SMALL DATA SCATTERING FOR SEMI-RELATIVISTIC EQUATIONS WITH HARTREE TYPE NONLINEARITY

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ABSTRACT. We prove that the initial value problem for the equation

$$-i\partial_t u + \sqrt{m^2 - \Delta}\, u = \left(\frac{e^{-\mu_0 |x|}}{|x|} * |u|^2 \right) u \text{ in } \mathbb{R}^{1+3}, \quad m \geq 0, \; \mu_0 > 0$$

is globally well-posed and the solution scatters to free waves asymptotically as $t \to \pm\infty$ if we start with initial data which is small in $H^s(\mathbb{R}^3)$ for $s > \frac{1}{2}$, and if $m > 0$. Moreover, if the initial data is radially symmetric we can improve the above result to $m \geq 0$ and $s > 0$, which is almost optimal, in the sense that $L^2(\mathbb{R}^3)$ is the critical space for the equation. The main ingredients in the proof are certain endpoint Strichartz estimates, $L^2(\mathbb{R}^{1+3})$ bilinear estimates for free waves and an application of the U^p and V^p function spaces.

1. INTRODUCTION

We consider the initial value problem (IVP) for the semi-relativistic equation with a cubic Hartree-type nonlinearity:

$$\begin{aligned} -i\partial_t u + \sqrt{m^2 - \Delta}\, u &= (V * |u|^2) u \quad \text{in } \mathbb{R}^{1+3}, \\ u(0, \cdot) &= f \in H^s(\mathbb{R}^3), \end{aligned} \tag{1.1}$$

where $\sqrt{m^2 - \Delta}$ is defined via its symbol $\sqrt{m^2 + |\xi|^2}$ in Fourier space, the constant $m \geq 0$ is a physical mass parameter, the symbol $*$ denotes convolution in $\mathbb{R}^3$ and V is a potential, typically,

$$V(x) = \frac{e^{-\mu_0 |x|}}{|x|}, \quad \mu_0 \geq 0, \tag{1.2}$$

which is called a *Coulomb potential* if $\mu_0 = 0$ and a *Yukawa potential* if $\mu_0 > 0$.

Equation (1.1) is used to describe the dynamics and gravitational collapse of relativistic boson stars and it is often referred to as *the boson star equation*; see [12, 14, 22, 24] and the references therein.

It is well-known that equation (1.1) exhibits the following conserved quantities of energy and L^2-mass, which are given by

$$\begin{aligned} E(u(t)) &= \frac{1}{2} \int_{\mathbb{R}^3} \bar{u} \sqrt{m^2 - \Delta} u \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \left(V * |u|^2 \right) |u|^2 \, dx \,, \\ M(u(t)) &= \int_{\mathbb{R}^3} |u|^2 \, dx. \end{aligned}$$

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From these conservation laws, we see that the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$ serve as the energy space for problem (1.1). Furthermore, in the case of $m = 0$, (1.1) is invariant under the scaling

$$u(t,x) \mapsto u_\lambda(t,x) = \lambda^{\frac{3}{2}} u(\lambda t, \lambda x)$$

for fixed $\lambda > 0$. This scaling symmetry leaves the L^2-mass $M(u(t)) = M(u_\lambda(t))$ invariant, and so equation (1.1) is L^2-*critical*.

There has been a considerable mathematical interest concerning the low regularity well-posedness (both local and global-in-time) and scattering theory of the initial value problem (1.1) in the past few years. A first well-posedness result was obtained by Lenzmann [21] for $s \geq \frac{1}{2}$ using energy methods. Moreover, he showed global well-posedness in $H^{\frac{1}{2}}$ for initial data sufficiently small in L^2. There has been further well-posedness and scattering results for equation (1.1) with a more general potential, namely, $V(x) = |x|^{-\gamma}$, $\gamma \in (0, n)$; see eg. [5, 6, 7, 8]. Recently, Pusateri [26] proved a modified scattering result in the case of the Coulomb potential in dimension $n = 3$, if $m > 0$.

Recently, Lenzmann and the first author [17] proved local well-posedness for initial data with $s > \frac{1}{4}$ (and $s > 0$ if the data is radially symmetric). Moreover, these results are optimal up to end points, i.e., up to $s = \frac{1}{4}$ (and $s = 0$), in the framework of perturbation methods. Strichartz estimates and space-time bilinear estimates were the main ingredients. This is in contrast with previous results where only energy methods and linear Strichartz estimates were used.

The aim of this paper is to prove global existence and scattering of solutions to the IVP (1.1). Our main result is the following.

Theorem 1.1 (Main Theorem)**.** *Let $\mu_0 > 0$ in (1.2), i.e., V is a Yukawa potential. Assume one of the following holds:*
(a) $m \geq 0$, $s > 0$ and f is radially symmetric,
(b) $m > 0$ and $s > \frac{1}{2}$.
Then, there exists $\delta > 0$ such that for all $f \in H^s(\mathbb{R}^3)$ satisfying

$$\|f\|_{H^s} < \delta,$$

the IVP (1.1) has a global solution (spatially radial solution if f is radial)

$$u \in C(\mathbb{R}, H^s(\mathbb{R}^3)).$$

Moreover, the solution depends continuously on f and scatters asymptotically as $t \to \pm\infty$. Furthermore, it is unique in some smaller subspace of $C(\mathbb{R}, H^s(\mathbb{R}^3))$.

If $m > 0$, due to the modified scattering result in [26] (see also [5, Theorem 4.1]), it is known that the scattering result in Theorem 1.1 does not carry over to the case $\mu_0 = 0$, i.e. if V is the Coulomb potential.

Remark 1.2. Let us consider the IVP for the nonlinear Dirac equation with Hartree type nonlinearity:

$$(1.3) \qquad \begin{aligned} (-i\partial_t + \boldsymbol{\alpha} \cdot D + m\beta)\psi &= \lambda(V * |\psi|^2)\psi \quad \text{in } \mathbb{R}^{1+3}, \\ \psi(0,\cdot) &= \psi_0 \in H^s(\mathbb{R}^3), \end{aligned}$$

where $D = -i\nabla$, $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ is the Dirac spinor regarded as a column vector, $\lambda \in \mathbb{C}$, and β and $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ are the Dirac matrices. We refer the reader to [2] for a representation of the Dirac matrices and a recent result for a related problem with a cubic nonlinearity with null-structure.

Equation (1.3), with a Coulomb potential V, was derived by Chadam and Glassey [3] by uncoupling the Maxwell-Dirac equations under the assumption of vanishing magnetic field. Then in two space dimensions they showed existence of a unique global solution for smooth initial data with compact support. They also conjectured [3, see pp. 507] equation (1.3) with a Yukawa potential V can be derived by uncoupling the Dirac-Klein-Gordon equations, see also [4, 1] for certain global and scattering results in this context. Later, Dias and Figueira [10, 11] proved existence of weak solution for (1.3) with a Yukawa potential in the massless case ($m = 0$). We now comment on how to conclude a similar result as in Theorem 1.1b for the IVP (1.3) with a Yukawa potential.

Following [2], we define the projections

$$P_\pm(\xi) = \frac{1}{2}\left(I \pm \frac{1}{\langle \xi \rangle_m}[\xi \cdot \boldsymbol{\alpha} + m\beta] \right),$$

where

$$\langle \xi \rangle_m = \sqrt{|\xi|^2 + m^2}.$$

Then $\psi = \psi_+ + \psi_-$, where $\psi_\pm = P_\pm(D)\psi$. Now, if we apply $P_\pm(D)$ to (1.3), the IVP transforms to

$$\begin{cases} \left(-i\partial_t + \langle D \rangle_m\right)\psi_+ = P_+(D)\left[(V * |\psi|^2)\psi\right], & \psi_+(0,\cdot) = \psi_0^+ \in H^s(\mathbb{R}^3), \\ \left(-i\partial_t - \langle D \rangle_m\right)\psi_- = P_-(D)\left[(V * |\psi|^2)\psi\right], & \psi_-(0,\cdot) = \psi_0^- \in H^s(\mathbb{R}^3), \end{cases} \tag{1.4}$$

where $\psi_0^\pm = P_\pm(D)\psi_0$. Notice that these equations are of the form (1.1), and hence an easy modification[1] of the proof of Theorem 1.1b will give the following result.

Corollary 1.3. *Let $\mu_0 > 0$ in (1.2) (i.e., V is a Yukawa potential). Assume $m > 0$ and $s > \frac{1}{2}$. Then, there exists $\delta > 0$ such that for all $(\psi_0^+, \psi_0^-) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ satisfying*

$$\|\psi_0^\pm\|_{H^s} < \delta,$$

the IVP (1.4) has a global solution

$$(\psi_+, \psi_-) \in C(\mathbb{R}, H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)).$$

Moreover, the solution depends continuously on (ψ_0^+, ψ_0^-) and scatters asymptotically as $t \to \pm\infty$. Furthermore, it is unique in some smaller subspace.

Recently, in [2, 1] small data scattering results for low regularity initial data have been proven for the nonlinear Dirac equation and a massive Dirac-Klein-Gordon system. These problems exhibit null-structure and a non-resonant behavior. Note that there is also null-structure in (1.4), so we expect that the regularity threshold in Corollary 1.3 can be lowered, but we do not pursue this here. We will use some ideas introduced in [2, 1], in particular localized Strichartz estimates in the non-radial case, but otherwise the analysis differs significantly.

