p-PARTS OF CHARACTER DEGREES

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ABSTRACT. We show that if p is an odd prime and G is a finite group satisfying the condition that p^2 divides the degree of no irreducible character of G, then $|G : \mathbf{O}_p(G)|_p \leq p^4$ where $\mathbf{O}_p(G)$ is the largest normal p-subgroup of G, and if P is a Sylow p-subgroup of G, then P'' is subnormal in G. Our investigations suggest that if p^a is the largest power of p dividing the irreducible characters of G, then $|G : \mathbf{O}_p(G)|_p$ is bounded by $p^{f(a)}$ where $f(a)$ is a function in a and $P^{(a+1)}$ is subnormal in G.

1. INTRODUCTION

Throughout this paper, G will be a finite group and p will be a prime. Let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G. The celebrated Itô-Michler theorem says that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if G has a normal abelian Sylow p-subgroup. Recently in [16], the first and second authors along with Tom Wolf proved that if G is a finite solvable group and every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$, then $|G : \mathbf{F}(G)|_p \leq p^2$ where $\mathbf{F}(G)$ is the Fitting subgroup of G and n_p is the p-part of n. That is, n_p is the largest power of p that divides the integer n. An immediate consequence of that result is that P' is subnormal in G, where P is a Sylow p-subgroup of G. In fact, the authors of that paper proved the slightly stronger result that $P\mathbf{F}(G)/\mathbf{F}(G)$ is an elementary abelian p-group. The authors then asked if there is any version of their result for nonsolvable groups.

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For $p = 2$, the first author had considered a stronger question in [15] where he showed that if 4 divides no degree of a nonsolvable group G, then $G \cong \mathrm{Alt}_7 \times R$ for some solvable group R satisfying 2 divides no degree in $\mathrm{cd}(R)$. Using this result, it is proved in [16] that if every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_2 \leq 2$, then $|G : \mathbf{F}(G)|_2 \leq 2^3$ and P'' is subnormal in G where P is a Sylow 2-subgroup of G. The simple group Alt_7 shows that this bound is best possible.

In this paper, we consider the situation where p is an odd prime and G is not necessarily solvable. In particular, we prove the following theorem.

Theorem 1.1. *Let G be a group, and let p be an odd prime. If every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$, then $|G : \mathbf{O}_p(G)|_p \leq p^4$. In addition, if P is a Sylow p-subgroup of G, then P'' is subnormal in G.*

Note that $|\mathbf{F}(G)|_p = |\mathbf{O}_p(G)|$, so $|G : \mathbf{O}_p(G)|_p = |G : \mathbf{F}(G)|_p$. Since p-groups of order p^4 have derived length at most 2, this shows that if p is an odd prime, P is a Sylow p-subgroup of G and every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$, then P'' is subnormal in G. This raises the following question. If every character $\chi \in \mathrm{Irr}\, G$ satisfies $\chi(1)_p \leq p^a$, then is it true that $P^{(a+1)}$ is subnormal in G?

Restricting our attention to p-solvable groups, we obtain the following stronger result. We write $\mathrm{sol}(G)$ for the solvable radical of G, this is the subgroup generated by all normal solvable subgroups of G. Note that for a finite group, it is the largest solvable normal subgroup in G.

Theorem 1.2. *Let p be an odd prime and let G be a p-solvable group. If every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$, then $|G/\mathrm{sol}(G)|_p \leq p$ and $|G/\mathbf{O}_p(G)|_p \leq p^3$.*

When G is p-solvable, we also obtain a stronger result regarding the Sylow p-subgroups of G. Note that the condition that $P/\mathbf{O}_p(G)$ is abelian is equivalent to the condition that P' is subnormal in G. We note that in [10], Isaacs has studied the Sylow p-subgroups of a solvable group G satisfying the condition that every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p^a$ showing that if P is a Sylow p-subgroup of G, then P has derived length at most $2a + 1$. Also, using the techniques in Theorem B of [25], under this same hypothesis that G is a solvable group and every character $\chi \in \mathrm{Irr}\, G$ satisfies $\chi(1)_p \leq p^a$ and additionally assuming that $p \geq 5$ one can show that $|G : O_p(G)|_p \leq p^{2.5a}$. To see that a similar result probably holds for nonsolvable groups, note that if S is a nonabelian simple group and $p \geq 5$, then there exists a character $\chi \in \mathrm{Irr}(S)$ such that $\chi(1)_p = |S|_p$ (this can be obtained from Corollary 2 of [9]).

To address the question of whether the Sylow p-subgroups of $G/\mathbf{F}(G)$ are abelian, we turn to p-Brauer characters. As usual, let $\mathrm{IBr}_p(G)$ be the set of irreducible p-Brauer characters of G. There are some significant differences between ordinary character degrees and p-Brauer character degrees; for example, the Brauer degrees need not divide the order of the group, and a Brauer character version of the Itô-Michler theorem only holds for the given prime p. That is, if p divides no p-Brauer degree of a finite group G, then G has a normal Sylow p-subgroup (see [19, Theorem 5.5]).

In 1990, Leisering [14] considered a similar variation of this result by proving that if G is a finite solvable group with $\mathbf{O}_p(G) = 1$ and p^2 does not divide $\beta(1)$ for every Brauer character $\beta \in \mathrm{IBr}_p(G)$, then G has an elementary abelian Sylow p-subgroup and every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$. Now, by Theorem 1 in [16], we have $|G : \mathbf{F}(G)|_p \leq p^2$ when G is solvable and every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$. However, since $\mathbf{O}_p(G) = 1$ and $\mathbf{F}(G)$ is a p'-group, we deduce that $|G|_p = |G : \mathbf{F}(G)|_p \leq p^2$ in this situation.

From [24, Corollary 2.6], we know that if G is a p-solvable group with $\mathbf{O}_p(G) = 1$ and p^2 dividing no Brauer character degree, then G has elementary abelian Sylow p-subgroups, so a similar result to Leisering's holds for p-solvable groups. If G is a p-solvable group and $\varphi \in \mathrm{IBr}(G)$, then the Fong-Swan Theorem [21, Theorem 10.1] asserts that $\varphi = \chi^\circ$, the restriction to G°, the set of p-regular elements of G, of some character $\chi \in \mathrm{Irr}(G)$. Hence, if G is p-solvable and every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$, then every Brauer character $\varphi \in \mathrm{IBr}(G)$ satisfies $\varphi(1)_p \leq p$. Combining this with Wang's result [24], we deduce that if G is a p-solvable group that has the property that every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$ and P is a Sylow p-subgroup of G, then $P/\mathbf{O}_p(G)$ is elementary abelian.

Turning to groups that are not p-solvable, Leisering's conclusion does not hold. For example, if one takes $G = \mathrm{M}_{22}$ and $p = 2$, then $\mathbf{O}_2(G) = 1$ and every Brauer character $\beta \in \mathrm{IBr}_p(G)$ satisfies $\beta(1)_2 \leq 2$, but $|G|_2 = 2^7$ and there exists a character $\chi \in \mathrm{Irr}(G)$ with $\chi(1)_2 = 2^3$. However, if we make the additional assumption that the group has an abelian Sylow p-subgroup, then we can make use of a recent result of Kessar and Malle in [13] on Brauer's height zero conjecture to prove the following.

Theorem 1.3. *Let p be a prime and let G be a finite group with $\mathbf{O}_p(G) = 1$. If G has an abelian Sylow p-subgroup and every Brauer character $\varphi \in \mathrm{IBr}_p(G)$ satisfies $\varphi(1)_p \leq p$, then every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$.*

As a corollary, we deduce that for a p-solvable group G, the condition that every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$ is equivalent to the condition that every Brauer character $\varphi \in \mathrm{IBr}(G)$ satisfies $\varphi(1)_p \leq p$.

Corollary 1.4. *Let p be a prime and G be a finite p-solvable group. Then every Brauer character $\varphi \in \mathrm{IBr}_p(G)$ satisfies $\varphi(1)_p \leq p$ if and only if every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$.*

Obviously, for arbitrary finite groups, we do not have such an equivalence. We can obtain the following conclusion from knowing that p^2 does divide any Brauer character degree.

