ON THE RADIUS OF SPATIAL ANALYTICITY FOR THE 1D DIRAC-KLEIN-GORDON EQUATIONS

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Abstract. We study the well-posedness of the Dirac-Klein-Gordon system in one space dimension with initial data that are analytic in a strip around the real axis. It is proved that for short times t the radius of analyticity $\sigma(t)$ of the solutions remains constant while for $|t| \to \infty$ we obtain a lower bound $\sigma(t) \ge c/|t|^{5+}$ in the case of positive Klein-Gordon mass and $\sigma(t) \ge c/|t|^{8+}$ in the massless case.

1. Introduction

Consider the Dirac-Klein-Gordon equations (DKG) on $\mathbb{R}^{1+1}$,

$$\begin{cases} \left(-i\gamma^0\partial_t - i\gamma^1\partial_x + M\right)\psi = \phi\psi, \\ \left(\partial_t^2 - \partial_x^2 + m^2\right)\phi = \psi^*\gamma^0\psi, \end{cases} \qquad (t, x \in \mathbb{R}) \tag{1}$$

with initial condition

$$\psi(0,x) = \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \partial_t\phi(0,x) = \phi_1(0,x). \tag{2}$$

Here the unknowns are $\phi\colon \mathbb{R}^{1+1} \to \mathbb{R}$ and $\psi\colon \mathbb{R}^{1+1} \to \mathbb{C}^2$, the latter regarded as a column vector with conjugate transpose ψ^*. The masses $M, m \ge 0$ are given constants. The 2×2 Dirac matrices γ^0, γ^1 should satisfy $\gamma^0\gamma^1+\gamma^1\gamma^0 = 0$, $(\gamma^0)^2 = I$, $(\gamma^1)^2 = -I$, $(\gamma^0)^* = \gamma^0$ and $(\gamma^1)^* = -\gamma^1$; we will work with the representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The well-posedness of this Cauchy problem with data in the family of Sobolev spaces $H^s = (1-\partial_x^2)^{-s/2}L^2(\mathbb{R})$, $s \in \mathbb{R}$, has been intensively studied recently; see [6, 3, 1, 4, 15, 13, 18, 17, 16, 21, 14, 19, 5]. Local well-posedness holds for data

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with $s > -1/2$ and $|s| \le r \le s+1$; see [14], where it is also proved that this is the optimal result, in the sense that for other (r,s) one either has ill-posedness or the solution map (if it exists) is not regular.

Moreover, when $s \ge 0$ there is conservation of charge,

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2},$$

implying that the solutions extend globally when $0 \le s \le r \le s+1$. Indeed, by the result on persistence of higher regularity from [19, Theorem 1.6] it suffices to prove the extension when $s = 0 \le r < 1/2$, but then energy estimates, Sobolev embedding

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and conservation of charge yield a bound $\|(\phi,\partial_t\phi)(t)\|_{H^r\times H^{r-1}} \le C(1+|t|)$, where C is proportional to the sum of $\|(\phi_0,\phi_1)\|_{H^r\times H^{r-1}}$ and $\|\psi_0\|_{L^2}^2$, hence the local result can be continued indefinitely.

In this paper we are interested in the spatial analyticity of the solutions. We use the following spaces of Gevrey type. For $\sigma \ge 0$ and $s\in\mathbb{R}$, let $G^{\sigma,s}$ be the Banach space with norm

$$\|f\|_{G^{\sigma,s}} = \left\| e^{\sigma|\xi|}\langle\xi\rangle^s \widehat{f}(\xi)\right\|_{L^2_\xi},$$

where $\widehat{f}(\xi)=\int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$ is the Fourier transform and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. So for $\sigma>0$ we have $G^{\sigma,s} = \{f\in L^2\colon e^{\sigma|\cdot|}\langle\cdot\rangle^s\widehat{f}\in L^2\}$ and for $\sigma=0$ we recover the Sobolev space $H^s = G^{0,s}$ with norm

$$\|f\|_{H^s} = \left\|\langle\xi\rangle^s\widehat{f}(\xi)\right\|_{L^2_\xi}.$$

Observe the inclusions

$$G^{\sigma,s}\subset H^{s'} \qquad \text{for } 0<\sigma \text{ and } s,s'\in\mathbb{R}, \tag{3}$$

$$G^{\sigma,s}\subset G^{\sigma',s'} \qquad \text{for } 0<\sigma'<\sigma \text{ and } s,s'\in\mathbb{R}. \tag{4}$$

Every function in $G^{\sigma,s}$ with $\sigma>0$ has an analytic extension to the strip

$$S_\sigma = \{x+iy\colon x,y\in\mathbb{R},\ |y|<\sigma\}.$$

Paley-Wiener Theorem. *Let $\sigma>0$, $s\in\mathbb{R}$. The following are equivalent:*

(i) $f\in G^{\sigma,s}$.
(ii) *f is the restriction to the real line of a function F which is holomorphic in the strip S_σ and satisfies* $\sup_{|y|<\sigma}\|F(x+iy)\|_{H^s_x}<\infty$.

The proof given for $s=0$ in [12, p. 209] applies also for $s\in\mathbb{R}$ with some obvious modifications.

2. Main results

Consider the Cauchy problem (1), (2) with data

$$(\psi_0,\phi_0,\phi_1)\in G^{\sigma_0,s}\times G^{\sigma_0,r}\times G^{\sigma_0,r-1}, \tag{5}$$

where $\sigma_0>0$ and $(r,s)\in\mathbb{R}^2$. By the inclusion (3) and the existing well-posedness theory we know that this problem has a unique, smooth solution for all time, regardless of the values of r and s. But is the solution analytic in some strip for each time t? Our first observation is that the radius of analyticity actually remains constant for short times, for a restricted set of (r,s).

Theorem 1. *Assume $\sigma_0>0$, $s>-1/4$, $r>0$ and $|s|\le r\le s+1$. Then for any data (5) there exists a time $\delta>0$, depending on the norm of the data, such that the solution of (1), (2) satisfies*

$$(\psi,\phi,\partial_t\phi)\in C\left([-\delta,\delta];G^{\sigma_0,s}\times G^{\sigma_0,r}\times G^{\sigma_0,r-1}\right). \tag{6}$$

Here (r,s) is in the region of local well-posedness obtained in [18]; this region was extended in [14], and it is possible that the same extension could be carried out in the present analytic setting. However, if we allow for an arbitrarily small loss in the radius of analyticity, we can in fact handle any $(r,s)\in\mathbb{R}^2$, since the embedding (4) immediately implies the following corollary to Theorem 1.