The rest of the paper is organized as follows. In the next Section, we give some notation, define the U^p and V^p-spaces and collect their properties. In Section 3, we prove some linear and bilinear estimates for solutions of free Klein-Gordon equation. In Section 4, we state a key proposition and then give the proof of Theorem 1.1. In Section 5–6, we give the proof of the key proposition.

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[1] One has to be careful in the radial case since the Dirac operator does not preserve spherical symmetry in the classical sense (see e.g. [23, 25]). However, we do not consider this case here.

2. Notation, U^p and V^p-spaces and their properties

2.1. Notation. We denote the spatial Fourier transform by $\widehat{\cdot}$ or $\mathcal{F}_x$ and the space-time Fourier transform by $\widetilde{\cdot}$. Frequencies will be denoted by Greek letters μ and λ, which we will assume to be dyadic, that is of the form 2^k for $k \in \mathbb{Z}$.

Consider an even function $\beta_1 \in C_0^\infty((-2,2))$ such that $\beta_1(s) = 1$ if $|s| \le 1$, and define for $\lambda > 1$,

$$\beta_\lambda(s) = \beta_1\left(\frac{s}{\lambda}\right) - \beta_1\left(\frac{2s}{\lambda}\right).$$

Thus, $\operatorname{supp}\beta_1 = \{s \in \mathbb{R} : |s| < 2\}$ whereas $\operatorname{supp}\beta_\lambda = \{s \in \mathbb{R} : \frac{\lambda}{2} < |s| < 2\lambda\}$ for $\lambda > 1$. We define

$$P_\lambda u := u_\lambda = \mathcal{F}_x^{-1}(\beta_\lambda(|\cdot|)\mathcal{F}_x u) \quad \text{and} \quad \widetilde{P}_\lambda = P_{\frac{\lambda}{2}} + P_\lambda + P_{2\lambda}.$$

For fixed $m \ge 0$, we denote $S_m(t) = e^{-it\langle D\rangle_m}$ to be the linear propagator of the Boson star equation (1.1) defined by

$$\mathcal{F}_x(S_m(t)f)(\xi) = e^{-it\langle\xi\rangle_m}\widehat{f}(\xi).$$

2.2. U^p and V^p spaces. These function spaces were originally introduced in the unpublished work of Tataru on the wave map problem and then in Koch-Tataru [19] in the context of NLS. The spaces have since been used to obtain critical results in different problems related to dispersive equations (see eg. [15, 16, 27, 18]) and they serve as a useful replacement of $X^{s,b}$-spaces in the limiting cases. For the convenience of the reader we list the definitions and some properties of these spaces.

Let $\mathcal{Z}$ be the collection of finite partitions $-\infty < t_0 < \cdots < t_K \le \infty$ of $\mathbb{R}$. If $t_K = \infty$, we use the convention $u(t_K) := 0$ for all functions $u : \mathbb{R} \to L^2$. We use χ_I to denote the sharp characteristic function of a set $I \subset \mathbb{R}$.

Definition 2.1. Let $1 \le p < \infty$. A U^p-atom is defined by a step function $a : \mathbb{R} \to L^2$ of the form

$$a(t) = \sum_{k=1}^{K} \chi_{[t_{k-1},t_k)(t)}\phi_{k-1},$$

where

$$\{t_k\}_{k=0}^K \in \mathcal{Z}, \quad \{\phi_k\}_{k=0}^{K-1} \subset L^2 \text{ with } \sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1.$$

The atomic space $U^p(\mathbb{R}; L^2)$ is defined to be the collection of functions $u : \mathbb{R} \to L^2$ of the form

$$u = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{where } a_j\text{'s are } U^p\text{-atoms and } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1, \tag{2.1}$$

with the norm

$$\|u\|_{U^p} := \inf_{\text{representation (2.1)}} \sum_{j=1}^{\infty} |\lambda_j|.$$

Definition 2.2. Let $1 \le p < \infty$.

(i) define $V^p(\mathbb{R}, L^2)$ as the space of all functions $v : \mathbb{R} \to L^2$ for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}} \tag{2.2}$$

is finite.

(ii) Likewise, let $V_-^p(\mathbb{R}, L^2)$ denote the normed space of all functions $v : \mathbb{R} \to L^2$ such that $\lim_{t\to-\infty} v(t) = 0$ and $\|v\|_{V^p} < \infty$, endowed with the norm (2.2).

(iii) We let $V_{rc}^p(\mathbb{R}, L^2)$ ($V_{-,rc}^p(\mathbb{R}, L^2)$) denote the closed subspace of all right-continuous $V^p(\mathbb{R}, L^2)$ functions ($V_-^p(\mathbb{R}, L^2)$ functions).

We collect some useful properties of these spaces. For more details about the spaces and proofs we refer to [15, 18].

Proposition 2.3. *Let $1 \le p < q < \infty$. Then we have the following:*

(i) $U^p(\mathbb{R}, L^2)$ is a Banach space.

(ii) The embeddings $U^p(\mathbb{R}, L^2) \subset U^q(\mathbb{R}, L^2) \subset L^\infty(\mathbb{R}; L^2)$ are continuous.

(iii) Every $u \in U^p(\mathbb{R}, L^2)$ is right-continuous. Moreover, $\lim_{t\to-\infty} u(t) = 0$.

Proposition 2.4. *Let $1 \le p < q < \infty$. Then we have the following:*

(i) The spaces $V^p(\mathbb{R}, L^2)$, $V_{rc}^p(\mathbb{R}, L^2)$, $V_-^p(\mathbb{R}, L^2)$ and $V_{-,rc}^p(\mathbb{R}, L^2)$ are Banach spaces.

(ii) The embedding $U^p(\mathbb{R}, L^2) \subset V_{-,rc}^p(\mathbb{R}, L^2)$ is continuous.

(iii) The embeddings $V^p(\mathbb{R}, L^2) \subset V^q(\mathbb{R}, L^2)$ and $V_-^p(\mathbb{R}, L^2) \subset V_-^q(\mathbb{R}, L^2)$ are continuous.

(iv) The embedding $V_{-,rc}^p(\mathbb{R}, L^2) \subset U^q(\mathbb{R}, L^2)$ is continuous.

Lemma 2.5. [20] *Let $p > 2$ and $v \in V^2(\mathbb{R}, L^2)$. There exists $\kappa = \kappa(p) > 0$ such that for all $M \ge 1$, there exist $w \in U^2(\mathbb{R}, L^2)$ and $z \in U^p(\mathbb{R}, L^2)$ with*

$$v = w + z$$

and

$$\frac{\kappa}{M}\|w\|_{U^2} + e^M \|z\|_{U^p} \lesssim \|v\|_{V^2}.$$

We now introduce U^p, V^p-type spaces that are adapted to the linear propagator $S_m(t) = e^{it\langle D\rangle_m}$ of equation (1.1):

Definition 2.6. We define $U_m^p(\mathbb{R}, L^2)$ (and $V_m^p(\mathbb{R}, L^2)$, respectively) to be the spaces of all functions $u : \mathbb{R} \mapsto L^2(\mathbb{R}^3)$ such that $t \to S_m(-t)u$ is in $U^p(\mathbb{R}, L^2)$ (resp. $V^p(\mathbb{R}, L^2)$), with the respective norms:

$$\|u\|_{U_m^p} = \|S_m(-t)u\|_{U^p},$$
$$\|u\|_{V_m^p} = \|S_m(-t)u\|_{V^p}.$$

We use $V_{\mathrm{rc},m}^p(\mathbb{R}, L^2)$ to denote the subspace of right-continuous functions in $V_m^p(\mathbb{R}, L^2)$.

Remark 2.7. Proposition 2.3, Proposition 2.4 and Lemma 2.5 naturally extends to the spaces $U_m^p(\mathbb{R}, L^2)$ and $V_m^p(\mathbb{R}, L^2)$.

Lemma 2.8. (Transfer principle) *Let*

$$T : L^2 \times \cdots \times L^2 \to L_{loc}^1(\mathbb{R}^n; \mathbb{C})$$

be a multilinear operator and suppose that we have

$$\big\|T(S_m(t)\phi_1,\dots,S_m(t)\phi_k)\big\|_{L^p_tL^r_x(\mathbb{R}\times\mathbb{R}^n)}\lesssim\prod_{j=1}^k\|\phi_j\|_{L^2_x(\mathbb{R}^n)}$$

for some $1\le p,r\le\infty$. *Then*

$$\big\|T(u_1,\dots,u_k)\big\|_{L^p_tL^r_x(\mathbb{R}\times\mathbb{R}^n)}\lesssim\prod_{j=1}^k\|u_j\|_{U^p_m}.$$

3. Linear and bilinear estimates

In this section, we prove linear and bilinear estimates for free solutions of the Klein-Gordon equation both for radial and non-radial data, which are key to prove our main Theorem.