Theorem 1.5. *Let p be a prime and G be a finite group with $\mathbf{O}_p(G) = 1$. If every Brauer character $\beta \in \mathrm{IBr}_p(G)$ satisfies $\beta(1)_p \leq p$, then the following hold.*

(1) *If $p = 2$, then $|G|_2 \leq 2^9$.*
(2) *if $p \geq 5$ or if $p = 3$ and Alt_7 is not involved in G, then $|G|_p \leq p^4$.*
(3) *If $p = 3$ and Alt_7 is involved in G, then $|G|_3 \leq 3^5$.*

It seems to us that the bounds in Theorem 1.5 are probably not best possible. We conjecture that the correct bounds in Theorem 1.5 should be 2^7 in (1), p^2 in (2), and 3^3 in (3). Notice every Brauer character $\varphi \in \mathrm{IBr}(\mathrm{M}_{22})$ satisfies $\varphi(1)_2 \leq 2$ and $|\mathrm{M}_{22}|_2 = 2^7$. Returning to the case of the ordinary character degrees, we know from [16] that for every odd prime p there is a solvable group H so that every character $\chi \in \mathrm{Irr}(H)$ satisfies $\chi(1)_p \leq p$ and $|H : \mathbf{O}_p(H)|_p = p^2$, and there is a simple group S so that p does not divide $|S|$. Taking $G = H \times S$, we see that G is a p-solvable group where every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$ and $|G : \mathbf{O}_p(G)|_p = p^2$. Since we do not have any examples where the p-part of the index of $\mathbf{O}_p(G)$ is larger, we believe in both Theorem 1.1 and Theorem 1.2 that the correct bound is p^2, however, it appears to be a daunting task to prove this. Note that the only non p-solvable example where $|G : \mathbf{O}_p(G)|_p > p$ is $G = \mathrm{Alt}_7$ and $p = 3$ which has $|G|_p = |G : \mathbf{O}_3(G)|_3 = 9$. Also, we would like to know whether the hypothesis that G is p-solvable is necessary for G satisfying the property that every character $\chi \in \mathrm{Irr}(G)$ satisfies $\chi(1)_p \leq p$ to imply that the Sylow p-subgroups of $G/\mathbf{F}(G)$ are elementary abelian.

2. p-solvable groups

Let $\mathrm{cd}(G)$ be the set of all (ordinary) degrees of G, that is, $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. In this section, we bound $|G : \mathbf{F}(G)|_p$ when G is p-solvable and $a_p \leq p$ for all $a \in \mathrm{cd}(G)$ for some odd prime p. Consider a finite set Ω. If $H < \mathrm{Sym}(\Omega)$ and $\Omega_1, \dots, \Omega_m$ are subsets of Ω, we let $\mathrm{Stab}_H(\Omega_1, \dots, \Omega_m)$ denote the subgroup consisting of all elements in H that fixes each set Ω_i, $1 \le i \le m$, set-wise.

Lemma 2.1. *Let Ω be a finite set, and let $H < \mathrm{Sym}(\Omega)$ be a subgroup of order divisible by p.*

(i) *There exist disjoint (possibly empty) subsets Ω_1, Ω_2 of Ω such that p divides $[H : \mathrm{Stab}_H(\Omega_1, \Omega_2)]$.*

(ii) *If H is solvable, then there exist disjoint (possibly empty) subsets Ω_1, Ω_2 of Ω such that $\mathrm{Stab}_H(\Omega_1, \Omega_2)$ is a 2-group.*

Proof. Conclusion (i) is a partial case of [3, Lemma 8], and Conclusion (ii) is [5, Corollary 4]. □

We next consider a transitive subgroup of $\mathrm{Sym}(\Omega)$.

Lemma 2.2. *Let Ω be a finite set and let $H < \mathrm{Sym}(\Omega)$ be a transitive subgroup of order divisible by $p > 2$. Suppose that $H = \mathbf{O}^{p'}(H)$ is p-solvable.*

(i) *Suppose that H is primitive, and either H is non-solvable or $p > 3$. Then H has a regular orbit on the set of all subsets of Ω.*

(ii) *Suppose that p^2 divides $|H|$. Then there exist s disjoint (possibly empty) subsets $\Omega_1, \dots, \Omega_s$ of Ω such that p^2 divides*

$$[H : \mathrm{Stab}_H(\Omega_1, \dots, \Omega_s)],$$

where $s = 3$ of $p > 3$ and $s = 4$ if $p = 3$.

Proof. (i) The assumption implies that $H \not\geq \mathrm{Alt}(\Omega)$. Now the main result of [22] lists all the 46 cases where H fails to have a regular orbit on the subsets of Ω. Among them, there are 35 cases where the group H is non-solvable, but either H is not p-solvable, or $H \neq \mathbf{O}^{p'}(H)$. Next, in 8 of the 11 solvable cases, $H \neq \mathbf{O}^{p'}(H)$. In the remaining 3 cases, we have that $p = 3$ (and $(|\Omega|, H)$ is one of the following: $(4, \mathrm{Alt}_4)$, $(8, \mathrm{A\Gamma L}_1(8))$, $(9, \mathrm{ASL}_2(3))$). Hence, conclusion (i) follows.

(ii) First we consider the case H is primitive. If $p > 3$ or H is non-solvable, we are done by conclusion (i) (taking $s = 1$), whereas if $p = 3$ and H is solvable then we are done by Lemma 2.1 (ii) (taking $s = 2$).

We may now assume that H is imprimitive, and so, permutes transitively k disjoint subsets $\Delta_1, \dots, \Delta_k$ of Ω, with $|\Delta_i| = |\Omega|/k$ and $1 < k < |\Omega|$. Let

$$B := \mathrm{Stab}_H(\Delta_1, \dots, \Delta_k)$$

so that H/B is a transitive subgroup of Sym_k. Choosing the smallest k possible, we have that H/B is a primitive subgroup of Sym_k. Certainly

H/B is p-solvable and $H/B = \mathbf{O}^{p'}(H/B)$. Since $k > 1$, the latter implies that p divides $|H/B|$. Again applying conclusion (i) and Lemma 2.1 (ii) to H/B, we see that there are t disjoint subsets $\Omega_1, \dots, \Omega_t$ of Ω, each being a union of some Δ_j's, so that

$$p \nmid |\,\mathrm{Stab}_{H/B}(\Omega_1, \dots, \Omega_t)|. \tag{2.1}$$

Here we can take $t = 1$ if $p > 3$ and $t \leq 2$ if $p = 3$. We may furthermore assume that $\cup_{i=1}^{t}\Omega_i \neq \Omega$. (Indeed, this is obvious if $t = 1$ as $p \mid |H/B|$. If $t = 2$ and $\Omega_1 \cup \Omega_2 = \Omega$, then $\mathrm{Stab}_{H/B}(\Omega_1) = \mathrm{Stab}_{H/B}(\Omega_1, \dots, \Omega_t)$, and so, we can again take $t = 1$.)

