

# Harnack inequalities for Hunt processes with Green function

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## Abstract

Let  $(X, \mathcal{W})$  be a balayage space,  $1 \in \mathcal{W}$ , or – equivalently – let  $\mathcal{W}$  be the set of excessive functions of a Hunt process on a locally compact space  $X$  with countable base such that  $\mathcal{W}$  separates points, every function in  $\mathcal{W}$  is the supremum of its continuous minorants and there exist strictly positive continuous  $u, v \in \mathcal{W}$  such that  $u/v \rightarrow 0$  at infinity. We suppose that there is a Green function  $G > 0$  for  $X$ , a metric  $\rho$  on  $X$  and a decreasing function  $g: [0, \infty) \rightarrow (0, \infty]$  having the doubling property and a mild upper decay near 0 such that  $G \approx g \circ \rho$  (which is equivalent to a  $3G$ -inequality).

Then the corresponding capacity for balls of radius  $r$  is bounded by a constant multiple of  $1/g(r)$ . Assuming that reverse inequalities hold as well and that jumps of the process, when starting at neighboring points, are related in a suitable way, it is proven that positive harmonic functions satisfy scaling invariant Harnack inequalities. Provided that the Ikeda-Watanabe formula holds, sufficient conditions for this relation are given. This shows that rather general Lévy processes are covered by this approach.

Keywords: Harnack inequality; Hunt process; balayage space; Lévy process; Green function;  $3G$ -property; equilibrium potential; capacity.

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## 1 Setting and main result

Our basic setting will be almost as in [7], but assuming that points are polar:

Let  $X$  be a locally compact space with countable base. Let  $\mathcal{C}(X)$  denote the set of all continuous real functions on  $X$  and let  $\mathcal{B}(X)$  be the set of all Borel measurable numerical functions on  $X$ . The set of all (positive) Radon measures on  $X$  will be denoted by  $\mathcal{M}(X)$ .

Moreover, let  $\mathcal{W}$  be a convex cone of positive lower semicontinuous numerical functions on  $X$  such that  $1 \in \mathcal{W}$  and  $(X, \mathcal{W})$  is a balayage space (see [2], [5] or [9, Appendix]). In particular, the following holds:

(C)  $\mathcal{W}$  separates the points of  $X$ , for every  $w \in \mathcal{W}$ ,

$$w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) : v \leq w\},$$

and there are strictly positive  $u, v \in \mathcal{W} \cap \mathcal{C}(X)$  such that  $u/v \rightarrow 0$  at infinity.

Then there exists a Hunt process  $\mathfrak{X}$  on  $X$  such that  $\mathcal{W}$  is the set  $E_{\mathbb{P}}$  of excessive functions for the transition semigroup  $\mathbb{P} = (P_t)_{t>0}$  of  $\mathfrak{X}$  (see [2, IV.7.6]), that is,

$$\mathcal{W} = \{v \in \mathcal{B}^+(X) : \sup_{t>0} P_t v = v\}.$$

We note that, conversely, given any sub-Markov right-continuous semigroup  $\mathbb{P} = (P_t)_{t>0}$  on  $X$  such that (C) is satisfied by its convex cone  $E_{\mathbb{P}}$  of excessive functions,  $(X, E_{\mathbb{P}})$  is a balayage space, and  $\mathbb{P}$  is the transition semigroup of a Hunt process (see [5, Corollary 2.3.8] or [9, Corollary A.5]).

For every subset  $A$  of  $X$ , we have reduced functions  $R_u^A$ ,  $u \in \mathcal{W}$ , and reduced measures  $\varepsilon_x^A$ ,  $x \in X$ , defined by

$$R_u^A := \inf\{v \in \mathcal{W} : v \geq u \text{ on } A\} \quad \text{and} \quad \int u d\varepsilon_x^A = R_u^A(x).$$

Of course,  $R_u^A \leq u$  on  $X$  and  $R_u^A = u$  on  $A$ . The greatest lower semicontinuous minorant  $\hat{R}_1^A$  of  $R_1^A$  (which is also the greatest finely lower semicontinuous minorant of  $R_1^A$ ) is contained in  $\mathcal{W}$ , and  $\hat{R}_1^A = R_1^A$  on  $A^c$  (see [2, VI.2.3]). If  $A$  is not thin at any of its points (see [2, VI.4]) for the definition), in particular, if  $A$  is open, then  $R_u^A \in \mathcal{W}$ . If  $A$  is Borel measurable, then

$$(1.1) \quad R_1^A(x) = P^x[T_A < \infty], \quad x \in X,$$

where  $T_A(\omega) := \inf\{t \geq 0 : X_t(\omega) \in A\}$  (see [2, VI.3.14]) and, for every Borel measurable set  $B$  in  $X$ ,

$$\varepsilon_x^A(B) = P^x[X_{T_A} \in B; T_A < \infty].$$

For every open set  $U$  in  $X$ , let  $\mathcal{H}^+(U)$  denote the set of all functions  $h \in \mathcal{B}^+(X)$  which are *harmonic on  $U$*  (in the sense of [2]), that is, such that  $h|_U \in \mathcal{C}(U)$  and

$$(1.2) \quad H_V h(x) := \varepsilon_x^{V^c}(h) := \int h d\varepsilon_x^{V^c} = h(x) \quad \text{if } V \text{ is open and } x \in V \subset\subset U.$$

If, for example,  $A$  is a Borel measurable set in  $X$  and  $u \in \mathcal{W}$ , then, by [2, VI.2.6],

$$(1.3) \quad R_u^A \in \mathcal{H}^+(X \setminus \bar{A}) \quad \text{provided } u \leq w \text{ for some } w \in \mathcal{W} \cap \mathcal{C}(X).$$

We note that  $U \mapsto \mathcal{H}^+(U)$  has the following sheaf property: If  $U_i$ ,  $i \in I$ , are open sets in  $X$ , then

$$\bigcap_{i \in I} \mathcal{H}^+(U_i) = \mathcal{H}^+\left(\bigcup_{i \in I} U_i\right).$$

In fact, given an open set  $U$  in  $X$ , a function  $h \in \mathcal{B}^+(X)$  which is continuous on  $U$  is already contained in  $\mathcal{H}^+(U)$ , if, for every  $x \in U$ , there exists a fundamental system of relatively compact open neighborhoods  $V$  of  $x$  in  $U$  such that  $\varepsilon_x^{V^c}(h) = h(x)$  (see [2, III.4.4 and III.4.5] or [5, Corollary 5.2.8 and Corollary 5.2.9]).

Moreover, let  $\tilde{\mathcal{H}}^+(U)$  be the set of all  $h \in \mathcal{B}^+(X)$  such that  $h < \infty$  on  $U$  and (1.2) holds. Then every function in  $\tilde{\mathcal{H}}^+(U)$  is lower semicontinuous on  $U$ , and

$$(1.4) \quad \tilde{\mathcal{H}}_b^+(U) = \mathcal{H}_b^+(U).$$

Indeed, let  $V$  be an open set such that  $V \subset\subset U$ . By (1.3),  $H_V 1 = R_1^{V^c}$  is harmonic on  $V$ . Moreover, for every  $f \in \mathcal{B}_b^+(X)$  with compact support, the function  $H_V f$  is continuous on  $V$  (see [2, III.2.8]). So, for every  $f \in \mathcal{B}^+(X)$ , the function  $H_V f$  is lower semicontinuous. Assuming that  $f$  is bounded, both  $H_V f$  and  $H_V(\|f\| - f)$  are lower semicontinuous on  $V$ , and hence (due to the continuity of the sum) both are continuous on  $V$ .