Corollary 1. *Let $\sigma_0 > 0$ and $(r,s) \in \mathbb{R}^2$. Then for any data (5) there exists $\delta > 0$ such that (6) holds with σ_0 replaced by any $\sigma_0' < \sigma$.*

The proof of Theorem 1 is a straightforward modification of the contraction argument used to prove local well-posedness in [18]; instead of contracting in the usual Bourgain spaces $X^{s,b}$ one uses Gevrey-modified spaces $X^{\sigma,s,b}$. Such spaces were introduced by Grujić and Kalisch in [10, 11] in the context of the KdV equation and the nonlinear Schrödinger equation to prove analogues of our Theorem 1.

While the local result is easy to obtain, a much more difficult and interesting question is what happens for large times. Our approach is to prove a more precise version of the local result, where we determine the dependence of the existence time δ on the data norm and then estimate the growth of the solution in the time interval $[-\delta, \delta]$, measured in the data norm. Here it is important that although the conservation of charge does not hold exactly in the Gevrey space $G^{\sigma,0}$, it does hold in an approximate sense. Iterating the local result we are then able to prove our main result:

Theorem 2. *Let $\sigma_0 > 0$ and $(r,s) \in \mathbb{R}^2$. Then for any data (5) the solution of the Cauchy problem (1), (2) satisfies*

$$(\psi, \phi, \partial_t \phi)(t) \in G^{\sigma(t),s} \times G^{\sigma(t),r} \times G^{\sigma(t),r-1} \quad \textit{for all } t \in \mathbb{R},$$

where the radius of analyticity $\sigma(t) > 0$ satisfies an asymptotic lower bound

$$\left.\begin{aligned} &\textit{if } m > 0\textit{:} \quad \sigma(t) \geq \frac{c_\varepsilon}{|t|^{5+\varepsilon}} \\ &\textit{if } m = 0\textit{:} \quad \sigma(t) \geq \frac{c_\varepsilon}{|t|^{8+\varepsilon}} \end{aligned}\right\} \quad \textit{as } |t| \to \infty,$$

for any $\varepsilon > 0$. Here the constant $c_\varepsilon > 0$ depends on ε and also on (σ_0, r, s) and the norm of the initial data.

Analogous results for the KdV equation were proved by Bona, Grujić and Kalisch in [2], but the method used there is different: They estimate in Gevrey-modified Bourgain spaces directly on any large time interval $[-T, T]$ while we iterate a precise local result. If we adapt the method from [2], we only get a rate $1/|t|^{8+}$, while our method gives $1/|t|^{5+}$. The reason for this loss is twofold: (i) the Bourgain norms involve integration in time, which is a disadvantage for large time intervals, and (ii) it is not clear how to get any kind of approximate charge conservation in this large-time set-up. Our short-time iterative approach is more inspired by the ideas developed by Colliander, Holmer and Tzirakis in [9] in the context of global well-posedness in the standard Sobolev spaces. Moreover, the idea of using almost conservation laws goes back to [7]; for DKG an almost conservation conservation law first appeared in [21].

We now turn to the proofs. Leaving the case $m = 0$ until the very end of the paper, we will assume $m > 0$ for now. By a rescaling we may assume $m = 1$.

3. Reformulation of the system

It will be convenient to rewrite the DKG system as follows. Write $\psi = (\psi_+, \psi_-)^T$ and $\phi = \phi_+ + \phi_-$ with $\phi_\pm = \frac{1}{2}\left(\phi \pm i\langle D_x\rangle^{-1}\partial_t \phi\right)$, where $D_x = -i\partial_x$, hence D_x and $\langle D_x \rangle$ are Fourier multipliers with symbols ξ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ respectively.

Writing also $D_t = -i\partial_t$, then (1), (5) with $m = 1$ is equivalent to

$$
(7) \qquad \begin{cases} (D_t + D_x)\,\psi_+ = -M\psi_- + \phi\psi_-, & \psi_+(0) = f_+ \in G^{\sigma_0,s}, \\ (D_t - D_x)\,\psi_- = -M\psi_+ + \phi\psi_+, & \psi_-(0) = f_- \in G^{\sigma_0,s}, \\ (D_t + \langle D_x\rangle)\,\phi_+ = -\langle D_x\rangle^{-1}\,\mathrm{Re}\left(\overline{\psi_+}\psi_-\right), & \phi_+(0) = g_+ \in G^{\sigma_0,r}, \\ (D_t - \langle D_x\rangle)\,\phi_- = +\langle D_x\rangle^{-1}\,\mathrm{Re}\left(\overline{\psi_+}\psi_-\right), & \phi_-(0) = g_- \in G^{\sigma_0,r}, \end{cases}
$$

where $\psi_0 = (f_+, f_-)^T$ and $g_\pm = \frac{1}{2}\left(\phi_0 \pm i\langle D_x\rangle^{-1}\phi_1\right)$. We remark that $\overline{\phi_+} = \phi_-$, since ϕ is real-valued.

Remark 1. Applying $e^{\sigma|D_x|}$ to the system and writing $\Psi_\pm = e^{\sigma|D_x|}\psi_\pm$, $\Phi_\pm = e^{\sigma|D_x|}\phi_\pm$, $\mathfrak{f}_\pm = e^{\sigma|D_x|}f_\pm$ and $\mathfrak{g}_\pm = e^{\sigma|D_x|}g_\pm$ one obtains

$$
\begin{cases} (D_t \pm D_x)\,\Psi_\pm = -M\Psi_\mp + e^{\sigma|D_x|}\left(\phi\psi_\mp\right), & \Psi_\pm(0) = \mathfrak{f}_\pm \in H^s, \\ (D_t \pm \langle D_x\rangle)\,\Phi_\pm = \mp\langle D_x\rangle^{-1}\,\mathrm{Re}\, e^{\sigma|D_x|}\left(\overline{\psi_+}\psi_-\right), & \Phi_\pm(0) = \mathfrak{g}_\pm \in H^r, \end{cases}
$$