3.1. Estimates in the radial case.

Lemma 3.1. *Let* $m\ge 0$. *Consider* $u(t)=S_m(t)f$, *where* f *is radial. Then*

$$\|u_\lambda\|_{L^2_tL^\infty_x(\mathbb{R}^{1+3})}\lesssim\lambda\|f_\lambda\|_{L^2_x(\mathbb{R}^3)}. \tag{3.1}$$

Moreover, for all radial function $u_\lambda\in U^2_m$, *we have*

$$\|u_\lambda\|_{L^2_tL^\infty_x(\mathbb{R}^{1+3})}\lesssim\lambda\|u_\lambda\|_{U^2_m}. \tag{3.2}$$

Proof. For the proof of (3.1), see for example [28, Theorem 1.3]. Then (3.2) follows from (3.1) by applying the transfer principle in Lemma 2.8. □

The following Lemma extends the result of Foschi-Klainerman for $m=0$ [13, Lemma 4.4] to the massive case.

Lemma 3.2. *Let* $m\ge 0$ *and consider the integral*

$$I(\phi,\psi)(\tau,\xi)=\int\phi(|\eta|)\psi(|\xi-\eta|)\delta(\tau-\langle\eta\rangle_m+\langle\xi-\eta\rangle_m)\,d\eta.$$

Then

$$I(\phi,\psi)(\tau,\xi)\simeq\frac{1}{|\xi|}\int_{\frac{\tau+|\xi|}{2}}^\infty\phi(\rho)\psi(\varphi(\tau,\rho))\rho(\langle\rho\rangle_m-\tau)\,d\rho, \tag{3.3}$$

where

$$\varphi(\tau,\rho)=\sqrt{(\langle\rho\rangle_m-\tau)^2-m^2}.$$

Proof. The proof given here is a modification of the argument for $m=0$ from [13, Lemma 4.4]. For a smooth function g, define the hypersurface

$$S=\{x\in\mathbb{R}^3:g(x)=0\}.$$

If $\nabla g\neq 0$ for $x\in S\cap\operatorname{supp}f$, then

$$\int f(x)\delta(g(x))\,dx=\int_S\frac{f(x)}{|\nabla g(x)|}\,dS_x. \tag{3.4}$$

For a nonnegative smooth function h which does not vanish on S, (3.4) also implies

$$\delta(g(x))=h(x)\delta\left(h(x)g(x)\right). \tag{3.5}$$

Now using (3.5), we can write

$$\begin{aligned}\delta(\tau-\langle\eta\rangle_m+\langle\xi-\eta\rangle_m) &= [-(\tau-\langle\eta\rangle_m)+\langle\xi-\eta\rangle_m]\delta\left(-(\tau-\langle\eta\rangle_m)^2+\langle\xi-\eta\rangle_m^2\right)\\ &= 2(\langle\eta\rangle_m-\tau)\delta\left(|\xi|^2-\tau^2+2\tau\langle\eta\rangle_m-2\xi\cdot\eta\right),\end{aligned}$$

where in the first line we multiplied the argument of the delta function on the left by $-(\tau-\langle\eta\rangle_m)+\langle\xi-\eta\rangle_m$.

Introduce polar coordinate $\eta=\rho\omega$, where $\omega\in\mathbb{S}^2$. Then

$$|\eta|=\rho,\quad d\eta=\rho^2 dS_w d\rho.$$

If we also set $a=\omega\cdot\xi/|\xi|$, then

$$dS_w=dS_{w'}da,\quad d\eta=\rho^2 dS_{w'}d\rho da,$$

where $\omega'\in\mathbb{S}^1$. In the change of variables, we obtain $\langle\xi-\eta\rangle_m=\langle\eta\rangle_m-\tau=\langle\rho\rangle_m-\tau$, which in turn implies $|\xi-\eta|^2=(\langle\rho\rangle_m-\tau)^2-m^2$. With these transformations our integral becomes

$$I(\phi,\psi)(\tau,\xi)\simeq\int_0^\infty\int_{-1}^1\phi(\rho)\psi\left(\varphi(\tau,\rho)\right)\rho^2(\langle\rho\rangle_m-\tau)\delta\left(|\xi|^2-\tau^2+2\tau\langle\rho\rangle_m-2|\xi|\rho a\right)\,da d\rho.$$

The delta function sets the value of a to

$$a=\frac{|\xi|^2-\tau^2+2\tau\langle\rho\rangle_m}{2|\xi|\rho},\tag{3.6}$$

which implies $a\geq\frac{\tau}{|\xi|}$, and thus we are restricted to $\frac{\tau}{|\xi|}\leq a\leq 1$. This forces us to integrate over

$$\left\{\rho>0:\rho\geq\frac{\tau+|\xi|}{2}\right\}.$$

Using these facts and (3.4) gives the desired estimate. □

Lemma 3.3. *Let $m\geq 0$. Consider $u^+(t)=S_m(t)f$ and $v^-(t)=S_m(-t)g$, where f and g are radial. Then for any $\mu,\lambda_1,\lambda_2\geq 1$, we have*

$$\|P_\mu(u^+_{\lambda_1}v^-_{\lambda_2})\|_{L^2_{t,x}(\mathbb{R}^{1+3})}\lesssim\mu\|f_{\lambda_1}\|_{L^2_x(\mathbb{R}^3)}\|g_{\lambda_2}\|_{L^2_x(\mathbb{R}^3)}.$$

Then Lemma 2.8 and Lemma 3.3 imply the following Corollary.

Corollary 3.4. *Let $m\geq 0$. Suppose u_{λ_1} and v_{λ_2} are radial functions such that $u_{\lambda_1},v_{\lambda_2}\in U^2_m$. Then for any $\mu,\lambda_1,\lambda_2\geq 1$, we have*

$$\|P_\mu(u_{\lambda_1}\overline{v}_{\lambda_2})\|_{L^2_{t,x}(\mathbb{R}^{1+3})}\lesssim\mu\|u_{\lambda_1}\|_{U^2_m}\|v_{\lambda_2}\|_{U^2_m}.$$

Proof of Lemma 3.3. First assume $\min(\lambda_1,\lambda_2)=1$. By symmetry we may assume $\lambda_1\leq\lambda_2$. Then by Hölder, (3.1) and the energy inequality, we obtain

$$\begin{aligned}\|P_\mu(u^+_{\lambda_1}v^-_{\lambda_2})\|_{L^2_{t,x}} &\lesssim\|u^+_{\lambda_1}\|_{L^2_tL^\infty_x}\|v^-_{\lambda_2}\|_{L^\infty_tL^2_x}\\ &\lesssim\|f_{\lambda_1}\|_{L^2_x(\mathbb{R}^3)}\|g_{\lambda_2}\|_{L^2_x(\mathbb{R}^3)}.\end{aligned}$$

Therefore, we assume from now on that $\lambda_1,\lambda_2>1$. If f and g are radial, then their Fourier transforms will also be radial. Let $\widehat{f}(\xi)=\phi(|\xi|)$ and $\widehat{g}(\xi)=\psi(|\xi|)$. Applying the space-time Fourier transform, we write

$$P_\mu\widetilde{(u^+_{\lambda_1}v^-_{\lambda_2})}(\tau,\xi)=\beta_\mu(|\xi|)\int_{\mathbb{R}^3}\beta_{\lambda_1}(|\eta|)\phi(|\eta|)\beta_{\lambda_2}(|\xi-\eta|)\psi(|\xi-\eta|)\delta(\tau-\langle\eta\rangle_m+\langle\xi-\eta\rangle_m)\,d\eta.$$

Note that

$$\tau^2 = [\langle\eta\rangle_m - \langle\xi-\eta\rangle_m]^2 = \left[\frac{|\eta|^2 - |\xi-\eta|^2}{\langle\eta\rangle_m + \langle\xi-\eta\rangle_m}\right]^2 = \left[\frac{|\xi|^2 - 2|\xi||\eta|\cos(\angle(\xi,\eta))}{\langle\eta\rangle_m + \langle\xi-\eta\rangle_m}\right]^2,$$

which implies

$$|\tau| \le \frac{|\xi|\,[|\xi| + 2|\eta|]}{\langle\eta\rangle_m + \langle\xi-\eta\rangle_m} \lesssim \mu.$$

Then using Lemma 3.2, we obtain

$$\begin{aligned}
&\|P_\mu(u^+_{\lambda_1} v^-_{\lambda_2})\|^2_{L^2_{t,x}(\mathbb{R}^{1+3})} \\
&\quad\lesssim \int_{\mathbb{R}^3}\int_{|\tau|\lesssim\mu} \frac{\beta_\mu^2(|\xi|)}{|\xi|^2}\left|\int_{\frac{\tau+|\xi|}{2}}^\infty [\beta_{\lambda_1}(\rho)\rho\phi(\rho)][\beta_{\lambda_2}(\varphi(\tau,\rho))(\langle\rho\rangle_m - \tau)\psi(\varphi(\tau,\rho))]\,d\rho\right|^2 d\tau d\xi \\
&\quad\simeq \int_0^\infty \beta_\mu^2(r)\int_{|\tau|\lesssim\mu}\left|\int_{\frac{\tau+r}{2}}^\infty [\beta_{\lambda_1}(\rho)\rho\phi(\rho)][\beta_{\lambda_2}(\varphi(\tau,\rho))(\langle\rho\rangle_m - \tau)\psi(\varphi(\tau,\rho))]\,d\rho\right|^2 d\tau dr \\
&\quad\lesssim \mu\int_{|\tau|\lesssim\mu}\left(\int_{\mathbb{R}} |\beta_{\lambda_1}(\rho)\rho\phi(\rho)|^2\,d\rho\right)\left(\int_{\mathbb{R}} |\beta_{\lambda_2}(\varphi(\tau,\rho))(\langle\rho\rangle_m - \tau)\psi\left(\varphi(\tau,\rho)\right)|^2\,d\rho\right) d\tau,
\end{aligned}$$

where to get the third line we used the change of variable $\xi = r\omega$, for $\omega \in S^2$, and to obtain the fourth inequality we used Cauchy-Schwarz with respect to ρ and the fact that $\int_0^\infty \beta_\mu^2(r)\,dr \lesssim \mu$.