For our main result we shall assume that we have a metric  $\rho$  for  $X$ , a Green function  $G$  on  $X$ , a decreasing function  $g$  on  $[0, \infty]$ , and  $0 < R_0 \leq \infty$  such that

$$G \approx g \circ \rho,$$

$g$  having the doubling property and weak upper decay on  $(0, R_0)$  (Assumption 2.1). Defining balls  $B(x, r) := \{y \in X : \rho(y, x) < r\}$ ,  $x \in X$ ,  $0 < r < R_0$ , we suppose that, for some  $c_0 \geq 1$ , the corresponding capacities satisfy

$$(1.5) \quad \text{cap } B(x, r) \geq c_0^{-1} g(r)^{-1}$$

(Assumption 4.2; a reverse estimate is a consequence of the previous assumption). Finally, we shall suppose (Assumption 5.1) that there are constants  $0 < R_1 \leq \infty$ ,  $c_J \geq 1$  and  $0 < \alpha_0 < 1$  such that, for all  $x \in X$ ,  $0 < r < R_1$  and  $y \in B(x, \alpha_0 r)$ ,

$$(1.6) \quad \varepsilon_x^{B(x, \alpha_0 r)^c} \leq c_J \varepsilon_y^{B(x, r)^c} \quad \text{on } B(x, r)^c.$$

Then our main result is the following (see also Remarks 2.2,6 and 6.2).

**THEOREM 1.1.** (1) *For every open set  $U$  in  $X$ ,  $\tilde{\mathcal{H}}^+(U) = \mathcal{H}^+(U)$ .*

(2) *Scaling invariant Harnack inequalities: There exists constants  $\alpha \in (0, 1)$  and  $K \in (1, \infty)$  such that the following holds: For all  $x_0 \in X$ ,  $0 < R < R_0 \wedge R_1$  such that  $\overline{B}(x_0, R)$  is a proper compact subset of  $X$ , and all  $h \in \mathcal{B}^+(X)$  which are harmonic in a neighborhood of  $\overline{B}(x_0, R)$ ,*

$$(1.7) \quad \sup h(B(x_0, \alpha R)) \leq K \inf h(B(x_0, \alpha R)).$$

In Section 2 we shall shortly discuss a Green function for  $(X, \mathcal{W})$  and the related capacity. In Section 3, the probability of hitting a subset  $A$  of a ball before leaving a much larger ball is estimated in terms of the capacity of  $A$ . And in Section 4 we repeat basic facts on the relation between a lower estimate of the equilibrium potential of a ball by the Green function and (1.5).

Having prepared it by two crucial lemmas in Section 5, the proof of Theorem 1.1 is given in Section 6. Sufficient conditions for the validity of (1.6) are discussed in Section 7. In a final Section 8, we prove Harnack inequalities under intrinsic and local assumptions.

## 2 Green function and capacity

By definition, a *potential on  $X$*  is a function  $p \in \mathcal{W}$  such that, for every relatively compact open set  $V$  in  $X$ , the function  $H_V p = R_p^{V^c}$  is continuous and real on  $V$  (and hence harmonic on  $V$ ) and

$$\inf\{R_p^{V^c} : V \text{ relatively compact open in } X\} = 0.$$

By [5, Proposition 4.2.10], a function  $p \in \mathcal{W} \cap \mathcal{C}(X)$  is a potential if and only if there exists a strictly positive  $q \in \mathcal{W} \cap \mathcal{C}(X)$  such that  $p/q$  vanishes at infinity. Let  $\mathcal{P}$  denote the set of all continuous real potentials on  $X$ .

Unless stated otherwise we assume from now on the following:

**ASSUMPTION 2.1.** *There exists a Borel measurable function  $G: X \times X \rightarrow (0, \infty]$  such that  $G = \infty$  on the diagonal and the following holds:*

- (i) *For every  $y \in X$ ,  $G(\cdot, y)$  is a potential which is harmonic on  $X \setminus \{y\}$ .*
- (ii) *For every potential  $p$  on  $X$ , there exists a measure  $\mu$  on  $X$  such that*

$$(2.1) \quad p = G\mu := \int G(\cdot, y) d\mu(y).$$

- (iii) *There exist a metric  $\rho$  for  $X$  (compatible with the topology of  $X$ ), a decreasing numerical function  $g > 0$  on  $[0, \infty)$ , and constants  $c \geq 1$ ,  $c_D > 1$ ,  $0 < \alpha_0 < 1$ ,  $0 < \eta_0 < 1$ ,  $0 < R_0 \leq \infty$  such that*

$$(2.2) \quad c^{-1}g \circ \rho \leq G \leq cg \circ \rho,$$

and, for every  $0 < r < R_0$ ,

$$(2.3) \quad g(r/2) \leq c_D g(r) \quad \text{and} \quad g(r) \leq \eta_0 g(\alpha_0 r).$$

**REMARKS 2.2.** 1. Having (i), each of the following properties implies (ii).

- $G$  is lower semicontinuous on  $X \times X$ , continuous outside the diagonal, the potential kernel  $V_0 := \int_0^\infty P_t dt$  of  $\mathfrak{X}$  is proper, and there is a measure  $\mu$  on  $X$  such that  $V_0 f = \int G(\cdot, y) f(y) d\mu(y)$ ,  $f \in \mathcal{B}^+(X)$  (see [12] and [2, III.6.6]).
- $G$  is locally bounded off the diagonal, each function  $G(x, \cdot)$  is lower semicontinuous on  $X$  and continuous on  $X \setminus \{x\}$ , and there exists a measure  $\nu$  on  $X$  such that  $G\nu \in \mathcal{C}(X)$  and  $\nu(U) > 0$ , for every finely open  $U \neq \emptyset$  (the latter holds, for example, if  $V_0(x, \cdot) \ll \nu$ ,  $x \in X$ ). See [8, Theorem 4.1].

2. The measure in (2.1) is uniquely determined and, given any measure  $\mu$  on  $X$  such that  $p := G\mu$  is a potential, the complement of the support of  $\mu$  is the largest open set, where  $p$  is harmonic (see, for example, [8, Proposition 5.2 and Lemma 2.1]).

3. For the special case  $X = \mathbb{R}^d$  with  $\rho(x, y) = |x - y|$  and isotropic unimodular Green function, covering rather general Lévy processes, see [9, Section 6] and [6].

4. Of course, (2.3) implies that, for any  $\eta > 0$ , there exists  $\alpha \in (0, 1)$  such that  $g(r) \leq \eta g(\alpha r)$  for every  $0 < r < R_0$  (it suffices to choose  $m \in \mathbb{N}$  such that  $\eta_0^m \leq \eta$  and to take  $\alpha := \alpha_0^m$ ). Moreover, we see that  $\lim_{r \rightarrow 0} g(r) = \infty$ .

5. We may (and shall) assume without loss of generality that  $g$  is continuous on  $(0, R_0)$ . Indeed, if  $R_0 = \infty$ , let  $Z := \{2^n : n \in \mathbb{Z}\}$  and let  $\tilde{g}$  be the continuous function on  $(0, \infty)$  such that  $\tilde{g} = g$  on  $Z$  and  $\tilde{g}$  is locally affinely linear on  $(0, \infty) \setminus Z$ . Then  $g$  is decreasing, and it is easily verified that  $c_D^{-1}g \leq \tilde{g} \leq c_D g$ .

If  $R_0 < \infty$ , we may proceed similarly using  $Z := \{2^{-n}R_0 : n \in \mathbb{N}\}$  and defining  $\tilde{g}(R_0) := \lim_{s \rightarrow R_0^-} g(s)$ .

6. Let us mention that it is rather easy and straightforward to show the following. If all balls are relatively compact and the doubling property  $g(r/2) \leq c_D g(r)$  holds for all  $0 < r < \infty$ , then

$$\sup v(B(x_0, R/2)) \leq (cc_D)^2 \inf v(B(x_0, R/2))$$

for all  $v \in \mathcal{W}$ ,  $x_0 \in X$  and  $R > 0$  such that  $v$  is contained in  $\tilde{\mathcal{H}}^+(U)$  for some open neighborhood  $U$  of  $\overline{B}(x_0, R)$ . If, in addition, the function 1 is harmonic on  $X$ , then every function in  $\tilde{\mathcal{H}}^+(X)$  is constant (Liouville property; see [7, Section 2]).