and if it were not for the fact that $e^{\sigma|D_x|}(fg)$ is not equal to $e^{\sigma|D_x|}f \cdot e^{\sigma|D_x|}g$, we could now have applied directly the local well-posedness theory in the standard Sobolev spaces to get Theorem 1. But for the purpose of proving product estimates in norms which depend only on the size of the Fourier transform, it turns out that $e^{\sigma|D_x|}$ does in effect distribute, since its symbol satisfies

$$
e^{\sigma|\xi+\eta|} \le e^{\sigma|\xi|}e^{\sigma|\eta|}
$$

by the triangle inequality. So for example any Sobolev product estimate

$$
\|fg\|_{H^a} \le C\,\|f\|_{H^b}\,\|g\|_{H^c}
$$

immediately implies

$$
\|fg\|_{G^{\sigma,a}} \le C\,\|f\|_{G^{\sigma,b}}\,\|g\|_{G^{\sigma,c}}
$$

for all $\sigma > 0$, and the same mechanism applies for product estimates in the Bourgain norms, defined below.

4. Function spaces

For any continuous function $h\colon \mathbb{R} \to \mathbb{R}$ of polynomial growth (which for us will be either $h(\xi) = \pm\xi$ or $h(\xi) = \pm\langle\xi\rangle$) the Bourgain space $X^{s,b}_{h(\xi)}$ for $(s,b) \in \mathbb{R}^2$ is the Banach space with norm

$$
\|u\|_{X^{s,b}_{h(\xi)}} = \left\|\langle\xi\rangle^s\langle\tau + h(\xi)\rangle^b\widetilde{u}(\tau,\xi)\right\|_{L^2_{\tau,\xi}},
$$

where $\widetilde{u}(\tau,\xi) = \iint e^{-i(t\tau + x\xi)}u(t,x)\,dt\,dx$ is the space-time Fourier transform. For $\sigma \ge 0$ the Gevrey-modified space $X^{\sigma,s,b}$ is then given by the norm

$$
\|u\|_{X^{\sigma,s,b}_{h(\xi)}} = \left\|e^{\sigma|D_x|}u\right\|_{X^{s,b}_{h(\xi)}} = \left\|e^{\sigma|\xi|}\langle\xi\rangle^s\langle\tau + h(\xi)\rangle^b\widetilde{u}(\tau,\xi)\right\|_{L^2_{\tau,\xi}},
$$

so for $\sigma = 0$ it reduces to the standard space.

All the well-known properties of the standard $X^{s,b}$ spaces carry over to the Gevrey-modified spaces by the substitution $u \to e^{\sigma|D_x|}u$; the properties we need are given in the following four lemmas. Proofs of Lemmas 1, 2 and 4 can be found, for instance, in [20, Section 2.6], whereas the less well known Lemma 3 follows by the argument used to prove [8, Lemma 3.1].

Lemma 1. *If $b > 1/2$, then* $\sup_{t\in\mathbb{R}}\|u(t)\|_{G^{\sigma,s}} \le C_b\,\|u\|_{X^{\sigma,s,b}_{h(\xi)}}$.

The restriction of $X^{\sigma,s,b}_{h(\xi)}$ to a time slab $(-\delta,\delta)\times\mathbb{R}$ is denoted $X^{\sigma,s,b}_{h(\xi)}(\delta)$; by Lemma 1 it embeds into $C([-\delta,\delta];G^{\sigma,s})$ if $b>1/2$.

Lemma 2. *Let $\sigma\ge 0$, $s\in\mathbb{R}$, $-1/2<b<b'<1/2$ and $0<\delta\le 1$. Then*

$$\|u\|_{X^{\sigma,s,b}_{h(\xi)}(\delta)}\le C\delta^{b'-b}\|u\|_{X^{\sigma,s,b'}_{h(\xi)}(\delta)},$$

where C depends only on b and b'.

Lemma 3. *Let $\sigma\ge 0$, $s\in\mathbb{R}$, $-1/2<b<1/2$ and $\delta>0$. Then for any time interval $I\subset[-\delta,\delta]$ we have*

$$\|\chi_I u\|_{X^{\sigma,s,b}_{h(\xi)}}\le C\|u\|_{X^{\sigma,s,b}_{h(\xi)}(\delta)},$$

where C depends only on b.

Next we note that the solution u of the Cauchy problem

$$(D_t+h(D_x))\,u=F,\quad u(0)=f$$

for sufficiently regular f and F is given by Duhamel's formula

$$u(t)=W_{h(\xi)}(t)f+i\int_0^t W_{h(\xi)}(t-t')F(t')\,dt',$$

where $W_{h(\xi)}(t)=e^{-ith(D)}$ is the solution group. This implies the energy inequality

$$\|u(t)\|_{G^{\sigma,s}}\le\|f\|_{G^{\sigma,s}}+\left|\int_0^t\|F(t')\|_{G^{\sigma,s}}\,dt'\right|\qquad(t\in\mathbb{R}),$$

and we recall also the following counterpart in the Bourgain norms:

Lemma 4. *Let $\sigma\ge 0$, $s\in\mathbb{R}$, $1/2<b\le 1$ and $0<\delta\le 1$. Then for all $f\in G^{\sigma,s}$ and $F\in X^{\sigma,s,b-1+\epsilon}(\delta)$ we have the estimates*

$$\left\|W_{h(\xi)}(t)f\right\|_{X^{\sigma,s,b}_{h(\xi)}(\delta)}\le C\|f\|_{G^{\sigma,s}},$$

$$\left\|\int_0^t W_{h(\xi)}(t-t')F(t')\,dt'\right\|_{X^{\sigma,s,b}_{h(\xi)}(\delta)}\le C\|F\|_{X^{\sigma,s,b-1}_{h(\xi)}(\delta)},$$

where C depends only on b.