We now use the change of variable $\rho \mapsto \sigma = \varphi(\tau,\rho)$, which implies

$$\langle\rho\rangle_m - \tau = \langle\sigma\rangle_m, \quad d\rho = \frac{\sigma\langle\rho\rangle_m}{\rho\langle\sigma\rangle_m}d\sigma,$$

(where $\rho \sim \lambda_1$, since $\lambda_1 > 1$) to write

$$\begin{aligned}
\int_{\mathbb{R}} |\beta_{\lambda_2}(\varphi(\tau,\rho))(\langle\rho\rangle_m - \tau)\psi\left(\varphi(\tau,\rho)\right)|^2\,d\rho &\simeq \frac{\langle\lambda_1\rangle_m}{\lambda_1}\int_{\mathbb{R}} |\beta_{\lambda_2}(\sigma)\langle\sigma\rangle_m\psi\left(\sigma\right)|^2 \frac{\sigma}{\langle\sigma\rangle_m}\,d\sigma \\
&= \frac{\langle\lambda_1\rangle_m}{\lambda_1}\int_{\mathbb{R}} |\beta_{\lambda_2}(\sigma)\sigma\psi\left(\sigma\right)|^2 \frac{\langle\sigma\rangle_m}{\sigma}\,d\sigma \\
&\simeq \frac{\langle\lambda_1\rangle_m\langle\lambda_2\rangle_m}{\lambda_1\lambda_2}\int_{\mathbb{R}} |\beta_{\lambda_2}(\sigma)\sigma\psi\left(\sigma\right)|^2\,d\sigma \\
&\lesssim \int_{\mathbb{R}} |\beta_{\lambda_2}(\sigma)\sigma\psi\left(\sigma\right)|^2\,d\sigma.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\|P_\mu(u^+_{\lambda_1} v^-_{\lambda_2})\|^2_{L^2_{t,x}(\mathbb{R}^{1+3})} &\lesssim \mu^2\|\beta_{\lambda_1}(\rho)\rho\phi(\rho)\|^2_{L^2_\rho(\mathbb{R})}\|\beta_{\lambda_2}(\rho)\rho\psi(\rho)\|^2_{L^2_\rho(\mathbb{R})} \\
&= \mu^2\|f_{\lambda_1}\|^2_{L^2_x(\mathbb{R}^3)}\|g_{\lambda_2}\|^2_{L^2_x(\mathbb{R}^3)},
\end{aligned}$$

where to get the last equality we used the identities

$$\|\beta_{\lambda_1}(\rho)\rho\phi(\rho)\|_{L^2_\rho(\mathbb{R})} = \|f_{\lambda_1}\|_{L^2_x(\mathbb{R}^3)}, \quad \|\beta_{\lambda_2}(\rho)\rho\psi(\rho)\|_{L^2_\rho(\mathbb{R})} = \|g_{\lambda_2}\|_{L^2_x(\mathbb{R}^3)}.$$

□

3.2. **Estimates in the non-radial case.** The first part of the next Lemma is well-known.

Lemma 3.5 (KG-Strichartz)**.** *Let $m>0$ and $u(t)=S_m(t)f$. Then*

$$\|u_\lambda\|_{L_t^2L_x^6(\mathbb{R}^{1+3})} \lesssim \lambda^{\frac{5}{6}}\|f_\lambda\|_{L_x^2(\mathbb{R}^3)}. \tag{3.7}$$

Moreover, for all $u_\lambda \in U_m^2$, we have

$$\|u_\lambda\|_{L_t^2L_x^6(\mathbb{R}^{1+3})} \lesssim \lambda^{\frac{5}{6}}\|u_\lambda\|_{U_m^2}. \tag{3.8}$$

Proof. For the proof of (3.7), see for example [9]. The estimate (3.8) follows from (3.7) by applying the transfer principle in Lemma 2.8. □

Lemma 3.6 (KG-Localized Strichartz)**.** *Let $m>0$. Consider $u_{\lambda,\mu}(t)=S_m(t)f_{\lambda,\mu}$, where $\widehat{f}_{\lambda,\mu}$ is supported in a cube of side length μ at a distance $\sim\lambda$ from the origin, with $1\lesssim\mu\lesssim\lambda$. Then for all $\varepsilon>0$, we have*

$$\|u_{\lambda,\mu}\|_{L_t^2L_x^\infty(\mathbb{R}^{1+3})} \lesssim (\mu\lambda)^{\frac{1}{2}}\lambda^\varepsilon\|f_{\lambda,\mu}\|_{L_x^2(\mathbb{R}^3)}. \tag{3.9}$$

Moreover, for all $u_{\lambda,\mu}\in U_m^2$, we have

$$\|u_{\lambda,\mu}\|_{L_t^2L_x^\infty(\mathbb{R}^{1+3})} \lesssim (\mu\lambda)^{\frac{1}{2}}\lambda^\varepsilon\|u_{\lambda,\mu}\|_{U_m^2}. \tag{3.10}$$

Proof. The estimate (3.10) follows from (3.9) by applying the transfer principle in Lemma 2.8. So we only prove (3.9).

Without loss of generality, we take $m=1$. By a standard TT^* argument, (3.9) is equivalent to the estimate

$$\|K_{\lambda,\mu}\star F\|_{L_t^2L_x^\infty(\mathbb{R}^{1+3})} \lesssim \mu\lambda^{1+2\varepsilon}\|F\|_{L_t^2L_x^1(\mathbb{R}^{1+3})}, \tag{3.11}$$

where

$$K_{\lambda,\mu}(t,x) = \int e^{i(x\cdot\xi - t\langle\xi\rangle)}|\beta_{\lambda,\mu}(\xi)|^2\,d\xi \tag{3.12}$$

and $\star$ defines the space-time convolution. The kernel $K_{\lambda,\mu}$ satisfies the following estimates (see [2, Lemma 2.2. Eq. (2.8)] and [1, Eq. (3.2)]):

$$|K_{\lambda,\mu}(t,x)| \lesssim \lambda^4\left(1+\lambda|t|\right)^{-\frac{3}{2}},$$
$$|K_{\lambda,\mu}(t,x)| \lesssim \mu^3\left(1+\frac{\mu^2}{\lambda}|t|\right)^{-1}.$$

We then interpolate these estimates, for $\theta\in(0,1)$, to obtain

$$\begin{aligned}|K_{\lambda,\mu}(t,x)| &\lesssim \lambda^{4\theta}\left(1+\lambda|t|\right)^{-\frac{3}{2}\theta}\mu^{3-3\theta}\left(1+\frac{\mu^2}{\lambda}|t|\right)^{-(1-\theta)}\\ &\lesssim \mu^{3-3\theta}\lambda^{4\theta}\left(1+\frac{\mu^2}{\lambda}|t|\right)^{\frac{-\theta-2}{2}},\end{aligned}$$

where we used $\mu\lesssim\lambda$. Hence

$$\begin{aligned}\|K_{\lambda,\mu}\|_{L_t^1L_x^\infty} &\lesssim \mu^{3-3\theta}\lambda^{4\theta}\int_0^\infty\left(1+\frac{\mu^2}{\lambda}t\right)^{\frac{-\theta-2}{2}}dt\\ &\lesssim \mu^{3-3\theta}\lambda^{4\theta}\cdot\frac{\lambda}{\mu^2}\lesssim \mu\lambda^{1+4\theta},\end{aligned}$$

which implies (applying also Young's inequality in t and x)

$$\|K_{\lambda,\mu} \star F\|_{L^2_t L^\infty_x} \lesssim \|K_{\lambda,\mu}\|_{L^1_t L^\infty_x} \|F\|_{L^2_t L^1_x} \lesssim \mu\lambda^{1+4\theta} \|F\|_{L^2_t L^1_x}.$$

Choosing $\theta = \frac{\varepsilon}{2}$ yields the desired estimate (3.11). □

Remark 3.7. Let $C_z = \mu z + [0,\mu)^3$, $z \in \mathbb{Z}^3$, be the collection of cubes (as in [17]) which induce a disjoint covering of $\mathbb{R}^3$. Then we have

$$\sum_{z\in\mathbb{Z}^3} \|P_{C_z} u_\lambda\|^2_{L^2_t L^\infty_x} \lesssim (\mu\lambda)\lambda^{2\varepsilon} \|u_\lambda\|^2_{U^2_m}.$$