Suppose that  $A$  is a subset of  $X$  such that  $\hat{R}_1^A$  is a potential. Then there is a unique measure  $\mu_A$  on  $X$ , the *equilibrium measure for  $A$* , such that

$$\hat{R}_1^A = G\mu_A.$$

If  $A$  is open, then  $\hat{R}_1^A = R_1^A \in \mathcal{H}(X \setminus \overline{A})$  and  $\mu_A$  is supported by  $\overline{A}$ . For a general balayage space this may already fail if  $A$  is compact (see [2, V.9.1]).

We define inner capacities for open sets  $U$  in  $X$  by

$$(2.4) \quad \text{cap}_* U := \sup \{ \|\mu\| : \mu \in \mathcal{M}(X), \mu(X \setminus U) = 0, G\mu \leq 1 \}$$

and outer capacities for arbitrary sets  $A$  in  $X$  by

$$(2.5) \quad \text{cap}^* A := \inf \{ \text{cap}_* U : U \text{ open neighborhood of } A \}.$$

Obviously,  $\text{cap}^* A = \text{cap}_* A$ , if  $A$  is open. If  $\text{cap}_* A = \text{cap}^* A$ , we might simply write  $\text{cap} A$  and speak of the capacity of  $A$ . It is easily seen that  $U \mapsto \text{cap} U$  is subadditive and  $\text{cap} U_n \uparrow \text{cap} U$ , for any sequence  $(U_n)$  of open sets in  $X$  with  $U_n \uparrow U$ .

The capacity of open sets  $U$  is essentially determined by the total mass of equilibrium measures for relatively compact open sets in  $U$  (see [7, Lemma 1.6]):

**LEMMA 2.3.** *For every open set  $U$  in  $X$ ,*

$$\text{cap} U \geq \sup \{ \|\mu_V\| : V \text{ open and } \overline{V} \text{ compact in } U \} \geq c^{-2} \text{cap} U.$$

### 3 Hitting of sets before leaving large balls

Let us first recall the following simple fact (see [6]), where, as usual,

$$\tau_U := T_{U^c}.$$

**LEMMA 3.1.** *Let  $A$  be a Borel measurable set in an open set  $U \subset X$  and  $\gamma > 0$ . If  $R_1^A \leq \gamma$  on  $U^c$ , then*

$$P^x[T_A < \tau_U] \geq R_1^A(x) - \gamma, \quad \text{for every } x \in U.$$

Using Lemma 2.3, this leads to a lower estimate for the probability of hitting a subset of a ball before leaving a much larger ball (cf. [6, Proposition 4]).

**PROPOSITION 3.2.** *Let  $\eta := (2c^3 c_D^2)^{-1}$  and let  $0 < \alpha < 1/2$ ,  $0 < r < R_0$  be such that  $g((1 - 2\alpha)r) \leq c\eta g(\alpha r)$ . Then, for all  $x_0 \in X$ ,  $x \in B := B(x_0, 2\alpha r)$ , and Borel measurable sets  $A$  in  $B(x_0, 2\alpha r)$ ,*

$$(3.1) \quad P^x[T_A < \tau_{B(x_0, r)}] \geq \eta g(\alpha r) \text{cap}^*(A).$$

*Proof.* To prove (3.1) we may assume without loss of generality that  $A$  is open (see [2, VI.3.14]). Let  $V$  be an open set such that  $\bar{V}$  is compact in  $A$ . Since  $\rho(x, \cdot) \leq 4\alpha r$  on  $\bar{V}$ , we have

$$R_1^V(x) = \int G(x, z) d\mu_V(z) \geq c^{-1}g(4\alpha r)\|\mu_V\| \geq 2c^2\eta g(\alpha r)\|\mu_V\|.$$

If  $y \in X \setminus B(x_0, r)$ , then  $\rho(y, \cdot) \geq (1 - 2\alpha)r$  on  $\bar{V}$ , and therefore

$$R_1^V(y) = \int G(y, z) d\mu_V(z) \leq cg((1 - 2\alpha)r)\|\mu_V\| \leq c^2\eta g(\alpha r)\|\mu_V\|.$$

So, using Lemma 3.1,

$$P^x[T_A < \tau_{B(x_0, r)}] \geq P^x[T_V < \tau_{B(x_0, r)}] \geq c^2\eta g(\alpha r)\|\mu_V\|.$$

An application of Lemma 2.3 completes the proof.  $\square$

**REMARK 3.3.** Let us note that our probabilistic statements and proofs can be replaced by analytic ones using that, for all Borel measurable sets  $A, B$  in an open set  $U$ ,

$$P^x[X_{T_A} \in B; T_A < \tau_U] = \varepsilon_x^{A \cup U^c}(B)$$

(see [2, VI.2.9]) and, for all Borel measurable sets  $B$  in  $X$  and  $B \subset A \subset X$ ,

$$(3.2) \quad \varepsilon_x^B = \varepsilon_x^A|_B + (\varepsilon_x^A|_{B^c})^B.$$

(If  $x \in B$ , then (3.2) holds trivially. If  $x \notin B$  and  $p \in \mathcal{P}$ , then, by [2, VI.9.1],

$$\hat{R}_p^B(x) = R_p^B(x) = \int R_p^B d\varepsilon_x^A = \int_B p d\varepsilon_x^A + \int_{B^c} \hat{R}_p^B d\varepsilon_x^A.)$$

## 4 Equilibrium potential and capacity of balls

The following estimates are [7, Proposition 1.7].

**PROPOSITION 4.1.** *Let  $x \in X$ ,  $r > 0$ , and  $B := B(x, r)$ . Then the reduced function  $R_1^B$  is a potential (in fact, bounded by a potential  $p \in \mathcal{P}$ ),*

$$R_1^B \leq c \frac{G(\cdot, x)}{g(r)} \leq c^2 \frac{g(\rho(\cdot, x))}{g(r)}, \quad \|\mu_B\| \vee \text{cap } B \leq c \frac{1}{g(r)},$$

$$R_1^B \geq c^{-1} \text{cap } B \cdot g(\rho(\cdot, x) + r).$$

For the next three sections we assume, in addition, the following.

**ASSUMPTION 4.2.** *There exists  $c_0 \geq 1$  such that, for all  $x \in X$  and  $0 < r < R_0$ ,*

$$(4.1) \quad \text{cap } B(x, r) \geq c_0^{-1}g(r)^{-1}.$$

Then, by Proposition 4.1, for all  $x \in X$  and  $0 < r < R_0$ ,

$$(4.2) \quad R_1^{B(x, r)} \geq (cc_0)^{-1} \frac{g(\rho(\cdot, x) + r)}{g(r)}.$$

**EXAMPLES 4.3.** 1. Assume for the moment that  $(X, \mathcal{W})$  is a harmonic space, that is,  $\mathfrak{X}$  is a diffusion. Moreover, suppose that  $X$  is non-compact, but balls are relatively compact. Then (4.1) holds with  $c_0 := c^3 c_D$  if  $0 < r < R_0/2$ .

Indeed, let  $x \in X$ ,  $0 < r < R_0/2$ ,  $B := B(x, r)$ . Given  $\tilde{c} > c$ , we have  $G(\cdot, x) \leq cg(r) < \tilde{c}g(r)$  on  $X \setminus B$ , hence, for some  $0 < \tilde{r} < r$ ,  $G(\cdot, x) < \tilde{c}g(r)$  on  $X \setminus B(x, \tilde{r})$ , and we see, by the minimum principle ([2, III.6.6]), that  $G(\cdot, x) \leq \tilde{c}g(r)R_1^B$  on  $X \setminus B(x, \tilde{r})$ . So  $G(\cdot, x) \leq cg(r)R_1^B$  on  $X \setminus B$ .

