p-PARTS OF CHARACTER DEGREES

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Abstract. We show that if \( p \) is an odd prime and \( G \) is a finite group satisfying the condition that \( p^2 \) divides the degree of no irreducible character of \( G \), then \( |G : O_p(G)|_p \leq p^4 \) where \( O_p(G) \) is the largest normal \( p \)-subgroup of \( G \), and if \( P \) is a Sylow \( p \)-subgroup of \( G \), then \( P'' \) is subnormal in \( G \). Our investigations suggest that if \( p^a \) is the largest power of \( p \) dividing the irreducible characters of \( G \), then \( |G : O_p(G)|_p \) is bounded by \( p^{f(a)} \) where \( f(a) \) is a function in \( a \) and \( P^{(a+1)} \) is subnormal in \( G \).

1. Introduction

Throughout this paper, \( G \) will be a finite group and \( p \) will be a prime. Let \( \text{Irr}(G) \) be the set of all complex irreducible characters of \( G \). The celebrated Itô-Michler theorem says that \( p \) does not divide \( \chi(1) \) for all \( \chi \in \text{Irr}(G) \) if and only if \( G \) has a normal abelian Sylow \( p \)-subgroup. Recently in [16], the first and second authors along with Tom Wolf proved that if \( G \) is a finite solvable group and every character \( \chi \in \text{Irr}(G) \) satisfies \( \chi(1)_p \leq p \), then \( |G : F(G)|_p \leq p^2 \) where \( F(G) \) is the Fitting subgroup of \( G \) and \( n_p \) is the \( p \)-part of \( n \). That is, \( n_p \) is the largest power of \( p \) that divides the integer \( n \). An immediate consequence of that result is that \( P' \) is subnormal in \( G \), where \( P \) is a Sylow \( p \)-subgroup of \( G \). In fact, the authors of that paper proved the slightly stronger result that \( PF(G)/F(G) \) is an elementary abelian \( p \)-group. The authors then asked if there is any version of their result for nonsolvable groups.

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For \( p = 2 \), the first author had considered a stronger question in [15] where he showed that if 4 divides no degree of a nonsolvable group \( G \), then \( G \cong \text{Alt}_7 \times R \) for some solvable group \( R \) satisfying 2 divides no degree in \( \text{cd}(R) \). Using this result, it is proved in [16] that if every character \( \chi \in \text{Irr}(G) \) satisfies \( \chi(1)_p \leq 2 \), then \( |G : \text{O}_p(G)|_p \leq 2^3 \) and \( P'' \) is subnormal in \( G \) where \( P \) is a Sylow 2-subgroup of \( G \). The simple group \( \text{Alt}_7 \) shows that this bound is best possible.

In this paper, we consider the situation where \( p \) is an odd prime and \( G \) is not necessarily solvable. In particular, we prove the following theorem.

**Theorem 1.1.** Let \( G \) be a group, and let \( p \) be an odd prime. If every character \( \chi \in \text{Irr}(G) \) satisfies \( \chi(1)_p \leq p \), then \( |G : \text{O}_p(G)|_p \leq p^4 \). In addition, if \( P \) is a Sylow \( p \)-subgroup of \( G \), then \( P'' \) is subnormal in \( G \).

Note that \( |F(G)|_p = |\text{O}_p(G)| \), so \( |G : \text{O}_p(G)|_p = |G : F(G)|_p \). Since \( p \)-groups of order \( p^4 \) have derived length at most 2, this shows that if \( p \) is an odd prime, \( P \) is a Sylow \( p \)-subgroup of \( G \) and every character \( \chi \in \text{Irr}(G) \) satisfies \( \chi(1)_p \leq p \), then \( P'' \) is subnormal in \( G \). This raises the following question. If every character \( \chi \in \text{Irr}G \) satisfies \( \chi(1)_p \leq p^a \), then is it true that \( P^{(a+1)} \) is subnormal in \( G \)?

Restricting our attention to \( p \)-solvable groups, we obtain the following stronger result. We write \( \text{sol}(G) \) for the solvable radical of \( G \), this is the subgroup generated by all normal solvable subgroups of \( G \). Note that for a finite group, it is the largest solvable normal subgroup in \( G \).

**Theorem 1.2.** Let \( p \) be an odd prime and let \( G \) be a \( p \)-solvable group. If every character \( \chi \in \text{Irr}(G) \) satisfies \( \chi(1)_p \leq p \), then \( |G/\text{sol}(G)|_p \leq p^a \) and \( |G/\text{O}_p(G)|_p \leq p^3 \).

When \( G \) is \( p \)-solvable, we also obtain a stronger result regarding the Sylow \( p \)-subgroups of \( G \). Note that the condition that \( P/\text{O}_p(G) \) is abelian is equivalent to the condition that \( P'' \) is subnormal in \( G \). We note that in [10], Isaacs has studied the Sylow \( p \)-subgroups of a solvable group \( G \) satisfying the condition that every character \( \chi \in \text{Irr}(G) \) satisfies \( \chi(1)_p \leq p^a \) showing that if \( P \) is a Sylow \( p \)-subgroup of \( G \), then \( P \) has derived length at most \( 2a + 1 \). Also, using the techniques in Theorem B of [25], under this same hypothesis that \( G \) is a solvable group and every character \( \chi \in \text{Irr}G \) satisfies \( \chi(1)_p \leq p^a \) and additionally assuming that \( p \geq 5 \) one can show that \( |G : \text{O}_p(G)|_p \leq p^{2.5a} \). To see that a similar result probably holds for nonsolvable groups, note that if \( S \) is a nonabelian simple group and \( p \geq 5 \), then there exists a character \( \chi \in \text{Irr}(S) \) such that \( \chi(1)_p = |S|_p \) (this can be obtained from Corollary 2 of [9]).
To address the question of whether the Sylow $p$-subgroups of $G/F(G)$ are abelian, we turn to $p$-Brauer characters. As usual, let $\text{IBr}_p(G)$ be the set of irreducible $p$-Brauer characters of $G$. There are some significant differences between ordinary character degrees and $p$-Brauer character degrees; for example, the Brauer degrees need not divide the order of the group, and a Brauer character version of the Itô-Michler theorem only holds for the given prime $p$. That is, if $p$ divides no $p$-Brauer degree of a finite group $G$, then $G$ has a normal Sylow $p$-subgroup (see [19, Theorem 5.5]).

In 1990, Leisering [14] considered a similar variation of this result by proving that if $G$ is a finite solvable group with $O_p(G) = 1$ and $p^2$ does not divide $\beta(1)$ for every Brauer character $\beta \in \text{IBr}_p(G)$, then $G$ has an elementary abelian Sylow $p$-subgroup and every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p^2$. Now, by Theorem 1 in [16], we have $|G:F(G)| = p^2$ when $G$ is solvable and every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p^2$. However, since $O_p(G) = 1$ and $F(G)$ is a $p'$-group, we deduce that $|G| = |G:F(G)|_p \leq p^2$ in this situation.

From [24, Corollary 2.6], we know that if $G$ is a $p$-solvable group with $O_p(G) = 1$ and $p^2$ dividing no Brauer character degree, then $G$ has elementary abelian Sylow $p$-subgroups, so a similar result to Leisering’s holds for $p$-solvable groups. If $G$ is a $p$-solvable group and $\varphi \in \text{IBr}(G)$, then the Fong-Swan Theorem [21, Theorem 10.1] asserts that $\varphi = \chi^\circ$, the restriction to $G^\circ$, the set of $p$-regular elements of $G$, of some character $\chi \in \text{Irr}(G)$. Hence, if $G$ is $p$-solvable and every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p$, then every Brauer character $\varphi \in \text{IBr}(G)$ satisfies $\varphi(1)_p \leq p$. Combining this with Wang’s result [24], we deduce that if $G$ is a $p$-solvable group that has the property that every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p$ and $P$ is a Sylow $p$-subgroup of $G$, then $P/O_p(G)$ is elementary abelian.

Turning to groups that are not $p$-solvable, Leisering’s conclusion does not hold. For example, if one takes $G = M_{22}$ and $p = 2$, then $O_2(G) = 1$ and every Brauer character $\beta \in \text{IBr}_p(G)$ satisfies $\beta(1)_2 \leq 2$, but $|G|_2 = 2^7$ and there exists a character $\chi \in \text{Irr}(G)$ with $\chi(1)_2 = 2^3$. However, if we make the additional assumption that the group has an abelian Sylow $p$-subgroup, then we can make use of a recent result of Kessar and Malle in [13] on Brauer’s height zero conjecture to prove the following.

