section 2.1 we study centralisers and introduce the concept of similarity class shadows for arbitrary degree n. In Section 2.2 we specialise to $n = 3$ and state Theorem 2.8, concerning shadows and branching rules, which plays a crucial role in the computation of similarity class and representation zeta functions in the following sections. The purpose of Section 2.3 is to produce a complete set of representatives for the similarity classes of 3×3 matrices over a discrete valuation ring; we emphasise that the ring in question may be of positive characteristic. The proofs are technically involved and may be skipped at first reading. Section 2.4 contains a proof of our main result, Theorem 2.8.

2.1. **Centralisers and shadows.** Let $\ell \in \mathbb{N}_0$ and let $A \in \mathfrak{gl}_n(\mathfrak{o}_\ell)$. The centraliser $\mathrm{C}_{\mathsf{GL}_n(\mathfrak{o})}(A)$ of A in the group $\mathsf{GL}_n(\mathfrak{o})$ is the stabiliser of A under the adjoint action of $\mathsf{GL}_n(\mathfrak{o})$. The centraliser $\mathrm{C}_{\mathfrak{gl}_n(\mathfrak{o})}(A)$ of A in the $\mathfrak{o}$-Lie lattice $\mathfrak{gl}_n(\mathfrak{o})$ is the stabiliser of A under the adjoint action of $\mathfrak{gl}_n(\mathfrak{o})$.

Definition 2.2. Let $\ell \in \mathbb{N}_0$. The *group centraliser shadow* $\mathrm{sh}_{\mathsf{GL}}(A)$ of an element $A \in \mathfrak{gl}_n(\mathfrak{o}_\ell)$ is the image $\overline{\mathrm{C}_{\mathsf{GL}_n(\mathfrak{o})}(A)} \leq \mathsf{GL}_n(\mathbf{k})$ of $\mathrm{C}_{\mathsf{GL}_n(\mathfrak{o})}(A)$ under reduction modulo $\mathfrak{p}$. The *Lie centraliser shadow* $\mathrm{sh}_{\mathfrak{gl}}(A)$ of an element $A \in \mathfrak{gl}_n(\mathfrak{o}_\ell)$ is the image $\overline{\mathrm{C}_{\mathfrak{gl}_n(\mathfrak{o})}(A)} \leq \mathfrak{gl}_n(\mathbf{k})$ of $\mathrm{C}_{\mathfrak{gl}_n(\mathfrak{o})}(A)$ under reduction modulo $\mathfrak{p}$.

For each similarity class $\mathcal{C}$ in $\mathfrak{gl}_n(\mathfrak{o}_\ell)$ we define the (*similarity class*) *shadow*

$$\mathrm{sh}_{\mathsf{GL}}(\mathcal{C}) = \{(\mathrm{sh}_{\mathsf{GL}}(A), \mathrm{sh}_{\mathfrak{gl}}(A)) \mid A \in \mathcal{C}\},$$

of $\mathcal{C}$, and we denote the collection of all shadows by

$$\mathfrak{Sh}_{\mathsf{GL}_n(\mathfrak{o})} = \{\mathrm{sh}_{\mathsf{GL}}(\mathcal{C}) \mid \mathcal{C} \in \mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},\ell} \text{ for some } \ell \in \mathbb{N}_0\}. \tag{2.1}$$

For $\sigma \in \mathfrak{Sh}_{\mathsf{GL}_n(\mathfrak{o})}$ we set

$$\|\sigma\| = |\mathrm{sh}_{\mathsf{GL}}(A)| \qquad \text{and} \qquad \dim(\sigma) = \dim_{\mathbf{k}}(\mathrm{sh}_{\mathfrak{gl}}(A)),$$

where $A \in \mathcal{C} \in \mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},\ell}$, for some $\ell \in \mathbb{N}_0$, with $\sigma = \mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$; furthermore, it is convenient to select one group centraliser shadow $\mathrm{sh}_{\mathsf{GL}}(A)$, where $A \in \mathcal{C} \in \mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},\ell}$, for some $\ell \in \mathbb{N}_0$, with $\sigma = \mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$, and to denote it by $\sigma(\mathbf{k})$. We only use properties of $\sigma(\mathbf{k})$ that are independent of the arbitrary choice involved in its definition.

The definition of a shadow reflects the idea that a shadow is a group object together with an associated Lie structure, independently of the choices for the parameters ℓ, $\mathcal{C}$ and A. Indeed one may think of a shadow σ as a conjugacy class of subgroups of $\mathsf{GL}_n(\mathbf{k})$, represented by $\sigma(\mathbf{k})$, together with corresponding Lie subalgebras of $\mathfrak{gl}_n(\mathbf{k})$. Formally, there is some built-in redundancy. Of course, every shadow σ is completely determined by any of its 'representatives' $(\mathrm{sh}_{\mathsf{GL}}(A), \mathrm{sh}_{\mathfrak{gl}}(A))$. Furthermore, the second coordinate $\mathrm{sh}_{\mathfrak{gl}}(A)$ suffices to pin down σ, as

$\mathrm{sh}_{\mathsf{GL}}(A)$ simply consists of the units of the ring $\mathrm{sh}_{\mathsf{gl}}(A)$. Similarly, the first coordinate $\mathrm{sh}_{\mathsf{GL}}(A)$ determines $\mathrm{sh}_{\mathsf{gl}}(A)$, at least for $q > 2$, by the next lemma.

Lemma 2.3. *Suppose that $q > 2$. Let $\ell \in \mathbb{N}_0$ and $A \in \mathsf{gl}_n(\mathfrak{o}_\ell)$. Then the Lie centraliser shadow $\mathrm{sh}_{\mathsf{gl}}(A)$ is equal to the additive span of the group centraliser shadow $\mathrm{sh}_{\mathsf{GL}}(A)$.*

Proof. We only need to prove that $\mathrm{sh}_{\mathsf{gl}}(A)$ is contained in the additive span of $\mathrm{sh}_{\mathsf{GL}}(A)$, because the other inclusion is clear. Let $\overline{X} \in \mathrm{sh}_{\mathsf{gl}}(A)$ be the image of $X \in \mathrm{C}_{\mathsf{gl}_n(\mathfrak{o})}(A)$. Illustrating the idea for the general case treated below, we observe that under the extra assumption $q > n$ we may choose $a \in \mathfrak{o}$ so that $\overline{X - a\,\mathrm{Id}_n}$ has no zero eigenvalues and therefore $\overline{X} = \overline{X - a\,\mathrm{Id}_n} + \overline{a\,\mathrm{Id}_n}$ lies in the additive span of $\mathrm{sh}_{\mathsf{GL}}(A)$.

In general we may assume, by the Primary Decomposition Theorem, that

$$\overline{X} = \mathrm{diag}(\overline{Y_1}, \ldots, \overline{Y_r}) = \overline{X_1} + \ldots + \overline{X_r},$$

where the $\overline{Y_i}$ are block matrices and $\overline{X_i} = \mathrm{diag}(0, \ldots, 0, \overline{Y_i}, 0, \ldots, 0)$, for $1 \le i \le r$. The blocks correspond to the generalised eigenspaces of $\overline{X}$ or, equivalently, the factors in the factorization $f_{\overline{X}}(t) = \overline{f_X}(t) = \prod_{i=1}^r f_{\overline{X_i}}(t)^{e_i}$ of the characteristic polynomial into a product of pairwise co-prime powers of irreducible polynomials. Applying Hensel's Lemma (cf. [43, Theorem 8.3]) to this factorization, we may lift the decomposition to $\mathsf{gl}_n(\mathfrak{o})$, and hence assume that

$$X = \mathrm{diag}(Y_1, \ldots, Y_r) = X_1 + \ldots + X_r,$$

where the block matrices $Y_i \in \mathsf{gl}_n(\mathfrak{o})$ are lifts of the block matrices $\overline{Y_i}$, and each matrix $X_i = \mathrm{diag}(0, \ldots, 0, Y_i, 0, \ldots, 0)$ is a lift of $\overline{X_i}$, such that

- each X_i is a polynomial expression in X, and hence $\overline{X_i} \in \mathrm{sh}_{\mathsf{gl}}(A)$,
- the minimal polynomial of $\overline{X_i}$ over $\mathbf{k}$ is a power of an irreducible polynomial.

For each $i \in \{1, \ldots, r\}$ the matrix $\overline{Y_i}$ has at most one eigenvalue in $\mathbf{k}$. Since $q > 2$, we may choose $\overline{a_i} \in \mathbf{k} \smallsetminus \{0\}$ so that $\overline{X_i} = \overline{X_i - a_i\,\mathrm{Id}_n} + \overline{a_i\,\mathrm{Id}_n} \in \mathrm{sh}_{\mathsf{GL}}(A) + \mathrm{sh}_{\mathsf{GL}}(A)$. This shows that $\overline{X}$ can be expressed as the sum of at most $2n$ elements from $\mathrm{sh}_{\mathsf{GL}}(A)$. $\square$

The next proposition shows that group centralisers can naturally be identified as groups of $\mathbf{k}$-rational points of certain algebraic groups. For $\mathrm{char}(\mathfrak{o}) = 0$, the Greenberg transform of level ℓ associates to an $\mathfrak{o}_\ell$-scheme X of finite type a $\mathbf{k}$-scheme $\mathcal{X}$ of finite type in such a way that $X(\mathfrak{O}_\ell) \simeq \mathcal{X}(\mathbf{K})$ for unramified finite extensions $\mathfrak{O}$ of $\mathfrak{o}$ with residue field $\mathbf{K}$; the construction makes use of Witt vectors; see [26]. For $\mathrm{char}(\mathfrak{o}) > 0$, the residue field $\mathbf{k}$ can be regarded as a subfield of $\mathfrak{o}_\ell$ and Weil restriction associates, in a similar but simpler way, to any $\mathfrak{o}_\ell$-scheme X a $\mathbf{k}$-scheme $\mathcal{X}$ by 'restriction of scalars'.

Proposition 2.4. *Let $\mathcal{X}^{\mathsf{gl}}_{n,\ell}$ be the Greenberg transform of level ℓ (for $\mathrm{char}(\mathfrak{o}) = 0$) or the Weil restriction (for $\mathrm{char}(\mathfrak{o}) > 0$) of the $\mathfrak{o}_\ell$-scheme gl_n to $\mathbf{k}$-schemes so that $\mathsf{gl}_n(\mathfrak{o}_\ell) \simeq \mathcal{X}^{\mathsf{gl}}_{n,\ell}(\mathbf{k})$. For $A \in \mathsf{gl}_n(\mathfrak{o}_\ell)$ there is a linear subvariety $\mathbf{V} \subset \mathcal{X}^{\mathsf{gl}}_{n,\ell}$ such that*

$$\mathrm{C}_{\mathsf{gl}_n(\mathfrak{o}_\ell)}(A) \simeq \mathbf{V}(\mathbf{k}) \subset \mathcal{X}^{\mathsf{gl}}_{n,\ell}(\mathbf{k}).$$

Furthermore, $\mathbf{V}$ contains a Zariski-open connected algebraic group $\mathbf{C}$ such that

$$\mathrm{C}_{\mathsf{GL}_n(\mathfrak{o}_\ell)}(A) \simeq \mathbf{C}(\mathbf{k}).$$

Proof. Upon choosing an $\mathfrak{o}_\ell$-basis for the $\mathfrak{o}_\ell$-module $\mathsf{gl}_n(\mathfrak{o}_\ell)$ we may identify $\mathsf{gl}_n(\mathfrak{o}_\ell)$ with $\mathfrak{o}_\ell^N$, where $N = n^2$, and there exists a matrix $B \in \mathsf{gl}_N(\mathfrak{o}_\ell)$ representing the $\mathfrak{o}_\ell$-linear map $\mathrm{ad}(A)$. Then we may identify $\mathrm{C}_{\mathsf{gl}_n(\mathfrak{o}_\ell)}(A)$ with the set

$$C(A) = \{\mathbf{x} \in \mathfrak{o}_\ell^N \mid B\mathbf{x} \equiv_{\mathfrak{p}} 0\}.$$

After performing an $\mathfrak{o}_\ell$-linear change of bases, if necessary, we may assume that B is in standard elementary divisor form, that is

$$B = \mathrm{diag}(\pi^{e_1}, \dots, \pi^{e_N})$$

for suitable integers $0 \le e_1 \le e_2 \le \ldots \le e_N \le \ell$. Then

$$C(A) = \{(x_1, \dots, x_N)^{\mathrm{tr}} \in \mathfrak{o}_\ell^N \mid \forall i : v(x_i) \ge \ell - e_i\}.$$

Clearly, this corresponds to the set of $\mathbf{k}$-rational points of a linear subvariety $\mathbf{V} \subset \mathcal{X}^{\mathsf{gl}}_{n,\ell}$. The elements of $C(A)$ in bijection to elements of $\mathrm{C}_{\mathsf{GL}_n(\mathfrak{o}_\ell)}(A)$ are determined by imposing the open condition that $\mathbf{x}$ corresponds to an invertible element in $\mathsf{gl}_n(\mathfrak{o}_\ell)$. This defines a Zariski-open algebraic group $\mathbf{C}$ in $\mathbf{V}$. Being a linear variety, $\mathbf{V}$ is irreducible and hence $\mathbf{C}$ is connected. $\square$

Proposition 2.5. *Let $\sigma, \tau \in \mathfrak{Sh}_{\mathsf{GL}_n(\mathfrak{o})}$. Let $\ell \in \mathbb{N}_0$ and suppose that $\widetilde{\mathcal{C}} \in \mathcal{Q}^{\mathsf{gl}_n}_{\mathfrak{o},\ell+1}$ is a class with $\mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}}) = \tau$ which lies above a class $\mathcal{C} \in \mathcal{Q}^{\mathsf{gl}_n}_{\mathfrak{o},\ell}$ with $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C}) = \sigma$. Then*

$$\frac{|\widetilde{\mathcal{C}}|}{|\mathcal{C}|} = q^{\dim \mathsf{gl}_n - \dim(\sigma)} \frac{\|\sigma\|}{\|\tau\|}.$$

In particular, the ratio $|\widetilde{\mathcal{C}}|/|\mathcal{C}|$ depends only on the shadows σ, τ and not on ℓ, $\mathcal{C}$ or $\widetilde{\mathcal{C}}$.

Proof. Let $A \in \mathcal{C}$, $\widetilde{A} \in \widetilde{\mathcal{C}}$ such that $A \equiv \widetilde{A}$ modulo $\mathfrak{p}^\ell$. If $\ell = 0$, then $\mathcal{C} = \{0\}$ and $\|\sigma\| = |\mathsf{GL}_n(\mathbf{k})|$ so that

$$\frac{|\widetilde{\mathcal{C}}|}{|\mathcal{C}|} = |\widetilde{\mathcal{C}}| = [\mathsf{GL}_n(\mathbf{k}) : \mathrm{C}_{\mathsf{GL}_n(\mathbf{k})}(\widetilde{A})] = \frac{\|\sigma\|}{\|\tau\|}.$$

Now suppose that $\ell \ge 1$, and put $A_{\ell+1} = \widetilde{A}$ and $A_\ell = A$. For $i \in \{\ell, \ell+1\}$ set $C_i = \mathrm{C}_{\mathsf{GL}_n(\mathfrak{o}_i)}(A_i)$. Then $|\mathsf{GL}_n(\mathfrak{o}_{\ell+1})|/|\mathsf{GL}_n(\mathfrak{o}_\ell)| = [\mathsf{GL}_n^\ell(\mathfrak{o}) : \mathsf{GL}_n^{\ell+1}(\mathfrak{o})] = q^{\dim \mathsf{gl}_n}$ implies that

$$\frac{|\widetilde{\mathcal{C}}|}{|\mathcal{C}|} = \frac{[\mathsf{GL}_n(\mathfrak{o}_{\ell+1}) : C_{\ell+1}]}{[\mathsf{GL}_n(\mathfrak{o}_\ell) : C_\ell]} = q^{\dim \mathsf{gl}_n} \frac{|C_\ell|}{|C_{\ell+1}|}. \tag{2.2}$$

Writing $A_{\ell-1}$ for A modulo $\mathfrak{p}^{\ell-1}$ and setting $Z_{i-1} = \mathrm{C}_{\mathsf{gl}_n(\mathfrak{o}_{i-1})}(A_{i-1})$ for $i \in \{\ell, \ell+1\}$, we observe that the reduction map modulo $\mathfrak{p}$ yields exact sequences

$$0 \to Z_{i-1} \cap \mathsf{gl}_n^1(\mathfrak{o}_{i-1}) \to Z_{i-1} \to \mathrm{sh}_{\mathsf{gl}}(A_{i-1}) \to 0,$$

$$1 \to C_i \cap \mathsf{GL}_n^1(\mathfrak{o}_i) \to C_i \to \mathrm{sh}_{\mathsf{GL}}(A_i) \to 1,$$

and that, furthermore, the maps

$$\begin{aligned} \mathsf{gl}_n(\mathfrak{o}) &\to \mathsf{gl}_n^1(\mathfrak{o}), \quad X \mapsto \pi X, \\ \mathsf{gl}_n(\mathfrak{o}) &\to \mathsf{GL}_n^1(\mathfrak{o}), \quad X \mapsto \mathrm{Id}_n + \pi X \end{aligned} \tag{2.3}$$

induce bijections (of sets)

$$Z_{i-1} \simeq Z_i \cap \mathsf{gl}_n^1(\mathfrak{o}_i) \qquad \text{and} \qquad Z_{i-1} \simeq C_i \cap \mathsf{GL}_n^1(\mathfrak{o}_i).$$

Thus we conclude from (2.2) that

$$\begin{aligned}\frac{|\widetilde{\mathcal{C}}|}{|\mathcal{C}|} &= q^{\dim \mathsf{gl}_n} \frac{|C_\ell \cap \mathsf{GL}_n^1(\mathfrak{o}_\ell)|}{|C_{\ell+1} \cap \mathsf{GL}_n^1(\mathfrak{o}_{\ell+1})|} \frac{|\mathrm{sh}_{\mathsf{GL}}(A_\ell)|}{|\mathrm{sh}_{\mathsf{GL}}(A_{\ell+1})|} \\ &= q^{\dim \mathsf{gl}_n} \frac{|Z_{\ell-1}|}{|Z_\ell|} \frac{\|\sigma\|}{\|\tau\|} \\ &= q^{\dim \mathsf{gl}_n} \frac{|Z_\ell \cap \mathsf{gl}_n^1(\mathfrak{o}_\ell)|}{|Z_\ell \cap \mathsf{gl}_n^1(\mathfrak{o}_\ell)||\mathrm{sh}_{\mathsf{gl}}(A_\ell)|} \frac{\|\sigma\|}{\|\tau\|} \\ &= q^{\dim \mathsf{gl}_n - \dim(\sigma)} \frac{\|\sigma\|}{\|\tau\|}. \end{aligned}$$

□

Proposition 2.5 highlights the relevance of the shadows to the computation of the sizes of similarity classes, and motivates the following definition.

Definition 2.6. For $\sigma, \tau \in \mathfrak{Sh}_{\mathsf{GL}_n(\mathfrak{o})}$ let

$$b_{\sigma,\tau}^{(1)}(q) = q^{\dim \mathsf{gl}_n - \dim(\sigma)} \frac{\|\sigma\|}{\|\tau\|}. \tag{2.4}$$

Remark 2.7. The quantities $b_{\sigma,\tau}^{(1)}(q)$ which are relevant for us are the ones where $\sigma = \mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ and $\tau = \mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}})$ with $\widetilde{\mathcal{C}} \in \mathcal{Q}_{\mathfrak{o},\ell+1}^{\mathsf{gl}_n}$ lying above $\mathcal{C} \in \mathcal{Q}_{\mathfrak{o},\ell}^{\mathsf{gl}_n}$. In the case $n = 3$ these turn out to be integral polynomials in q, independent of $\mathfrak{o}$; see Table 2.2. It is an interesting open question whether this is true more generally.

For a uniform treatment of integral $\mathfrak{p}$-adic matrices and anti-hermitian integral $\mathfrak{p}$-adic matrices, leading to a uniform description of the representation zeta functions of general/special linear groups and general/special unitary groups, it is convenient to write $b_{\sigma,\tau}^{(1)}$ and to use $b_{\sigma,\tau}^{(-1)}$ to denote similar polynomials occurring in Section 3, where we treat similarity classes of anti-hermitian matrices. Table 2.2 already incorporates the complementary information from Section 3.

2.2. **Shadows and branching rules for $\mathsf{gl}_3(\mathfrak{o}_\ell)$.** In Table 2.1 we list ten shadows in $\mathfrak{Sh}_{\mathsf{GL}_3(\mathfrak{o})}$, classified by (shadow) types; compare (1.5). For $q > 2$, all but the last two of these types already arise from $\ell = 1$: they are conjugacy classes of centralisers of elements $A \in \mathsf{gl}_3(\mathbf{k})$ and are therefore classified by the shape of the minimal polynomials of such A over $\mathbf{k}$. The last two shadows, of types $\mathcal{K}_0$ and $\mathcal{K}_\infty$, come from $\ell = 2$ and higher: they are the conjugacy classes of the reductions modulo $\mathfrak{p}$ of the centralisers of the matrices $A_0 = \left[\begin{smallmatrix} 0 & \pi & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ and $A_\infty = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \pi & 0 & 0 \end{smallmatrix}\right]$ in $\mathsf{gl}_3(\mathfrak{o}_2)$, as explained below. The types $\mathcal{K}_0$ and $\mathcal{K}_\infty$ can be thought of as 'shears' of the type $\mathcal{N}$; see the proof of Theorem 2.8. The third column of Table 2.1 describes the isomorphism types of the group centraliser shadows as algebraic groups; compare Table 1.1. In the table, we write $\mathbf{k}_2$ and $\mathbf{k}_3$ for the quadratic and cubic extensions of the finite field $\mathbf{k}$. Furthermore, Heis stands for the Heisenberg group of upper uni-triangular 3×3 matrices.

Let us take a closer look at the types $\mathcal{K}_0$ and $\mathcal{K}_\infty$ that do not arise for $\ell = 1$. The group centraliser shadows of the matrices $A_0, A_\infty \in \mathsf{gl}_3(\mathfrak{o}_2)$ are equal to

$$H_0 = \left\{ \left[\begin{smallmatrix} t & 0 & s_1 \\ 0 & t & s_3 \\ 0 & 0 & t \end{smallmatrix}\right] \mid s_1, s_3 \in \mathbf{k} \text{ and } t \in \mathbf{k}^\times \right\} \text{ and } H_\infty = \left\{ \left[\begin{smallmatrix} t & 0 & 0 \\ s_2 & t & s_3 \\ 0 & 0 & t \end{smallmatrix}\right] \mid s_2, s_3 \in \mathbf{k} \text{ and } t \in \mathbf{k}^\times \right\};$$

see Proposition 2.16, where the notation $A_0 = E_2(1, 0, 0, 0, 0)$ and $A_\infty = E_2(\infty, \pi, 0, 0, 0)$ is employed. Comparing orders of groups, we see that the corresponding shadows σ_0 and σ_∞ cannot be of type $\mathcal{G}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ or $\mathcal{M}$. To rule out type $\mathcal{N}$, we observe that $(h - t\,\mathrm{Id}_3)^2 = 0$ for any $h \in H_0 \cup H_\infty$ with diagonal entries t. Thus neither H_0 nor H_∞ contains a matrix whose minimal polynomial is of degree 3 over $\mathbf{k}$. Consequently, σ_0 and σ_∞ cannot be of type $\mathcal{N}$.

Finally, we verify that the subgroups H_0 and H_∞ are not conjugate in $\mathsf{GL}_3(\mathbf{k})$ so that $\sigma_0 \neq \sigma_\infty$. Assume, for a contradiction, that $g = (g_{ij}) \in \mathsf{GL}_3(\mathbf{k})$ satisfies $gH_0 = H_\infty g$. As scalar matrices are invariant under conjugation, it suffices to compare matrices

$$g \begin{bmatrix} 0 & 0 & s_1 \\ 0 & 0 & s_3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ \tilde{s}_2 & 0 & \tilde{s}_3 \\ 0 & 0 & 0 \end{bmatrix} g$$

for $s_1, s_3, \tilde{s}_2, \tilde{s}_3 \in \mathbf{k}$. Inspecting the $(1,3)$- and $(3,3)$-entries, we deduce that for all $s_1, s_3 \in \mathbf{k}$,

$$g_{11}s_1 + g_{12}s_3 = 0 \qquad \text{and} \qquad g_{31}s_1 + g_{32}s_3 = 0.$$

This implies that $g_{11} = g_{12} = g_{31} = g_{32} = 0$ and hence g cannot be invertible. This completes the discussion of Table 2.1.

TABLE 2.1. Shadows σ in $\mathsf{GL}_3(\mathbf{k})$

Type	Minimal polynomial in $\mathbf{k}[t]$		Isomorphism type of $\sigma(\mathbf{k})$	$\dim(\sigma)$
$\mathcal{G}$	$t-\alpha$	$\alpha \in \mathbf{k}$	$\mathsf{GL}_3(\mathbf{k})$	9
$\mathcal{L}$	$(t-\alpha_1)(t-\alpha_2)$	$\alpha_1, \alpha_2 \in \mathbf{k}$ distinct	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_2(\mathbf{k})$	5
$\mathcal{J}$	$(t-\alpha)^2$	$\alpha \in \mathbf{k}$	$\mathsf{Heis}(\mathbf{k}) \rtimes (\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k}))$	5
$\mathcal{T}_1$	$\prod_{i=1}^3 (t-\alpha_i)$	$\alpha_1, \alpha_2, \alpha_3 \in \mathbf{k}$ distinct	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k})$	3
$\mathcal{T}_2$	$(t-\alpha)f(t)$	$\alpha \in \mathbf{k}$, f irred. quadratic	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k}_2)$	3
$\mathcal{T}_3$	$f(t)$	f irred. cubic	$\mathsf{GL}_1(\mathbf{k}_3)$	3
$\mathcal{M}$	$(t-\alpha_1)(t-\alpha_2)^2$	$\alpha_1, \alpha_2 \in \mathbf{k}$ distinct	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k}[t]/(t^2))$	3
$\mathcal{N}$	$(t-\alpha)^3$	$\alpha \in \mathbf{k}$	$\mathsf{GL}_1(\mathbf{k}[t]/(t^3))$	3
$\mathcal{K}_0$	not applicable		$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{G}_\mathrm{a}(\mathbf{k}) \times \mathsf{G}_\mathrm{a}(\mathbf{k})$	3
$\mathcal{K}_\infty$	not applicable		$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{G}_\mathrm{a}(\mathbf{k}) \times \mathsf{G}_\mathrm{a}(\mathbf{k})$	3

Theorem 2.8 (Classification of shadows and branching rules)**.**

(1) *The set of shadows $\mathfrak{Sh}_{\mathsf{GL}_3(\mathfrak{o})}$ consists of ten elements, classified by the types*

$$\mathcal{G},\ \mathcal{L},\ \mathcal{J},\ \mathcal{T}_1,\ \mathcal{T}_2,\ \mathcal{T}_3,\ \mathcal{M},\ \mathcal{N},\ \mathcal{K}_0,\ \mathcal{K}_\infty$$

described in Table 2.1*.*

(2) *For all $\sigma, \tau \in \mathfrak{Sh}_{\mathsf{GL}_3(\mathfrak{o})}$ there exists a polynomial $a_{\sigma,\tau} \in \mathbb{Z}[\frac{1}{6}][t]$ such that the following holds: for every $\ell \in \mathbb{N}$ and every $\mathcal{C} \in \mathcal{Q}^{\mathfrak{gl}_3}_{\mathfrak{o},\ell}$ with $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C}) = \sigma$ the number of classes $\widetilde{\mathcal{C}} \in \mathcal{Q}^{\mathfrak{gl}_3}_{\mathfrak{o},\ell+1}$ with $\mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}}) = \tau$ lying above $\mathcal{C}$ is equal to $a_{\sigma,\tau}(q)$.*

Remark 2.9. Of course, many of the polynomials $a_{\sigma,\tau}$ are simply zero. The non-zero $a_{\sigma,\tau}$ give the local branching behaviour of the directed graph $\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o}}$, which we introduced in Definition 2.1, and thus determine $\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o}}$ completely. Moreover, together with the corresponding polynomials $b^{(1)}_{\sigma,\tau}$, defined in (2.4), the $a_{\sigma,\tau}$ determine recursively the numbers and sizes of similarity classes in $\mathfrak{gl}_3(\mathfrak{o}_\ell)$ for all $\ell \in \mathbb{N}$; cf. Section 4.2. We refer to these two sets of polynomials as *branching rules* and record them in Table 2.2.

While the polynomials $a_{\sigma,\tau}$ are determined in the course of the proof of Theorem 2.8, the polynomials $b^{(1)}_{\sigma,\tau}$ can already be easily computed with the aid of Table 2.1.

TABLE 2.2. Branching rules for $\mathcal{Q}_{\mathfrak{o}}^{\mathsf{gl}_3}$ ($\varepsilon = 1$) and for $\mathcal{Q}_{\mathfrak{o}}^{\mathsf{gu}_3}$ ($\varepsilon = -1$)

#	Type of σ	Type of τ	$a_{\sigma,\tau}(q)$	$b_{\sigma,\tau}^{(\varepsilon)}(q)$
1	$\mathcal{G}$	$\mathcal{G}$	q	1
2	$\mathcal{G}$	$\mathcal{L}$	$(q-1)q$	$(q^2+\varepsilon q+1)q^2$
3	$\mathcal{G}$	$\mathcal{J}$	q	$(q^3-\varepsilon)(q+\varepsilon)$
4	$\mathcal{G}$	$\mathcal{T}_1$	$\frac{1}{6}(q-1)(q-2)q$	$(q^2+\varepsilon q+1)(q+\varepsilon)q^3$
5	$\mathcal{G}$	$\mathcal{T}_2$	$\frac{1}{2}(q-1)q^2$	$(q^3-\varepsilon)q^3$
6	$\mathcal{G}$	$\mathcal{T}_3$	$\frac{1}{3}(q^2-1)q$	$(q+\varepsilon)(q-\varepsilon)^2q^3$
7	$\mathcal{G}$	$\mathcal{M}$	$(q-1)q$	$(q^3-\varepsilon)(q+\varepsilon)q^2$
8	$\mathcal{G}$	$\mathcal{N}$	q	$(q^3-\varepsilon)(q^2-1)q$
9	$\mathcal{L}$	$\mathcal{L}$	q^2	q^4
10	$\mathcal{L}$	$\mathcal{T}_1$	$\frac{1}{2}(q-1)q^2$	$(q+\varepsilon)q^5$
11	$\mathcal{L}$	$\mathcal{T}_2$	$\frac{1}{2}(q-1)q^2$	$(q-\varepsilon)q^5$
12	$\mathcal{L}$	$\mathcal{M}$	q^2	$(q^2-1)q^4$
13	$\mathcal{J}$	$\mathcal{J}$	q^2	q^4
14	$\mathcal{J}$	$\mathcal{M}$	$(q-1)q^2$	q^6
15	$\mathcal{J}$	$\mathcal{N}$	$(q-1)q$	$(q-\varepsilon)q^5$
16*	$\mathcal{J}$	$\mathcal{K}_0$	q	$(q-1)q^5$
17*	$\mathcal{J}$	$\mathcal{K}_\infty$	q	$(q-1)q^5$
18*	other	same as σ	q^3	q^6

* Rows involving types $\mathcal{K}_0$ and $\mathcal{K}_\infty$ only apply if $\varepsilon = 1$.

Remark 2.10. One reads off rows (2) and (3) in Table 2.2 that exactly

$$(q-1)q \cdot (q^2+q+1)q^2 + q \cdot (q^3-1)(q+1) = q(q-1)|\mathbb{P}^2(\mathbb{F}_q)|^2$$

of the q^9 elements of $\mathsf{gl}_3(\mathbb{F}_q)$ have adjoint orbits of dimension 4. The other elements are either scalar or have 6-dimensional adjoint orbits. This reflects the fact that $(q-1)|\mathbb{P}^2(\mathbb{F}_q)|^2$ of the q^8-1 non-zero elements of $\mathsf{sl}_3(\mathbb{F}_q)$ elements are irregular in the sense of [5, Section 6.1]. Similar considerations hold for $\mathsf{gu}_3(\mathbb{F}_q)$. Sorting matrices in $\mathsf{gl}_3(\mathbb{F}_q)$, respectively $\mathsf{gu}_3(\mathbb{F}_q)$, by their shadows therefore yields a partition refining the stratification by centraliser dimension or (in type A_2 equivalently) by sheets; cf. Remark 6.5.

The proof of Theorem 2.8 is given in Section 2.4.

2.3. **Similarity classes of 3×3 matrices.** Let $\ell \in \mathbb{N}$ be fixed. In preparation for the proof of Theorem 2.8 we introduce some notation and refine several results from [7]. For elements

a, b, c, d in $\mathfrak{o}$, or its finite quotient $\mathfrak{o}_\ell$, and $m \in \mathbb{N} \cup \{\infty\}$, let

$$D(a,b,c) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \quad \text{and} \quad E = E(m,a,b,c,d) = \begin{bmatrix} d & \pi^m & 0 \\ 0 & d & 1 \\ a & b & c+d \end{bmatrix}.$$

A matrix $C \in \mathsf{gl}_n(\mathfrak{o}_\ell)$ is called *cyclic* if $\mathfrak{o}_\ell^n$ is cyclic as an $\mathfrak{o}_\ell[C]$-module, i.e. if there exists $v \in \mathfrak{o}_\ell^n$ such that the $\mathfrak{o}_\ell$-span of $\{C^i v \mid 0 \leq i < n\}$ is equal to $\mathfrak{o}_\ell^n$. Equivalently, a matrix is cyclic if it is similar to the companion matrix of its characteristic polynomial.

For every $\nu \in \mathbb{N}_0$, fix a set of representatives for $\mathfrak{o}_\nu = \mathfrak{o}/\mathfrak{p}^\nu$ in $\mathfrak{o}$,

$$\varsigma(\mathfrak{o}_\nu) = \varsigma_\nu(\mathfrak{o}_\nu) \subset \mathfrak{o},$$

including π^ν as a representative for 0, so that $0 \leq v(\varsigma(a)) \leq \nu$ for all $a \in \mathfrak{o}_\nu$. The particular choice $\varsigma(0) = \pi^\nu$, rather than the less pretentious convention $\varsigma(0) = 0$, plays a role in the subcase (III_∞) in Theorem 2.11 below; otherwise it has no significance. The valuation map naturally extends to $\mathfrak{o}_\nu$ via $v(a) := v(\varsigma(a))$.

In the formulation of Theorem 2.11 and some of the proofs below, we slightly abuse notation in two ways. Firstly, we write $\varsigma(\mathfrak{o}_\nu) \subset \mathfrak{o}_\mu$ for $\nu < \mu$ to denote the reduction of $\varsigma(\mathfrak{o}_\nu)$ modulo $\mathfrak{p}^\mu$. Secondly, we write $\pi^\nu \mathfrak{o}_{\ell-\nu} \subset \mathfrak{o}_\ell$ for the reduction of $\pi^\nu \varsigma(\mathfrak{o}_{\ell-\nu})$ modulo π^ℓ. These conventions are also applied in the obvious way to expressions involving matrices.

The next theorem gives a complete description of the similarity classes in $\mathsf{gl}_3(\mathfrak{o}_\ell)$. In addition it describes the resulting shadows.

Theorem 2.11. *Let $\mathcal{C} \subset \mathsf{gl}_3(\mathfrak{o}_\ell)$ be a similarity class. Then $\mathcal{C}$ contains exactly one of the following matrices.*

(i) $d\,\mathrm{Id}_3$, *where $d \in \mathfrak{o}_\ell$; the shadow $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{G}$.*

(ii) $d\,\mathrm{Id}_3 + \pi^i D(a,0,0)$, *where $0 \leq i < \ell$, $d \in \mathfrak{o}_\ell$ and $a \in \mathfrak{o}_{\ell-i}^\times$; the shadow $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{L}$.*

(iii)
$$d\,\mathrm{Id}_3 + \pi^i D(a,0,0) + \pi^j \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix},$$
where $0 \leq i < j < \ell$, $d \in \varsigma(\mathfrak{o}_j)$, $a \in \mathfrak{o}_{\ell-i}^\times$ and $C \in \mathsf{gl}_2(\mathfrak{o}_{\ell-j})$ a companion matrix; the shadow $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{M}$, depending on C.

(iv) $d\,\mathrm{Id}_3 + \pi^i C$, *where $0 \leq i < \ell$, $d \in \varsigma(\mathfrak{o}_i)$ and $C \in \mathsf{gl}_3(\mathfrak{o}_{\ell-i})$ a companion matrix; the shadow $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{M}$ or $\mathcal{N}$, depending on C.*

(v) $d'\,\mathrm{Id}_3 + \pi^i E$, *where $0 \leq i < \ell$, $d' \in \varsigma(\mathfrak{o}_i)$ and $E \in \mathsf{gl}_3(\mathfrak{o}_{\ell-i})$ is one of the following matrices:*
- (I) $E(\ell - i, 0, 0, c, d)$, *where $c, d \in \mathfrak{o}_{\ell-i}$ with $v(c) > 0$,*
- (II) $E(\mu, a, b, c, d)$, *where $1 \leq \mu < \ell - i$, $a, b \in \mathfrak{o}_{\ell-i}$ with $\mu = v(b) \leq v(a)$, $c \in \mathfrak{o}_{\ell-i}$ with $v(c) > 0$ and $d \in \varsigma(\mathfrak{o}_\mu)$,*
- (III_1) $E(\mu, a, b, c, d)$, *where $1 \leq \mu < \ell - i$, $a, b \in \mathfrak{o}_{\ell-i}$ with $\mu = v(a) < v(b)$, $c \in \mathfrak{o}_{\ell-i}$ with $v(c) > 0$ and $d \in \varsigma(\mathfrak{o}_\mu)$,*
- (III_0) $E(\mu, a, b, c, d)$, *where $1 \leq \mu < \ell - i$, $a, b \in \mathfrak{o}_{\ell-i}$ with $\mu < v(a)$ and $\mu < v(b)$, $c \in \mathfrak{o}_{\ell-i}$ with $v(c) > 0$ and $d \in \varsigma(\mathfrak{o}_\mu)$,*
- (III_∞) $E(m, a, b, c, d)$, *where $1 \leq \mu < \ell - i$, $\mu < m \leq \ell - i$, $a \in \varsigma(\mathfrak{o}_{\ell-i-m+\mu})$ and $b \in \mathfrak{o}_{\ell-i}$ with $\mu = v(a) < v(b)$, $c \in \mathfrak{o}_{\ell-i}$ with $v(c) > 0$ and $d \in \varsigma(\mathfrak{o}_\mu)$;*

the shadow $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ in these subcases has type $\mathcal{J}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{K}_0$ and $\mathcal{K}_\infty$, respectively.

Similarity classes in $\mathsf{gl}_3(\mathfrak{o}_\ell)$ were already studied in [7], where the following is proved.

Theorem 2.12 ([7, §3 §4])**.** *Every matrix $A \in \mathfrak{gl}_3(\mathfrak{o}_\ell)$ is $\mathrm{Ad}(\mathsf{GL}_3(\mathfrak{o}))$-conjugate to the reduction modulo $\mathfrak{p}^\ell$ of at least one matrix of one of five types described below. Matrices of different types are not $\mathrm{Ad}(\mathsf{GL}_3(\mathfrak{o}))$-conjugate.*

The types (i)–(v) *consist of matrices of the following form, where $a, b, c, d \in \mathfrak{o}$:*

(i) $d\,\mathrm{Id}_3$,

(ii) $d\,\mathrm{Id}_3 + \pi^i D(a,b,b)$, *where* $0 \le i < \ell$ *and* $a \not\equiv_\mathfrak{p} b$,

(iii)
$$d\,\mathrm{Id}_3 + \pi^i D(a,b,b) + \pi^j \begin{bmatrix} 0 & 0 \\ 0 & \varsigma(C) \end{bmatrix},$$
where $0 \le i < j < \ell$, $a \not\equiv_\mathfrak{p} b$ *and* $C \in \mathfrak{gl}_2(\mathfrak{o}_{\ell-j})$ *is cyclic,*

(iv) $d\,\mathrm{Id}_3 + \pi^i \varsigma(C)$, *where* $0 \le i < \ell$ *and* $C \in \mathfrak{gl}_3(\mathfrak{o}_{\ell-i})$ *is cyclic,*

(v) $d\,\mathrm{Id}_3 + \pi^i E(m,a,b,c,0)$, *where* $0 \le i < \ell$ *and* $m, v(a), v(b), v(c) > 0$.

In each of the families (i)–(iv) it is easy to specify a set of pairwise non-conjugate representatives, using Lemma 2.13 below. In contrast, it is less clear how to manufacture a set of pairwise non-conjugate representatives of the family (v), and the problem remained unsolved in [7]. Theorem 2.20 below removes this stumbling block and leads to a proof of Theorem 2.11.

Lemma 2.13. *Let $(n_1, \dots, n_r)$ be a composition of $n \in \mathbb{N}$. Let*
$$A = \mathrm{diag}(A_1, \dots, A_r), \quad A' = \mathrm{diag}(A'_1, \dots, A'_r) \in \mathfrak{gl}_n(\mathfrak{o}_\ell)$$
be block diagonal matrices with blocks $A_i, A'_i \in \mathfrak{gl}_{n_i}(\mathfrak{o}_\ell)$ such that $A_i \equiv_\mathfrak{p} A'_i \equiv_\mathfrak{p} a_i \,\mathrm{Id}_{n_i}$, where $a_1, \dots, a_r \in \mathfrak{o}_\ell$ with $a_i \not\equiv_\mathfrak{p} a_j$ if $i \ne j$. Let $X \in \mathfrak{gl}_n(\mathfrak{o}_\ell)$.

Then $XA = A'X$ if and only if $X = \mathrm{diag}(X_1, \dots, X_r)$ is block diagonal with blocks $X_i \in \mathfrak{gl}_{n_i}(\mathfrak{o}_\ell)$ satisfying $X_iA_i = A'_iX_i$ for $1 \le i \le r$. In particular, $X \in \mathrm{C}_{\mathfrak{gl}_n(\mathfrak{o}_\ell)}(A)$ if and only if $X = \mathrm{diag}(X_1, \dots, X_r)$ with $X_i \in \mathrm{C}_{\mathfrak{gl}_{n_i}(\mathfrak{o}_\ell)}(A_i)$ for $1 \le i \le r$.

Proof. The *if* part is clear. For the *only if* part, write X as a block matrix $(X_{ij})_{ij}$ with $X_{ij} \in \mathfrak{gl}_{n_i,n_j}(\mathfrak{o}_\ell)$ for $1 \le i, j \le r$. Reducing the equation $XA = A'X$ modulo $\mathfrak{p}$, and looking at the (i,j)-th block for $i \ne j$, we get $(a_j - a_i)X_{ij} \equiv_\mathfrak{p} 0$. It follows that $X_{ij} \equiv_\mathfrak{p} 0$. Repeating the same argument with a sequence of reductions modulo $\mathfrak{p}^m$ for $2 \le m \le \ell$ gives $X_{ij} = 0$. The equality $XA = A'X$ now implies $X_{ii}A_i = A'_iX_{ii}$. □

Proof of Theorem 2.11 (*modulo Proposition* 2.16 *and Theorem* 2.20). Applying Theorem 2.12, we single out unique representatives for similarity classes in each of the cases (i)–(v). To determine the shadow types it suffices to pin down the Lie centraliser shadows of these representatives.

(i) The assertion for scalar matrices is immediate.

(ii) Any matrix of the form $d\,\mathrm{Id}_3 + \pi^i D(a,b,b)$ with $0 \le i < \ell$ and $a \not\equiv_\mathfrak{p} b$ can be written as $d'\,\mathrm{Id}_3 + \pi^i D(a',0,0)$, where $d' \in \mathfrak{o}_\ell$ and $a \in \mathfrak{o}_{\ell-i}^\times$. Moreover, two matrices of the latter form are conjugate if and only if they have the same eigenvalues, or equivalently, have the same parameters. By Lemma 2.13, the centraliser of such a matrix is block diagonal with blocks of sizes 1×1 and 2×2, modulo $\mathfrak{p}^{\ell-i}$. Hence the corresponding shadow is of type $\mathcal{L}$; cf. Table 2.1.

(iii) Clearly, any matrix of the form

$$d\,\mathrm{Id}_3 + \pi^i D(a,b,b) + \pi^j \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}, \tag{2.5}$$

where $0 \le i < j < \ell$, $a \not\equiv_\mathfrak{p} b$ and $C \in \mathfrak{gl}_2(\mathfrak{o}_{\ell-j})$ is cyclic, can be written as

$$A = d'\,\mathrm{Id}_3 + \pi^i D(a',0,0) + \pi^j \begin{bmatrix} 0 & 0 \\ 0 & C' \end{bmatrix} \tag{2.6}$$

with $d' \in \varsigma(\mathfrak{o}_j)$, $a' \in \mathfrak{o}_{\ell-i}^\times$ and $C' \in \mathfrak{gl}_2(\mathfrak{o}_{\ell-j})$ cyclic. After a further conjugation by an appropriate block diagonal matrix, with blocks 1×1 and 2×2, we may assume that C' is a companion matrix. It follows that matrices of the form (2.6), with C' a companion matrix, represent as many classes as matrices of the form (2.5). At the same time they are pairwise non-conjugate by Lemma 2.13. The same lemma implies that the centraliser of any matrix of the form (2.6) is block diagonal, modulo $\mathfrak{p}^{\ell-i}$. Combined with the cyclicity of the 2×2-block C', we deduce that the Lie centraliser shadow $\mathrm{sh}_{\mathfrak{gl}}(A)$ is of the form $\mathfrak{gl}_1(\mathbf{k}) \times \mathbf{k}[\overline{C'}]$, because the algebra $\mathbf{k}[\overline{C'}]$ generated by the reduction $\overline{C'}$ modulo $\mathfrak{p}$ is equal to the reduction modulo $\mathfrak{p}$ of the centraliser of C' in $\mathfrak{gl}_2(\mathfrak{o})$. Consequently, the shadow $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{M}$ according to $\overline{C'}$ being split semisimple, non-split semisimple or a scalar translate of a nilpotent matrix.

(iv) Any matrix of the form $d\,\mathrm{Id}_3 + \pi^i C \in \mathfrak{gl}_3(\mathfrak{o}_\ell)$, with $0 \le i < \ell$ and $C \in \mathfrak{gl}_3(\mathfrak{o}_{\ell-i})$ cyclic, can be written as $d'\,\mathrm{Id}_3 + \pi^i C'$ with $d' \in \varsigma(\mathfrak{o}_i)$ and $C' \in \mathfrak{gl}_3(\mathfrak{o}_{\ell-i})$ again cyclic. Such a matrix has a unique conjugate A of the same form where the cyclic matrix is a companion matrix, proving the first part of the assertion. The image of the centraliser of A in $\mathfrak{gl}_3(\mathfrak{o}_{\ell-i})$ is the same as the centraliser of C. Hence the shadow $\mathrm{sh}_{\mathsf{GL}}(C)$ has type equal to one of $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{N}, \mathcal{M}$, depending on C; for instance, see [5, Lemma 7.5].

(v) The assertion follows from Proposition 2.16 and Theorem 2.20 below. □

It remains to establish Proposition 2.16 and Theorem 2.20 below: our goal is to produce a complete and irredundant list of representatives for the similarity classes in case (v) of Theorem 2.12 and to compute their shadows. Interestingly, shadows will play a crucial role in producing the list of representatives in the first place.