Choosing  $y \in X \setminus \overline{B}(x, 2r)$ , we know that the potential  $G(\cdot, y)$  is strictly positive and harmonic on  $B(x, 2r)$ , hence  $\partial B(x, 2r) \neq \emptyset$ . Let  $z \in \partial B(x, 2r)$ . Then

$$R_1^B(z) \geq c^{-1}G(z, x)/g(r) \geq c^{-2}g(2r)/g(r) \geq (c^2 c_D)^{-1}.$$

Let  $a < R_1^B(z)$ . By [2, VI.1.2], there exists  $0 < s < r$  such that  $V := B(x, s)$  satisfies  $a < R_1^V(z)$ . Since  $\rho(z, \cdot) > r$  on  $B$ ,

$$R_1^V(z) = \int G(z, y) d\mu_V(y) \leq cg(r)\|\mu_V\| \leq cg(r) \text{cap } B.$$

Thus  $(c^3 c_D)^{-1}g(r)^{-1} \leq \text{cap } B$ .

2. If  $X = \mathbb{R}^d$  and  $\rho(x, y) = |x - y|$ , then Assumption 4.2 is satisfied provided there exists  $C_G \geq 1$  such that  $d \int_0^r s^{d-1}g(s) ds \leq C_G r^d g(r)$  for all  $0 < r < R_0$ , since then the normalized Lebesgue measure  $\lambda_{B(x, r)}$  on  $B(x, r)$  satisfies  $G\lambda_{B(x, r)} \leq G\lambda_{B(x, r)}(x) \leq cC_G g(r)$  (see [6]). So Assumption 4.2 is satisfied for rather general isotropic unimodular Lévy processes.

## 5 Two crucial lemmas

In this and the following section, we assume the following on the jumps.

**ASSUMPTION 5.1.** *There exist  $0 < R_1 \leq \infty$ ,  $c_J > 0$  and  $0 < \alpha < 1/2$  such that, for all  $x \in X$ ,  $0 < r < R_1$  and  $y \in B(x, \alpha r)$ ,*

$$(5.1) \quad \varepsilon_x^{B(x, \alpha r)^c} \leq c_J \varepsilon_y^{B(y, r)^c} \quad \text{on } B(y, r)^c.$$

**REMARKS 5.2.** 1. If  $\mathfrak{X}$  is a diffusion or – equivalently – if  $(X, \mathcal{W})$  is a harmonic space, then Assumption 5.1 holds trivially, since the measures  $\varepsilon_x^{B(x, \alpha r)^c}$  do not charge the complement of  $B(y, r)$ .

2. If  $0 < \alpha' \leq \alpha$ , then  $B(x, \alpha r)^c \subset B(x, \alpha' r)^c$ , and hence, by (3.2),

$$(5.2) \quad \varepsilon_x^{B(x, \alpha' r)^c} |_{B(x, \alpha r)^c} \leq \varepsilon_x^{B(x, \alpha r)^c}.$$

Therefore we may replace  $\alpha$  in (5.1) by any smaller  $\alpha'$ .

3. Similarly, (3.2) implies that, for every  $y \in B(x, \alpha r)$ ,  $\varepsilon_y^{B(y, r)^c} |_{B(x, 2r)^c} \leq \varepsilon_y^{B(x, 2r)^c}$  and  $\varepsilon_y^{B(x, r/2)^c} |_{B(y, r)^c} \leq \varepsilon_y^{B(y, r)^c}$ . Hence Assumption 5.1 is equivalent to the assumption, where (5.1) is replaced by

$$(5.3) \quad \varepsilon_x^{B(x, \alpha r)^c} \leq c_J \varepsilon_y^{B(x, r)^c} \quad \text{on } B(x, r)^c.$$

For a proof of Theorem 1.1, we employ essential ideas from [1]. However, not assuming the existence of a volume measure and not having any information on the expectation of hitting times, we shall rely entirely on capacities of sets.

A very similar approach has been used in [14], where the Lévy process on  $\mathbb{R}^d$ ,  $d \geq 3$ , with characteristic exponent  $\phi(\xi) = |\xi|^2 \ln^{-1}(1 + |\xi|^2) - 1$  is considered, and  $g(r) \approx r^{2-d} \ln(1/r)$  as  $r \rightarrow 0$ .

As in Proposition 3.2, let  $\eta := (2c^3c_D^2)^{-1}$ . We may choose  $0 < \alpha < 1/4$  such that Assumption 5.1 holds with (5.3) in place of (5.1) and

$$(5.4) \quad g(r) \leq cc_D^{-1}\eta g(\alpha r) \quad \text{for every } 0 < r < R_0.$$

Since  $1 - 2\alpha \geq 1/2$ , we know that  $g((1 - 2\alpha)r) \leq c_D g(r) \leq c\eta g(\alpha r)$  for every  $0 < r < R_0$ .

Moreover, let

$$\beta := \frac{\eta}{6c_0}, \quad \gamma := \frac{1}{6} \wedge \frac{\beta}{c_J}, \quad \kappa := 3\beta\gamma = \frac{\eta\gamma}{2c_0}.$$

We choose  $j_0, m_0, m_1 \in \mathbb{N}$  such that

$$(1 + \beta)^{j_0} > c_D, \quad 2^{m_0} > 2j_0, \quad 2^{m_1}\alpha^2 > 1,$$

and define

$$(5.5) \quad K := \kappa^{-1}c_D^{m_0+m_1}.$$

Now we fix  $x_0 \in X$  and  $0 < R < R_0 \wedge R_1$  such that  $B(x_0, R)$  is relatively compact. Since  $\lim_{r \rightarrow 0} g(r) = \infty$  and  $g$  is decreasing and continuous on  $(0, R_0)$ , we may choose  $r_j > 0$ , such that

$$(5.6) \quad g(r_j) = c_D^{m_0}(1 + \beta)^{j-1}g(\alpha^4 R), \quad j \in \mathbb{N}.$$

The following two lemmas are crucial for the proof of Theorem 1.1.

**LEMMA 5.3.** *The sum of all  $r_j$ ,  $j \in \mathbb{N}$ , is less than  $\alpha^4 R$ .*

*Proof.* If  $1 \leq k \leq j_0$  and  $m \geq 0$ , then  $g(r_{mj_0+k}) > c_D^{m_0+m}g(\alpha^4 R) \geq g(2^{-(m_0+m)}\alpha^4 R)$ , and hence  $r_{mj_0+k} < 2^{-(m_0+m)}\alpha^4 R$ . Thus

$$\sum_{j=1}^{\infty} r_j < \sum_{m=0}^{\infty} j_0 2^{-(m_0+m)}\alpha^4 R = j_0 2^{-m_0+1}\alpha^4 R < \alpha^4 R.$$

□

**LEMMA 5.4.** *Let  $h \in \mathcal{H}_b^+(B(x_0, R))$  such that  $h(y_0) = 1$  for some  $y_0 \in B(x_0, \alpha^2 R)$ . If  $j \in \mathbb{N}$  and  $x \in B(x_0, 2\alpha^2 R)$  such that*

$$h(x) > (1 + \beta)^{j-1}K,$$

*then there exists  $x' \in B(x, r_j/\alpha^2)$  such that*

$$h(x') > (1 + \beta)^j K.$$

*Proof.* Let  $j \in \mathbb{N}$ ,  $r := r_j$ , and  $x \in B(x_0, 2\alpha^2 R)$  with  $h(x) > (1 + \beta)^{j-1} K$ . Let

$$U_1 := B(x, r) \cap \{h > \gamma h(x)\} \quad \text{and} \quad U_2 := B(x, r) \cap \{h < 2\gamma h(x)\}.$$