**Theorem 1.3.** Let $p$ be a prime and let $G$ be a finite group with $O_p(G) = 1$. If $G$ has an abelian Sylow $p$-subgroup and every Brauer character $\varphi \in \text{IBr}_p(G)$ satisfies $\varphi(1)_p \leq p$, then every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p$. 

As a corollary, we deduce that for a $p$-solvable group $G$, the condition that every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p$ is equivalent to the condition that every Brauer character $\varphi \in \text{IBr}(G)$ satisfies $\varphi(1)_p \leq p$.

**Corollary 1.4.** Let $p$ be a prime and $G$ be a finite $p$-solvable group. Then every Brauer character $\varphi \in \text{IBr}_p(G)$ satisfies $\varphi(1)_p \leq p$ if and only if every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p$.

Obviously, for arbitrary finite groups, we do not have such an equivalence. We can obtain the following conclusion from knowing that $p^2$ does divide any Brauer character degree.

**Theorem 1.5.** Let $p$ be a prime and $G$ be a finite group with $O_p(G) = 1$. If every Brauer character $\beta \in \text{IBr}_p(G)$ satisfies $\beta(1)_p \leq p$, then the following hold.

1. If $p = 2$, then $|G|_2 \leq 2^9$.
2. If $p \geq 5$ or if $p = 3$ and $\text{Alt}_7$ is not involved in $G$, then $|G|_p \leq p^4$.
3. If $p = 3$ and $\text{Alt}_7$ is involved in $G$, then $|G|_3 \leq 3^5$.

It seems to us that the bounds in Theorem 1.5 are probably not best possible. We conjecture that the correct bounds in Theorem 1.5 should be $2^7$ in (1), $p^2$ in (2), and $3^5$ in (3). Notice every Brauer character $\varphi \in \text{IBr}(M_{22})$ satisfies $\varphi(1)_2 \leq 2$ and $|M_{22}|_2 = 2^7$. Returning to the case of the ordinary character degrees, we know from [16] that for every odd prime $p$ there is a solvable group $H$ so that every character $\chi \in \text{Irr}(H)$ satisfies $\chi(1)_p \leq p$ and $|H : O_p(H)|_p = p^2$, and there is a simple group $S$ so that $p$ does not divide $|S|$. Taking $G = H \times S$, we see that $G$ is a $p$-solvable group where every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p$ and $|G : O_p(G)|_p = p^2$. Since we do not have any examples where the $p$-part of the index of $O_p(G)$ is larger, we believe in both Theorem 1.1 and Theorem 1.2 that the correct bound is $p^2$, however, it appears to be a daunting task to prove this. Note that the only non-$p$-solvable example where $|G : O_p(G)|_p > p$ is $G = \text{Alt}_7$ and $p = 3$ which has $|G|_p = |G : O_3(G)|_3 = 9$. Also, we would like to know whether the hypothesis that $G$ is $p$-solvable is necessary for $G$ satisfying the property that every character $\chi \in \text{Irr}(G)$ satisfies $\chi(1)_p \leq p$ to imply that the Sylow $p$-subgroups of $G/F(G)$ are elementary abelian.

2. $p$-Solvable Groups

Let $\text{cd}(G)$ be the set of all (ordinary) degrees of $G$, that is, $\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$. In this section, we bound $|G : F(G)|_p$ when $G$ is $p$-solvable and $a_p \leq p$ for all $a \in \text{cd}(G)$ for some odd prime $p$. Consider
a finite set $\Omega$. If $H < \text{Sym}(\Omega)$ and $\Omega_1, \ldots, \Omega_m$ are subsets of $\Omega$, we let $\text{Stab}_H(\Omega_1, \ldots, \Omega_m)$ denote the subgroup consisting of all elements in $H$ that fixes each set $\Omega_i$, $1 \leq i \leq m$, set-wise.

**Lemma 2.1.** Let $\Omega$ be a finite set, and let $H < \text{Sym}(\Omega)$ be a subgroup of order divisible by $p$. 

(i) There exist disjoint (possibly empty) subsets $\Omega_1, \Omega_2$ of $\Omega$ such that $p$ divides $[H : \text{Stab}_H(\Omega_1, \Omega_2)]$.

(ii) If $H$ is solvable, then there exist disjoint (possibly empty) subsets $\Omega_1, \Omega_2$ of $\Omega$ such that $\text{Stab}_H(\Omega_1, \Omega_2)$ is a 2-group.

**Proof.** Conclusion (i) is a partial case of [3, Lemma 8], and Conclusion (ii) is [5, Corollary 4]. □

We next consider a transitive subgroup of $\text{Sym}(\Omega)$.

**Lemma 2.2.** Let $\Omega$ be a finite set and let $H < \text{Sym}(\Omega)$ be a transitive subgroup of order divisible by $p > 2$. Suppose that $H = O^p(H)$ is $p$-solvable.

(i) Suppose that $H$ is primitive, and either $H$ is non-solvable or $p > 3$. Then $H$ has a regular orbit on the set of all subsets of $\Omega$.

(ii) Suppose that $p^2$ divides $|H|$. Then there exist $s$ disjoint (possibly empty) subsets $\Omega_1, \ldots, \Omega_s$ of $\Omega$ such that $p^2$ divides $[H : \text{Stab}_H(\Omega_1, \ldots, \Omega_s)]$, where $s = 3$ of $p > 3$ and $s = 4$ if $p = 3$.

**Proof.** (i) The assumption implies that $H \nleq \text{Alt}(\Omega)$. Now the main result of [22] lists all the 46 cases where $H$ fails to have a regular orbit on the subsets of $\Omega$. Among them, there are 35 cases where the group $H$ is non-solvable, but either $H$ is not $p$-solvable, or $H \neq O^p(H)$. Next, in 8 of the 11 solvable cases, $H \neq O^p(H)$. In the remaining 3 cases, we have that $p = 3$ (and $(|\Omega|, H)$ is one of the following: $(4, \text{Alt}_4)$, $(8, \text{AGL}_1(8))$, $(9, \text{ASL}_2(3))$). Hence, conclusion (i) follows.

(ii) First we consider the case $H$ is primitive. If $p > 3$ or $H$ is non-solvable, we are done by conclusion (i) (taking $s = 1$), whereas if $p = 3$ and $H$ is solvable then we are done by Lemma 2.1 (ii) (taking $s = 2$).

We may now assume that $H$ is imprimitive, and so, permutes transitively $k$ disjoint subsets $\Delta_1, \ldots, \Delta_k$ of $\Omega$, with $|\Delta_i| = |\Omega|/k$ and $1 < k < |\Omega|$. Let

$B := \text{Stab}_H(\Delta_1, \ldots, \Delta_k)$

so that $H/B$ is a transitive subgroup of $\text{Sym}_k$. Choosing the smallest $k$ possible, we have that $H/B$ is a primitive subgroup of $\text{Sym}_k$. Certainly
$H/B$ is $p$-solvable and $H/B = \mathcal{O}_p'(H/B)$. Since $k > 1$, the latter implies that $p$ divides $|H/B|$. Again applying conclusion (i) and Lemma 2.1 (ii) to $H/B$, we see that there are $t$ disjoint subsets $\Omega_1, \ldots, \Omega_t$ of $\Omega$, each being a union of some $\Delta_j$’s, so that

\begin{equation}
2.1 \quad p \nmid |\text{Stab}_{H/B}(\Omega_1, \ldots, \Omega_t)|.
\end{equation}

Here we can take $t = 1$ if $p > 3$ and $t \leq 2$ if $p = 3$. We may furthermore assume that $\bigcup_{i=1}^t \Omega_i \neq \Omega$. (Indeed, this is obvious if $t = 1$ as $p \mid |H/B|$. If $t = 2$ and $\Omega_1 \cup \Omega_2 = \Omega$, then $\text{Stab}_{H/B}(\Omega_1) = \text{Stab}_{H/B}(\Omega_1, \ldots, \Omega_t)$, and so, we can again take $t = 1$.)