For $m \in \mathbb{N} \cup \{\infty\}$ and $a, b, c, d \in \mathfrak{o}$, we write $E_\ell(m, a, b, c, d)$ for the reduction of $E(m, a, b, c, d)$ modulo $\mathfrak{p}^\ell$. Furthermore, it will be convenient to keep track of the parameter ℓ by writing E_ℓ, or more generally A_ℓ, for elements of $\mathfrak{gl}_3(\mathfrak{o}_\ell)$. It follows from [7, Proposition 3.5] that

$$\mathcal{E}_\ell := \{E_\ell(m, a, b, c, d) \mid m \in \mathbb{N} \text{ and } a, b, c, d \in \mathfrak{o} \text{ with } v(a), v(b), v(c) > 0\} \subset \mathfrak{gl}_3(\mathfrak{o}_\ell)$$

is an exhaustive, but redundant set of representatives for the similarity classes of all matrices in $\mathfrak{gl}_3(\mathfrak{o}_\ell)$ such that the minimal polynomial of their reduction modulo $\mathfrak{p}$ takes the form $(X - \alpha)^2$, $\alpha \in \mathbf{k}$. The parameter

$$\mu_\ell(m, a, b) = \min\{m, v(a), v(b), \ell\} \in \{1, \dots, \ell\}$$

allows us to partition the set $\mathcal{E}_\ell$ into disjoint subsets

$$\begin{aligned}
\mathcal{E}_\ell^{\mathrm{I}} &= \{E_\ell(m, a, b, c, d) \in \mathcal{E}_\ell \mid \mu_\ell(m, a, b) = \ell\},\\
\mathcal{E}_\ell^{\mathrm{II}} &= \{E_\ell(m, a, b, c, d) \in \mathcal{E}_\ell \mid \mu_\ell(m, a, b) < \ell \text{ and } \mu_\ell(m, a, b) = v(b) \le \min\{m, v(a)\}\},\\
\mathcal{E}_\ell^{\mathrm{III}} &= \{E_\ell(m, a, b, c, d) \in \mathcal{E}_\ell \mid \mu_\ell(m, a, b) < \ell \text{ and } \mu_\ell(m, a, b) = \min\{m, v(a)\} < v(b)\}.
\end{aligned}$$

The third set can be divided further into three disjoint subsets

$$\begin{aligned}
\mathcal{E}_\ell^{\mathrm{III},1} &= \{E_\ell(m, a, b, c, d) \in \mathcal{E}_\ell^{\mathrm{III}} \mid \mu_\ell(m, a, b) = m = v(a)\},\\
\mathcal{E}_\ell^{\mathrm{III},0} &= \{E_\ell(m, a, b, c, d) \in \mathcal{E}_\ell^{\mathrm{III}} \mid \mu_\ell(m, a, b) = m < v(a)\},\\
\mathcal{E}_\ell^{\mathrm{III},\infty} &= \{E_\ell(m, a, b, c, d) \in \mathcal{E}_\ell^{\mathrm{III}} \mid \mu_\ell(m, a, b) = v(a) < m\}.
\end{aligned}$$

The division of $\mathcal{E}_\ell$ into subsets according to the parameter $\mu_\ell(m, a, b)$ is motivated by the following observation.

Lemma 2.14. *Let $E_\ell = E_\ell(m, a, b, c, d) \in \mathcal{E}_\ell$. Then $\mu := \mu_\ell(m, a, b)$ and the reduction of d modulo $\mathfrak{p}^\mu$ are invariants of the similarity class of E_ℓ, i.e. if $E_\ell(m, a, b, c, d)$ is similar to $E_\ell(m', a', b', c', d')$ then $\mu_\ell(m, a, b) = \mu_\ell(m', a', b')$ and $d \equiv_{\mathfrak{p}^\mu} d'$.*

Proof. The first claim follows from the fact that

$$\begin{aligned}\mu_\ell(m,a,b) &= \max\left\{ \min\{m, v(a), v(b), v(\tilde{d}), v(\tilde{d}(c+\tilde{d})-b), \ell\} \mid \tilde{d} \in \mathfrak{o}\right\} \\ &= \max\left\{ \min\left(\{v(\det(M)) \mid M \text{ a } 2\times 2\text{-submatrix of } E(m,a,b,c,\tilde{d})\} \cup \{\ell\}\right) \mid \tilde{d} \in \mathfrak{o}\right\}.\end{aligned}$$

As for the second claim, the reduction of E_ℓ modulo $\mathfrak{p}^\mu$ has eigenvalues congruent to d, d and $c+d$ modulo $\mathfrak{p}^\mu$. Hence d modulo $\mathfrak{p}^\mu$ is the unique eigenvalue of multiplicity at least 2 of this matrix and consequently an invariant of the similarity class of E_ℓ. □

In Section 2.3.1 we determine the centralisers and shadows of matrices in each of the sets $\mathcal{E}_\ell^{\mathrm{I}}$, $\mathcal{E}_\ell^{\mathrm{II}}$, $\mathcal{E}_\ell^{\mathrm{III},1}$, $\mathcal{E}_\ell^{\mathrm{III},0}$ and $\mathcal{E}_\ell^{\mathrm{III},\infty}$. Corollary 2.17 shows that the five sets cover disjoint sets of similarity classes in $\mathsf{gl}_3(\mathfrak{o}_\ell)$. In Section 2.3.2 we extend Lemma 2.14 and determine, in Theorem 2.20, for each of the five sets, explicit representatives for the similarity classes covered by that set.

2.3.1. *Centralisers and shadows.*

Proposition 2.15. *Let $E_\ell = E_\ell(m,a,b,c,d) \in \mathcal{E}_\ell$. Then the centraliser $\mathrm{C}_{\mathsf{gl}_3(\mathfrak{o})}(E_\ell)$ consists of all matrices in $\mathsf{gl}_3(\mathfrak{o})$ which are congruent modulo $\mathfrak{p}^\ell$ to a matrix of the form*

$$F_{m,a,b,c}(t_1,t_2,s_1,s_2,s_3) = \begin{bmatrix} t_1 & \pi^m s_3 - cs_1 & s_1 \\ s_2 & t_2 & s_3 \\ as_3 & \pi^m s_2 + bs_3 & t_2 + cs_3 \end{bmatrix},$$

where $t_1,t_2,s_1,s_2,s_3 \in \mathfrak{o}$ satisfy the congruences

$$\begin{aligned} as_1 &\equiv_{\mathfrak{p}^\ell} \pi^m s_2, \\ bs_1 &\equiv_{\mathfrak{p}^\ell} \pi^m(t_2-t_1), \\ bs_2 &\equiv_{\mathfrak{p}^\ell} a(t_2-t_1). \end{aligned} \tag{2.7}$$

The centraliser $\mathrm{C}_{\mathsf{GL}_3(\mathfrak{o})}(E_\ell)$ consists of the same matrices subject to the additional condition that $t_1,t_2 \in \mathfrak{o}^\times$.

Proof. This is a straightforward computation; see [7, §4.1]. The additional condition for invertible matrices is obtained by considering $F_{m,a,b,c}(t_1,t_2,s_1,s_2,s_3)$ modulo $\mathfrak{p}$. □

For the following proposition recall the shadow types listed in Table 2.1.

Proposition 2.16. *Let $\mathcal{C}$ be the similarity class of a matrix $E_\ell = E_\ell(m,a,b,c,d) \in \mathcal{E}_\ell$. Then the shadow $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ is classified as follows.*

1. *If $E_\ell \in \mathcal{E}_\ell^{\mathrm{I}}$, then $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{J}$.*
2. *If $E_\ell \in \mathcal{E}_\ell^{\mathrm{II}}$, then $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{M}$.*
3. *If $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},1}$, then $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{N}$.*
4. *If $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},0}$, then $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{K}_0$.*
5. *If $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},\infty}$, then $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{K}_\infty$.*

Proof. We put $\mu = \mu_\ell(m,a,b)$, and throughout we use Proposition 2.15.

(1) Suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{I}}$. Then the congruences (2.7) hold trivially and $\mathrm{C}_{\mathsf{gl}_3(\mathfrak{o})}(E_\ell)$ consists of all the matrices which are congruent modulo $\mathfrak{p}^\ell$ to matrices of the form

$$F_{\infty,0,0,c}(t_1,t_2,s_1,s_2,s_3) = \begin{bmatrix} t_1 & -cs_1 & s_1 \\ s_2 & t_2 & s_3 \\ 0 & 0 & t_2 + cs_3 \end{bmatrix}.$$

Since $v(c) > 0$, we deduce that the collection of the reductions modulo $\mathfrak{p}$ of these matrices, i.e. the Lie centraliser shadow $\mathrm{sh}_{\mathsf{gl}}(E_\ell)$, is equal to the centraliser of the matrix $\left[\begin{smallmatrix} 0&0&0\\0&0&1\\0&0&0\end{smallmatrix}\right] \in \mathsf{gl}_3(\mathbf{k})$. Hence $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{J}$.

(2) Suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{II}}$. By [7, Lemma 4.3], we may assume that $\mu = m = v(b) \le v(a)$. Setting $\alpha = a/\pi^\mu$ and $\beta = b/\pi^\mu$, we have $v(\alpha) \ge 0$ and $v(\beta) = 0$. Moreover, the congruences (2.7) are equivalent to the conditions $\alpha s_1 \equiv_{\mathfrak{p}^{\ell-\mu}} s_2$ and $\beta s_1 \equiv_{\mathfrak{p}^{\ell-\mu}} t_2 - t_1$. From this we deduce that the Lie centraliser shadow of E_ℓ is

$$\mathrm{sh}_{\mathsf{gl}}(E_\ell) = \left\{ \left[\begin{smallmatrix} t_1 & 0 & (t_2-t_1)/\bar\beta \\ \bar\alpha(t_2-t_1)/\bar\beta & t_2 & s_3 \\ 0 & 0 & t_2 \end{smallmatrix}\right] \mid t_1, t_2, s_3 \in \mathbf{k} \right\}.$$

Left-conjugation by $\left[\begin{smallmatrix} 1 & 0 & -1/\bar\beta \\ \bar\alpha/\bar\beta & 1 & 0 \\ 0 & 0 & 1\end{smallmatrix}\right]$ maps $\mathrm{sh}_{\mathsf{gl}}(E_\ell)$ onto the centraliser of $\left[\begin{smallmatrix} 0&0&0\\0&1&1\\0&0&1\end{smallmatrix}\right] \in \mathsf{gl}_3(\mathbf{k})$. Hence $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{M}$.

(3),(4),(5) Suppose $E_\ell \in \mathcal{E}_\ell^{\mathrm{III}}$ so that $\mu = \min\{m, v(a)\} < v(b)$. Setting $\alpha = a/\pi^\mu$, $\beta = b/\pi^\mu$ and $\rho = \pi^m/\pi^\mu$, we have $\min\{v(\alpha), v(\rho)\} = 0$ and $v(\beta) > 0$.

The congruences (2.7) are equivalent to the conditions $\alpha s_1 \equiv_{\mathfrak{p}^{\ell-\mu}} \rho s_2$, $\beta s_1 \equiv_{\mathfrak{p}^{\ell-\mu}} \rho(t_2 - t_1)$ and $\beta s_2 \equiv_{\mathfrak{p}^{\ell-\mu}} \alpha(t_2 - t_1)$. From this we deduce that

$$\mathrm{sh}_{\mathsf{gl}}(E_\ell) = \left\{ \left[\begin{smallmatrix} t & 0 & s_1 \\ s_2 & t & s_3 \\ 0 & 0 & t\end{smallmatrix}\right] \mid t, s_1, s_2, s_3 \in \mathbf{k} \text{ such that } \bar\alpha s_1 = \bar\rho s_2 \right\}.$$

Consideration of $(\bar\alpha : \bar\rho) \in \mathbb{P}^1(\mathbf{k})$ leads naturally to the distinction into subcases $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},1}$, $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},0}$ and $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},\infty}$.

First suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},1}$. Then $(\bar\alpha : \bar\rho) \notin \{(0:1),(1:0)\}$ and left-conjugation by $\left[\begin{smallmatrix} 0 & \bar\alpha/\bar\rho & 0 \\ \bar\alpha/\bar\rho & 0 & 0 \\ 0 & 0 & 1\end{smallmatrix}\right]$ maps $\mathrm{sh}_{\mathsf{gl}}(E_\ell)$ onto the centraliser of $\left[\begin{smallmatrix} 0&1&0\\0&0&1\\0&0&0\end{smallmatrix}\right] \in \mathsf{gl}_3(\mathbf{k})$. Hence $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$ has type $\mathcal{N}$. Now suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},0} \cup \mathcal{E}_\ell^{\mathrm{III},\infty}$. Then $(\bar\alpha : \bar\rho) \in \{(0:1),(1:0)\}$, where $(0:1)$ matches the subscript 0 and $(1:0)$ matches ∞, and $\mathrm{sh}_{\mathsf{gl}}(E_\ell)$ is equal to

$$\left\{ \left[\begin{smallmatrix} t & 0 & s_1 \\ 0 & t & s_3 \\ 0 & 0 & t\end{smallmatrix}\right] \mid t, s_1, s_3 \in \mathbf{k} \right\} \quad \text{or} \quad \left\{ \left[\begin{smallmatrix} t & 0 & 0 \\ s_2 & t & s_3 \\ 0 & 0 & t\end{smallmatrix}\right] \mid t, s_2, s_3 \in \mathbf{k} \right\}$$

accordingly. The corresponding shadows are of types $\mathcal{K}_0$ and $\mathcal{K}_\infty$; see Section 2.2. $\square$

Corollary 2.17. *Each similarity class of a matrix in $\mathcal{E}_\ell$ intersects precisely one of the five sets $\mathcal{E}_\ell^{\mathrm{I}}$, $\mathcal{E}_\ell^{\mathrm{II}}$, $\mathcal{E}_\ell^{\mathrm{III},1}$, $\mathcal{E}_\ell^{\mathrm{III},0}$ and $\mathcal{E}_\ell^{\mathrm{III},\infty}$.*

2.3.2. *Representatives.* We extend Lemma 2.14 as follows.

Proposition 2.18. *Let $E_\ell = E_\ell(m,a,b,c,d) \in \mathcal{E}_\ell$ and $\mu = \mu_\ell(m,a,b)$. Then μ and the reduction of d modulo π^μ are invariants of the similarity class of E_ℓ. Moreover, E_ℓ is similar to a matrix $E'_\ell = E_\ell(m',a',b',c',d')$ with $d' \in \varsigma(\mathfrak{o}_\mu)$, and $m' = \mu$ if $E_\ell \in \mathcal{E}_\ell^{\mathrm{I}} \cup \mathcal{E}_\ell^{\mathrm{II}}$.*

Proof. The first part was proved in Lemma 2.14. It remains to show that E_ℓ is similar to a matrix $E'_\ell = E_\ell(m',a',b',c',d')$ with $d' \in \varsigma(\mathfrak{o}_\mu)$. By Corollary 2.17, we may treat the cases $E_\ell \in \mathcal{E}_\ell^{\mathrm{I}}$, $E_\ell \in \mathcal{E}_\ell^{\mathrm{II}}$ and $E_\ell \in \mathcal{E}_\ell^{\mathrm{III}}$ one by one.

(1) Suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{I}}$. Then $\mu = \ell$ and clearly we may assume that $d \in \varsigma(\mathfrak{o}_\mu)$.

(2) Suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{II}}$. By [7, Lemma 4.3], we may assume that $\mu = m = v(b)$. Consider the one-parameter subgroup

$$X \colon \mathfrak{o} \to \mathsf{GL}_3(\mathfrak{o}), \quad x \mapsto X(x) = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ \pi^m x^2 & 2\pi^m x & 1 \end{bmatrix}$$

A straightforward computation shows that for any $x \in \mathfrak{o}$ we have

$$X(x)E(m,a,b,c,d) = E(m',a',b',c',d')X(x),$$

where

$$\begin{array}{llll} (1) & a' = a - bx + \pi^{2m}x^3 + c\pi^m x^2, & (4) & d' = d - \pi^m x, \\ (2) & b' = b - 3\pi^{2m}x^2 - 2\pi^m cx, & (5) & m' = m. \\ (3) & c' = c + 3\pi^m x, & & \end{array}$$

Since $\mu = m$, we can choose x such that $d' \in \varsigma(\mathfrak{o}_\mu)$ as claimed.

(3) Suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{III}}$ so that $\mu = \min\{m, v(a)\}$. If $\mu = m$, then we can argue as in (2). Now suppose that $\mu = v(a) < m$. Consider the one-parameter subgroup

$$Y \colon \mathfrak{o} \to \mathsf{GL}_3(\mathfrak{o}), \quad y \mapsto Y(y) = \begin{bmatrix} e_0(y)^{-1} & e_0(y)^{-1}(ay^2 - cy) & e_0(y)^{-1}y \\ 0 & 1 & 0 \\ 0 & -ay & 1 \end{bmatrix},$$

where $e_0(y) = e(y)/\pi^{v(e(y))}$ with $e(y) = \pi^m + by + acy^2 - a^2y^3$. A straightforward computation shows that for any $y \in \mathfrak{o}$ we have

$$Y(y)E(m,a,b,c,d) = E(m',a',b',c',d')Y(y),$$

where

$$\begin{array}{llll} (1) & a' = ae_0(y), & (4) & d' = d + ay, \\ (2) & b' = b + 2acy - 3a^2y^2, & (5) & m' = v(e(y)). \\ (3) & c' = c - 3ay, & & \end{array}$$

Since $\mu = v(a)$, we can choose y such that $d' \in \varsigma(\mathfrak{o}_\mu)$ as claimed. □

Proposition 2.18 has the following immediate consequence.

Corollary 2.19. *Let $1 \le \mu \le \ell$, and suppose that $\mathcal{R}$ is a complete and irredundant set of representatives for similarity classes intersected with $\{E_\ell(m,a,b,c,0) \in \mathcal{E}_\ell \mid \mu_\ell(m,a,b) = \mu\}$. Then*

$$\{d' \,\mathrm{Id}_3 + R \mid d' \in \varsigma(\mathfrak{o}_\mu) \text{ and } R \in \mathcal{R}\}$$

is a complete and irredundant set of representatives for similarity classes intersected with the set $\{E_\ell(m,a,b,c,d) \in \mathcal{E}_\ell \mid \mu_\ell(m,a,b) = \mu\}$.

Theorem 2.20. *A complete and irredundant set of representatives for the similarity classes intersected with the set $\mathcal{E}_\ell$ is obtained as follows.*

1. *Every $E_\ell \in \mathcal{E}_\ell^{\mathrm{I}}$ is similar to a unique matrix of the form $E'_\ell = E_\ell(\ell,0,0,c',d')$, where $c' \in \varsigma(\mathfrak{o}_\ell)$ with $v(c') > 0$ and $d' \in \varsigma(\mathfrak{o}_\ell)$.*
2. *Every $E_\ell \in \mathcal{E}_\ell^{\mathrm{II}}$ is similar to a unique matrix of the form $E'_\ell = E_\ell(\mu,a',b',c',d')$, where $1 \le \mu < \ell$, $a',b' \in \varsigma(\mathfrak{o}_\ell)$ with $\mu = v(b') \le v(a')$, $c' \in \varsigma(\mathfrak{o}_\ell)$ with $v(c') > 0$ and $d' \in \varsigma(\mathfrak{o}_\mu)$.*
3. *Every $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},1}$ is similar to a unique matrix of the form $E'_\ell = E_\ell(\mu,a',b',c',d')$, where $1 \le \mu < \ell$, $a',b' \in \varsigma(\mathfrak{o}_\ell)$ with $\mu = v(a') < v(b')$, $c' \in \varsigma(\mathfrak{o}_\ell)$ with $v(c') > 0$ and $d' \in \varsigma(\mathfrak{o}_\mu)$.*
4. *Every $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},0}$ is similar to a unique matrix of the form $E'_\ell = E_\ell(\mu,a',b',c',d')$, where $1 \le \mu < \ell$, $a',b' \in \varsigma(\mathfrak{o}_\ell)$ with $\mu < v(a')$ and $\mu < v(b')$, $c' \in \varsigma(\mathfrak{o}_\ell)$ with $v(c') > 0$ and $d' \in \varsigma(\mathfrak{o}_\mu)$.*
5. *Every $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},\infty}$ is similar to a unique matrix of the form $E'_\ell = E_\ell(m,a',b',c',d')$, where $1 \le \mu < \ell$, $\mu < m \le \ell$, $a' \in \varsigma(\mathfrak{o}_{\ell-m+\mu})$ and $b' \in \varsigma(\mathfrak{o}_\ell)$ with $\mu = v(a') < v(b')$, $c' \in \varsigma(\mathfrak{o}_\ell)$ with $v(c') > 0$ and $d' \in \varsigma(\mathfrak{o}_\mu)$.*

Proof. By Corollary 2.19 it suffices to consider matrices $E_\ell(m,a,b,c,d) \in \mathcal{E}_\ell$ with $d=0$. Fix $E_\ell = E_\ell(m,a,b,c,0) \in \mathcal{E}_\ell$ and set $\mu = \mu_\ell(m,a,b)$. Without loss of generality we may assume that $m, v(a), v(b), v(c) \le \ell$. We need to understand for which m', a', b', c' the given matrix E_ℓ is similar to $E'_\ell = E_\ell(m',a',b',c',0)$. Lemma 2.14 shows that μ is an invariant of the similarity class of E_ℓ and, by Corollary 2.17, we can investigate each of the five subsets $\mathcal{E}_\ell^{\mathrm{I}}$, $\mathcal{E}_\ell^{\mathrm{II}}$, $\mathcal{E}_\ell^{\mathrm{III},1}$, $\mathcal{E}_\ell^{\mathrm{III},0}$ and $\mathcal{E}_\ell^{\mathrm{III},\infty}$ separately. The characteristic polynomial of E_ℓ is $t^3 - ct^2 - bt - a\pi^m \in \mathfrak{o}_\ell[t]$. Hence, modulo $\mathfrak{p}^\ell$, the parameters b, c and $a\pi^m$ are invariants of the similarity class of E_ℓ.

(1) Suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{I}}$. Since $\mu = \min\{m, v(a), v(b), \ell\}$ and c modulo $\mathfrak{p}^\ell$ are invariants of the similarity class of E_ℓ, the claim follows.

(2) Suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{II}}$. By [7, Lemma 4.3], we may assume that $\mu = m = v(b) \le v(a)$ and we may restrict our attention to possible conjugates $E'_\ell = E_\ell(\mu, a', b, c, 0)$, where $a' \in \mathfrak{o}$ with $v(a') \ge \mu$ and $v(a'-a) \ge \ell - m$. Part of the analysis for cases (3),(4) below, which only requires $\mu = m \le v(a)$, shows that – in the present situation – for E'_ℓ to be similar to E_ℓ it is necessary that $a' \equiv_{\mathfrak{p}^\ell} a$ and the claim follows.

(3),(4),(5) Suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{III}}$. We already observed that we can investigate the subsets $\mathcal{E}_\ell^{\mathrm{III},1}$, $\mathcal{E}_\ell^{\mathrm{III},0}$ and $\mathcal{E}_\ell^{\mathrm{III},\infty}$ separately. Inspection shows that the elementary divisors of E_ℓ are 1, π^m and $\pi^{v(a)}$. Hence not only μ, but even m and $v(a)$ are invariants of the similarity class of E_ℓ.

Thus we may restrict our attention to possible conjugates $E'_\ell = E_\ell(m, a', b, c, 0)$, where $a' \in \mathfrak{o}$ with $v(a') = v(a)$ and $v(a'-a) \ge \ell - m$. Writing $a' = a + y\pi^k$ with $y \in \mathfrak{o}^\times$ and $k \ge \max\{0, \ell - m\}$, we study the equation

$$E_\ell X = X E'_\ell, \qquad \text{for } X = (x_{ij}) \in \mathsf{GL}_3(\mathfrak{o}). \tag{2.8}$$

During the rest of the proof we abbreviate $\equiv_{\mathfrak{p}^\ell}$ to $\equiv$; in all other congruences we continue to write the modulus explicitly. Comparing individual matrix entries, as in [7, Section 4], we obtain the following collection of necessary and sufficient conditions for (2.8) to hold:

$$\begin{array}{llll}
(2,1) & x_{31} \equiv (a + y\pi^k) x_{23}, & (3,1) & a x_{11} + b x_{21} + c x_{31} - (a + y\pi^k) x_{33} \equiv 0, \\
(2,3) & x_{33} \equiv x_{22} + c x_{23}, & (1,2) & \pi^m x_{11} \equiv \pi^m x_{22} - b x_{13}, \\
(2,2) & x_{32} \equiv \pi^m x_{21} + b x_{23}, & (1,1) & \pi^m x_{21} \equiv (a + y\pi^k) x_{13}, \\
(3,3) & x_{32} \equiv a x_{13} + b x_{23}, & (3,2) & a x_{12} + b x_{22} + c x_{32} - \pi^m x_{31} - b x_{33} \equiv 0, \\
(1,3) & x_{12} \equiv \pi^m x_{23} - c x_{13}, & (X \text{ invertible}) & x_{11}, x_{22} \in \mathfrak{o}^\times.
\end{array}$$

We claim that these conditions are equivalent to the modified conditions 2.9 below. Indeed, using $(2,2)$, we can replace $(3,3)$ by $(3,3)'$. Using $(3,3)'$, we can replace $(1,1)$ by $(1,1)'$. Using $(2,1)$ and $(2,3)$, we can replace $(3,1)$ by $(3,1)'$. Using $(2,1)$, $(2,2)$, $(2,3)$, $(1,3)$ and $(3,3)'$, we see that $(3,2)$ can replaced by $\pi^m y \pi^k x_{23} \equiv 0$ which holds automatically due to $k \ge \ell - m$.

$$\begin{array}{llll}
(2,1) & x_{31} \equiv (a + y\pi^k) x_{23}, & (3,1)' & a x_{11} \equiv (a + y\pi^k) x_{22} - b x_{21}, \\
(2,3) & x_{33} \equiv x_{22} + c x_{23}, & (1,2) & \pi^m x_{11} \equiv \pi^m x_{22} - b x_{13}, \\
(2,2) & x_{32} \equiv \pi^m x_{21} + b x_{23}, & (1,1)' & y\pi^k x_{13} \equiv 0, \\
(3,3)' & \pi^m x_{21} \equiv a x_{13}, & & \\
(1,3) & x_{12} \equiv \pi^m x_{23} - c x_{13}, & (X \text{ invertible}) & x_{11}, x_{22} \in \mathfrak{o}^\times.
\end{array} \tag{2.9}$$

First suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},1} \cup \mathcal{E}_\ell^{\mathrm{III},0}$, corresponding to cases (3),(4). Then $\mu = m \le \min\{v(a), v(b)\}$ and $(3,3)'$ is equivalent to

$$(3,3)'' \quad x_{21} \equiv_{\mathfrak{p}^{\ell-\mu}} a\pi^{-m} x_{13}.$$

Multiplying $(1,2)$ by $a\pi^{-m}$ and using $(3,3)''$, we obtain

$$ax_{11} \equiv ax_{22} - a\pi^{-m}bx_{13} \equiv ax_{22} - bx_{21}.$$

Comparing with $(3,1)'$, we obtain the necessary condition $y\pi^k \equiv 0$, hence $k \geq \ell$. This means that E_ℓ is similar to $E'_\ell = E_\ell(m, a', b, c, 0)$ if and only if $a' \equiv a$.

Finally suppose that $E_\ell \in \mathcal{E}_\ell^{\mathrm{III},\infty}$, corresponding to case (5). Then $\mu = v(a) < \min\{m, v(b)\}$ and $(3,3)'$ is equivalent to

$$(3,3)''' \quad x_{13} \equiv_{\mathfrak{p}^{\ell-\mu}} \pi^m a^{-1} x_{21}.$$

Multiplying $(3,1)'$ by $\pi^m a^{-1}$ and using $(3,3)'''$, we obtain

$$\pi^m x_{11} \equiv \pi^m x_{22} + \pi^m a^{-1} y\pi^k x_{22} - \pi^m a^{-1} b x_{21} \equiv \pi^m x_{22} - bx_{13} + \pi^m a^{-1} y \pi^k x_{22}.$$

Comparing with $(1,2)$, we obtain the necessary condition $\pi^m a^{-1} y\pi^k \equiv 0$, hence

$$k \geq \ell - m + \mu.$$

Conversely, if this inequality holds, then $(3,1)'$ implies $(1,2)$ and $(3,3)'''$ implies $(1,1)'$. The remaining conditions can easily be satisfied. This means that E_ℓ is similar to $E'_\ell = E_\ell(m, a', b, c, 0)$ if and only if $v(a') = v(a) = \mu$ and $a' \equiv_{\mathfrak{p}^{\ell-m+\mu}} a$. □

2.4. **Proof of Theorem 2.8.** Part (1) of Theorem 2.8 follows from collecting the shadow types in Theorem 2.11. To prove part (2), we consider a matrix $A_\ell \in \mathfrak{gl}_3(\mathfrak{o}_\ell)$ of the form given in Theorem 2.11. Let $\mathcal{C}$ denote the similarity class of A_ℓ. Starting from the shadow σ of $\mathcal{C}$, we determine the shadows τ associated to similarity classes $\widetilde{\mathcal{C}}$ of lifts of A_ℓ to matrices $\widetilde{A}_{\ell+1} \in \mathfrak{gl}_3(\mathfrak{o}_{\ell+1})$. We also keep track of the multiplicities of such lifts. The claim then follows from the observation that in all cases the multiplicities depend only on the shadows involved and are as listed in Table 2.2.

($\mathcal{G}$) Suppose that σ has type $\mathcal{G}$. Then $A_\ell = d \operatorname{Id}_3$ with $d \in \mathfrak{o}_\ell$. Let $\widetilde{A}_{\ell+1} = \varsigma(d) \operatorname{Id}_3 + \pi^\ell X \in \mathfrak{gl}_3(\mathfrak{o}_{\ell+1})$ be a lift of A_ℓ with $X \in \mathfrak{gl}_3(\mathbf{k})$. Then $\mathrm{sh}_{\mathsf{GL}}(\widetilde{A}_{\ell+1}) = \mathrm{sh}_{\mathsf{GL}}(X)$ implies that the type of $\tau = \mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}})$ is not $\mathcal{K}_0$ or $\mathcal{K}_\infty$. Indeed, it is one of $\mathcal{G}$, $\mathcal{L}$, $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{J}$, according to the shape of the minimal polynomial of X. For τ not of type $\mathcal{L}$ the number $a_{\sigma,\tau}(q)$ of distinct lifts with shadow τ is the number of distinct minimal polynomials of the corresponding shape. For τ of type $\mathcal{L}$ the number $a_{\sigma,\tau}(q)$ is the number of distinct minimal polynomials of the prescribed shape, paired with a compatible characteristic polynomial. The numbers $a_{\sigma,\tau}(q)$ are easily computed from Table 2.1 and can be found in Table 2.2.

($\mathcal{L}$) Suppose that σ has type $\mathcal{L}$. By Theorem 2.11 we may assume that A_ℓ is of the form $A_\ell = d \operatorname{Id}_3 + \pi^i D(a, 0, 0)$ with $0 \leq i < \ell$, $d \in \mathfrak{o}_\ell$ and $a \in \mathfrak{o}_{\ell-i}^\times$. Any lift of A_ℓ is conjugate to a matrix of the form described in parts (ii) or (iii) of Theorem 2.11, that is, conjugate to a matrix of the form

$$\widetilde{A}_{\ell+1} = \varsigma(d) \operatorname{Id}_3 + \pi^i D(\varsigma(a), 0, 0) + \pi^\ell \begin{bmatrix} f & 0 \\ 0 & F \end{bmatrix} \quad \text{with } f \in \mathbf{k},\ F \in \mathfrak{gl}_2(\mathbf{k}),$$

where F scalar corresponds to case (ii) and F a companion matrix corresponds to case (iii). We classify the similarity classes depending on the form that F takes. The shadow $\tau = \mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}})$ has one of four types:

- τ has type $\mathcal{L}$ if and only if F is scalar. There are $a_{\sigma,\tau}(q) = q^2$ choices for (f, F).
- τ has type $\mathcal{T}_1$ if and only if the characteristic polynomial of F is separable and reducible over $\mathbf{k}$. There are $a_{\sigma,\tau}(q) = \frac{1}{2}(q-1)q^2$ choices for (f, F).
- τ has type $\mathcal{T}_2$ if and only if the characteristic polynomial of F is irreducible over $\mathbf{k}$. There are $a_{\sigma,\tau}(q) = \frac{1}{2}(q-1)q^2$ choices for (f, F).

- τ has type $\mathcal{M}$ if and only if the minimal polynomial of F is of the form $(x-\alpha)^2$ for some $\alpha \in \mathbf{k}$. There are $a_{\sigma,\tau}(q) = q^2$ choices for (f, F).

($\mathcal{J}$) Suppose that σ has type $\mathcal{J}$. In this case we may assume, by Theorem 2.11, that

$$A_\ell = d\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & c \end{bmatrix}, \quad \text{where } d \in \mathfrak{o}_\ell,\ 0 \le i < \ell \text{ and } c \in \mathfrak{o}_{\ell-i} \text{ with } v(c) > 0.$$

Theorem 2.11 yields a complete list of representatives for the similarity classes $\widetilde{\mathcal{C}}$ of matrices in $\mathfrak{gl}_3(\mathfrak{o}_{\ell+1})$ lying above A_ℓ. The shadow $\tau = \mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}})$ has one of five types:

- τ is of type $\mathcal{J}$ if and only if the lift of A_ℓ is conjugate to

$$\widetilde{A}_{\ell+1} = d'\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & c' \end{bmatrix},$$

  where $c' \in \mathfrak{o}_{\ell-i+1}$, $d' \in \mathfrak{o}_{\ell+1}$ are arbitrary lifts of c, d. Consequently there are $a_{\sigma,\tau}(q) = q^2$ choices.
- τ is of type $\mathcal{M}$ if and only if the lift of A_ℓ is conjugate to

$$\widetilde{A}_{\ell+1} = \varsigma(d)\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & \pi^{\ell-i} & 0 \\ 0 & 0 & 1 \\ a'\pi^{\ell-i} & b'\pi^{\ell-i} & c' \end{bmatrix},$$

  where $a' \in \mathbf{k}$, $b' \in \mathbf{k}^\times$, and $c' \in \mathfrak{o}_{\ell-i+1}$ is an arbitrary lift of c. There are $a_{\sigma,\tau}(q) = (q-1)q^2$ choices.
- τ is of type $\mathcal{N}$ if and only if the lift of A_ℓ is conjugate to

$$\widetilde{A}_{\ell+1} = \varsigma(d)\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & \pi^{\ell-i} & 0 \\ 0 & 0 & 1 \\ a'\pi^{\ell-i} & 0 & c' \end{bmatrix},$$

  where $a' \in \mathbf{k}^\times$ and $c' \in \mathfrak{o}_{\ell-i+1}$ is an arbitrary lift of c. There are $a_{\sigma,\tau}(q) = (q-1)q$ choices.
- τ is of type $\mathcal{K}_0$ or $\mathcal{K}_\infty$ if and only if the lift $\widetilde{A}_{\ell+1}$ of A_ℓ is conjugate to a matrix of the form

$$\widetilde{A}_{\ell+1}(0) = \varsigma(d)\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & \pi^{\ell-i} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & c' \end{bmatrix} \quad \text{or} \quad \widetilde{A}_{\ell+1}(\infty) = \varsigma(d)\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \pi^{\ell-i} & 0 & c' \end{bmatrix},$$

  where $c' \in \mathfrak{o}_{\ell-i+1}$ is a lift of c; recall that $\pi^{\ell-i} = \varsigma(0)$ for $0 \in \mathfrak{o}_{\ell-i}$. Matrices of the forms $\widetilde{A}_{\ell+1}(0)$ and $\widetilde{A}_{\ell+1}(\infty)$ are never conjugate and we have $a_{\sigma,\tau}(q) = q$; see Section 2.2.

($\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{M}, \mathcal{N}, \mathcal{K}_0, \mathcal{K}_\infty$) Suppose that σ has type equal to one of $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{K}_0$, $\mathcal{K}_\infty$. From Table 2.1 we observe that all these cases are minimal in the sense that $\mathrm{sh}_{\mathfrak{gl}}(A_\ell)$ cannot properly contain the Lie centraliser shadow of any other type. This implies that the shadow associated to any lift $\widetilde{A}_{\ell+1}$ of the matrix A_ℓ satisfies $\mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}}) = \mathrm{sh}_{\mathsf{GL}}(\mathcal{C})$. Therefore, in all the cases under consideration Proposition 2.5 and Definition 2.6 yield

$$a_{\sigma,\sigma}(q) = q^{\dim \mathfrak{gl}_3} / b^{(1)}_{\sigma,\sigma}(q) = q^{\dim(\sigma)} = q^3.$$

3. Similarity classes of anti-hermitian integral $\mathfrak{p}$-adic matrices

Let $\mathfrak{o}$ be a compact discrete valuation ring, with valuation ideal $\mathfrak{p}$ and finite residue field $\mathbf{k}$ such that $p := \mathrm{char}(\mathbf{k}) \neq 2$. Otherwise we impose no restriction on the characteristic of $\mathfrak{o}$ or $\mathbf{k}$. Put $q = |\mathbf{k}|$ and fix a uniformiser π of $\mathfrak{o}$; we observe that $q > 2$.

Let $\mathfrak{O}$ be an unramified quadratic extension of $\mathfrak{o}$, with valuation ideal $\mathfrak{P}$ and residue field $\mathbf{k}_2$, a quadratic extension of $\mathbf{k}$. Then $\mathfrak{O} = \mathfrak{o}[\delta]$, where $\delta = \sqrt{\rho}$ for an element $\rho \in \mathfrak{o}$ whose reduction modulo $\mathfrak{p}$ is a non-square in $\mathbf{k}$, and $\mathfrak{P} = \pi\mathfrak{O}$. For $\ell \in \mathbb{N}_0$, we write δ_ℓ, or even δ, for the image of δ modulo $\mathfrak{p}^\ell$. Let $\boldsymbol{\mathfrak{O}}$ denote the integral closure of $\mathfrak{O}$ in some fixed algebraic closure of its fraction field, and choose an $\mathfrak{o}$-automorphism $\circ$ of $\boldsymbol{\mathfrak{O}}$ restricting to the non-trivial Galois automorphism of the quadratic extension $\mathfrak{O} \mid \mathfrak{o}$. In particular, $(a + b\delta)^\circ = a - b\delta$ for all $a, b \in \mathfrak{o}$.

Let $n \in \mathbb{N}$. We extend $\circ$ to obtain the *standard* $(\mathfrak{O}, \mathfrak{o})$*-involution* 'conjugate transpose' on the $\mathfrak{O}$-algebra $\mathsf{Mat}_n(\mathfrak{O})$, i.e.

$$A^\circ = \left((a_{ij}^\circ)\right)^{\mathrm{tr}} = (a_{ji}^\circ)_{ij} \qquad \text{for } A = (a_{ij}) \in \mathfrak{gl}_n(\mathfrak{O}). \tag{3.1}$$

A matrix $A \in \mathfrak{gl}_n(\mathfrak{O})$ is *hermitian* if $A^\circ = A$ and *anti-hermitian* if $A^\circ = -A$. The *standard unitary group* over $\mathfrak{o}$ and the corresponding *standard unitary* $\mathfrak{o}$*-Lie lattice* are

$$\mathsf{GU}_n(\mathfrak{o}) = \{A \in \mathsf{GL}_n(\mathfrak{O}) \mid A^\circ A = \mathrm{Id}_n\} \quad \text{and} \quad \mathfrak{gu}_n(\mathfrak{o}) = \{A \in \mathfrak{gl}_n(\mathfrak{O}) \mid A^\circ + A = 0\}. \tag{3.2}$$

The associated *special unitary group* and *special unitary Lie lattice* are

$$\mathsf{SU}_n(\mathfrak{o}) = \mathsf{GU}_n(\mathfrak{o}) \cap \mathsf{SL}_n(\mathfrak{O}) \quad \text{and} \quad \mathfrak{su}_n(\mathfrak{o}) = \mathfrak{gu}_n(\mathfrak{o}) \cap \mathfrak{sl}_n(\mathfrak{O}).$$

For $\ell \in \mathbb{N}$, we write $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$, $\mathfrak{O}_\ell = \mathfrak{O}/\mathfrak{P}^\ell$ and correspondingly $\mathsf{GU}_n(\mathfrak{o}_\ell)$, $\mathfrak{gu}_n(\mathfrak{o}_\ell)$ et cetera. A matrix $A \in \mathfrak{gl}_n(\mathfrak{O}_\ell)$ is called *hermitian*, respectively *anti-hermitian*, if it is the image of a hermitian, respectively anti-hermitian matrix, modulo $\mathfrak{P}^\ell$.

Eigenvalues of matrices $A \in \mathfrak{gl}_n(\mathfrak{O})$ are taken in the fixed extension $\boldsymbol{\mathfrak{O}}$ so that $\circ$ can be applied to them. Throughout we shall also use $\circ$ to denote the induced action on finite quotients $\mathfrak{gl}_n(\mathfrak{O}_\ell)$ obtained by reduction modulo $\mathfrak{P}^\ell$.

3.1. Preliminaries.

We collect some auxiliary results regarding hermitian and anti-hermitian matrices over discrete valuation rings, starting with an analogue of Proposition 2.4.

Proposition 3.1. *Let $\mathcal{X}_{n,\ell}^{\mathsf{GU}}$ be the Greenberg transform of level ℓ or the Weil restriction of the $\mathfrak{o}_\ell$-scheme GU_n to $\mathbf{k}$-schemes, depending on whether* $\mathrm{char}(\mathfrak{o}) = 0$ *or* $\mathrm{char}(\mathfrak{o}) > 0$, *so that $\mathcal{X}_{n,\ell}^{\mathsf{GU}}(\mathbf{k}) \simeq \mathsf{GU}_n(\mathfrak{o}_\ell)$. Let $A \in \mathfrak{gu}_n(\mathfrak{o}_\ell) \subset \mathfrak{gl}_n(\mathfrak{O}_\ell)$. Then there exists a $\mathbf{k}$-defined connected algebraic subgroup $\mathbf{C}$ of $\mathcal{X}_{n,\ell}^{\mathsf{GU}}$ such that*

$$\mathbf{C}(\mathbf{k}) \simeq \mathrm{C}_{\mathsf{GU}_n(\mathfrak{o}_\ell)}(A).$$

Proof. Let $\mathcal{X}_{n,\ell}^{\mathsf{GL}}$ and $\mathbf{C}^{\mathsf{GL}}$ denote the connected $\mathbf{k}_2$-algebraic groups, supplied by the Greenberg functor (respectively Weil restriction) from $\mathfrak{O}$-schemes to $\mathbf{k}_2$-schemes, such that $\mathcal{X}_{n,\ell}^{\mathsf{GL}}(\mathbf{k}_2) \simeq \mathsf{GL}_n(\mathfrak{O}_\ell)$ and $\mathbf{C}^{\mathsf{GL}}(\mathbf{k}_2) \simeq \mathrm{C}_{\mathsf{GL}_n(\mathfrak{O}_\ell)}(A)$; compare Proposition 2.4.

The existence of a $\mathbf{k}$-algebraic group $\mathbf{C}$ such that $\mathbf{C}(\mathbf{k}) \simeq \mathrm{C}_{\mathsf{GU}_n(\mathfrak{o}_\ell)}(A)$ is guaranteed by the general properties of the Greenberg transform (respectively Weil restriction) from $\mathfrak{o}$-schemes to $\mathbf{k}$-schemes. To see that $\mathbf{C}$ is connected, it suffices to observe that $\mathbf{C} \simeq \mathbf{C}^{\mathsf{GL}}$ over $\mathbf{k}_2$. $\square$

An important consequence of Proposition 3.1 is the following.

Proposition 3.2. *Let $A, B \in \mathfrak{gu}_n(\mathfrak{o})$ be similar, i.e.* $\mathrm{Ad}(\mathsf{GL}_n(\mathfrak{O}))$*-conjugate. Then A, B are already* $\mathrm{Ad}(\mathsf{GU}_n(\mathfrak{o}))$*-conjugate.*

Proof. It suffices to prove the claim modulo $\mathfrak{P}^\ell$, i.e. that the images $A_\ell, B_\ell \in \mathsf{gu}_n(\mathfrak{o}_\ell)$ of A, B are $\mathrm{Ad}(\mathsf{GU}_n(\mathfrak{o}))$-conjugate, for $\ell \in \mathbb{N}$. Let $\mathbf{G} = \mathfrak{X}^{\mathsf{GU}}_{n,\ell}$ and $\mathbf{C}$ be as in Proposition 3.1 so that $\mathbf{G}(\mathbf{k}) \simeq \mathsf{GU}_n(\mathfrak{o}_\ell)$ and $\mathbf{C}(\mathbf{k}) \simeq \mathrm{C}_{\mathsf{GU}_n(\mathfrak{o}_\ell)}(A)$. Furthermore, let $\mathfrak{X}^{\mathsf{gu}}_{n,\ell}$ be the Greenberg transform of level ℓ (for $\mathrm{char}(\mathfrak{o}) = 0$) or the Weil restriction (for $\mathrm{char}(\mathfrak{o}) > 0$) of the $\mathfrak{o}_\ell$-scheme gu_n to $\mathbf{k}$-schemes so that $\mathfrak{X}^{\mathsf{gu}}_{n,\ell}(\mathbf{k}) \simeq \mathsf{gu}_n(\mathfrak{o}_\ell)$. Write $\mathsf{A}, \mathsf{B} \in \mathfrak{X}^{\mathsf{gu}}_{n,\ell}(\mathbf{k})$ for the elements corresponding to $A_\ell, B_\ell \in \mathsf{gu}_n(\mathfrak{o}_\ell)$, and let $\mathfrak{o}^{\mathrm{unr}}$ denote the maximal unramified extension of $\mathfrak{o}$.

Let $\mathbf{K}$ denote an algebraic closure of $\mathbf{k}$. By definition, $\mathbf{G}(\mathbf{K}) \simeq \mathsf{GL}_n(\mathfrak{o}_\ell^{\mathrm{unr}})$ acts transitively via the adjoint action on the orbit $\mathrm{Ad}(\mathbf{G}(\mathbf{K}))\mathsf{A}$ in $\mathfrak{X}^{\mathsf{gu}}_{n,\ell}(\mathbf{K}) \simeq \mathsf{gl}_n(\mathfrak{o}_\ell^{\mathrm{unr}})$. Furthermore, $\mathsf{B} \in \mathrm{Ad}(\mathbf{G}(\mathbf{K}))\mathsf{A} \cap \mathfrak{X}^{\mathsf{gu}}_{n,\ell}(\mathbf{k})$. As the stabiliser $\mathbf{C}$ of A is connected, the Lang–Steinberg theorem implies that $\mathrm{Ad}(\mathbf{G}(\mathbf{k})) \simeq \mathrm{Ad}(\mathsf{GU}_n(\mathfrak{o}_\ell))$ acts transitively on $\mathrm{Ad}(\mathbf{G}(\mathbf{K}))\mathsf{A} \cap \mathfrak{X}^{\mathsf{gu}}_{n,\ell}(\mathbf{k})$; see [21, Proposition 4.3.2]. Whence there is $g \in \mathsf{GU}_n(\mathfrak{o}_\ell)$ such that $\mathrm{Ad}(g)A_\ell = B_\ell$. □

Lemma 3.3. *Let $\Gamma \in \mathsf{GL}_n(\mathfrak{O})$. Then Γ is hermitian if and only if there exists $g \in \mathsf{GL}_n(\mathfrak{O})$ such that $\Gamma = g^\circ g$.*

Proof. If $\Gamma = g^\circ g$ for $g \in \mathsf{GL}_n(\mathfrak{O})$ then clearly $\Gamma^\circ = \Gamma$. For the converse direction, suppose that Γ is hermitian. It suffices to construct recursively a sequence $g_\ell \in \mathsf{GL}_n(\mathfrak{O})$, $\ell \in \mathbb{N}$, such that $\Gamma \equiv_{\mathfrak{P}^\ell} g_\ell^\circ g_\ell$ for every $\ell \in \mathbb{N}$. The existence of g_1 is guaranteed by the theory of hermitian matrices over finite fields; e.g. see [16, p. 16]. Now suppose that g_ℓ has been constructed for some $\ell \in \mathbb{N}$. Then $\Gamma - g_\ell^\circ g_\ell = \pi^\ell \Delta$, where $\Delta \in \mathsf{gl}_n(\mathfrak{O})$ is hermitian. Thus $g_{\ell+1} = g_\ell + \frac{1}{2}\pi^\ell (g_\ell^\circ)^{-1}\Delta$ satisfies

$$\begin{aligned} g_{\ell+1}^\circ g_{\ell+1} &= \left(g_\ell^\circ + \tfrac{1}{2}\pi^\ell \Delta g_\ell^{-1}\right)\left(g_\ell + \tfrac{1}{2}\pi^\ell (g_\ell^\circ)^{-1}\Delta\right) \\ &\equiv_{\mathfrak{P}^{\ell+1}} g_\ell^\circ g_\ell + \pi^\ell \Delta = \Gamma. \end{aligned}$$

□

Proposition 3.4. *Let $A \in \mathsf{gl}_n(\mathfrak{O}_\ell)$. Then A is $\mathrm{Ad}(\mathsf{GL}_n(\mathfrak{O}))$-conjugate to an anti-hermitian matrix if and only if there exists $\Gamma \in \mathsf{GL}_n(\mathfrak{O}_\ell)$ such that $\Gamma^\circ = \Gamma$ and $A^\circ \Gamma + \Gamma A = 0$.*

Proof. First suppose that $g \in \mathsf{GL}_n(\mathfrak{O}_\ell)$ is such that $B = \mathrm{Ad}(g)A = gAg^{-1}$ is anti-hermitian. Then $\Gamma = g^\circ g$ is hermitian and

$$A^\circ \Gamma + \Gamma A = A^\circ g^\circ g + g^\circ g A = g^\circ B^\circ g + g^\circ B g = g^\circ (B^\circ + B) g = 0.$$

For the reverse implication, suppose that $\Gamma \in \mathsf{GL}_n(\mathfrak{O}_\ell)$ satisfies $\Gamma^\circ = \Gamma$ and $A^\circ \Gamma + \Gamma A = 0$. Then by Lemma 3.3 there exists $g \in \mathsf{GL}_n(\mathfrak{O}_\ell)$ such that $\Gamma = g^\circ g$, and hence $B = gAg^{-1}$ satisfies

$$B^\circ + B = (g^\circ)^{-1} A^\circ g^\circ + gAg^{-1} = (g^\circ)^{-1}(A^\circ \Gamma + \Gamma A) g^{-1} = 0.$$

Thus B is anti-hermitian. □

Lemma 3.5. *Let $A \in \mathsf{gl}_n(\mathfrak{O}_\ell)$ with characteristic polynomial $f_A = t^n + \sum_{i=0}^{n-1} c_i t^i \in \mathfrak{O}_\ell[t]$. If A is $\mathrm{Ad}(\mathsf{GL}_n(\mathfrak{O}))$-conjugate to an anti-hermitian matrix, then $c_i^\circ = (-1)^{n-i} c_i$ for $0 \le i < n$. Conversely, if A is cyclic then the latter condition on the coefficients of f_A implies that A is $\mathrm{Ad}(\mathsf{GL}_n(\mathfrak{O}))$-conjugate to an anti-hermitian matrix.*

Proof. If A is $\mathrm{Ad}(\mathsf{GL}_n(\mathfrak{O}))$-conjugate to an anti-hermitian matrix B then, denoting the characteristic polynomial of B by f_B, we deduce from $f_A = f_B$ and $B^\circ + B = 0$ that

$$t^n + \sum_{i=0}^{n-1} c_i^\circ t^i = (f_B)^\circ = f_{B^\circ} = f_{-B} = (-1)^n f_B(-t) = t^n + \sum_{i=0}^{n-1} (-1)^{n-i} c_i t^i.$$

Now suppose that A is cyclic and that $c_i^\circ = (-1)^{n-i} c_i$ for $0 \le i < n$. Without loss of generality $A = (a_{ij})$ is a companion matrix for f_A, i.e. $a_{ij} = 1$ if $i = j+1$, $a_{ij} = -c_{i-1}$ if $j = n$, and $a_{ij} = 0$ in all other cases. Define $\Gamma = (\gamma_{ij}) \in \mathsf{gl}_n(\mathfrak{O}_\ell)$ as follows: γ_{ij} is the coefficient of

$\bar{t}^{n-1}$ in the expression of $(-\bar{t})^{i-1}\bar{t}^{j-1}$ as an $\mathfrak{O}_\ell$-linear combination of $\bar{1}, \bar{t}, \ldots, \bar{t}^{n-1}$ modulo f_A. A short computation shows that $\Gamma \in \mathsf{GL}_n(\mathfrak{O}_\ell)$ with $\Gamma^\circ = \Gamma$ and $A^\circ\Gamma + \Gamma A = 0$; thus A is $\mathsf{GL}_n(\mathfrak{O})$-conjugate to an anti-hermitian matrix by Proposition 3.4.