Let us write P^{σ}_{lo} for the projection onto frequencies $|\xi|\le 1/\sigma$. Thus, P^{σ}_{lo} is the Fourier multiplier with symbol $\chi_{\sigma|\xi|\le 1}$. Set $P^{\sigma}_{\mathrm{hi}}=1-P^{\sigma}_{\mathrm{lo}}$, so that for any $f\in G^{\sigma,s}$ we have $f=P^{\sigma}_{\mathrm{lo}}f+P^{\sigma}_{\mathrm{hi}}f$. The following estimates are obviously true for any $\sigma,p>0$ and $s\in\mathbb{R}$:

$$\|P^{\sigma}_{\mathrm{lo}}f\|_{G^{\sigma,s}}\le e\|f\|_{H^s},\tag{8}$$

$$\|P^{\sigma}_{\mathrm{hi}}f\|_{G^{\sigma,s}}\le\sigma^p\|f\|_{G^{\sigma,s+p}}.\tag{9}$$

5. Product estimates

We shall need some product estimates of the form

$$\left|\iint uvw\,dt\,dx\right|\le C\|u\|_{X^{s_1,b_1}_{+\xi}}\|v\|_{X^{s_2,b_2}_{-\xi}}\|w\|_{X^{s_3,b_3}_{\pm\langle\xi\rangle}}.\tag{10}$$

Sufficient and necessary conditions on $s_j,b_j\in\mathbb{R}$ for this to hold, up to borderline cases, are given in [5, Corollary 2]. Special cases have appeared earlier in [18], including some borderline cases not included in [5]. One important borderline case is the second estimate in the following lemma, proved in [18, Lemma 2] while the first and third estimates in the lemma are special cases of [5, Corollary 2].

Lemma 5. *For $\varepsilon > 0$ sufficiently small,*

$$\left| \iint uvw \, dt \, dx \right| \le C_\varepsilon \|u\|_{X^{0,2\varepsilon}_{+\xi}} \|v\|_{X^{0,2\varepsilon}_{-\xi}} \|w\|_{X^{1/2-\varepsilon,1/2-2\varepsilon}_{\pm\langle\xi\rangle}} ,$$

$$\left| \iint uvw \, dt \, dx \right| \le C_\varepsilon \|u\|_{X^{0,1/2+\varepsilon}_{+\xi}} \|v\|_{X^{0,1/2+\varepsilon}_{-\xi}} \|w\|_{X^{0,0}_{\pm\langle\xi\rangle}} ,$$

$$\left| \iint uvw \, dt \, dx \right| \le C_\varepsilon \|u\|_{X^{0,1/2-\varepsilon}_{+\xi}} \|v\|_{X^{0,1/2+\varepsilon}_{-\xi}} \|w\|_{X^{2\varepsilon,\varepsilon}_{\pm\langle\xi\rangle}} .$$

Some remarks are in order. First, the different signs in the first two norms on the right-hand side of (10) imply a null structure in the product uv. Second, (10) is equivalent, by duality, to the product estimates

$$\begin{aligned}
\|w\|_{X^{-s_3,-b_3}_{\pm\langle\xi\rangle}} &\le C \|u\|_{X^{s_1,b_1}_{+\xi}} \|v\|_{X^{s_2,b_2}_{-\xi}} , \\
\|vw\|_{X^{-s_1,-b_1}_{+\xi}} &\le C \|v\|_{X^{s_2,b_2}_{-\xi}} \|w\|_{X^{s_3,b_3}_{\pm\langle\xi\rangle}} , \\
\|uw\|_{X^{-s_2,-b_2}_{-\xi}} &\le C \|u\|_{X^{s_1,b_1}_{+\xi}} \|w\|_{X^{s_3,b_3}_{\pm\langle\xi\rangle}} ,
\end{aligned}$$

each of which implies a corresponding estimate in the Gevrey-modified norms by the mechanism in Remark 1. Moreover, each of these estimates implies the corresponding estimate restricted to a time slab $(-\delta, \delta) \times \mathbb{R}$.

6. Local well-posedness

In this section we prove Theorem 1. As Remark 1 suggests, we can in fact reduce to the case $\sigma_0 = 0$ proved in [18]. By iteration in Bourgain norms, that case reduces to the following product estimates:

Lemma 6 ([18]). *Assume $s > -1/4$, $r > 0$ and $|s| \le r \le s + 1$. Then for $a > 0$ sufficiently small and some $b, b' \in (1/2, 1)$ we have the product estimates*

$$\begin{aligned}
\|uv\|_{X^{r-1,b'-1+a}_{\pm\langle\xi\rangle}} &\le C \|u\|_{X^{s,b}_{+\xi}} \|v\|_{X^{s,b}_{-\xi}} , \\
\|vw\|_{X^{s,b-1+a}_{+\xi}} &\le C \|v\|_{X^{s,b}_{-\xi}} \|w\|_{X^{r,b'}_{\pm\langle\xi\rangle}} .
\end{aligned}$$

As in Remark 1 we immediately obtain the same estimates in the Gevrey-modified norms and Theorem 1 follows. We include here the details concerning how the iteration is set up, since this will in any case be needed in the next section, where we begin the proof of Theorem 2.

Consider the iterates $\psi_\pm^{(n)}$, $\phi_\pm^{(n)}$ given inductively by

$$\begin{aligned}
\psi_\pm^{(0)}(t) &= W_{\pm\xi}(t) f_\pm, \qquad \phi_\pm^{(0)}(t) = W_{\pm\langle\xi\rangle}(t) g_\pm, \\
\psi_\pm^{(n+1)}(t) &= \psi_\pm^{(0)}(t) + i \int_0^t W_{\pm\xi}(t-t') \left(-M\psi_\mp^{(n)} + \phi^{(n)}\psi_\mp^{(n)} \right)(t') \, dt', \\
\phi_\pm^{(n+1)}(t) &= \phi_\pm^{(0)}(t) \mp i \int_0^t W_{\pm\langle\xi\rangle}(t-t') \langle D_x \rangle^{-1} \operatorname{Re} \left(\overline{\psi_+^{(n)}} \psi_-^{(n)} \right)(t') \, dt'
\end{aligned}$$

for $n \in \{0, 1, 2, \dots\}$. Here $\phi^{(n)} = \phi_+^{(n)} + \phi_-^{(n)}$.