4. Proof of the Main Theorem

Similarly to [15, 19], we define X^s to be the complete space of all functions $u : \mathbb{R} \to L^2$ such that $P_\mu u \in U^2_m(\mathbb{R}, L^2)$ for all $\mu \geq 1$, with the norm

$$\|u\|_{X^s} = \left(\sum_{\mu\geq 1} \mu^{2s} \|P_\mu u\|^2_{U^2_m} \right)^{\frac{1}{2}} < \infty,$$

where

$$\|f\|_{U^2_m} = \|S_m(-t) f\|_{U^2}.$$

We also define by Y^s the corresponding space where U^2_m is replaced by V^2_m with a norm

$$\|f\|_{V^2_m} = \|S_m(-t) f\|_{V^2}.$$

On the time interval $I = [0,\infty)$, we define the restricted space X^s_I by

$$X^s_I = \left\{ u \in C(I, H^s) \mid \tilde{u} = \chi_I(t) u(t) \in X^s \right\}$$

with norm

$$\|u\|_{X^s_I} = \|\chi_I u\|_{X^s},$$

and define Y^s_I analogously. Note the embedding

$$X^s \subset Y^s.$$

The Duhamel representation of (1.1) is given by

$$u(t) = S_m(t) f - i J_m(u)(t) \tag{4.1}$$

where

$$J_m(u)(t) = \int_0^t S_m(t - t')[(V * |u|^2)u](t') dt'. \tag{4.2}$$

The linear part satisfies the following estimate ($m \geq 0$):

$$\begin{aligned} \|S_m(t) f\|^2_{X^s_I} &= \sum_{\mu\geq 1} \mu^{2s} \|\chi_I S_m(t) P_\mu f\|^2_{U^2_m} \\ &= \sum_{\mu\geq 1} \mu^{2s} \|\chi_I P_\mu u\|^2_{U^2} \sim \|f\|^2_{H^s}. \end{aligned}$$

Theorem 1.1 will follow by contraction argument from the above linear estimate and following nonlinear estimates.

Proposition 4.1. *For $J_m(u)$ as in (4.2), we have the following:*

(a) Let $m \ge 0$ *and* $s > 0$. *For all spatially radial* $u \in X_I^s$,

$$\|J_m(u)\|_{X_I^s} \lesssim \|u\|_{X_I^s}^3.$$

(b) Let $m > 0$ *and* $s > \frac{1}{2}$. *For all* $u \in X_I^s$,

$$\|J_m(u)\|_{X_I^s} \lesssim \|u\|_{X_I^s}^3.$$

The proof for this proposition is given in the last section.

Proof of Theorem 1.1. We solve the integral equation (4.1) by contraction mapping techniques as follows. Define the mapping

$$u(t) = \Phi(u)(t) := S_m(t)f + iJ_m(u)(t). \tag{4.3}$$

We look for the solution in the set

$$D_\delta = \{u \in X_I^s : \ \|u\|_{X_I^s} \le \delta\}.$$

For $u \in D_\delta$ and initial data of size $\|f\|_{H^s} \le \varepsilon \ll \delta$, we have by Proposition 4.1,

$$\|\Phi(u)\|_{X_I^s} \lesssim \varepsilon + \delta^3 \le \delta$$

for small enough δ. Moreover, for solutions u and v with the same data, one can show the difference estimate

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{X_I^s} &\lesssim (\|u\|_{X_I^s} + \|v\|_{X^s})^2 \|u - v\|_{X_I^s} \\ &\lesssim \delta^2 \|u - v\|_{X_I^s}\end{aligned}$$

whenever $u,\ v \in D_\delta$. Hence Φ is a contraction on D_δ when $\delta \ll 1$, which implies the existence of a unique fixed point in D_δ solving the integral equation (4.3).

It thus remains to show scattering of solution of (4.3) to a free solution as $t \to \infty$. By Proposition 2.4 and Proposition 4.1, we have for each μ

$$S_m(-t)P_\mu J_m(u) \in V^2_{-,\mathrm{rc}}$$

and hence the limit as $t \to \infty$ exists for each μ. Combining this with

$$\sum_{\mu \ge 1} \mu^{2s} \|P_\mu J_m(u)\|_{V^2}^2 \lesssim 1$$

implies that

$$\lim_{t \to \infty} S_m(-t) P_\mu J_m(u) =: \phi^+ \in H^s.$$

Hence for the solution u we have that

$$\|S_m(t)\phi^+ - u(t)\|_{H^s} \to 0 \ \text{ as } t \to \infty.$$

5. Proof of Proposition 4.1

We may assume that $u(t) = 0$ for $-\infty < t < 0$.

By duality (see e.g. [15]),

$$\begin{aligned}\|P_\lambda J_m(u)\|_{U_m^2} = \|S_m(-t)P_\lambda J_m(u)\|_{U^2} &= \|P_\lambda \int_0^t S_m(-t')[(V * |u|^2)u](t')dt'\|_{U^2} \\ &= \sup_{\|v\|_{V^2}=1} \Big| \iint [(V * |u|^2)u](t)\ \overline{S_m(t)P_\lambda v(t)}\ dtdx \Big| \\ &= \sup_{\|v_\lambda\|_{V_m^2}=1} \Big| \iint [(V * |u|^2)u](t) \overline{v}_\lambda(t)\ dtdx \Big|,\end{aligned}$$

Then
(5.1)

$$
\begin{aligned}
\|J_m(u)\|_{X^s}^2 &= \sum_{\lambda\geq 1} \lambda^{2s} \|P_\lambda J_m(u)\|_{U_m^2}^2 \\
&= \sum_{\lambda\geq 1} \lambda^{2s} \sup_{\|v_\lambda\|_{V_m^2}=1} \Big| \iint [V*(u\overline{u})]u\ \overline{v}_\lambda\ dtdx\Big|^2 \\
&\lesssim \sum_{\lambda\geq 1} \lambda^{2s} \sup_{\|v_\lambda\|_{V_m^2}=1} \left[\sum_{\lambda_1,\lambda_2,\lambda_3\geq 1} \Big| \iint [V*(u_{\lambda_1}\overline{u}_{\lambda_2})]u_{\lambda_3}\ \overline{v}_\lambda\ dtdx\Big|\right]^2
\end{aligned}
$$

To finish the proof of Proposition 4.1 (a), we use the following Lemma which we shall prove in the last section.

Lemma 5.1. *Let $\lambda_{\min}$ and λ_{med} denote the minimum and median of $(\lambda_1,\lambda_2,\lambda_3)$.*
(a) Let $m\geq 0$, $\varepsilon>0$. For all spatially radial $u_{j,\lambda_j}\in U_m^2$, $v_\lambda\in V_m^2$ and $\lambda_j,\lambda\geq 1$,

$$
\Big| \iint [V*(u_{1,\lambda_1}\overline{u}_{2,\lambda_2})]u_{3,\lambda_3}\ \overline{v}_\lambda\ dtdx\Big| \lesssim (\lambda_{\min}\lambda_{\mathrm{med}})^{\varepsilon} \prod_{j=1}^{3} \|u_{j,\lambda_j}\|_{U_m^2}\|v_\lambda\|_{V_m^2},
$$

(b) Let $m>0$ and assume $u_{j,\lambda_j}\in U_m^2$ and $v\in V_m^2$. Then, for all $\lambda_j,\lambda\geq 1$ and $\varepsilon>0$,

$$
\Big| \iint [V*(u_{1,\lambda_1}\overline{u}_{2,\lambda_2})]u_{3,\lambda_3}\ \overline{v}_\lambda\ dtdx\Big| \lesssim (\lambda_{\min}\lambda_{\mathrm{med}})^{\frac{1}{2}+\varepsilon} \prod_{j=1}^{3} \|u_{j,\lambda_j}\|_{U_m^2}\|v_\lambda\|_{V_m^2}.
$$

We now finish the proof of Proposition 4.1a by using Lemma 5.1a. The proof of Proposition 4.1b follows from a similar argument using Lemma 5.1b.

Applying Lemma 5.1a to (5.1), we obtain

$$
\begin{aligned}
\|J_m(u)\|_{X^s}^2 &\lesssim \sum_{\lambda\geq 1} \lambda^{2s} \sup_{\|v_\lambda\|_{V_m^2}=1} \left[\sum_{\lambda_1,\lambda_2,\lambda_3\geq 1} (\lambda_{\min}\lambda_{\mathrm{med}})^{\varepsilon} \prod_{j=1}^{3} \|u_{\lambda_j}\|_{U_m^2}\|v_\lambda\|_{V_m^2}\right]^2 \\
&:= S_1+S_2+S_3,
\end{aligned}
$$

where

$$
S_1 = \sum_{\substack{\lambda,\lambda_1,\lambda_2,\lambda_3\geq 1\\ \lambda_3\sim\lambda}}, \quad S_2 = \sum_{\substack{\lambda,\lambda_1,\lambda_2,\lambda_3\geq 1\\ \lambda_3\ll\lambda}}, \quad S_3 = \sum_{\substack{\lambda,\lambda_1,\lambda_2,\lambda_3\geq 1\\ \lambda_3\gg\lambda}}.
$$

First Consider S_1. We have

$$
\begin{aligned}
S_1 &\lesssim \sum_{\lambda\geq 1} \lambda^{2s} \left[\sum_{\substack{\lambda_1,\lambda_2,\lambda_3\geq 1\\ \lambda_3\sim\lambda}} (\lambda_1\lambda_2)^{\varepsilon} \|u_{\lambda_1}\|_{U_m^2}\|u_{\lambda_2}\|_{U_m^2}\|u_{\lambda_3}\|_{U_m^2}\right]^2 \\
&\lesssim \|u_1\|_{X^s}^2\|u_2\|_{X^s}^2 \sum_{\lambda\geq 1} \lambda^{2s} \left[\sum_{\lambda_3\sim\lambda} \|u_{\lambda_3}\|_{U_m^2}\right]^2 \\
&\lesssim \|u\|_{X^s}^2\|u\|_{X^s}^2\|u\|_{X^s}^2,
\end{aligned}
$$

where to obtain the second inequality, we used Cauchy-Schwarz in λ_1 and in λ_2, and the fact that $\sum_{\lambda_j\geq 1} \lambda_j^{-2(s-\varepsilon)} \lesssim 1$, if we assume $s>\varepsilon$.