Now, if $p^2 \mid |H/B|$, then we are done by (2.1), taking $s = t$. We will now assume that p^2 does not divide $|H/B|$ (and so $p \mid |B|$), and consider a Sylow p-subgroup Q of B. Then there must be some Δ_j such that Q does not act trivially on Δ_j. Since $B \lhd H$ and H permutes $\Delta_1, \dots, \Delta_k$ transitively, replacing Q by an H-conjugate, we may assume that

$$\Delta_j \subseteq \Omega \setminus \cup_{i=1}^{t}\Omega_i.$$

Thus, the action of B on Δ_j induces a subgroup $\bar{B} \leq \mathrm{Sym}(\Delta_j)$ of order divisible by p. By Lemma 2.1 (i), there are disjoint subsets Ω_{t+1} and Ω_{t+2} of Δ_j such that $p \mid [\bar{B} : \mathrm{Stab}_{\bar{B}}(\Omega_{t+1}, \dots, \Omega_{t+2})]$, whence

$$p|[B : \mathrm{Stab}_B(\Omega_{t+1}, \dots, \Omega_{t+2})]. \tag{2.2}$$

It remains to show that $p^2 \mid |H : J|$, where $J := \mathrm{Stab}_H(\Omega_1, \dots, \Omega_{t+2})$. Assume the contrary. Note that $K/B = \mathrm{Stab}_{H/B}(\Omega_1, \dots, \Omega_t)$ for $K := \mathrm{Stab}_H(\Omega_1, \dots, \Omega_t)$. As $p \mid |H/B|$ we have by (2.1) that $p \mid |H : K|$. Since $K \geq J$, it follows that p does not divide $|K : J|$, and so, p does not divide $|B : J \cap B|$. But this contradicts (2.2), as $J \cap B = \mathrm{Stab}_B(\Omega_{t+1}, \dots, \Omega_{t+2})$. □

The first two statements of the following lemma are well known:

Lemma 2.3. *Let S be a finite non-abelian simple group that admits an automorphism σ of prime order p where p does not divide $|S|$.*

(i) *Then S is a simple group of Lie type defined over a field $\mathbb{F}_q$ with $q = r^f$, $r \neq p$ a prime, and $p \mid f$.*
(ii) *If $P \in \mathrm{Syl}_p(\mathrm{Out}(S))$ then $P \leq \mathbf{Z}(\mathrm{Out}(S))$ and P is cyclic.*
(iii) *There exists $\alpha \in \mathrm{Irr}(S)$ such that p does not divide $|I_{\mathrm{Aut}(S)}(\alpha)|$. In fact, P has at least two regular orbits of such characters.*

Proof. (i) The condition on σ implies that S is a simple group of Lie type defined over $\mathbb{F}_q$ with $q = r^f$ and $r \neq p$. Now, by [8, Theorem 2.5.12], we have that $\mathrm{Out}(S) = \mathrm{Outdiag}(S) \rtimes A(S)$, where $\mathrm{Outdiag}(S)$ is either cyclic or a Klein four-group, all prime divisors of which divide $|S|$, and $A(S) = \Phi_S\Gamma_S$ is abelian. Here, Φ_S is a cyclic group of field automorphisms, and $|A(S):\Phi_S|$ divides $|S|$. It follows that $p \mid |\Phi_S|$ and so $p \mid f$.

(ii) We have shown in conclusion (i) that we can take $P \leq \Phi_S$; in particular, $[P, A(S)] = 1$ and P is cyclic. If $\mathrm{Outdiag}(S)$ is a Klein four-group, then it is centralized by Φ_S by [8, Theorem 2.5.12(h)]. Assume that $\mathrm{Outdiag}(S) \cong C_d$ is cyclic but not centralized by P; in particular, $p \mid \varphi(d)$ (the Euler function of d). Since p does not divide d, it follows that d is divisible by a prime s with $p \mid (s-1)$. As $p > 2$ by the Feit-Thompson theorem, we have that $s \geq 5$. Using the description of $\mathrm{Outdiag}(S)$ in [8, Theorem 2.5.12], we see that $S \cong PSL_n^{\epsilon}(q)$ with $\epsilon = +$ for $S \cong PSL_n(q)$ and $\epsilon = -$ for $S = PSU_n(q)$, and $d = \gcd(n, q-\epsilon)$. Clearly, $n \geq s > p$ as $s \mid d$. As p does not divide q, we have that $p \mid ((\epsilon q)^{p-1} - 1)$ and so $p||S|$, a contradiction. This proves conclusion (ii).

(iii) Note that the cyclic group P acts coprimely and faithfully on S. Hence, by [20, Proposition 2.6], P has at least two regular orbits on $\mathrm{Irr}(S)$. Let α belong to such an orbit, and assume that $p \mid |I_{\mathrm{Aut}(S)}(\alpha)|$. Then α is fixed by some automorphism $\tau \in \mathrm{Out}(S)$ of order p. As $P \lhd \mathrm{Out}(S)$, we have that $\tau \in P$ and $\alpha^{\tau} = \alpha$, a contradiction. (Explicit constructions of such α can be found in [18, §4] and [23, §6].) □

We now consider the case where G is p-solvable, satisfies $a_p \leq p$ for all $a \in \mathrm{cd}(G)$, and has a nonabelian minimal normal subgroup N.

Proposition 2.4. *Let $p > 2$ and let G be a p-solvable group and satisfy $a_p \leq p$ for all $a \in \mathrm{cd}(G)$. Suppose that $G = \mathbf{O}^{p'}(G)$ and $\mathbf{O}_p(G) = 1$. Suppose also that G admits a minimal normal subgroup $N = T_1 \times \ldots \times T_n \cong T^n$, where T is a non-abelian simple group. Then $|G|_p \leq p$.*

Proof. (i) Assume the contrary: $|G|_p \geq p^2$. Let K be the kernel of the transitive action of G on $\Omega := \{T_1, \ldots, T_n\}$. Then $G/K \leq \mathrm{Sym}_n$ is p-solvable and $G/K = \mathbf{O}^{p'}(G/K)$. Denote $p^a := |G/K|_p$ and $p^b := |K/C|_p = |K/CN|_p$ for $C := \mathbf{C}_G(N)$. Note that $a + b \geq 1$. Indeed, otherwise p does not divide $|G/C|$, and so, $N \leq G = C$ as $G = \mathbf{O}^{p'}(G)$, contradicting the assumption that N is non-abelian.

By Lemma 2.1 (i) and Lemma 2.2 (ii) applied to G/K, we can find $s \geq 1$ disjoint, non-empty subsets $\Omega_1, \ldots, \Omega_s$ of Ω such that

$$[G : \mathrm{Stab}_G(\Omega_1, \ldots, \Omega_s)]_p \geq p^{\min(a,2)}. \tag{2.3}$$

Here, we can take $s \leq 2$ if $a = 1$, $s \leq 3$ if $a \geq 2$ and $p > 3$, and $s \leq 4$ if $a \geq 2$ and $p = 3$. Note that $a \geq 1$ if $n > 1$ as $G/K = \mathbf{O}^{p'}(G/K)$. So in the case $a = 0$, we have that $\Omega = \{T_1\}$ and can then take $s = 1$ and $\Omega_1 = \Omega$. We may also assume that $T_1 \in \Omega_1$.

Recall that T is a non-abelian simple p'-group. If $p > 3$, then we can find at least 3 nontrivial characters $\alpha_1, \ldots, \alpha_3 \in \mathrm{Irr}(T)$ of pairwise distinct degrees. If $p = 3$, then $T \cong {}^2B_2(q)$ for some $q = 2^{2f+1}$ (this is a well-known result proved in unpublished work of J. G. Thompson) and so we can find at least 4 nontrivial characters $\alpha_1, \ldots, \alpha_4 \in \mathrm{Irr}(T)$ of pairwise distinct degrees.

(ii) Suppose that $a \geq 1$. Consider

$$\theta = \beta_1 \times \beta_2 \times \ldots \times \beta_n \in \mathrm{Irr}(N),$$

where $\beta_j \in \mathrm{Irr}(T_j)$, $\beta_j(1) = \alpha_i(1)$ if $T_j \in \Omega_i$ for some $i \leq s$, and $\beta_j = 1_{T_j}$ if $T_j \notin \cup_{i=1}^s \Omega_i$. Then $I_G(\theta) \leq \mathrm{Stab}_G(\Omega_1, \ldots, \Omega_s)$ and so (2.3) implies that p^a divides $|G : I_G(\theta)|$. In particular, if $a \geq 2$ then it follows by Clifford's theorem that any $\chi \in \mathrm{Irr}(G \mid \theta)$ has degree divisible by p^2, a contradiction. We have shown that $a \leq 1$.