Then  $U_1$  and  $U_2$  are open sets and  $U_1 \cup U_2 = B(x, r) \subset B(x_0, 2\alpha^2 R)$ . In particular,

$$(5.7) \quad \text{cap } B(x, r) \leq \text{cap } U_1 + \text{cap } U_2.$$

If  $V$  is an open set with  $\bar{V} \subset U_1$ , then, by Proposition 3.2,

$$\begin{aligned} 1 = h(y_0) &= \varepsilon_{y_0}^{\bar{V} \cup B(x_0, \alpha R)^c}(h) \geq \gamma h(x) \varepsilon_{y_0}^{\bar{V} \cup B(x_0, \alpha R)^c}(\bar{V}) \\ &\geq \gamma h(x) P^{y_0}[T_V < \tau_{B(x_0, \alpha R)}] \geq \eta \gamma h(x) g(\alpha^2 R) \text{cap } V. \end{aligned}$$

So  $\text{cap } U_1 \leq (\eta \gamma h(x) g(\alpha^2 R))^{-1}$ . By Assumption 4.2,  $\text{cap } B(x, r) \geq (c_0 g(r))^{-1}$ . Since  $g(\alpha^4 R) \leq g(2^{-m_1} \alpha^2 R) \leq c_D^{m_1} g(\alpha^2 R)$ , we conclude, by (5.6) and (5.5), that

$$\frac{\text{cap } U_1}{\text{cap } B(x, r)} \leq \frac{c_0 g(r)}{\eta \gamma h(x) g(\alpha^2 R)} = \frac{c_D^{m_0} (1 + \beta)^{j-1} g(\alpha^4 R)}{2\kappa h(x) g(\alpha^2 R)} \leq \frac{(1 + \beta)^{j-1} K}{2h(x)} < \frac{1}{2}.$$

By (5.7), we obtain that

$$(5.8) \quad \text{cap } U_2 > (1/2) \text{cap } B(x, r) \geq (2c_0 g(r))^{-1}.$$

We choose an open set  $W$  such that  $\bar{W} \subset U_2$ ,  $\text{cap } W > (2c_0 g(r))^{-1}$ , and define

$$L := \bar{W}, \quad \nu := \varepsilon_x^{L \cup B(x, r/\alpha)^c}.$$

Then, by Proposition 3.2,

$$\nu(L) = P^x[T_L < \tau_{B(x, r/\alpha)}] \geq P^x[T_W < \tau_{B(x, r/\alpha)}] \geq \eta g(r) \text{cap } W > \frac{\eta}{2c_0} = 3\beta.$$

By Lemma 5.3,  $r/\alpha^2 < \alpha^2 R$ , and hence  $\bar{B}(x, r/\alpha^2) \subset B(x_0, 3\alpha^2 R) \subset B(x_0, R)$ . We claim that  $H := 1_{B(x, r/\alpha^2)^c} h$  satisfies

$$(5.9) \quad \varepsilon_x^{B(x, r/\alpha)^c}(H) \leq \beta h(x).$$

Indeed, if not, then (5.3) implies that, for every  $y \in B(x, r)$ ,

$$h(y) = \varepsilon_y^{B(x, r/\alpha^2)^c}(h) = \varepsilon_y^{B(x, r/\alpha^2)^c}(H) \geq c_J^{-1} \varepsilon_x^{B(x, r/\alpha)^c}(H) > c_J^{-1} \beta h(x) \geq \gamma h(x),$$

contradicting the fact that  $U_1$  is a proper subset of  $B(x, r)$ .

Finally, let  $a := \sup h(B(x, r/\alpha^2))$ . Then

$$h(x) = \nu(h) \leq 2\gamma h(x) \nu(L) + \int_{X \setminus B(x, r/\alpha)} h \, d\nu,$$

where

$$\int_{B(x, r/\alpha^2) \setminus B(x, r/\alpha)} h \, d\nu \leq a \nu(B(x, r/\alpha^2) \setminus B(x, r/\alpha)) \leq a(1 - \nu(L))$$

and, by (3.2) and (5.9),

$$\int_{B(x,r/\alpha^2)^c} h d\nu = \nu(H) \leq \varepsilon_x^{B(x,r/\alpha)^c}(H) \leq \beta h(x).$$

Therefore

$$h(x) \leq 2\gamma h(x)\nu(L) + a(1 - \nu(L)) + \beta h(x),$$

and

$$(5.10) \quad a \geq \frac{1 - \beta - 2\gamma\nu(L)}{1 - \nu(L)} h(x) > (1 + \beta)h(x) > (1 + \beta)^j K$$

completing the proof (since  $1 - 2\gamma \geq 2/3$  and  $\nu(L) > 3\beta$ , we have  $(1 - 2\gamma)\nu(L) > 2\beta$ , hence  $1 - \beta - 2\gamma\nu(L) > 1 + \beta - \nu(L) \geq (1 + \beta)(1 - \nu(L))$ ).  $\square$

Finally, we shall use the following little observation.

**LEMMA 5.5.** *Let  $U := B(x, R)$ ,  $x \in X$ ,  $R > 0$ , such that  $\bar{U}$  is a proper compact subset of  $X$ . Then there exists a function  $h \in \mathcal{H}_b^+(U)$  such that  $h > 0$  on  $\bar{U}$ .*

*Proof.* Let  $y \in X \setminus \bar{U}$  and let  $V$  be a relatively compact open neighborhood of  $\bar{U}$  such that  $y \notin \bar{V}$ . Then, for every  $n \in \mathbb{N}$ ,

$$h_n := H_V(G(\cdot, y) \wedge n) \in \mathcal{H}_b^+(V) \quad \text{and} \quad h_n \uparrow H_V G(\cdot, y) = G(\cdot, y),$$

as  $n \rightarrow \infty$ . Since  $G(\cdot, y) > 0$ , there exists  $n \in \mathbb{N}$  such that  $h_n > 0$  on  $\bar{U}$ .  $\square$

## 6 Proof of Theorem 1.1

Let us first give a complete statement of Theorem 1.1.

**THEOREM 6.1.** *Let  $(X, \mathcal{W})$  be a balayage space,  $1 \in \mathcal{W}$ , suppose that the Assumptions 2.1, 4.2, 5.1 are satisfied and let  $\alpha, K$  be as in Section 5 (see (5.4), (5.5)). Then the following hold.*

- (1) For every open set  $U$  in  $X$ ,  $\tilde{\mathcal{H}}^+(U) = \mathcal{H}^+(U)$ .
- (2) Scaling invariant Harnack inequalities: Let  $x_0 \in X$ ,  $0 < R < R_0 \wedge R_1$ , and  $B := B(x_0, R)$  such that  $\bar{B}$  is a proper compact subset of  $X$ . Then, for all functions  $h \in \mathcal{B}^+(X)$  which are harmonic in a neighborhood of  $\bar{B}$ ,

$$(6.1) \quad \sup h(B(x_0, \alpha^2 R)) \leq K \inf h(B(x_0, \alpha^2 R)).$$

*Proof.* (a) To prove (2), let  $B_0 := B(x_0, \alpha^2 R)$ , and let us first consider  $h \in \mathcal{H}_b^+(B)$  with  $h(y_0) = 1$  for some point  $y_0 \in B_0$ . Then

$$(6.2) \quad h \leq K \quad \text{on } B_0.$$

Indeed, suppose that  $h(x_1) > K$  for some  $x_1 \in B_0$ . Then, by Lemmas 5.3 and 5.4, there exist points  $x_2, x_3, \dots$  in  $B(x_0, 2\alpha^2 R)$  such that  $h(x_j) > (1 + \beta)^{j-1} K$ ,  $j \in \mathbb{N}$ . This contradicts the boundedness of  $h$ .

(b) Next let  $h$  be an arbitrary function in  $\mathcal{H}_b^+(B)$ . By Lemma 5.5, there exists  $h_0 \in \mathcal{H}_b^+(B)$  such that  $h_0 > 0$  on  $\overline{B}$ . Let  $y \in B_0$  and  $\varepsilon > 0$ . Applying (6.2) to  $(h + \varepsilon h_0)/(h + \varepsilon h_0)(y)$  we get that  $h + \varepsilon h_0 \leq K(h + \varepsilon h_0)(y)$  on  $B_0$ . Thus (6.1) holds.