Next we consider S_2. We have $S_2 \leq S_{21} + S_{22} + S_{23}$, where

$$S_{21} = \sum_{\substack{\lambda_1 \ll \lambda_2 \\ \lambda_3 \ll \lambda}}, \quad S_{22} = \sum_{\substack{\lambda_1 \gg \lambda_2 \\ \lambda_3 \ll \lambda}}, \quad S_{23} = \sum_{\substack{\lambda_1 \sim \lambda_2 \\ \lambda_3 \ll \lambda}}$$

If $\lambda_1 \ll \lambda_2$, then $\lambda_2 \sim \lambda$. Then we can apply Cauchy-Schwarz in λ_1 and in λ_3 to obtain

$$\begin{aligned} S_{21} &\sim \sum_{\lambda} \lambda^{2s} \left[\sum_{\substack{\lambda_1 \ll \lambda_2 \\ \lambda_3 \ll \lambda \sim \lambda_2}} (\lambda_1 \lambda_3)^{\varepsilon} \|u_{\lambda_1}\|_{U^2_m} \|u_{\lambda_2}\|_{U^2_m} \|u_{\lambda_3}\|_{U^2_m} \right]^2 \\ &\lesssim \|u\|^2_{X^s} \|u\|^2_{X^s} \sum_{\lambda} \lambda^{2s} \left[\sum_{\lambda_2 \sim \lambda} \|u_{\lambda_2}\|_{U^2_m} \right]^2 \\ &\lesssim \|u\|^2_{X^s} \|u\|^2_{X^s} \|u\|^2_{X^s}, \end{aligned}$$

The estimate for S_{22} is similar.

If $\lambda_1 \sim \lambda_2$,then $\lambda \lesssim \lambda_1 \sim \lambda_2$. We apply Cauchy-Schwarz in $\lambda_1 \sim \lambda_2$ and in λ_3 to obtain

$$\begin{aligned} S_{23} &\lesssim \sum_{\lambda \geq 1} \lambda^{2s} \left[\sum_{\substack{\lambda_1 \sim \lambda_2 \\ \lambda_3 \ll \lambda \lesssim \lambda_2}} (\lambda_1 \lambda_3)^{\varepsilon} \|u_{\lambda_1}\|_{U^2_m} \|u_{\lambda_2}\|_{U^2_m} \|u_{\lambda_3}\|_{U^2_m} \right]^2 \\ &\lesssim \|u\|^2_{X^s} \|u\|^2_{X^s} \|u\|^2_{X^s} \sum_{\lambda \geq 1} \lambda^{2s} \left[\lambda^{\varepsilon - 2s} \right]^2 \\ &\lesssim \|u\|^2_{X^s} \|u\|^2_{X^s} \|u\|^2_{X^s}. \end{aligned}$$

Finally, we consider S_3. As in the previous case, we have $S_3 \leq S_{31} + S_{32} + S_{33}$, where

$$S_{31} = \sum_{\substack{\lambda_1 \ll \lambda_2 \\ \lambda_3 \gg \lambda}}, \quad S_{32} = \sum_{\substack{\lambda_1 \gg \lambda_2 \\ \lambda_3 \gg \lambda}}, \quad S_{33} = \sum_{\substack{\lambda_1 \sim \lambda_2 \\ \lambda_3 \gg \lambda}}$$

If $\lambda_1 \ll \lambda_2$, then $\lambda_2 \sim \lambda_3$. We apply Cauchy-Schwarz first in λ_1 and then in $\lambda_2 \sim \lambda_3$ to obtain

$$\begin{aligned} S_{31} &\lesssim \sum_{\lambda \geq 1} \lambda^{2s} \left[\sum_{\substack{\lambda_1 \ll \lambda_2 \\ \lambda \ll \lambda_3 \sim \lambda_2}} (\lambda_1 \lambda_2)^{\varepsilon} \|u_{\lambda_1}\|_{U^2_m} \|u_{\lambda_2}\|_{U^2_m} \|u_{\lambda_3}\|_{U^2_m} \right]^2 \\ &\lesssim \|u\|^2_{X^s} \|u\|^2_{X^s} \|u\|^2_{X^s} \sum_{\lambda} \lambda^{2s} \left[\lambda^{\varepsilon - 2s} \right]^2 \\ &\lesssim \|u\|^2_{X^s} \|u\|^2_{X^s} \|u\|^2_{X^s}. \end{aligned}$$

The estimate for S_{32} is similar.

If $\lambda_1 \sim \lambda_2$, then $\lambda_3 \lesssim \lambda_2$. We apply Cauchy-Schwarz in $\lambda_1 \sim \lambda_2$ and in λ_3 to obtain

$$\begin{aligned} S_{33} &\lesssim \sum_{\lambda} \lambda^{2s} \left[\sum_{\substack{\lambda_1 \sim \lambda_2 \\ \lambda_3 \gg \lambda}} (\lambda_1\lambda_3)^{\varepsilon} \|u_{\lambda_1}\|_{U^2_m} \|u_{\lambda_2}\|_{U^2_m} \|u_{\lambda_3}\|_{U^2_m} \right]^2 \\ &\lesssim \|u\|^2_{X^s} \|u\|^2_{X^s} \|u\|^2_{X^s} \sum_{\lambda \geq 1} \lambda^{2s} \left[\lambda^{\varepsilon - 2s}\right]^2 \\ &\lesssim \|u\|^2_{X^s} \|u\|^2_{X^s} \|u\|^2_{X^s}. \end{aligned}$$

6. Proof of Lemma 5.1

6.1. **Proof of Lemma 5.1a.** First, we claim the following estimates for spatially radial functions $u_{j,\lambda_j} \in U^2_m$, $v_\lambda \in V^2_m$ and $\lambda, \lambda_j > 0$, $\varepsilon > 0$ and $p > 2$:

$$\left| \iint [V * (u_{1,\lambda_1} \overline{u}_{2,\lambda_2})] u_{3,\lambda_3} \; \overline{v}_\lambda \; dtdx \right| \lesssim \min(\lambda_1, \lambda_2)^{\varepsilon} \prod_{j=1}^{3} \|u_{j,\lambda_j}\|_{U^2_m} \|v_\lambda\|_{U^2_m}, \tag{6.1}$$

$$\left| \iint [V * (u_{1,\lambda_1} \overline{u}_{2,\lambda_2})] u_{3,\lambda_3} \; \overline{v}_\lambda \; dtdx \right| \lesssim \lambda_{\min}\lambda_{\text{med}} \prod_{j=1}^{3} \|u_{j,\lambda_j}\|_{U^2_m} \|v_\lambda\|_{U^p_m}, \tag{6.2}$$

where $\varepsilon > 0$ and $p > 2$.

Let us for the time being assume that the claims (6.1) and (6.2) hold. Given $M \geq 1$, we use Lemma 2.5 to decompose v_λ into $v_\lambda = w_\lambda + z_\lambda$, where $w_\lambda \in U^2_m$ and $z_\lambda \in U^p_m$, such that

$$\frac{\kappa}{M} \|w_\lambda\|_{U^2_m} + e^M \|z_\lambda\|_{U^p_m} \lesssim \|v_\lambda\|_{V^2_m} \tag{6.3}$$

for some $\kappa > 0$. We can assume that $\widetilde{P}_\lambda w_\lambda = w_\lambda$ and $\widetilde{P}_\lambda z_\lambda = z_\lambda$.