Given $\sigma_0 > 0$, $s > -1/4$ and $r > 0$ satisfying $|s| \le r \le s+1$ we choose $a > 0$ and $b, b' \in (1/2, 1)$ as in Lemma 6. Writing

$$A_n(\delta) = \|\psi_+^{(n)}\|_{X_{+\xi}^{\sigma_0,s,b}(\delta)} + \|\psi_-^{(n)}\|_{X_{-\xi}^{\sigma_0,s,b}(\delta)} + \|\phi_+^{(n)}\|_{X_{+\langle\xi\rangle}^{\sigma_0,r,b'}(\delta)} + \|\phi_-^{(n)}\|_{X_{-\langle\xi\rangle}^{\sigma_0,r,b'}(\delta)},$$
$$R = \|f_+\|_{G^{\sigma_0,s}} + \|f_-\|_{G^{\sigma_0,s}} + \|g_+\|_{G^{\sigma_0,r}} + \|g_-\|_{G^{\sigma_0,r}},$$

and applying Lemmas 4 and 2 and the Gevrey-modification of Lemma 6 we get

$$A_0(\delta) \le CR, \qquad A_{n+1}(\delta) \le CR + C\delta^a A_n(\delta)\big(M + A_n(\delta)\big) \quad \text{for } n \ge 0, \tag{11}$$

where C depends only on (r, s). By induction it follows that $A_n(\delta) \le 2CR$ for all n if $\delta > 0$ is so small that $\delta^a 2C(M + 2CR) \le 1$. Applying the same arguments to

$$B_n(\delta) = \sum_{\pm} \left(\|\psi_\pm^{(n)} - \psi_\pm^{(n-1)}\|_{X_{\pm\xi}^{\sigma_0,s,b}(\delta)} + \|\phi_\pm^{(n)} - \phi_\pm^{(n-1)}\|_{X_{\pm\langle\xi\rangle}^{\sigma_0,r,b'}(\delta)} \right),$$

one finds that $B_{n+1}(\delta) \le C\delta^a B_n(\delta)\big(M + A_n(\delta)\big) \le (1/2)B_n(\delta)$, so the iterates converge and Theorem 1 follows.

7. A more precise local result for $s = 0$, $r = 1/2-$.

Our next aim is to prove Theorem 2. By the embedding (4) it suffices to prove the theorem for a single choice of (r, s). Throughout the rest of the paper we let $\varepsilon > 0$ denote a fixed, small number and we take $s = 0$ and $r = 1/2 - \varepsilon$.

We use $X \lesssim Y$ as shorthand for $X \le CY$, where $C > 0$ is a constant depending only on ε and the Dirac mass M. The notation $X \sim Y$ stands for $X \lesssim Y \lesssim X$. If $a \in \mathbb{R}$, then $a+$ stands for $a + \eta$ where $\eta = \eta(\varepsilon) > 0$ and satisfies $\eta \to 0$ as $\varepsilon \to 0$.

The first step in the proof of Theorem 2 is the following:

Theorem 3. *If $s = 0$ and $r = 1/2 - \varepsilon$, then the existence time in Theorem 1 satisfies $\delta \sim 1/(1+R)^{1+}$, where $R = \sum_\pm (\|f_\pm\|_{G^{\sigma_0,0}} + \|g_\pm\|_{G^{\sigma_0,1/2-\varepsilon}})$.*

From the last section we know that $\delta \sim (1+R)^{-1/a}$, so it suffices to check that the iteration estimate (11) holds with $a = 1-$. In the definition of $A_n(\delta)$ we now take $s = 0$, $r = 1/2 - \varepsilon$ and $b = b' = 1/2 + \varepsilon$.

Observe that by the first estimate in Lemma 5,

$$\|uv\|_{X_{\pm\langle\xi\rangle}^{\sigma_0,-1/2+\varepsilon,-1/2+2\varepsilon}(\delta)} \lesssim \|u\|_{X_{+\xi}^{\sigma_0,0,2\varepsilon}(\delta)} \|v\|_{X_{-\xi}^{\sigma_0,0,2\varepsilon}(\delta)}, \tag{12}$$

$$\|vw\|_{X_{+\xi}^{\sigma_0,0,-2\varepsilon}(\delta)} \lesssim \|v\|_{X_{-\xi}^{\sigma_0,0,2\varepsilon}(\delta)} \|w\|_{X_{\pm\langle\xi\rangle}^{\sigma_0,1/2-\varepsilon,1/2-2\varepsilon}(\delta)}. \tag{13}$$

By Lemma 4, Lemma 2 and (13) we now get

$$\begin{aligned}
&\left\| \int_0^t W_{+\xi}(t-t') \left(-M\psi_-^{(n)} + \phi^{(n)}\psi_-^{(n)} \right)(t')\, dt' \right\|_{X_{+\xi}^{\sigma_0,0,b}(\delta)} \\
&\lesssim \left\| M\psi_-^{(n)} \right\|_{X_{+\xi}^{\sigma_0,0,b-1}(\delta)} + \left\| \phi^{(n)}\psi_-^{(n)} \right\|_{X_{+\xi}^{\sigma_0,0,b-1}(\delta)} \\
&\lesssim M\delta^{1/2-\varepsilon} \left\| \psi_-^{(n)} \right\|_{X_{+\xi}^{\sigma_0,0,0}(\delta)} + \delta^{1/2-3\varepsilon} \left\| \phi^{(n)}\psi_-^{(n)} \right\|_{X_{+\xi}^{\sigma_0,0,-2\varepsilon}(\delta)} \\
&\lesssim M\delta^{1/2-\varepsilon} \left\| \psi_-^{(n)} \right\|_{X_{-\xi}^{\sigma_0,0,0}(\delta)} + \delta^{1/2-3\varepsilon} \left\| \phi^{(n)} \right\|_{X^{\sigma_0,r,b}(\delta)} \left\| \psi_-^{(n)} \right\|_{X_{-\xi}^{\sigma_0,0,2\varepsilon}(\delta)} \\
&\lesssim M\delta^{1-2\varepsilon} \left\| \psi_-^{(n)} \right\|_{X_{-\xi}^{\sigma_0,0,b}(\delta)} + \delta^{1/2-3\varepsilon} \left\| \phi^{(n)} \right\|_{X^{\sigma_0,r,b}(\delta)} \delta^{1/2-3\varepsilon} \left\| \psi_-^{(n)} \right\|_{X_{-\xi}^{\sigma_0,0,b}(\delta)}
\end{aligned}$$

and similarly for the reverse signs. Here $\|\phi\|_{X^{\sigma_0,r,b}(\delta)} = \sum_\pm \|\phi_\pm\|_{X^{\sigma_0,r,b}_{\pm\langle\xi\rangle}(\delta)}$; in fact, the two terms on the right-hand side are identical, since $\overline{\phi_+} = \phi_-$.