Indeed, the free $\mathfrak{O}_\ell$-module $\mathfrak{O}_\ell[t]/f_A\mathfrak{O}_\ell[t]$ with $(\mathfrak{O}_\ell, \mathfrak{o}_\ell)$-involution $\circ$ admits the non-degenerate $\circ$-hermitian form $\langle \cdot, \cdot \rangle$, where $\langle g, h \rangle$ is the coefficient of $\bar{t}^{n-1}$ in the expression of $g^\circ(-\bar{t})h(\bar{t})$ as an $\mathfrak{O}_\ell$-linear combination of the basis $\bar{1}, \bar{t}, \ldots, \bar{t}^{n-1}$. The matrix Γ is the structure matrix of this hermitian form and A is the coordinate matrix of the endomorphism given by multiplication by $\bar{t}$, with respect to the basis $\bar{1}, \bar{t}, \ldots, \bar{t}^{n-1}$. $\square$

3.2. **Similarity class tree, centralisers, and unitary shadows.** The following concepts are analogous to the ones introduced in Definitions 2.1 and 2.2.

Definition 3.6. For $\ell \in \mathbb{N}_0$ let $\mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o},\ell} = \mathrm{Ad}(\mathsf{GU}_n(\mathfrak{o}))\backslash \mathfrak{gu}_n(\mathfrak{o}_\ell)$ denote the set of $\mathrm{Ad}(\mathsf{GU}_n(\mathfrak{o}))$-orbits in $\mathfrak{gu}_n(\mathfrak{o}_\ell)$; by Proposition 3.2, this is the same as the collection of $\mathsf{GL}_n(\mathfrak{O})$-similarity classes in $\mathfrak{gu}_n(\mathfrak{o}_\ell) \subset \mathfrak{gl}_n(\mathfrak{O})$, obtained by intersection. We endow

$$\mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o}} = \coprod_{\ell=0}^{\infty} \mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o},\ell}$$

with the structure of a directed graph, induced by reduction modulo powers of $\mathfrak{P}$: vertices $\mathcal{C} \in \mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o},\ell}$ and $\widetilde{\mathcal{C}} \in \mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o},\ell+1}$ are connected by a directed edge $(\mathcal{C}, \widetilde{\mathcal{C}})$ if the reduction of $\widetilde{\mathcal{C}}$ modulo $\mathfrak{P}^\ell$ is equal to $\mathcal{C}$. In this way $\mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o}}$ becomes an infinite rooted subtree of $\mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{O}}$. We refer to $\mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o}}$ as the *anti-hermitian similarity class tree* in degree n over $\mathfrak{o}$.

Let $\ell \in \mathbb{N}_0$ and let $A \in \mathfrak{gu}_n(\mathfrak{o}_\ell)$. The centraliser $\mathrm{C}_{\mathsf{GU}_n(\mathfrak{o})}(A)$ of A in the group $\mathsf{GU}_n(\mathfrak{o})$ is the stabiliser of A under the adjoint action of $\mathsf{GU}_n(\mathfrak{o})$. The centraliser $\mathrm{C}_{\mathfrak{gu}_n(\mathfrak{o})}(A)$ of A in the $\mathfrak{o}$-Lie lattice $\mathfrak{gu}_n(\mathfrak{o})$ is the stabiliser of A under the adjoint action of $\mathfrak{gu}_n(\mathfrak{o})$.

Definition 3.7. Let $\ell \in \mathbb{N}_0$. The *group centraliser shadow* $\mathrm{sh}_{\mathsf{GU}}(A)$ of an element $A \in \mathfrak{gu}_n(\mathfrak{o}_\ell)$ is the image $\overline{\mathrm{C}_{\mathsf{GU}_n(\mathfrak{o})}(A)} \le \mathsf{GU}_n(\mathbf{k})$ of $\mathrm{C}_{\mathsf{GU}_n(\mathfrak{o})}(A)$ under reduction modulo $\mathfrak{P}$. The *Lie centraliser shadow* $\mathrm{sh}_{\mathfrak{gu}}(A)$ of an element $A \in \mathfrak{gu}_n(\mathfrak{o}_\ell)$ is the image $\overline{\mathrm{C}_{\mathfrak{gu}_n(\mathfrak{o})}(A)} \le \mathfrak{gu}_n(\mathbf{k})$ of $\mathrm{C}_{\mathfrak{gu}_n(\mathfrak{o})}(A)$ under reduction modulo $\mathfrak{P}$.

For each similarity class $\mathcal{C}$ in $\mathfrak{gu}_n(\mathfrak{o}_\ell)$ we define the *unitary* (*similarity class*) *shadow*

$$\mathrm{sh}_{\mathsf{GU}}(\mathcal{C}) = \{(\mathrm{sh}_{\mathsf{GU}}(A), \mathrm{sh}_{\mathfrak{gu}}(A)) \mid A \in \mathcal{C}\},$$

of $\mathcal{C}$ and we denote the collection of all unitary shadows by

$$\mathfrak{Sh}_{\mathsf{GU}_n(\mathfrak{o})} = \{\mathrm{sh}_{\mathsf{GU}}(\mathcal{C}) \mid \mathcal{C} \in \mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o},\ell} \text{ for some } \ell \in \mathbb{N}_0\}. \tag{3.3}$$

For $\sigma \in \mathfrak{Sh}_{\mathsf{GU}_n(\mathfrak{o})}$ we set

$$\|\sigma\| = |\mathrm{sh}_{\mathsf{GU}}(A)| \qquad \text{and} \qquad \dim(\sigma) = \dim_{\mathbf{k}}(\mathrm{sh}_{\mathfrak{gu}}(A)),$$

where $A \in \mathcal{C} \in \mathcal{Q}^{\mathfrak{gu}_n}_{\mathfrak{o},\ell}$ for some $\ell \in \mathbb{N}_0$ with $\sigma = \mathrm{sh}_{\mathsf{GU}}(\mathcal{C})$; furthermore, it is convenient to select one group centraliser shadow $\mathrm{sh}_{\mathsf{GU}}(A)$, where $A \in \mathcal{C} \in \mathcal{Q}^{\mathfrak{gl}_n}_{\mathfrak{o},\ell}$ with $\sigma = \mathrm{sh}_{\mathsf{GU}}(\mathcal{C})$, and to denote it by $\sigma(\mathbf{k})$. We only use properties of $\sigma(\mathbf{k})$ that are independent of the arbitrary choice involved in its definition.

Comments similar to those in connection with Definition 2.2 apply. In order to see that a unitary shadow σ is, in fact, completely determined by the $\mathsf{GU}_n(\mathbf{k})$-conjugacy class of $\sigma(\mathbf{k})$, or alternatively the associated Lie algebra, we recall the Cayley maps.

Definition 3.8. For any subset $Y \subset \mathsf{gl}_n(\mathfrak{O})$, we denote by $Y_{\mathrm{gen}} \subset Y$ the set of elements that do not have an eigenvalue congruent to -1 modulo $\mathfrak{P}$. The *Cayley maps*

$$\mathrm{cay}\colon \mathsf{GU}_n(\mathfrak{o})_{\mathrm{gen}} \to \mathsf{gu}_n(\mathfrak{o})_{\mathrm{gen}} \qquad \text{and} \qquad \mathrm{Cay}\colon \mathsf{gu}_n(\mathfrak{o})_{\mathrm{gen}} \to \mathsf{GU}_n(\mathfrak{o})_{\mathrm{gen}}$$

are both defined by the mapping rule

$$y \mapsto (\mathrm{Id}_n - y)(\mathrm{Id}_n + y)^{-1} = (\mathrm{Id}_n + y)^{-1}(\mathrm{Id}_n - y). \tag{3.4}$$

The Cayley maps are easily seen to be mutual inverses of each other; see [58, II.10 and VI.2]. Furthermore, they commute with the adjoint action and preserve congruence levels. Thus they induce Cayley maps between the finite quotients $\mathsf{GU}_n(\mathfrak{o}_\ell)_{\mathrm{gen}}$ and $\mathsf{gu}_n(\mathfrak{o}_\ell)_{\mathrm{gen}}$ for each $\ell \in \mathbb{N}$.

The next lemma can be regarded as a 'unitary' version of Lemma 2.3; recall that throughout $\mathfrak{o}$ does not have residue characteristic 2 so that, in particular, $q > 2$.

Lemma 3.9. *Let $\ell \in \mathbb{N}_0$ and $A \in \mathsf{gu}_n(\mathfrak{o}_\ell)$. Then the group centraliser shadow $\mathrm{sh}_{\mathsf{GU}}(A)$ and the Lie centraliser shadow $\mathrm{sh}_{\mathsf{gu}}(A)$ determine one another in the following way:*

$$\mathrm{sh}_{\mathsf{gu}}(A) = \langle \mathrm{cay}(\mathrm{sh}_{\mathsf{GU}}(A)_{\mathrm{gen}}) \cup \{\overline{a\,\mathrm{Id}_n} \mid a \in \mathsf{gu}_1(\mathfrak{o})\}\rangle_{+\text{-span}}, \tag{3.5}$$

$$\mathrm{sh}_{\mathsf{GU}}(A) = \langle \mathrm{Cay}(\mathrm{sh}_{\mathsf{gu}}(A)_{\mathrm{gen}}) \cup \{\overline{a\,\mathrm{Id}_n} \mid a \in \mathsf{GU}_1(\mathfrak{o})\}\rangle. \tag{3.6}$$

Proof. A direct computation yields $\mathrm{cay}(\mathrm{sh}_{\mathsf{GU}}(A)_{\mathrm{gen}}) \subset \mathrm{sh}_{\mathsf{gu}}(A)$ and $\mathrm{Cay}(\mathrm{sh}_{\mathsf{gu}}(A)_{\mathrm{gen}}) \subset \mathrm{sh}_{\mathsf{GU}}(A)$. Thus the left-hand side contains the right-hand side in (3.5) and (3.6), and it suffices to prove the reverse inclusions. First consider (3.5). Let $\overline{X} \in \mathrm{sh}_{\mathsf{gu}}(A)$ be the image of $X \in \mathrm{C}_{\mathsf{gu}_n(\mathfrak{o})}(A)$. We argue below that, as in the proof of Lemma 2.3, it suffices to consider the situation

$$X = \mathrm{diag}(Y_1, Y_2) = X_1 + X_2, \tag{3.7}$$

where

- $X_1 = \mathrm{diag}(Y_1, 0), X_2 = \mathrm{diag}(0, Y_2) \in \mathrm{C}_{\mathsf{gu}_n(\mathfrak{o})}(A)$ are anti-hermitian,
- the eigenvalues of $\overline{X_1} \in \mathrm{sh}_{\mathsf{gu}}(A)$ are in $\{0, 1, -1\}$ and $\overline{X_2}$ does not have eigenvalue -1.

As $\delta \in \mathsf{gu}_1(\mathfrak{o}) \smallsetminus \{0\}$, we obtain

$$\overline{X} = \overline{\mathrm{cay}(\mathrm{Cay}(X_1 - \delta\,\mathrm{Id}_n))} + \overline{\delta\,\mathrm{Id}_n} + \overline{\mathrm{cay}(\mathrm{Cay}(X_2))}.$$

To justify (3.7), observe that X acts, by left multiplication, as an anti-hermitian operator on $V = \mathfrak{O}^n$, equipped with the standard hermitian form $\langle v_1, v_2 \rangle = v_1^\circ v_2$. By Hensel's Lemma, the characteristic polynomial $f_X \in \mathfrak{O}[t]$ factorises as a product $f_X = f_1 f_2$ of coprime monic polynomials so that $\overline{f_1} = \gcd(\overline{f_X}, \overline{(t^2-1)^n})$. The roots $\lambda_1, \ldots, \lambda_n$ of f_X are, up to permutation, equal to $-\lambda_1^\circ, \ldots, -\lambda_n^\circ$. Consequently, the roots of f_1 come in pairs $\mu, -\mu^\circ$ so that $f_1(X)^\circ + f_1(X) = 0$, and $f_1(X)$ is skew-adjoint as an operator on V. Hence V decomposes as a direct orthogonal sum of the X-invariant spaces $U = \ker f_1(X)$ and $W = \mathrm{Im}\, f_1(X)$. The standard form $\langle \cdot, \cdot \rangle$ restricts to non-degenerate, hence standard forms on U and W; see Lemma 3.3. Concatenating suitable bases for U and W, we may assume that $X = X_1 + X_2$, where $X_1 = \mathrm{diag}(Y_1, 0)$, $X_2 = \mathrm{diag}(0, Y_2)$ and the anti-hermitian matrices Y_1, Y_2 describe the restrictions of X to U, W. Since X_1, X_2 can be expressed as polynomials in X we deduce that $X_1, X_2 \in \mathrm{C}_{\mathsf{gu}_n(\mathfrak{o})}(A)$.

Next consider the pending inclusion in (3.6). Let $\overline{B} \in \mathrm{sh}_{\mathsf{GU}}(A)$ be the image of $B \in \mathrm{C}_{\mathsf{GU}_n(\mathfrak{o})}(A)$. We argue below that it suffices to consider the situation

$$B = \mathrm{diag}(C_1, C_2) = B_1 B_2, \tag{3.8}$$

where

- $B_1 = \mathrm{diag}(C_1, \mathrm{Id}_{n-m}), B_2 = \mathrm{diag}(\mathrm{Id}_m, C_2) \in \mathrm{C}_{\mathsf{GU}_n(\mathfrak{o})}(A)$ are unitary,
- the eigenvalues of $\overline{B_1} \in \mathrm{sh}_{\mathsf{GU}}(A)$ are in $\{1, -1\}$ and $\overline{B_2}$ does not have eigenvalue -1.

Choosing $a \in \mathsf{GU}_1(\mathfrak{o})$ such that $a \not\equiv_{\mathfrak{P}} \pm 1$, we obtain

$$\overline{B} = \overline{\mathrm{Cay}(\mathrm{cay}(B_1 \cdot a^\circ \, \mathrm{Id}_n))} \; \overline{(a \, \mathrm{Id}_n)} \; \overline{\mathrm{Cay}(\mathrm{cay}(B_2))}.$$

It remains to justify (3.8). Observe that B acts, by left multiplication, as a unitary operator on $V = \mathfrak{O}^n$, equipped with the standard hermitian form $\langle v_1, v_2 \rangle = v_1^\circ v_2$. By Hensel's Lemma, the characteristic polynomial $f_B \in \mathfrak{O}[t]$ factorises as a product $f_B = f_1 f_2$ of coprime monic polynomials so that $\overline{f_1} = \gcd(\overline{f_B}, \overline{(t+1)^n})$. Suppose that $f_B = \prod_{i=1}^n (t - \lambda_i)$ and $f_1 = \prod_{i=1}^m (t - \lambda_i)$. Then $\lambda_1^\circ, \ldots, \lambda_n^\circ$ are, up to permutation, equal to $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$, and thus $\lambda_1^\circ, \ldots, \lambda_m^\circ$ are, up to permutation, equal to $\lambda_1^{-1}, \ldots, \lambda_m^{-1}$. Putting $D = \prod_{i=1}^m \lambda_i^\circ \, \mathrm{Id}_n \in \mathsf{GU}_n(\mathfrak{o})$, we deduce that

$$f_1(B)^\circ = \prod\nolimits_{i=1}^m (B^\circ - \lambda_i^\circ \, \mathrm{Id}_n) = ((-B^\circ)^m D) \prod\nolimits_{i=1}^m (B - \lambda_i) = ((-B^\circ)^m D) f_1(B).$$

Hence V decomposes as a direct orthogonal sum of the B-invariant spaces $U = \ker f_1(B) = \ker f_1(B)^\circ$ and $W = \mathrm{Im}\, f_1(B)$. The standard form $\langle \cdot, \cdot \rangle$ restricts to non-degenerate, hence standard forms on U and W; see Lemma 3.3. Concatenating suitable bases for U and W, we may assume that $B = B_1 B_2$, where $B_1 = \mathrm{diag}(C_1, \mathrm{Id}_{n-m})$, $B_2 = \mathrm{diag}(\mathrm{Id}_m, C_2)$ and the unitary matrices C_1, C_2 describe the restrictions of B to U, W. Since B_1, B_2 can be expressed as polynomials in B we deduce that $B_1, B_2 \in \mathrm{C}_{\mathsf{GU}_n(\mathfrak{o})}(A)$. ☐

The following result is analogous to Proposition 2.5.

Proposition 3.10. *Let $\sigma, \tau \in \mathfrak{Sh}_{\mathsf{GU}_n(\mathfrak{o})}$. Let $\ell \in \mathbb{N}_0$ and suppose that $\widetilde{\mathcal{C}} \in \mathcal{Q}_{\mathfrak{o},\ell+1}^{\mathsf{gu}_n}$ is a class with $\mathrm{sh}_{\mathsf{GU}}(\widetilde{\mathcal{C}}) = \tau$ which lies above a class $\mathcal{C} \in \mathcal{Q}_{\mathfrak{o},\ell}^{\mathsf{gu}_n}$ with $\mathrm{sh}_{\mathsf{GU}}(\mathcal{C}) = \sigma$. Then*

$$\frac{|\widetilde{\mathcal{C}}|}{|\mathcal{C}|} = q^{\dim \mathsf{gu}_n - \dim(\sigma)} \frac{\|\sigma\|}{\|\tau\|}.$$

In particular, the ratio $|\widetilde{\mathcal{C}}|/|\mathcal{C}|$ depends only on the shadows σ, τ and not on ℓ, $\mathcal{C}$ or $\widetilde{\mathcal{C}}$.

Proof. The proof proceeds along the same lines as the proof of Proposition 2.5. The second map in (2.3) is replaced by

$$\mathsf{gu}_n(\mathfrak{o}) \to \mathsf{GU}_n^1(\mathfrak{o}), \quad X \mapsto \mathrm{Cay}(\pi X) = (\mathrm{Id}_n - \pi X)/(\mathrm{Id}_n + \pi X).$$

The other necessary translations are straightforward. ☐

In analogy with Definition 2.6, we introduce the following functions.

Definition 3.11. For $\sigma, \tau \in \mathfrak{Sh}_{\mathsf{GU}_n(\mathfrak{o})}$ let

$$b_{\sigma,\tau}^{(-1)}(q) = q^{\dim \mathsf{gu}_n - \dim(\sigma)} \frac{\|\sigma\|}{\|\tau\|}. \tag{3.9}$$

This variation of the earlier defined functions $b_{\sigma,\tau}^{(1)}(q)$ was already hinted at in Remark 2.7. Table 2.2 gives the explicit values of $b_{\sigma,\tau}^{(-1)}$ in the case $n = 3$, which can be computed with the aid of Table 3.1.

3.3. **Unitary shadows and branching rules for $\mathsf{gu}_3(\mathfrak{o}_\ell)$.** We list eight shadows in $\mathfrak{Sh}_{\mathsf{GU}_3(\mathfrak{o})}$, classified by types; compare (1.5). Recalling that $q > 2$, one sees that these are all unitary shadows arising from $\ell = 1$, i.e. arising from the centralisers of elements $A \in \mathsf{gu}_3(\mathbf{k})$. These shadows σ and the isomorphism types of $\sigma(\mathbf{k})$ are easily extracted from [5, Appendix C].

The following theorem is the counterpart of Theorem 2.8 for anti-hermitian matrices.

TABLE 3.1. Shadows in $\mathsf{GU}_3(\mathbf{k})$

Type	Minimal polynomial in $\mathbf{k}_2[t]$		Isomorphism type of $\sigma(\mathbf{k})$	$\dim(\sigma)$
$\mathcal{G}$	$t-\alpha$	$\alpha \in \mathsf{gu}_1(\mathbf{k})$	$\mathsf{GU}_3(\mathbf{k})$	9
$\mathcal{L}$	$(t-\alpha_1)(t-\alpha_2)$	$\alpha_1,\alpha_2 \in \mathsf{gu}_1(\mathbf{k})$ distinct	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_2(\mathbf{k})$	5
$\mathcal{J}$	$(t-\alpha)^2$	$\alpha \in \mathsf{gu}_1(\mathbf{k})$	$\mathsf{Heis}(\mathbf{k}) \rtimes (\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k}))$	5
$\mathcal{T}_1$	$\prod_{i=1}^3 (t-\alpha_i)$	$\alpha_1,\alpha_2,\alpha_3 \in \mathsf{gu}_1(\mathbf{k})$ distinct	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k})$	3
$\mathcal{T}_2$	$\prod_{i=1}^3 (t-\alpha_i)$	$\alpha_1 \in \mathsf{gu}_1(\mathbf{k}), \alpha_2 = -\alpha_3^\circ$ distinct	$\mathsf{GU}_1(\mathbf{k}) \times \mathbf{k}_2^\times$	3
$\mathcal{T}_3$	$f(t)$	f a suitable irreducible cubic*	$\mathsf{GU}_1(\mathbf{k}_3)$	3
$\mathcal{M}$	$(t-\alpha_1)(t-\alpha_2)^2$	$\alpha_1,\alpha_2 \in \mathsf{gu}_1(\mathbf{k})$ distinct	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k}) \times \mathsf{G}_\mathrm{a}(\mathbf{k})$	3
$\mathcal{N}$	$(t-\alpha)^3$	$\alpha \in \mathsf{gu}_1(\mathbf{k})$	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{G}_\mathrm{a}(\mathbf{k}) \times \mathsf{G}_\mathrm{a}(\mathbf{k})$	3

* We require: $f = t^3 + \sum_{i=0}^2 c_i t^i \in \mathbf{k}_2[t]$ satisfies $c_i^\circ = (-1)^{i+1} c_i$ for $0 \le i < 3$; cf. Lemma 3.5. The number of such polynomials is equal to $\frac{1}{3}|\mathsf{gu}_1(\mathbf{k}_3) \smallsetminus \mathsf{gu}_1(\mathbf{k})| = \frac{1}{3}(q^2-1)q$.

Theorem 3.12 (Classification of unitary shadows and branching rules)**.**

(1) *The set of shadows $\mathfrak{Sh}_{\mathsf{GU}_3(\mathfrak{o})}$ consists of eight elements, classified by the types*

$$\mathcal{G},\ \mathcal{L},\ \mathcal{J},\ \mathcal{T}_1,\ \mathcal{T}_2,\ \mathcal{T}_3,\ \mathcal{M},\ \mathcal{N}$$

described in Table 3.1*.*

(2) *For all $\sigma,\tau \in \mathfrak{Sh}_{\mathsf{GU}_3(\mathfrak{o})}$ there exists a polynomial $a_{\sigma,\tau} \in \mathbb{Z}[\frac{1}{6}][t]$ such that the following holds: for every $\ell \in \mathbb{N}$ and every $\mathcal{C} \in \mathcal{Q}^{\mathsf{gl}_3}_{\mathfrak{o},\ell}$ with $\mathrm{sh}_{\mathsf{GU}}(\mathcal{C}) = \sigma$ the number of classes $\widetilde{\mathcal{C}} \in \mathcal{Q}^{\mathsf{gl}_3}_{\mathfrak{o},\ell+1}$ with $\mathrm{sh}_{\mathsf{GU}}(\widetilde{\mathcal{C}}) = \tau$ lying above $\mathcal{C}$ is equal to $a_{\sigma,\tau}(q)$.*

Remark 3.13. We emphasise that the types $\mathcal{K}_0$ and $\mathcal{K}_\infty$ do not occur in the unitary setting and we refer to Table 2.2 for the explicit values of the polynomials $a_{\sigma,\tau}$ which turn out to be the same as in the general linear case.

Similar to the procedure in Section 2, we first produce a complete parametrisation for the $\mathrm{Ad}(\mathsf{GU}_3(\mathfrak{o}_\ell))$-orbits in $\mathsf{gu}_3(\mathfrak{o}_\ell)$. The proof of Theorem 3.12 is given in Section 3.5.

3.4. **Similarity classes of anti-hermitian 3×3 matrices.** For $\ell \in \mathbb{N}$, let $\mathcal{R}_3(\mathfrak{O}_\ell) \subset \mathsf{gl}_3(\mathfrak{O}_\ell)$ denote the set of representatives for similarity classes in $\mathsf{gl}_3(\mathfrak{O}_\ell)$ provided by Theorem 2.11. We use Proposition 3.4 to check for each $A \in \mathcal{R}_3(\mathfrak{O}_\ell)$ whether A is $\mathsf{GL}_3(\mathfrak{O}_\ell)$-conjugate to an anti-hermitian matrix. In this way we obtain a parametrisation $\mathcal{R}_3'(\mathfrak{O}_\ell) \subset \mathcal{R}_3(\mathfrak{O}_\ell)$ of the set of similarity classes of anti-hermitian 3×3 matrices. This is enough for our purposes, but note that only in some cases representatives $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$ are themselves anti-hermitian.

For every $\nu \in \mathbb{N}_0$ construct a 'strongly $\circ$-compatible' set of representatives

$$\varsigma(\mathfrak{O}_\nu) \subset \mathfrak{O} \qquad \text{for } \mathfrak{O}_\nu = \mathfrak{O}/\mathfrak{P}^\nu$$

in the following way. First choose a set of representatives $\varsigma_0(\mathfrak{o}_\nu) \subset \mathfrak{o}$ for $\mathfrak{o}_\nu = \mathfrak{o}/\mathfrak{p}^\nu$ such that $\varsigma_0(-a) = -\varsigma_0(a)$ for $a \in \mathfrak{o}_\nu$. Then extend ς_0, by setting $\varsigma(a_1 + a_2\delta_\nu) = \varsigma_0(a_1) + \varsigma_0(a_2)\delta$ for $a_1, a_2 \in \mathfrak{o}_\nu$. In particular, this ensures that $\varsigma(a^\circ) = \varsigma(a)^\circ$ for all $a \in \mathfrak{O}_\nu$ and $\varsigma(\mathsf{gu}_1(\mathfrak{o}_\nu)) \subset \mathsf{gu}_1(\mathfrak{o})$.

In contrast to the convention favoured in Section 2.3, this means that $\varsigma(0) = 0$; as we will see, the types $\mathcal{K}_0$ and $\mathcal{K}_\infty$ do not occur in the unitary setting so that we do not run into any conflicts. Similar to the custom in Section 2.3, we employ the notation in a flexible way; e.g. we write $\varsigma(\mathfrak{O}_\nu) \subset \mathfrak{O}_\mu$ for $\nu < \mu$ to denote the reduction of $\varsigma(\mathfrak{O}_\nu)$ modulo $\mathfrak{P}^\mu$, and we sometimes write $\pi^\nu \mathfrak{O}_{\ell-\nu}$ rather than $\pi^\nu \varsigma(\mathfrak{O}_{\ell-\nu})$. These conventions are also applied to matrices.

The next theorem gives a complete description of the similarity classes in $\mathsf{gu}_3(\mathfrak{o}_\ell)$ and their unitary shadows; it is the counterpart of Theorem 2.11 for anti-hermitian matrices.

Theorem 3.14. *The set*

$$\mathcal{R}_3'(\mathfrak{O}_\ell) = \{A \in \mathcal{R}_3(\mathfrak{O}_\ell) \mid A \text{ is } \mathsf{GL}_3(\mathfrak{O}_\ell)\text{-conjugate to an anti-hermitian matrix}\},$$

parametrising $\mathcal{Q}^{\mathsf{gu}_3}_{\mathfrak{o},\ell}$, *consists of the following matrices:*

(i) $d \operatorname{Id}_3$, *where* $d \in \mathsf{gu}_1(\mathfrak{o}_\ell)$*; the associated unitary shadow has type* $\mathcal{G}$.

(ii) $d \operatorname{Id}_3 + \pi^i D(a, 0, 0)$, *where* $0 \le i < \ell$, $d \in \mathsf{gu}_1(\mathfrak{o}_\ell)$ *and* $a \in \mathfrak{O}^\times_{\ell-i}$ *with* $a^\circ + a = 0$*; the associated unitary shadow has type* $\mathcal{L}$.

(iii)
$$d \operatorname{Id}_3 + \pi^i D(a, 0, 0) + \pi^j \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix},$$
where $0 \le i < j < \ell$, $d \in \varsigma(\mathsf{gu}_1(\mathfrak{o}_j))$, $a \in \mathfrak{O}^\times_{\ell-i}$ *with* $a^\circ + a = 0$, *and* $C \in \mathsf{gl}_2(\mathfrak{O}_{\ell-j})$ *a companion matrix with characteristic polynomial* $t^2 + b_1 t + b_0$ *such that* $b_1^\circ = -b_1$ *and* $b_0^\circ = b_0$*; the associated unitary shadows have types* $\mathcal{T}_1$, $\mathcal{T}_2$ *or* $\mathcal{M}$, *depending on* C.

(iv) $d \operatorname{Id}_3 + \pi^i C$, *where* $0 \le i < \ell$, $d \in \varsigma(\mathsf{gu}_1(\mathfrak{o}_i))$ *and* $C \in \mathsf{gl}_3(\mathfrak{O}_{\ell-i})$ *a companion matrix with characteristic polynomial* $t^3 + c_2 t^2 + c_1 t + c_0$ *such that* $c_k^\circ + (-1)^k c_k = 0$ *for* $k \in \{0, 1, 2\}$*; the associated unitary shadows have types* $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{M}$ *or* $\mathcal{N}$, *depending on* C.

(v) $d' \operatorname{Id}_3 + \pi^i E$, *where* $0 \le i < \ell$, $d' \in \varsigma(\mathsf{gu}_1(\mathfrak{O}_{\ell-i}))$ *and* $E \in \mathsf{gl}_3(\mathfrak{O}_{\ell-i})$ *is one of the following matrices:*
  - (I) $E(\ell - i, 0, 0, c, d)$, *where* $c, d \in \mathsf{gu}_1(\mathfrak{o}_{\ell-i})$ *with* $v(c) > 0$,
  - (II) $E(\mu, a, b, c, d)$, *where* $1 \le \mu < \ell - i$, $a, b \in \mathfrak{O}_{\ell-i}$ *with* $\mu = v(b) \le v(a)$ *and* $a^\circ + a = b^\circ - b = 0$, $c \in \mathsf{gu}_1(\mathfrak{o}_{\ell-i})$ *with* $v(c) > 0$ *and* $d \in \varsigma(\mathsf{gu}_1(\mathfrak{o}_\mu))$,
  - (III$_1$) $E(\mu, a, b, c, d)$, *where* $1 \le \mu < \ell - i$, $a, b \in \mathfrak{O}_{\ell-i}$ *with* $\mu = v(a) < v(b)$ *and* $a^\circ + a = b^\circ - b$, $c \in \mathsf{gu}_1(\mathfrak{o}_{\ell-i})$ *with* $v(c) > 0$ *and* $d \in \varsigma(\mathsf{gu}_1(\mathfrak{o}_\mu))$*;*

  the associated unitary shadows in these subcases have types $\mathcal{J}$, $\mathcal{M}$ *and* $\mathcal{N}$.

Proof. The proof is based on Theorem 2.11: we go through the cases described there and keep track of which similarity classes in $\mathsf{gl}_3(\mathfrak{O}_\ell)$ intersect $\mathsf{gu}_3(\mathfrak{o}_\ell)$ non-trivially. The latter is achieved by using the criterion provided by Proposition 3.4. On the way we pin down in each case the Lie centraliser shadow of $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$, with respect to a suitable hermitian form, to determine the unitary shadow associated to A; see Lemma 3.9.

Let $A \in \mathcal{R}_3(\mathfrak{O}_\ell)$ be one of the matrices representing a similarity class in $\mathsf{gl}_3(\mathfrak{O}_\ell)$.

(i) The similarity class $\{A\}$ of $A = d \operatorname{Id}_3 \in \mathsf{gl}_3(\mathfrak{O}_\ell)$ intersects $\mathsf{gu}_3(\mathfrak{o}_\ell)$ if and only $d^\circ + d = 0$, that is $d \in \mathsf{gu}_1(\mathfrak{o}_\ell)$. In this case the associated unitary shadow has type $\mathcal{G}$.

(ii) Let $A = d \operatorname{Id}_3 + \pi^i D(a, 0, 0) \in \mathsf{gl}_3(\mathfrak{O}_\ell)$, with $0 \le i < \ell$, $d \in \mathfrak{O}_\ell$ and $a \in \varsigma(\mathfrak{O}^\times_{\ell-i})$. Then $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$ if and only if its eigenvalues $\lambda_1 = d + \pi^i a$ and $\lambda_2 = \lambda_3 = d$ are anti-hermitian, that is, $d + d^\circ = \pi^i(a + a^\circ) = 0$. Indeed, if $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$ then its eigenvalues satisfy: $\lambda_1^\circ, \lambda_2^\circ, \lambda_3^\circ$ are equal to $-\lambda_1, -\lambda_2, -\lambda_3$, up to a permutation. As $\lambda_1 \ne \lambda_2 = \lambda_3$, we deduce that $\lambda_k^\circ = -\lambda_k$ for $k \in \{1, 2, 3\}$. For $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$ the associated unitary shadow has type $\mathcal{L}$.

(iii) Let
$$A = d \operatorname{Id}_3 + \pi^i D(a, 0, 0) + \pi^j \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \in \mathsf{gl}_3(\mathfrak{O}_\ell),$$
where $0 \le i < j < \ell$, $d \in \varsigma(\mathfrak{O}_j)$, $a \in \mathfrak{O}^\times_{\ell-i}$ and $C \in \mathsf{gl}_2(\mathfrak{O}_{\ell-j})$ a companion matrix. Applying (ii) to A modulo $\mathfrak{P}^j$, we see that $d \in \varsigma(\mathsf{gu}_1(\mathfrak{o}_j))$ whenever $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$. Let us assume that this condition is satisfied.

Proposition 3.4 shows that $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$ if and only if there exist $\Gamma \in \mathsf{GL}_3(\mathfrak{O}_\ell)$ such that $\Gamma^\circ = \Gamma$ and $A^\circ\Gamma + \Gamma A = 0$. Applying Lemma 2.13, for $A' = -A^\circ$ and $X = \Gamma$, one may further demand that Γ is block diagonal for blocks of sizes 1×1 and 2×2. Consequently $a^\circ + a = 0$ is a necessary condition and, as C is a companion matrix, the full assertion follows from Lemma 3.5.

Suppose that $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$. Using Lemma 2.13, the associated Lie centraliser shadow is built from the Lie centraliser shadow of a, i.e. $\mathfrak{gu}_1(\mathbf{k})$, and the Lie centraliser shadow of the reduction of C modulo $\mathfrak{P}$; cf. [5, Proof of Corollary 7.7]. Table 3.1 shows that the resulting unitary shadow has type $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{M}$, depending on C.

(iv) Let $A = d\,\mathrm{Id}_3 + \pi^i C \in \mathfrak{gl}_3(\mathfrak{O}_\ell)$, where $0 \le i < \ell$, $d \in \varsigma(\mathfrak{O}_i)$ and $C \in \mathfrak{gl}_3(\mathfrak{O}_{\ell-i})$ is a companion matrix with characteristic polynomial $t^3 + c_2t^2 + c_1t + c_0$. Applying (i) to A modulo $\mathfrak{P}^i$, we see that a $d \in \varsigma(\mathfrak{gu}_1(\mathfrak{o}_i))$ is a necessary condition for $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$. Let us assume that this condition is satisfied. Then $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$ if and only if C is similar to an anti-hermitian matrix. Furthermore, C is similar to an anti-hermitian matrix if and only if $c_k^\circ + (-1)^k c_k = 0$ for $k \in \{0, 1, 2\}$, by Lemma 3.5.

Suppose that $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$ so that C is similar to an anti-hermitian matrix. The associated Lie centraliser shadow is equal to the centraliser of the reduction of C modulo $\mathfrak{P}$; cf. [5, Proof of Corollary 7.7]. Inspection of Table 3.1 shows that the unitary shadows occurring are of type $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{M}$, $\mathcal{N}$, corresponding to different kinds of minimal polynomials of degree 3.

(v) Consider
$$A = d\,\mathrm{Id}_3 + \pi^i E \in \mathfrak{gl}_3(\mathfrak{O}_\ell),$$
where $0 \le i < \ell$, $d \in \mathfrak{O}_\ell$ and
$$E = E(m, a, b, c, 0) = \begin{bmatrix} 0 & \pi^m & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix} \in \mathfrak{gl}_3(\mathfrak{O}_{\ell-i})$$
with $1 \le m \le \ell - i$ and $a, b, c \in \mathfrak{O}_{\ell-i}$ such that $v(a), v(b), v(c) > 0$. Writing $\mu = \mu_{\ell-i}(m, a, b) = \min\{m, v(a), v(b), \ell-i\}$, we may assume further that $d \in \varsigma(\mathfrak{O}_{i+\mu})$; see Theorem 2.11. According to Proposition 3.4, one has $A \in \mathcal{R}_3'(\mathfrak{O}_\ell)$ if and only if there exists
$$\Gamma = \begin{bmatrix} x & y & z \\ y^\circ & u & w \\ z^\circ & w^\circ & r \end{bmatrix} \in \mathsf{GL}_3(\mathfrak{O}_\ell), \quad \text{with } x, u, r \in \mathfrak{o}_\ell, \text{ hence } \Gamma = \Gamma^\circ,$$
such that $A^\circ\Gamma + \Gamma A = 0$. Comparing matrix entries, we obtain the following equivalent system of equations over $\mathfrak{O}_\ell$:
$$\begin{aligned}
&(1,1) && 0 = (d^\circ + d)x + \pi^i(a^\circ z^\circ + az), \\
&(1,2) && 0 = (d^\circ + d)y + \pi^i(a^\circ w^\circ + \pi^m x + bz), \\
&(2,2) && 0 = (d^\circ + d)u + \pi^i(\pi^m y + \pi^m y^\circ + b^\circ w^\circ + bw), \\
&(1,3) && 0 = (d^\circ + d)z + \pi^i(a^\circ r + y + cz), \\
&(2,3) && 0 = (d^\circ + d)w + \pi^i(\pi^m z + b^\circ r + u + cw), \\
&(3,3) && 0 = (d^\circ + d)r + \pi^i(w + w^\circ + cr + c^\circ r).
\end{aligned} \tag{3.10}$$
Assume that such a matrix Γ exists. Since Γ is invertible, at least one of x, y, u is invertible. Reducing equations (1,1), (1,2), (2,2) modulo $\mathfrak{P}^{i+\mu}$, we deduce that $d^\circ + d = 0$. This leads to the following observation. By reducing equations (1,3), (2,3) and (3,3) modulo $\mathfrak{P}^{i+1}$ we deduce

that y, u, and $w + w^\circ$ are 0 modulo $\mathfrak{P}$. Consequently, x is invertible, and w, w° are invertible; otherwise the second column of Γ would be congruent to 0 modulo $\mathfrak{P}$.

Since $d^\circ + d = 0$, we deduce that $A \in \mathcal{R}'_3(\mathfrak{O}_\ell)$ if and only if E is similar to an anti-hermitian matrix. As the characteristic polynomial of E is equal to $t^3 - ct^2 + bt + \pi^m a \in \mathfrak{O}_{\ell-i}[t]$, Lemma 3.5 supplies necessary conditions for E being similar to an anti-hermitian matrix:

$$\pi^m(a^\circ + a) = b^\circ - b = c^\circ + c = 0. \tag{3.11}$$

With these at hand, (3.10) reduces to the following system of equations over $\mathfrak{O}_{\ell-i}$:

$$\begin{array}{llll} (1,1) & 0 = a^\circ z^\circ + az, & (1,3) & 0 = a^\circ r + y + cz, \\ (1,2) & 0 = a^\circ w^\circ + \pi^m x + bz, & (2,3) & 0 = \pi^m z + br + u + cw, \\ (2,2) & 0 = \pi^m(y + y^\circ), & (3,3) & 0 = w + w^\circ. \end{array} \tag{3.12}$$

Recall also that $x, u, r \in \mathfrak{o}_\ell$ are $\circ$-invariant. From now on, all computations will be carried out over $\mathfrak{O}_{\ell-i}$. We strengthen the first of the necessary conditions (3.11) to:

$$a^\circ + a = 0 \qquad \text{if } v(b) \geq m. \tag{3.13}$$

Indeed, from the equalities $(2,3)$ and $(3,3)$ in (3.12) we deduce that $\pi^m z^\circ = \pi^m z$, hence $(bz)^\circ = bz$ if $v(b) \geq m$. But then equalities $(1,2)$ and $(3,3)$ in (3.12), together with $w \in \mathfrak{O}_{\ell-i}^\times$ imply $a^\circ w^\circ = aw$ and hence $a^\circ = -a$. Below we will show that one can always arrange $v(b) \geq m$ so that the conclusion holds, in effect, unconditionally.

It is time to pin down not only necessary but, also sufficient conditions for E to be similar to an anti-hermitian matrix. From Proposition 2.15 we adapt the notation

$$F_{m,a,b,c}(t_1, t_2, s_1, s_2, s_3) = \begin{bmatrix} t_1 & \pi^m s_3 - cs_1 & s_1 \\ s_2 & t_2 & s_3 \\ as_3 & \pi^m s_2 + bs_3 & t_2 + cs_3 \end{bmatrix} \in \mathfrak{gl}_3(\mathfrak{O}_{\ell-i})$$

for $t_1, t_2, s_1, s_2, s_3 \in \mathfrak{O}_{\ell-i}$. Recall further that $\mathfrak{O} = \mathfrak{o}[\delta]$. Similar to the situation described in Theorem 2.11, we distinguish three subcases (I), (II), (III).

($\mathrm{v_I}$) Suppose that $\mu = \ell - i$, that is $\pi^m = a = b = 0$. Subject to the necessary condition $c^\circ + c = 0$ that we identified above, the matrix

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -c\delta & \delta \\ 0 & -\delta & 0 \end{bmatrix} \in \mathsf{GL}_3(\mathfrak{O}_{\ell-i})$$

is hermitian and satisfies $E^\circ\Gamma + \Gamma E = 0$. Proposition 3.4 shows that E is similar to an anti-hermitian matrix.

To determine the unitary shadow we compute the Lie centraliser of E in

$$\mathfrak{gu}_3(\mathfrak{o}_{\ell-i}; \Gamma_0) := \{Y \in \mathfrak{gl}_n(\mathfrak{O}) \mid Y^\circ\Gamma_0 + \Gamma_0 Y = 0\},$$

the unitary $\mathfrak{o}$-Lie lattice with respect to Γ_0, and subsequently reduce modulo $\mathfrak{P}$. By Proposition 2.15, the centraliser $\mathrm{C}_{\mathfrak{gl}_3(\mathfrak{O}_{\ell-i})}(E)$ consists of all matrices of the form

$$Y = F_{\infty,0,0,c}(t_1, t_2, s_1, s_2, s_3) = \begin{bmatrix} t_1 & -cs_1 & s_1 \\ s_2 & t_2 & s_3 \\ 0 & 0 & t_2 + cs_3 \end{bmatrix}.$$

The intersection $\mathfrak{u} = \mathfrak{gu}_3(\mathfrak{o}_{\ell-i};\Gamma_0) \cap \mathrm{C}_{\mathfrak{gl}_3(\mathfrak{O}_{\ell-i})}(E)$ is easily determined:

$$\mathfrak{u} = \{F_{\infty,0,0,c}(t_1,t_2,s_1,s_2,s_3) \mid t_1^\circ + t_1 = t_2^\circ + t_2 = s_3^\circ + s_3 = s_1 + \delta s_2^\circ = 0\}.$$

The reduction $\overline{\mathfrak{u}}$ modulo $\mathfrak{P}$ has dimension 5 and thus coincides with the centraliser in $\mathfrak{gu}_3(\mathbf{k};\overline{\Gamma_0})$ of the reduction $\overline{E}$ modulo $\mathfrak{P}$; see Table 3.1. Moreover, one easily computes the isomorphism type of $\overline{\mathfrak{u}}$ and deduces that the associated unitary shadow has type $\mathcal{J}$; cf. [5, Appendix C].

(v_{II}) Suppose that $1 \le \mu < \ell - i$ and $\mu = m = v(b) \le v(a)$. Recalling the necessary conditions (3.11) and (3.13), we choose $\alpha \in \mathfrak{O}_{\ell-i}$ and $\beta,\gamma \in \mathfrak{O}_{\ell-i}^\times$ with $\alpha^\circ + \alpha = \beta^\circ - \beta = \gamma^\circ + \gamma = 0$ such that $a = \pi^\mu\alpha$, $b = \pi^\mu\beta$ and $c = \pi^{v(c)}\gamma$. Furthermore, we choose $e \in \mathfrak{o}_{\ell-i}$ such that

$$x_e = \beta f - (1+\pi^\mu)\delta\alpha \in \mathfrak{o}_{\ell-i}^\times, \qquad \text{where } f = e + \delta c \in \mathfrak{o}_{\ell-i}.$$

In particular, this implies $\pi^m x_e = bf - (1+\pi^\mu)\delta a$, and the matrix

$$\Gamma_0 = \begin{bmatrix} x_e & cf & -f \\ -cf & (1+\pi^\mu)e - f & (1+\pi^\mu)\delta \\ -f & -(1+\pi^\mu)\delta & 0 \end{bmatrix} \in \mathsf{GL}_3(\mathfrak{O}_{\ell-i})$$

is hermitian and satisfies $E^\circ\Gamma_0 + \Gamma_0 E = 0$. Thus Proposition 3.4 shows that E is similar to an anti-hermitian matrix.