Now, if $p^2 \mid |H/B|$, then we are done by (2.1), taking $s = t$. We will now assume that $p^3$ does not divide $|H/B|$ (and so $p \mid |B|$), and consider a Sylow $p$-subgroup $Q$ of $B$. Then there must be some $\Delta_j$ such that $Q$ does not act trivially on $\Delta_j$. Since $B < H$ and $H$ permutes $\Delta_1, \ldots, \Delta_k$ transitively, replacing $Q$ by an $H$-conjugate, we may assume that

$$\Delta_j \subseteq \Omega \setminus \bigcup_{i=1}^t \Omega_i.$$ 

Thus, the action of $B$ on $\Delta_j$ induces a subgroup $\bar{B} \leq \text{Sym}(\Delta_j)$ of order divisible by $p$. By Lemma 2.1 (i), there are disjoint subsets $\Omega_{t+1}$ and $\Omega_{t+2}$ of $\Delta_j$ such that $p \mid [\bar{B} : \text{Stab}_{\bar{B}}(\Omega_{t+1}, \ldots, \Omega_{t+2})]$, whence

\begin{equation}
2.2 \quad p \mid [B : \text{Stab}_{B}(\Omega_{t+1}, \ldots, \Omega_{t+2})] .
\end{equation}

It remains to show that $p^2 \mid |H : J|$, where $J := \text{Stab}_H(\Omega_1, \ldots, \Omega_{t+2})$. Assume the contrary. Note that $K/B = \text{Stab}_{H/B}(\Omega_1, \ldots, \Omega_t)$ for $K := \text{Stab}_H(\Omega_1, \ldots, \Omega_t)$. As $p \mid |H/B|$ we have by (2.1) that $p \mid |K : J|$. Since $K \geq J$, it follows that $p$ does not divide $|K : J|$, and so, $p$ does not divide $|B : J \cap B|$. But this contradicts (2.2), as $J \cap B = \text{Stab}_{B}(\Omega_{t+1}, \ldots, \Omega_{t+2})$. \hfill \square

The first two statements of the following lemma are well known:

**Lemma 2.3.** Let $S$ be a finite non-abelian simple group that admits an automorphism $\sigma$ of prime order $p$ where $p$ does not divide $|S|$, then

(i) Then $S$ is a simple group of Lie type defined over a field $\mathbb{F}_q$ with $q = r^f$, $r \neq p$ a prime, and $p \mid f$.

(ii) If $P \in \text{Syl}_p(\text{Out}(S))$ then $P \leq Z(\text{Out}(S))$ and $P$ is cyclic.

(iii) There exists $\alpha \in \text{Irr}(S)$ such that $p$ does not divide $|\text{Irr}(\text{Aut}(S))(\alpha)|$.

In fact, $P$ has at least two regular orbits of such characters.

**Proof.** (i) The condition on $\sigma$ implies that $S$ is a simple group of Lie type defined over $\mathbb{F}_q$ with $q = r^f$ and $r \neq p$. Now, by [8, Theorem 2.5.12], we have that $\text{Out}(S) = \text{Outdiag}(S) \rtimes A(S)$, where $\text{Outdiag}(S)$ is either cyclic or a Klein four-group, all prime divisors of which divide $|S|$, and $A(S) = \Phi_S \Gamma_S$ is abelian. Here, $\Phi_S$ is a cyclic group of field
automorphisms, and \(|A(S) : \Phi_S|\) divides \(|S|\). It follows that \(p \mid |\Phi_S|\) and so \(p \mid f\).

(ii) We have shown in conclusion (i) that we can take \(P \leq \Phi_S\); in particular, \([P, A(S)] = 1\) and \(P\) is cyclic. If \(\text{Outdiag}(S)\) is a Klein four-group, then it is centralized by \(\Phi_S\) by [8 Theorem 2.5.12(h)]. Assume that \(\text{Outdiag}(S) \cong C_d\) is cyclic but not centralized by \(P\); in particular, \(p \mid \varphi(d)\) (the Euler function of \(d\)). Since \(p\) does not divide \(d\), it follows that \(d\) is divisible by a prime \(s\) with \(p \mid (s - 1)\). As \(p > 2\) by the Feit-Thompson theorem, we have that \(s \geq 5\). Using the description of \(\text{Outdiag}(S)\) in [8 Theorem 2.5.12], we see that \(S \cong PSL_n(q)\) with \(\epsilon = +\) for \(S \cong PSL_n(q)\) and \(\epsilon = -\) for \(S = PSU_n(q)\), and \(d = \gcd(n, q - \epsilon)\). Clearly, \(n \geq s > p\) as \(s \mid d\). As \(p\) does not divide \(q\), we have that \(p \mid ((\epsilon q)^{p-1} - 1)\) and so \(p|\langle S\rangle\), a contradiction. This proves conclusion (ii).

(iii) Note that the cyclic group \(P\) acts coprimely and faithfully on \(S\). Hence, by [20 Proposition 2.6], \(P\) has at least two regular orbits on \(\text{Irr}(S)\). Let \(\alpha\) belong to such an orbit, and assume that \(p \mid |I_{\text{Aut}(S)}(\alpha)|\). Then \(\alpha\) is fixed by some automorphism \(\tau \in \text{Out}(S)\) of order \(p\). As \(P < \text{Out}(S)\), we have that \(\tau \in P\) and \(\alpha\tau = \alpha\), a contradiction. (Explicit constructions of such \(\alpha\) can be found in [18 §4] and [23 §6].) \(\square\)

We now consider the case where \(G\) is \(p\)-solvable, satisfies \(a_p \leq p\) for all \(a \in \text{cd}(G)\), and has a nonabelian minimal normal subgroup \(N\).

**Proposition 2.4.** Let \(p > 2\) and let \(G\) be a \(p\)-solvable group and satisfy \(a_p \leq p\) for all \(a \in \text{cd}(G)\). Suppose that \(G = O_p^\alpha(G)\) and \(O_p(G) = 1\). Suppose also that \(G\) admits a minimal normal subgroup \(N = T_1 \times \ldots \times T_n \cong T^n\), where \(T\) is a non-abelian simple group. Then \(|G|_p \leq p\).

**Proof.** (i) Assume the contrary: \(|G|_p \geq p^2\). Let \(K\) be the kernel of the transitive action of \(G\) on \(\Omega := \{T_1, \ldots, T_n\}\). Then \(G/K \leq \text{Sym}_n\) is \(p\)-solvable and \(G/K = O_p^\beta(G/K)\). Denote \(p^a := |G/K|_p\) and \(p^b := |K/C|_p = |K/CN|_p\) for \(C := C_G(N)\). Note that \(a + b \geq 1\). Indeed, otherwise \(p\) does not divide \(|G/C|\), and so, \(N \leq G = C\) as \(G = O_p^\beta(G)\), contradicting the assumption that \(N\) is non-abelian.

By Lemma 2.1(i) and Lemma 2.2(ii) applied to \(G/K\), we can find \(s \geq 1\) disjoint, non-empty subsets \(\Omega_1, \ldots, \Omega_s\) of \(\Omega\) such that

\[
(2.3) \quad |G : \text{Stab}_G(\Omega_1, \ldots, \Omega_s)|_p \geq p^{\min(a, 2)}.
\]

Here, we can take \(s \leq 2\) if \(a = 1\), \(s \leq 3\) if \(a \geq 2\) and \(p > 3\), and \(s \leq 4\) if \(a \geq 2\) and \(p = 3\). Note that \(a \geq 1\) if \(n > 1\) as \(G/K = O_p^\beta(G/K)\). So in the case \(a = 0\), we have that \(\Omega = \{T_1\}\) and can then take \(s = 1\) and \(\Omega_1 = \Omega\). We may also assume that \(T_1 \in \Omega_1\).
Recall that $T$ is a non-abelian simple $p'$-group. If $p > 3$, then we can find at least 3 nontrivial characters $\alpha_1, \ldots, \alpha_3 \in \text{Irr}(T)$ of pairwise distinct degrees. If $p = 3$, then $T \cong B_2(q)$ for some $q = 2^{2j+1}$ (this is a well-known result proved in unpublished work of J. G. Thompson) and so we can find at least 4 nontrivial characters $\alpha_1, \ldots, \alpha_4 \in \text{Irr}(T)$ of pairwise distinct degrees.