From (12) with ε replaced by $\varepsilon/2$ we obtain

$$\begin{aligned}
&\left\| \int_0^t W_{\pm\langle\xi\rangle}(t-t')\langle D_x\rangle^{-1} \operatorname{Re}\left(\overline{\psi_+^{(n)}}\psi_-^{(n)}\right)(t')\,dt' \right\|_{X^{\sigma_0,r,b}_{\pm\langle\xi\rangle}(\delta)} \\
&\lesssim \left\| \operatorname{Re}\left(\overline{\psi_+^{(n)}}\psi_-^{(n)}\right)\right\|_{X^{\sigma_0,-1/2-\varepsilon,-1/2+\varepsilon}_{\pm\langle\xi\rangle}(\delta)} \lesssim \left\|\psi_+^{(n)}\right\|_{X^{\sigma_0,0,\varepsilon}_{+\xi}(\delta)} \left\|\psi_-^{(n)}\right\|_{X^{\sigma_0,0,\varepsilon}_{-\xi}(\delta)} \\
&\lesssim \delta^{1-4\varepsilon} \left\|\psi_+^{(n)}\right\|_{X^{\sigma_0,0,b}_{+\xi}(\delta)} \left\|\psi_-^{(n)}\right\|_{X^{\sigma_0,0,b}_{-\xi}(\delta)}.
\end{aligned}$$

Thus (11) holds with $a = 1-6\varepsilon$ and this proves Theorem 3.

8. Growth estimate and almost conservation of charge

Next we estimate the growth in time of

$$\begin{aligned}
M_\sigma(t) &= \|\psi_+(t)\|^2_{G^{\sigma,0}} + \|\psi_-(t)\|^2_{G^{\sigma,0}}, \\
N_\sigma(t) &= \|\phi_+(t)\|_{G^{\sigma,1/2-\varepsilon}} + \|\phi_-(t)\|_{G^{\sigma,1/2-\varepsilon}},
\end{aligned}$$

where $\sigma \in (0,\sigma_0]$ is considered a parameter. By the local theory developed so far we know that $M_\sigma(t)$ and $N_\sigma(t)$ remain finite for times $t \in [-\delta,\delta]$, where

$$\delta = \delta(\sigma) \sim \frac{1}{(1+M_\sigma(0)^{1/2}+N_\sigma(0))^{1+}}. \tag{14}$$

Moreover, from the iteration we know that

$$\sum_\pm \left(\|\psi_\pm\|_{X^{\sigma,0,b}_{\pm\xi}(\delta)} + \|\phi_\pm\|_{X^{\sigma,r,b}_{\pm\langle\xi\rangle}(\delta)} \right) \lesssim M_\sigma(0)^{1/2} + N_\sigma(0), \tag{15}$$

where $r = 1/2-\varepsilon$ and $b = 1/2+\varepsilon$.

Theorem 4. *With hypotheses as in Theorem 3, then for $\sigma \in (0,\sigma_0]$ and $\delta = \delta(\sigma)$ as in (14), then*

$$\begin{aligned}
&\sup_{|t|\le\delta} M_\sigma(t) \le M_\sigma(0) + C\sigma^{1/2-}\delta^{1/2-}\left(M_\sigma(0)^{1/2}+N_\sigma(0)\right)^3, \\
&\sup_{|t|\le\delta} N_\sigma(t) \le N_\sigma(0) + C\delta\,\|\psi_0\|^2_{L^2} + C\sigma^{1/2+}\delta^{1/2-}\left(M_\sigma(0)^{1/2}+N_\sigma(0)\right)^2,
\end{aligned}$$

where C depends only on ε.

It suffices to prove these estimates at the endpoint $t=\delta$. Recall that $r = 1/2-\varepsilon$ and $b = 1/2+\varepsilon$.

We start with N_σ. From Duhamel's formula applied to the third equation in (7), and using also Lemmas 1 and 4 and the low/high frequency estimates (8) and (9), we obtain

$$\begin{aligned}
N_\sigma(\delta) &\le N_\sigma(0) + \sum_{\pm} \left\| \int_0^\delta W_{\pm\langle\xi\rangle}(\delta - t)\langle D_x\rangle^{-1} \operatorname{Re}\left(\overline{\psi_+}\psi_-\right)(t)\,dt \right\|_{G^{\sigma,r}} \\
&\le N_\sigma(0) + 2\delta \left\| P^\sigma_{\mathrm{lo}}\left(\overline{\psi_+}\psi_-\right)\right\|_{L^\infty_t G^{\sigma,r-1}} + \sum_{\pm} C \left\| P^\sigma_{\mathrm{hi}}\left(\overline{\psi_+}\psi_-\right)\right\|_{X^{\sigma,r-1,b-1}_{\pm\langle\xi\rangle}(\delta)} \\
&\le N_\sigma(0) + 2\delta e \left\|\overline{\psi_+}\psi_-\right\|_{L^\infty_t H^{r-1}} + \sum_{\pm} C\delta^{1/2-\varepsilon}\sigma^{1/2+\varepsilon}\left\|\overline{\psi_+}\psi_-\right\|_{X^{\sigma,0,0}_{\pm\langle\xi\rangle}(\delta)} \\
&\le N_\sigma(0) + 2\delta e \left\|\psi_0\right\|^2_{L^2} + 2C\delta^{1/2-\varepsilon}\sigma^{1/2+\varepsilon}\left\|\psi_+\right\|_{X^{\sigma,0,b}_{+\xi}(\delta)}\left\|\psi_-\right\|_{X^{\sigma,0,b}_{-\xi}(\delta)},
\end{aligned}$$

and now (15) implies the claimed estimate for N_σ. Here C changes from line to line but depends only on ε. Moreover, we used (i) $\|f\|_{H^{-1/2-\varepsilon}} \lesssim \|f\|_{L^1}$, (ii) the fact that $\|\psi_\pm\|_{L^\infty_t L^2_x} \le \|\psi_0\|_{L^2}$ by conservation of charge, and (iii) the null form estimate

$$\|uv\|_{X^{0,0}_{\pm\langle\xi\rangle}} \lesssim \|u\|_{X^{0,1/2+\varepsilon}_{+\xi}} \|v\|_{X^{0,1/2+\varepsilon}_{-\xi}}$$

from Lemma 5.

