

# Hunt's hypothesis (H) and triangle property of the Green function

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## Abstract

Let  $X$  be a locally compact abelian group with countable base and let  $\mathcal{W}$  be a convex cone of positive numerical functions on  $X$  which is invariant under the group action and such that  $(X, \mathcal{W})$  is a balayage space or (equivalently, if  $1 \in \mathcal{W}$ ) such that  $\mathcal{W}$  is the set of excessive functions of a Hunt process on  $X$ ,  $\mathcal{W}$  separates points, every function in  $\mathcal{W}$  is the supremum of its continuous minorants in  $\mathcal{W}$ , and there exist strictly positive continuous  $u, v \in \mathcal{W}$  such that  $u/v \rightarrow 0$  at infinity.

Assuming that there is a Green function  $G > 0$  for  $X$  which locally satisfies the triangle inequality  $G(x, z) \wedge G(y, z) \leq CG(x, y)$  (true for many Lévy processes), it is shown that Hunt's hypothesis (H) holds, that is, every semipolar set is polar.

Keywords: Hunt process; Lévy process; balayage space; Green function; 3G-property; continuity principle; polar set; semipolar set; hypothesis (H).

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The purpose of this short paper is to show that in the settings considered in [6, 7, 9, 10] Hunt's hypothesis (H) holds, that is, semipolar sets are polar provided the underlying space  $X$  is an abelian group and the set  $\mathcal{W}$  of positive hyperharmonic functions on  $X$  (the set of excessive functions of a corresponding Hunt process) is invariant under the group action. The essential property we use is a local triangle property of a Green function for  $(X, \mathcal{W})$ . Our results constitute a contribution to the long-lasting discussion of Gettoor's conjecture, that is, of the validity of (H) for all "reasonable" Lévy processes (see [3, 12] and Example 3).

Let  $X$  be a locally compact space with countable base. Let  $\mathcal{C}(X)$  denote the set of all continuous real functions on  $X$  and let  $\mathcal{B}(X)$  be the set of all Borel measurable numerical functions on  $X$ . The set of all (positive) Radon measures on  $X$  will be denoted by  $\mathcal{M}(X)$ .

Moreover, let  $\mathcal{W}$  be a convex cone of positive lower semicontinuous numerical functions on  $X$  such that  $(X, \mathcal{W})$  is a balayage space (see [2], [5] or [9, Appendix]). In particular, the following holds:

(C)  $\mathcal{W}$  linearly separates the points of  $X$ , for every  $w \in \mathcal{W}$ ,

$$w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) : v \leq w\},$$

and there are strictly positive  $u, v \in \mathcal{W} \cap \mathcal{C}(X)$  such that  $u/v \rightarrow 0$  at infinity.

**REMARKS 1.** 1. If  $1 \in \mathcal{W}$ , then there exists a Hunt process  $\mathfrak{X}$  on  $X$  such that  $\mathcal{W}$  is the set  $E_{\mathbb{P}}$  of excessive functions for the transition semigroup  $\mathbb{P} = (P_t)_{t>0}$  of  $\mathfrak{X}$  (see [5, Proposition 1.2.1] and [2, IV.8.1]), that is,

$$\mathcal{W} = \{v \in \mathcal{B}^+(X) : \sup_{t>0} P_t v = v\}.$$

2. Let us note that the condition  $1 \in \mathcal{W}$  is not very restrictive. Indeed, if  $(X, \mathcal{W})$  is a balayage space,  $w_0 \in \mathcal{W} \cap C(X)$  is strictly positive, and  $\widetilde{\mathcal{W}} := \{w/w_0 : w \in \mathcal{W}\}$ , then  $(X, \widetilde{\mathcal{W}})$  is a balayage space such that  $1 \in \widetilde{\mathcal{W}}$ , and results for  $(X, \widetilde{\mathcal{W}})$  yield results for  $(X, \mathcal{W})$ .

3. Moreover, given any sub-Markov right-continuous semigroup  $\mathbb{P} = (P_t)_{t>0}$  on  $X$  such that (C) is satisfied by its convex cone  $E_{\mathbb{P}}$  of excessive functions,  $(X, E_{\mathbb{P}})$  is a balayage space, and  $\mathbb{P}$  is the transition semigroup of a Hunt process (see [5, Corollary 2.3.8] or [9, Corollary A.5]).