Next we determine the unitary shadow, using the same strategy as in case (v_{I}). By Proposition 2.15, the centraliser of E in $\mathfrak{gl}_3(\mathfrak{O}_{\ell-i})$ is given by

$$\text{(3.14)} \quad \mathrm{C}_{\mathfrak{gl}_3(\mathfrak{O}_{\ell-i})}(E) = \{F_{m,a,b,c}(t_1,t_2,s_1,s_2,s_3) \mid as_1 = \pi^m s_2, bs_1 = \pi^m(t_2 - t_1), bs_2 = a(t_2 - t_1).\}$$

We are interested in those $Y \in \mathrm{C}_{\mathfrak{gl}_3(\mathfrak{O}_{\ell-i})}(E)$ that satisfy the $\mathfrak{o}_{\ell-i}$-linear equation $Y^\circ\Gamma_0 + \Gamma_0 Y = 0$. A straightforward computation shows that parameters (t_1,t_2,s_1,s_2,s_3) leading to such Y must satisfy the following congruences modulo $\mathfrak{P}$,

$$t_1^\circ + t_1 \equiv_{\mathfrak{P}} t_2^\circ + t_2 \equiv_{\mathfrak{P}} 0, \qquad s_1 \equiv_{\mathfrak{P}} \beta^{-1}(t_2 - t_1), \qquad s_2 \equiv_{\mathfrak{P}} \alpha\beta^{-1}(t_2 - t_1).$$

We deduce that the Lie centraliser shadow of E in $\mathfrak{gu}_3(\mathbf{k};\overline{\Gamma_0})$ is at most 3-dimensional. Conversely, E itself, the scalar matrix $\delta\,\mathrm{Id}_3$, and the matrix

$$Y_0 = F_{m,a,b,c}(0,\beta\delta,\delta,\alpha\delta,0) = \begin{bmatrix} 0 & -c\delta & \delta \\ \alpha\delta & \beta\delta & 0 \\ 0 & a\delta & \beta\delta \end{bmatrix}$$

centralise E and satisfy the condition $Y^\circ\Gamma_0 + \Gamma_0 Y = 0$; to verify the assertion for Y_0, observe that $\pi^\mu c\delta^{-1}Y_0 = E^3 - (b + c^2)E - \pi^\mu a\,\mathrm{Id}_3$. Hence the Lie centraliser shadow of E in $\mathfrak{gu}_3(\mathbf{k};\overline{\Gamma_0})$ has dimension 3, and by inspection of Table 3.1 we deduce that the unitary shadow associated to E has type $\mathcal{M}$.

($\mathrm{v}_{\mathrm{III}}$) Suppose that $1 \le \mu < \ell - i$ and $\mu = \min\{m, v(a)\} < v(b) \le \ell$. Recall the necessary conditions (3.11) and (3.13). In Theorem 2.11 there is a subdivision into three cases

$$(\mathrm{III}_1)\ \ \mu = m = v(a), \qquad (\mathrm{III}_0)\ \ \mu = m < v(a), \qquad (\mathrm{III}_\infty)\ \ \mu = v(a) < m.$$

We claim that matrices corresponding to (III_0) and (III_∞) are not similar to anti-hermitian matrices. Indeed, equality $(1,2)$ in (3.12) yields $a^\circ w^\circ \equiv_{\mathfrak{P}^{\mu+1}} -\pi^m x$. Since both w° and

x are already required to be invertible, a necessary condition for the solubility of this congruence is $\mu = m = v(a^\circ) = v(a)$.

We now focus on (III$_1$); the procedure is similar to case (v$_{\mathrm{II}}$). We choose $\alpha, \gamma \in \mathfrak{O}_{\ell-i}^\times$ and $\beta \in \mathfrak{O}_{\ell-i}$ with $\alpha^\circ + \alpha = \beta^\circ - \beta = \gamma^\circ + \gamma = 0$ such that $a = \pi^\mu \alpha$, $b = \pi^\mu \beta$ and $c = \pi^{v(c)}\gamma$. Furthermore, we put $\xi = \alpha^{-1}$ so that $\xi^\circ + \xi = 0$. Subject to the necessary conditions collected above, the matrix

$$\Gamma_0 = \begin{bmatrix} 1 & -c & 1 \\ c & \xi(1+\beta)c - \pi^\mu & -\xi(1+\beta) \\ 1 & \xi(1+\beta) & 0 \end{bmatrix} \in \mathsf{GL}_3(\mathfrak{O}_{\ell-i})$$

is hermitian and satisfies $E^\circ \Gamma_0 + \Gamma_0 E = 0$. Thus Proposition 3.4 shows that E is similar to an anti-hermitian matrix.

To determine the unitary shadow, we look for solutions $Y \in \mathrm{C}_{\mathsf{gl}_3(\mathfrak{O}_{\ell-i})}(E)$, see (3.14), to the equation $Y^\circ \Gamma_0 + \Gamma_0 Y = 0$. A straightforward computation shows that parameters $(t_1, t_2, s_1, s_2, s_3)$ leading to such Y must satisfy the following congruences modulo $\mathfrak{P}$,

$$t_1 + t_1^\circ \equiv_{\mathfrak{P}} t_2 - t_1 \equiv_{\mathfrak{P}} 0, \qquad s_1 + s_1^\circ \equiv_{\mathfrak{P}} s_3 - s_3^\circ \equiv_{\mathfrak{P}} 0, \qquad s_1 \equiv_{\mathfrak{P}} \xi s_2.$$

As in the case (v$_{\mathrm{II}}$) one shows that the Lie centraliser shadow of E in $\mathsf{gu}_3(\mathbf{k}; \overline{\Gamma_0})$ is 3-dimensional. By inspection of Table 3.1 we deduce that the unitary shadow associated to B has type $\mathcal{N}$. □

3.5. **Proof of Theorem 3.12.** Part (1) of Theorem 3.12 follows from collecting the types in Theorem 3.14. To prove part (2), we proceed along the same lines as in the proof of part (2) of Theorem 2.8. Interestingly, we get the same polynomials $a_{\sigma,\tau}$ as in Theorem 2.8, with the exception of types $\mathcal{K}_0$ and $\mathcal{K}_\infty$ which do not occur in the present setting.

Let $A_\ell \in \mathsf{gl}_3(\mathfrak{O}_\ell)$ be of one of the matrices specified in Theorem 3.14 and $\mathcal{C}$ the intersection of its similarity class with $\mathsf{gu}_3(\mathfrak{o}_\ell)$. Starting from the shadow σ of $\mathcal{C}$, we determine the shadows τ associated to $\widetilde{\mathcal{C}}$, the intersections with $\mathsf{gu}_3(\mathfrak{o}_\ell)$ of the similarity classes of lifts of A_ℓ to matrices $\widetilde{A}_{\ell+1} \in \mathsf{gl}_3(\mathfrak{O}_{\ell+1})$. We also keep track of the multiplicities of such lifts: these depend only on the shadows involved and the non-zero values $a_{\sigma,\tau}(q)$ are as listed in Table 2.2.

($\mathcal{G}$) Suppose that σ has type $\mathcal{G}$. Then $A_\ell = d \operatorname{Id}_3$ with $d \in \mathsf{gu}_1(\mathfrak{o})$. Consider $\widetilde{A}_{\ell+1} = d \operatorname{Id}_3 + \pi^\ell X \in \mathsf{gl}_3(\mathfrak{O}_{\ell+1})$, with $X \in \mathsf{gl}_3(\mathbf{k}_2)$. Without loss of generality we may assume that $X \in \mathsf{gu}_3(\mathbf{k}_2)$. Then $\mathrm{sh}_{\mathsf{GU}}(\widetilde{A}_{\ell+1}) = \mathrm{sh}_{\mathsf{GU}}(X)$ and $\mathrm{sh}_{\mathsf{gu}}(\widetilde{A}_{\ell+1}) = \mathrm{sh}_{\mathsf{gu}}(X)$; furthermore, these shadows can be classified according to the shape of the minimal polynomial of X as listed in Table 3.1. The number $a_{\sigma,\tau}(q)$ of distinct lifts with shadow τ is the number of distinct minimal polynomials of the shape given in Table 3.1, paired with a compatible characteristic polynomial for type $\mathcal{L}$. Explicit formulae for the $a_{\sigma,\tau}(q)$ are given in Table 2.2.

($\mathcal{L}$) Suppose that σ has type $\mathcal{L}$. By Theorem 3.14 we may assume that $A_\ell = d \operatorname{Id}_3 + \pi^i D(a, 0, 0)$, where $0 \le i < \ell$, $d \in \mathsf{gu}_1(\mathfrak{o}_\ell)$ and $a \in \mathfrak{O}_{\ell-i}^\times$ with $a^\circ + a = 0$. Any lift of A_ℓ that is conjugate to an anti-hermitian matrix is conjugate to a matrix of the form

$$\widetilde{A}_{\ell+1} = d \operatorname{Id}_3 + \pi^i D(a, 0, 0) + \pi^\ell \begin{bmatrix} f & 0 \\ 0 & F \end{bmatrix} \quad \text{with } f \in \mathsf{gu}_1(\mathbf{k}),\ F \in \mathsf{gl}_2(\mathbf{k}_2),$$

where F scalar corresponds to case (ii) and F a companion matrix corresponds to case (iii). By the analysis of cases (ii) and (iii) in the proof of Theorem 3.14, we classify the similarity classes depending on the minimal polynomial of F. The shadow $\tau = \mathrm{sh}_{\mathsf{GU}}(\widetilde{\mathcal{C}})$ has one of four types:

- τ has type $\mathcal{L}$ if and only if F is an anti-hermitian scalar matrix. There are $a_{\sigma,\tau}(q) = q^2$ choices for (f, F).
- τ has type $\mathcal{T}_1$ if and only if F has a reducible separable minimal polynomial over $\mathbf{k}_2$ with anti-hermitian roots c_1, c_2. There are $a_{\sigma,\tau}(q) = \frac{1}{2}(q-1)q^2$ choices for (f, F).
- τ has type $\mathcal{T}_2$ if and only if F has a reducible separable minimal polynomial over $\mathbf{k}_2$ with roots satisfying $c_1 + c_2^\circ = 0$. There are $a_{\sigma,\tau}(q) = \frac{1}{2}(q-1)q^2$ choices for (f, F).
- τ has type $\mathcal{M}$ if and only if F has minimal polynomial $(x-\alpha)^2$ for some $\alpha \in \mathsf{gu}_1(\mathbf{k})$. There are $a_{\sigma,\tau}(q) = q^2$ choices for (f, F).

($\mathcal{J}$) Suppose that σ has type $\mathcal{J}$. In this case we may assume, by Theorem 3.14, that

$$A_\ell = d\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & c \end{bmatrix},$$

where $0 \leq i < \ell$, $d \in \mathsf{gu}_1(\mathfrak{O}_\ell)$ and $c \in \mathsf{gu}_1(\mathfrak{o}_{\ell-i})$ with $v(c) > 0$. Theorem 3.14 yields a complete parametrisation for the intersections $\widetilde{\mathcal{C}}$ of $\mathsf{gu}_3(\mathfrak{o}_{\ell+1})$ with similarity classes of matrices lying above A_ℓ in $\mathsf{gl}_3(\mathfrak{O}_{\ell+1})$. The shadow $\tau = \mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}})$ has one of three types:

- τ is of type $\mathcal{J}$ if and only if the lift of A_ℓ is conjugate to

$$\widetilde{A}_{\ell+1} = d'\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & c' \end{bmatrix},$$

  where c' and d' are arbitrary anti-hermitian lifts of c and d, respectively. Consequently there are $a_{\sigma,\tau}(q) = q^2$ choices.
- τ is of type $\mathcal{M}$ if and only if the lift of A_ℓ is conjugate to

$$\widetilde{A}_{\ell+1} = \varsigma(d)\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & \pi^{\ell-i} & 0 \\ 0 & 0 & 1 \\ a'\pi^{\ell-i} & b'\pi^{\ell-i} & c' \end{bmatrix},$$

  where $a' \in \mathsf{gu}_1(\mathbf{k})$, $b' \in \mathbf{k}^\times$ and $c' \in \mathsf{gu}_1(\mathfrak{o}_{\ell-i+1})$ is a lift of c. Therefore, there are $a_{\sigma,\tau}(q) = (q-1)q^2$ choices.
- τ is of type $\mathcal{N}$ if and only if the lift of A_ℓ is conjugate to

$$\widetilde{A}_{\ell+1} = \varsigma(d)\,\mathrm{Id}_3 + \pi^i \begin{bmatrix} 0 & \pi^{\ell-i} & 0 \\ 0 & 0 & 1 \\ a'\pi^{\ell-i} & 0 & c' \end{bmatrix},$$

  where $a' \in \mathsf{gu}_1(\mathbf{k})$ and $c' \in \mathsf{gu}_1(\mathfrak{o}_{\ell-i+1})$ is a lift of c. There are $a_{\sigma,\tau}(q) = (q-1)q$ choices.

($\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{M}, \mathcal{N}$) Suppose that σ has type equal to one of $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{M}$, $\mathcal{N}$. From Table 3.1 we observe that all these cases are minimal in the sense that $\mathrm{sh}_{\mathsf{gu}}(A_\ell)$ cannot properly contain the Lie centraliser shadow of any other type. This implies that the shadow associated to any lift $\widetilde{A}_{\ell+1}$ of the matrix A_ℓ satisfies $\mathrm{sh}_{\mathsf{GU}}(\widetilde{\mathcal{C}}) = \mathrm{sh}_{\mathsf{GU}}(\mathcal{C})$. Therefore, in all the cases under consideration Proposition 3.10 and Definition 3.11 yield

$$a_{\sigma,\sigma}(q) = q^{\dim \mathsf{gu}_3}/b^{(-1)}_{\sigma,\sigma}(q) = q^{\dim(\sigma)} = q^3.$$

4. Similarity class zeta functions

Let $\mathfrak{o}$ be a compact discrete valuation ring with valuation ideal $\mathfrak{p}$ and finite residue field $\mathbf{k}$ of cardinality q. In the context of anti-hermitian matrices we assume that $\mathrm{char}(\mathbf{k}) \neq 2$. There is no other restriction on the characteristic of $\mathfrak{o}$ or $\mathbf{k}$. In this section we define similarity class zeta functions of the finite spaces $\mathsf{gl}_n(\mathfrak{o}_\ell)$ and $\mathsf{gu}_n(\mathfrak{o}_\ell)$ for $\ell \in \mathbb{N}_0$, and suitable limit objects as $\ell \to \infty$. We employ the results from Sections 2 and 3 to compute, in Section 4.2, all of these functions for $n = 3$. From these we deduce Theorem E and Corollary F.

4.1. **Similarity class zeta functions and shadow graphs.** Let $n \in \mathbb{N}$. The two cases

$$\mathsf{g} = \mathsf{gl}_n \text{ and } \mathsf{G} = \mathsf{GL}_n, \qquad \mathsf{g} = \mathsf{gu}_n \text{ and } \mathsf{G} = \mathsf{GU}_n$$

of pairs of $\mathfrak{o}$-schemes are very similar and we treat them in parallel. Fix $\ell \in \mathbb{N}_0$. We write $\mathcal{Q}_\ell$ for the finite set of similarity classes $\mathcal{Q}^{\mathsf{g}}_{\mathfrak{o},\ell} = \mathrm{Ad}(\mathsf{G}(\mathfrak{o}))\backslash \mathsf{g}(\mathfrak{o}_\ell)$, introduced in Definitions 2.1 and 3.6, and let $\mathfrak{Sh}$ stand for the shadow set $\mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$; cf. (2.1) and (3.3).

Definition 4.1. The *similarity class zeta function* of $\mathsf{g}(\mathfrak{o}_\ell)$ is the Dirichlet polynomial

$$\gamma_\ell(s) := \sum_{\mathcal{C} \in \mathcal{Q}_\ell} |\mathcal{C}|^{-s}.$$

For $\sigma \in \mathfrak{Sh}$, we set

$$\gamma^\sigma_\ell(s) = \sum_{\substack{\mathcal{C} \in \mathcal{Q}_\ell \\ \mathrm{sh}(\mathcal{C}) = \sigma}} |\mathcal{C}|^{-s}, \tag{4.1}$$

yielding the natural decomposition $\gamma_\ell(s) = \sum_{\sigma \in \mathfrak{Sh}} \gamma^\sigma_\ell(s)$.

Proposition 4.2. *Let $\sigma \in \mathfrak{Sh}$ and write $\gamma^\sigma_\ell(s) = \sum_{m=1}^\infty c^\sigma_{\ell,m} m^{-s}$. For each $m \in \mathbb{N}$, the sequence $(q^{-\ell} c^\sigma_{\ell,m})_{\ell \in \mathbb{N}_0}$ is eventually constant. In particular, the normalised Dirichlet polynomials $q^{-\ell}\gamma^\sigma_\ell(s)$ converge coefficientwise to a Dirichlet series*

$$\gamma^\sigma(s) := \lim_{\ell \to \infty} q^{-\ell} \gamma^\sigma_\ell(s)$$

with non-negative rational coefficients. If $\mathrm{char}(\mathbf{k})$ does not divide n, then the numbers $c^\sigma_{\ell,m}$ are all divisible by q^ℓ, whence $\gamma^\sigma(s)$ has integral coefficients.

Proof. In principle, the coefficients $c^\sigma_{\ell,m}$ can be computed by induction on ℓ; this requires consideration of other shadows τ and class sizes $m' \leq m$. (In the special case $n = 3$ we can carry out the procedure effectively; see Lemma 4.6 and Proposition 4.7 below.) To prove that $q^{-\ell} c^\sigma_{\ell,m}$ becomes constant as $\ell \to \infty$, we observe that the q^ℓ scalar matrices are the only matrices in $\mathsf{g}(\mathfrak{o}_\ell)$ whose lifts to matrices in $\mathsf{g}(\mathfrak{o}_{\ell+1})$ do not all give rise to larger similarity classes. Indeed, Propositions 2.5 and 3.10 imply that if $\widetilde{\mathcal{C}} \subset \mathsf{g}(\mathfrak{o}_{\ell+1})$ is a class with $\mathrm{sh}_{\mathsf{G}}(\widetilde{\mathcal{C}}) = \widetilde{\tau}$ which lies above a class $\mathcal{C} \subset \mathsf{g}(\mathfrak{o}_\ell)$ with $\mathrm{sh}_{\mathsf{G}}(\mathcal{C}) = \tau$, then the quotient $|\widetilde{\mathcal{C}}|/|\mathcal{C}|$ is greater than 1 unless τ is of type $\mathcal{G}$, i.e. a scalar matrix. The branching process by which one arrives from a scalar matrix in $\mathsf{g}(\mathfrak{o}_\ell)$ to similarity classes of the given size m and shadow σ in $\mathsf{g}(\mathfrak{o}_{\ell'})$ for $\ell' > \ell$ is independent of ℓ. Due to the normalisation, the numbers $q^{-\ell} c^\sigma_{\ell,m}$ thus stabilise as $\ell \to \infty$.

Finally we argue for the integrality. Let $m \in \mathbb{N}$. For every similarity class $\mathcal{C} \in \mathcal{Q}_\ell$ of size m and shadow σ, the scalar shifts $d\,\mathrm{Id}_n + \mathcal{C}$, where $d\,\mathrm{Id}_n \in \mathsf{g}(\mathfrak{o}_\ell)$, form q^ℓ similarity classes in $\mathcal{Q}_\ell$, each of size m and shadow σ. They are all distinct since the traces of any two such shifts by $d\,\mathrm{Id}_n$ and $d'\,\mathrm{Id}_n$, say, differ by $n(d - d')$ and n is invertible in $\mathfrak{o}_\ell$. Thus each coefficient $c^\sigma_{\ell,m}$ is divisible by q^ℓ. $\square$

Remark 4.3. Corollary 4.8 below shows that the assumption $\mathrm{char}(\mathbf{k}) \nmid n$ may be necessary and that the limit functions $\gamma^\sigma(s)$, $\sigma \in \mathfrak{Sh}$, are rational functions for $n = 3$. Proposition A.2 implies the analogous fact for $n = 2$. It is an interesting question whether rationality also holds for $n \geq 4$. It is perceivable that the methods developed in the paper [27], which proves rationality of zeta functions enumerating classes of certain definable equivalence relations, are applicable in this context.

From now on let $n = 3$ so that g, G are either $\mathsf{gl}_3, \mathsf{GL}_3$ or $\mathsf{gu}_3, \mathsf{GU}_3$. It is convenient to use the parameter $\varepsilon = \varepsilon_{\mathsf{g}} = \varepsilon_{\mathsf{G}} \in \{1, -1\}$ to distinguish between the non-unitary and the unitary setting; see (1.4). Recall also the definition of shadow types $\mathbb{T}^{(\varepsilon)}$; see (1.5). For 3×3 matrices, it turned out that the similarity class trees $\mathcal{Q}_{\mathfrak{o}}^{\mathsf{gl}_3}$ and $\mathcal{Q}_{\mathfrak{o}}^{\mathsf{gu}_3}$ have a structure that can be described uniformly for different choices of $\mathfrak{o}$. This motivates the following definition.

Definition 4.4. The *shadow graph* associated to the scheme pair (g, G) is the finite directed graph $\Gamma = \Gamma^{(\varepsilon)}$ with the following vertex and edge sets

$$\begin{aligned}
&V(\Gamma) = \mathbb{T}^{(\varepsilon)},\\
&E(\Gamma) = \{(\mathcal{S}, \mathcal{T}) \in \mathbb{T}^{(\varepsilon)} \times \mathbb{T}^{(\varepsilon)} \mid \exists \sigma, \tau \in \mathfrak{Sh} \text{ of types } \mathcal{S}, \mathcal{T} : a_{\sigma,\tau} \neq 0\}.
\end{aligned}$$

In the following it is convenient to refer to $\dot{V}(\Gamma) := \mathfrak{Sh}$ and $\dot{E}(\Gamma) := \{(\sigma, \tau) \in \mathfrak{Sh} \times \mathfrak{Sh} \mid a_{\sigma,\tau} \neq 0\}$, suppressing the implicit dependency on $\mathfrak{o}$.

Remark 4.5. Recall that the polynomials $a_{\sigma,\tau} \in \mathbb{Z}[\frac{1}{6}][t]$ are defined in Theorems 2.8 and 3.12, and tabulated in Table 2.2. The results in Sections 2 and 3 imply that Γ is naturally isomorphic to the directed graph $\dot{\Gamma}$ with vertex set $V(\dot{\Gamma}) = \dot{V}(\Gamma)$ and edge set $E(\dot{\Gamma}) = \dot{E}(\Gamma)$. The graph $\dot{\Gamma}$ in turn is nothing but the quotient graph of the rooted tree $\mathcal{Q} := \mathcal{Q}_{\mathfrak{o}}^{\mathsf{g}}$ (cf. Definitions 2.1 and 3.6) induced by the map $V(\mathcal{Q}) \to \mathfrak{Sh}$, $\mathcal{C} \mapsto \mathrm{sh}(\mathcal{C})$.

By Theorems 2.8 and 3.12, the shadow graph Γ and the data $a_\xi(q)$, $\xi \in \dot{E}(\Gamma)$, determine the tree $\mathcal{Q}$ completely, whereas the graph Γ and the data $b_\xi^{(\varepsilon)}(q)$, $\xi \in \dot{E}(\Gamma)$, determine the sizes of the similarity classes, which correspond to the vertices of $\mathcal{Q}$. Indeed, if $\mathcal{C} \in \mathcal{Q}_\ell$, then Propositions 2.5 and 3.10 show that

$$|\mathcal{C}| = \prod_{i=0}^{\ell-1} b^{(\varepsilon)}_{\mathrm{sh}(\mathcal{C}_i),\mathrm{sh}(\mathcal{C}_{i+1})}(q),$$

where $\mathcal{C}_i$ is the reduction of $\mathcal{C}$ modulo $\mathfrak{p}^i$. Figure 4.1 displays the shadow graph Γ associated to the scheme pair (g, G). The edge labels in Figure 4.1 match with the row numbers in Table 2.2.

In particular, the graph Γ together with the data $a_\xi(q)$ and $b_\xi^{(\varepsilon)}(q)$, $\xi \in \dot{E}(\Gamma)$, provides recursive formulae for the similarity class zeta functions of $\mathsf{g}(\mathfrak{o}_\ell)$, $\ell \in \mathbb{N}_0$.

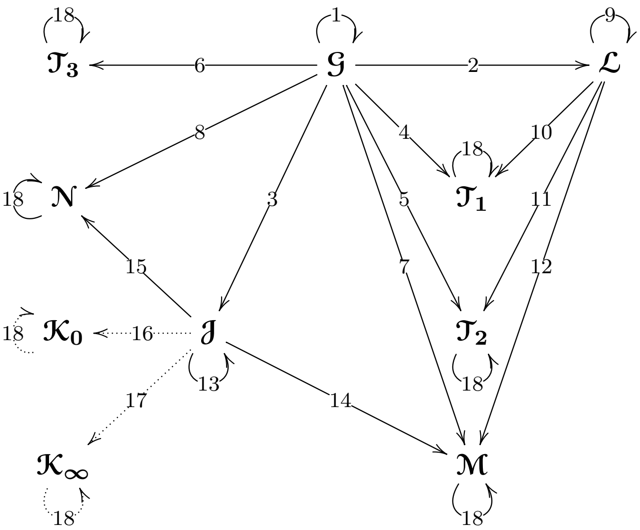
Lemma 4.6. *For $\ell \in \mathbb{N}_0$ and $\tau \in \mathfrak{Sh}$ we have*

$$\gamma_{\ell+1}^\tau(s) = \sum_{(\sigma,\tau) \in \dot{E}(\Gamma^{(\varepsilon)})} a_{\sigma,\tau}(q)\, b^{(\varepsilon)}_{\sigma,\tau}(q)^{-s}\, \gamma_\ell^\sigma(s).$$

Proof. For $\varepsilon = 1$, the claim follows from Proposition 2.5 and Theorem 2.8; for $\varepsilon = -1$, from Proposition 3.10 and Theorem 3.12:

$$\gamma_{\ell+1}^\tau(s) = \sum_{\substack{\widetilde{\mathcal{C}} \in \mathcal{Q}_{\ell+1} \\ \mathrm{sh}(\widetilde{\mathcal{C}}) = \tau}} |\widetilde{\mathcal{C}}|^{-s} = \sum_{\substack{\mathcal{C} \in \mathcal{Q}_\ell \\ \mathrm{sh}(\mathcal{C}) = \sigma}} \sum_{\substack{(\sigma,\tau) \in \\ \dot{E}(\Gamma^{(\varepsilon)})}} a_{\sigma,\tau}(q) \left(b^{(\varepsilon)}_{\sigma,\tau}(q)|\mathcal{C}|\right)^{-s} = \sum_{\substack{(\sigma,\tau) \in \\ \dot{E}(\Gamma^{(\varepsilon)})}} a_{\sigma,\tau}(q) b^{(\varepsilon)}_{\sigma,\tau}(q)^{-s} \gamma_\ell^\sigma(s). \qquad \square$$

FIGURE 4.1. The shadow graph $\Gamma^{(\varepsilon)}$ for $\mathsf{gl}_3, \mathsf{GL}_3$ ($\varepsilon = 1$) and $\mathsf{gu}_3, \mathsf{GU}_3$ ($\varepsilon = -1$)*



* For $\varepsilon = -1$ the vertices $\mathcal{K}_0$ and $\mathcal{K}_\infty$ with the incident edges are to be omitted.

4.2. **Explicit formulae for similarity class zeta functions for type A_2.** In order to state explicit formulae for the Dirichlet generating function $\gamma_\ell^\sigma(s)$ we define, for $r \in \mathbb{N}$, the auxiliary polynomials

$$f_\ell^r(a_1, \dots, a_r) := \sum_{\substack{(j_1,\dots,j_r)\in\mathbb{N}_0^r \\ \sum_{i=1}^r j_i \le \ell - r}} a_1^{j_1} \cdots a_r^{j_r} \in \mathbb{Z}[a_1, \dots, a_r]. \tag{4.2}$$

We shall only make use of f_ℓ^1 and f_ℓ^2. Note that, as rational functions in $\mathbb{Q}(a_1, \dots, a_r)$,

$$f_\ell^1(a_1) = \frac{1 - a_1^\ell}{1 - a_1} \qquad \text{and} \qquad f_\ell^2(a_1, a_2) = \frac{a_1 a_2^\ell - a_1^\ell a_2 + a_1^\ell - a_2^\ell + a_2 - a_1}{(a_2 - a_1)(1 - a_1)(1 - a_2)}.$$

We set

$$A_{q,\ell}(s) = f_\ell^1(q^{1-4s}), \quad B_{q,\ell}(s) = f_\ell^1(q^{2-6s}), \quad C_{q,\ell}(s) = f_\ell^2(q^{1-4s}, q^{2-6s}). \tag{4.3}$$

Proposition 4.7. *For $\sigma \in \mathfrak{Sh}$ of type $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$ and $\ell \in \mathbb{N}_0$,*

$$\gamma_\ell^\sigma(s) = q^\ell \Gamma^{\mathcal{S}}_{\varepsilon,q,\ell}(s),$$

where the function $\Gamma^{\mathcal{S}}_{\varepsilon,q,\ell}(s) := \Gamma^{\mathcal{S}}_{\mathsf{A}_2,\varepsilon,q,\ell}(s)$ is defined as

$$\begin{array}{ll}
1 & \text{if } \mathcal{S} = \mathcal{G}, \\
(q-1)\left((q^2+\varepsilon q+1)q^2\right)^{-s} A_{q,\ell}(s) & \text{if } \mathcal{S} = \mathcal{L}, \\
\left((q-\varepsilon)^3(q+\varepsilon)\right)^{-s} A_{q,\ell}(s) & \text{if } \mathcal{S} = \mathcal{J}, \\
\frac{1}{6}(q-1)\left((q+\varepsilon)(q^2+\varepsilon q+1)q^3\right)^{-s}\left[(q-2)B_{q,\ell}(s) + 3(q-1)q^{1-4s}C_{q,\ell}(s)\right] & \text{if } \mathcal{S} = \mathcal{T}_1, \\
\frac{1}{2}(q-1)\left((q^3-\varepsilon)q^3\right)^{-s}\left[qB_{q,\ell}(s) + (q-1)q^{1-4s}C_{q,\ell}(s)\right] & \text{if } \mathcal{S} = \mathcal{T}_2, \\
\frac{1}{3}(q^2-1)\left((q+\varepsilon)(q-\varepsilon)^2q^3\right)^{-s} B_{q,\ell}(s) & \text{if } \mathcal{S} = \mathcal{T}_3, \\
(q-1)\left((q-\varepsilon)^3(q+\varepsilon)q^2\right)^{-s}\left[B_{q,\ell}(s) + 2q^{1-4s}C_{q,\ell}(s)\right] & \text{if } \mathcal{S} = \mathcal{M}, \\
\left((q^2-1)(q^3-\varepsilon)q\right)^{-s}\left[B_{q,\ell}(s) + (1-q^{-1})q^{1-4s}C_{q,\ell}(s)\right] & \text{if } \mathcal{S} = \mathcal{N}, \\
\left((q^2-1)(q^3-\varepsilon)q^5\right)^{-s} C_{q,\ell}(s) & \text{if } \mathcal{S} \in \{\mathcal{K}_0, \mathcal{K}_\infty\}.
\end{array}$$

Proof. The proof is a straightforward induction on ℓ, using: Lemma 4.6, the explicit formulae for the polynomials $a_{\sigma,\tau}(q)$ and $b^{(\varepsilon)}_{\sigma,\tau}(q)$ from Table 2.2, and the definitions (4.3).

The case $\ell = 0$ is clear as $\Gamma^{\mathcal{S}}_{\varepsilon,q,0}(s) = 0$ unless $\mathcal{S} = \mathcal{G}$. For the induction step we assume that the proposition is proved for $\ell \in \mathbb{N}_0$. We give exemplary proofs for the types $\mathcal{G}$, $\mathcal{L}$, and $\mathcal{T}_1$, which are representative of the shadow graph's local complexities; cf. Figure 4.1. The computations for the other types are similar.

Let $\sigma, \tau, \upsilon \in \mathfrak{Sh}$ be shadows of types $\mathcal{G}, \mathcal{L}, \mathcal{T}_1$, respectively. For σ we have

$$\gamma^{\sigma}_{\ell+1}(s) = a_{\sigma,\sigma}(q)\, b^{(\varepsilon)}_{\sigma,\sigma}(q)^{-s}\, \gamma^{\sigma}_{\ell}(s) = q\gamma^{\sigma}_{\ell}(s) = q^{\ell+1}$$

as claimed. For τ we have

$$\begin{aligned}
\gamma^{\tau}_{\ell+1}(s) &= a_{\sigma,\tau}(q)\, b^{(\varepsilon)}_{\sigma,\tau}(q)^{-s}\, \gamma^{\sigma}_{\ell}(s) + a_{\tau,\tau}(q)\, b^{(\varepsilon)}_{\tau,\tau}(q)^{-s}\, \gamma^{\tau}_{\ell}(s) \\
&= q^{\ell+1}(q-1)\left((q^2+\varepsilon q+1)q^2\right)^{-s} + q^{1-4s}q^{\ell+1}(q-1)\left((q^2+\varepsilon q+1)q^2\right)^{-s}\sum_{j=0}^{\ell-1} q^{(1-4s)j} \\
&= q^{\ell+1}(q-1)\left((q^2+\varepsilon q+1)q^2\right)^{-s}\sum_{j=0}^{\ell} q^{(1-4s)j}
\end{aligned}$$

as claimed. We now argue for the shadow υ. Set

$$\begin{aligned}
u_{q,\ell}(s) &= \tfrac{1}{6}q^{\ell}(q-1)(q-2)\left((q^2+\varepsilon q+1)(q+\varepsilon)q^3\right)^{-s}\sum_{j=0}^{\ell-1} q^{(2-6s)j}, \\
v_{q,\ell}(s) &= \tfrac{1}{2}q^{\ell+1}(q-1)^2\left((q^2+\varepsilon q+1)(q+\varepsilon)q^7\right)^{-s}\sum_{\substack{(j_1,j_2)\in\mathbb{N}_0^2\\ j_1+j_2\leq \ell-2}} q^{(1-4s)j_1+(2-6s)j_2}
\end{aligned}$$

so that

$$\gamma^{\upsilon}_{\ell}(s) = u_{q,\ell}(s) + v_{q,\ell}(s).$$

Then

$$\begin{aligned}
\gamma^{\upsilon}_{\ell+1}(s) &= a_{\sigma,\upsilon}(q)\, b^{(\varepsilon)}_{\sigma,\upsilon}(q)^{-s}\, \gamma^{\sigma}_{\ell}(s) + a_{\tau,\upsilon}(q)\, b^{(\varepsilon)}_{\tau,\upsilon}(q)^{-s}\, \gamma^{\tau}_{\ell}(s) + a_{\upsilon,\upsilon}(q)\, b^{(\varepsilon)}_{\upsilon,\upsilon}(q)^{-s}\, \gamma^{\upsilon}_{\ell}(s) \\
&= \tfrac{1}{6}q(q-1)(q-2)\left((q^2+\varepsilon q+1)(q+\varepsilon)q^3\right)^{-s}\gamma^{\sigma}_{\ell}(s) \\
&\quad + \tfrac{1}{2}q^2(q-1)\left((q+\varepsilon)q^5\right)^{-s}\gamma^{\tau}_{\ell}(s) + q^{3-6s}\left(u_{q,\ell}(s)+v_{q,\ell}(s)\right) \\
&= \tfrac{1}{6}q^{\ell+1}(q-1)(q-2)\left((q^2+\varepsilon q+1)(q+\varepsilon)q^3\right)^{-s} + q^{3-6s}u_{q,\ell}(s) \\
&\quad + \tfrac{1}{2}q^2(q-1)\left((q+\varepsilon)q^5\right)^{-s}q^{\ell}(q-1)\left((q^2+\varepsilon q+1)q^2\right)^{-s}\sum_{j=0}^{\ell-1}q^{(1-4s)j} + q^{3-6s}v_{q,\ell}(s) \\
&= u_{q,\ell+1}(s) + v_{q,\ell+1}(s). \qquad \square
\end{aligned}$$

Corollary 4.8. *For $\sigma \in \mathfrak{Sh}$ of type $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$,*

$$\gamma^{\sigma}(s) = \lim_{\ell\to\infty} q^{-\ell}\gamma^{\sigma}_{\ell}(s) = \Gamma^{\mathcal{S}}_{\varepsilon,q}(s),$$

where the function $\Gamma^{\mathcal{S}}_{\varepsilon,q}(s) := \Gamma^{\mathcal{S}}_{\mathsf{A}_2,\varepsilon,q}(s)$ *is given by*

$$\begin{array}{ll}
1 & \text{if } \mathcal{S} = \mathcal{G}, \\
(q-1)\left((q^2+\varepsilon q+1)q^2\right)^{-s}(1-q^{1-4s})^{-1} & \text{if } \mathcal{S} = \mathcal{L}, \\
\left((q-\varepsilon)^3(q+\varepsilon)\right)^{-s}(1-q^{1-4s})^{-1} & \text{if } \mathcal{S} = \mathcal{J}, \\
\frac{1}{6}(q-1)\left((q+\varepsilon)(q^2+\varepsilon q+1)q^3\right)^{-s}\frac{q-2+2q^{2-4s}-q^{1-4s}}{(1-q^{1-4s})(1-q^{2-6s})} & \text{if } \mathcal{S} = \mathcal{T}_1, \\
\frac{1}{2}(q-1)\left((q^3-\varepsilon)q^3\right)^{-s}(q-q^{1-4s})\left((1-q^{1-4s})(1-q^{2-6s})\right)^{-1} & \text{if } \mathcal{S} = \mathcal{T}_2, \\
\frac{1}{3}(q^2-1)\left((q+\varepsilon)(q-\varepsilon)^2q^3\right)^{-s}(1-q^{2-6s})^{-1} & \text{if } \mathcal{S} = \mathcal{T}_3, \\
(q-1)\left((q-\varepsilon)^3(q+\varepsilon)q^2\right)^{-s}(1+q^{1-4s})\left((1-q^{1-4s})(1-q^{2-6s})\right)^{-1} & \text{if } \mathcal{S} = \mathcal{M}, \\
\left((q^2-1)(q^3-\varepsilon)q\right)^{-s}(1-q^{-4s})\left((1-q^{1-4s})(1-q^{2-6s})\right)^{-1} & \text{if } \mathcal{S} = \mathcal{N}, \\
\left((q^2-1)(q^3-\varepsilon)q^5\right)^{-s}\left((1-q^{1-4s})(1-q^{2-6s})\right)^{-1} & \text{if } \mathcal{S} \in \{\mathcal{K}_0, \mathcal{K}_\infty\}.
\end{array}$$

We conclude this section with the proofs of two main results stated in the introduction.

Proof of Theorem E. The claimed formula is a direct consequence of Proposition 4.7, upon noting that $s_\ell(\mathfrak{g}(\mathfrak{o})) = \gamma_\ell(0) = \sum_{\sigma\in\mathfrak{S}\mathfrak{h}} \gamma^\sigma_\ell(0)$ for $\ell \in \mathbb{N}_0$. Using

$$A_{q,\ell}(s) \to \frac{1}{1-q}, \quad B_{q,\ell}(s) \to \frac{1}{1-q^2}, \quad C_{q,\ell}(s) \to \frac{1}{(1-q)(1-q^2)} \qquad \text{as } \ell\to\infty,\ s\to 0$$

the computation becomes routine. □

Proof of Corollary F. The corollary is formulated in such a way that, given a place $v \notin S$ of the number field k, the Euler factor $\zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_v)}(s)$ of $\zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_S)}(s)$ in (1.14) is equal to $\zeta^{\mathrm{sc}}_{\mathfrak{gu}_3(\mathcal{O}_v)}(s)$ if $\varepsilon_{\mathbf{G}} = -1$ and v is not decomposed in the quadratic extension of $K \mid k$ defining $\mathbf{G} = \mathsf{GU}_3(K,f)$; in all other cases $\zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_v)}(s) = \zeta^{\mathrm{sc}}_{\mathfrak{gl}_3(\mathcal{O}_v)}(s)$. The claimed formula thus follow from (1.12) via the Euler product decomposition of the Dedekind zeta function $\zeta_k(s)$, the fact that the abscissa of convergence of $\zeta_k(s)$ is 1, and the Tauberian Theorem 7.2 stated in Section 7. □

Part 2. Representation zeta functions of groups of type A_2

5. The Kirillov orbit method and Clifford theory

Let $\mathfrak{o}$ be a compact discrete valuation ring of residue characteristic p. For the main part, we focus in this section on the case $\mathrm{char}(\mathfrak{o}) = 0$; we also exhibit analogues of some results in positive characteristic. Fix $n \in \mathbb{N}_{\geq 2}$ and let G be one of the four $\mathfrak{o}$-group schemes GL_n, GU_n, SL_n, SU_n, assuming $p > 2$ in the unitary cases. Write $G = \mathsf{G}(\mathfrak{o})$ and $N = \mathsf{G}^1(\mathfrak{o})$, the 1st principal congruence subgroup. We develop techniques to study the irreducible complex characters of G, in relation to the irreducible complex characters of N. Given a character $\chi \in \mathrm{Irr}(N)$, we write $S_\chi = \mathrm{I}_G(\chi)$ for the inertia group of χ and we denote by R_χ the maximal normal pro-p subgroup of S_χ. Observe that $N \triangleleft R_\chi \triangleleft S_\chi \leq G$.

Under suitable assumptions, relating p to n and $e = e(\mathfrak{o}, \mathbb{Z}_p)$, the pro-$p$ groups N and R_χ are guaranteed to belong to the class of saturable and potent groups; see Sections 5.1 and 5.2. This makes them amenable to the Kirillov orbit method, a machinery to describe the irreducible complex characters in terms of co-adjoint orbits; cf. Section 5.3. In Section 5.4 we provide the setup to apply the Kirillov orbit method to the principal congruence subgroups $\mathsf{G}^m(\mathfrak{o})$. The transition from similarity class zeta functions (see Section 4) to representation zeta functions of groups of the form $\mathsf{G}^m(\mathfrak{o})$ and their finite quotients $\mathsf{G}^m(\mathfrak{o})/\mathsf{G}^\ell(\mathfrak{o})$ is set out in Section 5.5. In Section 5.6 we discuss circumstances under which the character χ extends from N to the pro-p group R_χ; in Section 5.7 we provide cohomological criteria for the character to extend further, from R_χ to S_χ. By Clifford theory, any extension of χ to its inertia group S_χ induces irreducibly to G, completing the transition from $\mathrm{Irr}(N)$ to $\mathrm{Irr}(G)$.

To a certain degree the procedure works also over compact discrete valuation rings of positive characteristic; we will indicate the necessary modifications on the way. The corresponding starred remarks can be skipped if one wants to focus on the main situation.

5.1. **Saturable $\mathbb{Z}_p$-Lie lattices and pro-p groups.** We recall some results from p-adic Lie theory. The notions we require originate from Lazard's pioneering work [39] and were put into a group theoretic setting by Lubotzky and Mann; see [18]. They were developed further and refined in [35, 22].

Let $\mathfrak{r}$ be a $\mathbb{Z}_p$-Lie lattice. A Lie sublattice $\mathfrak{n}$ of $\mathfrak{r}$ is said to be *PF-embedded* in $\mathfrak{r}$ if there exists a *potent filtration* starting at $\mathfrak{n}$, i.e. a descending series of Lie sublattices $\mathfrak{n} = \mathfrak{n}_1 \supseteq \mathfrak{n}_2 \supseteq \ldots$ with $\bigcap_i \mathfrak{n}_i = 0$ such that (i) $[\mathfrak{r}, \mathfrak{n}_i] \subset \mathfrak{n}_{i+1}$ and (ii) $[\mathfrak{r}\ _{(p-1)}, \mathfrak{n}_i] \subset p\mathfrak{n}_{i+1}$ for all $i \geq 1$. Here and in the following we use right-normed Lie brackets so that $[X\ _{(n)}, Y] = [X, [\cdots [X, [X, Y]] \cdots]]$, with X occurring n times. Observe that every PF-embedded Lie sublattice is a Lie ideal in $\mathfrak{r}$. By [22, Theorem 4.1], the Lie lattice $\mathfrak{r}$ is *saturable* in the sense of Lazard if and only if it is PF-embedded in itself. Given a saturable Lie lattice $\mathfrak{r}$ one introduces, by means of the Hausdorff series

$$\Phi_{\mathrm{Hd}}(X, Y) = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}\big([X, [X, Y]] + [Y, [Y, X]]\big) + \ldots \in \mathbb{Q}\langle\langle X, Y\rangle\rangle$$

a group multiplication on the set $\mathfrak{r}$. This yields a *saturable* pro-p group $R = \exp(\mathfrak{r})$, again in the sense of Lazard. Moreover, the map $\mathfrak{r} \mapsto \exp(\mathfrak{r})$ yields an isomorphism between the category of saturable $\mathbb{Z}_p$-Lie lattices and saturable pro-p groups. (Sometimes the term 'saturable' is applied to groups that are not necessarily finitely generated. In this paper, we agree that saturable pro-p groups are by definition finitely generated.) We denote the inverse isomorphism by log, writing $\mathfrak{r} = \log(R)$. We write $e^X = \exp(X)$ for $X \in \mathfrak{r}$ and $\log(x)$ for $x \in R$ to denote the corresponding elements in the associated structure. Conjugation in R is linked to the *adjoint action* of R on $\mathfrak{r}$ via

$$\mathrm{Ad}(x)Y = \log(xe^Y x^{-1}) \qquad \text{for } x \in R \text{ and } Y \in \mathfrak{r}.$$

Lemma 5.1. *Let $\mathfrak{r}$ be a saturable $\mathbb{Z}_p$-Lie lattice. Let $\mathfrak{n}$ be a PF-embedded Lie ideal of $\mathfrak{r}$, with potent filtration $\mathfrak{n} = \mathfrak{n}_1 \supseteq \mathfrak{n}_2 \supseteq \ldots$. Let $X \in \mathfrak{r}$, $i \in \mathbb{N}$ and $Y \in \mathfrak{n}_i$. Then there exists $Z \in \mathfrak{n}_{i+1}$ such that*

$$\mathrm{Ad}(e^X)Y = Y + \mathrm{ad}(X)Y + \mathrm{ad}(X)Z.$$

Proof. The proof is similar to that of [23, Lemma 2.3(4)]. We have

$$\mathrm{Ad}(e^X)Y = Y + \mathrm{ad}(X)Y + \sum_{j=2}^{\infty} \frac{\mathrm{ad}(X)^j Y}{j!}.$$

Hence it suffices to check that $Z = \sum_{j=2}^{\infty} \mathrm{ad}(X)^{j-1}Y/j!$ is an element of $\mathfrak{n}_{i+1}$. Let $j \geq 2$. Writing $j - 1 = (p-1)k + l$ with $k \geq 0$ and $0 \leq l \leq p-2$, we conclude that

$$\mathrm{ad}(X)^{j-1}Y = \mathrm{ad}(X)^{(p-1)k}(\mathrm{ad}(X)^l Y) \in [\mathfrak{r}\ _{(p-1)k}, \mathfrak{n}_{i+l}] \subset p^k \mathfrak{n}_{i+k+l}.$$

On the other hand, the p-valuation of $j!$ is at most $\lfloor (j-1)/(p-1) \rfloor = k$. Thus $\mathrm{ad}(X)^{j-1}y/j! \in \mathfrak{n}_{i+k+l}$. Since $k + l \geq 1$ tends to infinity with j, the claim follows. $\square$

Lemma 5.2. *Let $\mathfrak{r}$ be a saturable $\mathbb{Z}_p$-Lie lattice with a PF-embedded Lie ideal $\mathfrak{n}$. Writing $N = \exp(\mathfrak{n})$, we have $\log(e^X N) = X + \mathfrak{n}$ for every $X \in \mathfrak{r}$.*

Proof. Let $\mathfrak{n} = \mathfrak{n}_1 \supseteq \mathfrak{n}_2 \supseteq \ldots$ be a potent filtration starting at $\mathfrak{n}$, and fix $X \in \mathfrak{r}$. We observe that $\log(e^X N) = \{\Phi_{\mathrm{Hd}}(X,Y) \mid Y \in \mathfrak{n}\} \subset X + \mathfrak{n}$. To obtain the reverse inclusion, it suffices to show that $X + \mathfrak{n} \subset \{\Phi_{\mathrm{Hd}}(X,Y) \mid Y \in \mathfrak{n}\} + \mathfrak{n}_i$ for all $i \in \mathbb{N}$. We argue by induction. Clearly, the claim is true for $i = 1$, as $\Phi_{\mathrm{Hd}}(X,0) = X$. For the induction step, let $i \geq 2$ and consider an arbitrary element $Z \in \{\Phi_{\mathrm{Hd}}(X,Y) \mid Y \in \mathfrak{n}\} + \mathfrak{n}_{i-1}$, that is $Z = \Phi_{\mathrm{Hd}}(X,Y) + U$ with $Y \in \mathfrak{n}$ and $U \in \mathfrak{n}_{i-1}$. Then $Z = \Phi_{\mathrm{Hd}}(\Phi_{\mathrm{Hd}}(X,Y),U) + U' = \Phi_{\mathrm{Hd}}(X,Y') + U'$ for suitable $U' \in [\mathfrak{r}, \mathfrak{n}_{i-1}] \subset \mathfrak{n}_i$ and $Y' = \Phi_{\mathrm{Hd}}(Y,Z) \in \mathfrak{n}$. ☐

Corollary 5.3. *Let $\mathfrak{r}$ be a saturable $\mathbb{Z}_p$-Lie lattice with a PF-embedded Lie ideal $\mathfrak{n}$ and a saturable Lie sublattice $\mathfrak{h}$. Write $R = \exp(\mathfrak{r})$, $N = \exp(\mathfrak{n})$, and $H = \exp(\mathfrak{h})$ for the corresponding saturable pro-p groups. Then $\mathfrak{r} = \mathfrak{n} + \mathfrak{h}$ if and only if $R = NH$.*

A pro-p group R is called *potent* if $\gamma_{p-1}(R) \subset R^p$. This notion is closely linked to saturability: if R is finitely generated, torsion-free and potent then R is saturable; see [22, Corollary 5.4]. Conversely, a saturable pro-p group need not be potent.