(c) Let us now consider an open neighborhood  $W$  of  $\overline{B}$  and  $h \in \mathcal{H}^+(W)$ . By (1.4),

$$(6.3) \quad h_n := H_W(h \wedge n) \in \mathcal{H}_b^+(W),$$

$n \in \mathbb{N}$  (the relation  $H_V H_W = H_W$  for open  $V \subset\subset W$  is a special case of (3.2)). By (b),  $\sup h_n(B_0) \leq K \inf h_n(B_0)$ . Clearly,  $h_n \uparrow h$  as  $n \rightarrow \infty$ . So  $h$  satisfies (6.1).

(d) To prove (1), let  $U$  be an open set in  $X$ ,  $h \in \mathcal{H}^+(U)$ , and  $x_0 \in U$ . We choose  $0 < R < R_0 \wedge R_1$  such that the closure of  $B = B(x_0, R)$  is a proper compact subset of  $U$ . Let  $W$  be a relatively compact open neighborhood of  $\overline{B}$  in  $U$  and let  $h_n$ ,  $n \in \mathbb{N}$ , be as in (6.3). Then  $h - h_n \in \mathcal{H}^+(W)$  for every  $n \in \mathbb{N}$ , and hence, by (c),

$$h - h_n \leq K(h - h_n)(x_0) \quad \text{on } B_0 = B(x_0, \alpha^2 R).$$

So the functions  $h_n$  (which are continuous on  $W$ ) converge to  $h$  uniformly on  $B_0$ . Therefore  $h|_{B_0} \in \mathcal{C}(B_0)$ . Thus  $h|_U \in \mathcal{C}(U)$  completing the proof.  $\square$

**REMARK 6.2.** The preceding proofs show that, given  $x_0 \in X$  and  $R > 0$  such that  $\overline{B}(x_0, R)$  is a proper compact subset of  $X$ , we still obtain (6.1) with some  $K \in (1, \infty)$ , which may depend on  $x_0$  and  $R$ , provided there exist  $c_0, c_J \in (0, \infty)$  and  $\alpha \in (0, 1)$  such that (4.1) and (5.1) hold for all  $x \in B(x_0, R)$  and  $0 < r < R$ . For an application see Section 8.

## 7 Sufficient conditions for Assumption 5.1

For relatively compact open sets  $V$  in  $X$ , let  $G_V$  denote the associated Green function on  $V$ , that is,

$$G_V(\cdot, y) := G(\cdot, y) - R_{G(\cdot, y)}^{V^c}, \quad y \in V.$$

We shall need the following simple statement.

**LEMMA 7.1.** *There exists  $0 < \alpha < 1/4$  such that, for all  $y \in X$  and  $0 < r < R_0$ ,*

$$(7.1) \quad G_{B(y, r)}(\cdot, y) \geq \frac{1}{2} G(\cdot, y) \quad \text{on } B(y, 2\alpha r).$$

*Proof.* Let  $0 < \alpha < 1/4$  such that  $g(r) \leq (2c^2 c_D)^{-1} g(\alpha r)$  for every  $0 < r < R_0$ . Let  $y \in X$  and  $0 < r < R_0$ . Since  $G(\cdot, y) \leq cg(r)$  on  $B(y, r)^c$ , we obtain that  $R_{G(\cdot, y)}^{B(y, r)^c} \leq cg(r) \leq (2cc_D)^{-1} g(\alpha r)$ , whereas  $G(\cdot, y) \geq c^{-1} g(2\alpha r) \geq (cc_D)^{-1} g(\alpha r)$  on  $B(y, 2\alpha r)$ . So (7.1) holds.  $\square$

In this section, let us assume the following estimate of Ikeda-Watanabe type, which by [11, Example 1 and Theorem 1] holds, with  $C_N = 1$  and on  $X \setminus \overline{B}(x, r)$ , for all (temporally homogeneous) Lévy processes.

**ASSUMPTION 7.2.** *There exist a measure  $\lambda$  on  $X$ , a kernel  $N$  on  $X$ ,  $M_N \geq 3$ , and  $C_N \geq 1$  such that, for all  $x \in X$  and  $0 < r < R_0$ ,*

$$(7.2) \quad C_N^{-1} \varepsilon_x^{B(x, r)^c} \leq \int G_{B(x, r)}(x, z) N(z, \cdot) d\lambda(z) \leq C_N \varepsilon_x^{B(x, r)^c} \quad \text{on } B(x, M_N r)^c.$$

**PROPOSITION 7.3.** *Suppose that there exist  $C \geq 1$ ,  $a \geq 3$  and  $R > 0$  such that, for all  $x \in X$ ,  $0 < r < R$  and  $y \in B(x, r)$ ,*

$$(7.3) \quad N(x, \cdot) \leq CN(y, \cdot) \quad \text{on } B(y, ar)^c$$

and

$$(7.4) \quad \int_{B(x,r)} g(\rho(x, z)) d\lambda(z) \leq C \int_{B(y,2r)} g(\rho(y, z)) d\lambda(z).$$

Then Assumption 5.1 is satisfied.

*Proof.* Let  $M \geq M_N \vee (a + 2)$  such that Lemma 7.1 holds with  $\alpha := 1/M$ . Let  $x \in X$ ,  $0 < r < \alpha(R \wedge R_0)$ ,  $y \in B(x, r)$ , and let  $E$  be a Borel measurable set in  $B(y, Mr)^c$ . If  $z \in B(y, 2r)$ , then  $E \subset B(z, ar)^c$  and, by (7.3),

$$C^{-1}N(z, E) \leq N(y, E) \leq CN(z, E).$$

Since  $B(x, r) \subset B(y, 2r)$ , we obtain that

$$\begin{aligned} \varepsilon_x^{B(x,r)^c}(E) &\leq C_N \int G_{B(x,r)}(x, z) N(z, E) d\lambda(z) \\ &\leq cCC_N N(y, E) \int_{B(x,r)} g(\rho(x, z)) d\lambda(z) \\ &\leq cC^2 C_N N(y, E) \int_{B(y,2r)} g(\rho(y, z)) d\lambda(z) \\ &\leq 2c^2 C^3 C_N \int G_{B(y,Mr)}(y, z) N(z, E) d\lambda(z) \\ &\leq 2c^2 C^3 C_N^2 \varepsilon_y^{B(y,Mr)^c}(E). \end{aligned}$$

Thus Assumption 5.1 holds taking  $R_1 := R \wedge R_0$ . □

For simplicity, let us now assume that  $X = \mathbb{R}^d$ ,  $\rho(x, y) = |x - y|$ , the measure  $\lambda$  in Assumption 7.2 is Lebesgue measure (a case, where clearly (7.4) holds) and that there exists a constant  $C_G \geq 1$  such that

$$(7.5) \quad G\lambda_{B(x,r)} \leq C_G g(r), \quad x \in X, 0 < r < R_0.$$

We might recall that (7.5) implies that Assumption 4.2 is satisfied (see [7, (1.14)]).

**PROPOSITION 7.4.** *Suppose that there exist a measure  $\tilde{\lambda}$  on  $\mathbb{R}^d$ , a function  $n: \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$  and  $C \geq 1$ ,  $a \geq 3$  such that  $N(y, \cdot) = n(y, \cdot)\tilde{\lambda}$  for every  $y \in X$  and, for all  $x \in X$ ,  $0 < r < R_0$ ,  $y \in B(x, r)$  and  $\tilde{z} \in B(x, ar)^c$ ,*

$$(7.6) \quad n(x, \tilde{z}) \leq Cn(y, \tilde{z}) \quad \text{provided } |x - \tilde{z}| \geq |y - \tilde{z}|.$$

Then Assumption 5.1 is satisfied.