(iii) Consider the case $b \geq 1$. Note that $K/CN \hookrightarrow \mathrm{Out}(T)^n$. Hence the condition $b \geq 1$ implies that $p \mid |\mathrm{Out}(T)|$, but p does not divide $|T|$. By Lemma 2.3 (iii), one can find a character $\alpha \in \mathrm{Irr}(T)$ such that p does not divide $|\mathrm{Stab}_{\mathrm{Out}(T)}(\alpha)|$. In the notation of (i), we can set $\alpha_1 = \alpha$. Using the transitivity of G on Ω and that $K \lhd G$, we may assume that the action of K on T_1 induces a subgroup X of $\mathrm{Out}(T_1)$ of order divisible by p.

Recall we now have $a \leq 1$, and so, $s \leq 2$ by our construction of Ω_i. Consider

$$\varphi = \beta_1 \times \beta_2 \times \ldots \times \beta_n \in \mathrm{Irr}(N),$$

where $\beta_j \in \mathrm{Irr}(T_j)$, $\beta_1 = \alpha_1$, $\beta_j(1) = \alpha_2(1)$ if $T_j \in \Omega_1 \smallsetminus \{T_1\}$, $\beta_j(1) = \alpha_3(1)$ if $s > 1$ and $T_j \in \Omega_2$, and $\beta_j = 1_{T_j}$ if $T_j \notin \cup_{i=1}^s \Omega_i$. Then

$$J := I_G(\varphi) \leq H := \mathrm{Stab}_G(\Omega_1, \ldots, \Omega_s).$$

We claim that

$$p^c := [G : J]_p \geq p^{\min(a+b,2)}. \tag{2.4}$$

Assume the contrary: $c < \min(a + b, 2)$. Note that $[G : H]_p \geq p^a$ by (2.3). Hence, if $a = 1$ then we obtain that p does not divide $|H : J|$ and so, as p does not divide $|H/K|$, J contains a Sylow p-subgroup Q of K. By the construction of θ, J normalizes T_1, and so does Q. As $Q \in \mathrm{Syl}_p(K)$, the action of Q on T_1 must induce a (non-trivial) Sylow p-subgroup of X; also, Q fixes $\alpha_1 = \alpha$. Thus, p divides $|\mathrm{Stab}_{\mathrm{Out}(T)}(\alpha)|$, contradicting the choice of α. So we must have that $a = 0$, whence $n = 1$ (as mentioned above) and $G = K$. As $c < b$ and $G/CN \hookrightarrow \mathrm{Out}(T_1)$, we again see that α is fixed by some outer automorphism of T of order p, again a contradiction.

(iv) Now if $a+b \geq 2$, then (2.4) implies that $p^2 \mid \chi(1)$ for any $\chi \in \mathrm{Irr}(G \mid \varphi)$, a contradiction. Hence, we have that $a+b=1$. Now, as shown in (ii) and (iii), there is a character $\gamma \in \mathrm{Irr}(N)$ such that $p \mid |G : I_G(\gamma)|$ (namely, $\gamma = \theta$ if $a=1$ and $\gamma = \varphi$ if $b=1$). Since $|G/C|_p = p$ and $|G|_p \geq p^2$, we have that $p \mid |C|$. Also, $\mathbf{O}_p(C) \leq \mathbf{O}_p(G) = 1$. It follows by the Itô-Michler theorem that $p \mid \delta(1)$ for some $\delta \in \mathrm{Irr}(C)$. Clearly, $G \rhd NC \cong N \times C$, $\gamma \times \delta \in \mathrm{Irr}(NC)$, and $I_G(\gamma \times \delta) \leq I_G(\gamma)$ has index divisible by p in G. By the Clifford theorem, p^2 divides the degree of any $\chi \in \mathrm{Irr}(G \mid \gamma \times \delta)$, a contradiction. □

Denote by $\mathbf{E}(G)$ the subgroup of G generated by all components of G, that is, quasisimple subnormal subgroups of G. Often, $\mathbf{E}(G)$ is called the *layer* of G. Then the *generalized Fitting subgroup* of G is the subgroup $\mathbf{F}^*(G) = \mathbf{E}(G)\mathbf{F}(G) \trianglelefteq G$. It is well known that $\mathbf{C}_G(\mathbf{F}^*(G)) \leq \mathbf{F}^*(G)$. Note that this next result includes Theorem 1.2.

Theorem 2.5. *Let p be an odd prime, and let G be any p-solvable finite group. If $a_p \leq p$ for all $a \in \mathrm{cd}(G)$, then the following statements hold.*

(i) $|G/\mathrm{sol}(G)|_p \leq p$ *and* $|G/\mathbf{F}(\mathrm{sol}(G))|_p \leq p^3$.

(ii) $|G/\mathbf{F}(G)|_p \leq p^3$.

(iii) *Either* $|G/\mathbf{F}(G)|_p \leq p$, *or* $\mathbf{F}^*(\mathbf{O}^{p'}(G)) = \mathbf{F}(\mathbf{O}^{p'}(G))$.

Proof. Let $L := \mathbf{O}^{p'}(G)$. Then $|L|_p = |G|_p$, $\mathrm{sol}(L) \lhd \mathrm{sol}(G)$, $\mathbf{F}(L) \leq \mathbf{F}(G)$, and $\mathbf{F}(\mathrm{sol}(L)) \leq \mathbf{F}(\mathrm{sol}(G))$. Furthermore, $a_p \leq p$ for all $a \in \mathrm{cd}(G)$ if and only if $a_p \leq p$ for all $a \in \mathrm{cd}(L)$. Hence, we may replace G by L, and assume that $G = \mathbf{O}^{p'}(G)$.

(i) Let $R := \mathrm{sol}(G)$. Then G/R is p-solvable, $G/R = \mathbf{O}^{p'}(G/R)$, and $a_p \leq p$ for all $a \in \mathrm{cd}(G/R)$. Furthermore, $\mathrm{sol}(G/R) = 1$; in particular, $\mathbf{O}_p(G/R) = 1$. We are done if $G = R$. So let $N/R \neq 1$ be a minimal normal subgroup of G/R. As $\mathrm{sol}(G/R) = 1$, we have that N/R is non-abelian. Now we can apply Proposition 2.4 and get that $|G/R|_p \leq p$. Obviously, R is solvable and $a_p \leq p$ for all $a \in \mathrm{cd}(R)$. Hence, $|R/\mathbf{F}(R)|_p \leq p^2$ by the main result of [16]. It follows that $|G/\mathbf{F}(R)|_p \leq p^3$.

(ii) This follows from (i) as $\mathbf{F}(\mathrm{sol}(G)) \leq \mathbf{F}(G)$.

(iii) Assume that $\mathbf{E}(G) \not\leq \mathbf{F}(G)$. Let $Z := \mathbf{Z}(\mathbf{E}(G))$ and let M/Z be a (non-abelian) minimal normal subgroup of G/Z. Also, let $L := M^{(\infty)} \lhd G$, so that $M = LZ$, L is perfect, $L \cap Z = \mathbf{Z}(L)$, and $L/\mathbf{Z}(L) \cong M/Z$ is a non-abelian minimal normal subgroup of $G/\mathbf{Z}(L)$. Set $P/\mathbf{Z}(L) = \mathbf{O}_p(G/\mathbf{Z}(L))$. Then $P \cap L = \mathbf{Z}(L)$, $[P, L] \leq \mathbf{Z}(L)$, and so by the three-subgroup lemma,

$$[P, L] = [P, [L, L]] \leq [[P, L], L] = 1.$$

In particular, $[P, \mathbf{Z}(L)] = 1$. Writing $\mathbf{Z}(L) = \mathbf{O}_{p'}(\mathbf{Z}(L)) \times \mathbf{O}_p(\mathbf{Z}(L))$, we now have that $P = \mathbf{O}_{p'}(\mathbf{Z}(L))Q = \mathbf{O}_{p'}(\mathbf{Z}(L)) \times Q$ for $Q \in \mathrm{Syl}_p(P)$, and so $Q \leq \mathbf{O}_p(G)$.