Let us recall that, for all  $A \subset X$  and  $u \in \mathcal{W}$ , the function  $R_u^A$  is the infimum of all  $v \in \mathcal{W}$  such that  $v \geq u$  on  $A$ , and  $\hat{R}_u^A(x) := \liminf_{y \rightarrow x} R_u^A(y)$ ,  $x \in X$ .

A set  $P$  in  $X$  is *polar*, if  $\hat{R}_v^P = 0$  for some (every) function  $v > 0$  in  $\mathcal{W}$ . A set  $T$  in  $X$  is *totally thin*, if  $\hat{R}_v^T < v$  for some  $v \in \mathcal{W}$ , and *semipolar*, if it is a countable union of totally thin sets. For example, the sets  $\{\hat{R}_u^A < R_u^A\}$ ,  $A \subset X$ ,  $u \in \mathcal{W}$ , are semipolar (and subsets of  $A \cap \partial A$ ; see [2, VI.5.11 and VI.2.3]).

A function  $h \in \mathcal{B}^+(X)$  is *harmonic* on an open set  $U$  in  $X$  if  $h|_U \in \mathcal{C}(U)$  and  $\int h d\varepsilon_x^{V^c} = h(x)$ , for all  $x \in U$  and open  $V$  such that  $x \in V$  and  $\bar{V}$  is compact in  $U$  (the measures  $\varepsilon_x^{V^c}$  are given by  $\int u d\varepsilon_x^{V^c} = R_u^{V^c}(x)$ ,  $u \in \mathcal{W}$ ).

**ASSUMPTION A.** *Let us assume that  $G: X \times X \rightarrow (0, \infty]$  is a Borel measurable function,  $G = \infty$  on the diagonal, such that  $G$  is a Green function for  $(X, \mathcal{W})$ , that is, the following holds (see [2, 5] for the definition of potentials for  $(X, \mathcal{W})$ ):*

- (i) *For every  $y \in X$ ,  $G(\cdot, y)$  is a potential which is harmonic on  $X \setminus \{y\}$ .*
- (ii) *For every potential  $p$  on  $X$ , there exists a measure  $\mu$  on  $X$  such that*

$$(1) \quad p = G\mu := \int G(\cdot, y) d\mu(y).$$

**REMARKS 2.** 1. Having (i), each of the following properties implies (ii).

- $G$  is locally bounded off the diagonal, each function  $G(x, \cdot)$  is lower semicontinuous on  $X$  and continuous on  $X \setminus \{x\}$ , and there exists a measure  $\nu$  on  $X$  such that  $G\nu \in \mathcal{C}(X)$  and  $\nu(U) > 0$  for every finely open  $U \neq \emptyset$  (see [8, Theorem 4.1]).
- $G$  is lower semicontinuous on  $X \times X$ , continuous outside the diagonal, and  $\mathcal{W} = E_{\mathbb{P}}$  for some sub-Markov semigroup  $\mathbb{P} = (P_t)_{t>0}$  such that the potential kernel  $V_0 := \int_0^\infty P_t dt$  is proper, and there is a measure  $\mu$  on  $X$  such that  $V_0 f := \int G(\cdot, y) f(y) d\mu(y)$ ,  $f \in \mathcal{B}^+(X)$  (see [13, p. 114], where (c) follows from [2, VI.2.6 and III.6.6]).

2. The measure in (1) is uniquely determined and, given any measure  $\mu$  on  $X$  such that  $p := G\mu$  is a potential, the complement of the support of  $\mu$  is the largest open set, where  $p$  is harmonic (see, for example, [8, Proposition 5.2 and Lemma 2.1]).