*Proof* (cf. the proof of [3, Proposition 6]). Again, let  $M \geq M_N \vee (a + 2)$  such that Lemma 7.1 holds with  $\alpha := 1/M$ , let  $x \in X$ ,  $0 < r < \alpha(R \wedge R_0)$ ,  $y \in B(x, r)$ , and let  $E$  be a Borel measurable set in  $B(y, Mr)^c$ . By (7.2),

$$\varepsilon_x^{B(x,r)^c}(E) \leq cC_N \int_{B(x,r)} \int_E g(|x-z|)n(z, \tilde{z}) d\tilde{\lambda}(\tilde{z}) d\lambda(z).$$

Similarly, since  $B(x, r) \subset B(y, 2r)$  and  $|y-z| \leq 2r$  for every  $z \in B(x, r)$ ,

$$\begin{aligned} \varepsilon_y^{B(y,Mr)^c}(E) &\geq C_N^{-1} \int G_{B(y,Mr)}(y, z)N(z, E) d\lambda(z) \\ &\geq (2C_N)^{-1} \int_{B(x,r)} G(y, z)N(z, E) d\lambda(z) \\ &\geq (2cc_D C_N)^{-1} g(r) \int_{B(x,r)} \int_E n(z, \tilde{z}) d\tilde{\lambda}(\tilde{z}) d\lambda(z). \end{aligned}$$

Hence it will be sufficient to show that, for every  $\tilde{z} \in B(y, Mr)^c$ ,

$$(7.7) \quad \int_{B(x,r)} g(|x-z|)n(z, \tilde{z}) d\lambda(z) \leq C'g(r) \int_{B(x,r)} n(z, \tilde{z}) d\lambda(z)$$

(with some constant  $C' > 0$ ). So let  $\tilde{z} \in B(y, Mr)^c$ .

Let  $B := B(x, r/2)$ . Since  $g(|x-z|) \leq g(r/2) \leq c_D g(r)$  for every  $z \in B^c$ ,

$$\int_{B(x,r) \setminus B} g(|x-z|)n(z, \tilde{z}) d\lambda(z) \leq c_D g(r) \int_{B(x,r)} n(z, \tilde{z}) d\lambda(z).$$

Moreover, let

$$x' := x + \frac{3}{4} \frac{\tilde{z} - x}{|\tilde{z} - x|} r \quad \text{and} \quad B' := B(x', r/4),$$

so that  $B' \subset B(x, r) \setminus B$ . If  $z \in B$  and  $z' \in B'$ , then  $|z - \tilde{z}| \geq |z' - \tilde{z}|$ , and therefore, by (7.6),

$$n(z, \tilde{z}) \leq \frac{C}{\lambda(B')} \int_{B'} n(z', \tilde{z}) d\lambda(z') = \frac{2^d C}{\lambda(B)} \int_{B'} n(z', \tilde{z}) d\lambda(z').$$

Hence

$$\begin{aligned} \int_B g(|x-z|)n(z, \tilde{z}) d\lambda(z) &\leq 2^d C \left( \int_{B'} n(z', \tilde{z}) d\lambda(z') \right) \cdot \left( \frac{1}{\lambda(B)} \int_B g(|x-z|) d\lambda(z) \right), \end{aligned}$$

where

$$\frac{1}{\lambda(B)} \int_B g(|x-z|) d\lambda(z) \leq cG\lambda_B(x) \leq cC_G g(r/2) \leq cc_D C_G g(r).$$

Thus (7.7) holds with  $C' := c_D(1 + 2^d cC_G)$ .  $\square$

If  $y \in B(x, r)$  and  $\tilde{z} \in B(x, 2r)^c$ , then  $|x - \tilde{z}| \leq 2|x - y| + 2|y - \tilde{z}| - |x - \tilde{z}| < 2|y - \tilde{z}|$ . Hence we have the following result.

**COROLLARY 7.5.** *Suppose that there exists a measure  $\tilde{\lambda}$  on  $\mathbb{R}^d$  such that  $N(y, \cdot) = n(y, \cdot)\tilde{\lambda}$ ,  $y \in X$ , where  $n(x, y) \approx n_0(|x - y|)$ , and that there exists  $C_0 \geq 1$  such that*

$$(7.8) \quad n_0(s) \leq C_0 n_0(r), \quad \text{whenever } 0 < r < s < 2r.$$

*Then Assumption 5.1 holds.*

Thus rather general Lévy processes may serve as examples for our approach (see, for example, [3, 13, 14, 15, 16, 17]).

## 8 Harnack inequalities under intrinsic assumptions

In this section, let us assume that  $(X, \mathcal{W})$  is a balayage space,  $1 \in \mathcal{W}$ , and that we have a Borel measurable function  $G: X \times X \rightarrow (0, \infty]$  such that  $G = \infty$  on the diagonal and the following holds:

- (i) For every  $y \in X$ ,  $G(\cdot, y)$  is a potential which is harmonic on  $X \setminus \{y\}$ .
- (ii) For every potential  $p$  on  $X$ , there exists a measure  $\mu$  on  $X$  such that

$$p = G\mu := \int G(\cdot, y) d\mu(y).$$

Moreover, we assume that there is a function  $w \in \mathcal{W} \cap \mathcal{C}(X)$ ,  $0 < w \leq 1$ , such that each function  $G(\cdot, x)/w$ ,  $x \in X$ , is bounded at infinity and  $G$  has the  $(w, w)$ -triangle property (see [4]), that is, for some constant  $C_w > 1$ , the function

$$\tilde{G}: (x, y) \mapsto G(x, y)/(w(x)w(y))$$

satisfies

$$(8.1) \quad \tilde{G}(x, z) \wedge \tilde{G}(y, z) \leq C_w \tilde{G}(x, y), \quad x, y, z \in X.$$

For  $x \in X$  and  $r > 0$ , we define open neighborhoods  $V(x, r)$  of  $x$  by

$$(8.2) \quad V(x, r) := \{G(\cdot, x) > 1/r\}.$$

We intend to prove the following result (where  $\tilde{\mathcal{H}}^+(U)$  has the same meaning as in Section 1).

**THEOREM 8.1.** *Let  $U$  be an open set which is covered by open sets  $V$  having the following property: There are real numbers  $R_1 \in (0, \infty]$ ,  $C, c_J \in (1, \infty)$  and  $\alpha \in (0, 1)$  (which may depend on  $V$ ) such that, for all  $0 < r < R_1$  and  $x \in V$ ,*

$$(8.3) \quad R_1^{V(x, r)} \geq C^{-1} G(\cdot, x)r \quad \text{on } V(x, r)^c$$

*and, for all  $y \in V(x, \alpha r)$ ,*

$$(8.4) \quad \varepsilon_x^{V(x, \alpha r)^c} \leq c_J \varepsilon_y^{V(y, r)^c} \quad \text{on } V(y, r)^c.$$

*Then  $\tilde{\mathcal{H}}^+(U) = \mathcal{H}^+(U)$  and, for every  $x \in U$ , there exists a compact neighborhood  $L$  of  $x$  in  $U$  and a constant  $K \geq 1$  such that*

$$(8.5) \quad \sup h(L) \leq K \inf h(L) \quad \text{for every } h \in \mathcal{H}^+(U).$$

**REMARKS 8.2.** 1. Of course, similar properties as in Section 7, locally in  $x$ , will be sufficient for (8.4).

2. If  $U$  is arcwise connected, then standard arguments show that, for every compact  $L$  in  $U$ , there exists  $K \geq 1$  such that (8.5) holds.