(ii) Suppose that $a \geq 1$. Consider

$$\theta = \beta_1 \times \beta_2 \times \ldots \times \beta_n \in \text{Irr}(N),$$

where $\beta_j \in \text{Irr}(T_j)$, $\beta_j(1) = \alpha_i(1)$ if $T_j \in \Omega_i$ for some $i \leq s$, and $\beta_j = 1$ if $T_j \notin \bigcup_{i=1}^s \Omega_i$. Then $I_G(\theta) \leq \text{Stab}_G(\Omega_1, \ldots, \Omega_s)$ and so (2.3) implies that $p^a$ divides $|G : I_G(\theta)|$. In particular, if $a \geq 2$ then it follows by Clifford’s theorem that any $\chi \in \text{Irr}(G | \theta)$ has degree divisible by $p^2$, a contradiction. We have shown that $a \leq 1$.

(iii) Consider the case $b \geq 1$. Note that $K/CN \hookrightarrow \text{Out}(T)^n$. Hence the condition $b \geq 1$ implies that $p \mid |\text{Out}(T)|$, but $p$ does not divide $|T|$. By Lemma 2.3 (iii), one can find a character $\alpha \in \text{Irr}(T)$ such that $p$ does not divide $|\text{Stab}_{\text{Out}(T)}(\alpha)|$. In the notation of (i), we can set $\alpha_1 = \alpha$. Using the transitivity of $G$ on $\Omega$ and that $K < G$, we may assume that the action of $K$ on $T_1$ induces a subgroup $X$ of $\text{Out}(T_1)$ of order divisible by $p$.

Recall we now have $a \leq 1$, and so, $s \leq 2$ by our construction of $\Omega_i$. Consider

$$\varphi = \beta_1 \times \beta_2 \times \ldots \times \beta_n \in \text{Irr}(N),$$

where $\beta_j \in \text{Irr}(T_j)$, $\beta_1 = \alpha_1$, $\beta_j(1) = \alpha_2(1)$ if $T_j \in \Omega_1 \setminus \{T_1\}$, $\beta_j(1) = \alpha_3(1)$ if $s > 1$ and $T_j \in \Omega_2$, and $\beta_j = 1$ if $T_j \notin \bigcup_{i=1}^s \Omega_i$. Then

$$J := I_G(\varphi) \leq H := \text{Stab}_G(\Omega_1, \ldots, \Omega_s).$$

We claim that

$$p^c := [G : J]_p \geq p^{\min(a+b,2)}.$$  \hspace{1cm} (2.4)

Assume the contrary: $c < \min(a + b, 2)$. Note that $[G : H]_p \geq p^a$ by (2.3). Hence, if $a = 1$ then we obtain that $p$ does not divide $|H : J|$ and so, as $p$ does not divide $|H/K|$, $J$ contains a Sylow $p$-subgroup $Q$ of $K$. By the construction of $\theta$, $J$ normalizes $T_1$, and so does $Q$. As $Q \in \text{Syl}_p(K)$, the action of $Q$ on $T_1$ must induce a (non-trivial) Sylow $p$-subgroup of $X$; also, $Q$ fixes $\alpha_1 = \alpha$. Thus, $p$ divides $|\text{Stab}_{\text{Out}(T)}(\alpha)|$, contradicting the choice of $\alpha$. So we must have that $a = 0$, whence $n = 1$ (as mentioned above) and $G = K$. As $c < b$ and $G/CN \hookrightarrow \text{Out}(T_1)$, we again see that $\alpha$ is fixed by some outer automorphism of $T$ of order $p$, again a contradiction.
shown in (ii) and (iii), there is a character \( \gamma \in \operatorname{Irr}(N) \) such that \( p \mid |G : I_G(\gamma)| \) (namely, \( \gamma = \theta \) if \( a = 1 \) and \( \gamma = \varphi \) if \( b = 1 \)). Since \( |G/C|_p = p \) and \( |G|_p \geq p^2 \), we have that \( p \mid |C|. \) Also, \( \mathcal{O}_p(C) \leq \mathcal{O}_p(G) = 1. \) It follows by the Itô-Michler theorem that \( p \mid (1) \) for some \( \delta \in \operatorname{Irr}(C). \) Clearly, \( G \triangleright NC \cong N \times C, \gamma \times \delta \in \operatorname{Irr}(NC), \) and \( I_G(\gamma \times \delta) \leq I_G(\gamma) \) has index divisible by \( p \) in \( G. \) By the Clifford theorem, \( p^2 \) divides the degree of any \( \chi \in \operatorname{Irr}(G \mid \gamma \times \delta), \) a contradiction. \( \square \)

Denote by \( E(G) \) the subgroup of \( G \) generated by all components of \( G, \) that is, quasisimple subnormal subgroups of \( G. \) Often, \( E(G) \) is called the layer of \( G. \) Then the generalized Fitting subgroup of \( G \) is the subgroup \( F^*(G) = E(G)F(G) \leq G. \) It is well known that \( C_G(F^*(G)) \leq F^*(G). \) Note that this next result includes Theorem 1.2.

**Theorem 2.5.** Let \( p \) be an odd prime, and let \( G \) be any \( p \)-solvable finite group. If \( a_p \leq p \) for all \( a \in \text{cd}(G), \) then the following statements hold.

(i) \( |G / \text{sol}(G)|_p \leq p \) and \( |G / \text{F}(\text{sol}(G))|_p \leq p^3. \)

(ii) \( |G / \text{F}(G)|_p \leq p^3. \)

(iii) Either \( |G / \text{F}(\text{sol}(G))|_p \leq p, \) or \( \text{F}^*(\mathcal{O}^p(G)) = \text{F}(\mathcal{O}^p(G)). \)

**Proof.** Let \( L := \mathcal{O}^p(G). \) Then \( |L|_p = |G|_p = \text{sol}(L) \leq \text{sol}(G), \) \( F(L) \leq F(G), \) and \( \text{F}^*(\text{sol}(L)) \leq \text{F}^*(\text{sol}(G)). \) Furthermore, \( a_p \leq p \) for all \( a \in \text{cd}(G) \) if and only if \( a_p \leq p \) for all \( a \in \text{cd}(L). \) Hence, we may replace \( G \) by \( L, \) and assume that \( G = \mathcal{O}^p(G). \)

(i) Let \( R := \text{sol}(G). \) Then \( G / R \) is \( p \)-solvable, \( G / R = \mathcal{O}^p(G / R), \) and \( a_p \leq p \) for all \( a \in \text{cd}(G / R). \) Furthermore, \( \text{sol}(G / R) = 1; \) in particular, \( \mathcal{O}_p(G / R) = 1. \) We are done if \( G = R. \) So let \( N / R \neq 1 \) be a minimal normal subgroup of \( G / R. \) As \( \text{sol}(G / R) = 1, \) we have that \( N / R \) is non-abelian. Now we can apply Proposition 2.4 and get that \( |G / R|_p \leq p. \) Obviously, \( R \) is solvable and \( a_p \leq p \) for all \( a \in \text{cd}(R). \) Hence, \( |R / \text{F}(R)|_p \leq p^2 \) by the main result of [16]. It follows that \( |G / \text{F}(R)|_p \leq p^3. \)

(ii) This follows from (i) as \( \text{F}(\text{sol}(G)) \leq \text{F}(G). \)

(iii) Assume that \( E(G) \nleq \text{F}(G). \) Let \( Z := \text{Z}(E(G)) \) and let \( M / Z \) be a (non-abelian) minimal normal subgroup of \( G / Z. \) Also, let \( L := M^{(\infty)} \triangleleft G, \) so that \( M = LZ, L \) is perfect, \( L \cap Z = \text{Z}(L), \) and \( L / \text{Z}(L) \cong M / Z \) is a non-abelian minimal normal subgroup of \( G / \text{Z}(L). \) Set \( P / \text{Z}(L) = \mathcal{O}_p(G / \text{Z}(L)). \) Then \( P \cap L = \text{Z}(L), \) \( |P, L| \leq \text{Z}(L), \) and so by the
three-subgroup lemma,

\[ [P, L] = [P, [L, L]] \leq [[P, L], L] = 1. \]

In particular, \([P, Z(L)] = 1\). Writing \(Z(L) = O_{p'}(Z(L)) \times O_p(Z(L))\), we now have that \(P = O_{p'}(Z(L))Q = O_{p'}(Z(L)) \times Q\) for \(Q \in \text{Syl}_p(P)\), and so \(Q \leq O_p(G)\).