We then use (6.1)–(6.3) to obtain

$$\begin{aligned} \left| \iint [V * (u_{1,\lambda_1} \overline{u}_{2,\lambda_2})] u_{3,\lambda_3} \; \overline{v}_\lambda \; dtdx \right| &\leq \left| \iint [V * (u_{1,\lambda_1} \overline{u}_{1,\lambda_2})] u_{1,\lambda_3} \; \overline{w}_\lambda \; dtdx \right| \\ &\quad + \left| \iint [V * (u_{1,\lambda_1} \overline{u}_{1,\lambda_2})] u_{1,\lambda_3} \; \overline{z}_\lambda \; dtdx \right| \\ &\lesssim \min(\lambda_1, \lambda_2)^{\varepsilon} \prod_{j=1}^{3} \|u_{j,\lambda_j}\|_{U^2_m} \|w_\lambda\|_{U^2_m} \\ &\quad + \lambda_{\min}\lambda_{\text{med}} \prod_{j=1}^{3} \|u_{j,\lambda_j}\|_{U^2_m} \|z_\lambda\|_{U^p_m} \\ &\lesssim C_{\lambda,\kappa,M} \prod_{j=1}^{3} \|u_{j,\lambda_j}\|_{U^2_m} \|u_\lambda\|_{V^2_m}, \end{aligned}$$

where

$$C_{\lambda,M} = \left\{ \frac{M}{\kappa} \min(\lambda_1, \lambda_2)^{\varepsilon} + e^{-M} \lambda_{\min}\lambda_{\text{med}} \right\}.$$

Now if we choose

$$M = \ln\left(\lambda_{\min}\lambda_{\text{med}}\right),$$

then

$$C_{\lambda,\kappa,M} = \frac{1}{\kappa}\ln\left(\lambda_{\min}\lambda_{\text{med}}\right)\min(\lambda_1,\lambda_2)^{\varepsilon} + 1 \lesssim (\lambda_{\min}\lambda_{\text{med}})^{\varepsilon},$$

and therefore the desired estimate follows.

So it remains to prove (6.1) and (6.2), which we do in the following subsections.

6.1.1. *Proof of* (6.1). In $\mathbb{R}^3$ convolution with V is (up to a multiplicative constant) the Fourier-multiplier $\langle D\rangle^{-2}$ with symbol $\langle\xi\rangle^{-2}$. Using Littlewood-Paley decomposition and Cauchy-Schwarz, we obtain

$$\begin{aligned}\text{LHS of (6.1)} &\lesssim \sum_{\mu\geq 1}\langle\mu\rangle^{-2}\|P_\mu(u_{1,\lambda_1}\overline{u}_{2,\lambda_2})\|_{L^2}\|P_\mu(u_{3,\lambda_3}\ \overline{v}_\lambda)\|_{L^2}\\ &:= S_1+S_2+S_3,\end{aligned}$$

where

$$S_1 = \sum_{1\leq\mu\ll\lambda_1}, \quad S_2 = \sum_{1\leq\mu\sim\lambda_1}, \quad S_3 = \sum_{\mu\gg\lambda_1},$$

We now use Corollary 3.4 to obtain

$$\begin{aligned} S_1 &\lesssim \sum_{1\leq\mu\ll\lambda_1}\frac{\mu^2}{\langle\mu\rangle^2}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{U^2_m}\\ &\lesssim \lambda_1^{\varepsilon}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{U^2_m},\\ S_2 &\lesssim \sum_{1\leq\mu\sim\lambda_1}\frac{\mu^2}{\langle\mu\rangle^2}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{U^2_m}\\ &\lesssim \prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{U^2_m}, \end{aligned}$$

where in the case of $\mu \ll \lambda_1$ we used

$$\sum_{1\leq\mu\ll\lambda_1}\frac{\mu}{\langle\mu\rangle}\lesssim \ln(\lambda_1)+1\lesssim\lambda_1^{\varepsilon}$$

whereas if $\mu\sim\lambda_1$ the sum is finite, i.e., $\sum_{1\leq\mu\sim\lambda_1}\frac{\mu}{\langle\mu\rangle}\lesssim 1$. Similarly,

$$S_3 \lesssim \prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{U^2_m},$$

6.1.2. *Proof of* (6.2). By symmetry (we do not exploit complex conjugation), we assume $\lambda_1\leq\lambda_2\leq\lambda_3$. Since V is bounded in $L^1(\mathbb{R}^3)$, we have by Hölder inequality, (3.2) and Proposition 2.3, we have for all $p\geq 2$,

$$\begin{aligned}\text{LHS of (6.2)} &\lesssim \|u_{1,\lambda_1}\|_{L^2_tL^\infty_x}\|u_{2,\lambda_2}\|_{L^2_tL^\infty_x}\|u_{3,\lambda_3}\|_{L^\infty_tL^2_x}\|v_\lambda\|_{L^\infty_tL^2_x}.\\ &\lesssim \lambda_1\lambda_2\|u_{1,\lambda_1}\|_{U^2_m}\|u_{2,\lambda_2}\|_{U^2_m}\|u_{3,\lambda_3}\|_{U^2_m}\|v_\lambda\|_{U^p_m}.\end{aligned}$$

6.2. **Proof of Lemma 5.1b.** Let $m>0$ and assume $u_{j,\lambda_j}\in U^2_m$ and $v\in V^2_m$. Using Littlewood-Paley decomposition, we get

$$\Big|\iint [V*(u_{1,\lambda_1}\overline{u}_{2,\lambda_2})]u_{3,\lambda_3}\ \overline{v}_\lambda\ dtdx\Big|\lesssim K:=\sum_{\mu\geq 1}\langle\mu\rangle^{-2}K_\mu,$$

where

$$K_\mu=\|P_\mu(u_{1,\lambda_1}\overline{u}_{2,\lambda_2})P_\mu(u_{3,\lambda_3}\overline{v}_\lambda)\|_{L^1_{t,x}}.$$

By symmetry, we may assume $\lambda_1\leq\lambda_2$.

6.2.1. *Case 1:* $\lambda_1\leq\lambda_2\leq\lambda_3$.

(a) Assume $\lambda_1\ll\lambda_2$. Then $\mu\sim\lambda_2$. By Sobolev, Hölder, (3.10) and Lemma 2.8,

$$\begin{aligned}K_\mu&\lesssim\|P_\mu(u_{1,\lambda_1}\overline{u}_{2,\lambda_2})\|_{L^1_tL^\infty_x}\|u_{3,\lambda_3}\|_{L^\infty_tL^2_x}\|v_\lambda\|_{L^\infty_tL^2_x}\\&\lesssim\mu\|u_{1,\lambda_1}\|_{L^2_tL^6_x}\|u_{2,\lambda_2}\|_{L^2_tL^6_x}\|u_{3,\lambda_3}\|_{L^\infty_tL^2_x}\|v_\lambda\|_{L^\infty_tL^2_x}\\&\lesssim\mu(\lambda_1\lambda_2)^{\frac56}\prod_{j=1}^3\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{V^2_m}\end{aligned}$$

Then (using $\mu\sim\lambda_2$)

$$\begin{aligned}K&\lesssim\sum_{\mu\geq1}\langle\mu\rangle^{-2}\mu^{2-2\varepsilon}(\lambda_1\lambda_2)^{\frac13+\varepsilon}\prod_{j=1}^3\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{V^2_m}\\&\lesssim(\lambda_1\lambda_2)^{\frac13+\varepsilon}\prod_{j=1}^3\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{V^2_m}.\end{aligned}$$

(b) Assume $\lambda_1\sim\lambda_2\gg\mu$. In this case, we consider (as in [17]) the collection of cubes $C_z=\mu z+[0,\mu)^3$, $z\in\mathbb{Z}^3$, which induce a disjoint covering of $\mathbb{R}^3$. By the triangle inequality, we have

$$K_\mu\lesssim\sum_{z,z'\in\mathbb{Z}^3}\|P_\mu(P_{C_z}u_{1,\lambda_1}P_{C_{z'}}\overline{u}_{2,\lambda_2})P_\mu(u_{3,\lambda_3}\overline{v}_\lambda)\|_{L^1_{t,x}}$$

The point of this decomposition is that the term $P_\mu(P_{C_z}u_{1,\lambda_1}P_{C_{z'}}\overline{u}_{2,\lambda_2})$ is zero for most of $z,z'\in\mathbb{Z}^3$, i.e., for each $z\in\mathbb{Z}^3$, only those $z'\in\mathbb{Z}^3$ with $|z-z'|\lesssim1$ yield a nontrivial contribution on the sum. We use this fact, Hölder, (3.10) and Remark 3.7 to obtain

$$\begin{aligned}K_\mu&\lesssim\sum_{z,z'\in\mathbb{Z}^3}\|P_{C_z}u_{1,\lambda_1}\|_{L^2_tL^\infty_x}\|P_{C_{z'}}u_{2,\lambda_2}\|_{L^2_tL^\infty_x}\|u_{3,\lambda_3}\|_{L^\infty_tL^2_x}\|v_\lambda\|_{L^\infty_tL^2_x}\\&\lesssim\Big(\sum_{z\in\mathbb{Z}^3}\|P_{C_z}u_{1,\lambda_1}\|^2_{L^2_tL^\infty_x}\Big)^{\frac12}\Big(\sum_{z'\in\mathbb{Z}^3}\|P_{C_{z'}}u_{2,\lambda_2}\|^2_{L^2_tL^\infty_x}\Big)^{\frac12}\|u_{3,\lambda_3}\|_{U^2_m}\|v_\lambda\|_{V^2_m}\\&\lesssim\mu(\lambda_1\lambda_2)^{\frac12+\varepsilon}\prod_{j=1}^3\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{V^2_m}.\end{aligned}$$

Hence

$$\begin{aligned} K &\lesssim \sum_{\mu\geq 1}\langle\mu\rangle^{-2}\mu(\lambda_1\lambda_2)^{\frac{1}{2}+\varepsilon}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{U^4_m} \\ &\lesssim (\lambda_1\lambda_2)^{\frac{1}{2}+\varepsilon}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{V^2_m} \\ &\lesssim (\lambda_1\lambda_2)^{\frac{1}{2}+\varepsilon}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{V^2_m}. \end{aligned}$$

6.2.2. *Case 2: $\lambda_1 \leq \lambda_3 \leq \lambda_2$ or $\lambda_3 \leq \lambda_1 \leq \lambda_2$.* First assume $\lambda_1 \leq \lambda_3 \leq \lambda_2$. If $\lambda_3 \ll \lambda_2$ then we follow the same argument as in Case 1 (a). If $\lambda_3 \sim \lambda_2$, then by following the same argument in Case 1 (b), we can obtain a maximum of $(\lambda_1\lambda_3)^{\frac{1}{2}+\varepsilon}$ factor in the estimate for K.