Now, we can embed $L/\mathbf{Z}(L)$ in G/P as a non-abelian minimal normal subgroup. As $\mathbf{O}_p(G/P) = 1$, $G/P = \mathbf{O}^{p'}(G/P)$, and G/P is p-solvable and $a_p \leq p$ for all $a \in \mathrm{cd}(G/P)$, we have $|G/P|_p \leq p$. It follows that $|G/\mathbf{F}(G)|_p \leq p$. □

3. Non p-solvable groups

We continue to assume that $a_p \leq p$ for all $a \in \mathrm{cd}(G)$, but we now drop the assumption that G is p-solvable. All degrees in this section are ordinary (complex) character degrees. We first begin with the following result due to Gagola [7].

Lemma 3.1. *Let S be a nonabelian simple group and let p be a prime diving $|S|$. Then $|S|_p > |\mathrm{Out}(S)|_p$.*

Proof. If S is a sporadic simple group, an alternating group Alt_n with $n \geq 7$ or the Tits group, then $|\mathrm{Out}(S)|$ divides 2. Since S is nonabelian simple, we have $|S|_2 \geq 4$. Therefore, the inequality $|S|_p > |\mathrm{Out}(S)|_p$ is trivially true for odd prime p and it also holds for $p = 2$. Notice that $\mathrm{Alt}_6 \cong \mathrm{PSL}_2(9)$ and $\mathrm{Alt}_5 \cong \mathrm{PSL}_2(5)$. Finally, if S is a finite simple group of Lie type, then the result follows from Equation 3.3 in [7]. □

The next result is also due to Gagola [7] which can be proved directly.

Lemma 3.2. *Let S be a nonabelian simple group and let G be an almost simple group with $S \trianglelefteq G \leq \mathrm{Aut}(S)$. Let p be an odd prime. Suppose that p divides $|S|$ and $a_p \leq p$ for all $a \in \mathrm{cd}(G)$. Then either $S \cong \mathrm{Alt}_7$, $p = 3$, and $|G|_p = p^2$; or $|G|_p = p$. In both cases, we have $|G|_p = |S|_p \leq p^2$.*

Proof. If $S \cong \mathrm{Alt}_7$ and $p = 3$, then the result follows easily by using [4]. Therefore, we can assume that $(S, p) \neq (\mathrm{Alt}_7, 3)$. Clearly, $a_p \leq p$ for all $a \in \mathrm{cd}(S)$. Since $p \mid |S|$, [7, Lemma 1.2] yields that $|\mathrm{Aut}(S)|_p < a_p^2 \leq p^2$ for some degree $a \in \mathrm{cd}(S)$, and thus, $|G|_p \leq |\mathrm{Aut}(S)|_p < p^2$ which forces $|G|_p = p$. Since $|S|$ divides $|G|$ and $p \mid |S|$, we have $|G|_p = |S|_p = p$ as required. □

For a fixed prime p, we denote by $\mathrm{sol}_p(G)$ the *p-solvable radical* of G which is the unique largest normal p-solvable subgroup of G. Notice that by the Feit-Thompson Odd-Order theorem, we have $\mathrm{sol}_2(G) = \mathrm{sol}(G)$.

Lemma 3.3. *Let G be a finite group, p be a prime, and L a normal subgroup of G. Then the following hold:*

(*i*) $\mathbf{F}(L) = L \cap \mathbf{F}(G)$.
(*ii*) *If p does not divide $|G : L|$, then $|G : \mathbf{F}(G)|_p = |L : \mathbf{F}(L)|_p$.*
(*iii*) $\mathrm{sol}_p(L) = L \cap \mathrm{sol}_p(G)$ *and* $\mathrm{sol}_p(G/\mathrm{sol}_p(G))$ *is trivial.*

The proof of this lemma is nearly trivial, so we omit it.

We are curious if one could mimic the proof of Proposition 2.4 to obtain a result in this next lemma when p does not divide $|N/M|$. In particular, we wonder if one could prove that $|G/C|_p \leq p$ in this situation. Unfortunately, at this times we have not been able to prove anything like that.

Lemma 3.4. *Let p be an odd prime, and let G be a finite group with $\mathbf{O}^{p'}(G) = G$ and $a_p \leq p$ for all $a \in \mathrm{cd}(G)$. Suppose N/M is a nonabelian chief factor of G with p dividing $|N/M|$, and set $C/M = \mathbf{C}_{G/M}(N/M)$. Then*

(1) *N/M is nonabelian simple.*
(2) *$G/M \cong C/M \times N/M$.*
(3) *C/M is an abelian p-group.*

Proof. As N/M is nonabelian, $N/M \cong S^k$ for some nonabelian simple group S and some integer $k \geq 1$. As p divides $|N/M| = |S|^k$, it follows that p divides $|S|$, and thus, by the Itô-Michler Theorem ([19, Theorem 5.4]), $p \mid \theta(1)$ for some character $\theta \in \mathrm{Irr}(S)$. Since $\theta^k \in \mathrm{Irr}(N/M)$ and $\theta(1)^k$ is divisible by p^k, we see that p^k divides some degree in $\mathrm{cd}(G)$. This forces $k = 1$. So $N/M \cong S$ with S a nonabelian simple group. It follows that G/C is an almost simple group with socle S. By Lemma 3.2, $|G/C|_p = |S|_p$ and G/NC is a p'-group. Since $\mathbf{O}^{p'}(G) = G$, we deduce that $G = NC$, and thus, $G/M \cong C/M \times N/M$ as $C \cap N = M$. Now, if $p \mid \lambda(1)$ for some character $\lambda \in \mathrm{Irr}(C/M)$, then $\lambda \times \theta \in \mathrm{Irr}(G/M)$ whose degree is divisible by p^2, a contradiction. Hence, p divides no degree of C/M, and thus, C/M possesses a normal abelian Sylow p-subgroup, say K/M. Clearly, $K \trianglelefteq G$, and thus, $KN \trianglelefteq G$. But then G/KN is a p'-group, so $G = KN$ and $C/M = K/M$ is an abelian p-group as wanted. □

Studying the proof of Theorem D in [17] carefully, one can see that the authors only used the condition that 9 divides no degree of the group. So, we obtain the following.

Lemma 3.5. *Let G be a finite perfect group and suppose that $a_3 \leq 3$ for all $a \in \mathrm{cd}(G)$. Suppose that $G/M \cong \mathrm{Alt}_7$ for some solvable normal subgroup M of G. Then $G \cong \mathrm{Alt}_7$ or $3 \cdot \mathrm{Alt}_7$.*

Recall that a group G is a central product of two normal subgroups A and B, which we denote by $G = A \circ B$, if $G = AB$ and $[A, B] = 1$.

Corollary 3.6. *Let G be a finite group with $\mathbf{O}^{3'}(G) = G$. Suppose that 3^2 divides no degree of G and that Alt_7 is involved in G. Then $G \cong N \circ C$, where C is an abelian 3-group and $N \cong \mathrm{Alt}_7$ or $3 \cdot \mathrm{Alt}_7$.*

Proof. Let N be the last term of the derived series of G. Then G/N is solvable. Also, N is perfect, and Alt_7 is involved in N. By Lemma 3.5, we have $N \cong \mathrm{Alt}_7$ or $3 \cdot \mathrm{Alt}_7$. Let $M = \mathbf{Z}(N)$. Then $M \trianglelefteq G$ and $N/M \cong \mathrm{Alt}_7$ is a chief factor of G with p dividing $|N/M|$. Let $C/M = \mathbf{C}_{G/M}(N/M)$. By Lemma 3.4, $G/M = C/M \times N/M$ and C/M is an abelian 3-group. It follows that C is a normal 3-subgroup of G and $G/C \cong N/M \cong \mathrm{Alt}_7$. If $M = 1$, then the result follows. So, we assume that M is nontrivial. Then $N \cong 3 \cdot \mathrm{Alt}_7$ and $|M| = 3$. We see that $N \cap C = M = \mathbf{Z}(N)$. As $[N, N] = N$ and $[N, C] \leq \mathbf{Z}(N)$, we apply the Three Subgroups Lemma to obtain $[N, C] = 1$. We deduce that $G = N \circ C$ is a central product. It remains to show that C is abelian. Suppose that C is nonabelian. Then we can find a character $\lambda \in \mathrm{Irr}(C)$ with $\lambda(1) > 1$, hence $3 \mid \lambda(1)$ since C is a 3-group. As C/M is abelian, there exists a character ν with $1_M \neq \nu \in \mathrm{Irr}(M)$ such that $\lambda \in \mathrm{Irr}(C \mid \nu)$. Using [4], we can find a character $\varphi \in \mathrm{Irr}(N \mid \nu)$ with $3 \mid \varphi(1)$. Now, by the representation theory of central products (see Lemma 5.1 of [11]), $\chi = \phi \cdot \lambda \in \mathrm{Irr}(G)$ and $\chi(1) = \varphi(1)\lambda(1)$ is divisible by 9, which is a contradiction. □

We now consider groups that are not p-solvable satisfying $a_p \leq p$ for all $a \in \mathrm{cd}(G)$.