5.2. **Application to pro-p subgroups of matrix groups.** Let $\mathfrak{o}$ be a compact discrete valuation ring of residue characteristic p. Fix a uniformiser π so that the valuation ideal of $\mathfrak{o}$ takes the form $\mathfrak{p} = \pi\mathfrak{o}$, and let $\mathfrak{O}$ be an unramified quadratic extension of $\mathfrak{o}$ with valuation ideal $\mathfrak{P} = \pi\mathfrak{O}$. For $p > 2$ we write $\mathfrak{O} = \mathfrak{o}[\delta]$, where $\delta = \sqrt{\rho}$ for an element $\rho \in \mathfrak{o}$ whose reduction modulo $\mathfrak{p}$ is a non-square in the residue field $\mathfrak{o}/\mathfrak{p}$.

Let $n \in \mathbb{N}_{\geq 2}$. In this section we consider the Sylow pro-p subgroups of $\mathsf{GL}_n(\mathfrak{o})$ and $\mathsf{SL}_n(\mathfrak{o})$, respectively $\mathsf{GU}_n(\mathfrak{o})$ and $\mathsf{SU}_n(\mathfrak{o})$, as well as the corresponding $\mathfrak{o}$-Lie lattices. As before, we assume throughout that $p > 2$ in the unitary setting. In particular, we are interested in the lower central series of these groups and Lie lattices.

In order to arrive at a uniform description, it is convenient to work with versions of the unitary groups and the unitary Lie lattices that are different from those used in Section 3. Let $W = (w_{ij}) \in \mathsf{GL}_n(\mathfrak{O})$ denote the matrix corresponding to the longest element in the Weyl group of permutation matrices, i.e. let $w_{ij} = \delta_{i,n+1-j}$ using the Kronecker delta. We equip the $\mathfrak{O}$-algebra $\mathsf{Mat}_n(\mathfrak{O})$ with the $(\mathfrak{O},\mathfrak{o})$-involution

$$A^\star = WA^\circ W^{-1} \qquad \text{for } A \in \mathsf{Mat}_n(\mathfrak{O}),$$

where $\circ$ denotes the standard $(\mathfrak{O},\mathfrak{o})$-involution 'conjugate transpose' as in (3.1). Then

$$\mathsf{GU}_n^\star(\mathfrak{o}) = \{A \in \mathsf{GL}_n(\mathfrak{O}) \mid A^\star A = \mathrm{Id}_n\} \quad \text{and} \quad \mathsf{gu}_n^\star(\mathfrak{o}) = \{A \in \mathsf{gl}_n(\mathfrak{O}) \mid A^\star + A = 0\}$$

are isomorphic to $\mathsf{GU}_n(\mathfrak{o})$ and $\mathsf{gu}_n(\mathfrak{o})$; cf. Lemma 3.3. Similarly, we are interested in

$$\mathsf{SU}_n^\star(\mathfrak{o}) = \mathsf{GU}_n^\star(\mathfrak{o}) \cap \mathsf{SL}_n(\mathfrak{O}) \quad \text{and} \quad \mathsf{su}_n^\star(\mathfrak{o}) = \mathsf{gu}_n^\star(\mathfrak{o}) \cap \mathsf{sl}_n(\mathfrak{O}),$$

which are isomorphic to $\mathsf{SU}_n(\mathfrak{o})$ and $\mathsf{su}_n(\mathfrak{o})$.

Observe that $A = (a_{ij}) \in \mathsf{gl}_n(\mathfrak{O})$ belongs to $\mathsf{gu}_n^\star(\mathfrak{o})$ if and only if its entries satisfy the conditions $a_{ij} + a_{n+1-j,n+1-i}^\circ = 0$ for $1 \leq i,j \leq n$. From this, one constructs natural $\mathfrak{o}$-bases for the Lie lattices $\mathsf{gl}_n(\mathfrak{o})$ and $\mathsf{gu}_n(\mathfrak{o})$. Denoting by E_{ij} the elementary $n \times n$ matrix with entry 1 in the (i,j)-position and entries 0 elsewhere, we define

$$\begin{aligned} E_{ij}^{(1)} &= E_{ij}, \\ E_{ij}^{(-1)} &= \begin{cases} E_{ij} - E_{n+1-j,n+1-i} & \text{if } i+j < n+1, \\ \delta E_{ij} & \text{if } i+j = n+1, \\ \delta E_{ij} + \delta E_{n+1-j,n+1-i} & \text{if } i+j > n+1. \end{cases} \end{aligned} \tag{5.1}$$

Using the parameter $\varepsilon \in \{1, -1\}$ to distinguish between the general linear and unitary settings, and defining

$$(5.2)\qquad \begin{aligned} \mathsf{G} = \mathsf{G}^{(\varepsilon)} &= \begin{cases} \mathsf{GL}_n \\ \mathsf{GU}_n^\star \end{cases} \text{and} \quad \mathsf{g} = \mathsf{g}^{(\varepsilon)} = \begin{cases} \mathsf{gl}_n & \text{for } \varepsilon = 1, \\ \mathsf{gu}_n^\star & \text{for } \varepsilon = -1, \end{cases} \\ \mathsf{H} = \mathsf{H}^{(\varepsilon)} &= \begin{cases} \mathsf{SL}_n \\ \mathsf{SU}_n^\star \end{cases} \text{and} \quad \mathsf{h} = \mathsf{h}^{(\varepsilon)} = \begin{cases} \mathsf{sl}_n & \text{for } \varepsilon = 1, \\ \mathsf{su}_n^\star & \text{for } \varepsilon = -1, \end{cases} \end{aligned}$$

we see that the matrices $E_{ij}^{(\varepsilon)}$, $1 \le i, j \le n$, form a basis for the $\mathfrak{o}$-Lie lattice scheme g.

For $m \in \mathbb{Z}$, the $\mathfrak{o}$-submodule schemes

$$\begin{aligned} \mathsf{b}_m = \mathsf{b}_m^{(\varepsilon)} &= \mathsf{h}^{(\varepsilon)} \cap \operatorname{span}\langle E_{i,j}^{(\varepsilon)} \mid j - i \ge m \rangle \\ &= \{(x_{ij}) \in \mathsf{h}^{(\varepsilon)} \mid x_{ij} = 0 \text{ for } j - i < m\} \end{aligned}$$

form a filtration

$$\{0\} = \ldots = \mathsf{b}_n \subset \mathsf{b}_{n-1} \subset \ldots \subset \mathsf{b}_{1-n} = \ldots = \mathsf{h}.$$

Moreover, identifying $\mathfrak{O} \otimes_\mathfrak{o} \mathsf{gl}_n(\mathfrak{o})$ with $\mathsf{gl}_n(\mathfrak{O})$ via the basis E_{ij}, $1 \le i, j \le n$, one checks easily that, for each $m \in \mathbb{Z}$, the $\mathfrak{O}$-submodule schemes $\mathfrak{O} \otimes_\mathfrak{o} \mathsf{b}_m^{(1)}$ and $\mathfrak{O} \otimes_\mathfrak{o} \mathsf{b}_m^{(-1)}$ are equal as subschemes of the $\mathfrak{O}$-Lie lattice scheme $\mathsf{sl}_n = \mathfrak{O} \otimes_\mathfrak{o} \mathsf{sl}_n = \mathfrak{O} \otimes_\mathfrak{o} \mathsf{su}_n^\star$. Consequently, the resulting $\mathfrak{o}$-submodule filtrations of $\mathsf{h}^{(1)}(\mathfrak{o})$ and $\mathsf{h}^{(-1)}(\mathfrak{o})$ produce, under extension of scalars, the same $\mathfrak{O}$-submodule filtration $\mathsf{b}_m^{(1)}(\mathfrak{O}) = \mathsf{b}_m^{(-1)}(\mathfrak{O})$, $m \in \mathbb{Z}$, of $\mathsf{sl}_n(\mathfrak{O}) = \mathsf{h}^{(1)}(\mathfrak{O}) = \mathsf{h}^{(-1)}(\mathfrak{O})$. This means that properties that are 'stable' under extension of scalars can be derived uniformly for both cases, $\varepsilon = 1$ and $\varepsilon = -1$.

Next we record an auxiliary lemma describing certain types of commutators of terms of the filtration described above, based on explicit matrix identities.

Lemma 5.4. *With the above notation, the following hold:*

(1) $\mathsf{b}_m \cdot \mathsf{b}_{m'} \subset \mathsf{b}_{m+m'}$, *in particular* $[\mathsf{b}_m, \mathsf{b}_{m'}] \subset \mathsf{b}_{m+m'}$ *for all* $m, m' \in \mathbb{Z}$;
(2) *if* $p > 2$ *or* $n \ge 3$ *then* $[\mathsf{b}_1, \mathsf{b}_m] = \mathsf{b}_{m+1}$ *for* $m \in \mathbb{Z}$ *with* $m \ge 1 - n$;
(3) *if* $p > 2$ *or* $n \ge 3$ *then* $[\mathsf{b}_0, \mathsf{h}] = \mathsf{h}$.

Proof. (1) Let $X = (x_{ij}) \in \mathsf{b}_m$ and $Y = (y_{ij}) \in \mathsf{b}_{m'}$ for $m, m' \in \mathbb{Z}$. Suppose that the (i,k)-entry of XY is non-zero, i.e. $\sum_{j=1}^n x_{ij} y_{jk} \ne 0$. Then $x_{ij} \ne 0$ and $y_{jk} \ne 0$ for some index j, and consequently $k - i = (j - i) + (k - j) \ge m + m'$.

(2) Let $m \in \mathbb{Z}$ with $m \ge 1 - n$. By (1), it remains to show that $\mathsf{b}_{m+1} \subset [\mathsf{b}_1, \mathsf{b}_m]$. This can be checked modulo π, and since extension of scalars preserves the dimension of vector spaces, it is enough to show that $\mathsf{b}_{m+1}(\mathfrak{O}) \subset [\mathsf{b}_1(\mathfrak{O}), \mathsf{b}_m(\mathfrak{O})]$. Thus we may assume without loss of generality that $\varepsilon = 1$; see the remark preceding the lemma.

The $\mathfrak{o}$-lattice b_{m+1} is spanned by elements of the form

(i) E_{ij}, where $1 \le i, j \le n$ with $i \ne j$ and $j - i \ge m + 1$,
(ii) $E_{ii} - E_{i+1,i+1}$, where $1 \le i < n$, if $m \le -1$.

First consider E_{ij} of type (i). If $j - i \ge 2$ or $j - i < 0$, we use the identities

$$E_{ij} = \begin{cases} [E_{i,i+1}, E_{i+1,j}] & \text{if } i < i + 1 < j \text{ or } j < i < i + 1 \le n, \\ [E_{i,j-1}, E_{j-1,j}] & \text{if } 1 \le j - 1 < j < i \end{cases}$$

to deduce that $E_{ij} \in [\mathsf{b}_1, \mathsf{b}_m]$; the case $(i,j) = (n,1)$ does not arise, as $j - i \geq m + 1 \geq 2 - n$. It remains to consider the case $j - i = 1$, i.e. $j = i + 1$. Then $m \leq 0$ and we use the identities

$$E_{ij} = \begin{cases} \frac{1}{2}[E_{ii} - E_{jj}, E_{ij}] & \text{if } p > 2, \\ [E_{ii} - E_{kk}, E_{ij}] & \text{if } k \notin \{i,j\} \end{cases} \tag{5.3}$$

for $j = i + 1$ to deduce that $E_{i,i+1} \in [\mathsf{b}_1, \mathsf{b}_m]$.

Finally, for $m \leq -1$ and $1 \leq i < n$ we see that $E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}] \in [\mathsf{b}_1, \mathsf{b}_m]$.

(3) Similar to part (2) we may assume without loss of generality that $\varepsilon = 1$ and, clearly, it suffices to show that $\mathsf{h} \subset [\mathsf{b}_0, \mathsf{h}]$. For $1 \leq i,j \leq n$ with $i \neq j$ we use the identities (5.3) to see that $E_{ij} \in [\mathsf{b}_0, \mathsf{h}]$. For $1 \leq i < n$ we have $E_{ii} - E_{i+1,i+1} = [E_{i,i+1}, E_{i+1,i}] \in [\mathsf{b}_0, \mathsf{h}]$. □

We are interested in the $\mathfrak{o}$-Lie lattice scheme

$$\mathsf{s} = \mathsf{s}^{(\varepsilon)} = \pi\mathsf{h} + \mathsf{b}_1^{(\varepsilon)}, \tag{5.4}$$

which is defined so that, under suitable assumptions detailed below, $\mathsf{s}(\mathfrak{o})$ is a saturable Lie lattice yielding a Sylow pro-p subgroup $\exp(\mathsf{s}(\mathfrak{o}))$ of $\mathsf{H}(\mathfrak{o})$.

Lemma 5.5. *Suppose that $p > 2$ or $n \geq 3$. The terms of the lower central series of the Lie lattice scheme s are:*

$$\gamma_{i+jn}(\mathsf{s}) = \pi^j \gamma_i(\mathsf{s}) = \pi^{j+2}\mathsf{h} + \pi^{j+1}\mathsf{b}_{i-n} + \pi^j \mathsf{b}_i,$$

where $1 \leq i \leq n$ and $j \geq 0$.

Proof. Note that $\mathsf{h} = \mathsf{b}_{1-n}$. For $(i,j) = (1,1)$, the formula on the right-hand side equals

$$\pi^{1+2}\mathsf{h} + \pi^{1+1}\mathsf{b}_{1-n} + \pi^1 \mathsf{b}_1 = \pi(\pi\mathsf{h} + \mathsf{b}_1) = \pi\mathsf{s}.$$

Hence it suffices to prove the formula up to the $(n+1)$th term of the lower central series. Clearly, $\gamma_1(\mathsf{s}) = \pi^2\mathsf{h} + \pi\mathsf{b}_{1-n} + \mathsf{b}_1 = \mathsf{s}$ holds true. Now suppose that $1 \leq i \leq n$. By induction and using Lemma 5.4 (1), we have

$$\begin{aligned} \gamma_{i+1}(\mathsf{s}) = [\mathsf{s}, \gamma_i(\mathsf{s})] &= [\pi\mathsf{h} + \mathsf{b}_1, \pi^2\mathsf{h} + \pi\mathsf{b}_{i-n} + \mathsf{b}_i] \\ &= \pi^3[\mathsf{h},\mathsf{h}] + \pi^2[\mathsf{h}, \mathsf{b}_1 + \mathsf{b}_{i-n}] + \pi([\mathsf{h}, \mathsf{b}_i] + [\mathsf{b}_1, \mathsf{b}_{i-n}]) + [\mathsf{b}_1, \mathsf{b}_i] \\ &\subset \pi^2\mathsf{h} + \pi\mathsf{b}_{i+1-n} + \mathsf{b}_{i+1}. \end{aligned}$$

Moreover, for $i = n$, we note that the last term indeed equals $\pi^2\mathsf{h} + \pi\mathsf{b}_1 = \pi\mathsf{s}$, as $\mathsf{b}_{n+1} = \{0\}$.

The required reverse inclusions

$$[\mathsf{h}, \mathsf{b}_1 + \mathsf{b}_{i-n}] \supset \mathsf{h}, \quad [\mathsf{h}, \mathsf{b}_i] + [\mathsf{b}_1, \mathsf{b}_{i-n}] \supset \mathsf{b}_{i+1-n}, \quad [\mathsf{b}_1, \mathsf{b}_i] \supset \mathsf{b}_{i+1}.$$

are obtained from Lemma 5.4 (2) and (3), upon noting that $\mathsf{b}_0 \subset \mathsf{b}_1 + \mathsf{b}_{i-n}$ and $i-n, i \geq 1-n$. □

Proposition 5.6. *Let $\mathfrak{o}$ be a compact discrete valuation ring with $\mathrm{char}(\mathfrak{o}) = 0$ and residue characteristic p. Let $n \in \mathbb{N}_{\geq 2}$ and put $e = e(\mathfrak{o}, \mathbb{Z}_p)$. Suppose that $p > en + n$. Let G be GL_n or GU_n, and accordingly let H be SL_n or SU_n. Put $G = \mathsf{G}(\mathfrak{o})$ and $N = \mathsf{G}^1(\mathfrak{o})$, or $G = \mathsf{H}(\mathfrak{o})$ and $N = \mathsf{H}^1(\mathfrak{o})$. Let R be any pro-p subgroup of G containing N.*

Then N and R are potent and saturable, $\mathfrak{n} = \log(N)$ is PF-embedded in $\mathfrak{r} = \log(R)$, and $\mathfrak{n}$ is naturally isomorphic to $\mathsf{g}^1(\mathfrak{o})$ or $\mathsf{h}^1(\mathfrak{o})$, respectively.

Proof. In the unitary setting, it is convenient to work with $\mathsf{G} = \mathsf{GU}_n^\star$ and $\mathsf{H} = \mathsf{SU}_n^\star$. Hence let G, g and H, h be as in (5.2), parametrised implicitly by $\varepsilon \in \{1, -1\}$. Without loss of generality we may assume that R is contained in a Sylow pro-p subgroup of our choice. Observe that

$$\tilde{S} = \{A \in \mathsf{GL}_n(\mathfrak{O}) \mid A \text{ is upper uni-triangular modulo } \pi\}$$

is a Sylow pro-p subgroup of $\mathsf{GL}_n(\mathfrak{O})$ and that $S = \tilde{S} \cap G$ is a Sylow pro-p subgroup of G. The inequality $p > en + 1$ guarantees that $\tilde{S}$ is saturable; moreover, $\tilde{S}$ embeds naturally into the associative algebra $\mathsf{Mat}_n(\mathfrak{O}')$, where $\mathfrak{O}'$ is a finite extension of $\mathfrak{O}$, such that the Lie lattice $\tilde{\mathfrak{s}} = \log(\tilde{S})$ can be identified with a Lie sublattice of $\mathsf{gl}_n(\mathfrak{O}')$ and the exp-log correspondence is achieved by applying the p-adic exponential and logarithm series to matrices over $\mathfrak{O}'$; see [39, III (3.2.7)]. In [35, Proposition 2.5], the argument is extended to show that Sylow pro-p subgroups of automorphism groups of p-adic vector spaces equipped with a bilinear form are saturable. The proof given there covers, mutatis mutandis, also hermitian forms, and we conclude that, as $p > en+1$, in all cases considered here, the group S is saturable. Furthermore the corresponding Lie lattice $\log(S)$ can be identified with the matrix Lie lattice

$$\mathfrak{s} = \{X \in \mathfrak{g} \mid X \text{ is strictly upper triangular modulo } \pi\},$$

where $\mathfrak{g} = \mathsf{g}(\mathfrak{o})$ or $\mathfrak{g} = \mathsf{h}(\mathfrak{o})$. Moreover applying the p-adic exponential and logarithm maps, defined by the series $\mathrm{Exp}(Z) = \sum_{j=0}^{\infty} Z^j/j!$ and $\mathrm{Log}(1+Z) = \sum_{j=1}^{\infty}(-1)^{j+1}Z^j/j$, one can translate between S and $\mathfrak{s}$ in place of the Hausdorff series construction.

It is convenient to deal first with the case $G = \mathsf{H}(\mathfrak{o})$ and $\mathfrak{g} = \mathsf{h}(\mathfrak{o})$. We observe that $\mathfrak{s} = \mathsf{s}(\mathfrak{o})$, where s is as in (5.4). We claim that the Lie sublattice $\mathfrak{n} := \mathsf{h}^1(\mathfrak{o})$ is PF-embedded in $\mathfrak{s}$. More precisely, we claim that

$$\mathfrak{n}_i := \gamma_i(\mathfrak{s}) \cap \mathfrak{n}, \quad i \in \mathbb{N},$$

forms a potent filtration of $\mathfrak{n}$ in $\mathfrak{s}$. Indeed, from Lemma 5.5 we obtain $\gamma_n(\mathfrak{s}) \subset \mathfrak{n}$ and $\gamma_{n+en}(\mathfrak{s}) \subset p\gamma_n(\mathfrak{s})$. In particular, $\mathfrak{n}_i = \gamma_i(\mathfrak{s})$ for $i \geq n$, and it suffices to observe that $p > en + n$ implies

$$[\mathfrak{s}_{\ (p-1)}, \mathfrak{n}] \subset [\mathfrak{s}_{\ (en)}, [\mathfrak{s}_{\ (n-1)}, \mathfrak{n}]] \subset [\mathfrak{s}_{\ (en)}, \gamma_n(\mathfrak{s})] = \gamma_{n+en}(\mathfrak{s}) = p\gamma_n(\mathfrak{s}) \subset p\mathfrak{n}.$$

Furthermore, $\mathrm{Exp}(\mathfrak{n}) \subset N$ and, comparing Haar measures, we deduce that $\mathrm{Exp}(\mathfrak{n}) = N$ so that $\mathfrak{n}$ is naturally identified with the Lie lattice $\log(N)$ corresponding to the saturable group N.

By [22, Theorem B], the terms of the lower central series of S and $\mathfrak{s}$ correspond to one another via the exp-log correspondence. From this we observe that

$$\gamma_{n+en}(S) = \mathrm{Exp}(\gamma_{n+ne}(\mathfrak{s})) = \mathrm{Exp}(p\gamma_n(\mathfrak{s})) \subset \mathrm{Exp}(p\mathfrak{n}) = N^p.$$

Since $p > en + n$, we deduce that for the given subgroup R of S,

$$\gamma_{p-1}(R) \subset \gamma_{n+ne}(S) \subset N^p \subset R^p.$$

Thus R is finitely generated, torsion-free, and potent, hence saturable.

It remains to treat the case $G = \mathsf{G}(\mathfrak{o})$ and $\mathfrak{g} = \mathsf{g}(\mathfrak{o})$. From $p > n$ we see that $\mathfrak{g} = \mathsf{h}(\mathfrak{o}) + \mathfrak{z}$, where $\mathfrak{z}$ denotes the centre of $\mathfrak{g}$, and hence $\mathfrak{s} = \mathsf{s}(\mathfrak{o}) + \pi\mathfrak{z}$. This implies that $\gamma_i(\mathfrak{s}) = \gamma_i(\mathsf{s}(\mathfrak{o}))$ for $i \geq 2$. Using this observation, it is easy to extend the arguments provided for $\mathsf{H}(\mathfrak{o})$ and $\mathsf{h}(\mathfrak{o})$ to conclude the proof. □

Remark 5.7. For the purpose of extending characters from N to G, Proposition 5.6 is only applied to pro-p subgroups $R = R_\chi$ that arise as maximal normal pro-p subgroups of inertia subgroups $S_\chi = \mathrm{I}_G(\chi)$, where $\chi \in \mathrm{Irr}(N)$. A priori this specific situation requires control over much fewer groups R and it is possible that with extra work the restrictions on p can be eased.

*Remark** 5.8. Let $n, \ell \in \mathbb{N}$ with $n \geq 2$. Suppose now that $\mathfrak{o}$ has arbitrary characteristic and residue characteristic $p \geq n\ell$. As in the proposition, let $G = \mathsf{G}(\mathfrak{o})$ and $N = \mathsf{G}^1(\mathfrak{o})$, or $G = \mathsf{H}(\mathfrak{o})$ and $N = \mathsf{H}^1(\mathfrak{o})$. In addition, we write $M = \mathsf{G}^\ell(\mathfrak{o})$ or $M = \mathsf{H}^\ell(\mathfrak{o})$. Let R be any pro-p subgroup of G containing N. Then R/M has nilpotency class at most $p-1$, as $p \geq n\ell$; cf. [35, Appendix A].

Consequently, we may argue similarly as in [23, Section 6.4]. There is a surjective homomorphism η from the free nilpotent pro-p group $\widetilde{R}$ of class $p-1$ on a certain number of generators onto R/M. We observe that both $\widetilde{R}$ and the pre-images $\widetilde{N} := \eta^{-1}(N)$ and $\widetilde{M} := \eta^{-1}(M)$ are potent and saturable. Moreover, $\log(\widetilde{N})$ is PF-embedded in $\log(\widetilde{R})$. Finally, the finite Lie ring $\log(\widetilde{N})/\log(\widetilde{M})$ is naturally isomorphic to $\mathsf{g}^1(\mathfrak{o})/\mathsf{g}^\ell(\mathfrak{o})$ or $\mathsf{h}^1(\mathfrak{o})/\mathsf{h}^\ell(\mathfrak{o})$.

5.3. **The Kirillov orbit method.** The Kirillov orbit method, as described below, applies to potent saturable pro-p groups and yields a description of their irreducible complex characters in terms of co-adjoint orbits; for details see [23].

Recall that the *Pontryagin dual* $\mathfrak{a}^\vee$ of a locally compact, abelian group $\mathfrak{a}$ consists of all continuous homomorphisms from $\mathfrak{a}$ to the circle group $\{z \in \mathbb{C} \mid |z| = 1\}$. Let G be a saturable pro-p subgroup and $\mathfrak{g} = \log(G)$ the corresponding $\mathbb{Z}_p$-Lie lattice. The adjoint action of G on $\mathfrak{g}$ induces an action of G on the Pontryagin dual $\mathfrak{g}^\vee$ of (the additive group) $\mathfrak{g}$. We call this action the *co-adjoint action* and denote it by Ad^*. In concrete terms, this action is given by

$$(\mathrm{Ad}^*(g)\omega)(X) = \omega(\mathrm{Ad}(g^{-1})X) = \omega(\log(g^{-1}e^X g)) \quad \text{for } g \in G,\, \omega \in \mathfrak{g}^\vee, \text{ and } X \in \mathfrak{g}.$$

Theorem 5.9. *Let G be a potent saturable pro-p group and let $\mathfrak{g} = \log(G)$. Then there is a one-to-one correspondence $\mathrm{Ad}^*(G)\backslash\mathfrak{g}^\vee \to \mathrm{Irr}(G)$, $\Omega \mapsto \chi_\Omega$ between $\mathrm{Ad}^*(G)$-orbits in $\mathfrak{g}^\vee$ and irreducible characters of G. Furthermore, the following hold.*

(1) *For every $\mathrm{Ad}^*(G)$-orbit Ω the character χ_Ω is given by*

$$\chi_\Omega(g) = \frac{1}{|\Omega|^{1/2}} \sum_{\omega \in \Omega} \omega(\log(g)) \qquad \text{for } g \in G.$$

In particular, the degree of the character χ_Ω is equal to $|\Omega|^{1/2}$.

(2) *Suppose that H is a potent open subgroup of G and let $\mathfrak{h} = \log(H)$. Let $\Omega = \mathrm{Ad}^*(H)\omega$ and $\Theta = \mathrm{Ad}^*(G)\vartheta$ be co-adjoint orbits of $\omega \in \mathfrak{h}^\vee$ and $\vartheta \in \mathfrak{g}^\vee$. Then χ_Ω is a constituent of $\mathrm{Res}^G_H(\chi_\Theta)$ if and only if there exists $g \in G$ such that $\omega = (\mathrm{Ad}^*(g)\vartheta)|_\mathfrak{h}$.*

Proof. The proof of the first half of the theorem is given in [23]. We now justify part (2). Observe that H is saturable by [22, Corollary 5.4]. The multiplicity of χ_Ω in $\mathrm{Res}^G_H(\chi_\vartheta)$ is given by the inner product of the two characters. Since the map $\exp\colon \mathfrak{h} \to H$ is a measure-preserving bijection, we deduce from part (1) that

$$\begin{aligned}\langle \chi_\Omega, \mathrm{Res}^G_H(\chi_\Theta)\rangle &= \int_H \chi_\Omega(h) \cdot \overline{\mathrm{Res}^G_H(\chi_\Theta)(h)}\, d\mu(h) \\ &= \frac{1}{|\Omega|^{1/2}|\Theta|^{1/2}} \sum_{\omega' \in \Omega} \sum_{\vartheta' \in \Theta} \int_\mathfrak{h} \omega'(X) \cdot \overline{\vartheta'|_\mathfrak{h}(X)}\, d\mu(X).\end{aligned}$$

All the terms ω' and $\vartheta'|_\mathfrak{h}$ in the above sum represent 1-dimensional characters of $\mathfrak{h}$. Hence by the orthogonality of characters we deduce that

$$\int_\mathfrak{h} \omega'(X) \cdot \overline{\vartheta'|_\mathfrak{h}(X)}\, d\mu(X) = \begin{cases} 1 & \text{if } \omega' = \vartheta'|_\mathfrak{h}, \\ 0 & \text{if } \omega' \neq \vartheta'|_\mathfrak{h}. \end{cases}$$

The claim follows immediately from this. □

Corollary 5.10. *Let G be a potent saturable pro-p group and let N be a potent open normal subgroup of G. Let $\mathfrak{g} = \log(G)$ and $\mathfrak{n} = \log(N)$. Then the Kirillov orbit map induces a one-to-one correspondence $\mathrm{Ad}^*(G/N)\backslash(\mathfrak{g}/\mathfrak{n})^\vee \to \mathrm{Irr}(G/N)$, $\Omega \mapsto \chi_\Omega$.*

Proof. By Theorem 5.9 the irreducible characters of G are of the form χ_Ω, where Ω runs through the $\mathrm{Ad}^*(G)$-orbits in $\mathfrak{g}^\vee$. Irreducible characters of G/N correspond to characters χ_Ω with $\chi_\Omega(g) = \chi_\Omega(1) = |\Omega|^{1/2}$ for $g \in N$. For $\omega \in \Omega$ this condition is equivalent to $\omega(\log(g)) = 1$ for $g \in N$, i.e. $\omega(X) = 1$ for all $X \in \mathfrak{n}$. □

*Remark** 5.11. Continuing in the setup and with the notation of Remark 5.8, we observe that the characters of the finite p-groups R/M and N/M can be described in terms of the Kirillov orbit method applied to the potent saturable pro-p groups $\widetilde{R}$, $\widetilde{N}$, and $\widetilde{M}$, using Corollary 5.10. This relies on the fact that the natural isomorphism between the finite group and Lie lattice sections is equivariant under the adjoint actions. Theorem 5.9 can be applied mutatis mutandis.

5.4. **Principal congruence subgroups.** Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic 0, with residue field $\mathbf{k}$ of cardinality q, and put $p = \mathrm{char}(\mathbf{k})$ and $e = e(\mathfrak{o}, \mathbb{Z}_p)$. Let π be a uniformiser of $\mathfrak{o}$ and let $\mathfrak{O} \supset \mathfrak{o}$ be an unramified quadratic extension and fix $n \in \mathbb{N}$. Let G be one of the $\mathfrak{o}$-group schemes $\mathsf{GL}_n, \mathsf{GU}_n$, assuming $p > 2$ in the unitary case, and let g denote the corresponding $\mathfrak{o}$-Lie lattice scheme $\mathsf{gl}_n, \mathsf{gu}_n$. Write $\mathfrak{Sh}$ for the shadow set $\mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$; see Definitions 2.2 and 3.7.

Let $\ell, m \in \mathbb{N}$ with $\ell \geq m$. Let $G = \mathsf{G}(\mathfrak{o})$ and let $G^m = \mathsf{G}^m(\mathfrak{o})$ denote its mth principal congruence subgroup. We write G^m_ℓ for the quotient G^m/G^ℓ. Put $\mathfrak{g} = \mathsf{g}(\mathfrak{o})$, and let $\mathfrak{g}^m = \mathsf{g}^m(\mathfrak{o})$ denote the mth principal congruence Lie sublattice. Write $\mathfrak{g}_\ell = \mathfrak{g}/\mathfrak{g}^\ell$ and $\mathfrak{g}^m_\ell = \mathsf{g}^m/\mathsf{g}^\ell$.

To ensure that the group G^m is potent and saturable, we assume that

$$p > 2 \quad \text{and} \quad m \geq e/(p-2);$$

cf. [5, Proposition 2.3]. Saturability gives a Lie correspondence between G^m and $\mathfrak{g}^m \simeq \log(G^m)$. Let $\mathfrak{F}$ denote the fraction field of $\mathfrak{O}$. We fix a non-trivial character of the additive group of $\mathfrak{F}$

$$\varphi\colon \mathfrak{F} \xrightarrow{\mathrm{Tr}_{\mathfrak{F}\,|\,\mathbb{Q}_p}} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\simeq} \mu_{p^\infty} \subset \mathbb{C}^\times. \tag{5.5}$$

One obtains an isomorphism

$$\mathfrak{F} \to \mathfrak{F}^\vee, \quad x \mapsto \varphi_x, \qquad \text{where} \quad \varphi_x\colon \mathfrak{F} \to \mathbb{C}^\times,\, \varphi_x(y) = \varphi(\pi^{-\nu}xy).$$

Here ν is the valuation of a generator of the different $\mathfrak{D}_{\mathfrak{F}|\mathbb{Q}_p}$, which is introduced in order to maintain the self-duality upon descent to finite quotients. For $A \in \mathsf{gl}_n(\mathfrak{O}_\ell)$ we consider the character

$$\omega_A\colon \mathsf{gl}_n(\mathfrak{O}_\ell) \to \mathbb{C}^\times, \quad \omega_A(X) = \varphi(\pi^{-\ell}\,\mathrm{tr}(AX))$$

and its restriction to $\mathfrak{g}_\ell$ which we also denote by ω_A for simplicity.

Lemma 5.12. *The map $\mathfrak{g}_\ell \to \mathfrak{g}_\ell^\vee$, $A \mapsto \omega_A$ is an isomorphism of finite abelian groups that is G-equivariant with respect to the adjoint action* Ad *on $\mathfrak{g}_\ell$ and the co-adjoint action* Ad^* *on $\mathfrak{g}_\ell^\vee$.*

Proof. The essential observation is that the trace induces a non-degenerate $\mathfrak{O}_\ell$-bilinear form

$$\beta\colon \mathsf{gl}_n(\mathfrak{O}_\ell) \times \mathsf{gl}_n(\mathfrak{O}_\ell) \to \mathfrak{O}_\ell, \quad (A, B) \mapsto \mathrm{tr}(AB).$$

Indeed, for every $A \in \mathsf{gl}_n(\mathfrak{O}_\ell)$, the $\mathfrak{O}_\ell$-linear map $B \mapsto \mathrm{tr}(AB)$ is the zero map if and only if $A = 0$ as one sees, for instance, by evaluating it on elementary matrices. Restriction of the form β to $\mathsf{gl}_n(\mathfrak{o}_\ell) \times \mathsf{gl}_n(\mathfrak{o}_\ell)$ establishes the isomorphism $\mathsf{gl}_n(\mathfrak{o}_\ell) \simeq \mathsf{gl}_n(\mathfrak{o}_\ell)^\vee$.

Similarly, restriction of β to $\mathsf{gu}_n(\mathfrak{o}_\ell) \times \mathsf{gu}_n(\mathfrak{o}_\ell)$ establishes the desired isomorphism in the unitary case. For this one notes that the image of the restriction lies in $\mathfrak{o}_\ell$ as $\mathrm{tr}(AB)^\circ = \mathrm{tr}(AB)$ for $A, B \in \mathsf{gu}_n(\mathfrak{o}_\ell)$. Furthermore, evaluation of the $\mathfrak{O}_\ell$-linear map $B \mapsto \mathrm{tr}(AB)$ on matrices of the form $E_{ij}^{(-1)}$ as described in (5.1) – which form a generating set of $\mathsf{gu}_n(\mathfrak{o}_\ell)$ as an $\mathfrak{o}_\ell$-module – shows that this map is the zero map if and only if $A = 0$.

In both cases, the G-equivariance is immediate. □

We also use the isomorphism

$$\mathfrak{g}_{\ell-m} \to (\mathfrak{g}^m_\ell)^\vee, \quad A \mapsto \omega^m_A, \qquad \text{where} \quad \omega^m_A\colon \mathfrak{g}^m_\ell \to \mathbb{C}^\times,\, \omega^m_A(X) = \omega_A(\pi^{-m}X). \tag{5.6}$$

As G^m is potent, the Kirillov orbit method sets up a correspondence between $\mathrm{Irr}(G^m)$ and $\mathrm{Ad}^*(G^m)\backslash(\mathfrak{g}^m)^\vee$. As $(\mathfrak{g}^m)^\vee = \lim_{\ell\to\infty}(\mathfrak{g}^m_\ell)^\vee$, the isomorphisms (5.6) lead us to consider the finite orbit spaces $\mathrm{Ad}(G^m)\backslash\mathfrak{g}_{\ell-m}$ for $\ell > m$. The analysis of these spaces in the next section utilises the similarity class zeta functions from Section 4.

*Remark** 5.13. The setup above can be adapted to study the characters of the groups G^m_ℓ in the case that $\mathrm{char}(\mathfrak{o})$ is arbitrary, provided $p \geq \ell/m$. The latter condition ensures that the nilpotency class of the finite p-group G^m_ℓ is at most $p-1$. Write $N = G^m$ and $M = G^\ell$. Similar to Remark 5.8, there is a homomorphism η from a free nilpotent pro-p group $\widetilde{N}$ of class $p-1$ on a certain number of generators onto N/M. Both $\widetilde{N}$ and $\widetilde{M} := \eta^{-1}(M)$ are potent and saturable. The finite Lie ring $\log(\widetilde{N})/\log(\widetilde{M})$ is naturally isomorphic to $\mathfrak{g}^m_\ell$. If $\mathfrak{o}$ has positive characteristic p, the non-trivial character

$$\varphi\colon \mathfrak{F} \xrightarrow{\mathrm{Tr}_{\mathfrak{F}\,|\,\mathbb{F}_p((t))}} \mathbb{F}_p((t)) \xrightarrow{\mathrm{Res}_0} \mathbb{F}_p \xrightarrow{\simeq} \mu_p \subset \mathbb{C}^\times,$$

replaces the character described in (5.5), with $\nu = 0$ as the different in this case is trivial. Here the residue map Res_0 picks out the coefficient of t^{-1}.

5.5. **From similarity class zeta functions to representation zeta functions.** We continue to use the notation set up in Section 5.4. In addition, we write H for the $\mathfrak{o}$-group scheme SL_n or SU_n, according to whether G is GL_n or GU_n. Put $H = \mathsf{H}(\mathfrak{o})$ and, for $m \in \mathbb{N}$, let $H^m = \mathsf{H}^m(\mathfrak{o})$ denote its mth principal congruence subgroup. For $\ell \in \mathbb{N}$ with $\ell \geq m$ write $H^m_\ell = H^m/H^\ell$. In Section 4.1 we introduced the similarity class zeta functions $\gamma^\sigma_\ell(s)$, $\sigma \in \mathfrak{Sh}$, and discussed the limit of their normalizations $q^{-\ell}\gamma^\sigma_\ell(s)$ as $\ell \to \infty$; cf. Proposition 4.2. The following variants of these functions play a central role in our derivation of formulae for representation zeta functions.

Definition 5.14. Let $\sigma \in \mathfrak{Sh}$. For $\ell \in \mathbb{N}_0$ we set

$$\xi^\sigma_\ell(s) = [\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})]^{1+s/2} q^{-\ell}\gamma^\sigma_\ell(s/2),$$

and we define $\xi^\sigma(s) = \lim_{\ell\to\infty}\xi^\sigma_\ell(s) = [\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})]^{1+s/2}\gamma^\sigma(s/2)$; see Proposition 4.2.

In Proposition 5.16 below we provide formulae for the zeta functions of groups of the form G^m_ℓ, H^m_ℓ, and H^m in terms of the functions ξ^σ_ℓ and ξ^σ. For this we require the following lemma.

Lemma 5.15. *Let $\ell, m \in \mathbb{N}$ with $\ell \geq m$. Then*

$$\sum_{\Omega \in \mathrm{Ad}^*(G^m)\backslash(\mathfrak{g}^m_\ell)^\vee} |\Omega|^{-s/2} = \begin{cases} q^{(\ell-m)\dim \mathsf{G}} & \text{if } \ell \leq 2m, \\ q^{(m-1)\dim\mathsf{G}} \sum_{\Omega \in \mathrm{Ad}^*(G^1)\backslash(\mathfrak{g}^1_{\ell-2m+2})^\vee} |\Omega|^{-s/2} & \text{if } \ell > 2m. \end{cases}$$

Proof. If $\ell \leq 2m$, the co-adjoint action of G^m on $(\mathfrak{g}^m_\ell)^\vee$ is trivial and the sum over the singletons equals $|\mathfrak{g}^m_\ell| = q^{(\ell-m)\dim\mathsf{G}}$, as claimed. Now suppose that $\ell > 2m$. Using the G-equivariant isomorphism (5.6) between $\mathfrak{g}_{\ell-m}$ and $(\mathfrak{g}^m_\ell)^\vee$, and similarly between $\mathfrak{g}_{\ell-2m+1}$ and $(\mathfrak{g}^1_{\ell-2m+2})^\vee$, it suffices to work with $\mathrm{Ad}(G^m)$-orbits in $\mathfrak{g}_{\ell-m}$ on the left hand side and $\mathrm{Ad}(G^1)$-orbits in $\mathfrak{g}_{\ell-2m+1}$ on the right hand side.

Suppose that $A \in \mathfrak{g}$, and for $k \in \mathbb{N}$ let $A_k \in \mathfrak{g}_k$ be the image of A under the natural projection. Consider the orbit $\mathcal{A} = \mathrm{Ad}(G^m)A_{\ell-m} \in \mathrm{Ad}(G^m)\backslash\mathfrak{g}_{\ell-m}$. If $\mathsf{G} = \mathsf{GL}_n$, then

$$G^m \to \mathfrak{g}^m, \quad \mathrm{Id}_3 + \pi^m X \mapsto \pi^m X \tag{5.7}$$

is a measure-preserving bijection, mapping the stabiliser $\mathrm{Stab}_{G^m}(A_k)$ onto the Lie centraliser $\mathrm{C}_{\mathfrak{g}^m}(A_k)$ for every $k \in \mathbb{N}$. Furthermore, as $\ell > 2m$, the map

$$\mathfrak{g}^m \to \mathfrak{g}^1, \quad \pi^m X \mapsto \pi X$$

is an isomorphism of abelian groups, mapping $\mathrm{C}_{\mathfrak{g}^m}(A_{\ell-m})$ onto $\mathrm{C}_{\mathfrak{g}^1}(A_{\ell-2m+1})$. If $\mathsf{G} = \mathsf{GU}_n$, we reach the same conclusions by using the Cayley map cay (cf. Definition 3.8) in place of (5.7); compare with the proof of Proposition 3.10. Thus in each case we deduce that

$$|\mathcal{A}| = |G^m : \mathrm{Stab}_{G^m}(A_{\ell-m})| = |\mathfrak{g}^m : \mathrm{C}_{\mathfrak{g}^m}(A_{\ell-m})| = |\mathfrak{g}^1 : \mathrm{C}_{\mathfrak{g}^1}(A_{\ell-2m+1})|,$$

and, using (5.7), we get

$$\begin{aligned}\sum_{\mathcal{A}\in\mathrm{Ad}(G^m)\backslash\mathfrak{g}_{\ell-m}} |\mathcal{A}|^{-s/2} &= \sum_{A_{\ell-m}\in\mathfrak{g}_{\ell-m}} |\mathfrak{g}^1 : \mathrm{C}_{\mathfrak{g}^1}(A_{\ell-2m+1})|^{-1-s/2} \\ &= q^{(m-1)\dim\mathfrak{g}} \sum_{A_{\ell-2m+1}\in\mathfrak{g}_{\ell-2m+1}} |G^1 : \mathrm{Stab}_{G^1}(A_{\ell-2m+1})|^{-1-s/2} \\ &= q^{(m-1)\dim\mathsf{G}} \sum_{\mathcal{A}\in\mathrm{Ad}(G^1)\backslash\mathfrak{g}_{\ell-2m+1}} |\mathcal{A}|^{-s/2}. \qquad \square\end{aligned}$$

Proposition 5.16. *Let $\ell, m \in \mathbb{N}$ with $\ell \geq m$, and suppose that $p \nmid 2n$ and $m \geq e/(p-2)$. Then*

$$\zeta_{H_\ell^m}(s) = \frac{\zeta_{G_\ell^m}(s)}{q^{\ell-m}} = \begin{cases} q^{(\ell-m)\dim\mathsf{H}} & \text{if } \ell \leq 2m, \\ q^{(m-1)\dim\mathsf{H}} \sum_{\sigma\in\mathfrak{Sh}} \xi^\sigma_{\ell-2m+1}(s) & \text{if } \ell > 2m. \end{cases} \tag{5.8}$$

Moreover,

$$\zeta_{H^m}(s) = q^{(m-1)\dim\mathsf{H}} \sum_{\sigma\in\mathfrak{Sh}} \xi^\sigma(s). \tag{5.9}$$

Proof. Observe that H_ℓ^m is the kernel of the determinant map on G_ℓ^m. Since $p \nmid n$, every element in the p-group $\det(G_\ell^m)$ admits an nth root. Thus the central subgroup $\mathrm{S}(G_\ell^m)$ of scalar matrices maps onto $\det(G_\ell^m)$, and G_ℓ^m decomposes as a direct product $G_\ell^m = H_\ell^m \times \mathrm{S}(G_\ell^m)$. This yields the first equality in (5.8).

The assumption $m \geq e/(p-2)$ guarantees that the pro-p group G^m is potent and saturable (see [5, Proposition 2.3]) so that the orbit method can be used to parametrise irreducible characters of the finite quotient G_ℓ^m; see Corollary 5.10. For $\ell \leq 2m$ the second equality in (5.8) follows directly from the observation that G_ℓ^m is abelian and $\dim\mathsf{G} = \dim\mathsf{H} + 1$. For $\ell > 2m$, we apply the orbit method, Lemma 5.15, the correspondence (5.6), and the fact

$$[G : \mathrm{Stab}_G(A)] = [\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})][G^1 : \mathrm{Stab}_{G^1}(A)] \quad \text{for } A \in \mathfrak{g}_{\ell-2m+1} \text{ with } \mathrm{sh}_{\mathsf{G}}(A) = \sigma \tag{5.10}$$

to obtain

$$\begin{aligned}\zeta_{G_\ell^m}(s) &= \sum_{\Omega\in\mathrm{Ad}^*(G^m)\backslash(\mathfrak{g}_\ell^m)^\vee} |\Omega|^{-s/2} \\ &= q^{(m-1)\dim\mathsf{G}} \sum_{\Omega\in\mathrm{Ad}^*(G^1)\backslash(\mathfrak{g}^1_{\ell-2m+2})^\vee} |\Omega|^{-s/2} \\ &= q^{(m-1)\dim\mathsf{G}} \sum_{\sigma\in\mathfrak{Sh}} \sum_{\substack{\mathcal{A}\in\mathrm{Ad}(G^1)\backslash\mathfrak{g}_{\ell-2m+1} \\ \forall A\in\mathcal{A}:\ \mathrm{sh}_{\mathsf{G}}(A)=\sigma}} |\mathcal{A}|^{-s/2} \\ &= q^{(m-1)\dim\mathsf{G}} \sum_{\sigma\in\mathfrak{Sh}} \sum_{\substack{\mathcal{A}\in\mathrm{Ad}(G)\backslash\mathfrak{g}_{\ell-2m+1} \\ \forall A\in\mathcal{A}:\ \mathrm{sh}_{\mathsf{G}}(A)=\sigma}} |\mathcal{A}|^{-s/2}\, [\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})]^{1+s/2} \\ &= q^{\ell-m} q^{(m-1)\dim\mathsf{H}} \sum_{\sigma\in\mathfrak{Sh}} \xi^\sigma_{\ell-2m+1}(s).\end{aligned}$$

As for equation (5.9), every continuous character of H^m factors through H^m_ℓ for sufficiently large ℓ. Therefore

$$\zeta_{H^m}(s) = \lim_{\ell\to\infty} \zeta_{H^m_\ell}(s) = q^{(m-1)\dim \mathsf{H}} \sum_{\sigma\in\mathfrak{S}\mathfrak{h}} \xi^\sigma(s). \qquad \square$$

Remark 5.17. In fact, the second formula on the right-hand side of (5.8) also holds for $\ell = 2m$, because $\sum_{\sigma\in\mathfrak{S}\mathfrak{h}} \xi^\sigma_1(s) = q^{\dim \mathsf{H}}$.