Next assume $\lambda_3 \leq \lambda_1 \leq \lambda_2$. The only issue in this case is when $\lambda_3 \ll \lambda_1 \sim \lambda_2$. If $\lambda_3 \lesssim \mu$, we can then use the same argument as in Case 1 (a) by putting both u_{1,λ_1} and u_{3,λ_3} in $L^2_tL^6_x$ and the others in $L^\infty_tL^2_x$. It remains to consider the case $\lambda_3 \gg \mu$, which also implies $\lambda \sim \lambda_3 \ll \lambda_1 \sim \lambda_2$. Let $\varepsilon' > 0$. Arguing as in Case 1 (b) (i.e., using the fact $\lambda_1 \sim \lambda_2 \gg \mu$), we have

$$K \lesssim (\lambda_1\lambda_2)^{\frac{1}{2}+\varepsilon'}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{V^2_m}. \tag{6.4}$$

Similarly, we apply the decomposition in $\lambda_3 \sim \lambda$ as in Case 1 (b) by putting both u_{3,λ_3} and v_λ in $L^2_tL^\infty_x$, whereas u_{1,λ_1} and u_{2,λ_2} in $L^\infty_tL^2_x$. Doing so we obtain

$$K \lesssim (\lambda\lambda_3)^{\frac{1}{2}+\varepsilon'}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|v_\lambda\|_{U^2_m}. \tag{6.5}$$

Now we argue as in the proof of Lemma 5.1a by decomposing $v_\lambda = w_\lambda + z_\lambda$, where $w_\lambda \in U^2_m$ and $z_\lambda \in U^p_m$, such that (6.3) is satisfied. Then using (6.4)–(6.5), we obtain

$$K \lesssim C_{\kappa,\lambda,M}\prod_{j=1}^{3}\|u_{j,\lambda_j}\|_{U^2_m}\|u_\lambda\|_{V^2_m},$$

where

$$C_{\lambda,M} = \left[\frac{M}{\kappa}(\lambda\lambda_3)^{\frac{1}{2}+\varepsilon'} + e^{-M}(\lambda_1\lambda_2)^{\frac{1}{2}+\varepsilon'}\right].$$

Now if we choose

$$M = \left(\frac{1}{2}+\varepsilon'\right)\ln\left(\lambda_1\lambda_2\right),$$

and $\varepsilon' = \varepsilon/2$, then

$$\begin{aligned} C_{\lambda,\kappa,M} &= \frac{1+\varepsilon}{2\kappa}[\ln\left(\lambda_1\lambda_2\right)](\lambda\lambda_3)^{\frac{1+\varepsilon}{2}} + 1 \\ &\lesssim \lambda_1^{\varepsilon}\lambda_3^{1+\varepsilon} \end{aligned}$$

which is in fact better than the desired estimate.

References

[1] Ioan Bejenaru and Sebastian Herr. On global well-posedness and scattering for the massive Dirac-Klein-Gordon system. arxiv:1409.177.

[2] Ioan Bejenaru and Sebastian Herr. The cubic Dirac equation: Small initial data in $H^1(\mathbb{R}^3)$. *Comm. Math. Phys.*, 2014.

[3] J. M. Chadam and R. T. Glassey. On the Maxwell-Dirac equations with zero magnetic field and their solution in two space dimensions. *J. Math. Anal. Appl.*, 53(3):495–507, 1976.

[4] John M. Chadam and Robert T. Glassey. On certain global solutions of the Cauchy problem for the (classical) coupled Klein-Gordon-Dirac equations in one and three space dimensions. *Arch. Rational Mech. Anal.*, 54:223–237, 1974.

[5] Yonggeun Cho and Tohru Ozawa. On the semirelativistic Hartree-type equation. *SIAM J. Math. Anal.*, 38(4):1060–1074 (electronic), 2006.

[6] Yonggeun Cho and Tohru Ozawa. Global solutions of semirelativistic Hartree type equations. *J. Korean Math. Soc.*, 44(5):1065–1078, 2007.

[7] Yonggeun Cho and Tohru Ozawa. On radial solutions of semi-relativistic Hartree equations. *Discrete Contin. Dyn. Syst. Ser. S*, 1(1):71–82, 2008.

[8] Yonggeun Cho, Tohru Ozawa, Hironobu Sasaki, and Yongsun Shim. Remarks on the semirelativistic Hartree equations. *Discrete Contin. Dyn. Syst.*, 23(4):1277–1294, 2009.

[9] Piero D'Ancona and Luca Fanelli. Strichartz and smoothing estimates of dispersive equations with magnetic potentials. *Comm. Partial Differential Equations*, 33(4-6):1082–1112, 2008.

[10] João-Paulo Dias and Mário Figueira. On the existence of weak solutions for a nonlinear time dependent Dirac equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 113(1-2):149–158, 1989.

[11] João-Paulo Dias and Mário Figueira. Solutions faibles du problème de Cauchy pour certaines équations de Dirac non linéaires. *Portugal. Math.*, 46(suppl.):475–484, 1989. Workshop on Hyperbolic Systems and Mathematical Physics (Lisbon, 1988).

[12] Alexander Elgart and Benjamin Schlein. Mean field dynamics of boson stars. *Comm. Pure Appl. Math.*, 60(4):500–545, 2007.

[13] Damiano Foschi and Sergiu Klainerman. Bilinear space-time estimates for homogeneous wave equations. *Ann. Sci. École Norm. Sup. (4)*, 33(2):211–274, 2000.

[14] J. Fröhlich, B. L. G. Jonsson, and E. Lenzmann. Effective dynamics for boson stars. *Nonlinearity*, 20(5):1031–1075, 2007.

[15] Martin Hadac, Sebastian Herr, and Herbert Koch. Well-posedness and scattering for the KP-II equation in a critical space. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(3):917–941, 2009.

[16] Martin Hadac, Sebastian Herr, and Herbert Koch. Erratum to "Well-posedness and scattering for the KP-II equation in a critical space" [Ann. I. H. Poincaré—AN 26 (3) (2009) 917–941] [2526409]. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3):971–972, 2010.

[17] Sebastian Herr and Enno Lenzmann. The Boson star equation with initial data of low regularity. *Nonlinear Anal.*, 97:125–137, 2014.

[18] Sebastian Herr, Daniel Tataru, and Nikolay Tzvetkov. Global well-posedness of the energy-critical nonlinear Schrödinger equation with small initial data in $H^1(\mathbb{T}^3)$. *Duke Math. J.*, 159(2):329–349, 2011.

[19] Herbert Koch and Daniel Tataru. A priori bounds for the 1D cubic NLS in negative Sobolev spaces. *Int. Math. Res. Not. IMRN*, (16):Art. ID rnm053, 36, 2007.

[20] Herbert Koch, Daniel Tataru, and Monica Visan, editors. *Dispersive Equations and Nonlinear Waves: Generalized Korteweg-de Vries, Nonlinear Schrödinger, Wave and Schrödinger Maps*, volume 45 of *Oberwolfach Seminars*. Birkhäuser, Basel, 2014.

[21] Enno Lenzmann. Well-posedness for semi-relativistic Hartree equations of critical type. *Math. Phys. Anal. Geom.*, 10(1):43–64, 2007.

[22] Elliott H. Lieb and Horng-Tzer Yau. The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. *Comm. Math. Phys.*, 112(1):147–174, 1987.

[23] Shuji Machihara, Makoto Nakamura, Kenji Nakanishi, and Tohru Ozawa. Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation. *J. Funct. Anal.*, 219(1):1–20, 2005.

[24] Alessandro Michelangeli and Benjamin Schlein. Dynamical collapse of boson stars. *Comm. Math. Phys.*, 311(3):645–687, 2012.

[25] Evgeni Y. Ovcharov. Radial Strichartz estimates with application to the 2-D Dirac-Klein-Gordon system. *Comm. Partial Differential Equations*, 37(10):1754–1788, 2012.
[26] Fabio Pusateri. Modified scattering for the boson star equation. *Comm. Math. Phys.*, 332(3):1203–1234, 2014.
[27] Tobias Schottdorf. Global existence without decay for quadratic Klein-Gordon equations. arXiv:1209.1518.
[28] Jacob Sterbenz. Angular regularity and Strichartz estimates for the wave equation. *Int. Math. Res. Not.*, (4):187–231, 2005. With an appendix by Igor Rodnianski.

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