**EXAMPLE 3.** Let  $\mathbb{P} = (P_t)_{t>0}$  be a right continuous sub-Markov semigroup on  $\mathbb{R}^d$ ,  $d \geq 1$ , such that the potential kernel  $V_0 := \int_0^\infty P_t dt$  is given by  $V_0 f = G_0 * f$ , where  $G_0 = g(|\cdot|)$  and  $g: [0, \infty) \rightarrow (0, \infty]$  is decreasing with  $\lim_{r \rightarrow 0} g(r) = g(0) = \infty$ ,  $\lim_{r \rightarrow \infty} g(r) = 0$ ,  $\int_0^1 g(r)r^{d-1} dr < \infty$ , and  $g(r) \leq Cg(2r)$  for small  $r > 0$ .

Then  $(\mathbb{R}^d, E_{\mathbb{P}})$  is a balayage space such that  $G(x, y) := G_0(x - y)$  satisfies Assumption A as well as the following Assumption B (see [9, Section 6] and [6]; cf. [7] for more general Lévy processes).

**ASSUMPTION B.** *We assume, in addition, that  $G$  has the local triangle property, that is,  $X$  is covered by open sets  $U$  for which there exists a constant  $C > 0$  such that*

$$(2) \quad G(x, z) \wedge G(y, z) \leq CG(x, y), \quad x, y, z \in U.$$

**PROPOSITION 4** (Continuity principle of Evans-Vasilesco). *Let  $\mu$  be a measure on  $X$ ,  $A := \text{supp}(\mu)$  and  $x_0 \in A$  such that  $G\mu$  is a potential and  $(G\mu)|_A$  is continuous at  $x_0$ . Then  $G\mu$  is continuous at  $x_0$ .*

*Proof* (cf. the proof of [2, V.4.11]). If  $G\mu(x_0) = \infty$ , then  $G\mu$  is continuous at  $x_0$ . So let  $G\mu(x_0) < \infty$ . Let  $U$  be a relatively compact open neighborhood of  $x_0$  such that (2) holds. If  $K$  is a compact neighborhood of  $x_0$  in  $U$  and  $\mu' := 1_K\mu$ ,  $\mu'' := 1_{K^c}\mu$ , then  $G\mu = G\mu' + G\mu''$ , where  $G\mu''$  is harmonic on the interior of  $K$ , and hence continuous at  $x_0$ . So we may assume without loss of generality that the support  $A$  of  $\mu$  is a non-empty compact in  $U$ .

By (2),  $G(y, x) \leq CG(x, y)$ ,  $x, y \in U$  (take  $z = x$ ). Further, for all  $x, y, z \in U$ ,

$$G(x, y)^{-1} \leq C(G(x, z) \wedge G(y, z))^{-1} \leq C(G(x, z)^{-1} + G(y, z)^{-1}).$$

Therefore  $(x, y) \mapsto G(x, y)^{-1} + G(y, x)^{-1}$  is a quasi-metric on  $U \times U$  and, by [11, Proposition 14.5], there exist  $c \geq 1$ ,  $\gamma > 0$  and a metric  $\rho$  on  $U$  such that

$$c^{-1}\rho^{-\gamma} \leq G \leq c\rho^{-\gamma} \quad \text{on } U \times U$$

(see also [4]). Since, for every  $y \in U$ , the sets  $\{x \in U: G(x, y) > a\}$ ,  $a > 0$ , form a fundamental system of neighborhoods of  $y$ , the metric  $\rho$  is compatible with the topology of  $U$ .

For  $x \in U$ , let  $y_x \in A$  be such that

$$\rho(x, y_x) = \min\{\rho(x, y): y \in A\}.$$

Then, for every  $y \in A$ ,

$$(3) \quad \rho(y_x, y) \leq \rho(y_x, x) + \rho(x, y) \leq 2\rho(x, y),$$

and hence, for all measures  $\nu$  on  $A$  and  $x \in U$ ,

$$G\nu(x) \leq c \int \rho(x, y)^{-\gamma} d\nu(y) \leq 2^\gamma c \int \rho(y_x, y)^{-\gamma} d\nu(y) \leq 2^\gamma c^2 G\nu(y_x).$$