We record, as a byproduct of the proof of Proposition 5.16, an explicit formula for the degrees of irreducible characters of the principal congruence subgroup quotients G^m_ℓ.

Corollary 5.18. *Let $\ell, m \in \mathbb{N}$ with $\ell \geq 2m$. Suppose that $A \in \mathfrak{g}_{\ell-m}$ corresponds, via the isomorphism* (5.6)*, to a character $\omega^m_A \in (\mathfrak{g}^m_\ell)^\vee$. Set $\mathcal{A} = \mathrm{Ad}(G)A \in \mathrm{Ad}(G)\backslash\mathfrak{g}_{\ell-m}$ and, for $1 \leq i \leq \ell-2m+1$, denote by $\mathcal{A}_i$ the reduction of $\mathcal{A}$ modulo π^i. Then the degree of the character $\chi_{\mathrm{Ad}^*(G^m)\omega^m_A} \in \mathrm{Irr}(G^m_\ell)$ associated to the co-adjoint orbit of ω^m_A is equal to*

$$\chi_{\mathrm{Ad}^*(G^m)\omega^m_A}(1) = q^{\frac{1}{2}\left((\ell-2m)\dim \mathsf{G} - \sum_{i=1}^{\ell-2m} \dim(\mathrm{sh}_{\mathsf{G}}(\mathcal{A}_i))\right)}.$$

Proof. For $\ell = 2m$ the degree is indeed 1. Suppose that $\ell > 2m$ and set $\sigma = \mathrm{sh}_{\mathsf{G}}(\mathcal{A}_{\ell-2m+1})$. Then

$$\begin{aligned}
\chi_{\mathrm{Ad}^*(G^m)\omega^m_A}(1)^2 &= [\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})]^{-1} |\mathcal{A}_{\ell-2m+1}| \\
&= [\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})]^{-1} \prod_{i=1}^{\ell-2m} q^{\dim \mathsf{G} - \dim(\mathrm{sh}_{\mathsf{G}}(\mathcal{A}_i))} \frac{\|\mathrm{sh}_{\mathsf{G}}(\mathcal{A}_i)\|}{\|\mathrm{sh}_{\mathsf{G}}(\mathcal{A}_{i+1})\|} \\
&= \prod_{i=1}^{\ell-2m} q^{\dim \mathsf{G} - \dim(\mathrm{sh}_{\mathsf{G}}(\mathcal{A}_i))} \\
&= q^{(\ell-2m)\dim \mathsf{G} - \sum_{i=1}^{\ell-2m} \dim(\mathrm{sh}_{\mathsf{G}}(\mathcal{A}_i))}. \qquad \square
\end{aligned}$$

*Remark** 5.19. As in the previous sections we comment on how to modify the results for G^m_ℓ and H^m_ℓ in case of arbitrary characteristic, provided that $p \geq \ell/m$. Definition 5.14 does not change. Lemma 5.15 and the first equality in (5.8) hold as stated, also in positive characteristic. To obtain the second equality in (5.8) and Corollary 5.18, we use the Kirillov orbit method for the finite p-groups G^m_ℓ and H^m_ℓ as indicated in Remark 5.13. This relies on the condition $p \geq \ell/m$.

5.6. **Extensions of characters via the orbit method.** Let G be a topological group, and let N be a normal open subgroup of G. The group G acts on N by conjugation, and hence acts on the set $\mathrm{Irr}(N)$ of irreducible complex characters of N:

$$^g\chi(h) = \chi(g^{-1}hg) \qquad \text{for } g \in G,\ \chi \in \mathrm{Irr}(N) \text{ and } h \in N.$$

The *inertia group* $\mathrm{I}_G(\chi)$ of $\chi \in \mathrm{Irr}(N)$ in G is the stabiliser of χ in G.

Suppose further that N is a potent saturable pro-p group, and write and $\mathfrak{n} = \log(N)$. Fix $\omega \in \mathfrak{n}^\vee$ with co-adjoint orbit $\Omega = \mathrm{Ad}^*(N)\omega$, and recall that $\chi_\Omega \in \mathrm{Irr}(N)$ denotes the character corresponding to Ω via the orbit method.

Lemma 5.20. *With the notation as above, the following hold:*

(a) $\mathrm{I}_G(\chi_\Omega) = \mathrm{Stab}_G(\Omega) = N\,\mathrm{Stab}_G(\omega)$,
(b) $\mathrm{Ad}^*(\mathrm{I}_G(\chi_\Omega))\omega = \Omega$.

Proof. From Theorem 5.9 we see that $\mathrm{I}_G(\chi_\Omega) = \mathrm{Stab}_G(\Omega)$. Clearly, one has $\mathrm{Stab}_G(\Omega) \supseteq N\,\mathrm{Stab}_G(\omega)$. For the reverse inclusion, assume that $g \in \mathrm{Stab}_G(\Omega)$. Then there exists $h \in N$ such that $\mathrm{Ad}^*(g)\omega = \mathrm{Ad}^*(h)\omega$ so that $h^{-1}g \in \mathrm{Stab}_G(\omega)$. Thus $g \in N\,\mathrm{Stab}_G(\omega)$. This proves part (a), and (b) is a direct consequence of (a). $\square$

In addition to the notation fixed already, suppose that $N \leq R \leq G$, where R is a potent saturable pro-p group, and set $\mathfrak{r} = \log(R)$.

Lemma 5.21. *With the notation as above, the character χ_Ω extends from N to R if and only if there exists $\vartheta \in \mathfrak{r}^\vee$, with co-adjoint orbit $\Theta = \mathrm{Ad}^*(R)\vartheta$, such that $\vartheta|_\mathfrak{n} = \omega$ and $|\Theta| = |\Omega|$; in this case $\chi_\Theta \in \mathrm{Irr}(R)$ is an extension of χ_Ω.*

Proof. Let $\Theta \subset \mathfrak{r}^\vee$ be an $\mathrm{Ad}^*(R)$-orbit. Then χ_Θ extends χ_Ω if and only if χ_Ω is a constituent of $\mathrm{Res}_N^R(\chi_\Theta)$ and $\chi_\Theta(1) = \chi_\Omega(1)$. By Theorem 5.9, these conditions are equivalent to: there exists $\vartheta \in \Theta$ such that $\vartheta|_\mathfrak{n} = \omega$ and $|\Theta| = |\Omega|$. □

Corollary 5.22. *With the notation as above, suppose that $R \subset \mathrm{I}_G(\chi_\Omega)$. Then the character χ_Ω extends to R if and only if ω extends to $\vartheta \in \mathfrak{r}^\vee$ such that $|\mathrm{Ad}^*(R)\omega| = |\Theta|$, where $\Theta = \mathrm{Ad}^*(R)\vartheta$. In this case $\chi_\Theta \in \mathrm{Irr}(R)$ is an extension of χ_Ω.*

Proof. The claim follows directly from the previous two lemmata. □

Next we develop a criterion for applying Corollary 5.22. Let $\mathfrak{r}$ be a $\mathbb{Z}_p$-Lie lattice, let $\mathfrak{n}$ be a Lie ideal of $\mathfrak{r}$ and let $\omega \in \mathfrak{n}^\vee$. The *radical* of ω in $\mathfrak{r}$ is

$$\mathrm{rad}_\mathfrak{r}(\omega) := \{X \in \mathfrak{r} \mid \forall Y \in \mathfrak{n} : \omega([X,Y]) = 1\};$$

see [23], but note the difference in notation. We observe that $\mathrm{rad}_\mathfrak{r}(\omega)$ is equal to

$$\mathrm{stab}_\mathfrak{r}(\omega) = \{X \in \mathfrak{r} \mid \mathrm{ad}^*(X)\omega = 1\},$$

the *stabiliser* of ω under the co-adjoint action given by

$$(\mathrm{ad}^*(X)\omega)(Y) = \omega(-\mathrm{ad}(X)Y) \qquad \text{for } X \in \mathfrak{r},\ \omega \in \mathfrak{n}^\vee, \text{ and } Y \in \mathfrak{n}.$$

Lemma 5.23. *Let $\mathfrak{r}$ be a saturable $\mathbb{Z}_p$-Lie lattice, and let $R = \exp(\mathfrak{r})$. Let $\mathfrak{n}$ be an open Lie ideal that is PF-embedded in $\mathfrak{r}$. Let $\omega \in \mathfrak{n}^\vee$. Then $\mathrm{stab}_\mathfrak{r}(\omega)$ is saturable and*

$$\exp(\mathrm{stab}_\mathfrak{r}(\omega)) = \mathrm{Stab}_R(\omega).$$

Proof. Suppose that $X \in \mathfrak{r}$ such that $e^X \in \mathrm{Stab}_R(\omega)$, i.e. such that

$$\omega(\mathrm{Ad}(e^X)Y) = \omega(Y) \qquad \text{for all } Y \in \mathfrak{n}.$$

Let $\mathfrak{n} = \mathfrak{n}_1 \supseteq \mathfrak{n}_2 \supseteq \ldots$ be a potent filtration for $\mathfrak{n}$. We claim that $\omega(\mathrm{ad}(X)Y) \in \omega(\mathrm{ad}(X)\mathfrak{n}_j)$ for all $j \in \mathbb{N}$. Since $\omega(\mathrm{ad}(X)\mathfrak{n}_j) = 1$ for sufficiently large j, this will imply that $X \in \mathrm{stab}_\mathfrak{r}(\omega)$. Clearly, one has $\omega(\mathrm{ad}(X)Y) \in \omega(\mathrm{ad}(X)\mathfrak{n}_1)$. Now let $j \geq 2$ and suppose inductively that $\omega(\mathrm{ad}(X)Y) = \omega(\mathrm{ad}(X)Y_{j-1})$ for some $Y_{j-1} \in \mathfrak{n}_{j-1}$. Then by Lemma 5.1 we have

$$\mathrm{Ad}(e^X)Y_{j-1} = Y_{j-1} + \mathrm{ad}(X)Y_{j-1} - \mathrm{ad}(X)Y_j,$$

where $Y_j \in \mathfrak{n}_j$. This yields

$$\omega(\mathrm{ad}(X)Y) = \omega(\mathrm{ad}(X)Y_{j-1}) = \omega(\mathrm{Ad}(e^X)Y_{j-1} - Y_{j-1})\omega(\mathrm{ad}(X)Y_j) = \omega(\mathrm{ad}(X)Y_j).$$

Conversely, suppose that $X \in \mathrm{stab}_\mathfrak{r}(\omega)$, i.e. that $\omega(\mathrm{ad}(X)Y) = 1$ for all $Y \in \mathfrak{n}$. Then by Lemma 5.1 we have

$$\omega(\mathrm{Ad}(e^X)Y) = \omega(Y)\omega(\mathrm{ad}(X)(Y+Z)) = \omega(Y)$$

for some $Z \in \mathfrak{n}$. Hence $e^X \in \mathrm{Stab}_R(\omega)$.

It follows that $\mathrm{stab}_\mathfrak{r}(\omega)$ is saturable and $\exp(\mathrm{stab}_\mathfrak{r}(\omega)) = \mathrm{Stab}_R(\omega)$. □

Proposition 5.24. *Let $\mathfrak{r}$ be a potent saturable $\mathbb{Z}_p$-Lie lattice with a potent open Lie ideal $\mathfrak{n}$ that is PF-embedded in $\mathfrak{r}$. Put $R = \exp(\mathfrak{r})$, $N = \exp(\mathfrak{n})$. Let $\vartheta \in \mathfrak{r}^\vee$, $\Theta = \mathrm{Ad}^*(R)\vartheta$ and $\omega = \vartheta|_\mathfrak{n} \in \mathfrak{n}^\vee$, $\Omega = \mathrm{Ad}^*(N)\omega$. Suppose that $\mathfrak{r} = \mathfrak{n} + \mathrm{stab}_\mathfrak{r}(\vartheta)$. Then*

(1) $\mathrm{stab}_{\mathfrak{r}}(\vartheta) = \mathrm{stab}_{\mathfrak{r}}(\omega)$;
(2) $\mathrm{Stab}_R(\vartheta) = \mathrm{Stab}_R(\omega)$;
(3) $\chi_\Omega \in \mathrm{Irr}(N)$ *extends to* $\chi_\Theta \in \mathrm{Irr}(R)$.

Proof. (1) Clearly, $\mathrm{stab}_{\mathfrak{r}}(\vartheta) \subset \mathrm{stab}_{\mathfrak{r}}(\omega)$. For the reverse inclusion, let $X \in \mathrm{stab}_{\mathfrak{r}}(\omega)$. As $\mathfrak{r} = \mathfrak{n} + \mathrm{stab}_{\mathfrak{r}}(\vartheta)$ it suffices to show that $\vartheta(-\mathrm{ad}(X)Y) = 1$ for $Y \in \mathfrak{n} \cup \mathrm{stab}_{\mathfrak{r}}(\vartheta)$. If $Y \in \mathfrak{n}$ then $\vartheta(-\mathrm{ad}(X)Y) = \omega(-\mathrm{ad}(X)Y) = 1$. If $Y \in \mathrm{stab}_{\mathfrak{r}}(\vartheta)$ then $\vartheta(-\mathrm{ad}(X)Y) = \vartheta(\mathrm{ad}(Y)X) = 1$.

(2) The claim follows immediately from (1) and Lemma 5.23.

(3) From $\mathfrak{r} = \mathfrak{n} + \mathrm{stab}_{\mathfrak{r}}(\vartheta)$, Lemma 5.23 and Corollary 5.3 we deduce that $R = N\,\mathrm{Stab}_R(\vartheta)$. From (2) we deduce that $\mathrm{I}_R(\chi_\Omega) = R$, and (2) with Corollary 5.22 shows that χ_Θ extends χ_Ω. □

5.7. **Cohomological criteria for extendability.** Let G be a group, N a finite-index normal subgroup of G and $\chi \in \mathrm{Irr}(N)$. Define $\mathrm{Irr}(G \mid \chi)$ to be the set of irreducible characters ψ of G such that χ is an irreducible constituent of $\mathrm{Res}^G_N(\psi)$. The *relative representation zeta function* of G with respect to χ is defined as

$$\zeta_{G \mid \chi}(s) = \sum_{\psi \in \mathrm{Irr}(G \mid \chi)} (\psi(1)/\chi(1))^{-s}. \tag{5.11}$$

In the notation introduced in [5, Section 7.2.1], we have $\zeta_{G \mid \chi}(s) = [G : \mathrm{I}_G(\chi)]^{-s} \zeta_{G,\chi}(s)$. Clifford theory yields the following proposition; see [29, Corollary 6.17].

Proposition 5.25. *Let G be a profinite group, N an open normal subgroup of G, and $\chi \in \mathrm{Irr}(N)$. Suppose that χ extends to a character of $\mathrm{I}_G(\chi)$. Then*

$$\zeta_{G \mid \chi}(s) = [G : \mathrm{I}_G(\chi)]^{-s}\, \zeta_{\mathrm{I}_G(\chi)/N}(s).$$

Corollary 5.26. *Using the same notation as in the proposition and supposing that every $\chi \in \mathrm{Irr}(N)$ extends to a character of $\mathrm{I}_G(\chi)$, we have*

$$\zeta_G(s) = \sum_{\chi \in \mathrm{Irr}(N)} |G : \mathrm{I}_G(\chi)|^{-1-s}\, \zeta_{\mathrm{I}_G(\chi)/N}(s)\, \chi(1)^{-s}. \tag{5.12}$$

Whether a character extends can be studied in the framework of the second cohomology group of the inertia quotient. Let S be a group with a finite-index normal subgroup $R \lhd S$, and let $\chi \in \mathrm{Irr}(R)$. Clearly, a necessary condition for the extendability of χ to S is that S fixes the character χ, i.e. that $\mathrm{I}_S(\chi) = S$. Assuming this, one constructs an element in the second cohomology group $\mathrm{H}^2(S/R, \mathbb{C}^\times)$, also known as the Schur multiplier, as follows; see [29, Chapter 11].

Let M be a left R-module affording the character χ. Choose a left transversal T for R inside S such that $1 \in T$. For every $t \in T$, the R-modules M and tM are isomorphic, because $t \in \mathrm{I}_S(\chi)$. For each $t \in T$, we choose an isomorphism $P_t \colon M \to tM$, selecting the identity $P_1 = \mathrm{Id}$ for $t = 1$. Every element of S can be written uniquely as th, where $t \in T$ and $h \in R$. We define $P_{th} \colon M \to tM$ by $P_{th}(m) = P_t(h \cdot m)$. It can be easily checked that for every pair $g_1, g_2 \in S$, the operator

$$P_{g_1 g_2}^{-1} \circ P_{g_1} \circ P_{g_2} \colon M \to M$$

is a non-zero endomorphism of the R-module M; because M is irreducible, this morphism is multiplication by a scalar $\alpha(g_1, g_2) \in \mathbb{C}^\times$, say. Note that the value of $\alpha(g_1, g_2)$ depends only on the cosets g_1R and g_2R. The function α is a 2-cocycle and, although it generally depends on the particular choices for T and P_t, the cohomology class $\beta \in \mathrm{H}^2(S/R, \mathbb{C}^\times)$ that it represents does not. By [29, Theorem 11.7], the character χ extends to S if and only if β is trivial.

In our setting S is a profinite group and R an open normal pro-p subgroup of S. In this case all finite-dimensional representations of R factor through finite p-groups.

Proposition 5.27. *Let S be a profinite group, $R \lhd S$ be an open normal pro-p subgroup of S and $\chi \in \mathrm{Irr}(R)$ such that $\mathrm{I}_S(\chi) = S$. If $\beta \in \mathrm{H}^2(S/R, \mathbb{C}^\times)$ is the cohomology class attached to (S, R, χ), then β is a p-element in $\mathrm{H}^2(S/R, \mathbb{C}^\times)$.*

Proof. We continue to use the notation set up above. Let ρ denote a representation associated with the R-module M. Fix volume forms on the R-modules tM, $t \in T$. Let $t_1, t_2 \in T$ and suppose that $h \in R$ is such that $t_1 t_2 h \in T$. Taking determinants, we get

$$\alpha(t_1, t_2)^{\dim(M)} = \det(P_{t_1 t_2 h})^{-1} \det(P_{t_1}) \det(P_{t_2}) \det(\rho(h)).$$

We may assume that the isomorphisms P_t, $t \in T$, have determinant 1. Since ρ is an irreducible representation of the pro-p group R, its dimension $\dim(M)$ is a power of p, and $\det \rho(h)$ is a p^nth root of unity for some $n \in \mathbb{N}$. Therefore $\alpha(t_1, t_2)$ is a p^m-root of unity for some $m \in \mathbb{N}$. It follows that the order of β is a power of p. ☐

5.8. **Extension of characters from N to their stabiliser in G.** From now on consider again $G = \mathsf{G}(\mathfrak{o})$, where G denotes one of the $\mathfrak{o}$-group schemes $\mathsf{GL}_n, \mathsf{GU}_n, \mathsf{SL}_n, \mathsf{SU}_n$, with 1st principal congruence subgroup $N = G^1 = \mathsf{G}^1(\mathfrak{o})$ and $\mathrm{char}(\mathfrak{o}) = 0$, as discussed at the beginning of Section 5. Let $\chi \in \mathrm{Irr}(N)$ with stabiliser $S_\chi = \mathrm{I}_G(\chi)$, and let R_χ denote a maximal normal pro-p subgroup of S_χ. Assume that $p > en + n$, and let $\mathfrak{n} = \log(N)$, $\mathfrak{r} = \log(R)$ denote the $\mathbb{Z}_p$-Lie lattice associated to N, R; see Proposition 5.6. Writing g for the $\mathfrak{o}$-Lie lattice scheme $\mathsf{gl}_n, \mathsf{gu}_n, \mathsf{sl}_n$, or su_n, according to our choice of G, we have $\mathfrak{n} = \mathsf{g}^1(\mathfrak{o})$; furthermore we put $\mathfrak{g} = \mathsf{g}(\mathfrak{o})$. For $\ell, m \in \mathbb{N}$ with $m \leq \ell$, we write $\mathfrak{g}_\ell = \mathfrak{g}/\mathfrak{g}^\ell$ and $\mathfrak{g}_\ell^m = \mathfrak{g}^m/\mathfrak{g}^\ell$. We also put $\mathfrak{n}_\ell = \mathfrak{g}_{\ell+1}^1$. Similarly to the situation in Section 5.4, the character φ in (5.5) induces an isomorphism

$$\mathfrak{g}_\ell \to \mathfrak{g}_\ell^\vee, \quad A \mapsto \omega_A, \quad \text{where} \quad \omega_A \colon \mathfrak{g}_\ell \to \mathbb{C}^\times, \, \omega_A(X) = \varphi\left(\pi^{-\ell} \operatorname{tr}(AX)\right), \tag{5.13}$$

of finite abelian groups which is G-equivariant with respect to the adjoint action Ad on $\mathfrak{g}_\ell$ and the co-adjoint action Ad^* on $\mathfrak{g}_\ell^\vee$. For the general linear and unitary cases this is Lemma 5.12. The special linear/unitary cases follow along the same lines noting that p does not divide n.

In addition there is, similar to (5.6), an isomorphism

$$\mathfrak{g}_\ell \to (\mathfrak{n}_\ell)^\vee, \quad A \mapsto \omega_A^1, \quad \text{where} \quad \omega_A^1 \colon (\mathfrak{n}_\ell)^\vee \to \mathbb{C}^\times, \, \omega_A^1(X) = \omega_A(\pi^{-1}X).$$

Definition 5.28. Let G and $\mathfrak{g}$ be as above. Let $A \in \mathfrak{g}_\ell$ and $\widetilde{A} \in \mathfrak{g}_{\ell+1}$. We say that $\widetilde{A}$ is a *shadow-preserving lift* of A if $A \equiv_{\pi^\ell} \widetilde{A}$ and $\overline{\mathrm{C}_G(A)} = \overline{\mathrm{C}_G(\widetilde{A})}$.

Theorem 5.29. *Let G, N and $\mathfrak{g}$ be as above. Suppose that $p > en + n$. Let $\chi \in \mathrm{Irr}(N)$, corresponding to the co-adjoint orbit $\mathrm{Ad}^*(N)\omega_A^1$ for $A \in \mathfrak{g}_\ell$, and put $S_\chi = \mathrm{I}_G(\chi)$ with maximal normal pro-p subgroup R_χ. If*

- *there exists a shadow-preserving lift $\widetilde{A} \in \mathfrak{g}_{\ell+1}$ of A and*
- *$\mathrm{H}^2(S_\chi/R_\chi, \mathbb{C}^\times) = 1$,*

then χ extends to S_χ.

Proof. As $\widetilde{A}$ is a shadow-preserving lift of A, we have that

$$S_\chi/N = \mathrm{Stab}_G(A)N/N = \mathrm{Stab}_G(\widetilde{A})N/N.$$

As $N \leq R_\chi \leq S_\chi$, this implies that

$$R_\chi = \mathrm{Stab}_{R_\chi}(A)\, N = \mathrm{Stab}_{R_\chi}(\widetilde{A})\, N,$$

and thus

$$R_\chi = \mathrm{Stab}_{R_\chi}(\omega_A^1)\, N = \mathrm{Stab}_{R_\chi}(\omega_{\widetilde{A}})\, N.$$

Setting $\vartheta = \omega_{\widetilde{A}}|_{\mathfrak{r}}$, we apply Lemma 5.3 to deduce that $\mathfrak{r} = \mathfrak{n} + \mathrm{stab}_{\mathfrak{r}}(\vartheta)$. We conclude from Proposition 5.24 that the character $\chi_\Theta \in \mathrm{Irr}(R)$ associated to $\Theta = \mathrm{Ad}^*(R)\vartheta$ extends χ.

To show that χ_Θ extends to S_χ, we first show that $\mathrm{I}_{S_\chi}(\chi_\Theta) = \mathrm{I}_{S_\chi}(\chi)$. The inclusion $\mathrm{I}_{S_\chi}(\chi_\Theta) \subset \mathrm{I}_{S_\chi}(\chi)$ is clear and the reverse inclusion follows from

$$\mathrm{I}_{S_\chi}(\chi) = \mathrm{Stab}_{S_\chi}(\omega_A^1)N = \mathrm{Stab}_{S_\chi}(A)N = \mathrm{Stab}_{S_\chi}(\widetilde{A})N \subset \mathrm{Stab}_{S_\chi}(\vartheta)N \subset \mathrm{I}_{S_\chi}(\chi_\Theta).$$

Since $\mathrm{H}^2(S_\chi/R_\chi, \mathbb{C}^\times) = 1$, the character χ_Θ extends to an irreducible character of S_χ; cf. Section 5.7. □

*Remark** 5.30. We indicate with what changes the results, in particular Theorem 5.29, remain true in case of arbitrary characteristic, provided that p is sufficiently large. Definition 5.28 remains the same in positive characteristic. Using all the previous remarks in Section 5, one sees that the conclusion of Theorem 5.29, for $\chi \in \mathrm{Irr}(N)$, holds true if the level $\ell - 1$ of χ satisfies $p \geq n\ell$. Here, the level $\ell - 1$ of χ is the minimal value of $\ell - 1$ for $\ell \in \mathbb{N}$ such that χ is trivial on the ℓ-th principal congruence subgroup; cf. Definition 6.10.

6. Zeta functions of groups of type A_2

In this section we apply some of the machinery developed in previous sections in the special case of groups of type A_2. In particular, we prove Theorem J in Section 6.1, Theorem C in Section 6.2, Corollary D in Section 6.3, and Theorems H and I in Section 6.4.

We use the notation introduced in Section 5, focusing now on the special case $n = 3$. In summary, $\mathfrak{o}$ denotes a compact discrete valuation ring with valuation ideal $\mathfrak{p}$ and finite residue field $\mathbf{k}$, where $p := \mathrm{char}(\mathbf{k})$ and $q := |\mathbf{k}|$. The letter G, accordingly H, stands for one of the $\mathfrak{o}$-group schemes $\mathsf{GL}_3, \mathsf{GU}_3$, accordingly $\mathsf{SL}_3, \mathsf{SU}_3$, assuming $p > 2$ in the unitary cases. Write $G = \mathsf{G}(\mathfrak{o})$, $H = \mathsf{H}(\mathfrak{o})$, and $G^m, H^m, G^m_\ell, H^m_\ell$ for the principal congruence subgroups and finite subquotients. The corresponding $\mathfrak{o}$-Lie lattice schemes, respectively Lie lattices, are denoted by g, h, $\mathfrak{g} = \mathsf{g}(\mathfrak{o}), \mathfrak{h} = \mathsf{h}(\mathfrak{o})$, $\mathfrak{g}^m_\ell, \mathfrak{h}^m_\ell$ et cetera. The parameter $\varepsilon = \varepsilon_{\mathsf{G}} = \varepsilon_{\mathsf{H}} \in \{1, -1\}$ facilitates the parallel treatment of the (general) linear and (general) unitary setting; compare (1.4). For $\ell \in \mathbb{N}_0$, we use the abbreviated notation $\mathcal{Q}_\ell := \mathcal{Q}^{\mathsf{g}}_{\mathfrak{o},\ell}$ (cf. Definitions (2.1) and (3.6)) and $\mathfrak{Sh} := \mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$ (cf. (2.1) and (3.3)).

6.1. Principal congruence subgroups.

6.1. Principal congruence subgroups. Proposition 5.16 provides general formulae for the representation zeta functions of (i) the finite groups G^m_ℓ, H^m_ℓ in terms of the functions ξ^σ_ℓ and (ii) the infinite groups H^m in terms of the limit functions $\xi^\sigma = \lim_{\ell\to\infty} \xi^\sigma_\ell$; cf. Definition 5.14. The definition of the functions ξ^σ_ℓ has two ingredients: firstly the indices $[\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})]$ and secondly the partial similarity class zeta function $\gamma^\sigma_\ell(s)$; cf. Definition 4.1. The former are tabulated in Table 6.1 while explicit formulae for the latter are provided in Proposition 4.7. We record expressions for the functions ξ^σ_ℓ, using the auxiliary functions $A_{q,\ell}(s), B_{q,\ell}(s), C_{q,\ell}(s)$ introduced in (4.3).

Proposition 6.1. *For $\sigma \in \mathfrak{Sh}$ of type $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$ and $\ell \in \mathbb{N}_0$,*

$$\xi^\sigma_\ell(s) = \Xi^{\mathcal{S}}_{\varepsilon,q,\ell}(s),$$

where the function $\Xi^{\mathcal{S}}_{\varepsilon,q,\ell}(s) := \Xi^{\mathcal{S}}_{\mathsf{A}_2,\varepsilon,q,\ell}(s)$ *is defined as*

$$\begin{array}{ll}
1 & \text{if } \mathcal{S} = \mathcal{G},\\
(q-1)(q^2+\varepsilon q+1)q^2 A_{q,\ell}(s/2) & \text{if } \mathcal{S} = \mathcal{L},\\
(q^3-\varepsilon)(q+\varepsilon)A_{q,\ell}(s/2) & \text{if } \mathcal{S} = \mathcal{J},\\
\frac{1}{6}(q-1)(q^2+\varepsilon q+1)(q+\varepsilon)q^3\left[(q-2)B_{q,\ell}(s/2)+3(q-1)q^{1-2s}C_{q,\ell}(s/2)\right] & \text{if } \mathcal{S} = \mathcal{T}_1,\\
\frac{1}{2}(q-1)(q^3-\varepsilon)q^3\left[qB_{q,\ell}(s/2)+(q-1)q^{1-2s}C_{q,\ell}(s/2)\right] & \text{if } \mathcal{S} = \mathcal{T}_2,\\
\frac{1}{3}(q^2-1)(q+\varepsilon)(q-\varepsilon)^2q^3B_{q,\ell}(s/2) & \text{if } \mathcal{S} = \mathcal{T}_3,\\
(q-1)(q^3-\varepsilon)(q+\varepsilon)q^2\left[B_{q,\ell}(s/2)+2q^{1-2s}C_{q,\ell}(s/2)\right] & \text{if } \mathcal{S} = \mathcal{M},\\
(q^3-\varepsilon)(q^2-1)\left[qB_{q,\ell}(s/2)+(q-1)q^{1-2s}C_{q,\ell}(s/2)\right] & \text{if } \mathcal{S} = \mathcal{N},\\
(q^3-\varepsilon)(q^2-1)q^{1-2s}C_{q,\ell}(s/2) & \text{if } \mathcal{S} \in \{\mathcal{K}_0,\mathcal{K}_\infty\}.
\end{array}$$

Corollary 6.2. *For* $\sigma \in \mathfrak{Sh}$ *of type* $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$,

$$\xi^\sigma(s) = \Xi^{\mathcal{S}}_{\varepsilon,q}(s),$$

where the function $\Xi^{\mathcal{S}}_{\varepsilon,q}(s) := \Xi^{\mathcal{S}}_{\mathsf{A}_2,\varepsilon,q}(s)$ *is given by*

$$\begin{array}{ll}
1 & \text{if } \mathcal{S} = \mathcal{G},\\
(q-1)(q^2+\varepsilon q+1)q^2\,(1-q^{1-2s})^{-1} & \text{if } \mathcal{S} = \mathcal{L},\\
(q^3-\varepsilon)(q+\varepsilon)\,(1-q^{1-2s})^{-1} & \text{if } \mathcal{S} = \mathcal{J},\\
\frac{1}{6}(q-1)(q^2+\varepsilon q+1)(q+\varepsilon)q^3\,(q-2+2q^{2-2s}-q^{1-2s})\left((1-q^{1-2s})(1-q^{2-3s})\right)^{-1} & \text{if } \mathcal{S} = \mathcal{T}_1,\\
\frac{1}{2}(q-1)(q^3-\varepsilon)(1-q^{-2s})q^4\,\left((1-q^{1-2s})(1-q^{2-3s})\right)^{-1} & \text{if } \mathcal{S} = \mathcal{T}_2,\\
\frac{1}{3}(q^2-1)(q+\varepsilon)(q-\varepsilon)^2q^3\,(1-q^{2-3s})^{-1} & \text{if } \mathcal{S} = \mathcal{T}_3,\\
(q^3-\varepsilon)(q-1)(q+\varepsilon)q^2\,(1+q^{1-2s})\left((1-q^{1-2s})(1-q^{2-3s})\right)^{-1} & \text{if } \mathcal{S} = \mathcal{M},\\
(q^3-\varepsilon)(q^3-q)\,(1-q^{-2s})\left((1-q^{1-2s})(1-q^{2-3s})\right)^{-1} & \text{if } \mathcal{S} = \mathcal{N},\\
(q^3-1)(q^2-1)q^{1-2s}\,\left((1-q^{1-2s})(1-q^{2-3s})\right)^{-1} & \text{if } \mathcal{S} \in \{\mathcal{K}_0,\mathcal{K}_\infty\}.
\end{array}$$

Proof. We may regard $\xi^\sigma(s)$ as a limit of $\xi^\sigma_\ell(s)$ as $\ell \to \infty$. Taking the formal limit

$$\Xi^{\mathcal{S}}_{\varepsilon,q}(s) = \lim_{\ell\to\infty} \Xi^{\mathcal{S}}_{\varepsilon,q,\ell}(s)$$

amounts to employing the substitutions

$$A_{q,\ell}(s/2) \to \frac{1}{1-q^{1-2s}}, \quad B_{q,\ell}(s/2) \to \frac{1}{1-q^{2-3s}}, \quad C_{q,\ell}(s/2) \to \frac{1}{(1-q^{1-2s})(1-q^{2-3s})}$$

to the formulae provided by the proposition. □

Proof of Theorem J. The explicit formulae for the representation zeta functions of the finite groups G^m_ℓ, H^m_ℓ as well as the infinite groups H^m follow directly from Proposition 5.16, Remark 5.19, Proposition 6.1, and Corollary 6.2. □

We now give an alternative proof of Corollary 6.2. It is based on Lemma 4.6 and finite recursion equations, bypassing the computation of the functions ξ^σ_ℓ. Recall the Definition 4.4 of the shadow graph $\Gamma = \Gamma^{(\varepsilon)}$.

Proposition 6.3. *Let* $\tau \in \mathfrak{Sh}$ *of type* $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$. *Then* $\xi^\tau(s) \equiv 1$ *if* $\mathcal{S} = \mathcal{G}$. *Otherwise,*

$$\xi^\tau(s) = \sum_{\substack{(\sigma,\tau)\in \dot{E}(\Gamma)\\ \sigma\neq\tau}} \frac{q^{-1}\left[\sigma(\mathbf{k}):\tau(\mathbf{k})\right] a_{\sigma,\tau}(q)\, q^{-\frac{1}{2}(\dim \mathsf{G}-\dim(\sigma))s}\,\xi^\sigma(s)}{1-q^{-1}\,a_{\tau,\tau}(q)\,b^{(\varepsilon)}_{\tau,\tau}(q)^{-s/2}}.$$

Proof. Note that the only edge in the shadow graph $\Gamma = \Gamma^{(\varepsilon)}$ with target $\mathcal{G}$ is $(\mathcal{G}, \mathcal{G})$. Applying Lemma 4.6 to the special case τ of type $\mathcal{G}$ and consulting Table 2.2, we use induction to find that $\gamma_\ell^\tau(s) = q^\ell$, and hence $\xi^\tau(s) \equiv 1$ for τ of type $\mathcal{G}$.

Now let τ be a shadow of type different from $\mathcal{G}$. Lemma 4.6 shows that for $\ell \in \mathbb{N}_0$ we have

$$\frac{\gamma_{\ell+1}^\tau(s/2)}{q^{\ell+1}} = q^{-1}\, a_{\tau,\tau}(q)\, b_{\tau,\tau}^{(\varepsilon)}(q)^{-s/2}\, \frac{\gamma_\ell^\tau(s/2)}{q^\ell} + \sum_{\substack{(\sigma,\tau)\in \dot{E}(\Gamma)\\ \sigma\neq\tau}} q^{-1}\, a_{\sigma,\tau}(q)\, b_{\sigma,\tau}^{(\varepsilon)}(q)^{-s/2}\, \frac{\gamma_\ell^\sigma(s/2)}{q^\ell}.$$

Multiplying both sides by $[\mathsf{G}(\mathbf{k}) : \tau(\mathbf{k})]^{1+s/2}$ and taking the limit as $\ell \to \infty$, we obtain

$$\xi^\tau(s) = q^{-1}\, a_{\tau,\tau}(q)\, b_{\tau,\tau}^{(\varepsilon)}(q)^{-s/2}\, \xi^\tau(s) + \sum_{\substack{(\sigma,\tau)\in \dot{E}(\Gamma)\\ \sigma\neq\tau}} q^{-1}\, a_{\sigma,\tau}(q)\, b_{\sigma,\tau}^{(\varepsilon)}(q)^{-s/2}\, \frac{[\mathsf{G}(\mathbf{k}) : \tau(\mathbf{k})]^{1+s/2}}{[\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})]^{1+s/2}} \xi^\sigma(s).$$

Substituting in the defining value for $b_{\sigma,\tau}^{(\varepsilon)}(q)$ from (2.4) resp. (3.9), using $[\sigma(\mathbf{k}) : \tau(\mathbf{k})] = \|\sigma\|/\|\tau\|$, and solving for $\xi^\tau(s)$ yields the desired formula. $\square$

6.2. **Expressions for zeta functions of groups of type A_2.** For groups of type A_2 we are in a position to apply the sufficient criteria developed in Section 5 for the extension of characters from the principal congruence subgroup G^1, respectively H^1, to G, respectively H. Indeed, the next two lemmata establish the existence of shadow-preserving lifts and the vanishing of the relevant cohomology groups.

Lemma 6.4. *Let $\ell \in \mathbb{N}_0$. Every $A \in \mathfrak{g}_\ell$ has a shadow-preserving lift $\widetilde{A} \in \mathfrak{g}_{\ell+1}$. Furthermore, if $p \neq 3$ then every $A \in \mathfrak{h}_\ell$ has a shadow-preserving lift $\widetilde{A} \in \mathfrak{h}_{\ell+1}$.*

Proof. The assertion for $\mathfrak{g}_\ell$ is equivalent to the claim that in the shadow graph $\Gamma = \Gamma^{(\varepsilon)}$, see Figure 4.1, there is a loop on every vertex or, equivalently, that $a_{\sigma,\sigma}(q) \neq 0$ for all $\sigma \in \mathfrak{S}\mathfrak{h}$. That this is the case follows from Theorems 2.8 and 3.12; cf. Table 2.2.

Assume now that $p \neq 3$ and consider $A \in \mathfrak{h}_\ell \leq \mathfrak{g}_\ell$. Let $\widetilde{B} \in \mathfrak{g}_{\ell+1}$ be a shadow-preserving lift of A within $\mathfrak{g}_{\ell+1}$, i.e. suppose that $A \equiv_{\pi^\ell} \widetilde{B}$ and $G^1\mathrm{C}_G(A) = G^1\mathrm{C}_G(\widetilde{B})$. Put $\widetilde{A} = \widetilde{B} - \frac{\mathrm{tr}(\widetilde{B})}{3}\,\mathrm{Id}_3$. Then $\widetilde{A} \in \mathfrak{h}_{\ell+1}$ and $G^1\mathrm{C}_G(A) = G^1\mathrm{C}_G(\widetilde{A})$.

We claim that $H^1\mathrm{C}_H(A) = H^1\mathrm{C}_H(\widetilde{A})$. The inclusion $\supset$ is obvious. For the reverse inclusion, let $h \in \mathrm{C}_H(A)$. There is $g \in G^1$ and $y \in \mathrm{C}_G(\widetilde{A})$ such that $h = gy$. From $\det h = 1$ and $\det g \equiv_\pi 1$ we deduce $\det y \equiv_\pi 1$. As $p \neq 3$, the pro-p group $\mathsf{GL}_1^1(\mathfrak{o})$ for $\varepsilon = 1$, or $\mathsf{GU}_1^1(\mathfrak{o})$ for $\varepsilon = -1$, contains a cube root of $\det y$. Hence there is a scalar matrix $s \in G^1$ such that $s^{-1}y \in \mathrm{C}_H(\widetilde{A})$ and $gs \in G^1 \cap H = H^1$. It follows that $h = (gs)(s^{-1}y) \in H^1\mathrm{C}_H(\widetilde{A})$. $\square$

Remark 6.5. The proof of the existence of shadow-preserving lifts of matrices in type A_2 in Lemma 6.4 resorts to the shadow graph Γ. We sketch here a geometric point of view on the existence of shadow-preserving lifts, which also pertains to other 'semisimple' Lie rings, say of type A_{n-1}, $n \geq 4$, or other classical types, where we currently do not know of an equally uniform description of the lifting behaviour of similarity classes.

In [5, Section 3.2] we presented zeta functions of groups such as $\mathsf{SL}_n^m(\mathfrak{o})$ in terms of $\mathfrak{p}$-adic integrals. The latter are, in general, defined in terms of polynomial ideals defining the rank varieties of certain matrices of linear forms. In [5, Section 5] we described a link between these rank varieties and stratifications of the (complexification of the) associated semisimple Lie algebras by quasi-affine varieties comprising elements of constant centraliser dimension, $\mathcal{V}_i \setminus \mathcal{V}_{i+1}$ in the parlance of [5].

It can be shown that shadow-preserving lifts exist if and only if, for all $\ell \in \mathbb{N}$ and any such variety, every point modulo $\mathfrak{p}^\ell$ has a lift to a point modulo $\mathfrak{p}^{\ell+1}$. The latter holds – essentially by Hensel's Lemma – if the relevant varieties are all smooth. In general, they are disjoint unions of finitely many subvarieties called *sheets*; see [11]. The complex Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, for instance, is the union of three sheets, consisting of elements of centraliser dimensions 8 (the null sheet), 6 (the subregular sheet), and 4 (the regular sheet), respectively. But even if the sheets are all smooth, as they are in type A_{n-1}, some of their unions might not be. In $\mathfrak{sl}_4(\mathbb{C})$ and $\mathfrak{sl}_5(\mathbb{C})$, the varieties of elements of constant centraliser dimension each consist of a single sheet; hence they are smooth, and shadow-preserving lifts exist. Already in $\mathfrak{sl}_6(\mathbb{C})$, however, the varieties of elements of centraliser dimension 17 and 11 both are the union of two sheets, respectively, of different dimensions. For a further discussion, also regarding the dimensions of sheets in other semisimple Lie algebras, see [45].

Lemma 6.6. *Suppose that $p > 3$ is prime and that q is a power of p. Then*

(a) *If $T \leq \mathsf{GL}_3(\mathbb{F}_q)$ or $T \leq \mathsf{GU}_3(\mathbb{F}_q)$ is contained in a torus, then $|\mathrm{H}^2(T, \mathbb{C}^\times)|$ is prime to p,*
(b) $\mathrm{H}^2(\mathsf{SL}_3(\mathbb{F}_q), \mathbb{C}^\times) = 1$ *and* $\mathrm{H}^2(\mathsf{SL}_2(\mathbb{F}_q), \mathbb{C}^\times) = 1$,
(c) $\mathrm{H}^2(\mathsf{SU}_3(\mathbb{F}_q), \mathbb{C}^\times) = 1$ *and* $\mathrm{H}^2(\mathsf{SU}_2(\mathbb{F}_q), \mathbb{C}^\times) = 1$,
(d) $\mathrm{H}^2(\mathsf{GL}_2(\mathbb{F}_q), \mathbb{C}^\times) = 1$,
(e) $\mathrm{H}^2(\mathsf{GU}_2(\mathbb{F}_q), \mathbb{C}^\times) = 1$.

Proof. The assertion (a) holds because every prime that divides the order of $\mathrm{H}^2(T, \mathbb{C}^\times)$ must also divide the order of T, which is not divisible by p. Assertions (b) and (c) are well known; cf. [32, 55] as well as [33, p. 246] and [15, Table 5].

Next we consider (d). From the Lyndon–Hochschild–Serre spectral sequence

$$\mathrm{H}^a(G/N, \mathrm{H}^b(N, M)) \Rightarrow \mathrm{H}^{a+b}(G, M),$$

applied to $a + b = 2$, $G = \mathsf{GL}_2(\mathbb{F}_q)$, $N = \mathsf{SL}_2(\mathbb{F}_q)$ and $M = \mathbb{C}^\times$, we deduce that the order of $\mathrm{H}^2(\mathsf{GL}_2(\mathbb{F}_q), \mathbb{C}^\times)$ divides

$$|\mathrm{H}^0(\mathbb{F}_q^\times, \mathrm{H}^2(\mathsf{SL}_2(\mathbb{F}_q), \mathbb{C}^\times))| \cdot |\mathrm{H}^1(\mathbb{F}_q^\times, \mathrm{H}^1(\mathsf{SL}_2(\mathbb{F}_q), \mathbb{C}^\times))| \cdot |\mathrm{H}^2(\mathbb{F}_q^\times, \mathrm{H}^0(\mathsf{SL}_2(\mathbb{F}_q), \mathbb{C}^\times))|.$$

From (b) we see that $\mathrm{H}^2(\mathsf{SL}_2(\mathbb{F}_q), \mathbb{C}^\times)$ is trivial and hence the first factor is 1. Since $\mathsf{SL}_2(\mathbb{F}_q)$ is perfect for $q > 3$, also $\mathrm{H}^1(\mathsf{SL}_2(\mathbb{F}_q), \mathbb{C}^\times)$ is trivial and the second factor is 1. Finally, the third factor equals $|\mathrm{H}^2(\mathbb{F}_q^\times, \mathbb{C}^\times)| = 1$, because $\mathbb{F}_q^\times$ is cyclic.

Noting that $\mathsf{SU}_2(\mathbb{F}_q) \simeq \mathsf{SL}_2(\mathbb{F}_q)$ (e.g., see [28, II.8.8]), a similar argument yields (e). ☐

We have collected all the necessary information to deal with characters of G. For H we prove an additional lemma, enabling us to employ the theory of shadows also in this context.

Lemma 6.7. *With the notation as above, suppose that the Kirillov orbit method is applicable to H^1 and that $p \neq 3$. Let $\chi \in \mathrm{Irr}(H^1)$ and $\omega \in \mathfrak{h}^\vee$ a representative of the corresponding co-adjoint orbit. Let $S_\chi = I_H(\chi)$ be the inertia subgroup. Then*

$$S_\chi/H^1 = \mathrm{Stab}_H(\omega)H^1/H^1 \simeq \big(\mathrm{Stab}_G(\omega)G^1/G^1\big) \cap \mathsf{H}(\mathbf{k}),$$

where $\mathrm{Stab}_G(\omega)G^1/G^1$ is identified with its image in $G/G^1 \simeq \mathsf{G}(\mathbf{k})$.

Proof. The equality follows from Lemma 5.20, and we argue for the isomorphism. It suffices to show that every coset on the right-hand side is the image of a coset on the left-hand side under the natural inclusion map. Let $g \in \mathrm{Stab}_G(\omega)$ such that the reduction of g modulo G^1 is unimodular. Since $p \neq 3$, there is a scalar matrix $u \in G^1$ such that $h := gu \in \mathrm{Stab}_H(\omega)$; compare the proof of Lemma 6.4. It follows that hH^1 maps to gG^1. ☐

Proof of Theorem C. Consider one of the groups $\mathsf{GL}_3(\mathfrak{o}), \mathsf{GU}_3(\mathfrak{o}), \mathsf{SL}_3(\mathfrak{o}), \mathsf{SU}_3(\mathfrak{o})$, and its 1st principal congruence subgroup N. Based upon $p \geq 3e+3$, we check the hypotheses in Theorem 5.29 regarding characters $\chi \in \mathrm{Irr}(N)$. Lemma 6.4 guarantees the existence of shadow-preserving lifts. Lemma 6.6, in conjunction with Tables 6.1 and 6.2 as well as Lemma 6.7, shows that the relevant cohomology groups vanish. Thus Theorem 5.29 implies that every $\chi \in \mathrm{Irr}(N)$ extends to its inertia group S_χ and Corollary 5.26 is applicable.

Regarding the infinite groups $\mathsf{SL}_3(\mathfrak{o}), \mathsf{SU}_3(\mathfrak{o})$, formula (1.9) follows now directly by collecting the summands in (5.12) according to shadows.