Now, we can embed \(L/Z(L)\) in \(G/P\) as a non-abelian minimal normal subgroup. As \(O_p(G/P) = 1\), \(G/P = O_{p'}(G/P)\), and \(G/P\) is \(p\)-solvable and \(a_p \leq p\) for all \(a \in \text{cd}(G/P)\), we have \(|G/P|_p \leq p\). It follows that \(|G/F(G)|_p \leq p\).

\(\square\)

3. Non \(p\)-solvable groups

We continue to assume that \(a_p \leq p\) for all \(a \in \text{cd}(G)\), but we now drop the assumption that \(G\) is \(p\)-solvable. All degrees in this section are ordinary (complex) character degrees. We first begin with the following result due to Gagola [7].

**Lemma 3.1.** Let \(S\) be a nonabelian simple group and let \(p\) be a prime dividing \(|S|\). Then \(|S|_p > |\text{Out}(S)|_p\).

*Proof.* If \(S\) is a sporadic simple group, an alternating group \(\text{Alt}_n\) with \(n \geq 7\) or the Tits group, then \(|\text{Out}(S)|\) divides 2. Since \(S\) is nonabelian simple, we have \(|S|_2 \geq 4\). Therefore, the inequality \(|S|_p > |\text{Out}(S)|_p\) is trivially true for odd prime \(p\) and it also holds for \(p = 2\). Notice that \(\text{Alt}_6 \cong \text{PSL}_2(9)\) and \(\text{Alt}_5 \cong \text{PSL}_2(5)\). Finally, if \(S\) is a finite simple group of Lie type, then the result follows from Equation 3.3 in [7]. \(\square\)

The next result is also due to Gagola [7] which can be proved directly.

**Lemma 3.2.** Let \(S\) be a nonabelian simple group and let \(G\) be an almost simple group with \(S \leq G \leq \text{Aut}(S)\). Let \(p\) be an odd prime. Suppose that \(p\) divides \(|S|\) and \(a_p \leq p\) for all \(a \in \text{cd}(G)\). Then either \(S \cong \text{Alt}_7\), \(p = 3\), and \(|G|_p = p^2\); or \(|G|_p = p\). In both cases, we have \(|G|_p = |S|_p \leq p^2\).

*Proof.* If \(S \cong \text{Alt}_7\) and \(p = 3\), then the result follows easily by using [4]. Therefore, we can assume that \((S, p) \neq (\text{Alt}_7, 3)\). Clearly, \(a_p \leq p\) for all \(a \in \text{cd}(S)\). Since \(p \mid |S|\), [7, Lemma 1.2] yields that \(|\text{Aut}(S)|_p < a_p^2 \leq p^2\) for some degree \(a \in \text{cd}(S)\), and thus, \(|G|_p \leq |\text{Aut}(S)|_p < p^2\) which forces \(|G|_p = p\). Since \(|S|\) divides \(|G|\) and \(p \mid |S|\), we have \(|G|_p = |S|_p = p\) as required. \(\square\)

For a fixed prime \(p\), we denote by \(\text{sol}_p(G)\) the \(p\)-solvable radical of \(G\) which is the unique largest normal \(p\)-solvable subgroup of \(G\). Notice
that by the Feit-Thompson Odd-Order theorem, we have \( \text{sol}_2(G) = \text{sol}(G) \).

**Lemma 3.3.** Let \( G \) be a finite group, \( p \) be a prime, and \( L \) a normal subgroup of \( G \). Then the following hold:

(i) \( F(L) = L \cap F(G) \).

(ii) If \( p \) does not divide \( |G : L| \), then \( |G : F(G)|_p = |L : F(L)|_p \).

(iii) \( \text{sol}_p(L) = L \cap \text{sol}_p(G) \) and \( \text{sol}_p(G/\text{sol}_p(G)) \) is trivial.

The proof of this lemma is nearly trivial, so we omit it.

We are curious if one could mimic the proof of Proposition 2.4 to obtain a result in this next lemma when \( p \) does not divide \( |N/M| \). In particular, we wonder if one could prove that \( |G/C|_p \leq p \) in this situation. Unfortunately, at this time we have not been able to prove anything like that.

**Lemma 3.4.** Let \( p \) be an odd prime, and let \( G \) be a finite group with \( O_p'(G) = G \) and \( a_p \leq p \) for all \( a \in \text{cd}(G) \). Suppose \( N/M \) is a nonabelian chief factor of \( G \) with \( p \) dividing \( |N/M| \), and set \( C/M = C_{G/M}(N/M) \). Then

1. \( N/M \) is nonabelian simple.
2. \( G/M \cong C/M \times N/M \).
3. \( C/M \) is an abelian \( p \)-group.

**Proof.** As \( N/M \) is nonabelian, \( N/M \cong S^k \) for some nonabelian simple group \( S \) and some integer \( k \geq 1 \). As \( p \) divides \( |N/M| = |S|^k \), it follows that \( p \) divides \( |S| \), and thus, by the Itô-Michler Theorem [19, Theorem 5.4], \( p \mid \theta(1) \) for some character \( \theta \in \text{Irr}(S) \). Since \( \theta^k \in \text{Irr}(N/M) \) and \( \theta(1)^k \) is divisible by \( p^k \), we see that \( p^k \) divides some degree in \( \text{cd}(G) \). This forces \( k = 1 \). So \( N/M \cong S \) with \( S \) a nonabelian simple group. It follows that \( G/C \) is an almost simple group with socle \( S \). By Lemma 3.2, \( |G/C|_p = |S|_p \) and \( G/NC \) is a \( p' \)-group. Since \( O_p'(G) = G \), we deduce that \( G = NC \), and thus, \( G/M \cong C/M \times N/M \) as \( C \cap N = M \). Now, if \( p \mid \lambda(1) \) for some character \( \lambda \in \text{Irr}(C/M) \), then \( \lambda \times \theta \in \text{Irr}(G/M) \) whose degree is divisible by \( p^2 \), a contradiction. Hence, \( p \) divides no degree of \( C/M \), and thus, \( C/M \) possesses a normal abelian Sylow \( p \)-subgroup, say \( K/M \). Clearly, \( K \leq G \), and thus, \( KN \leq G \). But then \( G/KN \) is a \( p' \)-group, so \( G = KN \) and \( C/M = K/M \) is an abelian \( p \)-group as wanted. \( \square \)

Studying the proof of Theorem D in [17] carefully, one can see that the authors only used the condition that \( 9 \) divides no degree of the group. So, we obtain the following.
Lemma 3.5. Let $G$ be a finite perfect group and suppose that $a_3 \leq 3$ for all $a \in \text{cd}(G)$. Suppose that $G/M \cong \text{Alt}_7$ for some solvable normal subgroup $M$ of $G$. Then $G \cong \text{Alt}_7$ or $3 \cdot \text{Alt}_7$.

Recall that a group $G$ is a central product of two normal subgroups $A$ and $B$, which we denote by $G = A \circ B$, if $G = AB$ and $[A,B] = 1$.

Corollary 3.6. Let $G$ be a finite group with $\text{O}^3(G) = G$. Suppose that $3^2$ divides no degree of $G$ and that $\text{Alt}_7$ is involved in $G$. Then $G \cong N \circ C$, where $C$ is an abelian 3-group and $N \cong \text{Alt}_7$ or $3 \cdot \text{Alt}_7$.