To estimate M_σ we do not apply Duhamel's formula. This would be too crude since it does not take into account the mechanism of conservation of charge, which holds exactly for $\sigma = 0$ and in an approximate way for $\sigma > 0$. To see this we proceed as in the proof of conservation of charge. Applying $e^{\sigma|D_x|}$ to each side of the Dirac equation $(D_t \pm D_x)\psi_\pm = -M\psi_\mp + \phi\psi_\mp$ gives

$$(D_t \pm D_x)\Psi_\pm = (\phi - M)\Psi_\mp + F_\mp,$$

where

$$\Psi_\pm = e^{\sigma|D_x|}\psi_\pm, \qquad F_\pm = e^{\sigma|D_x|}(\phi\psi_\pm) - \phi e^{\sigma|D_x|}\psi_\pm.$$

Since ϕ and M are real-valued we get

$$\begin{aligned}
\frac{d}{dt}M_\sigma(t) &= 2\operatorname{Re}\int \partial_t\Psi_+(t,x)\overline{\Psi_+(t,x)} + \partial_t\Psi_-(t,x)\overline{\Psi_-(t,x)}\,dx \\
&= \int \partial_x\left(-\Psi_+(t,x)\overline{\Psi_+(t,x)} + \Psi_-(t,x)\overline{\Psi_-(t,x)}\right)dx \\
&\quad + 2\operatorname{Re}\int i(\phi(t,x) - M)\left(\Psi_-(t,x)\overline{\Psi_+(t,x)} + \Psi_+(t,x)\overline{\Psi_-(t,x)}\right)dx \\
&\quad + 2\operatorname{Re}\int iF_-(t,x)\overline{\Psi_+(t,x)} + iF_+(t,x)\overline{\Psi_-(t,x)}\,dx \\
&= -2\operatorname{Im}\int F_-(t,x)\overline{\Psi_+(t,x)} + F_+(t,x)\overline{\Psi_-(t,x)}\,dx.
\end{aligned}$$

In the last step we used the fact that we may assume that $\Psi_\pm(t)$ decays to zero at spatial infinity. Indeed, if we want to prove the estimates in Theorem 4 for a given σ, then by the monotone convergence theorem it suffices to prove it for all $\sigma' < \sigma$, and then we get the decay by the Riemann-Lebesgue lemma.

Integration in time yields

$$\begin{aligned} M_\sigma(\delta) &= M_\sigma(0) - 2\,\mathrm{Im} \iint \chi_{[0,\delta]}(t) \left(F_-(t,x)\overline{\Psi_+(t,x)} + F_+(t,x)\overline{\Psi_-(t,x)} \right) \, dx \, dt \\ &\le M_\sigma(0) + 2 \left\| \chi_{[0,\delta]} F_- \right\|_{X_{+\xi}^{0,-1/2+\varepsilon}} \left\| \chi_{[0,\delta]} \Psi_+ \right\|_{X_{+\xi}^{0,1/2-\varepsilon}} \\ &\qquad + 2 \left\| \chi_{[0,\delta]} F_+ \right\|_{X_{-\xi}^{0,-1/2+\varepsilon}} \left\| \chi_{[0,\delta]} \Psi_- \right\|_{X_{-\xi}^{0,1/2-\varepsilon}} \\ &\le M_\sigma(0) + C \left\| F_- \right\|_{X_{+\xi}^{0,-1/2+\varepsilon}(\delta)} \left\| \Psi_+ \right\|_{X_{+\xi}^{0,1/2-\varepsilon}(\delta)} \\ &\qquad + C \left\| F_+ \right\|_{X_{-\xi}^{0,-1/2+\varepsilon}(\delta)} \left\| \Psi_- \right\|_{X_{-\xi}^{0,1/2-\varepsilon}(\delta)} , \end{aligned}$$

where C only depends on ε and we used Lemma 3.

It remains to estimate (the reverse choice of signs is handled similarly)

$$\left\| F_- \right\|_{X_{+\xi}^{0,-1/2+\varepsilon}(\delta)} = \left\| e^{\sigma|D_x|}(\phi\psi_-) - \phi e^{\sigma|D_x|}\psi_- \right\|_{X_{+\xi}^{0,-1/2+\varepsilon}(\delta)} .$$

Here there is a crucial cancellation given by the following symbol estimate.

Lemma 7. *For all $\sigma > 0$, $\xi, \eta \in \mathbb{R}$ and $0 \le \theta \le 1$ we have*

$$\left| e^{\sigma|\xi+\eta|} - e^{\sigma|\eta|} \right| \le (\sigma|\xi|)^\theta \, e^{\sigma|\xi|} e^{\sigma|\eta|}.$$

Proof. If $|\xi + \eta| \ge |\eta|$, the left-hand side equals

$$e^{\sigma|\xi+\eta|} - e^{\sigma|\eta|} \le e^{\sigma(|\xi|+|\eta|)} - e^{\sigma|\eta|} = e^{\sigma|\eta|} \left(e^{\sigma|\xi|} - 1 \right) \le e^{\sigma|\eta|} (\sigma|\xi|)^\theta e^{\sigma|\xi|},$$

where the last inequality follows since $e^x - 1 \le e^x$ and $e^x - 1 \le x e^x$ for $x \ge 0$.

If $|\xi + \eta| < |\eta|$, the left-hand side equals

$$e^{\sigma|\eta|} - e^{\sigma|\xi+\eta|} \le e^{\sigma|\xi+\eta|} \left(e^{\sigma|\xi|} - 1 \right) \le e^{\sigma|\eta|} \left(e^{\sigma|\xi|} - 1 \right) \le e^{\sigma|\eta|} (\sigma|\xi|)^\theta e^{\sigma|\xi|}.$$

□

Combining this with the product estimate

$$\left\| vw \right\|_{X_{+\xi}^{0,-1/2+\varepsilon}} \lesssim \left\| v \right\|_{X_{-\xi}^{0,1/2+\varepsilon}} \left\| w \right\|_{X_{\pm\langle\xi\rangle}^{2\varepsilon,\varepsilon}}$$

from Lemma 5, we obtain

$$\left\| e^{\sigma|D_x|}(\phi\psi_-) - \phi e^{\sigma|D_x|}\psi_- \right\|_{X_{+\xi}^{0,-1/2+\varepsilon}} \lesssim \sigma^\theta \left\| \phi \right\|_{X^{\sigma,2\varepsilon+\theta,\varepsilon}} \left\| \psi_- \right\|_{X_{-\xi}^{\sigma,0,1/2+\varepsilon}} \tag{16}$$

for any $\theta \in [0,1]$, and this implies the same estimate restricted to the time slab $(-\delta, \delta) \times \mathbb{R}$. Taking $\theta = 1/2 - 3\varepsilon$ and using also Lemma 2 we finally conclude that

$$M_\sigma(\delta) \le M_\sigma(0) + C\sigma^{1/2-}\delta^{1/2-} \left\| \phi \right\|_{X^{\sigma,r,b}(\delta)} \left\| \psi_+ \right\|_{X_{+\xi}^{\sigma,0,b}(\delta)} \left\| \psi_- \right\|_{X_{-\xi}^{\sigma,0,b}(\delta)} ,$$

where C only depends on ε. By (15) we then obtain the claimed estimate for M_σ, and this concludes the proof of Theorem 4.