Regarding the finite groups $\mathsf{GL}_3(\mathfrak{o}_\ell), \mathsf{GU}_3(\mathfrak{o}_\ell), \mathsf{SL}_3(\mathfrak{o}_\ell), \mathsf{SU}_3(\mathfrak{o}_\ell)$, formulae (1.7) and (1.8) are obtained by restricting the relevant sums to characters factoring over the ℓth principal congruence subgroup. Once more, Lemma 6.7 is used to deal with the special linear/unitary groups. □

6.3. **Zeta functions of the shadows.** In order to derive from Theorem C explicit formulae such as the one in Corollary D we need, in addition to the functions $\xi^\sigma_{\ell-1} = \Xi^{\mathcal{S}}_{\varepsilon,q,\ell-1}$ and $\xi^\sigma = \Xi^{\mathcal{S}}_{\varepsilon,q}$ given in Proposition 6.1 and Corollary 6.2, the respective shadows' indices and zeta functions. In fact, the former may be expressed in terms of the latter, because the order of any finite group is equal to the value of its zeta function at $s = -2$. The isomorphism classes of the groups whose zeta functions we need to compute are listed in Tables 6.1 and 6.2. As before, $\mathbf{k}_m$ denotes a degree m extension of $\mathbf{k}$. For $m \mid n$, we write $N_{\mathbf{k}_n \mid \mathbf{k}_m} : \mathbf{k}_n^\times \to \mathbf{k}_m^\times$ for the norm map. Furthermore, $\varepsilon = 1$ for $\mathsf{G} = \mathsf{GL}_3, \mathsf{H} = \mathsf{SL}_3$ and $\varepsilon = -1$ for $\mathsf{G} = \mathsf{GU}_3, \mathsf{H} = \mathsf{SU}_3$; we write $q = |\mathbf{k}|$ and $\iota(\varepsilon, q) = \gcd(q - \varepsilon, 3)$.

TABLE 6.1. Shadows in $\mathsf{GL}_3(\mathbf{k})$, for $\varepsilon = 1$, and $\mathsf{GU}_3(\mathbf{k})$, for $\varepsilon = -1$

Type	$\sigma(\mathbf{k}) \subset \mathsf{GL}_3(\mathbf{k})$	$\sigma(\mathbf{k}) \subset \mathsf{GU}_3(\mathbf{k})$	Order $\lvert\sigma(\mathbf{k})\rvert$
$\mathcal{G}$	$\mathsf{GL}_3(\mathbf{k})$	$\mathsf{GU}_3(\mathbf{k})$	$(q-\varepsilon)(q^2-1)(q^3-\varepsilon)q^3$
$\mathcal{L}$	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_2(\mathbf{k})$	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_2(\mathbf{k})$	$(q^2-1)(q-\varepsilon)^2 q$
$\mathcal{J}$	$\mathsf{Heis}(\mathbf{k}) \rtimes \mathsf{GL}_1(\mathbf{k})^2$	$\mathsf{Heis}(\mathbf{k}) \rtimes \mathsf{GU}_1(\mathbf{k})^2$	$(q-\varepsilon)^2 q^3$
$\mathcal{T}_1$	$\mathsf{GL}_1(\mathbf{k})^3$	$\mathsf{GU}_1(\mathbf{k})^3$	$(q-\varepsilon)^3$
$\mathcal{T}_2$	$\mathsf{GL}_1(\mathbf{k}_2) \times \mathsf{GL}_1(\mathbf{k})$	$\mathsf{GL}_1(\mathbf{k}_2) \times \mathsf{GU}_1(\mathbf{k})$	$(q^2-1)(q-\varepsilon)$
$\mathcal{T}_3$	$\mathsf{GL}_1(\mathbf{k}_3)$	$\mathsf{GU}_1(\mathbf{k}_3)$	$q^3-\varepsilon$
$\mathcal{M}$	$\mathsf{GL}_1(\mathbf{k}[x]/(x^2)) \times \mathsf{GL}_1(\mathbf{k})$	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k})$	$(q-\varepsilon)^2 q$
$\mathcal{N}$	$\mathsf{GL}_1(\mathbf{k}[x]/x^3)$	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k})$	$(q-\varepsilon)q^2$
$\mathcal{K}_0, \mathcal{K}_\infty$*	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k})$	—	$(q-\varepsilon)q^2$

* Only applies if $\varepsilon = 1$.

Definition 6.8. Let $\sigma \in \mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$. Recalling that $\sigma(\mathbf{k})$ denotes a subgroup of $\mathsf{G}(\mathbf{k})$ representing the shadow σ, we put
$$\sigma'(\mathbf{k}) := \sigma(\mathbf{k}) \cap \mathsf{H}(\mathbf{k}).$$

Proposition 6.9. *Let $\sigma \in \mathfrak{Sh}$ be of type $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$. Then*
$$\zeta_{\sigma(\mathbf{k})}(s) = (q-\varepsilon) Z^{\mathcal{S}}_{\varepsilon,1,q}(s),$$
$$\zeta_{\sigma'(\mathbf{k})}(s) = Z^{\mathcal{S}}_{\varepsilon,\iota(\varepsilon,q),q}(s),$$

TABLE 6.2. Shadows in $\mathsf{SL}_3(\mathbf{k})$, for $\varepsilon = 1$, and $\mathsf{SU}_3(\mathbf{k})$, for $\varepsilon = -1$

Type	$\sigma'(\mathbf{k}) = \sigma(\mathbf{k}) \cap \mathsf{SL}_3(\mathbf{k})$	$\sigma'(\mathbf{k}) = \sigma(\mathbf{k}) \cap \mathsf{SU}_3(\mathbf{k})$	Order $\lvert\sigma'(\mathbf{k})\rvert$
$\mathcal{G}$	$\mathsf{SL}_3(\mathbf{k})$	$\mathsf{SU}_3(\mathbf{k})$	$(q^2-1)(q^3-\varepsilon)q^3$
$\mathcal{L}$	$\mathsf{GL}_2(\mathbf{k})$	$\mathsf{GU}_2(\mathbf{k})$	$(q^2-1)(q-\varepsilon)q$
$\mathcal{J}$	$\mathsf{Heis}(\mathbf{k}) \rtimes \mathsf{GL}_1(\mathbf{k})$	$\mathsf{Heis}(\mathbf{k}) \rtimes \mathsf{GU}_1(\mathbf{k})$	$(q-\varepsilon)q^3$
$\mathcal{T}_1$	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k})$	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k})$	$(q-\varepsilon)^2$
$\mathcal{T}_2$	$\mathsf{GL}_1(\mathbf{k}_2)$	$\mathsf{GL}_1(\mathbf{k}_2)$	q^2-1
$\mathcal{T}_3$	$\ker(N_{\mathbf{k}_3 \mid \mathbf{k}})$	$\ker(N_{\mathbf{k}_6 \mid \mathbf{k}_3}) \cap \ker(N_{\mathbf{k}_6 \mid \mathbf{k}_2})$	$q^2+\varepsilon q+1$
$\mathcal{M}$	$\mathsf{GL}_1(\mathbf{k}[x]/(x^2))$	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k})$	$(q-\varepsilon)q$
$\mathcal{N}$	$\mu_3(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k})$	$(\mu_3(\mathbf{k}_2) \cap \ker(N_{\mathbf{k}_2 \mid \mathbf{k}})) \times (\mathsf{G}_a(\mathbf{k}))^2$	$\iota(\varepsilon,q)\,q^2$
$\mathcal{K}_0, \mathcal{K}_\infty$ *	$\mu_3(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k})$	—	$\iota(\varepsilon,q)\,q^2$

* Only applies if $\varepsilon = 1$.

where $Z^{\mathcal{S}}_{\varepsilon,i,q} := Z^{\mathcal{S}}_{\mathsf{A}_2,\varepsilon,i,q}$ *for* $i \in \{1,3\}$ *is given by*

$$\begin{aligned}
Z^{\mathcal{G}}_{\varepsilon,i,q}(s) = {} & 1 + (q^2+\varepsilon q)^{-s} + (q-1-\varepsilon)(q^2+\varepsilon q+1)^{-s} \\
& + \tfrac{1}{2}(q^2-q-1+\varepsilon)(q^3-\varepsilon)^{-s} + q^{-3s} + (q-1-\varepsilon)(q^3+\varepsilon q^2+q)^{-s} \\
& + \tfrac{1}{3}(q^2+\varepsilon q-2)((q+\varepsilon)(q-\varepsilon)^2)^{-s} + \tfrac{2}{3}i^2((q+\varepsilon)(q-\varepsilon)^2/i)^{-s} \\
& + \tfrac{1}{6}(q-\varepsilon)(q-3-\varepsilon)((q^2+\varepsilon q+1)(q+\varepsilon))^{-s} + \tfrac{1}{3}i^2((q^2+\varepsilon q+1)(q+\varepsilon)/i)^{-s},
\end{aligned}$$

if $\mathcal{S} = \mathcal{G}$, *and in the remaining cases* $\mathcal{S} \neq \mathcal{G}$ *defined as*

$$\begin{array}{ll}
(q-\varepsilon)\left(1+q^{-s}+\tfrac{1}{2}(q-2)(q+1)^{-s}+\tfrac{1}{2}q(q-1)^{-s}\right) & \textit{if } \mathcal{S}=\mathcal{L}, \\
(q-\varepsilon)+(q+\varepsilon)i^2((q-\varepsilon)/i)^{-s}+(q-1)(q-\varepsilon)q^{-s} & \textit{if } \mathcal{S}=\mathcal{J}, \\
(q-\varepsilon)^2 & \textit{if } \mathcal{S}=\mathcal{T}_1, \\
q^2-1 & \textit{if } \mathcal{S}=\mathcal{T}_2, \\
q^2+\varepsilon q+1, & \textit{if } \mathcal{S}=\mathcal{T}_3, \\
q(q-\varepsilon) & \textit{if } \mathcal{S}=\mathcal{M}, \\
iq^2 & \textit{if } \mathcal{S}\in\{\mathcal{N},\mathcal{K}_0,\mathcal{K}_\infty\}.
\end{array}$$

Proof. The isomorphism types of the groups in question appear in Tables 6.1 and 6.2. The formula for $\zeta_{\mathsf{G}(\mathbf{k})}(s)$ is extracted from the character tables in [54], for $\varepsilon = 1$, and [20, §7], for $\varepsilon = -1$. The formulae for $\zeta_{\mathsf{H}(\mathbf{k})}(s)$ are obtained from the data provided in [52]. For groups of type $\mathcal{L}$ the formula follows, for example, from [17, § 15.9], for $\varepsilon = 1$, and [20, §6], for $\varepsilon = -1$. It remains to discuss shadows of type $\mathcal{J}$, the only other non-abelian cases.

Groups in the shadow $\sigma \in \mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$ of type $\mathcal{J}$ are isomorphic to $J_\varepsilon = E_\varepsilon \rtimes D_\varepsilon$, where $E_\varepsilon \simeq \mathsf{Heis}(\mathbf{k})$ is given explicitly by

$$E_1 := \left\{ \begin{bmatrix} 1 & s_1 & z \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{bmatrix} \mid s_1, s_2, z \in \mathbf{k} \right\} \quad \text{and} \quad E_{-1} := \left\{ \begin{bmatrix} 1 & s & z \\ 0 & 1 & s^\circ \\ 0 & 0 & 1 \end{bmatrix} \mid s, z \in \mathbf{k}_2, s^\circ s = z + z^\circ \right\}.$$

This representation of the shadow $\mathcal{J}$ uses the centraliser of the elementary matrix e_{13} that has its non-zero entry in the $(1,3)$-position, highlighting the appearance of the Heisenberg group. The convention in the rest of the present paper, using the centraliser of e_{23}, is consistent with [7]; the centraliser of e_{12} is used in [5]. The groups D_ε are given explicitly by

$$D_1 := \{\mathrm{diag}(u,v,u) \mid u,v \in \mathbf{k}^\times\} \simeq \mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k}),$$
$$D_{-1} := \{\mathrm{diag}(u,v,u) \mid u,v \in \ker(N_{\mathbf{k}_2\,|\,\mathbf{k}})\} \simeq \mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k}).$$

Furthermore, the intersection of J_ε with $\mathsf{H}(\mathbf{k})$ is equal to $J'_\varepsilon = E_\varepsilon \rtimes D'_\varepsilon$, where

$$D'_\varepsilon := D_\varepsilon \cap \mathsf{H}(\mathbf{k}) = \{(u,v,u) \in D_\varepsilon \mid v = u^{-2}\} \simeq \begin{cases} \mathsf{GL}_1(\mathbf{k}) & \text{if } \varepsilon = 1, \\ \mathsf{GU}_1(\mathbf{k}) & \text{if } \varepsilon = -1. \end{cases}$$

Let

$$Z_\varepsilon := \left\{ \left[\begin{smallmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] \mid z \in \mathbf{k} \right\}$$

denote the centre of E_ε. The group $E_\varepsilon \simeq \mathsf{Heis}(\mathbf{k})$ has $q-1$ irreducible characters of degree q which correspond bijectively to the non-trivial characters of the centre, and q^2 linear characters factoring through its abelianisation $Q_\varepsilon := E_\varepsilon/Z_\varepsilon \simeq \mathbf{k} \times \mathbf{k}$.

The group D_ε acts trivially on Z_ε and hence stabilises all the q-dimensional irreducible characters of E_ε. As q is prime to $|D_\varepsilon| = (q-\varepsilon)^2$, they all extend to irreducible characters of J_ε. We get $(q-\varepsilon)^2(q-1)$ distinct q-dimensional irreducible characters of J_ε, and similarly $(q-\varepsilon)(q-1)$ such characters of J'_ε.

The remaining irreducible characters of J_ε, respectively J'_ε, factor through its quotient by Z_ε, viz. $Q_\varepsilon \rtimes D_\varepsilon$, respectively $Q_\varepsilon \rtimes D'_\varepsilon$. We consider separately the cases $\varepsilon = 1$ and $\varepsilon = -1$.

First suppose that $\varepsilon = 1$. It is convenient to identify Q_1 and its dual $Q_1^\vee$ with the additive group $\mathbf{k}\times\mathbf{k}$. With this identification, the action of $\mathrm{diag}(u,v,u) \in D_1$ on $Q_1^\vee$ is given by $(s_1,s_2) \mapsto (u^{-1}vs_1, uv^{-1}s_2)$. We use Mackey's method for semi-direct products; cf. [50, Section 8.2]. The orbits of D_1, respectively D'_1, on $Q_1^\vee \simeq \mathbf{k} \times \mathbf{k}$ are classified as follows. For D_1 we obtain

Orbit	Parameter	Stabiliser in D_1
$[0,0]$	—	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k})$
$[0,\mathbf{k}^\times]$	—	$\mathsf{GL}_1(\mathbf{k})$
$[\mathbf{k}^\times,0]$	—	$\mathsf{GL}_1(\mathbf{k})$
$\mathbf{k}^\times \cdot [s_1,s_2]$	$(s_1,s_2) \in \mathbf{k}^\times \times_{\mathbf{k}^\times} \mathbf{k}^\times$	$\mathsf{GL}_1(\mathbf{k})$

yielding $|\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k})| = (q-1)^2$ linear characters of J_1 lying above the trivial orbit $[0,0]$ and $q+1$ irreducible characters of degree $q-1$ lying above the remaining orbits. Similarly, for D'_1 we obtain

Orbit	Parameter	Stabiliser in D'_1
$[0,0]$	—	$\mathsf{GL}_1(\mathbf{k})$
$[0,(\mathbf{k}^\times)^3 \cdot s_2]$	$s_2 \in \mathbf{k}^\times/(\mathbf{k}^\times)^3$	$\mu_3(\mathbf{k})$
$[(\mathbf{k}^\times)^3 \cdot s_1, 0]$	$s_1 \in \mathbf{k}^\times/(\mathbf{k}^\times)^3$	$\mu_3(\mathbf{k})$
$(\mathbf{k}^\times)^3 \cdot [s_1,s_2]$	$(s_1,s_2) \in \mathbf{k}^\times \times_{(\mathbf{k}^\times)^3} \mathbf{k}^\times$	$\mu_3(\mathbf{k})$

yielding $|\mathsf{GL}_1(\mathbf{k})| = q-1$ linear characters and $(|\mathbf{k}|+1)|\mathbf{k}^\times/(\mathbf{k}^\times)^3||\mu_3(\mathbf{k})| = (q+1)\iota(\varepsilon,q)^2$ irreducible characters of degree $|\mathbf{k}^\times/\mu_3(\mathbf{k})| = (q-1)/\iota(\varepsilon,q)$ of J'_1.

Now suppose that $\varepsilon = -1$. In this case we identify Q_{-1} and its dual $Q_{-1}^\vee$ with the additive group $\mathbf{k}_2$. The action of $\mathrm{diag}(u,v,u) \in D_{-1}$ is given by $s \mapsto u^{-1}vs$. To use Mackey's method for semi-direct products we classify the orbits of D_{-1}, respectively D'_{-1}, on $Q_{-1}^\vee \simeq \mathbf{k}_2$. For D_1 we obtain

Orbit	Parameter	Stabiliser in D_{-1}
$[0]$	—	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k})$
$[s]$	$s \in \mathbf{k}_2^\times/\mathsf{GU}_1(\mathbf{k})$	$\mathsf{GU}_1(\mathbf{k})$

yielding $|\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k})| = (q+1)^2$ linear characters and $|\mathbf{k}_2^\times/\mathsf{GU}_1(\mathbf{k})||\mathsf{GU}_1(\mathbf{k})| = (q^2-1)$ irreducible characters of degree $|\mathsf{GU}_1(\mathbf{k})| = (q+1)$ of J_{-1}. Similarly for D_1' we obtain

Orbit	Parameter	Stabiliser in D'_{-1}
$[0]$	—	$\mathsf{GU}_1(\mathbf{k})$
$[s]$	$s \in \mathbf{k}_2^\times/\mathsf{GU}_1(\mathbf{k})^3$	$\mu_3(\mathbf{k}_2) \cap \mathsf{GU}_1(\mathbf{k})$

yielding $|\mathsf{GU}_1(\mathbf{k})| = q+1$ linear characters and $|\mathbf{k}_2^\times/\mathsf{GU}_1(\mathbf{k})^3||\mu_3(\mathbf{k}_2) \cap \mathsf{GU}_1(\mathbf{k})| = (q-1)\iota(\varepsilon,q)^2$ irreducible characters of degree $|\mathsf{GU}_1(\mathbf{k})/\mu_3(\mathbf{k}_2) \cap \mathsf{GU}_1(\mathbf{k})| = (q+1)/\iota(\varepsilon,q)$ of J'_{-1}.

In summary, for σ of type $\mathcal{J}$ we showed that

$$\zeta_{\sigma(\mathbf{k})}(s) = (q-\varepsilon)\left((q-\varepsilon) + (q+\varepsilon)(q-\varepsilon)^{-s} + (q-1)(q-\varepsilon)q^{-s}\right),$$
$$\zeta_{\sigma'(\mathbf{k})}(s) = (q-\varepsilon) + (q+\varepsilon)\iota(\varepsilon,q)^2\left((q-\varepsilon)/\iota(\varepsilon,q)\right)^{-s} + (q-1)(q-\varepsilon)q^{-s}. \qquad \square$$

Proof of Corollary D. The corollary is obtained from formula (1.9) in Theorem C and the explicit formulae provided in Table 6.2, Proposition 6.9, and Corollary 6.2. $\square$

6.4. **Character degrees and Ennola duality.** In this section we prove Theorems H and I. Consider the finite groups $G_\ell = \mathsf{G}(\mathfrak{o}_\ell)$ and $H_\ell = \mathsf{H}(\mathfrak{o}_\ell)$, for $\ell \in \mathbb{N}$. The conditions on p in the two theorems ensure that the Kirillov orbit method is available to describe the characters of the finite principal congruence subgroups G_ℓ^1 and H_ℓ^1 and that these characters extend to their respective stabilizers in G_ℓ and H_ℓ; see Theorem 5.29, Remark 5.30, and compare with the proof of Theorem C. An irreducible character χ of G_ℓ, respectively H_ℓ, therefore determines, and is determined by, the following data:

- a shadow σ of type $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$,
- a G_ℓ-orbit, respectively H_ℓ-orbit, of an irreducible character φ_σ of G_ℓ^1, respectively H_ℓ^1, whose inertia subgroup in G_ℓ, respectively H_ℓ, gives rise to the shadow σ,
- a choice of an extension $\hat{\varphi}_\sigma$ to its inertia subgroup in G_ℓ, respectively H_ℓ, that we will not mention further,
- an irreducible character ψ_σ of $\sigma(\mathbf{k})$, respectively $\sigma'(\mathbf{k})$.

Moreover, to any such φ_σ one associates

- a unique path $\Delta(\varphi_\sigma) \in \mathrm{Path}^{\ell-1}(\mathcal{G},\mathcal{S})$ of length $\ell-1$ in the shadow graph $\Gamma^{(\varepsilon)}$, see Figure 4.1, starting at $\mathcal{G}$ and ending at $\mathcal{S}$.

By Corollary 5.18, the degree of φ_σ is determined by the path $\Delta = \Delta(\varphi_\sigma)$:

$$\varphi_\sigma(1) = \prod_{(\tau,\upsilon)\in\Delta} q^{\frac{1}{2}(\dim \mathsf{G} - \dim(\tau))}, \tag{6.1}$$

feeding into the degree formula (cf. (5.12))

$$\chi(1) = \varphi_\sigma(1)\psi_\sigma(1)[\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})], \qquad \text{respectively} \quad \chi(1) = \varphi_\sigma(1)\psi_\sigma(1)[\mathsf{H}(\mathbf{k}) : \sigma'(\mathbf{k})]. \tag{6.2}$$

Proof of Theorem H. We are required to give an Ennola-type description of the p'-part of the character degrees of G_ℓ. Let $\sigma \in \mathfrak{Sh}$ be of type $\mathcal{S}$. We use (6.2) to control the character degrees of $\chi \in \mathrm{Irr}(G_\ell)$ associated to σ.

Formula (6.1) shows that the contribution $\varphi_\sigma(1)$ to $\chi(1)$ is a q-power, depending only on $\Delta = \Delta(\varphi_\sigma)$.

The shadow graphs $\Gamma^{(1)}$ and $\Gamma^{(-1)}$ are almost identical: there is a natural correspondence between paths in $\Gamma^{(1)}$ not ending in $\mathcal{K}_0, \mathcal{K}_\infty$ and paths in $\Gamma^{(-1)}$. Moreover, paths in $\Gamma^{(1)}$ of the same length $\ell - 1$ and containing at the same position one of the edges $(\mathcal{J}, \mathcal{K}_0), (\mathcal{J}, \mathcal{K}_\infty), (\mathcal{J}, \mathcal{N})$ lead to the same character degrees. Finally, shadows σ of types $\mathcal{K}_0, \mathcal{K}_\infty, \mathcal{N}$ yield isomorphic groups $\sigma(\mathbf{k})$. Thus, for our purposes, we may simply ignore $\mathcal{K}_0, \mathcal{K}_\infty$.

By Proposition 6.9 the p'-parts of character degrees of $\sigma(\mathbf{k})$ are of the form $g(q)$, for polynomials $g \in \mathbb{Z}[t]$, involving ε as a parameter in such a way that the Ennola transform $g(t) \mapsto (-1)^{\deg g} g(-t)$ translates between the cases $\mathsf{G} = \mathsf{GL}_3$ and $\mathsf{G} = \mathsf{GU}_3$. From Table 6.1 we see that the same holds for the indices $[\mathsf{G}(\mathbf{k}) : \sigma(\mathbf{k})]$. Thus (1.17) follows from (6.2) and the fact that the Ennola transform is multiplicative.

The explicit descriptions of the sets $\mathrm{cd}(\mathsf{G}(\mathfrak{o}_\ell))_{p'}$ are easily obtained, using e.g. Proposition 6.9. We note that the two additional terms for $\ell > 1$ are owed to shadows of types $\mathcal{J}$, $\mathcal{M}$, and $\mathcal{N}$. $\square$

Definition 6.10. The *level* of a character χ of $H = \mathsf{H}(\mathfrak{o})$ or $H^1 = \mathsf{H}^1(\mathfrak{o})$ is equal to $\ell - 1$, where $\ell \in \mathbb{N}$ is minimal such that χ is trivial, i.e. equal to the constant function $\chi(1)$, on the principal congruence subgroup H^ℓ. The terminology extends in a natural way to characters of H_ℓ, respectively H^1_ℓ, by implicitly lifting them to H, respectively H^1.

Proof of Theorem I. In the special case $\ell = 1$, Proposition 6.9 provides the necessary information about character degrees of the group $\mathsf{H}(\mathbf{k})$. We thus focus on the case $\ell \geq 2$.

As explained above, a character $\chi \in \mathrm{Irr}(H)$ of level $\ell - 1 \geq 1$ can be connected with a shadow σ of type $\mathcal{S} \in \mathbb{T}^{(\varepsilon)}$, a character $\varphi_\sigma \in \mathrm{Irr}(H_\ell)$ of level $\ell - 1$ and a path $\Delta = \Delta(\varphi_\sigma) \in \mathrm{Path}^{\ell-1}(\mathcal{G}, \mathcal{S})$ of length $\ell - 1$ in the shadow graph $\Gamma^{(\varepsilon)}$. Observe that Δ does not begin with a loop $(\mathcal{G}, \mathcal{G})$; in particular, $\mathcal{S} \neq \mathcal{G}$. Furthermore, we have

$$\chi(1) \geq \min_{\substack{\mathcal{S} \in \mathbb{T}^{(\varepsilon)}, \\ \sigma \text{ of type } \mathcal{S}}} \; \min_{\substack{\varphi_\sigma \text{ such that} \\ (\mathcal{G},\mathcal{G}) \notin \Delta(\varphi_\sigma) \in \mathrm{Path}^{\ell-1}(\mathcal{G},\mathcal{S})}} \; \min_{\psi_\sigma \in \mathrm{Irr}(\sigma'(\mathbf{k}))} \varphi_\sigma(1)\, \psi_\sigma(1)\, [\mathsf{H}(\mathbf{k}) : \sigma'(\mathbf{k})],$$

and similarly

$$\chi(1) \leq \max_{\substack{\mathcal{S} \in \mathbb{T}^{(\varepsilon)}, \\ \sigma \text{ of type } \mathcal{S}}} \; \max_{\substack{\varphi_\sigma \text{ such that} \\ (\mathcal{G},\mathcal{G}) \notin \Delta(\varphi_\sigma) \in \mathrm{Path}^{\ell-1}(\mathcal{G},\mathcal{S})}} \; \max_{\psi_\sigma \in \mathrm{Irr}(\sigma'(\mathbf{k}))} \varphi_\sigma(1)\, \psi_\sigma(1)\, [\mathsf{H}(\mathbf{k}) : \sigma'(\mathbf{k})];$$

cf. (6.2). To control the degree $\varphi_\sigma(1)$, given by (6.1), we argue as follow. From (5.8) and Remark 5.17 we see that the Dirichlet polynomial

$$\partial\xi^\sigma_{\ell-1}(s) := \xi^\sigma_{\ell-1}(s) - \xi^\sigma_{\ell-2}(s)$$

enumerates the irreducible characters of H^1_ℓ of level $\ell - 1$ and shadow σ. We set

$$D_{q,\ell}(s) = q^{(1-2s)(\ell-2)} f^1_{\ell-2}(q^{1-s});$$

cf. (4.2). Proposition 6.1 shows that, for $\sigma \in \mathfrak{Sh}$ of type $\mathcal{S}$, the function $\partial\xi^\sigma_{\ell-1}(s)$ equals

$$\begin{array}{ll}
(q-1)(q^2+\varepsilon q+1)\, q^{(1-2s)(\ell-2)+2} & \text{if } \mathcal{S} = \mathcal{L}, \\
(q^3-\varepsilon)(q+\varepsilon)\, q^{(1-2s)(\ell-2)} & \text{if } \mathcal{S} = \mathcal{J}, \\
\frac{1}{6}(q-1)(q^2+\varepsilon q+1)(q+\varepsilon)q^3 \left[(q-2)q^{(2-3s)(\ell-2)} + 3(q-1)D_{q,\ell}(s)\right] & \text{if } \mathcal{S} = \mathcal{T}_1, \\
\frac{1}{2}(q-1)(q^3-\varepsilon)q^3 \left[q^{(2-3s)(\ell-2)+1} + (q-1)D_{q,\ell}(s)\right] & \text{if } \mathcal{S} = \mathcal{T}_2, \\
\frac{1}{3}(q^2-1)(q+\varepsilon)(q-\varepsilon)^2\, q^{(2-3s)(\ell-2)+3} & \text{if } \mathcal{S} = \mathcal{T}_3, \\
(q-1)(q^3-\varepsilon)(q+\varepsilon)q^2 \left[q^{(2-3s)(\ell-2)} + 2D_{q,\ell}(s)\right] & \text{if } \mathcal{S} = \mathcal{M}, \\
(q^2-1)(q^3-\varepsilon) \left[q^{(2-3s)(\ell-2)+1} + (q-1)D_{q,\ell}(s)\right] & \text{if } \mathcal{S} = \mathcal{N}, \\
(q^2-1)(q^3-\varepsilon)\, D_{q,\ell}(s) & \text{if } \mathcal{S} \in \{\mathcal{K}_0, \mathcal{K}_\infty\}.
\end{array}$$

These functions being Dirichlet polynomials in q^{-s}, we define

$$P^{\sigma}_{\ell-1} = \{m \in \mathbb{N} \mid \text{the coefficient of } q^{-ms} \text{ in } \partial\xi^{\sigma}_{\ell-1}(s) \text{ is non-zero}\}.$$

Clearly,

$$\begin{aligned} P^{\sigma}_{\ell-1} &= \{2\ell-4\} && \text{if } \mathcal{S} \in \{\mathcal{L}, \mathcal{J}\}, \\ P^{\sigma}_{\ell-1} &= \{3\ell-6\} && \text{if } \mathcal{S} = \mathcal{T}_3, \\ P^{\sigma}_{\ell-1} &= \{2\ell-4, 2\ell-3, \ldots, 3\ell-7, 3\ell-6\} && \text{if } \mathcal{S} \in \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{M}, \mathcal{N}\}, \\ P^{\sigma}_{\ell-1} &= \{2\ell-4, 2\ell-3, \ldots, 3\ell-7\} && \text{if } \mathcal{S} \in \{\mathcal{K}_0, \mathcal{K}_\infty\}, \end{aligned}$$

Setting

$$C_\sigma(q) := \frac{[\mathsf{H}(\mathbf{k}) : \sigma'(\mathbf{k})]}{q^{\dim \mathsf{G} - \dim(\sigma)}}$$

we see, using (6.2), that

$$C_\sigma(q) \cdot q^{2\ell-4+\dim \mathsf{G}-\dim(\sigma)} \leq \chi(1) \leq C_\sigma(q) \cdot q^{3\ell-6+\dim \mathsf{G}-\dim(\sigma)} \max_{\psi_\sigma \in \mathrm{Irr}(\sigma'(\mathbf{k}))} \psi_\sigma(1), \tag{6.3}$$

Table 6.2 allows us to write $C_\sigma(q)$ explicitly in terms of ε and q; in particular, we see that $C_\sigma(q) = 1 + \mathrm{o}(q^{-1})$. To obtain the bounds for $\chi(1)$ given in the theorem, it thus suffices to bound the remaining factors in (6.3). The minimum on the left-hand side is $q^{2\ell}$, attained for shadows σ of type $\mathcal{L}$ and $\mathcal{J}$. Inspecting the explicit formulae for the shadow zeta functions $\zeta_{\sigma'(\mathbf{k})}(s)$ given in Proposition 6.9, one deduces easily that the maximum on the right-hand side is $q^{3\ell}$ and occurs, for example, for σ of type $\mathcal{T}_1$ for $\varepsilon = 1$ and $\mathcal{T}_3$ for $\varepsilon = -1$, both necessarily with $\psi_\sigma(1) = 1$ as the respective groups $\sigma'(\mathbf{k})$ are abelian. □

7. Adèlic zeta functions for type A_2 and their analytic properties

Theorem A and Corollary B are established in Section 7.1. Theorem G is proved in Section 7.2.

7.1. **Zeta functions of adèlic and arithmetic groups.** Let $\mathbf{H}(\widehat{\mathcal{O}_S})$ be an adèlic profinite group as in Theorem A. This means that $\mathbf{H}$ is a connected, simply-connected absolutely almost simple algebraic group of type A_2 defined over a number field k, with S-integers $\mathcal{O}_S$ for a finite set $S \subset \mathcal{V}_k$ of places including all the archimedean ones. Here $\mathcal{V}_k$ denotes the collection of all places of k, and we write $\mathcal{V}_k^\infty$ for the set of archimedean places. The starting point for our study of the analytic properties of the zeta function $\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s)$ is the Euler product

$$\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s) = \prod_{v \in \mathcal{V}_k \smallsetminus S} \zeta_{\mathbf{H}(\mathcal{O}_v)}(s), \tag{7.1}$$

arising from the isomorphism $\mathbf{H}(\widehat{\mathcal{O}_S}) \simeq \prod_{v \in \mathcal{V}_k \smallsetminus S} \mathbf{H}(\mathcal{O}_v)$.

The classification of absolutely almost simple algebraic groups over number fields implies that $\mathbf{H}$ is either an inner form, i.e. of type ${}^1\mathsf{A}_2$, arising from a matrix algebra over a central division algebra over k, or an outer form, i.e. of type ${}^2\mathsf{A}_2$, arising from a matrix algebra over a central division algebra over a quadratic extension K of k, equipped with an involution and with reference to a suitable hermitian form; see [48, Propositions 2.17 and 2.18] and the summary in [5, Appendix A]. The crucial point for us is that there is a finite set $T \subset \mathcal{V}_k$ with $S \subset T$ such that, for all v in

$$\mathcal{V}_0 := \mathcal{V}_k \smallsetminus T,$$

the completion $\mathbf{H}(\mathcal{O}_v)$ featuring in (7.1), is of the form $\mathsf{SL}_3(\mathcal{O}_v)$ or $\mathsf{SU}_3(\mathcal{O}_v)$ and, in the latter case, v is not dyadic and does not divide the (relative) discriminant $\Delta_{K \mid k}$ of $K \mid k$. Set

$$\mathcal{V}_{\mathsf{SL}} = \{v \in \mathcal{V}_0 \mid \mathbf{H}(\mathcal{O}_v) \simeq \mathsf{SL}_3(\mathcal{O}_v)\} \quad \text{and} \quad \mathcal{V}_{\mathsf{SU}} = \{v \in \mathcal{V}_0 \mid \mathbf{H}(\mathcal{O}_v) \simeq \mathsf{SU}_3(\mathcal{O}_v)\}. \tag{7.2}$$

We know, e.g. from [5, Theorem B], that all of the finitely many 'exceptional' factors of (7.1), indexed by the non-archimedean places in T, converge to a holomorphic function on the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 2/3\}$ without any zeros. Hence the abscissa of convergence of

$$Z(s) := \prod_{v \in \mathcal{V}_0} \zeta_{\mathbf{H}(\mathcal{O}_v)}(s) = \prod_{v \in \mathcal{V}_{\mathsf{SL}}} \zeta_{\mathsf{SL}_3(\mathcal{O}_v)}(s) \cdot \prod_{v \in \mathcal{V}_{\mathsf{SU}}} \zeta_{\mathsf{SU}_3(\mathcal{O}_v)}(s), \tag{7.3}$$

is equal to the abscissa of convergence of $\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s)$, which is known to be 1; cf. [5, Theorem C]. Moreover, it suffices to prove the first statement of Theorem A for $Z(s)$ instead of $\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s)$.

The set $\mathcal{V}_{\mathsf{SU}}$ is finite if and only if $\mathbf{H}$ is an inner form. If $\mathbf{H}$ is an outer form, then $\mathcal{V}_{\mathsf{SU}}$ has positive analytic density; see [5, Lemma A.1]. In this case, the distinction whether $\mathbf{H}(\mathcal{O}_v) \simeq \mathsf{SL}_3(\mathcal{O}_v)$ or $\mathbf{H}(\mathcal{O}_v) \simeq \mathsf{SU}_3(\mathcal{O}_v)$ is, for all $v \in \mathcal{V}_0$, dictated by the decomposition behaviour of the prime ideal $\mathfrak{p}_v$ of $\mathcal{O}$ associated to v in the ring of integers $\mathcal{O}_K$ of K. This behaviour, in turn, is described by the *Artin symbol* of the quadratic extension $K \mid k$. Indeed, the value of the Artin symbol at a place $v \in \mathcal{V}_k \smallsetminus \mathcal{V}_k^\infty$ not dividing the discriminant $\Delta_{K \mid k}$ is given by

$$\varepsilon(v) = \left(\frac{K \mid k}{v}\right) = \begin{cases} 1 & \text{if } \mathfrak{p}_v \text{ is decomposed in } \mathcal{O}_K, \\ -1 & \text{if } \mathfrak{p}_v \text{ is inert in } \mathcal{O}_K; \end{cases}$$

cf., for instance, [46, Chapter VI, § 7]. The Artin symbol thus defines the key parameter (1.4) in a global setting. For $v \in \mathcal{V}_k \smallsetminus \mathcal{V}_k^\infty$, with residue field $\mathbf{k}_v$ of cardinality q_v, we write $\iota(v) := \gcd(q_v - 1, 3) \in \{1, 3\}$ for the number of roots of unity in $\mathbf{k}_v$, as in (1.10).

Equation (1.9) presents each factor $\zeta_{\mathbf{H}(\mathcal{O}_v)}(s)$ of (7.3) as a finite sum of rational functions, indexed by shadow types and each depending on the parameters q_v, $\varepsilon(v)$, and $\iota(v)$. Furthermore, $(1 - q_v^{1-2s})(1 - q_v^{2-3s})$ is a common denominator for these summands. Informally speaking, we will show that clearing this common denominator strictly improves the abscissa of convergence of the Euler product defining $Z(s)$, from 1 to at least 5/6. More precisely, we claim that

$$\begin{aligned} \eta(s) &:= Z(s) \prod_{v \in \mathcal{V}_0} (1 - q_v^{1-2s})(1 - q_v^{2-3s}) \\ &= \prod_{v \in \mathcal{V}_{\mathsf{SL}}} (1 - q_v^{1-2s})(1 - q_v^{2-3s}) \zeta_{\mathsf{SL}_3(\mathcal{O}_v)}(s) \cdot \prod_{v \in \mathcal{V}_{\mathsf{SU}}} (1 - q_v^{1-2s})(1 - q_v^{2-3s}) \zeta_{\mathsf{SU}_3(\mathcal{O}_v)}(s) \end{aligned}$$

converges and does not vanish on the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 5/6\}$. As the Dedekind zeta function $\zeta_k(s) = \prod_{v \in \mathcal{V}_k \smallsetminus \mathcal{V}_k^\infty} (1 - q_v^{-s})^{-1}$ converges on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$, this yields a new proof of the fact that the abscissa of convergence of $\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s)$ is equal to 1. It also shows that $\zeta_{\mathbf{H}(\widehat{\mathcal{O}_S})}(s)$ has meromorphic continuation to at least $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 5/6\}$ and that the continued function has a unique singularity in this domain, namely a double pole at $s = 1$; cf. [46, Chapter VII, Corollary 5.11]. This will establish the first part of Theorem A.

We now prove the claim about the convergence of $\eta(s)$, using the following well-known lemma.

Lemma 7.1. *Let* $\mathcal{W} \subset \mathcal{V}_k \smallsetminus \mathcal{V}_k^\infty$. *Let I be a finite index set and $f_i, g_i \in \mathbb{Q}[t]$, $i \in I$, be polynomials of degrees* $\deg(f_i) \geq 0$ *and* $\deg(g_i) \geq 1$, *respectively. Then the Euler product*

$$\prod_{v \in \mathcal{W}} \left(1 + \sum_{i \in I} f_i(q_v) g_i(q_v)^{-s}\right) \tag{7.4}$$

converges on $\left\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \max_{i \in I} \frac{\deg(f_i)+1}{\deg(g_i)}\right\}$.

Note that we make no assumption on the set $\mathcal{W}$ of places and no statement about the precise value of the abscissa of convergence of the product (7.4). We refer to $\frac{\deg(f_i)+1}{\deg(g_i)}$ as the *degree* *ratio* of the expression $f_i(q_v)g_i(q_v)^{-s}$, for $i \in I$. Fixing $\varepsilon \in \{1, -1\}$ and $\iota \in \{1, 3\}$, we set

$$\mathcal{W} = \mathcal{W}_{\varepsilon,\iota} = \{v \in \mathcal{V}_0 \mid \varepsilon(v) = \varepsilon, \iota(v) = \iota\}.$$

Let $v \in \mathcal{W}$. The factor of $\eta(s)$ indexed by v is a sum of terms of the form

$$[\mathbf{H}(\mathbf{k}_v) : \sigma'(\mathbf{k}_v)]^{-1-s}\, \zeta_{\sigma'(\mathbf{k}_v)}(s)\, \xi^\sigma(s)\, (1 - q_v^{1-2s})(1 - q_v^{2-3s}), \tag{7.5}$$

where σ ranges over the shadow set $\mathfrak{Sh}_{\mathsf{G}(\mathcal{O}_v)}$ for $\mathsf{G} = \mathsf{GL}_3$ or $\mathsf{G} = \mathsf{GU}_3$ according to ε; cf. (1.9). By construction of $\mathcal{W}$, we may write these terms as sums of polynomial expressions in q_v with constant coefficients as in the factors of the Euler product (7.4). More precisely, there exist a finite index set I and non-constant polynomials $f_i, g_i \in \mathbb{Q}[t]$ such that the sum over the expressions in (7.5) is of the form $1 + \sum_{i \in I} f_i(q_v)g_i(q_v)^{-s}$. By Lemma 7.1, it remains to analyse the degree ratios occurring for each shadow σ of type $\mathcal{S}$, say, and to verify that they are all bounded above by $5/6$. In the sequel we occasionally write q instead of q_v.

If $\mathcal{S} = \mathcal{G}$, then (7.5) equals

$$\zeta_{\sigma'(\mathbf{k}_v)}(s)\, (1 - q_v^{1-2s})(1 - q_v^{2-3s}),$$

where $\sigma'(\mathbf{k}_v)$ is the finite group of Lie type $\mathsf{SL}_3(\mathbf{k}_v)$ if $\varepsilon = 1$ or $\mathsf{SU}_3(\mathbf{k}_v)$ if $\varepsilon = -1$. Inspection of the formulae for these zeta functions, given in (1.11), shows that

$$\begin{aligned}\zeta_{\sigma'(\mathbf{k}_v)}(s)\, (1 - q^{1-2s})(1 - q^{2-3s}) &= 1 + \big(q(q^2 + \varepsilon q + 1)^{-s} - q^{1-2s}\big) \\ &+ \Big(\tfrac{1}{6}q^2\, \big((q^2 + \varepsilon q + 1)(q + \varepsilon)\big)^{-s} + \tfrac{1}{2}q^2(q^3 - \varepsilon)^{-s} + \tfrac{1}{3}q^2((q + \varepsilon)(q - \varepsilon)^2)^{-s} - q^{2-3s}\Big) \\ &+ (\text{terms of degree ratios at most } 4/5)\,.\end{aligned} \tag{7.6}$$

For $s \in \mathbb{R}_{>0}$, the binomial series expansion implies that, for a suitable constant $C_1 \in \mathbb{R}_{>0}$ and q sufficiently large,

$$\left|q(q^2 + \varepsilon q + 1)^{-s} - q^{1-2s}\right| \le C_1 q^{-2s}$$

so that the relevant term on the right-hand side of (7.6) may be replaced by a polynomial expression of degree ratio $1/2$ without worsening the abscissa of convergence of the Euler product defining $\eta(s)$. A similar argument shows that, for $s \in \mathbb{R}_{>0}$, there is a constant $C_2 \in \mathbb{R}_{>0}$ such that, for all sufficiently large q,

$$\left|\tfrac{1}{6}q^2\, \big((q^2 + \varepsilon q + 1)(q + \varepsilon)\big)^{-s} + \tfrac{1}{2}q^2(q^3 - \varepsilon)^{-s} + \tfrac{1}{3}q^2\, \big((q + \varepsilon)(q - \varepsilon)^2\big)^{-s} - q^{2-3s}\right| \le C_2 q^{1-3s}, \tag{7.7}$$

so that the relevant term on the right-hand side of (7.6) may be replaced by a polynomial expression of degree ratio $2/3$ without worsening the abscissa of convergence of the Euler product.

If $\mathcal{S} = \mathcal{L}$ then (7.5) takes the form

$$\tfrac{1}{2}(q - 1)(q - \varepsilon)\, \big(2 + 2q^{-s} + (q - 2)(q + 1)^{-s} + q(q - 1)^{-s}\big)\, \big(q^2(q^2 + \varepsilon q + 1)\big)^{-s}\, (1 - q^{2-3s}).$$

The degree ratios occurring are at most $4/5$. The forms taken by the summand (7.5) in the remaining cases $\mathcal{S} \in \{\mathcal{J}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{M}, \mathcal{N}, \mathcal{K}_0, \mathcal{K}_\infty\}$, together with upper bounds for the occurring degree ratios, are listed in Table 7.1. Overall, the degree ratios occurring in these cases are bounded above by $5/6$.

This establishes the claim about the convergence of $\eta(s)$, and hence the first part of Theorem A. To prove the second part we recall the following Tauberian theorem.

Theorem 7.2 ([19, Theorem 4.20])**.** *Let the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ with non-negative real coefficients be convergent for $\mathrm{Re}(s) > \alpha > 0$. Assume that in a neighbourhood of α, one has $f(s) = g(s)(s - \alpha)^{-\beta} + h(s)$, where $g(s), h(s)$ are holomorphic functions, $g(\alpha) \neq 0$ and*

TABLE 7.1. Bounds for various degree ratios, where $q = q_v$

Type	Summand (7.5)	Deg. ratios
$\mathcal{J}$	$\big((q^3-\varepsilon)(q+\varepsilon)\big)^{-s}(1-q^{2-3s})$ $\cdot\big((q-\varepsilon)+(q+\varepsilon)\iota^2((q-\varepsilon)/\iota)^{-s}+(q-1)(q-\varepsilon)q^{-s}\big)$	$\leq 5/8$
$\mathcal{T}_1$	$\frac{1}{6}\left(q^3(q^2+\varepsilon q+1)(q+\varepsilon)\right)^{-s}(q-\varepsilon)^2(q-1)(q-2+2q^{2-2s}-q^{1-2s})$	$\leq 5/6$
$\mathcal{T}_2$	$\frac{1}{2}\left(q^3(q^3-\varepsilon)\right)^{-s}q(q-1)(q^2-1)(1-q^{-2s})$	$\leq 5/6$
$\mathcal{T}_3$	$\frac{1}{3}\left(q^3(q+\varepsilon)(q-\varepsilon)^2\right)^{-s}(q^2+\varepsilon q+1)(q^2-1)(1-q^{1-2s})$	$\leq 5/6$
$\mathcal{M}$	$\left(q^2(q^3-\varepsilon)(q+\varepsilon)\right)^{-s}q(q-\varepsilon)(q-1)(1+q^{1-2s})$	$\leq 2/3$
$\mathcal{N}$	$\iota^2\left(q(q^3-\varepsilon)(q^2-1)\iota\right)^{-s}q^2(1-q^{-2s})$	$\leq 1/2$
$\mathcal{K}_0, \mathcal{K}_\infty$*	$\iota^2\left(q(q^3-\varepsilon)(q^2-1)/\iota\right)^{-s}q^{2-2s}$	$\leq 3/8$

* Only applies if $\varepsilon = 1$.

$\beta > 0$. *Assume also that* $f(s)$ *can be holomorphically continued to the line* $\mathrm{Re}(s) = \alpha$ *except for the pole at* $s = \alpha$. *Then*

$$\frac{g(\alpha)}{\alpha\Gamma(\beta)} = \lim_{N\to\infty}\frac{\sum_{n=1}^N a_n}{N^\alpha(\log N)^{\beta-1}}.$$

Here, Γ denotes the Γ-function. The second claim of Theorem A follows from the first, with $\alpha = 1$, $\beta = 2$, and $c(\mathbf{H}(\widehat{\mathcal{O}_S})) = g(\alpha)/(\alpha\Gamma(\beta)) = g(1)$ for some holomorphic function g as in Theorem 7.2.