Proof. Let $N$ be the last term of the derived series of $G$. Then $G/N$ is solvable. Also, $N$ is perfect, and $\text{Alt}_7$ is involved in $N$. By Lemma 3.5, we have $N \cong \text{Alt}_7$ or $3 \cdot \text{Alt}_7$. Let $M = \mathbb{Z}(N)$. Then $M \triangleleft G$ and $N/M \cong \text{Alt}_7$ is a chief factor of $G$ with $p$ dividing $|N/M|$. Let $C/M = \mathbb{C}_G/M(N/M)$. By Lemma 3.4, $G/M = C/M \times N/M$ and $C/M$ is an abelian 3-group. It follows that $C$ is a normal 3-subgroup of $G$ and $G/C \cong N/M \cong \text{Alt}_7$. If $M = 1$, then the result follows. So, we assume that $M$ is nontrivial. Then $N \cong 3 \cdot \text{Alt}_7$ and $|M| = 3$. We see that $N \cap C = M = \mathbb{Z}(N)$. As $[N,N] = N$ and $[N,C] \leq \mathbb{Z}(N)$, we apply the Three Subgroups Lemma to obtain $[N,C] = 1$. We deduce that $G = N \circ C$ is a central product. It remains to show that $C$ is abelian. Suppose that $C$ is nonabelian. Then we can find a character $\lambda \in \text{Irr}(C)$ with $\lambda(1) > 1$, hence $3 \mid \lambda(1)$ since $C$ is a 3-group. As $C/M$ is abelian, there exists a character $\nu$ with $1_M \neq \nu \in \text{Irr}(M)$ such that $\lambda \in \text{Irr}(C \mid \nu)$. Using [4], we can find a character $\varphi \in \text{Irr}(N \mid \nu)$ with $3 \mid \varphi(1)$. Now, by the representation theory of central products (see Lemma 5.1 of [11]), $\chi = \varphi \cdot \lambda \in \text{Irr}(G)$ and $\chi(1) = \varphi(1)\lambda(1)$ is divisible by 9, which is a contradiction. \qed

We now consider groups that are not $p$-solvable satisfying $a_p \leq p$ for all $a \in \text{cd}(G)$.

Proposition 3.7. Let $p$ be an odd prime, and let $G$ be a finite non-$p$-solvable group with a trivial $p$-solvable radical and $\text{O}^d(G) = G$. If $a_p \leq p$ for all $a \in \text{cd}(G)$, then either $G$ is a nonabelian simple group with $|G|_p = p$ or $p = 3$ and $G \cong \text{Alt}_7$.

Proof. Since $\text{sol}_p(G) = 1$, we have $\mathbf{F}(G) = 1$, and thus, $\mathbf{F}^*(G) = \mathbf{E}(G) = T_1 \times T_2 \times \cdots \times T_k$, where $T_1, \ldots, T_k$ are nonabelian simple groups with $p$ dividing $|T_i|$ and $\mathbf{C}_G(\mathbf{F}^*(G)) = 1$. For each $i \in \{1, \ldots, k\}$, we have $p \mid |T_i|$ and $T_i$ is nonabelian simple. Thus, there exist characters $\theta_i \in \text{Irr}(T_i)$ with $p \mid \theta_i(1)$. Let $\psi = \theta_1 \times \cdots \times \theta_k \in \text{Irr}(\mathbf{F}^*(G))$. Then $\psi(1)$ is divisible by $p^k$ and divides some degree in $\text{cd}(G)$. This forces $k = 1$. Hence, $\mathbf{F}^*(G)$ is a nonabelian simple group, and thus, $G$ is an
almost simple group with socle $F^*(G)$. The result now follows from Lemma 3.2.

In the proof of Proposition 3.7, we can use the solvable radical $\text{sol}(G)$ rather than the $p$-solvable radical $\text{sol}_p(G)$. However, there are cases when $F^*(G)$ is a product of nonabelian simple groups whose order are coprime to $p$. For this, one would need to show that $|G|_p \leq p$.

We are now ready to prove our main result. We say that $H$ is a section of a group $G$ if there exist subgroup $A$ and $B$ in $G$ so that $B$ is subnormal in $G$ and $A$ is normal in $B$ with $B/A \cong H$. Observe that if $a_p \leq p$ for all $a \in \text{cd}(G)$, then every section of $G$ satisfies this same property. By Theorem 1.2, we know that the hypothesis of this next theorem will hold with $\nu = 3$. This will yield Theorem 1.1. However, we believe that the correct bound in Theorem 1.2 is that $|G : F(G)|_p \leq p^2$, this would allow the use of $\nu = 2$ in the next theorem, and we would obtain the conclusion that $|G : F(G)|_p \leq p^3$.

**Theorem 3.8.** Let $p$ be an odd prime, and suppose that $\nu$ is a positive integer and $G$ is a group such that every section $H$ of $G$ that is $p$-solvable satisfies $|H : F(H)|_p \leq p^\nu$. If $a_p \leq p$ for all $a \in \text{cd}(G)$, then $|G : F(G)|_p \leq p^{1+\nu}$.

**Proof.** If $G$ is $p$-solvable, then $|G : F(G)|_p \leq p^\nu < p^{1+\nu}$ and we are done. So, we assume that $G$ is not $p$-solvable. If $L = O_p^\nu(G)$, then $O_p^\nu(L) = L$ and by Lemma 3.3, $|G : F(G)|_p = |L : F(L)|_p$. By induction on $|G|$, one may assume that $G = O_p^\nu(G)$. Let $R_p$ be the $p$-solvable radical of $G$. Then $G/R_p$ has a trivial $p$-solvable radical and $O_p^\nu(G/R_p) = G/R_p$.

Proposition 3.7 yields $G/R_p \cong \text{Alt}_7$ and $p = 3$ or $G/R_p$ is a nonabelian simple group with $|G/R_p|_p = p$. Assume that the first case holds. By Corollary 3.6, we have $G = N \circ C$, where $N \cong \text{Alt}_7$ or $3 \cdot \text{Alt}_7$ and $C$ is a normal abelian 3-group. It follows that $C = R_p = O_3(G) = F(G)$, and clearly, $|G : F(G)|_3 = |\text{Alt}_7|_3 = 3^2 \leq 3^{1+\nu}$. Now assume that the latter case holds. Since $F(G) \leq R_p$, we have $F(R_p) = F(G) \cap R_p = F(G)$. Now, $R_p$ is a $p$-solvable group and is normal in $G$ and $a_p \leq p$ for all $a \in \text{cd}(R_p)$, so we obtain:

$$|G : F(G)|_p = |G : R_p|_p \cdot |R_p : F(R_p)|_p \leq p \cdot p^\nu = p^{1+\nu}.$$

\[\Box\]

**4. Brauer characters**

We now turn to the universe of Brauer characters and block theory. We follow the notation in [21] for block theory. Fix a prime $p$, and let $G$ be a finite group. For a $p$-block (or just block) $B$ of $G$, we denote
by $D = D(B)$ the defect group of $B$; also, we set $\text{Irr}(B) = B \cap \text{Irr}(G)$ and $\text{IBr}(B) = B \cap \text{IBr}(G)$. The defect $d(B)$ of $B$ is defined by

$$p^{a-d(B)} = \min\{\chi(1)_p \mid \chi \in \text{Irr}(B)\}$$

where $p^a = |G|_p$. The height $h(\chi)$ of a character $\chi \in \text{Irr}(B)$ is defined by the formula

$$p^{a-d(B)+h(\chi)} = \chi(1)_p.$$ 

By [21, Corollary 3.17], we have

$$p^{a-d(B)} = \min\{\varphi(1)_p \mid \varphi \in \text{IBr}(B)\}$$

and by [21, Theorem 4.6], $|D(B)| = p^{d(B)}$.

The following is the ‘if part’ of the famous Brauer’s height zero conjecture which was recently proved by R. Kessar and G. Malle.

**Lemma 4.1.** ([13, Theorem 1.1]) Let $p$ be a prime and $G$ be a group. Let $B$ be a $p$-block of $G$ with defect group $D$. If $D$ is abelian, then the degrees of the characters in $\text{Irr}(B)$ all have the same $p$-part, which is $|G : D|_p$.

We now are ready to prove Theorem 1.3 from the Introduction.

**Proof of Theorem 1.3.** Suppose that $p$ is a prime, $G$ is a finite group, and $\beta(1)_p \leq p$ for all $\beta \in \text{IBr}(G)$. Let $\chi \in \text{Irr}(G)$. If $\chi(1)_p = 1$, then we have nothing to prove. Suppose that $\chi(1)_p = p^m$ for some integer $m \geq 1$. As $\text{Irr}(G)$ is a disjoint union of $\text{Irr}(B)$, where $B$ runs over the set of all $p$-blocks of $G$, we see that $\chi \in \text{Irr}(B)$ for some block $B$ of $G$. As $G$ has an abelian Sylow $p$-subgroup of order, say $p^a$, we know that $D = D(B)$ is abelian, and thus, by Lemma 4.1 $\chi$ has height zero. That is, $\chi(1)_p = p^{a-d(B)}$. This implies that $a - d(B) = m \geq 1$. Since $p^{a-d(B)} = p^m$ divides all $\beta(1)$ with $\beta \in \text{IBr}(B)$ and $\beta(1)_p \leq p$, we conclude that $m = 1$. Therefore, $a_p \leq p$ for all $a \in \text{cd}(G)$ as desired.