Remark 2. If we had used Duhamel's formula directly to estimate M_σ, we would have ended up having to estimate

$$I = \left\| e^{\sigma|D_x|}(\phi\psi_-) \right\|_{X_{+\xi}^{0,-1/2+\varepsilon}}$$

instead of the left-hand side of (16). For the high frequency part $P_{\mathrm{hi}}^\sigma \phi$ of ϕ there is no problem estimating I, since obviously $e^{\sigma|\xi+\eta|} \le (\sigma|\xi|)^\theta \, e^{\sigma|\xi|} e^{\sigma|\eta|}$ when $\sigma|\xi| > 1$, so in this case I is dominated by the right-hand side of (16). Thus it is for the low frequency part of ϕ that the cancellation in the left-hand side of (16) is essential.

Remark 3. Another approach to estimating I directly could conceivably be to split into low and high frequencies for the product; then for the high frequency part of the product we have the symbol estimate $e^{\sigma|\xi+\eta|} \le (\sigma|\xi+\eta|)^\theta e^{\sigma|\xi|}e^{\sigma|\eta|}$ and to get the desired result we would need the bilinear estimate

$$\|vw\|_{X_{+\xi}^{\theta,-1/2+\varepsilon}} \lesssim \|v\|_{X_{-\xi}^{0,1/2+\varepsilon}} \|w\|_{X_{\pm\langle\xi\rangle}^{1/2-\varepsilon,\varepsilon}}$$

with $\theta = 1/2-$, but in fact this estimate fails for $\theta > \varepsilon$ by [5, Theorem 1].

9. Conclusion of the proof

All the tools required to complete the proof of the main result, Theorem 2, are now at hand.

We are given $\sigma_0 > 0$ and data such that $M_{\sigma_0}(0)$ and $N_{\sigma_0}(0)$ are finite. Our task is to prove that for any large time T, the solution has a positive radius of analyticity $\sigma(T) \sim 1/T^{5+}$ on the whole time interval $[-T, T]$. In fact we show this for $[0, T]$; by time reversal the same argument applies also to $[-T, 0]$.

Since we are interested in the behaviour as $T \to \infty$, we may assume

$$M_{\sigma_0}(0)^{1/2} + N_{\sigma_0}(0) \le C_1 T, \tag{17}$$

where $C_1 = C(1 + \|\psi_0\|_{L^2}^2)$ with C the constant from Theorem 4. Thus C_1 depends on ε and the initial charge. Now fix T and let $\sigma \in (0, \sigma_0]$ be a parameter to be chosen. Then obviously (17) holds also with σ_0 replaced by σ.

As long as $M_\sigma(t)^{1/2} + N_\sigma(t) \le M_\sigma(0)^{1/2} + N_\sigma(0) + 3C_1T$ we can apply the local results, Theorems 3 and 4, with a uniform time step (here we use (17) and the assumption $\sigma \le \sigma_0$)

$$\delta \sim \frac{1}{(C_1T)^{1+}},$$

which we can choose so that T/δ is an integer. Proceeding inductively we cover intervals $[(n-1)\delta, n\delta]$ for $n = 1, 2, \dots$, obtaining

$$M_\sigma(n\delta) \le M_\sigma(0) + nC\sigma^{1/2-}\delta^{1/2-}(4C_1T)^3,$$
$$N_\sigma(n\delta) \le N_\sigma(0) + nC_1\delta + nC\sigma^{1/2+}\delta^{1/2-}(4C_1T)^2,$$

and in order to reach the time $T = n\delta$ we require that

$$nC\sigma^{1/2-}\delta^{1/2-}(4C_1T)^3 \le (C_1T)^2,$$
$$nC_1\delta \le C_1T,$$
$$nC\sigma^{1/2+}\delta^{1/2-}(4C_1T)^2 \le C_1T.$$

The second condition holds trivially while the first and third both reduce to (using $T = n\delta$ and $\delta \sim 1/(C_1T)^{1+}$)

$$\sigma^{1/2-}T^{5/2+} \le C_2,$$

where C_2 depends on ε and the initial charge. Thus we can take

$$\sigma = \frac{C_3}{T^{5+}}$$

where C_3 again depends on ε and the initial charge. This completes the proof of Theorem 2 for the case $m > 0$.

10. The case $m = 0$

If $m = 0$, then we add ϕ to each side of the second equation in (1), hence the last two equations in (7) are replaced by

$$(D_t \pm \langle D_x \rangle)\, \phi_\pm = \mp \langle D_x \rangle^{-1} \left(\phi + \operatorname{Re}\left(\overline{\psi_+} \psi_- \right) \right)$$

This does not affect the conclusion in Theorem 3, but in Theorem 4 we get an extra term in right-hand side of the estimate for N_σ, namely $2\delta \left\| \phi \right\|_{L_t^\infty([0,\delta];H^{-1/2-\varepsilon})}$. But from the energy inequality for the wave equation and conservation of charge we have $\left\| \phi \right\|_{L_t^\infty([-T,T];L^2)} \lesssim T$ for any $T > 0$, where the implicit constant depends on $R = \left\| (\phi_0, \phi_1) \right\|_{L^2 \times H^{-1}} + \left\| \psi_0 \right\|_{L^2}^2$. Thus, in the induction argument in the previous section we just have to replace C_1 by $C_1 T$ throughout, and this results in the condition $\sigma \le C_4 / T^{8+}$, where C_4 depends on ε and R.

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