Proposition 3.7. *Let p be an odd prime, and let G be a finite non p-solvable group with a trivial p-solvable radical and $\mathbf{O}^{p'}(G) = G$. If $a_p \leq p$ for all $a \in \mathrm{cd}(G)$, then either G is a nonabelian simple group with $|G|_p = p$ or $p = 3$ and $G \cong \mathrm{Alt}_7$.*

Proof. Since $\mathrm{sol}_p(G) = 1$, we have $\mathbf{F}(G) = 1$, and thus, $\mathbf{F}^*(G) = \mathbf{E}(G) = T_1 \times T_2 \times \cdots \times T_k$, where $T_1, \ldots, T_k$ are nonabelian simple groups with p dividing $|T_i|$ and $\mathbf{C}_G(\mathbf{F}^*(G)) = 1$. For each $i \in \{1, \ldots k\}$, we have $p \mid |T_i|$ and T_i is nonabelian simple. Thus, there exist characters $\theta_i \in \mathrm{Irr}(T_i)$ with $p \mid \theta_i(1)$. Let $\psi = \theta_1 \times \cdots \times \theta_k \in \mathrm{Irr}(\mathbf{F}^*(G))$. Then $\psi(1)$ is divisible by p^k and divides some degree in $\mathrm{cd}(G)$. This forces $k = 1$. Hence, $\mathbf{F}^*(G)$ is a nonabelian simple group, and thus, G is an almost simple group with socle $\mathbf{F}^*(G)$. The result now follows from Lemma 3.2. □

In the proof of Proposition 3.7, we can use the solvable radical $\mathrm{sol}(G)$ rather than the p-solvable radical $\mathrm{sol}_p(G)$. However, there are cases when $\mathbf{F}^*(G)$ is a product of nonabelian simple groups whose order are coprime to p. For this, one would need to show that $|G|_p \leq p$.

We are now ready to prove our main result. We say that H is a *section* of a group G if there exist subgroup A and B in G so that B is subnormal in G and A is normal in B with $B/A \cong H$. Observe that if $a_p \leq p$ for all $a \in \mathrm{cd}(G)$, then every section of G satisfies this same property. By Theorem 1.2, we know that the hypothesis of this next theorem will hold with $\nu = 3$. This will yield Theorem 1.1. However, we believe that the correct bound in Theorem 1.2 is that $|G : \mathbf{F}(G)|_p \leq p^2$, this would allow the use of $\nu = 2$ in the next theorem, and we would obtain the conclusion that $|G : \mathbf{F}(G)|_p \leq p^3$.

Theorem 3.8. *Let p be an odd prime, and suppose that ν is a positive integer and G is a group such that every section H of G that is p-solvable satisfies $|H : \mathbf{F}(H)|_p \leq p^\nu$. If $a_p \leq p$ for all $a \in \mathrm{cd}(G)$, then $|G : \mathbf{F}(G)|_p \leq p^{1+\nu}$.*

Proof. If G is p-solvable, then $|G : \mathbf{F}(G)|_p \leq p^\nu < p^{1+\nu}$ and we are done. So, we assume that G is not p-solvable. If $L = \mathbf{O}^{p'}(G)$, then $\mathbf{O}^{p'}(L) = L$ and by Lemma 3.3, $|G : \mathbf{F}(G)|_p = |L : \mathbf{F}(L)|_p$. By induction on $|G|$, one may assume that $G = \mathbf{O}^{p'}(G)$. Let R_p be the p-solvable radical of G. Then G/R_p has a trivial p-solvable radical and $\mathbf{O}^{p'}(G/R_p) = G/R_p$. Proposition 3.7 yields $G/R_p \cong \mathrm{Alt}_7$ and $p = 3$ or G/R_p is a nonabelian simple group with $|G/R_p|_p = p$. Assume that the first case holds. By Corollary 3.6, we have $G = N \circ C$, where $N \cong \mathrm{Alt}_7$ or $3 \cdot \mathrm{Alt}_7$ and C is a normal abelian 3-group. It follows that $C = R_p = \mathbf{O}_3(G) = \mathbf{F}(G)$, and clearly, $|G : \mathbf{F}(G)|_3 = |\mathrm{Alt}_7|_3 = 3^2 \leq 3^{1+\nu}$. Now assume that the latter case holds. Since $\mathbf{F}(G) \leq R_p$, we have $\mathbf{F}(R_p) = \mathbf{F}(G) \cap R_p = \mathbf{F}(G)$. Now, R_p is a p-solvable group and is normal in G and $a_p \leq p$ for all $a \in \mathrm{cd}(R_p)$, so we obtain:

$$|G : \mathbf{F}(G)|_p = |G : R_p|_p \cdot |R_p : \mathbf{F}(R_p)|_p \leq p \cdot p^\nu = p^{1+\nu}.$$

□

4. Brauer characters

We now turn to the universe of Brauer characters and block theory. We follow the notation in [21] for block theory. Fix a prime p, and let G be a finite group. For a p-block (or just block) B of G, we denote by $D = D(B)$ the defect group of B; also, we set $\mathrm{Irr}(B) = B \cap \mathrm{Irr}(G)$ and $\mathrm{IBr}(B) = B \cap \mathrm{IBr}(G)$. The defect $d(B)$ of B is defined by

$$p^{a-d(B)} = \min\{\chi(1)_p \mid \chi \in \mathrm{Irr}(B)\}$$

where $p^a = |G|_p$. The height $h(\chi)$ of a character $\chi \in \mathrm{Irr}(B)$ is defined by the formula

$$p^{a-d(B)+h(\chi)} = \chi(1)_p.$$

By [21, Corollary 3.17], we have

$$p^{a-d(B)} = \min\{\varphi(1)_p \mid \varphi \in \mathrm{IBr}(B)\}$$

and by [21, Theorem 4.6], $|D(B)| = p^{d(B)}$.

The following is the 'if part' of the famous Brauer's height zero conjecture which was recently proved by R. Kessar and G. Malle.

Lemma 4.1. ([13, Theorem 1.1]) *Let p be a prime and G be a group. Let B be a p-block of G with defect group D. If D is abelian, then the degrees of the characters in* $\mathrm{Irr}(B)$ *all have the same p-part, which is $|G : D|_p$.*

We now are ready to prove Theorem 1.3 from the Introduction.

Proof of Theorem 1.3. Suppose that p is a prime, G is a finite group, and $\beta(1)_p \leq p$ for all $\beta \in \mathrm{IBr}(G)$. Let $\chi \in \mathrm{Irr}(G)$. If $\chi(1)_p = 1$, then we have nothing to prove. Suppose that $\chi(1)_p = p^m$ for some integer $m \geq 1$. As $\mathrm{Irr}(G)$ is a disjoint union of $\mathrm{Irr}(B)$, where B runs over the set of all p-blocks of G, we see that $\chi \in \mathrm{Irr}(B)$ for some block B of G. As G has an abelian Sylow p-subgroup of order, say p^a, we know that $D = D(B)$ is abelian, and thus, by Lemma 4.1, χ has height zero. That is, $\chi(1)_p = p^{a-d(B)}$. This implies that $a - d(B) = m \geq 1$. Since $p^{a-d(B)} = p^m$ divides all $\beta(1)$ with $\beta \in \mathrm{IBr}(B)$ and $\beta(1)_p \leq p$, we conclude that $m = 1$. Therefore, $a_p \leq p$ for all $a \in \mathrm{cd}(G)$ as desired. □

We next prove Corollary 1.4 from the Introduction.