We next prove Corollary 1.4 from the Introduction.

**Proof of Corollary 1.4.** If $p^2$ does not divide $\chi(1)$ for all $\chi \in \text{Irr}(G)$, then $p^3$ does not divide $\varphi(1)$ for all $\varphi \in \text{IBr}(G)$ since $\varphi = \psi^\phi$ for some $\psi \in \text{Irr}(G)$ by Fong-Swan Theorem [21, Theorem 10.1]. Conversely, suppose that $p^2$ does not divide $\varphi(1)$ for all $\varphi \in \text{IBr}(G)$. Since $\text{IBr}(G) = \text{IBr}(G/\mathcal{O}_p(G))$, we can assume that $\mathcal{O}_p(G) = 1$, and so, $G$ has abelian Sylow $p$-subgroups by [24, Corollary 2.6]. Now by Theorem 1.3 $a_p \leq p$ for all $a \in \text{cd}(G)$. 

□
We follow [21] for the notation of Brauer characters, and we refer to [12] for the representation theory of symmetric groups. Let $p$ be a prime, let $n$ be a natural number, and let $\lambda \vdash n$ be a partition of $n$; and write $|\lambda| = n$. The irreducible character of $\text{Sym}_n$ labeled by the partition $\lambda$ is denoted by $\chi^\lambda$. Let $\lambda_p(\rho)$ be the $p$-core of $\lambda$ obtained by repeatedly removing all $p$-hooks. The $p$-weight $\omega_p(\lambda)$ is the number of $p$-hooks we remove to obtain the $p$-core, and we have $|\lambda| = |\lambda_p(\rho)| + p\omega_p(\lambda)$. Nakayama’s conjecture asserts that two irreducible characters of $\text{Sym}_n$ labeled by two partitions $\lambda$ and $\mu$ of $n$ belong to the same $p$-block if and only if $\lambda$ and $\mu$ have the same $p$-core. Now, the $p$-blocks of $\text{Sym}_n$ are labeled by the $p$-core and the $p$-weight $w(B)$ of $B$ which is the common $p$-core $w_p(\lambda)$ for all partitions $\lambda \vdash n$ with $\chi^\lambda \in \text{Irr}(B)$.

Finally, the defect group $D = D(B)$ of $B$ is the Sylow $p$-subgroup of $\text{Sym}_{w(B)}$. (See [2].) Given the pair $(\alpha, w)$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a $p$-core partition and $w \in \mathbb{N}$, then $\lambda := (\alpha_1 + pw, \alpha_2, \ldots, \alpha_k)$ is a partition of $n := |\alpha| + pw$ with $p$-core $\lambda_p(\rho) = \alpha$ and $p$-weight $\omega_p(\lambda) = w$.

Recall that for a finite group $G$ and a prime $p$, an ordinary irreducible character $\chi \in \text{Irr}(G)$ is said to have $p$-defect zero if $\chi(1)_p = |G|_p$. [21] Theorem 3.8 yields that $\chi^\circ \in \text{IBr}(G)$ with $\chi^\circ(1) = \chi(1)$, where $G^\circ$ is the set of all $p$-regular elements of $G$.

**Lemma 4.2.** Let $p$ be a prime and let $G$ be an almost simple group with nonabelian simple socle $S$. Suppose that $p$ divides $|S|$ and $\beta(1)_p \leq p$ for all $\beta \in \text{IBr}(G)$. Then the following hold.

(i) If $p = 2$, then $G = S \cong M_{22}$ and $|G|_2 = 2^7$.

(ii) If $p = 3$ and $S \cong \text{Alt}_7$, then $|G|_3 = |S|_3 = 3^2$.

(iii) If $p \geq 3$ and $(S, p) \neq (\text{Alt}_7, 3)$, then $|G|_p = |S|_p = p$.

**Proof.** We observe first that if $S$ has an irreducible character $\chi$ with $p$-defect zero, then $\chi^\circ \in \text{IBr}(S)$ with $\chi^\circ(1) = \chi(1)$. Hence, $\chi^\circ(1)_p = |S|_p$. From the hypothesis, we have $|S|_p = p$. We consider the case $p = 2$ and $p > 2$ separately.

(1) $p = 2$. Since $S$ is nonabelian simple, we know that $|S|_2 \geq 4$, so $S$ does not have any irreducible character of 2-defect zero. By [9, Corollary 2], $S \cong \text{Alt}_n$ or $S$ is isomorphic to $M_{12}, M_{22}, M_{24}, J_2, HS, \text{Suz, Ru, Co}_1, \text{Co}_3$, or $B$. Assume that $S$ is a sporadic simple group. Except for the Baby Monster, one can use [4] to check that only $M_{22}$ satisfies the hypothesis with $|M_{22}|_2 = 2^7$. As $M_{22} \cdot 2$ does not satisfy the hypothesis, we deduce that $G = S \cong M_{22}$ with $|G|_2 = |S|_2 = 2^7$. For the Baby Monster, using [6], $B$ has a 2-block $B$ with defect $d(B) = 3$. Since $|B|_2 = 2^{41}$, we deduce that if $\phi \in \text{IBr}(B)$, then $2^{38} = 2^{41-3}$ divides $\phi(1)$. Thus, this case cannot happen.
Assume that \( S \cong \text{Alt}_n \) with \( n \geq 5 \). Using [6], we can assume that \( n \geq 12 \). From Proposition 5.2 and Corollary 4.2 in [1], \( \text{Alt}_n \) has a 2-Brauer irreducible character \( \beta \) which is the restriction of the irreducible 2-Brauer character \( D^{(n-3,3)} \) of \( \text{Sym}_n \) to \( \text{Alt}_n \) with degree

\[
\beta(1) = \begin{cases} 
\frac{1}{6} n(n-2)(n-7) & \text{if } n \equiv 0 \pmod{4}, \\
\frac{1}{5} n(n-1)(n-5) & \text{if } n \equiv 1 \pmod{4}, \\
\frac{1}{6} (n-1)(n-2)(n-6) & \text{if } n \equiv 2 \pmod{4}, \\
\frac{1}{6} (n+1)(n-1)(n-6) & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

Now, it is routine to check that \( 4 \mid \beta(1) \) in any of these cases.

(2) \( p \) is odd. If \( (S,p) = (\text{Alt}_7,3) \), then (ii) holds by using [4]. So, assume from now on that \( (S,p) \neq (\text{Alt}_7,3) \). We first claim that \( |S|_p = p \). If \( S \) has an irreducible character of \( p \)-defect zero, then we are done. So assume that \( S \) has no such character. By [9, Corollary 2], we have \( p = 3 \) and \( S \cong \text{Alt}_n \) or \( S \cong \text{Suz} \) or \( \text{Co}_3 \). If \( S \cong \text{Alt}_n \) with \( 5 \leq n \leq 19 \) or \( S \) is one of the two sporadic simple group above, then the result follows by using [4]. Thus we can assume that \( S \cong \text{Alt}_n \) with \( n \geq 20 \). As \( p \geq 3 \) and \( \text{Sym}_n / \text{Alt}_n \) is a cyclic group of order 2, we deduce that \( p^2 \) does not divide \( \varphi(1) \) for all \( \varphi \in \text{IBr}(\text{Sym}_n) \). Notice that if \( \varphi \in \text{IBr}(\text{Sym}_n \mid \beta) \) for some \( \beta \in \text{IBr}(\text{Alt}_n) \), then \( \varphi(1) / \beta(1) \) divides 2 = \( |\text{Sym}_n : \text{Alt}_n| \) (see [21, Theorem 8.30]) so \( \varphi(1)_p = \beta(1)_p \).

For each integer \( n \), we can choose two 3-regular partitions \( \lambda, \mu \) of \( n \) such that the corresponding 3-cores \( \lambda(3), \mu(3) \) are as follow:

- (a) \((3,1)\) and \((5,3,1^2)\), if \( n \equiv 1 \pmod{3}\);
- (b) \((3,1^2)\) and \((4,2,1^2)\), if \( n \equiv 2 \pmod{3}\);
- (c) \((4,2)\) and \((5,3,1)\), if \( n \equiv 0 \pmod{3}\).