Let us now fix  $\varepsilon > 0$ . Since  $G\mu(x_0) < \infty$ , there exists  $r > 0$  such that  $\overline{B}(x_0, r) \subset U$  (where, of course,  $B(x_0, r) := \{y \in U : \rho(y, x_0) < r\}$ ) and  $\nu := 1_{B(x_0, r)}\mu$  satisfies  $G\nu(x_0) < \varepsilon$ . Then, again by [8, Lemma 2.1],  $G(\mu - \nu)$  is continuous at  $x_0$ , and hence  $(G\nu)|_A$  is continuous at  $x_0$  as well. So there exists  $\delta > 0$  such that

$$\begin{aligned} |G(\mu - \nu) - G(\mu - \nu)(x_0)| &< \varepsilon && \text{on } B(x_0, \delta), \\ G\nu &< \varepsilon && \text{on } B(x_0, 2\delta) \cap A. \end{aligned}$$

If  $x \in B(x_0, \delta)$ , then  $\rho(y_x, x_0) \leq 2\rho(x, x_0) < 2\delta$ , by (3), and therefore

$$|G\nu(x) - G\nu(x_0)| \leq G\nu(x) + G\nu(x_0) \leq 2^\gamma c^2 G\nu(y_x) + G\nu(x_0) < (2^\gamma c^2 + 1)\varepsilon.$$

Thus  $|G\mu - G\mu(x_0)| < (2^\gamma c^2 + 2)\varepsilon$  on  $B(x_0, \delta)$ . □

**REMARK 5.** In the proof of Proposition 4 it would, of course, be sufficient to know that  $G \approx g \circ \rho$  for some metric  $\rho$  on  $U$  and some decreasing function  $g$  having the doubling property near 0 (cf. also [7, Proposition 1.7] for equivalences).

**PROPOSITION 6.** *For every real potential  $p$  on  $X$ , there exists a sequence  $(p_n)$  of continuous real potentials on  $X$  such that  $\sum_{n \in \mathbb{N}} p_n = p$  and the superharmonic supports of  $p_n$ ,  $n \in \mathbb{N}$ , are pairwise disjoint.*

*Proof* (cf. the proof of [2, V.4.12]). There exists a measure  $\mu$  on  $X$  such that  $G\mu = p$ . By Lusin's theorem, there exists a sequence  $(K_n)$  of pairwise disjoint compacts in  $X$  such that  $\mu(X \setminus \bigcup_{n \in \mathbb{N}} K_n) = 0$  and  $p|_{K_n}$  is continuous for every  $n \in \mathbb{N}$ . Let

$$\mu_n := 1_{K_n}\mu \quad \text{and} \quad p_n := G\mu_n.$$

Then, of course,  $\sum_{n \in \mathbb{N}} p_n = p$ . For every  $n \in \mathbb{N}$ ,  $p_n|_{K_n}$  is continuous, since  $p|_{K_n}$  is continuous and both  $p_n$ ,  $\sum_{m \in \mathbb{N}, m \neq n} p_m$  are lower semicontinuous. Thus  $p_n \in \mathcal{C}(X)$ , by Proposition 4. □

**COROLLARY 7** (Domination principle). *Let  $\mu$  be a measure on  $X$  and let  $A$  be a Borel measurable subset of  $X$  such that  $G\mu$  is a real potential on  $X$  and  $\mu(A^c) = 0$ . Then*

$$R_{G\mu}^A = G\mu.$$

*Proof.* See the proofs of [2, V.4.13 and V.4.14]. □

**ASSUMPTION C.** *From now on let us assume, in addition, that  $X$  is an abelian topological group and that  $\mathcal{W}$  is invariant under the translations  $x \mapsto x + y$ ,  $y \in X$ .*

See [1] for a general discussion of potential theory on locally compact abelian groups.

**PROPOSITION 8.** *Suppose that  $v \in \mathcal{W}$ ,  $w \in \mathcal{W} \cap \mathcal{C}(X)$ , and  $v \leq w$ . Then, for every  $A \subset X$ , the function  $\hat{R}_v^A$  is harmonic in  $X \setminus \overline{A}$ .*

*Proof.* By [2, VI.2.2], we may assume that  $A$  is a Borel set (there exists a  $G_\delta$ -set  $A'$  such that  $A \subset A' \subset \overline{A}$  and  $\hat{R}_v^{A'} = \hat{R}_v^A$ ). Let  $u := R_v^A$ . By [2, VI.2.4],  $u = \hat{u}$  on  $X \setminus A$ . In particular,  $u$  is Borel measurable (note that  $u = v$  on  $A$ ). By [2, VI.2.6],  $u$  is harmonic on  $X \setminus \overline{A}$ . We recall that, for general balayage spaces, this does not imply that  $\hat{u}$  is harmonic on  $X \setminus \overline{A}$ , not even if  $A$  is compact (see [2, V.9.1]).