For a proof of Theorem 8.1, let us first recall that, defining

$$(8.6) \quad \widetilde{\mathcal{W}} := \left\{ \frac{u}{w} : u \in \mathcal{W} \right\},$$

we have a balayage space  $(X, \widetilde{\mathcal{W}})$  such that  $1 \in \widetilde{\mathcal{W}}$  and, for every positive function  $f \geq 0$  on  $X$ ,

$$(8.7) \quad \tilde{R}_f := \inf \{ \tilde{v} \in \widetilde{\mathcal{W}} : \tilde{v} \geq f \} = \frac{1}{w} R_{fw}.$$

In particular, for all  $x \in X$  and  $A \subset X$ , the reduced measure  $\tilde{\varepsilon}_x^A$  with respect to  $(X, \widetilde{\mathcal{W}})$  is

$$(8.8) \quad \tilde{\varepsilon}_x^A = \frac{w}{w(x)} \varepsilon_x^A.$$

Therefore a function  $h$  is harmonic on  $U$  with respect to  $(X, \mathcal{W})$  if and only if the function  $h/w$  is harmonic with respect to  $(X, \widetilde{\mathcal{W}})$ . Moreover, it is easily verified that  $\tilde{G}$  is a Green function for  $(X, \widetilde{\mathcal{W}})$ : a function  $p$  on  $X$  is a potential for  $(X, \mathcal{W})$  if and only if  $p/w$  is a potential for  $(X, \widetilde{\mathcal{W}})$ . Clearly  $(1/w)G\mu = \tilde{G}(w\mu)$  for every measure  $\mu$  on  $X$ .

Since  $\tilde{G} = \infty$  on the diagonal, (8.1) implies that  $\tilde{G}(y, x) \leq C\tilde{G}(x, y)$  and  $\tilde{\rho}(x, y) := \tilde{G}(x, y)^{-1} + \tilde{G}(y, x)^{-1}$ ,  $x, y \in X$ , defines a quasi-metric on  $X$  which is equivalent to  $\tilde{G}^{-1}$ . By [10, Proposition 14.5] (see also [7, Proposition 6.1]), there exists a metric  $d$  on  $X$  and  $\gamma > 0$  such that  $\tilde{\rho} \approx d^\gamma$ . So there exists  $c \geq 1$  with

$$(8.9) \quad c^{-1}d^{-\gamma} \leq \tilde{G} \leq cd^{-\gamma}.$$

For  $x \in X$  and  $r > 0$ , let

$$(8.10) \quad B(x, r) := \{y \in X : d(y, x) < r\}.$$

Clearly,

$$(8.11) \quad B(x, r) \supset \{ \tilde{G}(\cdot, x) > cr^{-\gamma} \} \supset V(x, c^{-1}r^\gamma).$$

Further, if  $V$  is a relatively compact neighborhood of  $x$ , then, by assumption,  $G(\cdot, x)/w$  is bounded on  $X \setminus V$ ; so there exists  $M > 0$  such that  $\{ \tilde{G}(\cdot, x) > M \} \subset V$ , and hence  $B(x, (Mc)^{-1/\gamma}) \subset V$ . Therefore  $d$  is a metric for the topology of  $X$ .

Thus Assumption 2.1 is satisfied for  $(X, \widetilde{\mathcal{W}})$  and  $\tilde{G}$  taking

$$\rho := d, \quad g(r) := r^{-\gamma}, \quad R_0 := \infty, \quad c_D := 2^\gamma, \quad \eta_0 := \alpha_0^\gamma.$$

*Proof of Theorem 8.1.* Let us fix  $x_0 \in U$ , and let  $V$  be a relatively compact open neighborhood of  $x_0$  in  $U$  (with corresponding  $R_1, c_J, \alpha$ ) having the properties stated in Theorem 8.1. We choose  $0 < R \leq R_1 \wedge (cR_1)^{1/\gamma}$  such that  $B(x_0, 2R)$  is a proper subset of  $V$ , and define  $a, \beta \in (0, 1)$  by

$$(8.12) \quad a := \inf w(V) \quad \text{and} \quad \beta := (a/c)^{2/\gamma}.$$

Let  $x \in B(x_0, R)$ ,  $0 < r < R$ ,  $B := B(x, r)$ ,  $\tilde{r} := c^{-1}r^\gamma$ . Then

$$(8.13) \quad B(x, \beta r) \subset V(x, \tilde{r}) \subset B \subset V.$$

Indeed, of course,  $B \subset V$ , and, by (8.11),  $V(x, \tilde{r}) \subset B$ . And  $B(x, \beta r)$  is contained in  $V(x, \tilde{r})$ , since, for every  $y \in B(x, \beta r) \subset B \subset V$ ,

$$G(y, x) \geq a^2 \tilde{G}(y, x) \geq a^2 c^{-1} d(x, y)^{-\gamma} > a^2 c^{-1} (\beta r)^{-\gamma} = 1/\tilde{r}.$$

Since  $w \leq 1$  and  $\tilde{r} < R_1$ , we see, by (8.7), (8.13), and (8.3), that

$$(8.14) \quad \tilde{R}_1^B = \frac{1}{w} R_w^B \geq a R_1^{V(x, \tilde{r})} \geq a C^{-1} G(\cdot, x) \tilde{r} \quad \text{on } V(x, \tilde{r})^c.$$

In particular, fixing  $z \in V \setminus B(x_0, 2R)$ , we have

$$\tilde{R}_1^B(z) \geq a^3 C^{-1} \tilde{G}(z, x) \tilde{r} \geq a^3 (c^2 C)^{-1} g(d(z, x))/g(r).$$

Given  $\varepsilon > 0$ , there is  $0 < r' < r$  such that  $B' := B(x, r')$  satisfies  $\tilde{R}_1^{B'}(z) + \varepsilon > \tilde{R}_1^B(z)$ , where (denoting the capacity of  $B$  with respect to  $\tilde{G}$  by  $\widetilde{\text{cap}} B$ )

$$\tilde{R}_1^{B'}(z) = \int \tilde{G}(z, y) d\tilde{\mu}_{B'}(y) \leq c g(d(z, x)/2) \|\tilde{\mu}_{B'}\| \leq c c_D g(d(z, x)) \widetilde{\text{cap}} B,$$

since  $d(z, x)/2 \leq d(z, x) - r < d(z, \cdot)$  on  $B$  (cf. the proof of [7, Proposition 1.10, b]). So

$$\widetilde{\text{cap}} B \geq a^3 (c^3 c_D C)^{-1} g(r)^{-1}.$$

Next, let  $y \in B(x, \alpha\beta r)$ . By (3.2), (8.4), and (8.13) (applied to  $\beta r$  and  $r$ ),

$$\varepsilon_x^{B(x, \alpha\beta r)^c} \leq \varepsilon_x^{V(x, \alpha\tilde{r})^c} \leq c_J \varepsilon_y^{V(x, \tilde{r})^c} \leq c_J \varepsilon_y^{B(x, r)^c} \quad \text{on } B(x, r)^c.$$

Hence, by (8.8),

$$\tilde{\varepsilon}_x^{B(x, \alpha\beta r)^c} = \frac{w}{w(x)} \varepsilon_x^{B(x, \alpha\beta r)^c} \leq a^{-1} c_J \frac{w}{w(y)} \varepsilon_y^{B(x, r)^c} = a^{-1} c_J \tilde{\varepsilon}_y^{B(x, r)^c} \quad \text{on } B(x, r)^c.$$

Thus, by Remark 6.2, we conclude that there exist constants  $\tilde{\alpha} \in (0, 1/4)$  and  $\tilde{K} \geq 1$  such that, for every function  $\tilde{h} \geq 0$  which is harmonic on  $U$  with respect to  $(X, \widetilde{\mathcal{W}})$ ,

$$\sup \tilde{h}(\overline{B}(x_0, \tilde{\alpha}R)) \leq \tilde{K} \inf \tilde{h}(\overline{B}(x_0, \tilde{\alpha}R)).$$

Finally, if  $h \in \mathcal{H}^+(U)$ , then  $h/w$  is harmonic on  $U$  with respect to  $(X, \widetilde{\mathcal{W}})$ , and thus

$$\sup h(\overline{B}(x_0, \tilde{\alpha}R)) \leq a^{-1} \tilde{K} \inf h(\overline{B}(x_0, \tilde{\alpha}R)).$$

Of course, we obtain as well that  $\tilde{\mathcal{H}}^+(U) = \mathcal{H}^+(U)$ . □

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