Since \( \lambda \) and \( \mu \) have different 3-cores, we know that they belong to different 3-blocks, say \( B_1 \) and \( B_2 \) of \( \text{Sym}_n \). Because \( 3w(B_1) = n - |\lambda(3)| \) and \( 3w(B_2) = n - |\mu(3)| \) where \( |\mu(3)| - |\lambda(3)| \) is 3 or 6, we deduce that \( 3w(B_1) = 3w(B_2) + 3 \) or 6. This implies that the orders of the Sylow 3-subgroups of \( \text{Sym}_{3w(B_1)} \) and \( \text{Sym}_{3w(B_2)} \), namely, \( 3^{d(B_i)} \), which are also the orders of the defect group of the block \( B_i \), are distinct and strictly less than \( |\text{Sym}_n|_p = p^3 \). By [21, Corollary 3.7], we have \( 3^{a - d(B_i)} = \min\{ \varphi(1)_3 \mid \varphi \in \text{IBr}(B_i) \} \). As \( a > d(B_1) > d(B_2) \geq 1 \), we deduce that \( a - d(B_2) > a - d(B_1) \geq 1 \), and thus, \( a - d(B_2) \geq 2 \). So if \( \beta \in \text{IBr}(B_2) \), then \( p^2 \mid \beta(1) \), a contradiction.

We now claim that \( |G|_p = |S|_p = p \). By Lemma 3.1, we have that \( |S|_p > |\text{Out}(S)|_p \) whenever \( p \) divides \( |S| \). As \( |S|_p = p \), we deduce that \( |G : S|_p \) divides \( |\text{Out}(S)|_p \), where the latter is strictly less than \( p \). This forces \( |G : S|_p = 1 \), and thus, \( |G|_p = |S|_p = p \) as desired. \( \square \)
Combining the results of Leisering [14] with those of Lewis, Navarro and Wolf [16], we obtain:

**Lemma 4.3.** Let $p$ be a prime and let $G$ be a finite solvable group with $O_p(G) = 1$. If $\beta(1)_p \leq p$ for all $\beta \in IBr(G)$, then $|G|_p \leq p^2$ and a Sylow $p$-subgroup of $G$ is elementary abelian.

**Proof.** Let $P$ be a Sylow $p$-subgroup of $G$. By [14] Theorem 1, $P$ is elementary abelian and $a_p \leq p$ for all $a \in cd(G)$. Now, Theorem 1 in [16] yields that $|G : F(G)|_p \leq p^2$. Since $O_p(G) = 1$, we see that $F(G)$ is a $p'$-group, and thus, $|G|_p = |G : F(G)|_p \leq p^2$. □

We see when $O'p(G) = G$ and $O_p(G) = 1 = \text{sol}_p(G)$ that $G$ is simple.

**Lemma 4.4.** Let $p$ be a prime, and let $G$ be a non-$p$-solvable group. If $O'p(G) = G$, $O_p(G) = 1 = \text{sol}_p(G)$, and $\beta(1)_p \leq p$ for all $\beta \in IBr(G)$, then $G$ is isomorphic to one of the nonabelian simple groups in Lemma 4.2.

**Proof.** Since $\text{sol}_p(G) = 1$, we know that $F^*(G) = E(G) = T_1 \times T_2 \times \cdots \times T_k$, where $T_1, \ldots, T_k$ are nonabelian simple groups with $p$ dividing $|T_i|$ for all $1 \leq i \leq k$ and $C_G(F^*(G)) = 1$. By [19] Theorem 5.5, for each $i$, there exists a Brauer character $\theta_i \in IBr(T_i)$ with $p \mid \theta_i(1)$. Clearly, $\varphi = \theta_1 \times \cdots \times \theta_k \in IBr(F^*(G))$ and $\varphi(1)$ is divisible by $p^k$. If $\gamma \in IBr(G \mid \varphi)$, then $p^k$ divides $\gamma(1)$. This forces $k = 1$. Thus, $G$ is an almost simple group with the nonabelain simple socle $F^*(G)$. Lemma 4.2 now yields that $|G|_p = |F^*(G)|_p$. Since $O'p(G) = G$, we deduce that $G = F^*(G)$, and so, $G$ is a nonabelian simple group appearing in Lemma 4.2. □

We now prove Theorem 1.5 when $p$ is odd. As with Theorem 3.8, we know that $\nu = 3$ will hold, but this allows the possibility that $\nu = 2$.

**Theorem 4.5.** Let $G$ be a finite group, and let $p$ be an odd prime with $O_p(G) = 1$. Suppose that $\nu$ is a positive integer so that every $p$-solvable section $H$ of $G$ satisfies $|H : F(H)|_p \leq p^\nu$. If $\beta(1)_p \leq p$ for all $\beta \in IBr(G)$, then the following hold:

1. If $p \geq 5$ or $p = 3$ and $\text{Alt}_7$ is not involved in $G$, then $|G|_p \leq p^{1+\nu}$.
2. If $p = 3$ and $\text{Alt}_7$ is involved in $G$, then $|G|_3 \leq 3^{2+\nu}$.

**Proof.** We proceed by using induction on $|G|$. Notice that the hypotheses of the theorem are inherited by normal subgroups. Let $L = O'p(G)$, and let $R_p$ be the $p$-solvable radical of $G$. We have that $O_p(R_p) = 1$. If $L < G$, then $O_p(L) = 1$ and $|G|_p = |L|_p$ since $G/L$ is a $p'$-group.
So $L$ satisfies the hypotheses of the theorem, and by induction, either $|L|_p \leq p^{1+\nu}$ or $p = 3$, $\text{Alt}_7$ is involved in $L$, and $|L|_3 \leq 3^{2+\nu}$. Since $|G|_p = |L|_p$ and the fact that $\text{Alt}_7$ is involved in $G$ if and only if it is involved in $L$ (noting that $3 \mid |\text{Alt}_7|$), the conclusions of the theorem hold. Therefore, one can assume that $G = O^{2'}(G)$.

If $G$ is $p$-solvable, then $|G|_p \leq p^{\nu} < p^{1+\nu}$ and we are done. So we assume that $G$ is not $p$-solvable. Then $G/R_p$ satisfies the assumption of Lemma 4.4, and we deduce that either $|G/R_p|_p = p$ or $p = 3$ and $G/R_p \cong \text{Alt}_7$. Assume that $|G/R_p|_p = p$. By hypothesis since $R_p$ is $p$-solvable, we have $|R_p|_p \leq p^{\nu}$, and we conclude that $|G|_p = |G/R_p|_p \cdot |R_p|_p \leq p^{\nu} \cdot p = p^{1+\nu}$. Assume now that $G/R_p \cong \text{Alt}_7$ and $p = 3$. Then $|G|_3 = |G/R_3|_3 \cdot |R_3|_3 \leq 3^2 \cdot 3^\nu = 3^{2+\nu}$.

□

Finally, we prove Theorem 1.5 for $p = 2$.

**Theorem 4.6.** Let $G$ be a finite group with $O_2(G) = 1$, and let $P$ be a Sylow 2-subgroup of $G$. If $\beta_2(1) \leq 2$ for all $\beta \in \text{IBr}(G)$, then $P^{(4)} = 1$ and $|G|_2 \leq 2^9$.

**Proof.** By Lemma 4.3, we may assume that $G$ is nonsolvable. Using induction on $|G|$, we may assume that $O^{2'}(G) = G$. Let $R$ be the solvable radical of $G$. Lemmas 4.4 and 4.2 yield that $G/R \cong M_{22}$. Since $O_2(G) = 1$, we have $O_2(R) = 1$, and so, by Lemma 4.3, $|R|_2 \leq 2^2$. Hence, $|G|_2 = |G : R|_2 \cdot |R|_2 \leq 2^9$. Finally, $PR/R$ and $P \cap R$ are Sylow 2-subgroups of $G/R \cong M_{22}$ and $R$, respectively. From [4], we know that $PR/R$ has derived length 3 and $P \cap R$ is abelian by Lemma 4.3, therefore, $P''' \leq P \cap R$ and hence $P^{(4)} = 1$. This completes the proof.

□

It would be desirable for one to obtain a detailed structure of groups in Theorem 4.6.

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