Let  $x \in X \setminus \overline{A}$  and let  $U$  be a relatively compact open neighborhood of  $x$  such that  $\overline{U} \cap \overline{A} = \emptyset$ . We intend to show that

$$(4) \quad \int \hat{u} d\varepsilon_x^{U^c} = \hat{u}(x).$$

To that end we fix a relatively compact open neighborhood  $V$  of  $0$  such that  $\overline{U} + V$  does not intersect  $\overline{A}$ . Moreover, let  $\lambda$  be a measure on  $X$  such that  $G\lambda$  is a continuous real potential which is strict. By [2, VI.8.2 and VI.5.15],  $\lambda$  charges every Borel measurable non-empty finely open set, but does not charge semipolar sets. We define  $\nu := 1_V \lambda$  and

$$\mu := \varepsilon_x^{U^c} * \nu,$$

that is, for every Borel measurable set  $B$  in  $X$ ,

$$\mu(B) = \int \varepsilon_x^{U^c}(B - y) d\nu(y) = \int \nu(B - y) d\varepsilon_x^{U^c}(y).$$

In particular,  $\mu$  does not charge semipolar sets.

By translation invariance, the functions  $y \mapsto u(y + z)$ ,  $z \in X$ , are harmonic on  $X \setminus \overline{(A - z)}$ , and hence

$$\int u d\mu = \int \int u(y + z) d\varepsilon_x^{U^c}(y) d\nu(z) = \int u(x + z) d\nu(z).$$

The set  $\{\hat{u} < u\}$  is semipolar, by [2, VI.5.11], and hence the integrals do not change, if  $u$  is replaced by  $\hat{u}$ , that is,

$$\int \int \hat{u}(y + z) d\varepsilon_x^{U^c}(y) d\nu(z) = \int \hat{u} d\mu = \int \hat{u}(x + z) d\nu(z).$$

Since  $\hat{u} \in \mathcal{W}$ , we know that  $\int \hat{u}(y + z) d\varepsilon_x^{U^c}(y) \leq \hat{u}(x + z)$ , for every  $z \in X$ . Therefore

$$(5) \quad u_x(z) := \int \hat{u}(y + z) d\varepsilon_x^{U^c}(y) = \hat{u}(x + z)$$

for  $\nu$ -almost every  $z \in X$ , where also  $u_x \in \mathcal{W}$ . Since  $\nu$  charges every non-empty finely open set in  $V$ , we conclude that (5) holds for every  $z \in V$ . In particular, (4) holds.  $\square$

**COROLLARY 9.** *For every relatively compact set  $A$  in  $X$ , there exists a unique measure  $\mu$  on  $\overline{A}$  such that  $\hat{R}_{w_0}^A = G\mu$ .*

Let  $w_0 \in \mathcal{W} \cap \mathcal{C}(X)$ ,  $w_0 > 0$  (for example,  $w_0 = 1$  if  $1 \in \mathcal{W}$ ).

**COROLLARY 10.** *Every semipolar set in  $X$  is polar.*

*Proof.* Let  $K$  be a compact, nonpolar set in  $X$ . Then  $\hat{R}_{w_0}^K \neq 0$  and, by Proposition 8,  $\hat{R}_{w_0}^K$  is harmonic on  $K^c$ . Hence, by Proposition 6, there exists a continuous potential  $p \neq 0$  on  $X$  such that  $\hat{R}_{w_0}^K - p \in \mathcal{W}$ . Then both  $p$  and  $\hat{R}_{w_0}^K - p$  are harmonic on  $K^c$ . By [2, VI.5.15], this implies that  $K$  is not semipolar.  $\square$

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