Proof of Corollary 1.4. If p^2 does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, then p^2 does not divide $\varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ since $\varphi = \psi^\circ$ for some $\psi \in \mathrm{Irr}(G)$ by Fong-Swan Theorem [21, Theorem 10.1]. Conversely, suppose that p^2 does not divide $\varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$. Since $\mathrm{IBr}(G) = \mathrm{IBr}(G/\mathbf{O}_p(G))$, we can assume that $\mathbf{O}_p(G) = 1$, and so, G has abelian Sylow p-subgroups by [24, Corollary 2.6]. Now by Theorem 1.3, $a_p \leq p$ for all $a \in \mathrm{cd}(G)$. □

We follow [21] for the notation of Brauer characters, and we refer to [12] for the representation theory of symmetric groups. Let p be a prime, let n be a natural number, and let $\lambda \vdash n$ be a partition of n; and write $|\lambda| = n$. The irreducible character of Sym_n labeled by the partition λ is denoted by χ^λ. Let $\lambda_{(p)}$ be the p-core of λ obtained by repeatedly removing all p-hooks. The p-weight $\omega_p(\lambda)$ is the number of p-hooks we remove to obtain the p-core, and we have $|\lambda| = |\lambda_{(p)}| + p\omega_p(\lambda)$. Nakayama's conjecture asserts that two irreducible characters of Sym_n labeled by two partitions λ and μ of n belong to the same p-block if and only if λ and μ have the same p-core. Now, the p-blocks of Sym_n are labeled by the p-core and the p-weight $w(B)$ of B which is the common p-weight $w_p(\lambda)$ for all partitions $\lambda \vdash n$ with $\chi^\lambda \in \mathrm{Irr}(B)$. Finally, the defect group $D = D(B)$ of B is the Sylow p-subgroup of $\mathrm{Sym}_{pw(B)}$. (See [2].) Given the pair (α, w), where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is a p-core partition and $w \in \mathbf{N}$, then $\lambda := (\alpha_1 + pw, \alpha_2, \cdots, \alpha_k)$ is a partition of $n := |\alpha| + pw$ with p-core $\lambda_{(p)} = \alpha$ and p-weight $\omega_p(\lambda) = w$.

Recall that for a finite group G and a prime p, an ordinary irreducible character $\chi \in \mathrm{Irr}(G)$ is said to have p-defect zero if $\chi(1)_p = |G|_p$. [21, Theorem 3.8] yields that $\chi^\circ \in \mathrm{IBr}(G)$ with $\chi^\circ(1) = \chi(1)$, where G° is the set of all p-regular elements of G.

Lemma 4.2. *Let p be a prime and let G be an almost simple group with nonabelian simple socle S. Suppose that p divides $|S|$ and $\beta(1)_p \leq p$ for all $\beta \in \mathrm{IBr}(G)$. Then the following hold.*

(*i*) *If $p = 2$, then $G = S \cong \mathrm{M}_{22}$ and $|G|_2 = 2^7$.*
(*ii*) *If $p = 3$ and $S \cong \mathrm{Alt}_7$, then $|G|_3 = |S|_3 = 3^2$.*
(*iii*) *If $p \geq 3$ and $(S, p) \neq (\mathrm{Alt}_7, 3)$, then $|G|_p = |S|_p = p$.*

Proof. We observe first that if S has an irreducible character χ with p-defect zero, then $\chi^\circ \in \mathrm{IBr}(S)$ with $\chi^\circ(1) = \chi(1)$. Hence, $\chi^\circ(1)_p = |S|_p$. From the hypothesis, we have $|S|_p = p$. We consider the case $p = 2$ and $p > 2$ separately.

(1) $p = 2$. Since S is nonabelian simple, we know that $|S|_2 \geq 4$, so S does not has any irreducible character of 2-defect zero. By [9, Corollary 2], $S \cong \mathrm{Alt}_n$ or S is isomorphic to M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, or B. Assume that S is a sporadic simple group. Except for the Baby Monster, one can use [4] to check that only M_{22} satisfies the hypothesis with $|\mathrm{M}_{22}|_2 = 2^7$. As $\mathrm{M}_{22}\cdot 2$ does not satisfy the hypothesis, we deduce that $G = S \cong \mathrm{M}_{22}$ with $|G|_2 = |S|_2 = 2^7$. For the Baby Monster, using [6], B has a 2-block $\mathcal{B}$ with defect $d(\mathcal{B}) = 3$. Since $|\mathrm{B}|_2 = 2^{41}$, we deduce that if $\varphi \in \mathrm{IBr}(\mathcal{B})$, then $2^{38} = 2^{41-3}$ divides $\varphi(1)$. Thus, this case cannot happen.

Assume that $S \cong \mathrm{Alt}_n$ with $n \geq 5$. Using [6], we can assume that $n \geq 12$. From Proposition 5.2 and Corollary 4.2 in [1], Alt_n has a 2-Brauer irreducible character β which is the restriction of the irreducible 2-Brauer character $D^{(n-3,3)}$ of Sym_n to Alt_n with degree

$$\beta(1) = \begin{cases} \frac{1}{6}n(n-2)(n-7) & \text{if } n \equiv 0 \pmod 4, \\ \frac{1}{6}n(n-1)(n-5) & \text{if } n \equiv 1 \pmod 4, \\ \frac{1}{6}(n-1)(n-2)(n-6) & \text{if } n \equiv 2 \pmod 4, \\ \frac{1}{6}(n+1)(n-1)(n-6) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Now, it is routine to check that $4 \mid \beta(1)$ in any of these cases.

(2) p is odd. If $(S,p) = (\mathrm{Alt}_7, 3)$, then (ii) holds by using [4]. So, assume from now on that $(S,p) \neq (\mathrm{Alt}_7, 3)$. We first claim that $|S|_p = p$. If S has an irreducible character of p-defect zero, then we are done. So assume that S has no such character. By [9, Corollary 2], we have $p = 3$ and $S \cong \mathrm{Alt}_n$ or $S \cong \mathrm{Suz}$ or Co_3. If $S \cong \mathrm{Alt}_n$ with $5 \leq n \leq 19$ or S is one of the two sporadic simple group above, then the result follows by using [4]. Thus we can assume that $S \cong \mathrm{Alt}_n$ with $n \geq 20$. As $p \geq 3$ and $\mathrm{Sym}_n / \mathrm{Alt}_n$ is a cyclic group of order 2, we deduce that p^2 does not divide $\varphi(1)$ for all $\varphi \in \mathrm{IBr}(\mathrm{Sym}_n)$. Notice that if $\varphi \in \mathrm{IBr}(\mathrm{Sym}_n \mid \beta)$ for some $\beta \in \mathrm{IBr}(\mathrm{Alt}_n)$, then $\varphi(1)/\beta(1)$ divides $2 = |\mathrm{Sym}_n : \mathrm{Alt}_n|$ (see [21, Theorem 8.30]) so $\varphi(1)_p = \beta(1)_p$.

For each integer n, we can choose two 3-regular partitions λ, μ of n such that the corresponding 3-cores $\lambda_{(3)}, \mu_{(3)}$ are as follow:
(a) $(3,1)$ and $(5,3,1^2)$, if $n \equiv 1 \pmod 3$;
(b) $(3,1^2)$ and $(4,2,1^2)$, if $n \equiv 2 \pmod 3$;
(c) $(4,2)$ and $(5,3,1)$, if $n \equiv 0 \pmod 3$.