Proof of Corollary B. The group $\mathbf{H}(\mathcal{O}_S)$ contains a subgroup Γ of finite index such that $\widehat{\Gamma} \simeq \prod_{v\notin S}\Gamma_v$, where Γ_v is an open subgroup of $\mathbf{H}(\mathcal{O}_v)$ for all places v, with equality for all but finitely many v; if $\mathbf{H}(\mathcal{O}_S)$ has the sCSP we may take $\Gamma = \mathbf{H}(\mathcal{O}_S)$. It follows that $\zeta_\Gamma(s) = \zeta_{\mathbf{H}(\mathbb{C})}(s)^{|k:\mathbb{Q}|}\prod_{v\notin S}\zeta_{\Gamma_v}(s)$; cf. [38, Theorem 3.3]. The corollary is deduced from Theorem A; its proof shows how to deal with the finitely many 'exceptional' non-archimedean factors for which $\Gamma_v \neq \mathbf{H}(\mathcal{O}_v)$, and we only need to accommodate for the additional archimedean factors. In fact, they can be dealt with in a similar way: by [38, Theorem 5.1], each factor $\zeta_{\mathbf{H}(\mathbb{C})}(s)$ converges and does not vanish on the complex half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 2/3\}$. ☐

We add some remarks concerning the constant $c(\mathbf{H}(\mathcal{O}_S))$ in Corollary B in case $\mathbf{H}(\mathcal{O}_S)$ has the sCSP; similar comments apply to $c(\mathbf{H}(\widehat{\mathcal{O}_S}))$ in Theorem A. The invariant $c(\mathbf{H}(\mathcal{O}_S))$ is a rational multiple of the product of the following factors:

- the $|k:\mathbb{Q}|$-th power of the special value $\zeta_{\mathsf{SL}_3(\mathbb{C})}(1)$,
- the square of the residue $\lim_{s\to1}(s-1)\zeta_k(s)$ at $s = 1$ of the Dedekind zeta function $\zeta_k(s)$,
- an Euler product $\prod_{v\in\mathcal{V}'}(1-q_v^{-1})^2\zeta_{\mathbf{H}(\mathcal{O}_v)}(1)$ for a cofinite subset $\mathcal{V}' \subset \mathcal{V}_k \smallsetminus S$.

The residue $\lim_{s\to1}(s-1)\zeta_k(s)$ is, of course, well known and given by the classical class number formula; see [46, Chapter VII, Corollary 5.11].

For $\mathbf{H}(\mathcal{O}_S) = \mathsf{SL}_3(\mathbb{Z})$, for instance, we obtain

$$c(\mathsf{SL}_3(\mathbb{Z})) = \zeta_{\mathsf{SL}_3(\mathbb{C})}(1)\prod_{p \text{ prime}}\left((1-p^{-1})^2\zeta_{\mathsf{SL}_3(\mathbb{Z}_p)}(1)\right). \tag{7.8}$$

It is known that $\zeta_{\mathsf{SL}_3(\mathbb{C})}(s)$ is equal to the 'Mordell-Tornheim double series'

$$\zeta_{\mathrm{MT},2}(s) = \sum_{(m_1,m_2)\in\mathbb{N}^2}(m_1m_2(m_1+m_2))^{-s};$$

see, for instance, [37, p. 359]. In [44, p. 369], Mordell shows that

$$\zeta_{\mathsf{SL}_3(\mathbb{C})}(1) = \zeta_{\mathrm{MT},2}(1) = 2\zeta(3),$$

where $\zeta(s) = \zeta_{\mathbb{Q}}(s)$ denotes the Riemann zeta function. Furthermore, for $k = \mathbb{Q}$, the residue $\lim_{s\to 1}(s-1)\zeta(s) = 1$, and it remains to deal with the third factor listed above. Unfortunately, we are unable to determine $c(\mathsf{SL}_3(\mathbb{Z}))$ completely as we do not know $\zeta_{\mathsf{SL}_3(\mathbb{Z}_3)}(s)$ – or even just the special value $\zeta_{\mathsf{SL}_3(\mathbb{Z}_3)}(1)$ – explicitly; see [4, Theorem 1.4] for a formula for the related zeta function $\zeta_{\mathsf{SL}_3^m(\mathfrak{o})}(s)$ for unramified extensions $\mathfrak{o}$ of $\mathbb{Z}_3$. But we arrive at the following numerical fact regarding the third factors listed above.

Proposition 7.3. *Suppose that $\mathfrak{o}$ is a compact discrete valuation ring of characteristic zero, satisfying the conditions hypotheses of Corollary D, and that $\mathsf{H}(\mathfrak{o})$ is either $\mathsf{SL}_3(\mathfrak{o})$, for $\varepsilon = 1$, or $\mathsf{SU}_3(\mathfrak{o})$, for $\varepsilon = -1$, and put $\iota = \gcd(q-\varepsilon, 3)$. Then*

$$(1-q^{-1})^2\zeta_{\mathsf{H}(\mathfrak{o})}(1) = \frac{W_{\varepsilon,\iota}(q)}{(q^3-\varepsilon)(q^2-1)q^5},$$

where

$$\begin{aligned} W_{\varepsilon,\iota}(q) = q^{10} - (2\varepsilon+1)q^8 + (\iota^3-\varepsilon+2)q^7 + 4\varepsilon q^6 + ((\varepsilon+1)\iota^3 - (2\varepsilon+3))q^5 \\ - (2\iota^3-3)q^4 + (\iota^3+\varepsilon-3)q^3 + (\varepsilon+1)q^2 - \varepsilon(2q-1). \end{aligned}$$

Proof. The claim follows by inspection of the explicit formulae given in Theorem C. □

7.2. **Adèlic similarity class zeta functions.** We recall the setup of Theorem G. Let k be a number field and $\mathbf{G}$ one of the k-algebraic groups GL_3 or $\mathsf{GU}_3(K,f)$, where the unitary group is defined with respect to the standard hermitian form f associated to the non-trivial Galois automorphism of a quadratic extension K of k. We denote by $\mathfrak{g}$ the corresponding Lie algebra scheme gl_3 or $\mathsf{gu}(K,f)$, and we use $\varepsilon = \varepsilon_{\mathbf{G}} \in \{1,-1\}$ to distinguish between the general linear and unitary cases.

Let $S \subset \mathcal{V}_k$ be a finite set of places, including all the archimedean ones, and, if $\varepsilon = -1$, suppose that S includes all dyadic places as well as those places that ramify in the quadratic extension $K \mid k$. Put $\mathcal{V}_0 = \mathcal{V}_k \smallsetminus S$ and consider the Euler product

$$Z_{\mathfrak{g}(\mathcal{O}_S)}(s) = \prod_{v\in\mathcal{V}_0} Z_{\mathfrak{g}(\mathcal{O}_v)}(s), \qquad \text{where } Z_{\mathfrak{g}(\mathfrak{o})}(s) := \lim_{\ell\to\infty} q^{-\ell}\gamma_\ell(s)$$

for a compact discrete valuation ring $\mathfrak{o}$ of residue cardinality q; compare Definition 4.1 and Proposition 4.7. In analogy with (7.2), we set

$$\mathcal{V}_{\mathsf{GL}} = \{v \in \mathcal{V}_0 \mid \mathbf{G}(\mathcal{O}_v) \simeq \mathsf{GL}_3(\mathcal{O}_v)\} \quad \text{and} \quad \mathcal{V}_{\mathsf{GU}} = \{v \in \mathcal{V}_0 \mid \mathbf{G}(\mathcal{O}_v) \simeq \mathsf{GU}_3(\mathcal{O}_v)\}$$

so that $\mathcal{V}_0 = \mathcal{V}_{\mathsf{GL}} \cup \mathcal{V}_{\mathsf{GU}}$ and hence

$$Z_{\mathfrak{g}(\mathcal{O}_S)}(s) = \prod_{v\in\mathcal{V}_{\mathsf{GL}}} Z_{\mathsf{gl}_3(\mathcal{O}_v)}(s) \prod_{v\in\mathcal{V}_{\mathsf{GU}}} Z_{\mathsf{gu}_3(\mathcal{O}_v)}(s).$$

Similar to the proof of Theorem A, it suffices to show that the function

$$\begin{aligned} \eta_{\mathrm{sim}}(s) &:= Z_{\mathfrak{g}(\mathfrak{o})}(s) \prod_{v\in\mathcal{V}_0}(1-q_v^{1-4s})(1-q_v^{2-6s}) \\ &= \prod_{v\in\mathcal{V}_{\mathsf{GL}}}(1-q_v^{1-4s})(1-q_v^{2-6s})Z_{\mathsf{gl}(\mathcal{O}_v)}(s) \cdot \prod_{v\in\mathcal{V}_{\mathsf{GU}}}(1-q_v^{1-4s})(1-q_v^{2-6s})\zeta_{\mathsf{gu}_3(\mathcal{O}_v)}(s) \end{aligned} \tag{7.9}$$

converges and does not vanish on the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 2/5\}$. This will establish the first two parts of Theorem G; the third part follows via the Tauberian Theorem 7.2.

For each $v \in \mathcal{V}_0$, we have

$$(1 - q_v^{1-4s})(1 - q_v^{2-6s}) Z_{\mathfrak{g}(\mathcal{O}_v)}(s) = \sum_{\mathcal{S} \in \mathbb{T}^{(\varepsilon)}} (1 - q_v^{1-4s})(1 - q_v^{2-6s}) \Gamma^{\mathcal{S}}_{\varepsilon(v), q_v}(s), \tag{7.10}$$

where the functions $\Gamma^{\mathcal{S}}_{\varepsilon(v),q_v}(s)$ are given in Corollary 4.8. Fix $v \in \mathcal{V}_0$. We analyse the degree ratios occurring in each summand of (7.10). As in Section 7.1 we write q for q_v when we do not want to stress the dependence on the place.

If $\mathcal{S} = \mathcal{G}$, then the relevant summand in (7.10) is just

$$(1 - q_v^{1-4s})(1 - q_v^{2-6s}) = 1 - q_v^{1-4s} - q_v^{2-6s} + q_v^{3-10s}. \tag{7.11}$$

The term q_v^{3-10s} has degree ratio $2/5$. We will show that this is the maximal degree ratio occurring. In particular, the terms $-q_v^{1-4s}$ and $-q_v^{2-6s}$ "cancel" with terms occurring in types $\mathcal{L}$ and $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, respectively, in a way we shall explain.

Indeed, if $\mathcal{S} = \mathcal{L}$, then the relevant summand in (7.10) is

$$\begin{aligned}(q-1)\left((q^2+\varepsilon q+1)q^2\right)^{-s}(1-q^{2-6s})& \\ = q\left((q^2+\varepsilon q+1)q^2\right)^{-s} &+ (\text{terms of degree ratios at most } 3/10).\end{aligned}$$

By arguments akin to those used in the proof of Theorem A, for $s \in \mathbb{R}_{>0}$, there exists a constant $C_1 \in \mathbb{R}_{>0}$ such that for sufficiently large q,

$$\left| q\left((q^2+\varepsilon q+1)q^2\right)^{-s} - q^{1-4s}\right| \leq C_1 q^{-4s}.$$

This shows that the term $-q^{1-4s}$, occurring for $\mathcal{S} = \mathcal{G}$, and the term $q\left((q^2+\varepsilon q+1)q^2\right)^{-s}$, occurring for $\mathcal{S} = \mathcal{L}$, may be replaced by a polynomial expression of degree ratio $1/4$, without worsening the abscissa of convergence of the Euler product (7.9).

If $\mathcal{S} \in \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$, then the summands in (7.10) indexed by the relevant shadows are

$$\begin{array}{ll}(\mathcal{T}_1) & \frac{1}{6}(q-1)\left((q+\varepsilon)(q^2+\varepsilon q+1)q^3\right)^{-s}(q+2q^{2-4s}-2-q^{1-4s}) \\ (\mathcal{T}_2) & \frac{1}{2}(q-1)\left((q^3-\varepsilon)q^3\right)^{-s}(q-q^{1-4s}) \\ (\mathcal{T}_3) & \frac{1}{3}(q^2-1)\left((q+\varepsilon)(q-\varepsilon)^2q^3\right)^{-s}(1-q^{1-4s}).\end{array}$$

Modulo terms of degree ratios at most $3/10$, these read

$$\begin{array}{ll}(\mathcal{T}_1) & \frac{1}{6}q^2\left((q+\varepsilon)(q^2+\varepsilon q+1)q^3\right)^{-s} \\ (\mathcal{T}_2) & \frac{1}{2}q^2\left((q^3-\varepsilon)q^3\right)^{-s} \\ (\mathcal{T}_3) & \frac{1}{3}q^2\left((q+\varepsilon)(q-\varepsilon)^2q^3\right)^{-s}.\end{array}$$

Similar to the proof of Theorem A, we deduce that, for $s \in \mathbb{R}_{>0}$, there exists a constant $C_2 \in \mathbb{R}_{>0}$ such that for sufficiently large q,

$$\begin{aligned}\left|\tfrac{1}{6}q^2\left((q+\varepsilon)(q^2+\varepsilon q+1)q^3\right)^{-s} + \tfrac{1}{2}q^2\left((q^3-\varepsilon)q^3\right)^{-s} + \tfrac{1}{3}q^2\left((q+\varepsilon)(q-\varepsilon)^2q^3\right)^{-s} - q^{2-6s}\right|& \\ \leq C_2 q^{1-6s}&.\end{aligned}$$

This "cancels" the term $-q^{2-6s}$ from (7.11). The forms taken by the relevant summand in (7.10) in the remaining cases $\mathcal{S} \in \{\mathcal{J}, \mathcal{M}, \mathcal{N}, \mathcal{K}_0, \mathcal{K}_\infty\}$, together with an upper bound for the occurring degree ratios, is given in Table 7.2. This concludes the proof of Theorem G.

TABLE 7.2. Bounds for various degree ratios, where $q = q_v$

Type	Summand in (7.10)	Degree ratios
$\mathcal{J}$	$\left((q-\varepsilon)^3(q+\varepsilon)\right)^{-s}(1-q^{2-6s})$	$\leq 3/10$
$\mathcal{M}$	$(q-1)\left((q-\varepsilon)^3(q+\varepsilon)q^2\right)^{-s}(1+q^{1-4s})$	$\leq 1/3$
$\mathcal{N}$	$\left((q^2-1)(q^3-\varepsilon)q\right)^{-s}(1-q^{-4s})$	$\leq 1/6$
$\mathcal{K}_0, \mathcal{K}_\infty$*	$\left((q^2-1)(q^3-\varepsilon)q^5\right)^{-s}$	$\leq 1/10$

* Only applies if $\varepsilon = 1$.

Appendix and References

APPENDIX A. A MODEL VERSION: GROUPS OF TYPE A_1

The main ideas of this paper may be applied to groups of type A_1, such as groups of the form $\mathsf{GL}_2(\mathfrak{o})$ or $\mathsf{GU}_2(\mathfrak{o})$, where $\mathfrak{o}$ is a compact discrete valuation ring, and various subquotients of these groups. We record here – mainly without (detailed) proofs – results on similarity classes and associated zeta functions, as well as representation zeta functions of such groups. This leads, on the one hand, to new, unified computations for the known zeta functions of groups of the form $\mathsf{SL}_2(\mathfrak{o})$ and $\mathsf{GL}_2(\mathfrak{o}_\ell)$; cf. [30, 47], respectively. It also allows us to compute new zeta functions, such as the ones of groups of the form $\mathsf{GU}_2(\mathfrak{o}_\ell)$. Throughout we assume that $\mathfrak{o}$ is a compact discrete valuation ring with residue field $\mathbf{k}$ of cardinality q and residue characteristic p. Let G be one of the $\mathfrak{o}$-group schemes GL_2 and GU_2 and

$$\varepsilon = \varepsilon_{\mathsf{G}} = \begin{cases} +1 & \text{if } \mathsf{G} = \mathsf{GL}_2, \\ -1 & \text{if } \mathsf{G} = \mathsf{GU}_2, \end{cases} \tag{A.1}$$

analogous to (1.4). The $\mathfrak{o}$-group scheme GU_2 is defined with respect to the unramified quadratic extension of $\mathfrak{o}$; see the discussion at the beginning of Section 3 for details. We exclude $p = 2$ from our considerations in the unitary case. We write $\mathsf{g} \in \{\mathsf{gl}_2, \mathsf{gu}_2\}$ for the $\mathfrak{o}$-Lie lattice scheme associated to G and $\mathfrak{Sh}$ for the respective shadow set $\mathfrak{Sh}_{\mathsf{GL}_2(\mathfrak{o})}$ or $\mathfrak{Sh}_{\mathsf{GU}_2(\mathfrak{o})}$.

A.1. **Similarity classes and their shadows.** As in type A_2, similarity classes in $\mathsf{g}(\mathfrak{o}_\ell)$ are controlled by shadows and branching rules. The – rather simple – classification of similarity classes in $\mathrm{Ad}(\mathsf{GL}_2(\mathfrak{o}))\backslash\mathsf{gl}_2(\mathfrak{o}_\ell)$ is described in [7, Section 2.1]. The unitary case is analogous. It turns out that – as in the A_2-case – similarity classes and their lifting behaviour are governed by branching rules and shadow graphs. The following result is analogous to Theorems 2.8 and 3.12, formulated uniformly for both values of the parameter ε.

Theorem A.1 (Classification of shadows and branching rules)**.**

(1) *The shadow set* $\mathfrak{Sh}$ *consists of four elements, classified by the types*

$$\mathcal{G}',\ \mathcal{T}_1{}',\ \mathcal{T}_2{}',\ \mathcal{N}'$$

described in Table A.2.

(2) *For all* $\sigma, \tau \in \mathfrak{Sh}$ *there exists a polynomial* $a_{\sigma,\tau} \in \mathbb{Z}[\frac{1}{2}][t]$ *such that the following holds: for every* $\ell \in \mathbb{N}$ *and every* $\mathcal{C} \in \mathcal{Q}^{\mathsf{g}}_{\mathfrak{o},\ell}$ *with* $\mathrm{sh}_{\mathsf{GL}}(\mathcal{C}) = \sigma$ *the number of classes* $\widetilde{\mathcal{C}} \in \mathcal{Q}^{\mathsf{g}}_{\mathfrak{o},\ell+1}$ *with* $\mathrm{sh}_{\mathsf{GL}}(\widetilde{\mathcal{C}}) = \tau$ *lying above* $\mathcal{C}$ *is equal to* $a_{\sigma,\tau}(q)$.

Set $\mathbb{T}_{\mathsf{A}_1} = \{\mathcal{G}', \mathcal{T}_1{}', \mathcal{T}_2{}', \mathcal{N}'\}$. As in type A_2, it is remarkable that there are $\mathbf{k}$-forms of algebraic groups $\mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}$, for $\mathcal{S} \in \mathbb{T}_{\mathsf{A}_1}$, whose $\mathbf{k}$-rational points $\mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k})$ represent the shadow sets $\mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$. Similarly to the A_2-situation, given $\sigma \in \mathfrak{Sh}_{\mathsf{G}(\mathfrak{o})}$ we set $\sigma(\mathbf{k}) = \mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k})$ and $\sigma'(\mathbf{k}) = \sigma(\mathbf{k}) \cap \mathsf{SL}_2(\mathbf{k})$.

It is noteworthy that the quantities $b^{(\varepsilon)}_{\sigma,\tau}$ defined in Definitions 2.6 and 3.11 are in fact polynomials in q. Together with the polynomials $a_{\sigma,\tau}$ they determine recursively the numbers and sizes of similarity classes in $\mathsf{g}(\mathfrak{o}_\ell)$ for all $\ell \in \mathbb{N}$. These *branching rules* in type A_1 are collected in Table A.1. In analogy with the A_2-situation (cf. Definition 4.4) one may associate a *shadow graph* with each of the scheme pairs (g, G); cf. Figure A.1. In contrast to the A_2-situation, they coincide for $\varepsilon = 1$ and $\varepsilon = -1$.

Proof of Theorem A.1 (sketch). Instead of giving a proof from scratch, we indicate how the shadows and the polynomials $a_{\sigma,\tau}$ – and, in fact, $b^{(\varepsilon)}_{\sigma,\tau}$ – can be extracted from the A_2-case. Indeed, shadow type $\mathcal{L}$ corresponds to the groups $\mathsf{GL}_2 \times \mathsf{GL}_1$ and $\mathsf{GU}_2 \times \mathsf{GU}_1$. It follows that the shadow graph for groups of type A_1 is the subgraph of the A_2-shadow graph in Figure 4.1 consisting of vertices $\mathcal{L}, \mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{N}$ together with edges 9, 10, 11, 12, and with a loop around each vertex. The transition quantities are given by dividing the data $a_{\sigma,\tau}(q)$ of Table 2.2 by q to cancel the redundant GL_1 or GU_1 factor, and by dividing the data $b^{(\varepsilon)}_{\sigma,\tau}(q)$ in that table by q^4 to get the correct dimension. In the theorem's statement we used the labels $\mathcal{G}'$, $\mathcal{T}_1{}'$, $\mathcal{T}_2{}'$, and $\mathcal{N}'$ for the respective A_1-analogues of $\mathcal{L}$, $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{N}$. □

TABLE A.1. Branching rules for $\mathcal{Q}^{\mathfrak{gl}_2}_{\mathfrak{o}}$ ($\varepsilon = 1$) and for $\mathcal{Q}^{\mathfrak{gu}_2}_{\mathfrak{o}}$ ($\varepsilon = -1$)

#	Type of σ	Type of τ	$a_{\sigma,\tau}(q)$	$b^{(\varepsilon)}_{\sigma,\tau}(q)$
1	$\mathcal{G}'$	$\mathcal{G}'$	q	1
2	$\mathcal{G}'$	$\mathcal{T}_1{}'$	$\frac{1}{2}q(q-1)$	$(q+\varepsilon)q$
3	$\mathcal{G}'$	$\mathcal{T}_2{}'$	$\frac{1}{2}q(q-1)$	$(q-\varepsilon)q$
4	$\mathcal{G}'$	$\mathcal{N}'$	q	q^2-1
5	other	same as σ	q^2	q^2

FIGURE A.1. The shadow graph Γ for $(\mathfrak{gl}_2, \mathsf{GL}_2)$ and $(\mathfrak{gu}_2, \mathsf{GU}_2)$

$$\begin{tikzcd} \mathcal{T}_1{}' \arrow[loop left, "5"] & \mathcal{G}' \arrow[l, "2"'] \arrow[r, "3"] \arrow[d, "4"] \arrow[loop above, "1"] & \mathcal{T}_2{}' \arrow[loop right, "5"] \\ & \mathcal{N}' \arrow[loop below, "5"] & \end{tikzcd}$$

A.2. **Similarity class zeta functions.** As in the A_2-case, the classification of shadows and the associated branching rules allows us to compute various similarity class and representation zeta functions. Recall the definitions of the similarity class zeta functions $\gamma^\sigma_\ell(s)$ in (4.1) and the finite geometric progressions $A_{q,\ell}(s)$ in (4.3).

Proposition A.2. *For $\sigma \in \mathfrak{Sh}$ of type $\mathcal{S} \in \mathbb{T}_{\mathsf{A}_1}$ and $\ell \in \mathbb{N}_0$,*

$$\gamma^\sigma_\ell(s) = q^\ell \Gamma^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q,\ell}(s),$$

where the function $\Gamma^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q,\ell}(s)$ *is defined as*

$$\begin{array}{ll} 1 & \text{if } \mathcal{S} = \mathcal{G}', \\ \frac{1}{2}(q-1)\,(q(q+\varepsilon))^{-s}\,A_{q,\ell}(s/2) & \text{if } \mathcal{S} = \mathcal{T}_1{}', \\ \frac{1}{2}(q-1)\,(q(q-\varepsilon))^{-s}\,A_{q,\ell}(s/2) & \text{if } \mathcal{S} = \mathcal{T}_2{}', \\ (q^2-1)^{-s}A_{q,\ell}(s/2) & \text{if } \mathcal{S} = \mathcal{N}'. \end{array}$$

Proof. Analogous to Proposition 4.7. □

Recall further Definition 5.14 of the functions $\xi^\sigma_\ell(s)$ for $\sigma \in \mathfrak{Sh}$ and their limits ξ^σ as $\ell \to \infty$.

Proposition A.3. *For* $\sigma \in \mathfrak{Sh}$ *of type* $\mathcal{S} \in \mathbb{T}_{\mathsf{A}_1}$ *and* $\ell \in \mathbb{N}_0$,

$$\xi^\sigma_\ell(s) = \Xi^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q,\ell}(s),$$

where the function $\Xi^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q,\ell}(s)$ *is defined as*

$$\begin{array}{ll} 1 & \text{if } \mathcal{S} = \mathcal{G}', \\ \frac{1}{2}q(q-1)(q+\varepsilon)A_{q,\ell}(s/4) & \text{if } \mathcal{S} = \mathcal{T}_1{}', \\ \frac{1}{2}q(q-1)(q-\varepsilon)A_{q,\ell}(s/4) & \text{if } \mathcal{S} = \mathcal{T}_2{}', \\ (q^2-1)A_{q,\ell}(s/4) & \text{if } \mathcal{S} = \mathcal{N}'. \end{array}$$

Proof. Straightforward from the data collected in Tables A.2 and A.3. □

Corollary A.4. *For* $\sigma \in \mathfrak{Sh}$ *of type* $\mathcal{S} \in \mathbb{T}_{\mathsf{A}_1}$,

$$\xi^\sigma(s) = \Xi^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q}(s),$$

where the function $\Xi^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q}(s)$ *is defined as*

$$\begin{array}{ll} 1 & \text{if } \mathcal{S} = \mathcal{G}', \\ \frac{1}{2}q(q-1)(q+\varepsilon)(1-q^{1-s})^{-1} & \text{if } \mathcal{S} = \mathcal{T}_1{}', \\ \frac{1}{2}q(q-1)(q-\varepsilon)(1-q^{1-s})^{-1} & \text{if } \mathcal{S} = \mathcal{T}_2{}', \\ (q^2-1)(1-q^{1-s})^{-1} & \text{if } \mathcal{S} = \mathcal{N}'. \end{array}$$

TABLE A.2. Shadows in $\mathsf{GL}_2(\mathbf{k})$ for $\varepsilon = 1$, and $\mathsf{GU}_2(\mathbf{k})$, for $\varepsilon = -1$

Type	$\sigma(\mathbf{k}) \subset \mathsf{GL}_2(\mathbf{k})$	$\sigma(\mathbf{k}) \subset \mathsf{GU}_2(\mathbf{k})$	Order $\lvert\sigma(\mathbf{k})\rvert$
$\mathcal{G}'$	$\mathsf{GL}_2(\mathbf{k})$	$\mathsf{GU}_2(\mathbf{k})$	$q(q-\varepsilon)(q^2-1)$
$\mathcal{T}_1{}'$	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{GL}_1(\mathbf{k})$	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{GU}_1(\mathbf{k})$	$(q-\varepsilon)^2$
$\mathcal{T}_2{}'$	$\mathsf{GL}_1(\mathbf{k}_2)$	$\mathsf{GL}_1(\mathbf{k}_2)$	q^2-1
$\mathcal{N}'$	$\mathsf{GL}_1(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k})$	$\mathsf{GU}_1(\mathbf{k}) \times \mathsf{G}_a(\mathbf{k})$	$q(q-\varepsilon)$

For $\ell \in \mathbb{N}_0$, let

$$s_\ell(\mathfrak{g}(\mathfrak{o})) := \gamma_\ell(0) = \lvert \mathrm{Ad}\,\mathsf{G}(\mathfrak{o})\backslash\mathfrak{g}(\mathfrak{o}_\ell)\rvert$$

denote the number of similarity classes in $\mathfrak{g}(\mathfrak{o}_\ell)$. In analogy with Theorem E we obtain the following from our formulae for the functions $\gamma_\ell(s)$.

Theorem A.5. *Let* $\mathfrak{o}$, G, $\mathfrak{g}$ *and* $\varepsilon = \varepsilon_{\mathsf{G}}$ *be as above; if* $\varepsilon = -1$ *suppose that* $\mathfrak{o}$ *has odd residue characteristic. Then*

$$\sum_{\ell=0}^{\infty} s_\ell(\mathfrak{g}(\mathfrak{o}))t^\ell = \frac{1}{(1-qt)(1-q^2t)}. \tag{A.2}$$

TABLE A.3. Shadows in $\mathsf{SL}_2(\mathbf{k})$ for $\varepsilon = 1$, and $\mathsf{SU}_2(\mathbf{k})$, for $\varepsilon = -1$

Type	$\sigma'(\mathbf{k}) = \sigma(\mathbf{k}) \cap \mathsf{SL}_2(\mathbf{k})$	$\sigma'(\mathbf{k}) = \sigma(\mathbf{k}) \cap \mathsf{SU}_2(\mathbf{k})$	Order $\lvert\sigma(\mathbf{k})\rvert$
$\mathcal{G}'$	$\mathsf{SL}_2(\mathbf{k})$	$\mathsf{SU}_2(\mathbf{k})$	$q(q^2-1)$
$\mathcal{T}_1{}'$	$\mathsf{GL}_1(\mathbf{k})$	$\mathsf{GU}_1(\mathbf{k})$	$q-\varepsilon$
$\mathcal{T}_2{}'$	$\{a \in \mathbf{k}_2^\times \mid a^\circ a = 1\}$	$\{a \in \mathbf{k}_2^\times \mid a^\circ = a\}$	$q+\varepsilon$
$\mathcal{N}'$	$\mathbb{Z}/2\mathbb{Z} \times \mathsf{G}_a(\mathbf{k})$	$\mathbb{Z}/2\mathbb{Z} \times \mathsf{G}_a(\mathbf{k})$	$2q$

Remark A.6. For $\varepsilon = 1$ equation (A.2) was already computed in [7, Section 2]. It is remarkable that the same formula covers the case $\varepsilon = -1$.

As in the A_2-case, these results may be put in an adèlic respectively global context as follows. Let k be a number field with ring of integers $\mathcal{O}$. Let $\mathbf{G}$ be one of the k-algebraic groups GL_2 or $\mathsf{GU}_2(K,f)$, where the unitary group $\mathsf{GU}_2(K,f)$ is defined with respect to the standard hermitian form f associated to the non-trivial Galois automorphism of a quadratic extension K of k. Accordingly, let $\mathfrak{g}$ be one of the Lie algebra schemes $\mathfrak{gl}_2$ or $\mathfrak{gu}_2(K,f)$. Put $\varepsilon_{\mathbf{G}} = 1$ if $\mathbf{G} = \mathsf{GL}_2$, and $\varepsilon_{\mathbf{G}} = -1$ if $\mathbf{G}$ is unitary. For the ring of S-integers $\mathcal{O}_S$, where S is a finite set of places of k including all the archimedean ones, we consider the Dirichlet series $\zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_S)}(s)$ defined in (1.13).

Corollary A.7. *Let $\mathcal{O}_S \subset k$ and $\mathbf{G}$, $\mathfrak{g}$, $\varepsilon_{\mathbf{G}}$ be as above; if $\varepsilon_{\mathbf{G}} = -1$ suppose that S includes all dyadic places of k as well as those places which ramify in the quadratic extension K of k defining $\mathbf{G} = \mathsf{GU}_2(K,f)$. Then*

$$\zeta^{\mathrm{sc}}_{\mathfrak{g}(\mathcal{O}_S)}(s) = \zeta_{k,S}(s-1)\zeta_{k,S}(s-2).$$

In particular, there exists an invariant $\delta_1(\mathcal{O}_S) \in \mathbb{R}_{>0}$ such that

$$\delta_1(\mathcal{O}_S) = \lim_{N\to\infty} \frac{\sum_{n=1}^{\mathbb{N}} s_n(\mathfrak{g}(\mathcal{O}_S))}{N^3}.$$

We define $Z_{\mathfrak{g}(\mathcal{O}_S)}(s) = \sum_{n=1}^{\infty} \mathrm{sim}_n(\mathfrak{g}(\mathcal{O}_S))n^{-s} := \lim_{I \triangleleft \mathcal{O}_S} Z_{\mathfrak{g}(\mathcal{O}_S/I)}(s)$ as in (1.15). The formulae for the functions $\gamma^\sigma_\ell(s)$ provided in Proposition A.2 yield the following result.

Theorem A.8. *Let $\mathcal{O}_S \subset k$ and $\mathbf{G}$, $\mathfrak{g}$, $\varepsilon_{\mathbf{G}}$ be as above; if $\varepsilon_{\mathbf{G}} = -1$ suppose that S includes all dyadic places of k as well as those places which ramify in the quadratic extension K of k defining $\mathbf{G} = \mathsf{GU}_2(K,f)$. Then*

$$Z_{\mathfrak{g}(\mathcal{O}_S)}(s) = \prod_{v \notin S} \left(1 + \frac{\frac{1}{2}(q_v-1)((q_v(q_v+\varepsilon_v))^{-s} + (q_v(q_v-\varepsilon_v))^{-s}) + (q_v^2-1)^{-s}}{1-q_v^{1-2s}}\right),$$

where q_v is the residue cardinality at v and $\varepsilon_v = -1$ if $\varepsilon = -1$ and $\mathfrak{g}(\mathcal{O}_v) \simeq \mathfrak{gu}_2(\mathcal{O}_v)$, and $\varepsilon_v = 1$ otherwise. Consequently, the following hold.

1. *The abscissa of convergence of $Z_{\mathfrak{g}(\mathcal{O}_S)}(s)$ is equal to 1.*
2. *The function $Z_{\mathfrak{g}(\mathcal{O}_S)}(s)$ has meromorphic continuation to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1/2\}$. The only pole of $Z_{\mathfrak{g}(\mathcal{O}_S)}(s)$ in this domain is a single pole at $s = 1$.*
3. *There exists an invariant $\delta_2(\mathcal{O}_S) \in \mathbb{R}_{>0}$ such that*

$$\delta_2(\mathcal{O}_S) = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} \mathrm{sim}_n(\mathfrak{g}(\mathcal{O}_S))}{N}.$$

A.3. **Zeta functions of the shadows.** Recall that, for a shadow $\sigma \in \mathfrak{Sh}$, we set $\sigma'(\mathbf{k}) := \sigma(\mathbf{k}) \cap \mathsf{SL}_2(\mathbf{k})$. See Table A.3 for details on the groups occurring.

Proposition A.9. *Let $\sigma \in \mathfrak{Sh}$ be of type $\mathcal{S} \in \mathbb{T}_{\mathsf{A}_1}$. Then*

$$\zeta_{\sigma(\mathbf{k})}(s) = (q-\varepsilon) Z^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,1,q}(s) \text{ and}$$
$$\zeta_{\sigma'(\mathbf{k})}(s) = Z^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,2,q}(s),$$

where, for $i \in \{1,2\}$,

$$Z^{\mathcal{G}'}_{\mathsf{A}_1,\varepsilon,i,q}(s) = 1+q^{-s}+\tfrac{1}{2}(q-3)(q+1)^{-s}+\tfrac{1}{2}(q-1)(q-1)^{-s}+i^2/2\,((q+1)/i)^{-s}+i^2/2((q-1)/i)^{-s}.$$

In the remaining cases, the function $Z^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,i,q}(s)$ is defined as

$$\begin{array}{ll} q-\varepsilon & \text{if } \mathcal{S} = \mathcal{T}_1{}', \\ q+\varepsilon & \text{if } \mathcal{S} = \mathcal{T}_2{}', \\ iq & \text{if } \mathcal{S} = \mathcal{N}'. \end{array}$$

A.4. **Zeta functions of groups of type A_1.** Let H denote the $\mathfrak{o}$-group scheme SL_2 or SU_2, according to $\varepsilon = \varepsilon_{\mathsf{H}} \in \{-1,1\}$ as above. Recall that the ramification index of a compact discrete valuation ring $\mathfrak{o}$ of characteristic 0 is denoted by $e = e(\mathfrak{o}, \mathbb{Z}_p)$.

Theorem A.10. *Let $\mathfrak{o}$ be a compact discrete valuation ring of residue characteristic $p = \mathrm{char}(\mathbf{k})$. Let G, H be either GL_2, SL_2 or GU_2, SU_2 as above and $\ell \in \mathbb{N}$. Assume that $p \geq \min\{2\ell, 2e+2\}$ if $\mathrm{char}(\mathfrak{o}) = 0$, and $p \geq 2\ell$ if $\mathrm{char}(\mathfrak{o}) = p$. Then the following hold:*

$$\zeta_{\mathsf{G}(\mathfrak{o}_\ell)}(s) = q^{\ell-1} \sum_{\mathcal{S} \in \mathbb{T}_{\mathsf{A}_1}} [\mathsf{G}(\mathbf{k}) : \mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k})]^{-1-s} \zeta_{\mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k})}(s) \Xi^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q,\ell-1}(s), \tag{A.3}$$

$$\zeta_{\mathsf{H}(\mathfrak{o}_\ell)}(s) = \sum_{\mathcal{S} \in \mathbb{T}_{\mathsf{A}_1}} [\mathsf{H}(\mathbf{k}) : (\mathsf{H}(\mathbf{k}) \cap \mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k}))]^{-1-s} \zeta_{\mathsf{H}(\mathbf{k}) \cap \mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k})}(s) \Xi^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q,\ell-1}(s). \tag{A.4}$$

Moreover, if $\mathrm{char}(\mathfrak{o}) = 0$ and $p > 2e+2$, then

$$\zeta_{\mathsf{H}(\mathfrak{o})}(s) = \sum_{\mathcal{S} \in \mathbb{T}_{\mathsf{A}_1}} [\mathsf{H}(\mathbf{k}) : (\mathsf{H}(\mathbf{k}) \cap \mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k}))]^{-1-s} \zeta_{\mathsf{H}(\mathbf{k}) \cap \mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k})}(s) \Xi^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q}(s). \tag{A.5}$$

Remark A.11. Equation (A.5) confirms – in the cases where it is applicable – Jaikin-Zapirain's formula for the representation zeta function $\zeta_{\mathsf{SL}_2(\mathfrak{o})}(s)$. Recall the notational convention $\sigma(\mathbf{k}) = \mathsf{I}^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon}(\mathbf{k})$ and that the relevant information about these groups is recorded in Table A.2.

Remark A.12. It is noteworthy that the special values of the zeta functions $\zeta_{\mathsf{G}(\mathfrak{o}_\ell)}(s)$ – at least as far as they are given by (A.3) – at $s=-1$, i.e. the sums of character degrees of the groups $\mathsf{G}(\mathfrak{o}_\ell)$, coincide with the numbers of symmetric matrices in $\mathsf{G}(\mathfrak{o}_\ell)$, viz.

$$\zeta_{\mathsf{G}(\mathfrak{o}_\ell)}(-1) = (1-\varepsilon q^{-1}) q^{3\ell}.$$

This is in contrast to the situation in type A_2; cf. Remark 1.3.

Theorem A.13. *Let $\mathfrak{o}$ and G, H, $\varepsilon = \varepsilon_{\mathsf{G}} = \varepsilon_{\mathsf{H}}$ be as above. Let $\ell, m \in \mathbb{N}$ with $\ell \geq m$. Suppose that $p > 2$; suppose further that $m \geq \min\{\ell/p, e/(p-2)\}$ if $\mathrm{char}(\mathfrak{o}) = 0$, and $m \geq \ell/p$ if $\mathrm{char}(\mathfrak{o}) = p$.*

$$\zeta_{\mathsf{H}^m(\mathfrak{o})/\mathsf{H}^\ell(\mathfrak{o})}(s) = \begin{cases} q^{3(\ell-m)} & \text{if } \ell \leq 2m, \\ q^{3(m-1)} \left(1 + (q^3-1)\frac{1-(q^{1-s})^{\ell-2m+1}}{1-q^{1-s}}\right) & \text{if } \ell > 2m. \end{cases}$$

$$\zeta_{\mathsf{G}^m(\mathfrak{o})/\mathsf{G}^\ell(\mathfrak{o})}(s) = q^{\ell-m} \zeta_{\mathsf{H}^m(\mathfrak{o})/\mathsf{H}^\ell(\mathfrak{o})}(s).$$

Moreover, if $\mathrm{char}(\mathfrak{o}) = 0$ *then*

$$\zeta_{\mathsf{H}^m(\mathfrak{o})}(s) = q^{3(m-1)} \sum_{\mathcal{S}\in\mathbb{T}_{\mathsf{A}_1}} \Xi^{\mathcal{S}}_{\mathsf{A}_1,\varepsilon,q}(s) = q^{3m}\frac{1-q^{-2-s}}{1-q^{1-s}}. \tag{A.6}$$

Remark A.14. Equation (A.6) confirms [4, Theorem 1.2]. Despite appearance in (A.4), (A.5), and (A.6), and in contrast to their analogues in type A_2, the zeta functions $\zeta_{\mathsf{H}(\mathfrak{o}_\ell)}(s)$, $\zeta_{\mathsf{H}(\mathfrak{o})}(s)$, and $\zeta_{\mathsf{H}^m(\mathfrak{o})}(s)$ are independent of ε. This reflects the fact that the isomorphism $\mathsf{SL}_2(\mathbb{F}_q) \simeq \mathsf{SU}_2(\mathbb{F}_q)$ (cf. [28, II.8.8]) generalises to $\mathsf{SL}_2(\mathfrak{o}) \simeq \mathsf{SU}_2(\mathfrak{o})$.

References

[1] A. Aizenbud and N. Avni, *Representation growth and rational singularities of the moduli space of local systems*, Invent. math. (2015), DOI 10.1007/s00222-015-0614-8.

[2] N. Avni, *Arithmetic groups have rational representation growth*, Ann. of Math. (2) **174** (2011), no. 2, 1009–1056.

[3] N. Avni, B. Klopsch, U. Onn, and C. Voll, *Representation zeta functions of groups and a conjecture of Larsen-Lubotzky*, C. R. Math. Acad. Sci. Paris. **348** (2010), 363–367.

[4] ———, *Representation zeta functions of some compact p-adic analytic groups*, Zeta functions in algebra and geometry, Contemp. Math., vol. 566, Amer. Math. Soc., Providence, RI, 2012, pp. 295–330.

[5] ———, *Representation zeta functions of compact p-adic analytic groups and arithmetic groups*, Duke Math. J. **162** (2013), no. 1, 111–197.

[6] ———, *Arithmetic groups, base change, and representation growth*, arXiv:1110.6092v3, submitted, 2014.

[7] N. Avni, U. Onn, A. Prasad, and L. Vaserstein, *Similarity classes of* 3×3 *matrices over a local principal ideal ring*, Comm. Algebra **37** (2009), 2601–2615.

[8] H. Bass, A. Lubotzky, A. R. Magid, and S. Mozes, *The proalgebraic completion of rigid groups*, Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II (Haifa, 2000), vol. 95, 2002, pp. 19–58.

[9] H. Bass, J. Milnor, and J.-P. Serre, *Solution of the congruence subgroup problem for* $\mathrm{SL}_n\,(n \geq 3)$ *and* $\mathrm{Sp}_{2n}\,(n \geq 2)$, Inst. Hautes Études Sci. Publ. Math. (1967), no. 33, 59–137.

[10] M. N. Berman, J. Derakhshan, U. Onn, and P. Paajanen, *Uniform cell decomposition with applications to Chevalley groups*, J. Lond. Math. Soc. (2) **87** (2013), no. 2, 586–606.

[11] W. Borho, *Über Schichten halbeinfacher Lie-Algebren*, Invent. Math. **65** (1981/82), no. 2, 283–317.

[12] W. Borho and H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen*, Comment. Math. Helv. **54** (1979), no. 1, 61–104.

[13] A. Broer, *Lectures on decomposition classes*, Representation theories and algebraic geometry Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, pp. 39–83.

[14] F. Calegari and M. Emerton, *Completed cohomology—a survey*, Non-abelian fundamental groups and Iwasawa theory, London Mathematical Society Lecture Note Series, vol. 393, Cambridge University Press, Cambridge, 2012, pp. 239–257.

[15] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R.A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[16] J. Dieudonné, *La géométrie des groupes classiques*, Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 5, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

[17] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, vol. 21, Cambridge University Press, Cambridge, 1991.

[18] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, second ed., Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999.

[19] M. P. F. du Sautoy and F. J. Grunewald, *Analytic properties of zeta functions and subgroup growth*, Ann. of Math. (2) **152** (2000), 793–833.

[20] V. Ennola, *On the characters of the finite unitary groups*, Ann. Acad. Sci. Fenn. Ser. A I No. **323** (1963).

[21] M. Geck, *An introduction to algebraic geometry and algebraic groups*, Oxford Graduate Texts in Mathematics, vol. 20, Oxford University Press, Oxford, 2013.

[22] J. González-Sánchez, *On p-saturable groups*, J. Algebra **315** (2007), no. 2, 809–823.

[23] ———, *Kirillov's orbit method for p-groups and pro-p groups*, Comm. Algebra **37** (2009), no. 12, 4476–4488.

[24] J. González-Sánchez, A. Jaikin-Zapirain, and B. Klopsch, *The representation zeta function of a FAb compact p-adic Lie group vanishes at* -2, Bull. Lond. Math. Soc. **46** (2014), no. 2, 239–244.
[25] J. A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402–447.
[26] M. J. Greenberg, *Schemata over local rings. II*, Ann. of Math. (2) **78** (1963), 256–266.
[27] E. Hrushovski, B. Martin, S. Rideau, *Definable equivalence relations and zeta functions of groups*, with an appendix by R. Cluckers, arXiv:math/0701011, 2015.
[28] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967.
[29] I. M. Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006.
[30] A. Jaikin-Zapirain, *Zeta function of representations of compact p-adic analytic groups*, J. Amer. Math. Soc. **19** (2006), no. 19, 91–118.
[31] S. Jambor and W. Plesken, *Normal forms for matrices over uniserial rings of length two*, J. Algebra **358** (2012), 250–256.
[32] R. L. Griess Jr., *Schur multipliers of finite simple groups of Lie type*, Trans. Amer. Math. Soc. **183** (1973), 355–421.
[33] G. Karpilovsky, *The Schur multiplier*, London Mathematical Society Monographs. New Series, vol. 2, The Clarendon Press, Oxford University Press, New York, 1987.
[34] N. Kawanaka, *Generalized Gel′fand-Graev representations and Ennola duality*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 175–206.
[35] B. Klopsch, *On the Lie theory of p-adic analytic groups*, Math. Z. **249** (2005), no. 4, 713–730.
[36] ———, *Representation growth and representation zeta functions of groups*, Note Mat. **33** (2013), no. 1, 107–120.
[37] Y. Komori, K. Matsumoto, and H. Tsumura, *On Witten multiple zeta-functions associated with semi-simple Lie algebras IV*, Glasg. Math. J. **53** (2011), no. 1, 185–206.
[38] M. Larsen and A. Lubotzky, *Representation growth of linear groups*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 2, 351–390.
[39] M. Lazard, *Groupes analytiques p-adiques*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 26, 389–603.
[40] M. Liebeck and A. Shalev, *Character degrees and random walks in finite groups of Lie type*, Proc. London Math. Soc. (3) **90** (2005), no. 1, 61–86.
[41] A. Lubotzky and B. Martin, *Polynomial representation growth and the congruence subgroup growth*, Israel J. Math. **144** (2004), 293–316.
[42] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995.
[43] H. Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989.
[44] L. J. Mordell, *On the evaluation of some multiple series*, J. London Math. Soc. **33** (1958), 368–371.
[45] A. Moreau, *On the dimension of the sheets of a reductive Lie algebra*, J. Lie Theory **18** (2008), no. 3, 671–696.
[46] J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften, vol. 322, Springer-Verlag, Berlin, 1999.
[47] U. Onn, *Representations of automorphism groups of finite* $\mathfrak{o}$*-modules of rank two*, Adv. Math. **219** (2008), no. 6, 2058–2085.
[48] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., Boston, MA, 1994.
[49] A. Prasad, P. Singla, and S. Spallone, *Similarity of matrices over local rings of length two*, Indiana Univ. Math. J. **64** (2015), no. 2, 471–514.
[50] J.-P. Serre, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977.
[51] A. Shalev, *Applications of some zeta functions in group theory*, Zeta functions in algebra and geometry, Contemp. Math., vol. 566, Amer. Math. Soc., Providence, RI, 2012, pp. 331–344.
[52] W. A. Simpson and J. A. Frame, *The character tables for* $\mathrm{SL}(3, q)$*,* $\mathrm{SU}(3, q^2)$*,* $\mathrm{PSL}(3, q)$*,* $\mathrm{PSU}(3, q^2)$, Canad. J. Math. **25** (1973), 486–494.
[53] P. Singla, *On representations of general linear groups over principal ideal local rings of length two*, J. Algebra **324** (2010), no. 9, 2543–2563.
[54] R. Steinberg, *The representations of* $\mathrm{GL}(3,q), \mathrm{GL}(4,q), \mathrm{PGL}(3,q)$*, and* $\mathrm{PGL}(4,q)$, Canadian J. Math. **3** (1951), 225–235.
[55] ———, *Generators, relations and coverings of algebraic groups. II*, J. Algebra **71** (1981), no. 2, 527–543.

[56] N. Thiem and C. R. Vinroot, *On the characteristic map of finite unitary groups*, Adv. Math. **210** (2007), no. 2, 707–732.

[57] C. Voll, *Zeta functions of groups and rings – recent developments*, Groups St Andrews 2013 in St Andrews, London Mathematical Society Lecture Note Series, vol. 422, Cambridge University Press, Cambridge, 2015, 469 – 492.

[58] H. Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997.

[59] E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys. **141** (1991), no. 1, 153–209.

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