Since λ and μ have different 3-cores, we know that they belong to different 3-blocks, say B_1 and B_2 of Sym_n. Because $3w(B_1) = n - |\lambda_{(3)}|$ and $3w(B_2) = n - |\mu_{(3)}|$ where $|\mu_{(3)}| - |\lambda_{(3)}| = 3$ or 6, we deduce that $3w(B_1) = 3w(B_2) + 3$ or 6. This implies that the orders of the Sylow 3-subgroups of $\mathrm{Sym}_{3w(B_1)}$ and $\mathrm{Sym}_{3w(B_2)}$, namely, $3^{d(B_i)}$, which are also the orders of the defect group of the block B_i, are distinct and strictly less than $|\mathrm{Sym}_n|_p = p^a$. By [21, Corollary 3.7], we have $3^{a-d(B_i)} = \min\{\varphi(1)_3 \mid \varphi \in \mathrm{IBr}(B_i)\}$. As $a > d(B_1) > d(B_2) \geq 1$, we deduce that $a - d(B_2) > a - d(B_1) \geq 1$, and thus, $a - d(B_2) \geq 2$. So if $\beta \in \mathrm{IBr}(B_2)$, then $p^2 \mid \beta(1)$, a contradiction.

We now claim that $|G|_p = |S|_p = p$. By Lemma 3.1, we have that $|S|_p > |\mathrm{Out}(S)|_p$ whenever p divides $|S|$. As $|S|_p = p$, we deduce that $|G : S|_p$ divides $|\mathrm{Out}(S)|_p$, where the latter is strictly less than p. This forces $|G : S|_p = 1$, and thus, $|G|_p = |S|_p = p$ as desired. □

Combining the results of Leisering [14] with those of Lewis, Navarro and Wolf [16], we obtain:

Lemma 4.3. *Let p be a prime and let G be a finite solvable group with $\mathbf{O}_p(G) = 1$. If $\beta(1)_p \leq p$ for all $\beta \in \mathrm{IBr}(G)$, hen $|G|_p \leq p^2$ and a Sylow p-subgroup of G is elementary abelian.*

Proof. Let P be a Sylow p-subgroup of G. By [14, Theorem], P is elementary abelian and $a_p \leq p$ for all $a \in \mathrm{cd}(G)$. Now, Theorem 1 in [16] yields that $|G : \mathbf{F}(G)|_p \leq p^2$. Since $\mathbf{O}_p(G) = 1$, we see that $\mathbf{F}(G)$ is a p'-group, and thus, $|G|_p = |G : \mathbf{F}(G)|_p \leq p^2$. □

We see when $\mathbf{O}^{p'}(G) = G$ and $\mathbf{O}_p(G) = 1 = \mathrm{sol}_p(G)$ that G is simple.

Lemma 4.4. *Let p be a prime, and let G be a non-p-solvable group. If $\mathbf{O}^{p'}(G) = G$, $\mathbf{O}_p(G) = 1 = \mathrm{sol}_p(G)$, and $\beta(1)_p \leq p$ for all $\beta \in \mathrm{IBr}(G)$, then G is isomorphic to one of the nonabelian simple groups in Lemma 4.2.*

Proof. Since $\mathrm{sol}_p(G) = 1$, we know that $\mathbf{F}^*(G) = \mathbf{E}(G) = T_1 \times T_2 \times \cdots \times T_k$, where $T_1, \ldots, T_k$ are nonabelian simple groups with p dividing $|T_i|$ for all $1 \leq i \leq k$ and $\mathbf{C}_G(\mathbf{F}^*(G)) = 1$. By [19, Theorem 5.5], for each i, there exists a Brauer character $\theta_i \in \mathrm{IBr}(T_i)$ with $p \mid \theta_i(1)$. Clearly, $\varphi = \theta_1 \times \cdots \times \theta_k \in \mathrm{IBr}(\mathbf{F}^*(G))$ and $\varphi(1)$ is divisible by p^k. If $\gamma \in \mathrm{IBr}(G \mid \varphi)$, then p^k divides $\gamma(1)$. This forces $k = 1$. Thus, G is an almost simple group with the nonabelain simple socle $\mathbf{F}^*(G)$. Lemma 4.2 now yields that $|G|_p = |\mathbf{F}^*(G)|_p$. Since $\mathbf{O}^{p'}(G) = G$, we deduce that $G = \mathbf{F}^*(G)$, and so, G is a nonabelian simple group appearing in Lemma 4.2. □

We now prove Theorem 1.5 when p is odd. As with Theorem 3.8, we know that $\nu = 3$ will hold, but this allows the possibility that $\nu = 2$.

Theorem 4.5. *Let G be a finite group, and let p be an odd prime with $\mathbf{O}_p(G) = 1$. Suppose that ν is a positive integer so that every p-solvable section H of G satisfies $|H : \mathbf{F}(H)|_p \leq p^\nu$. If $\beta(1)_p \leq p$ for all $\beta \in \mathrm{IBr}(G)$, then the following hold:*

(1) *If $p \geq 5$ or $p = 3$ and Alt_7 is not involved in G, then $|G|_p \leq p^{1+\nu}$.*
(2) *If $p = 3$ and Alt_7 is involved in G, then $|G|_3 \leq 3^{2+\nu}$.*

Proof. We proceed by using induction on $|G|$. Notice that the hypotheses of the theorem are inherited by normal subgroups. Let $L = \mathbf{O}^{p'}(G)$, and let R_p be the p-solvable radical of G. We have that $\mathbf{O}_p(R_p) = 1$. If $L < G$, then $\mathbf{O}_p(L) = 1$ and $|G|_p = |L|_p$ since G/L is a p'-group.

So L satisfies the hypotheses of the theorem, and by induction, either $|L|_p \leq p^{1+\nu}$ or $p = 3$, Alt_7 is involved in L, and $|L|_3 \leq 3^{2+\nu}$. Since $|G|_p = |L|_p$ and the fact that Alt_7 is involved in G if and only if it is involved in L (noting that $3 \mid |\mathrm{Alt}_7|$), the conclusions of the theorem hold. Therefore, one can assume that $G = \mathbf{O}^{p'}(G)$. If G is p-solvable, then $|G|_p \leq p^{\nu} < p^{1+\nu}$ and we are done. So we assume that G is not p-solvable. Then G/R_p satisfies the assumption of Lemma 4.4, and we deduce that either $|G/R_p|_p = p$ or $p = 3$ and $G/R_p \cong \mathrm{Alt}_7$. Assume that $|G/R_p|_p = p$. By hypothesis since R_p is p-solvable, we have $|R_p|_p \leq p^{\nu}$, and we conclude that $|G|_p = |G/R_p|_p \cdot |R_p|_p \leq p^{\nu} \cdot p = p^{1+\nu}$. Assume now that $G/R_p \cong \mathrm{Alt}_7$ and $p = 3$. Then $|G|_3 = |G/R_3|_3 \cdot |R_3|_3 \leq 3^2 \cdot 3^{\nu} = 3^{2+\nu}$. □

Finally, we prove Theorem 1.5 for $p = 2$.

Theorem 4.6. *Let G be a finite group with $\mathbf{O}_2(G) = 1$, and let P be a Sylow 2-subgroup of G. If $\beta_2(1) \leq 2$ for all $\beta \in \mathrm{IBr}(G)$, then $P^{(4)} = 1$ and $|G|_2 \leq 2^9$.*

Proof. By Lemma 4.3, we may assume that G is nonsolvable. Using induction on $|G|$, we may assume that $\mathbf{O}^{2'}(G) = G$. Let R be the solvable radical of G. Lemmas 4.4 and 4.2 yield that $G/R \cong \mathrm{M}_{22}$. Since $\mathbf{O}_2(G) = 1$, we have $\mathbf{O}_2(R) = 1$, and so, by Lemma 4.3, $|R|_2 \leq 2^2$. Hence, $|G|_2 = |G : R|_2 \cdot |R|_2 \leq 2^9$. Finally, PR/R and $P \cap R$ are Sylow 2-subgroups of $G/R \cong \mathrm{M}_{22}$ and R, respectively. From [4], we know that PR/R has derived length 3 and $P \cap R$ is abelian by Lemma 4.3, therefore, $P''' \leq P \cap R$ and hence $P^{(4)} = 1$. This completes the proof. □

It would be desirable for one to obtain a detailed structure of groups in Theorem 4.6.

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